# Notes for Thesis

## Contents

# 1 Computations: In Progress

## (a) Cohomology Ring of $Z$

### (i) Additive Structure of $H^*(G_2/SO(4);\mathbb{Z})$

Since $Z$ is an $S^2$-bundle over $M = G_2/SO(4)$, we should start with the cohomology ring of $M$. It is shown on page 529 of Hirzebruch and Borel's classic paper on homogeneous spaces that $H^*(M;\mathbb{Z}_2)$ is generated by two elements $u_2, u_3$ in degree $2, 3$ respectively, subject to the relations $u_2^3 = u_3^2$ and $u_3^3 = 0$. Furthermore, the real cohomology is of rank one in degrees zero, four and eight. Poincaré duality shows that the generator in degree four must square to the generator in degree eight; up to orientation this can be taken to be the positive generator. This determines $H^*(M;\mathbb{R})$. Finally, it is known that $H^*(M;\mathbb{Z})$ has no torsion of odd order.

Now we want to determine $H^*(M;\mathbb{Z})$. We know some thing about the first few cohomology groups already. Since $M$ is simply connected, and $H_0(M;G)$ is always free of rank one (hence Ext vanishes), $H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M;\mathbb{Z});\mathbb{Z}) = 0$. Secondly, using the Hurewicz theorem we see that $\pi_2(M) \twoheadrightarrow H_2(M;\mathbb{Z})$. The long exact homotopy sequence of the fibration $SO(4) \to G_2 \to M$ contains the following piece:

$$\ldots \longrightarrow \pi_2(G_2) = 0 \longrightarrow \pi_2(M) \longrightarrow \pi_1(SO(4)) = \mathbb{Z}_2 \longrightarrow \pi_1(G_2) = 0$$

This shows that $\pi_2(M) = \mathbb{Z}_2$ and therefore $H_2(M;\mathbb{Z}) = \mathbb{Z}_2$. The universal coefficients theorem then shows that $H^2(M;\mathbb{Z}) = 0$.

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces a long exact sequence on cohomology. We find

$$0 \to H^2(M;\mathbb{Z}_2) = \mathbb{Z}_2 \to H^3(M;\mathbb{Z}) \xrightarrow{2\cdot} H^3(M;\mathbb{Z}) \to H^3(M;\mathbb{Z}_2) = \mathbb{Z}_2 \to \ldots$$

Thus, the map $H^3(M;\mathbb{Z}) \xrightarrow{2\cdot} H^3(M;\mathbb{Z})$ has kernel $\mathbb{Z}_2$ and cokernel contained in $\mathbb{Z}_2$. The fact that it has no free part or odd torsion then implies that $H^3(M;\mathbb{Z}) = \mathbb{Z}_{2^k}$. In fact, $H^3(M;\mathbb{Z}) = \mathbb{Z}_2$ and therefore the next piece of the long exact sequence yields

$$\ldots \longrightarrow \mathbb{Z}_2 \xrightarrow{0} H^4(M;\mathbb{Z}) \xrightarrow{2\cdot} H^4(M;\mathbb{Z}) \longrightarrow H^4(M;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow \ldots$$

$H^4(M;\mathbb{Z}) = \mathbb{Z} \oplus T_4$ where $T_4$ is the torsion in degree four, which only contains summands of the form $\mathbb{Z}_{2n}$ for $n \in \mathbb{N}$. But if $T_4 \neq \{0\}$, the multiplication map $2\cdot$ cannot be injective, hence $H^4(M;\mathbb{Z}) = \mathbb{Z}$. The map $H^4(M;\mathbb{Z}) \to H^4(M;\mathbb{Z}_2)$ is then onto and the next piece becomes

$$\ldots \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}_2 \xrightarrow{0} H^5(M;\mathbb{Z}) \xrightarrow{2\cdot} H^5(M;\mathbb{Z}) \longrightarrow H^5(M;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow \ldots$$

Thus, the map $2\cdot$ must be an injective map of the pure torsion group $H^5(M;\mathbb{Z})$. But that is only possible if $H^5(M;\mathbb{Z}) = 0$, since there is no odd torsion. We move on to the next degree:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H^6(M;\mathbb{Z}) \xrightarrow{2\cdot} H^6(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

where we used that $H^7(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = 0$ by simple connectedness. This once again establishes that $H^6(M;\mathbb{Z}) = \mathbb{Z}_{2^k}$ for some $k \in \mathbb{N}$ and in fact $H^6(M;\mathbb{Z}) = \mathbb{Z}_2$. We have obtained:

$$H^k(M;\mathbb{Z}) = \begin{cases} 0 & k = 1,2,5,7 \\ \mathbb{Z}_2 & k = 3,6 \\ \mathbb{Z} & k = 0,4,8 \end{cases}$$

### (ii) Ring Structure of $H^*(G_2/SO(4);\mathbb{Z})$

It follows from Poincaré duality that the generator of $H^4(M;\mathbb{Z})$ squares to a generator (without loss of generality, the positive one) of $H^8(M;\mathbb{Z})$. All that remains is to see whether the square of the generator in degree three vanishes or not. In fact, it squares to the generator of $H^6(M;\mathbb{Z})$.

### (iii) Additive Structure of $H^*(Z;\mathbb{Z})$

We know that $Z$ is an $S^2$-bundle over $M$, which is the sphere bundle of an oriented, rank three bundle. This allows us to formulate the Gysin sequence

$$\ldots \longrightarrow H^k(M;\mathbb{Z}) \stackrel{\smile e}{\longrightarrow} H^{k+3}(M;\mathbb{Z}) \stackrel{\pi^*}{\longrightarrow} H^{k+3}(Z;\mathbb{Z}) \longrightarrow H^{k+1}(M;\mathbb{Z}) \longrightarrow \ldots$$

where $e$ is the Euler class of the rank three bundle (which is 2-torsion). It is easy to see that $H^1(Z;\mathbb{Z}) = 0$ (either from the Gysin sequence or simple connectedness). We start in degree two:

$$0 \longrightarrow H^2(M;\mathbb{Z}) = 0 \longrightarrow H^2(Z;\mathbb{Z}) \longrightarrow H^0(M;\mathbb{Z}) \xrightarrow{\smile e} H^3(M;\mathbb{Z}) \longrightarrow \ldots$$

We know that $H^2(Z;\mathbb{Z})$ is not zero, since $Z$ is Kähler. The existence of an injective homomorphism $H^2(Z;\mathbb{Z}) \to \mathbb{Z}$ implies that $H^2(Z;\mathbb{Z}) = \mathbb{Z}$. For degree three, we find

$$\mathbb{Z} \xrightarrow{\smile e} \mathbb{Z}_2 \longrightarrow H^3(Z;\mathbb{Z}) \longrightarrow 0$$

Thus,

$$H^3(Z;\mathbb{Z}) = \begin{cases} 0 & e \neq 0 \\ \mathbb{Z}_2 & e = 0 \end{cases}$$

The next degree is easy:

$$0 \xrightarrow{\smile e} \mathbb{Z} \longrightarrow H^4(Z;\mathbb{Z}) \longrightarrow 0$$

and $H^4(Z;\mathbb{Z}) = \mathbb{Z}$ is forced. In degree five, we find the same ambiguity as in degree three:

$$0 \longrightarrow H^5(Z;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \xrightarrow{\smile e} \mathbb{Z}_2 \longrightarrow \ldots$$

If $e \neq 0$, it generates $H^3(M;\mathbb{Z})$ and therefore squares to the generator of $H^6(M;\mathbb{Z})$, i.e. $\smile e$ is an isomorphism in this case. This shows that

$$H^5(Z;\mathbb{Z}) = \begin{cases} 0 & e \neq 0 \\ \mathbb{Z}_2 & e = 0 \end{cases}$$

Continuing on, we find

$$\mathbb{Z}_2 \xrightarrow{\smile e} \mathbb{Z}_2 \longrightarrow H^6(Z;\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and a similar result:

$$H^6(Z;\mathbb{Z}) = \begin{cases} \mathbb{Z} & e \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & e = 0 \end{cases}$$

Degree seven is simple:

$$0 \longrightarrow H^7(Z;\mathbb{Z}) \longrightarrow 0$$

Next up, we determine

$$0 \xrightarrow{\smile e} \mathbb{Z} \longrightarrow H^8(Z;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

hence $H^8(Z;\mathbb{Z}) = \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. The universal coefficients theorem

$$0 \longrightarrow \mathrm{Ext}(H_7(Z),\mathbb{Z}) \longrightarrow H^8(Z;\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_8(Z),\mathbb{Z}) \longrightarrow 0$$

determines which it is, depending on the Euler class:

$$H^8(Z;\mathbb{Z}) = \begin{cases} \mathbb{Z} & e \neq 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & e = 0 \end{cases}$$

In fact, it is $\mathbb{Z}$. Finally, it is easy to check that $H^9(Z;\mathbb{Z}) = 0$. We conclude:

$$H^k(Z;\mathbb{Z}) = \begin{cases} \begin{cases} 0 & k = \text{odd} \\ \mathbb{Z} & k = \text{even}, \leq 10 \end{cases} & e \neq 0 \\ \\ \begin{cases} 0 & k = 1,7,9 \\ \mathbb{Z}_2 & k = 3,5 \\ \mathbb{Z} & k = 0,2,4,10 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k = 6,8 \end{cases} & e = 0 \end{cases}$$

Now, in our case, Salamon showed that $Z = S(S^2H)$ and furthermore $w_2(S^2H) = \epsilon \neq 0 \in H^2(M;\mathbb{Z}_2)$. It is a general fact that for a rank-3 oriented vector bundle $V$, $e(V) = \beta(w_2(V))$, where $\beta : H^2(M;\mathbb{Z}_2) \to H^3(M;\mathbb{Z})$ is the Bockstein homomorphism. Thus, since we know that $\beta(\epsilon) \neq 0$, we are in the case $e \neq 0$; the cohomology of $Z$ takes on the simplest possible form.

### (iv) Ring Structure of $H^*(Z;\mathbb{Z})$

The Gysin sequence shows that $\pi^* : H^4(M;\mathbb{Z}) \to H^4(Z;\mathbb{Z})$ is an isomorphism, i.e. if we denote the positive generator of $H^4(M;\mathbb{Z})$ by $g_4^M$, then the positive generator of $H^4(Z;\mathbb{Z})$ can be chosen to be $g_4 := \pi^* g_4^M$. Similarly, $\pi^*$ on degree eight is seen to be multiplication by 2, hence we set $g_8 := \frac{1}{2}\pi^* g_8^M$. But at the same time, $g_8^M = (g_4^M)^2$ and therefore naturality of pullback shows that $g_4^2 = 2g_8$.

Since there is no torsion, Poincaré duality yields a *non-degenerate* pairing

$$\begin{aligned} H^k(Z) \times H^{10-k} &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto (\alpha \smile \beta)(\mu) \end{aligned}$$

where $\mu$ is the fundamental class. We therefore have generators $g_{2k} \in H^{2k}(Z;\mathbb{Z})$ which satisfy $g_{2k} \smile g_{10-2k} = g_{10}$. In particular, we find $g_2 g_8 = \frac{1}{2} g_2 g_4^2 = g_4 g_6$ and therefore $g_6 = \frac{1}{2} g_2 g_4$. Now, the cohomology ring is completely determined by specifying the constant $\alpha$ in $g_2^2 = \alpha g_4$. This number is determined by the Chern number $c_1^5(Z)$, since if we write $c_1(Z) = d_2 g_2$ for some $d_2 \in \mathbb{Z}$, we find

$$c_1^5 = \langle d_2^5 g_2^5, Z\rangle = 2d_2^5 \alpha^2 \langle g_2 g_8, Z\rangle = 2d_2^5 \alpha^2$$

For the twistor space over $G_2/SO(4)$ the Hilbert polynomial is given by Semmelman & Weingart. It allows us to deduce that $c_1^5(Z) = 4374 = 2 \cdot 3^7$. $Z$, being a twistor space over a space of dimension $4n = 8$, is a holomorphic contact manifold. This implies that $c_1(Z)$ has divisibility at least $n+1 = 3$, thus $d_2 = 3$ and we find that $\alpha^2 = 9$, i.e. $\alpha = \pm 3$.

**Remark 1.1.** The same arguments go through for $Q$, but in this case one obtains $\alpha = \pm 1$.

The coefficient $\alpha$ can be determined using Pontryagin classes. Salamon showed that $T\pi \cong L^2$ and we saw that $c_1(Z) = 3g_2 = 3c_1(L^2)$, hence $c_1(L^2) = g_2$ and $p_1(L^2) = p_1(T\pi) = g_2^2 = \alpha g_4$. Using the fact that the normal bundle of $S^2 \subset \mathbb{R}^3$ is trivial, we see that as bundles on $S^2$, $TS^2 \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}} \oplus \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}$. We can apply this isomorphism fiberwise to the sphere bundle $Z = S(S^2H)$ to find that $\pi^*(S^2H) \cong T\pi \oplus \underline{\mathbb{R}}$. This implies that $\pi^* p_1(S^2H) = p_1(T\pi) = \alpha g_4$.

**Remark 1.2.** For any $SO(3)$-bundle, it also holds that $p_1 \equiv w_2^2 \mod 4$. This shows that in our case $\alpha$ must be odd.

Thus, the only thing left to do is to calculate $p_1(S^2H)$.

## (b) Chern Classes of $Z$

The Hilbert polynomial argument, using Semmelman & Weingart, shows that $c_1^5 = 4374$, $c_1^3 c_2 = 2106$. Furthermore, the Hodge numbers determine the Chern number $c_1 c_4 = 90$. Furthermore, recall that the cohomology ring of $Z$ depends on just one constant, $\alpha$, for which we found $\alpha^2 = 9$. The value of $\alpha$ can be found from $p_1(S^2H)$, but will not be needed here.

Let $c_k = d_{2k} g_{2k}$ for some $d_{2k} \in \mathbb{Z}$ and $g_{2k}$ the generator in degree $2k$. It is clear that $c_5 = 6g_{10}$ from the Euler characteristic. Using the ring structure, we have $c_1^5 = 2\alpha^2 d_2^5 = 18 \cdot d_2^5 \implies d_2 = 3$. Since $g_2 g_8 = g_{10}$, we then deduce that $c_4 = 30 g_8$ from $c_1 c_4 = 90$. Now, $c_1^3 c_2 = 27 \cdot 2\alpha d_4 = 2106$ shows that $\alpha d_4 = 3 \cdot 13$. Hence $d_4 = \pm 13$, depending on $\alpha$.

The final Chern class $c_3$ can be determined using the second Pontryagin class $p_2(Z)$. Note that it only depends on $\alpha^2$. We decompose the tangent bundle of $Z$ as $TZ = \pi^* TM \oplus T\pi$ where $M = G_2/SO(4)$ is the base space of the twistor fibration and $T\pi$ is the vertical tangent bundle (along the projection $\pi : Z \to M$). Since there is no 2-torsion in degrees $4n$, we have $p(Z) = \pi^* p(M) \cdot p(T\pi)$.

Because $T\pi$ has rank two, $p_1(T\pi) = c_1^2(T\pi)$ and Salamon's work shows that $T\pi \cong L^2$ (in Salamon's notation; $L$ itself is not globally defined) and that $c_1(Z) = 3g_2 = 3c_1(L^2)$, hence $p_1(T\pi) = 9\alpha g_4$. In addition, he shows that $7p_2(M) - p_1^2(M) = 45$ and $7p_1^2(M) - 4p_2(M) = 0$, whence $p_1(M) = \pm 2g_4^M$ and $p_2 = g_8^M$. Hence

$$p_2(Z) = c_1^2(T\pi)\pi^* p_1(M) + \pi^* p_2(M) = \pm 2g_2^2 g_4 + 2g_8 = (\pm 4\alpha + 2)g_8$$

and we find

$$p_2(Z) = 2c_4(Z) - 2c_3(Z)c_1(Z) + c_2^2(Z) = (\pm 4\alpha + 2)g_8$$

## 2 Side Questions

(i) Does the Todd genus of the nearly Kähler manifolds obtained by flipping fibers of twistor spaces always vanish?

(ii) What about the $S^2$-bundles over $G_2/SO(4)$ with $e \neq 0$ but different values of $\alpha$?

## 3 Finished Computations on $Q$

### (a) Is $Q$ 3-Symmetric?

Find $Q$ in some table by Wolf & Gray classifying 3-symmetric spaces. We need to do this to know that it has a canonical nearly Kähler structure. This, in turn, follows from some classification results by Nagy and Butruille which say that nearly Kähler structures come from products of just three families of examples:

(i) Complex 3-dimensional manifolds.

(ii) Twistor spaces of quaternionic Kähler manifolds.

(iii) 3-symmetric spaces.

We suspect $Q$ is an "irreducible" one, and it fails the first two criteria. Thus, it should be 3-symmetric. This is confirmed by page 111 from Wolf & Gray's paper "Homogeneous Spaces Defined by Lie Group Automorphisms, I", which has $G_2/U(2)$ in a list of 3-symmetric spaces. Furthermore, it is indicated that there are four invariant almost complex structures, hence it must be $Q$ (and not $Z$).

### (b) Chern Classes and Numbers of $Q$ with its Standard Complex Structure

$Q$ is a five-dimensional complex manifold; a quadric in $\mathbb{C}\mathrm{P}^6$. The normal bundle sequence

$$0 \longrightarrow TQ \longrightarrow T\mathbb{C}\mathrm{P}^6|_Q \longrightarrow \nu(Q) \longrightarrow 0$$

splits in the complex category due to the existence of Hermitian metrics (which, in turn, exist due to the existence of complex bump functions), i.e. $T\mathbb{C}\mathrm{P}^6|_Q \cong TQ \oplus \nu(Q)$. Let $\iota$ be the inclusion $Q \hookrightarrow \mathbb{C}\mathrm{P}^6$, and recall that $\nu(Q) = \iota^*\mathcal{O}(2)$, because $Q$ is defined by (the zero set of) a quadratic polynomial on $\mathbb{C}^7$, i.e. a section of $\mathcal{O}(2)$. Using naturality of Chern classes, we find

$$\begin{aligned} \iota^* c(\mathbb{C}\mathrm{P}^6) &= c(Q) \smile \iota^* c(\mathcal{O}(2)) \\ (1 + \iota^*\alpha)^7 &= c(Q)(1 + 2\iota^*\alpha) \\ &= (1 + c_1(Q) + \cdots + c_5(Q))(1 + 2\iota^*\alpha) \end{aligned}$$

where $\alpha$ is the standard generator of $H^2(\mathbb{C}\mathrm{P}^n)$. Set $x = \iota^*\alpha$. Looking at this level by level, we find

$$\begin{aligned}
7x &= c_1(Q) + 2x \implies c_1(Q) = 5x \\
21x^2 &= 2c_1(Q)x + c_2(Q) = 10x^2 + c_2(Q) \implies c_2(Q) = 11x^2 \\
35x^3 &= 2c_2(Q)x + c_3(Q) = 22x^3 + c_3(Q) \implies c_3(Q) = 13x^3 \\
35x^4 &= 2c_3(Q)x + c_4(Q) = 26x^4 + c_4(Q) \implies c_4(Q) = 9x^4 \\
21x^5 &= 2c_4(Q)x + c_5(Q) = 18x^5 + c_5(Q) \implies c_5(Q) = 3x^5
\end{aligned}$$

The total Chern class of $Q$ is therefore given by

$$c(Q) = 1 + 5x + 11x^2 + 13x^3 + 9x^4 + 3x^5 \tag{1}$$

Now, we can easily compute Chern numbers by evaluating on $[Q] \in H^{10}(\mathbb{C}\mathrm{P}^6)$. Because $Q$ is a quadric, $[Q] = 2[\mathbb{C}\mathrm{P}^5]$, i.e. the Chern numbers are given by *twice the coefficient of the Chern classes*:

| Chern Class | Chern Number |
|---|---|
| $c_5$ | 6 |
| $c_1^5$ | 6250 |
| $c_1^3c_2$ | 2750 |
| $c_1^2c_3$ | 650 |
| $c_1c_4$ | 90 |
| $c_1c_2^2$ | 1210 |
| $c_2c_3$ | 286 |

Table 1: Chern Numbers of $Q$ with standard complex structure.

This verifies that the first column of table 10 of Grama et al. is indeed $Q$.

## (c) Chern Classes and Numbers of $\mathbb{P}(TS^6)$ and $\mathbb{P}(T^*S^6)$

### (i) The Cohomology Ring of a Projectivized Bundle

This should be done using a computation analogous to Kotschick's paper on positive curvature. The projectivized tangent bundle $\mathbb{C}\mathrm{P}^5 \to \mathbb{P}(TS^6) \to S^6$ has a cohomology ring that can be computed using the Leray-Hirsch theorem, which states the following (for more information, see Bott & Tu, page 269 and onwards):

**Theorem 3.1** (Leray-Hirsch). *Let $E$ be a fiber bundle over $M$ with fiber $F$. Suppose $M$ has a finite, good cover (this is satisfied if $M$ is compact). If there are global cohomology classes $e_1, \ldots, e_n$ on $E$ which, when restricted to each fiber, freely generated the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, \ldots, e_n\}$, i.e.*

$$H^*(E) \cong H^*(M) \otimes \mathbb{Z}[e_1, \ldots, e_n] \cong H^*(M) \otimes H^*(F)$$

Furthermore, this yields a way of defining the Chern classes of $E$. Consider $\mathbb{P}(E)$, where $E$ is a complex vector bundle of rank $n$. It has certain universal bundles over it, namely the pullback bundle $\pi^*E = \{(\ell, v) \mid \pi_{\mathbb{P}(E)}\ell = \pi(v)\}$, i.e. the bundle whose fiber over $\ell_p$ (which maps to $p \in M$ under $\pi_{\mathbb{P}(E)}$) is $E_p$. It has a tautological subbundle $L = \{(\ell, v) \in \pi^*E \mid v \in \ell\}$. Its dual is, as usual, called the hyperplane bundle and denoted by $H$.

Set $y = c_1(H)$. Then $1, y, \ldots, y^{n-1}$ freely generate $H^*(\mathbb{P}(E_p))$, because $y$ restricts to the hyperplane class on each fiber. The Leray-Hirsch theorem tells us that $H^*(\mathbb{P}(E))$ is a free module over $H^*(M)$, generated by $1, y, \ldots, y^{n-1}$. In particular, $y^n$ can be expressed as a linear combination of the lower powers, with coefficients in $H^*(M)$.

**Definition 3.2.** The *Chern classes* of $E$ are the coefficients $c_1(E), \ldots, c_n(E)$ satisfying

$$y^n + c_1(E)y^{n-1} + \cdots + c_n(E) = 0 \qquad c_i(E) \in H^{2i}(M)$$

Thus, the ring structure of $H^*(\mathbb{P}(E))$ is given by

$$H^*(\mathbb{P}(E)) = H^*(M)[y]/\langle y^n + c_1(E)y^{n-1} + \cdots + c_n(E)\rangle$$

### (ii) Application to $\mathbb{P}(TS^6)$

Let $\alpha \in H^6(S^6)$ be an orientation class. Then $c_3(TS^6) = k \cdot \alpha$ for some $k \in \mathbb{Z}$. In fact, since the top Chern number is the Euler characteristic $\chi(S^6) = c_3(S^6) = 2$, i.e. $k = 2$. All the other classes must vanish because the cohomology of $S^6$ is so simple. As discussed before, $S^6$ admits an almost complex structure induced by viewing it as the unit sphere in $\operatorname{Im}\mathbb{O}$. This endows the tangent bundle with the structure of a complex vector bundle. Therefore, we can apply the above discussion to $TS^6$, viewed as a rank 3 complex vector bundle. We conclude:

**Proposition 3.3.** *The cohomology ring of $\mathbb{P}(TS^6)$ is generated by two elements: $x \in H^6(\mathbb{P}(TS^6))$ and $y \in H^2(\mathbb{P}(TS^6))$ which satisfy the relations*

$$x^2 = 0 \qquad y^3 = -2x$$

*$H^{10}(\mathbb{P}(TS^6))$ has positive generator $xy^2$.*

*Proof.* The only thing that we have not yet explained is $x^2 = 0$. But $x = \pi^*\alpha$, where $\pi : \mathbb{P}(TS^6) \to S^6$ is the obvious projection. The relation follows for dimensional reasons. The generator in top degree is $xy^2$, since $x$ is a (by definition positive) generator of the base space cohomology and $y$ is too, for each fiber. □

**Proposition 3.4.** *The total Chern class of $\mathbb{P}(TS^6)$ is given by*

$$c(\mathbb{P}(TS^6)) = 1 + 3y + 3y^2 + 2x + 6xy + 6xy^2$$

*Proof.* Since $T(\mathbb{P}(TM)) \cong T\pi \oplus \pi^*(TM)$ for any almost complex manifold $M$, we have $c(\mathbb{P}(TS^6)) = c(T\pi) \cdot c(\pi^*(TS^6) = c(T\pi) \cdot \pi^* c(S^6)$. Here, $T\pi$ is the vertical tangent bundle. Clearly, $c(\pi^*(TS^6)) = 1 + 2x = 1 - y^3$ by naturality of the Chern class. To calculate $c(T\pi)$, we use the *(relative) Euler sequence*

$$0 \longrightarrow L \longrightarrow \pi^*(TS^6) \longrightarrow L \otimes T\pi \longrightarrow 0$$

As complex bundles, every short exact sequence splits:

$$\pi^*(TS^6) = L \oplus (L \otimes T\pi) \Leftrightarrow H \otimes \pi^*(TS^6) = \underline{\mathbb{C}} \oplus T\pi$$

Therefore we have

$$c(T\pi) = c(\pi^*(TS^6) \otimes H)$$

For any complex vector bundle $E$ and complex line bundle $L$, the Chern classes of $E \otimes L$

are given—using exercise 4.4.6 from Huybrechts' book—by:

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{\operatorname{rank}_{\mathbb{C}} E - j}{i-j} c_j(E) c_1(L)^{i-j}$$

Again, $c(\pi^*(TS^6)) = 1 + 2x$ and $c(H) = 1 + y$, where $kx = c_3(\pi^*(TS^6))$ and of course $y = c_1(H)$. We compute (recall that $\pi : \mathbb{P}(TS^6) \to S^6$ is a rank two complex vector bundle):

$$\begin{aligned} c_1(T\pi) &= 3y \\ c_2(T\pi) &= 3y^2 \\ \implies c(T\pi) &= 1 + 3y + 3y^2 \end{aligned}$$

We conclude:

$$c(\mathbb{P}(TS^6)) = (1 + 3y + 3y^2)(1 + 2x) = 1 + 3y + 3y^2 + 2x + 6xy + 6xy^2$$

□

### (iii) Application to $\mathbb{P}(T^*S^6)$

Since $(T^*S^6) = (TS^6)^*$, its Chern classes are given by $c_k(T^*S^6) = (-1)^k c(TS^6)$, i.e. $c(T^*S^6) = 1 - 2x$.

**Proposition 3.5.** *The cohomology of $\mathbb{P}(T^*S^6)$ is generated by classes $x \in H^6(\mathbb{P}(T^*S^6))$ and $y \in H^2(\mathbb{P}(TS^6))$ which satisfy the relations*

$$x^2 = 0 \qquad\qquad z^3 = 2x$$

*and $H^{10}(\mathbb{P}(T^*S^6))$ has positive generator $xz^2$.*

*Proof.* Identical to the case $\mathbb{P}(TS^6)$. The positive generator is $xz^2$ for the same reasons as before. □

**Proposition 3.6.** *The total Chern class of $\mathbb{P}(T^*S^6)$ is given by*

$$c(\mathbb{P}(T^*S^6)) = 1 + 3z + 3z^2 + 2x + 6xz + 6xz^2$$

*Proof.* $c(\pi^*(T^*S^6)) = 1 + 2x$ and in this case $c(T\pi) = c(\pi^*(T^*S^6) \otimes H)$, but since it is of (complex) rank two, $c_3(T^*S^6)$ plays no role and the outcome is identical to the case $\mathbb{P}(TS^6)$. All in all, we find:

$$c(\mathbb{P}(T^*S^6)) = (1 + 3z + 3z^2)(1 + 2x) = 1 + 3z + 3z^2 + 2x + 6xz + 6xz^2$$

as claimed. □

The Chern numbers are summarized in table 2.

## (d) Flipping the Fibers

### (i) The Fibrations over $S^6$

We want the Chern numbers of $\mathbb{P}(TS^6)$, $\mathbb{P}(T^*S^6)$ and $Q$ with standard structure after flipping the fibers of the fibrations over $S^6$. The fiber has $\dim_{\mathbb{C}} = 2$, hence the orientation is not affected.

| Chern Number | $\mathbb{P}(TS^6)$ | $\mathbb{P}(T^*S^6)$ |
|---|---|---|
| $c_5$ | 6 | 6 |
| $c_1^5$ | $-486$ | 486 |
| $c_1^3c_2$ | $-162$ | 162 |
| $c_1^2c_3$ | 18 | 18 |
| $c_1c_4$ | 18 | 18 |
| $c_1c_2^2$ | $-54$ | 54 |
| $c_2c_3$ | 6 | 6 |

Table 2: Chern Numbers of $\mathbb{P}(TS^6)$.

Recall that we realized $Q$ as $\mathbb{P}(TS^6)$ (and $\mathbb{P}(T^*S^6)$). The Chern numbers were computed by writing $T(\mathbb{P}(TS^6)) = T\pi \oplus \pi^*TS^6$; the Whitney sum formula tells us $c(\mathbb{P}(TS^6)) = c(T\pi)c(\pi^*TS^6) = c(T\pi)(1+2x)$ where $x = \pi^*\alpha$ with $\alpha$ the positive generator of $H^6(S^6)$. Now we can simply flip the fiber, i.e. consider the conjugate complex structure on the fibers. Call the resulting almost complex manifold $R$. Then $TR = \overline{T\pi} \oplus \pi^*TS^6$ and we note that since the fibers have $\dim_{\mathbb{C}} = 2$, the orientation of $R$ is the same as that of $\mathbb{P}(TS^6)$. The Chern class of $\overline{T\pi}$ is $1 - c_1(T\pi) + c_2(T\pi)$, so we find that the total Chern class is

$$c(R) = c(\mathbb{P}(TS^6))\frac{c(\overline{T\pi})}{c(T\pi)} = c(\mathbb{P}(TS^6))\frac{1-3y+3y^2}{1+3y+3y^2}$$

Working out this calculation, we get:

$$c(R) = 1 - 3y + 3y^2 + 2x - 6xy + 6xy^2$$

which is also the result obtained by using $c_k(\overline{T\pi}) = (-1)^k c_k(T\pi)$. Recall the relation $y^3 = -2x$ and the fact that $xy^2$ is the positive generator. Similarly we can flip the fibers of $\mathbb{P}(T^*S^6)$ to obtain the space $S$ which again has Chern numbers that are the same up to sign, determined by the Chern class

$$c(S) = c(\mathbb{P}(T^*S^6))\frac{1-3z+3z^2}{1+3z+3z^2} = 1 - 3z + 3z^2 + 2x - 6xz + 6xz^2$$

For the calculation, remember that $z^3 = 2x$ and $xz^2$ is the positive generator. The Chern numbers of $R$ are those of $\mathbb{P}(T^*S^6)$ and those of $S$ correspond to $\mathbb{P}(TS^6)$.

When flipping the fiber for the standard structure on $Q$, we have to be more careful: The fibers are holomorphic submanifolds but the fibration is not holomorphic in any sense. Calling the fibration $\pi$, we thus have $TQ = T\pi \oplus D$ where $D$ is a complementary complex vector bundle (of rank three).

Recall that $c(Q) = 1 + 5x + 11x^2 + 13x^3 + 9x^4 + 3x^5$, where $x = \iota^*\alpha$ and $\alpha$ is the positive generator of $H^2(\mathbb{C}\mathrm{P}^n)$. Using the Whitney sum formula, we can determine the Chern classes of the subbundles. For the first Chern class, we have the equation

$$5x = c_1(T\pi) + c_1(D)$$

Of course $c_1(T\pi) = d \cdot x$ for some $d \in \mathbb{Z}$. From our earlier discussion on the cohomology of $Q$ we know that $x$ restricts to the positive generator of $H^2$ of the fiber. Since each fiber is just a copy of $\mathbb{C}\mathrm{P}^2$, this means that $c_1(T\pi) = 3x$. Thus, $c_1(D) = 2x$. We proceed similarly for $c_2$:

$$11x^2 = c_2(T\pi) + c_1(T\pi)c_1(D) + c_2(D) \Leftrightarrow 5x^2 = c_2(T\pi) + c_2(D)$$

As before, $x^2$ restricts to a positive generator of the fourth degree cohomology on the fiber and the Chern classes of $\mathbb{C}\mathrm{P}^2$ then tell us $c_2(T\pi) = 3x^2$. Hence $c_2(D) = 2x^2$. It is clear that the $c_3(D) = x^3$ since

$$1 + 5x + 11x^2 + 13x^3 + 9x^4 + 3x^5 = (1 + 3x + 3x^2)(1 + 2x + 2x^2 + x^3)$$

Now, we flip the fiber to obtain the almost complex manifold $P$ which has Chern class

$$c(P) = c(Q)\frac{1 - 3x + 3x^2}{1 + 3x + 3x^2} = 1 - x - x^2 + x^3 + 3x^4 + 3x^5$$

The Chern numbers for the "flipped fibrations" over $S^6$ are shown in table 3. Note that the Chern numbers of $P$ correspond to those of the last column of the first table of Grama et al.—this should be the nearly Kähler structure.

| Chern Number | $P$ | $R$ | $S$ |
|---|---|---|---|
| $c_5$ | 6 | 6 | 6 |
| $c_1^5$ | $-2$ | 486 | $-486$ |
| $c_1^3c_2$ | 2 | 162 | $-162$ |
| $c_1^2c_3$ | 2 | 18 | 18 |
| $c_1c_4$ | $-6$ | 18 | 18 |
| $c_1c_2^2$ | $-2$ | 54 | $-54$ |
| $c_2c_3$ | $-2$ | 6 | 6 |

Table 3: Chern Numbers of the "flipped fibrations" $P, P', Q'$ of the underlying manifold $G_2/U(2)$ over $S^6$.

### (ii) The Fibrations over $G_2/SO(4)$

We start by considering the projectivized tangent bundle of $S^6$. Recall that $c(\mathbb{P}(TS^6)) = 1 + 3y + 3y^2 - y^3 - 3y^4 - 3y^5$ and $y^5$ is $-2$ times the positive generator $xy^2$ of the top degree cohomology. Using polynomial long division one can show that $c(\mathbb{P}(TS^6))$ contains a factor of the form $1 + Cy$ ($C \in \mathbb{Z}$) only for $C = -1$. Hence

$$c(\mathbb{P}(TS^6)) = c(T\pi)c(D) = (1 - y)c(D)$$

which determines $c(D)$ as well. Flipping the fiber to obtain a space $R'$ and keeping in mind that the orientation is now reversed so that $\langle y^5, [R'] \rangle = 2$, we have:

$$c(R') = (1 + y)c(D) = c(\mathbb{P}(TS^6))\frac{1 + y}{1 - y} = 1 + 5y + 11y^2 + 13y^3 + 9y^4 + 3y^5$$

Clearly, the Chern numbers will be those of $Q$ with its standard structure.

Playing the same game for $\mathbb{P}(T^*S^6)$, which has Chern class $c(\mathbb{P}(T^*S^6)) = 1 + 3z + 3z^2 + z^3 + 3z^4 + 3z^5$ ($z$ is twice the positive generator $xz^2$ of top degree cohomology), we find that the fiber has Chern class $c(T\pi) = 1 + z$ and after flipping the fiber (and orientation, which ensures $\langle z^5, [S'] \rangle = -2$) to obtain $S'$, we find

$$c(S') = c(\mathbb{P}(T^*S^6))\frac{1 - z}{1 + z} = 1 + z - z^2 - z^3 + 3z^4 - 3z^5$$

The Chern numbers are easily computed.

We proceed in analogous fashion for the standard structure on $Q$: $c(Q)$ contains a factor $1+Cx$ ($C \in \mathbb{Z}$) only for $C=1$. Hence $c(T\pi) = 1+x$ and flipping the fiber will yield $P'$ with Chern class

$$c(P') = (1+5x+11x^2+13x^3+9x^4+3x^5)\frac{1-x}{1+x} = 1+3x+3x^2-x^3-3x^4-3x^5$$

It is easily checked that $C=1$ (in our earlier notation) is also the only option if one requires $c_5(P') = \chi(Q) = 6$ since we are not changing the homotopy type (this check also works for $R', S'$, of course). Keeping in mind that the orientation is now switched, hence $\langle x^5, [P'] \rangle = -2$, we find the Chern numbers of $\mathbb{P}(TS^6)$.

| Chern Number | $P'$ | $R'$ | $S'$ |
|---|---|---|---|
| $c_5$ | 6 | 6 | 6 |
| $c_1^5$ | $-486$ | 6250 | $-2$ |
| $c_1^3c_2$ | $-162$ | 2750 | 2 |
| $c_1^2c_3$ | 18 | 650 | 2 |
| $c_1c_4$ | 18 | 90 | $-6$ |
| $c_1c_2^2$ | $-54$ | 1210 | $-2$ |
| $c_2c_3$ | 6 | 286 | $-2$ |

Table 4: Chern Numbers of the "flipped fibrations" $P', R', S'$ over $G_2/SO(4)$.

### (e) Summary: The Homogeneous Almost Complex Structures on $Q$

The above computations can be summarized as follows: On the level of Chern numbers (classes?), we have established that there are four distinct possibilities that arise from different almost complex structures on $Q$. I conjecture that this classification and the resulting relations are true on the level of almost complex manifolds (i.e. those with the same Chern numbers *are* the same).
We denote the almost complex manifolds by $\mathbb{P}(TS^6)$, $\mathbb{P}(T^*S^6)$, $Q_{\text{std}}$ and $N$ (where the last one is the associated with the nearly Kähler structure). The following diagram summarizes our calculations:

$$\begin{tikzcd}
Q_{\text{std}} \arrow[rr, leftrightarrow, "\text{flip } S^6\text{-fibration}"] \arrow[d, leftrightarrow, "\text{flip } G_2/SO(4)\text{-fibration}"'] & & N \arrow[d, leftrightarrow, "\text{flip } G_2/SO(4)\text{-fibration}"] \\
\mathbb{P}(T^*S^6) \arrow[rr, leftrightarrow, "\text{flip } S^6\text{-fibration}"'] & & \mathbb{P}(TS^6)
\end{tikzcd}$$

## 4 Finished Computations on $Z$

### (a) Some Chern Numbers of $Z$ via the Hilbert Polynomial

For this, we use the Hilbert polynomial (from Semmelman and Weingart). On page 159, Semmelman and Weingart give the expression

$$\begin{aligned} P(r) &= \frac{1}{120}(r+2)(3r+5)(2r+3)(3r+4)(r+1) \\ &= \frac{3}{20}r^5 + \frac{9}{8}r^4 + \frac{10}{3}r^3 + \frac{39}{8}r^2 + \frac{211}{60}r + 1 \end{aligned}$$

for the Hilbert polynomial of $G_2/SO(4)$ (which is a so-called *Wolf space*). On the other hand, Kotschick & Terzic note that on a complex contact manifold (such as the twistor space $Z$) of complex dimension $2n+1$, there is a line bundle $L$ on $Z$ such that $c_1(Z) = (n+1)c_1(L) =: (n+1)a$. The Hilbert polynomial can then be computed as follows:

$$P(r) = \chi(Z, L^r) = \sum_{i=0}^{2n+1} (-1)^i \dim_{\mathbb{C}} H^i(Z, L^r)$$

The Hirzebruch-Riemann-Roch theorem tells us that

$$P(r) = \langle \mathrm{ch}(L^r)\,\mathrm{td}(Z), [Z] \rangle$$

The Chern characters and Todd classes can be expressed in terms of the Chern classes as follows:

$$\begin{aligned}
\mathrm{ch}_1 &= c_1 \\
\mathrm{ch}_2 &= \frac{1}{2}c_1^2 - c_2(Z) \\
\mathrm{ch}_3 &= \frac{1}{6}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3 \\
\mathrm{ch}_4 &= \frac{1}{24}c_1^4 - \frac{1}{6}c_1^2c_2 + \frac{1}{12}c_2^2 + \frac{1}{6}c_1c_3 - \frac{1}{6}c_4 \\
\mathrm{ch}_5 &= \frac{1}{120}c_1^5 - \frac{1}{24}c_1^3c_2 + \frac{1}{24}c_1c_2^2 + \frac{1}{24}c_1^2c_3 - \frac{1}{24}c_2c_3 - \frac{1}{24}c_1c_4 + \frac{1}{24}c_5 \\
\mathrm{td}_1 &= \frac{1}{2}c_1 \\
\mathrm{td}_2 &= \frac{1}{12}c_1^2 + \frac{1}{12}c_2 \\
\mathrm{td}_3 &= \frac{1}{24}c_1c_2 \\
\mathrm{td}_4 &= -\frac{1}{720}c_1^4 + \frac{1}{180}c_1^2c_2 + \frac{1}{240}c_2^2 + \frac{1}{720}c_1c_3 - \frac{1}{720}c_4 \\
\mathrm{td}_5 &= -\frac{1}{1440}c_1^3c_2 + \frac{1}{480}c_1c_2^2 + \frac{1}{1440}c_1^2c_3 - \frac{1}{1440}c_1c_4
\end{aligned}$$

Note that $\mathrm{ch}(A \otimes B) = \mathrm{ch}\,A \cdot \mathrm{ch}\,B$, i.e. $\mathrm{ch}(L^r) = (\mathrm{ch}\,L)^r$. Since $L$ is rank one, we find

$$\mathrm{ch}\,L = \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a \implies \mathrm{ch}\,L^r = e^{ra}$$

In particular, for the first five terms we have

$$\begin{aligned}
\mathrm{ch}\,L^r &= 1 + ra + \frac{r^2}{2}a^2 + \frac{r^3}{6}a^3 + \frac{r^4}{24}a^4 + \frac{r^5}{120}a^5 + \dots \\
&= 1 + \frac{c_1}{3}r + \frac{c_1^2}{18}r^2 + \frac{c_1^3}{162}r^3 + \frac{c_1^4}{1944}r^4 + \frac{c_1^5}{29160}r^5
\end{aligned}$$

Where we used that in our case $\dim_{\mathbb{C}} Z = 5$ and hence $c_1(Z) = 3a$, i.e. $a = c_1(Z)/3$. For the evaluation of $\mathrm{ch}(L^r)\,\mathrm{td}\,Z$ on $[Z]$, we are interested in the degree-ten component.

In terms of Chern numbers, we find:

$$\begin{aligned}\langle \operatorname{ch}(L^r)\operatorname{td}(Z),[Z]\rangle &= \frac{c_1^5}{29160}r^5 + \frac{c_1^5}{3888}r^4 + \frac{c_1^5+c_1^3c_2}{1944}r^3 + \frac{c_1^3c_2}{432}r^2 \\ &\quad + \frac{1}{2160}\Big(-c_1^5+4c_1^3c_2+3c_1c_2^2+c_1^2c_3-c_1c_4\Big)r \\ &\quad + \frac{1}{1440}\Big(-c_1^3c_2+3c_1c_2^2+c_1^2c_3-c_1c_4\Big)\end{aligned}$$

Matching coefficients with the expression from Semmelman and Weingart, we deduce the values of some Chern numbers:

$$\begin{aligned} c_1^5 &= \frac{29160\cdot 3}{20} = \frac{3888\cdot 9}{8} = 4374 \\ c_1^3c_2 &= \frac{1944\cdot 10}{3} - 4374 = \frac{432\cdot 39}{8} = 2106\end{aligned}$$

The lowest order terms give a relation between three remaining Chern numbers:

$$3c_1c_2^2 + c_1^2c_3 - c_1c_4 = \frac{2160\cdot 211}{60} + 4374 - 8424 = 1440 + 2106 = 3546$$

Note that if we fill in the entries given in the last table of Grama et al, this relation is satisfied.

## (b) The Cohomology Ring of $Z$ and $Q$ is Different

While computing the Chern numbers of $Q$, we showed that for $Q \subset \mathbb{C}\mathrm{P}^6$, $c_1 = 5x$ with $x = \iota^*\alpha$ and $\alpha$ the standard generator of $H^2(\mathbb{C}\mathrm{P}^6)$.

$$c_1^5(Q) = 6250 = 5^5 \cdot x^5[Q] = 5^5 * 2 \implies x^5[Q] = 2$$

We also computed the analogous quantities for $Z$. Recall that $c_1(Z) = 3a$, where $a = c_1(L)$ is a generator of $H^2(G_2/U(2))$. Now, we know from the calculation of (some of) the Chern numbers that

$$c_1^5(Z) = 4374 = 3^5 \cdot a^5[Q] = 3^5 \cdot 18 \implies a^5[Q] = 18$$

Hence the multiplicative structure on the cohomology is not the same. We deduce that $Q$ and $Z$ are not even homotopy equivalent.

## (c) Does the $S^2$-bundle $Z$ come from a complex bundle?

In our case, the answer is no, as explained in Salamon's paper "Quaternionic Kähler Manifolds".

## (d) Chern Classes of $Z$

We know that, additively, $Z$ has the cohomology of $\mathbb{C}\mathrm{P}^5$, i.e. the groups $H^{0,2,4,6,8,10}(Z) \cong \mathbb{Z}$ and the rest vanish.

Call the positive generators $g_2, \ldots, g_{10}$. Set $c_i = d_{2i}g_{2i}$. We will try to figure out the numbers $d_{2i}$ from the information we have on the Chern numbers. One of them is easy: $\langle c_5, [Z]\rangle = d_{10} = \chi(Z) = 6$. For the others, we use the prime decomposition of the Chern numbers.

### (i) Verified Chern Numbers

For example:

$$c_1^5 = d_2^5\langle g_2^5, [Z]\rangle = 4374 = 2\cdot 3^7$$

Since any factor of $d_2$ must appear with a power at least five, we see that $d_2 = 3$ or $d_2 = 1$. But $Z$ is the twistor space of a quaternionic Kähler manifold of dimension 8. Such a space must have first Chern class with divisibility a multiple of 3. (cf. Kotschick & Terzic, page 600). Therefore $d_2 = 3$. The sign is ensured to be positive since $\langle g_2^5, [Z]\rangle$ must be positive, hence 18. We continue in the same fashion:

$$c_1c_4 = 3d_8\langle g_2g_8, [Z]\rangle = 90 \implies d_8\langle g_2g_8, [Z]\rangle = 30$$

By our discussion of the pairing of cohomology classes we know that $\langle g_2g_8, [Z]\rangle = 1$, hence $d_8 = 30$.

$$c_1^3c_2 = 27d_4\langle g_2^3g_4, [Z]\rangle = 2106 \implies d_4\langle g_2^3g_4, [Z]\rangle = 78 = 2\cdot 3\cdot 13$$

This does not give much information since we do not know $g_2^3$ in terms of $g_6$. However, since $g_2^3$ is a positive rational multiple of $[\omega]^3$ and $g_4$ a positive rational multiple of $[\omega]^2$, we see that $d_4$ must be non-negative.

### (ii) Chern Numbers taken from Grama et al.

$$c_1c_2^2 = 3d_4^2\langle g_2g_4^2, [Z]\rangle = 1014 \implies d_4^2\langle g_2g_4^2, [Z]\rangle = 338 = 2\cdot 13^2$$

Since any factor of $d_4$ must appear with power at least two, we see that $d_4 \in \{1, 13\}$.

$$c_1^2c_3 = 9d_6\langle g_2^2g_6, [Z]\rangle = 594 \implies d_6\langle g_2^2g_6, [Z]\rangle = 66 = 2\cdot 3\cdot 11$$

Hence $d_6 \in \{1, 2, 3, 11\}$.

$$c_2c_3 = d_4d_6\langle g_4g_6, [Z]\rangle = 286 \implies d_4d_6 = 286 = 2\cdot 11\cdot 13$$

Since $d_6$ is non-negative, this shows that $d_4$ is too. If $d_4 = 1$, we see that $d_6 = 286$, which is impossible because $d_3\langle g_2^2g_6, [Z]\rangle \geq d_3 = 66$. Therefore, $d_4 = 13$ and $d_6 = 22$. Now, we have determined all the Chern classes; the result can be summarized in a table or by the total Chern class:

$$c(Z) = 1 + 3g_2 + 13g_4 + 22g_6 + 30g_8 + 6g_{10}$$

| Divisibility | Value |
|---|---|
| $d_{10}$ | 6 |
| $d_8$ | 30 |
| $d_6$ | 22 |
| $d_4$ | 13 |
| $d_2$ | 3 |

A posteriori, we also see that $g_2^2 = 3g_4$, $g_2^3 = 6g_6$, $g_2^4 = 18g_8$, $g_2^5 = 18g_{10}$ and $g_4^2 = 2g_8$.

## (e) Chern numbers on $Z$ with Nearly Kähler Structure

Since $Z$ is the twistor space of $G_2/SO(4)$, there is a hyperplane distribution $D$ such that $TZ \cong L \oplus D$ for a line bundle $L$. Let $N'$ denote the almost complex manifold obtained by replacing the complex structure on $Z$ with the nearly Kähler structure by flipping the fibers. Following Kotschick and Terzić (pages 600-601), we see that $TN' \cong L^{-1} \oplus D$. In particular, we find:

**Proposition 4.1.** *The total Chern class $c(N')$ is given by*

$$c(N') = c(Z)\frac{1-g_2}{1+g_2} = 1 + g_2 + g_4 - 6g_6 - 18g_8 - 6g_{10}$$

Proof. By the direct sum decompositions of the tangent bundles, we have $c(N) = (1 - c_1(L))c(D)$ and $c(Z) = (1 + c_1(L))c(D)$. Now recall $c_1(Z) = 3c_1(L)$. Note that $c_1(Z)$ is also $3g_2$, where $g_2$ is the positive generator of $H^2(Z)$. Since there is no torsion, the equality $3c_1(L) = 3g_2$ allows us to conclude $c_1(L) = g_2$. The rest is a computation. □

Computing the corresponding Chern numbers is now rather simple; one should observe that the almost complex structure $N'$ induces the opposite orientation on the underlying smooth manifold, i.e. $g_{10}$ is now a negative rather than a positive generator. The results are as follows:

| Chern Class | Chern Number |
|---|---|
| $c_5$ | 6 |
| $c_1^5$ | −18 |
| $c_1^3c_2$ | −6 |
| $c_1^2c_3$ | 18 |
| $c_1c_4$ | 18 |
| $c_1c_2^2$ | −2 |
| $c_2c_3$ | 6 |

Table 5: Chern Numbers of $Q$ with standard complex structure.

## (f) Consistency Checks for the Chern Classes of $Z$, $N'$

### (i) Pontryagin Classes

Since Pontryagin classes are oriented diffeomorphism invariants, they should be the same for $Z, N'$. $p_1 = c_1^2 - 2c_2$ and $p_2 = 2c_4 - 2c_1c_3 + c_2^2$. We compute them for $Z$ and $N'$.

For $Z$, we have

$$p_1(Z) = (3g_2)^2 - 2(13g_4) = 27g_4 - 26g_4 = g_4$$

and

$$p_2(Z) = 2(30g_8) - 2(3g_2)(22g_6) + (13g_4)^2 = 60g_8 - 132g_2g_6 + 169g_4^2 = 2g_8$$

while for $N'$ we find

$$p_1(N') = g_2^2 - 2g_4 = g_4$$

and

$$p_2(N') = 2(-18g_8) - 2g_2(-6g_6) + g_4^2 = -36g_8 + 12g_2g_6 + g_4^2 = 2g_8$$

which is consistent.

### (ii) Using the Hirzebruch-Riemann-Roch Theorem

Earlier, we computed the Hilbert polynomial of the twistor space and used

$$\begin{aligned}
[\mathrm{ch}(L^r)\,\mathrm{td}(Z)]_5 = &\frac{c_1^5}{29160}r^5 + \frac{c_1^5}{3888}r^4 + \frac{c_1^5 + c_1^3 c_2}{1944}r^3 + \frac{c_1^3 c_2}{432}r^2 \\
&+ \frac{1}{2160}\Big( - c_1^5 + 4c_1^3 c_2 + 3c_1 c_2^2 + c_1^2 c_3 - c_1 c_4 \Big) r \\
&+ \frac{1}{1440}\Big( - c_1^3 c_2 + 3c_1 c_2^2 + c_1^2 c_3 - c_1 c_4 \Big)
\end{aligned}$$

Where the $c_j$'s are the Chern *classes* (not numbers). Using our expression for the Chern class:

$$c(Z) = 1 + 3g_2 + 13g_4 + 22g_6 + 30g_8 + 6g_{10}$$

we get

$$\begin{aligned}
[\mathrm{ch}(L^r)\,\mathrm{td}(Z)]_5 = &\frac{g_2^5 r^5}{120} + \frac{g_2^5 r^4}{16} + \frac{g_2^5 r^3}{8} + \frac{13 g_2^3 g_4 r^3}{72} + \frac{13 g_2^3 g_4 r^2}{16} \\
&- \frac{9 g_2^5 r}{80} + \frac{13 g_2^3 g_4 r}{20} + \frac{169 g_2 g_4^2 r}{240} + \frac{11 g_2^2 g_6 r}{120} - \frac{g_2 g_8 r}{24} \\
&- \frac{39 g_2^3 g_4}{160} + \frac{169 g_2 g_4^2}{160} + \frac{11 g_2^2 g_6}{80} - \frac{g_2 g_8}{16} \\
= &\, g_{10}\Bigg( \frac{3r^5}{20} + \frac{9r^4}{8} + \frac{9r^3}{4} + \frac{13r^3}{12} + \frac{39r^2}{8} \\
&\quad - \frac{81r}{40} + \frac{39r}{10} + \frac{169r}{120} + \frac{11r}{40} - \frac{r}{24} \\
&\quad - \frac{117}{80} + \frac{169}{80} + \frac{33}{80} - \frac{1}{16} \Bigg) \\
= &\, g_{10}\left( \frac{3}{20}r^5 + \frac{9}{8}r^4 + \frac{10}{3}r^3 + \frac{39}{8}r^2 + \frac{211}{60}r + 1 \right)
\end{aligned}$$

which precisely yields the expression for the Hilbert polynomial given by Semmelman and Weingart. Now, we can compute the same polynomial in the Chern classes for the nearly Kähler structure, using

$$c(N') = 1 + g_2 + g_4 - 6g_6 - 18g_8 - 6g_{10}$$

We find

$$\begin{aligned}
&g_{10}\Bigg( \frac{1}{1620}r^5 + \frac{1}{216}r^4 + \frac{1}{81}r^3 + \frac{1}{72}r^2 \\
&+ \left( - \frac{1}{120} + \frac{1}{90} + \frac{1}{360} - \frac{1}{120} + \frac{1}{120} \right) r \\
&+ \left( - \frac{1}{240} + \frac{1}{240} - \frac{1}{80} + \frac{1}{80} \right) \Bigg) \\
=&g_{10}\left( \frac{1}{1620}r^5 + \frac{1}{216}r^4 + \frac{1}{81}r^3 + \frac{1}{72}r^2 + \frac{1}{180}r \right)
\end{aligned}$$

The main point of this calculation is actually the constant terms of these polynomials. This is simply the *Todd genus* of $Z$ or $N'$. On $Z$, it is known to equal 1 while with regards to $N'$, we can say that Kotschick observed that on many examples of flag manifolds equipped with a nearly Kähler structure coming from a twistor space construction, the Todd genus vanishes. This is all in accordance with our above calculations.