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Special Kähler Manifolds

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Abstract: We give an intrinsic definition of the special geometry which arises in global $N = 2$ supersymmetry in four dimensions. The base of an algebraic integrable system exhibits this geometry, and with an integrality hypothesis any special Kähler manifold is so related to an integrable system. The cotangent bundle of a special Kähler manifold carries a hyperkähler metric. We also define special geometry in supergravity in terms of the special geometry in global supersymmetry.

Constraints on Riemannian metrics occur in many places in supersymmetry. For example, the requirement of extended supersymmetry in a two dimensional σ-model constrains the target manifold to be Kähler or hyperkähler depending on the amount of supersymmetry. The scalars in supergravity theories are often constrained to live on a particular homogeneous Riemannian manifold. These sorts of special metrics – metrics with restricted holonomy group (such as Kähler and hyperkähler metrics) and homogeneous metrics – are much studied by Riemannian geometers, but there are situations in which we meet something new. One important example occurs in four dimensional gauge theories with $N = 2$ supersymmetry: the scalars in the vector multiplet lie in a *special Kähler manifold*. This is the case pertaining to global supersymmetry; when coupled to $N = 2$ supergravity in four dimensions the scalars lie in a *projective special Kähler manifold*.[1] Notice that $N = 1$ supersymmetry already constrains the scalars to lie

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[1] Physicists use the term “special Kähler manifold” for both cases, and use words like “rigid” and “local” to distinguish them. Since these words have other connotations in geometry, we adopt a different terminology.

in a Kähler manifold, which must be Hodge in the supergravity case. Special geometry is the additional constraint imposed by $N = 2$ supersymmetry.

Special geometry appeared in the physics literature in 1984 in both global supersymmetry [ST,G] and supergravity [WP]. Strominger [St] gave a coordinate-free definition in the supergravity case. Projective special Kähler manifolds are important in mirror symmetry, as explained by Candelas and de la Ossa [CO]. Special Kähler manifolds in global supersymmetry have received more attention recently due to their prominent role in the seminal work of Seiberg and Witten on $N = 2$ supersymmetric Yang–Mills theories [SW1,SW2]. See [F,CRTP] for recent discussions of special geometry and for extensive references.

In this paper we introduce an *intrinsic*[2] definition of special geometry: A special Kähler structure is a flat connection on the *tangent bundle* of a Kähler manifold. The crucial condition is expressed in (1.2). From it follow the usual equations for special coordinates, the holomorphic prepotential, the Kähler potential, etc. We recount this in Sect. 1, where we also define this geometry in terms of a holomorphic cubic form. In Sect. 2 we construct a hyperkähler metric on the cotangent bundle of a special Kähler manifold. A local version of this result appears in the physics literature [CFG]. It seems likely that there is actually a one parameter family of hyperkähler metrics of which the one we construct is a limiting case (see [SW3]), but we have not pursued that here. In Sect. 3 we prove the assertion made by Donagi and Witten [DW] that with a suitable integrality hypothesis a special Kähler manifold parametrizes an algebraic completely integrable system. As a consequence the total space of an algebraic integrable system carries a hyperkähler metric. The usual definition of a projective special Kähler manifold is based on a particular type of variation of Hodge structure, which was first studied by Bryant and Griffiths [BG]. Our main observation here is that a projective special Kähler structure on a Hodge manifold M of dimension n induces a special *pseudoKähler* structure of Lorentz type on a closely related manifold $\tilde{M}$ of dimension $n + 1$. ($\tilde{M}$ is the total space of the Hodge line bundle with the zero section omitted.) With a suitable integrality hypothesis the associated intermediate Jacobians are an integrable system and carry a hyperkähler metric, results obtained previously [DM2,C]. Finally, in Sect. 5 we make some brief comments on the physics (in the case of global supersymmetry). We explain that supersymmetry combined with the quantization of electric and magnetic charges leads to the conclusion that integrable systems must enter into the low energy description of $N = 2$ supersymmetric gauge theories.

As mentioned above, the base of an algebraic integrable system is a special Kähler manifold. This is, I believe, the proper context for special Kähler geometry. There are many examples of algebraic integrable systems, and hopefully this excuses the paucity of examples presented here.

As mentioned in the footnote on the previous page, our terminology differs from that in the physics literature. We include the following table to aid in translation:

Our Terminology	Physics Literature	
Special Kähler	Rigid Special Kähler	(vector multiplets in global $N = 2$ supersymmetry)
Projective Special Kähler	(Local) Special Kähler	(vector multiplets in $N = 2$ supergravity)

[2] *Intrinsic* geometry concerns the tangent bundle and associated bundles, whereas *extrinsic* geometry involves bundles not constructed directly from the coordinate charts of a manifold. Definitions of special geometry in the physics literature are not intrinsic in this sense.

This paper grew out of a seminar talk explaining [DW], and it had a long gestation period since. During that time I benefited from conversations and lectures by many colleagues, including Jacques Distler, Ron Donagi, Nigel Hitchin, Graeme Segal, Nathan Seiberg, Karen Uhlenbeck, and Edward Witten. From the first version of the paper I received helpful remarks from Vicente Cortés, James Gates, Zhiqin Lu, Simon Salamon, and the referees. I thank them all.

1. Definition and Basic Properties

We introduce the following definition.

Definition 1.1. *Let M be a Kähler manifold with Kähler form ω. A* **special Kähler** *structure on M is a real flat torsionfree symplectic connection ∇ satisfying*

$$d_\nabla I = 0, \tag{1.2}$$

where I is the complex structure on M.

First we examine the consequences of the connection on the underlying real symplectic structure on M. The connection ∇ determines an extension of the de Rham complex

$$0 \longrightarrow \Omega^0(TM) \xrightarrow{d_\nabla = \nabla} \Omega^1(TM) \xrightarrow{d_\nabla} \Omega^2(TM) \xrightarrow{d_\nabla} \cdots. \tag{1.3}$$

The flatness is the condition $d_\nabla^2 = 0$. Note that the Poincaré lemma holds for (1.3): a closed TM-valued form is locally exact. The torsionfree condition may be expressed by

$$d_\nabla(\mathrm{id}) = 0, \tag{1.4}$$

where $\mathrm{id} \in \Omega^1(TM)$ is the identity endomorphism of TM. Now if $\{\xi_\alpha\}$ is a flat local framing of M with dual coframing $\{\theta^\alpha\}$, then (1.4) implies $d\theta^\alpha = 0$, whence $\theta^\alpha = dt^\alpha$ for some local coordinate functions t^α.[3] Since $\nabla\omega = 0$ we can choose these coordinates to be *Darboux*; that is, the coordinate functions are x^i, y_j $(i, j = 1, \dots, n = \dim_{\mathbb{C}} M)$ with

$$\omega = dx^i \wedge dy_i. \tag{1.5}$$

Summarizing, a flat torsionfree symplectic connection ∇ is equivalent to a flat symplectic structure on M. This is a covering by *flat Darboux coordinate systems* $\{x^i, y_j\}$ whose transition functions are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad P \in Sp(2n; \mathbb{R}), \quad a, b \in \mathbb{R}^n. \tag{1.6}$$

(The coordinates are “flat” since $\nabla dx^i = \nabla dy_j = 0$.) Equation (1.5) is valid in any flat Darboux coordinate system.

[3] For simplicity we always choose our coordinate systems to be defined on *connected* open sets, and we allow the domains of the coordinate systems to shrink when necessary.

The compatibility with the complex structure is expressed[4] by (1.2), or equivalently by

$$d_\nabla \pi^{(1,0)} = 0, \tag{1.7}$$

where $\pi^{(1,0)} \in \Omega^{1,0}(T_{\mathbb{C}}M)$ is projection onto the $(1,0)$ part of the complexified tangent bundle. The Poincaré lemma ensures that locally we can find a complex vector field ζ with

$$\nabla \zeta = \pi^{(1,0)}. \tag{1.8}$$

Note that ζ is unique up to a flat complex vector field. Also, ζ is not necessarily holomorphic. Let $\{x^i, y_j\}$ be a flat Darboux coordinate system and write

$$\zeta = \frac{1}{2}\big(z^i \frac{\partial}{\partial x^i} - w_j \frac{\partial}{\partial y_j}\big) \tag{1.9}$$

for some complex functions z^i, w_j. (The choice of sign and the factor '1/2' yield standard formulas for $M = \mathbb{C}^n$.) Since $\pi^{(1,0)}$ has type $(1,0)$, Eq. (1.9) implies that z^i, w_j are holomorphic functions and

$$\pi^{(1,0)} = \frac{1}{2}\big(dz^i \otimes \frac{\partial}{\partial x^i} - dw_j \otimes \frac{\partial}{\partial y_j}\big). \tag{1.10}$$

It follows that

$$\mathrm{Re}(dz^i) = dx^i, \quad \mathrm{Re}(dw_j) = -dy_j. \tag{1.11}$$

In particular, $\{z^i\}$ is a local holomorphic coordinate system on M.[5] We easily compute

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\big(\frac{\partial}{\partial x^i} - \tau_{ij}\frac{\partial}{\partial y_j}\big), \tag{1.12}$$

where

$$\tau_{ij} = \frac{\partial w_j}{\partial z^i}. \tag{1.13}$$

Now the fact that ω has type $(1,1)$ implies that $\tau_{ij} = \tau_{ji}$, and so there is a (local) holomorphic function $\mathfrak{F}$, determined up to a constant, so that

$$w_j = \frac{\partial \mathfrak{F}}{\partial z^j}, \quad \tau_{ij} = \frac{\partial^2 \mathfrak{F}}{\partial z^i \partial z^j}. \tag{1.14}$$

$\mathfrak{F}$ is called the *holomorphic prepotential*. It determines a Kähler potential

$$K = \frac{1}{2}\,\mathrm{Im}\big(\frac{\partial \mathfrak{F}}{\partial z^i}\bar{z}^i\big) = \frac{1}{2}\,\mathrm{Im}(w_i \bar{z}^i), \tag{1.15}$$

[4] We give a characterization in terms of coordinates in Proposition 1.25 below.

[5] Of course, so is $\{w_j\}$. We call $\{z^i\}$ and $\{w_j\}$ *conjugate* coordinate systems (Definition 1.37).

and in terms of this data the Kähler form is

$$\omega = \sqrt{-1}\partial\bar{\partial}K = \frac{\sqrt{-1}}{2}\,\mathrm{Im}\big(\frac{\partial^2\mathfrak{F}}{\partial z^i\partial z^j}\big)dz^i\wedge\overline{dz}^j = \frac{\sqrt{-1}}{2}\,\mathrm{Im}(\tau_{ij})dz^i\wedge\overline{dz}^j. \qquad (1.16)$$

Formulas (1.14)–(1.16) are standard in the literature on special Kähler geometry; they show that our global Definition 1.1 reproduces the usual local characterization. We term $\{z^i\}$ a *special coordinate system*. We characterize special coordinate systems below in Definition 1.37.

Remark 1.17. Condition (1.2) does *not* mean that the complex structure I is flat. Indeed, if $\nabla I = 0$, then M is a flat Kähler manifold, locally isometric to $\mathbb{C}^n$. Such a manifold is special Kähler, but of a very special type. Note that the existence of a flat symplectic structure has nontrivial global topological consequences but gives no local restriction. Equation (1.2), on the other hand, is a stringent local condition.

Remark 1.18. Based on an earlier version of this paper, Zhigin Lu [L] proved that there are no nonflat *complete* special Kähler manifolds.

Remark 1.19. The special Kähler condition (1.7) automatically implies that ∇ is torsionfree, since (1.4) is twice the real part of (1.7).

Remark 1.20. Locally, we may specify a Kähler geometry by giving a holomorphic function $\mathfrak{F}(z^1,\dots,z^n)$ such that

$$\mathrm{Im}\big(\frac{\partial^2\mathfrak{F}}{\partial z^i\partial z^j}\big) > 0$$

is positive definite. The function

$$\mathfrak{F}(z^1,\dots,z^n) = \frac{\sqrt{-1}}{2}\big((z^1)^2+\cdots+(z^n)^2\big)$$

leads to the flat metric on $\mathbb{C}^n$. A nontrivial example in one dimension is provided by the holomorphic function

$$\mathfrak{F}(\tau) = \frac{\tau^3}{6},$$

defined on the upper half plane

$$\mathbb{H} = \{\tau : \mathrm{Im}\,\tau > 0\}.$$

The corresponding Kähler form

$$\omega = \frac{\sqrt{-1}}{2}\,\mathrm{Im}(\tau)d\tau\wedge\overline{d\tau}$$

has Gauss curvature $1/2(\mathrm{Im}\,\tau)^3$. Note that the coordinate conjugate to τ is $w = \partial\mathfrak{F}/\partial\tau = \tau^2/2$. An adapted[6] flat Darboux coordinate system $\{x, y\}$ is $x = \mathrm{Re}\,\tau,\ y = -\mathrm{Re}\,\tau^2/2$. In these coordinates the Riemannian metric is

$$g = \frac{2(x^2+y)\,dx^2 + 2x\,dxdy + dy^2}{\sqrt{x^2+2y}}.$$

It is the Hessian of the function $\phi = \frac{1}{3}(x^2+2y)^{3/2}$; see Proposition 1.24 below. This metric is incomplete; see Remark 1.18.

[6] See Definition 1.37.

Remark 1.21. Nowhere do we use the positive definiteness of ω. Hence our discussion applies also to *pseudo-Kähler* manifolds. (A pseudo-Kähler metric ω is nondegenerate and $d\omega = 0$, but it is not assumed positive definite.)

We have the following easy result.

Proposition 1.22. (a) *Let (M, ω, ∇) be a special Kähler manifold. The connection ∇ determines a horizontal distribution H in the real cotangent bundle T^*M. Then H is invariant under the complex structure of T^*M.*
(b) *The $(0,1)$ part of the connection ∇ on the complex tangent bundle TM equals the $\bar\partial$ operator.*

Proof. (a) Choose a flat Darboux coordinate system $\{x^i, y_j\}$. Then the local 1-forms dx^i, dy_j define sections of $T^*M \to M$ whose image is an integral manifold of H. Since dx^i and dy_j are the real parts of holomorphic differentials (see (1.11)) their graphs are complex submanifolds.

(b) From (1.12) we compute that $\nabla\, \partial/\partial z^i$ is a form of type $(1,0)$:

$$\nabla \frac{\partial}{\partial z^j} = -\frac{1}{2}\frac{\partial \tau_{j\ell}}{\partial z^k} dz^k \otimes \frac{\partial}{\partial y_\ell}. \tag{1.23}$$

Since $\partial/\partial z^i$ is a local basis of holomorphic sections, the desired assertion follows. □

The Riemannian metric has a very special form in flat real coordinates – it is the Hessian of a function. This observation is due to Nigel Hitchin.

Proposition 1.24. *Let (M, ω, ∇) be a special Kähler manifold. Suppose $\{u^\alpha\}$ is a ∇-flat coordinate system. (For example, it may be a flat Darboux coordinate system.) Then the Riemannian metric g is*

$$g = \frac{\partial^2 \phi}{\partial u^\alpha \partial u^\beta}\, du^\alpha \otimes du^\beta$$

for some real function ϕ. In fact, ϕ is a Kähler potential.

Proof. In these coordinates the symplectic form $\omega = \frac{1}{2}\omega_{\alpha\beta}\, du^\alpha \wedge du^\beta$ has constant coefficients. Now

$$g_{\alpha\beta} = \omega_{\alpha\gamma} I^\gamma_\beta$$

and the special Kähler condition (1.2) implies

$$\frac{\partial g_{\alpha\beta}}{\partial u^\gamma} = \frac{\partial g_{\alpha\gamma}}{\partial u^\beta}.$$

Hence $g_{\alpha\beta} = \partial\phi_\alpha/\partial u^\beta$ for some function ϕ_α. The symmetry of $g_{\alpha\beta}$ now implies that $\phi_\alpha = \partial\phi/\partial u^\alpha$ for some ϕ, as desired.

To see that ϕ is a Kähler potential, we compute

$$\begin{aligned}
\sqrt{-1}\,\partial\bar\partial\phi &= -\frac{1}{2} dI d\phi \\
&= -\frac{1}{2} d\big(\frac{\partial\phi}{\partial u^\alpha} I^\alpha_\gamma du^\gamma\big) \\
&= -\frac{1}{2}\big(\frac{\partial^2\phi}{\partial u^\alpha \partial u^\beta} I^\alpha_\gamma + \frac{\partial\phi}{\partial u^\alpha}\frac{\partial I^\alpha_\gamma}{\partial u^\beta}\big) du^\beta \wedge du^\gamma \\
&= -\frac{1}{2} g_{\alpha\beta} I^\alpha_\gamma du^\beta \wedge du^\gamma \\
&= \omega.
\end{aligned}$$

We use the special Kähler condition to pass from the third line to the fourth. $\square$

We next express the special Kähler condition (1.2) in terms of coordinates.

Proposition 1.25. *Let (M, ω) be a Kähler manifold of dimension n and ∇ a flat torsion-free symplectic connection. Suppose $\{z^i\}$ is any local holomorphic coordinate system on M and $\{x^i, y_j\}$ a flat Darboux coordinate system. Write*

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\big(\sigma_i^j \frac{\partial}{\partial x^j} - \tau_{ij} \frac{\partial}{\partial y_j}\big)$$

for functions σ_i^j, τ_{ij}. Then $d_\nabla I = 0$ if and only if σ_i^j, τ_{ij} are holomorphic functions of $z^1, \dots, z^n$ and

$$\frac{\partial \sigma_i^j}{\partial z^k} = \frac{\partial \sigma_k^j}{\partial z^i}, \quad \frac{\partial \tau_{ij}}{\partial z^k} = \frac{\partial \tau_{kj}}{\partial z^i}.$$

The proof is straightforward: Compute $d_\nabla(\pi^{(1,0)}) = d_\nabla\big(dz^i \otimes \frac{\partial}{\partial z^i}\big)$. Notice that τ_{ij} is not necessarily symmetric, but rather $\tau_{ik}\sigma_j^k$ is symmetric in i, j.

There is a holomorphic cubic form Ξ on a special Kähler manifold which encodes the extent to which ∇ fails to preserve the complex structure. Namely, set

$$\Xi = -\omega\big(\pi^{(1,0)}, \nabla\pi^{(1,0)}\big) \in H^0(M, Sym^3 T^*M). \tag{1.26}$$

That Ξ is symmetric follows from the fact that ω is skew-symmetric, $\nabla\omega = 0$, and the special Kähler condition (1.7) (which says that $\nabla\pi^{(1,0)}$ is symmetric). The holomorphicity follows from the computation (1.28) below. Note the alternative local expression

$$\Xi = -\omega(\nabla\zeta, \nabla^2\zeta), \tag{1.27}$$

where ζ is a local complex vector field satisfying (1.8). We compute (1.26) in special coordinates $\{z^i\}$ introduced above. From (1.23) and the fact that ω has type $(1,1)$, we have

$$\begin{aligned}\Xi &= -\omega\big(dz^i \otimes \frac{\partial}{\partial z^i}\ ,\ \nabla(dz^j \otimes \frac{\partial}{\partial z^j})\big) \\ &= -dz^i \otimes dz^j \ \ \omega\big(\frac{1}{2}(\frac{\partial}{\partial x^i} - \tau_{im}\frac{\partial}{\partial y_m})\ ,\ -\frac{1}{2}\frac{\partial \tau_{j\ell}}{\partial z^k}\ dz^k \otimes \frac{\partial}{\partial y_\ell}\big) \\ &= \frac{1}{4}\frac{\partial \tau_{ji}}{\partial z^k}\ dz^i \otimes dz^j \otimes dz^k \\ &= \frac{1}{4}\frac{\partial^3 \mathfrak{F}}{\partial z^i \partial z^j \partial z^k}\ dz^i \otimes dz^j \otimes dz^k.\end{aligned} \tag{1.28}$$

Here we use (1.14) as well. The cubic form Ξ can also be used[7] to relate the special Kähler connection ∇ to the Levi–Civita connection D. Write

$$\nabla = D + A_{\mathbb{R}}, \tag{1.29}$$

[7] I learned this from the account in [BCOV], though it also appears in many other works.

where $A_{\mathbb{R}} \in \Omega^1(M, \mathrm{End}_{\mathbb{R}} TM)$. Then since $D\pi^{(1,0)} = 0$, we have

$$\Xi = -\omega(\pi^{(1,0)}, [A_{\mathbb{R}}, \pi^{(1,0)}]). \tag{1.30}$$

Moreover, there is a complex tensor

$$A \in \Omega^{1,0}\big(\mathrm{Hom}(TM, \overline{TM})\big) \tag{1.31}$$

with

$$A_{\mathbb{R}} = A + \overline{A}.$$

To see this, note from (1.29) and Proposition 1.22(ii) that A_ξ vanishes on vectors of type $(0,1)$ if ξ is of type $(1,0)$. Then, since A_ξ is infinitesimal symplectic, for ζ of type $(1,0)$ and $\bar\eta$ of type $(0,1)$, we have

$$\omega(A_\xi\zeta, \bar\eta) = -\omega(\zeta, A_\xi\bar\eta) = 0.$$

Since ω has type $(1,1)$, this implies that $A_\xi\zeta$ is of type $(0,1)$. Therefore, A is as claimed in (1.31). Furthermore, A and Ξ determine each other. In particular, we recover the special Kähler structure from Ξ.

Conversely, we can start with a smooth cubic form $\Xi \in C^\infty(M, Sym^3T^*M)$ and ask for the conditions on Ξ which ensure that ∇ as defined by (1.29) and (1.30) is a special Kähler structure. Note 'T^*M' denotes the complex tangent bundle; we assume Ξ to be complex multilinear. The symmetry of Ξ implies that ∇ is symplectic, torsionfree, and satisfies (1.2). Setting the curvature of ∇ to zero from (1.29) yields the equation

$$0 = R + d_DA + A \wedge \overline{A} + \overline{A} \wedge A,$$

where R is the curvature of the Kähler metric on M. Here 'd_D' is the alternation of the Levi–Civita covariant derivative. Notice that as endomorphisms of the tangent bundle $R + A\wedge\overline{A} + \overline{A}\wedge A$ is complex linear, whereas d_DA is complex antilinear (1.31); whence these separately vanish. The (1,1) piece of d_DA is $\bar\partial A$, from which it follows that Ξ is holomorphic. The remaining equations are

$$\begin{aligned} \partial_D A &= 0, \\ R &= -(A \wedge \overline{A} + \overline{A} \wedge A). \end{aligned} \tag{1.32}$$

In any local coordinate system $\{z^i\}$ we write

$$\begin{aligned} \omega &= \sqrt{-1}\, h_{i\bar j}\, dz^i \wedge \overline{dz^j}, \\ R &= \left(R^j_{ik\bar\ell}\right)_{i,j}\, dz^k \wedge \overline{dz^\ell}, \\ \Xi &= \Xi_{ijk}\, dz^i \otimes dz^j \otimes dz^k, \\ (A_i)^{\bar k}_j &= \sqrt{-1}\, \Xi_{ij\ell} h^{\ell\bar k}. \end{aligned}$$

As usual, set $R_{i\bar j k\bar\ell} = h_{m\bar j} R^m_{ik\bar\ell}$. Then (1.32) is

$$\begin{aligned} D_iA_j &= D_jA_i, \\ R_{i\bar jk\bar\ell} &= -h^{\alpha\bar\beta}\, \Xi_{ik\alpha}\, \overline{\Xi_{j\ell\beta}}. \end{aligned} \tag{1.33}$$

We summarize this discussion as follows.

Proposition 1.34. (a) *If (M,ω,∇) is a special Kähler manifold, then there is an associated holomorphic cubic form $\Xi \in H^0(M, Sym^3T^*M)$, defined in* (1.26), *which satisfies* (1.32).

(b) *If (M,ω) is a Kähler manifold and $\Xi \in H^0(M, Sym^3T^*M)$ holomorphic cubic form which satisfies* (1.32), *then $\nabla = D + A$ is a special Kähler structure, where D is the Levi–Civita connection and A is defined from Ξ by* (1.30).

Remark 1.35. Lu [L] noticed that as a consequence of (1.33) any special Kähler manifold M has nonnegative scalar curvature ρ:

$$\begin{aligned}\rho &= -4h^{i\bar{j}}h^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}\\ &= 4h^{i\bar{j}}h^{k\bar{\ell}}h^{\alpha\bar{\beta}}\,\Xi_{ik\alpha}\overline{\Xi_{j\ell\beta}}\\ &= 4|\Xi|^2.\end{aligned}\tag{1.36}$$

Then he computes $\triangle\rho$ and uses a maximum principle to argue that if M is complete, then $\rho = 0$, from which Ξ and then R vanish.

Next, we discuss special coordinates.

Definition 1.37. *Let (M,ω,∇) be a special Kähler manifold.*

(a) *A holomorphic coordinate system $\{z^i\}$ is* **special** *if $\nabla\,\mathrm{Re}(dz^i) = 0$.*
(b) *We say that special coordinates $\{z^i\}$ and flat Darboux coordinates $\{x^i, y_j\}$ are* **adapted** *if $\mathrm{Re}(z^i) = x^i$.*
(c) *Special coordinate systems $\{z^i\}$, $\{w_j\}$ are said to be* **conjugate** *if there exists a flat Darboux coordinate system $\{x^i, y_j\}$ such that $\mathrm{Re}(z^i) = x^i$ and $\mathrm{Re}(w_j) = -y_j$.*

Given adapted special coordinates $\{z^i\}$ and flat Darboux coordinates $\{x^i, y_j\}$, conjugate special coordinates $\{w_j\}$ are determined up to translation by a purely imaginary constant. For adapted coordinate systems we have Eqs. (1.9)–(1.16), but note that $\{w_j\}$, τ_{ij}, $\mathfrak{F}$, and K are not completely determined by $\{z^i\}$ and $\{x^i, y_j\}$. The following proposition clarifies the choices involved.

Proposition 1.38. *Let (M,ω,∇) be a special Kähler manifold.*

(a) *Given a flat Darboux coordinate system $\{x^i, y_j\}$ there exists an adapted special coordinate system $\{z^i\}$. Any two choices $\{z^i\}$, $\{\tilde{z}^i\}$ satisfy $z^i = \tilde{z}^i + c^i$ for some purely imaginary constants c^i.*
(b) *Given a special coordinate system $\{z^i\}$ there exists an adapted flat Darboux coordinate system $\{x^i, y_j\}$. Any two choices differ by a change of variables*

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 0\\ A & 1\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} + \begin{pmatrix}0\\ b\end{pmatrix},$$

where A is a (real) symmetric matrix and $b \in \mathbb{R}^n$.
(c) *Given a special coordinate system $\{z^i\}$ the holomorphic prepotential $\mathfrak{F}$ is determined up to a change*

$$\mathfrak{F} \longrightarrow \mathfrak{F} + \frac{1}{2}A_{ij}z^iz^j + B_iz^i + C,$$

where $A = (A_{ij})$ is a real symmetric matrix, and $B_i, C \in \mathbb{C}$. So the conjugate coordinate system $\{w_j\}$ is determined up to a change

$$w_j \longrightarrow w_j + A_{jk}z^k + B_j$$

and the Kähler potential (1.15) *is determined up to a change*

$$K \longrightarrow K + \mathrm{Im}(B_i \bar{z}^i).$$

(d) *If $\{z^i\}$, $\{w_j\}$ are conjugate special coordinate systems, then any other pair $\{\tilde{z}^i\}$, $\{\tilde{w}_j\}$ of conjugate special coordinate systems are related by*

$$\begin{pmatrix} z \\ w \end{pmatrix} = P \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad P \in Sp(2n;\mathbb{R}), \quad a, b \in \mathbb{C}^n. \tag{1.39}$$

The corresponding matrices τ, $\tilde{\tau}$ are related by

$$\tau = (D\tilde{\tau} + C)(B\tilde{\tau} + A)^{-1}, \tag{1.40}$$

where $P = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

2. The Associated Hyperkähler Manifold

In this section we prove the following theorem, which (in local form) is due to Cecotti, Ferrara, and Girardello [CFG].[8]

Theorem 2.1. *The cotangent bundle T^*M of a special Kähler manifold (M, ω, ∇) carries a canonical hyperkähler structure.*

Recall that a Riemannian manifold (Y, g) is hyperkähler if it carries a triple of *integrable* almost complex structures I, J, K which satisfy the quaternion algebra and such that the associated 2-forms

$$\omega_T(\xi_1, \xi_2) = g(\xi_1, T\xi_2), \quad T = I, J, K, \tag{2.2}$$

are **closed**. A useful lemma of Hitchin [H, p. 64] asserts that if $\omega_I, \omega_J, \omega_K$ are closed, then I, J, K are integrable. If we consider (Y, ω_I) as a Kähler manifold with complex structure I, then

$$\eta = \omega_J + i\omega_K \tag{2.3}$$

is a holomorphic symplectic form.

[8] Equation (B.7) in [CFG] corresponds to our description of the metric in (2.4), where their Z^I are special coordinates on M and $\{Z^I, W_J\}$ the induced coordinate system on T^*M. Then (B.8b) describes the flat connection ∇.

Proof. Consider first a hermitian vector space V with complex structure I. The hermitian metric $\langle\cdot,\cdot\rangle$ determines a metric and symplectic form on the underlying real vector space $V_{\mathbb{R}}$:

$$\langle\xi_1,\xi_2\rangle = g(\xi_1,\xi_2) + i\omega(\xi_1,\xi_2), \quad \xi_1,\xi_2 \in V_{\mathbb{R}}.$$

Then $W = V \oplus V^* \cong V \oplus \overline{V}$ has a constant hyperkähler structure. The complex structure J is the antilinear map

$$\begin{aligned} J : V \oplus \overline{V} &\longrightarrow V \oplus \overline{V}, \\ v_1 \oplus \overline{v_2} &\longmapsto -v_2 \oplus \overline{v_1}. \end{aligned}$$

Now define $K = IJ$. Then I, J, K satisfy the quaternion algebra. The metric on $W_{\mathbb{R}}$ is

$$g_W(\xi_1 \oplus \alpha_1, \xi_2 \oplus \alpha_2) = g(\xi_1,\xi_2) + g^{-1}(\alpha_1,\alpha_2), \quad \xi_1,\xi_2 \in V_{\mathbb{R}}, \quad \alpha_1,\alpha_2 \in V_{\mathbb{R}}^*. \tag{2.4}$$

The forms $\omega_I, \omega_J, \omega_K$ are now determined by (2.2). It is straightforward to check that the holomorphic symplectic form η defined in (2.3) is the canonical form on $W = V \oplus V^* \cong T^*V$:

$$\eta(v_1 \oplus \ell_1, v_2 \oplus \ell_2) = \ell_1(v_2) - \ell_2(v_1), \quad v_1, v_2 \in V, \quad \ell_1,\ell_2 \in V^*.$$

Now let (M, ω, ∇) be special Kähler and let $Y = T^*M$. Consider the distribution of horizontal spaces on Y given by the connection ∇. Here 'horizontal' means relative to the projection map $\pi : Y \to M$. The horizontal space H_y at $y \in Y$ is a complex subspace of T_yY by Proposition 1.22. The projection π identifies $H_y \cong T_mM$, where $m = \pi(y)$, and so the splitting into horizontal and vertical is a splitting

$$T_yY \cong T_mM \oplus T_m^*M. \tag{2.5}$$

The linear algebra of the preceding paragraph gives global endomorphisms I, J, K which satisfy the quaternion algebra. According to Hitchin's lemma to check that this determines a hyperkähler structure we must only verify that $\omega_I, \omega_J, \omega_K$ are closed. First, since the canonical holomorphic symplectic form η on $Y = T^*M$ is closed, Eq. (2.3) implies that ω_J and ω_K are also closed. To see that ω_I is closed we choose a flat Darboux coordinate system $\{x^i, y_j\}$ on an open set $U \subset M$. This induces a local coordinate system $\{x^i, y_j; q_i, p^j\}$ on $\pi^{-1}U \subset Y$. Since the splitting (2.5) is induced by ∇, and dx^i, dy_j are ∇-flat by definition, it follows that

$$\omega_I = dx^i \wedge dy_i + dq_i \wedge dp^i.$$

This form is closed. $\square$

3. Integrable Systems

In the mathematical description of a (finite dimensional) classical mechanical system one meets a symplectic manifold X and a Hamiltonian function. It is an integrable system if there is a maximal set of Poisson commuting conserved momenta which includes the Hamiltonian. Under suitable hypotheses this leads to a foliation of X by lagrangian tori [GS, Sect. 44]. The complex analogue leads to the following definition [DM1], which we explain in the succeeding paragraphs.

Definition 3.1. *An* **algebraic integrable system** *is a holomorphic map $\pi : X \to M$ where*

(a) *X is a complex symplectic manifold with holomorphic symplectic form $\eta \in \Omega^{2,0}(X)$;*
(b) *The fibers of π are compact lagrangian submanifolds, hence affine tori;*
(c) *There is a family of smoothly varying cohomology classes $[\rho_m] \in H^{1,1}(X_m) \cap H^2(X_m;\mathbb{Z})$, $m \in M$, such that $[\rho_m]$ is a positive polarization of the fiber X_m. Hence X_m is an abelian torsor.*

The hypothesis that the fibers are compact lagrangian leads to the conclusion that they are affine tori. The fact that they are abelian torsors is an extra hypothesis. We assume that X and M are smooth.[9]

We now explain this definition and some consequences. Recall that a single[10] *abelian variety* is a quotient $A = V/\Lambda$ of a complex vector space V by a full real lattice Λ such that $H^{1,1}(A) \cap H^2(A;\mathbb{Z}) \neq 0$ and there is a positive class $[\rho]$ in this intersection. Such a class is called a *polarization* and is represented by a unique invariant positive closed $(1,1)$-form ρ on A. The polarization is *principal* if $\int_A \frac{\rho^n}{n!} = 1$. Note that ρ is a real symplectic form on A, and since it is invariant it is a symplectic form on $V_{\mathbb{R}}$ as well. Also, since ρ is an integral class, it induces a symplectic form on $\Lambda \cong H_1(A)$. Let $\{\gamma^i, \delta_j\} \subset \Lambda$ be a symplectic basis. Then there is a unique basis $\{\omega_i\}$ of holomorphic differentials on A with

$$\int_{\gamma_i} \omega_j = \delta^i_j, \tag{3.2}$$

where δ^i_j is the Kronecker symbol. In fact, we can identify $\{\omega_i\}$ as the complex basis of V^* dual to $\{\gamma^i\}$. Now

$$\int_{\delta_j} \omega_i = \tau_{ij} \tag{3.3}$$

defines the *period matrix* τ of A. The Riemann bilinear relations state that the matrix $\tau = (\tau_{ij})$ belongs to the Siegel upper half space

$$\mathbb{H}_n = \{\tau \text{ an } n \times n \text{ complex matrix} : \tau \text{ is symmetric and } \operatorname{Im}\tau \text{ is positive definite}\}.$$

The group $Sp(2n;\mathbb{R})$ acts transitively on $\mathbb{H}_n$. A change of symplectic basis $\{\gamma^i, \delta_j\}$ transforms τ by an element of a discrete subgroup $\Gamma \subset Sp(2n;\mathbb{R})$ which depends on the polarization. (For a principal polarization $\Gamma = Sp(2n;\mathbb{Z})$.) An abelian torsor X is a principal homogeneous space for an abelian variety $A = V/\Lambda$ with a polarization $[\rho]$. Here V is the space of invariant vector fields on X and $\Lambda \subset V$ the lattice of such vector fields which exponentiate to the identity map. We can identify A as the Albanese variety of X. Any point $x \in X$ determines an isomorphism $A \to X$, and the pullback of $[\rho]$ is a polarization $[\hat\rho]$ of A which is independent of the choice of x. The period matrix of X is equal to the period matrix of A.

An algebraic integrable system $\pi : X \to M$ leads to a parametrized version of the preceding discussion. First, the holomorphic symplectic form η gives an isomorphism

$$i : T^*M \xrightarrow{\cong} V,$$

[9] The singularities contain crucial physics, but for the geometry in this section we restrict to smooth points.

[10] For convenience we use the same notation for the single abelian varieties in this explanatory paragraph as we do in the rest of the text for families of abelian varieties.

where $V \to M$ is the bundle of invariant vector fields along the fibers of π. For a complex function $f : M \to \mathbb{C}$ and complex vector field ξ on X we have $\eta\big(i(df), \xi\big) = \pi^* df(\xi)$. This leads to a fiberwise action of $T^*M \cong V$ by exponentiation. Let Λ be the kernel of the action. A basic fact is that Λ is a complex lagrangian submanifold of T^*M, where T^*M has the canonical holomorphic symplectic structure. (See [GS, Sect. 44] for proofs of the assertions made here.) Furthermore, Λ intersects each fiber of T^*M in a full lattice. The quotient $A = T^*M/\Lambda$ is a family of abelian varieties parametrized by M; it is the bundle of Albanese varieties of $X \to M$. Since Λ is complex lagrangian, the canonical holomorphic symplectic form on T^*M passes to a holomorphic symplectic form $\hat{\eta}$ on the quotient A. Now a local lagrangian section of $\pi : X\big|_U \to U$ over an open set $U \subset M$ induces a local isomorphism $X\big|_U \cong A\big|_U$, and this isomorphism maps $\hat{\eta}$ to η. Such sections may not exist globally. Since any two choices of local section lead to isomorphisms which differ by a translation on each fiber, the family of polarizations $[\rho_m]$ on $X \to M$ define a family of polarizations $[\hat{\rho}_m]$ on $A \to M$.

To summarize: Every algebraic integrable system $X \to M$ has a canonically associated algebraic integrable system $A \to M$ whose fibers are abelian varieties. (An analogous assertion holds for real integrable systems.) Either system determines a well-defined *period map*

$$\tau : M \longrightarrow \mathcal{A}_n = \mathbb{H}_n / \Gamma$$

into the moduli space $\mathcal{A}_n$ of suitably polarized abelian varieties.

Now the bundle of lattices Λ determines a flat connection ∇ on T^*M, hence also on TM. Since Λ is lagrangian, ∇ is torsionfree. Also, the polarization $[\hat{\rho}_m]$ on $A_m = T^*_m M/\Lambda_m$ determines a real symplectic form on $T^*_m M$ which restricts to an integral symplectic form on the lattice Λ_m. The dual 2-form ω on M is flat – $\nabla\omega = 0$ – and since ∇ is torsionfree it follows that ω is closed. Thus ω is a real symplectic form on M. The holonomy group of the flat connection ∇ is contained in the *integral* symplectic group $Sp(\Lambda^*_m)$ at each $m \in M$, where Λ^*_m is the dual lattice to Λ_m. Furthermore, by the definition of a polarization ω is a (positive definite) Kähler form on M. If $\{\gamma^i, \delta_j\}$ is a local symplectic basis of sections of $\Lambda \subset T^*M$, then we can write

$$\omega = \gamma^i \wedge \delta_i .$$

There is also a global formula for ω. First, each polarization $[\hat{\rho}_m]$ is represented by a unique invariant closed form $\hat{\rho}_m \in \Omega^{1,1}(A_m)$. The family of forms $\{\hat{\rho}_m\}$ is flat with respect to ∇. Now the connection ∇ on T^*M induces an integrable distribution of horizontal planes on A, and we extend $\{\hat{\rho}_m\}$ to a form $\hat{\rho} \in \Omega^{1,1}(A)$ by requiring that $\hat{\rho}$ vanish on those horizontal planes. Then $d\hat{\rho} = 0$. The global formula for ω is expressed in terms of $\hat{\rho}$ and the holomorphic symplectic form $\hat{\eta} \in \Omega^{2,0}(A)$:

$$\omega = \frac{1}{4}\int_{A/M} \hat{\eta} \wedge \bar{\hat{\eta}} \wedge \frac{\hat{\rho}^{n-1}}{(n-1)!}.$$

The conclusion of this discussion is a result stated by Donagi and Witten [DW].

Theorem 3.4. (a) *Let $(X \to M, \eta, [\rho_m])$ be an algebraic integrable system. Then the Kähler form ω and the connection ∇ constructed above comprise a special Kähler structure on M. Furthermore, there is a lattice $\Lambda^* \subset TM$ whose dual $\Lambda \subset T^*M$ is a complex lagrangian submanifold, and the holonomy of ∇ is contained in the integral symplectic group defined by Λ^*.*

(b) *Conversely, suppose* (M, ω, ∇) *is a special Kähler manifold. Suppose further that there is a lattice* $\Lambda^* \subset TM$*, flat with respect to* ∇*, whose dual* $\Lambda \subset T^*M$ *is a complex lagrangian submanifold. Then* $A = T^*M/\Lambda \to M$ *admits a canonical holomorphic symplectic form* η *and a family of polarizations* $[\rho_m]$ *so that* $(A \to M, \eta, [\rho_m])$ *is an algebraic integrable system whose fibers are abelian varieties.*

Remark 3.5. The lattice Λ in (b) may be specified by a covering of distinguished flat Darboux coordinate systems $\{x^i, y_j\}$ whose transition functions satisfy (1.6) with $P \in Sp(2n; \mathbb{Z})$. In this case we also restrict the allowable special coordinate systems $\{z^i\}$ by requiring that $\{\mathrm{Re}(z^i)\}$ be part of a distinguished flat Darboux coordinate system.

Proof. For part (a) it remains to verify the special Kähler condition (1.2), or equivalently (1.7). We work locally. Let $\{\gamma^i, \delta_j\}$ be a local symplectic basis of sections of Λ. Since γ^i, δ_j are closed 1-forms we can find flat Darboux coordinates $\{x^i, y_j\}$ so that $\gamma^i = dx^i$ and $\delta_j = dy_j$. Now γ^i, δ_j also determine families of cycles on A and we can find holomorphic functions z^i, w_j such that

$$dz^i = \int_{\gamma^i} \hat{\eta}, \quad dw_j = -\int_{\delta_j} \hat{\eta}.$$

Here the integrals are over the families of cycles in the fibration $A \to M$, and Stokes' theorem shows that the integrals are holomorphic $(1, 0)$-forms. It is easy to check that $\mathrm{Re}(dz^i) = dx^i$ and $\mathrm{Re}(dw_j) = -dy_j$, so we can arrange that $\mathrm{Re}(z^i) = x^i$ and $\mathrm{Re}(w_j) = -y_j$. Then

$$\zeta = \frac{1}{2}\big(z^i \frac{\partial}{\partial x^i} - w_j \frac{\partial}{\partial y_j}\big)$$

is a local complex vector field which satisfies (1.8). This implies (1.7).

Notice that the vector fields $\omega_i = \frac{\partial}{\partial z^i}$ define local holomorphic differentials on the fibers of $A \to M$, and they satisfy (3.2). Thus Eq. (3.3) defines the period matrix (τ_{ij}) relative to $\{\gamma^i, \delta_j\}$. Equations (3.2) and (3.3) are equivalent to Eq. (1.12):

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\big(\frac{\partial}{\partial x^i} - \tau_{ij}\frac{\partial}{\partial y_j}\big).$$

By now the proof of (b) should be clear. Given $(M, \omega, \nabla, \Lambda)$, the family of polarizations on $A = T^*M/\Lambda \to M$ is represented by the dual of the Kähler form ω. Hence $A \to M$ is a family of abelian varieties. The symplectic form is induced from the canonical symplectic form on T^*M. The hypothesis that Λ is complex lagrangian makes the quotient T^*M/Λ complex symplectic. $\square$

Remark 3.6. An arbitrary family of abelian varieties $A \to M$ does *not* admit a symplectic form. For that the differential of the period map must come from a cubic form $c \in H^0(M, Sym^3 T^*M)$. (See [DM1, Sect. 7].) Here we assume a given identification of the bundle V with T^*M. (Recall that V is the bundle of constant vector fields along the fibers of $A \to M$.) The cubic condition on the period matrix is essentially the special Kähler condition (1.2), as is clear from Proposition 1.25. Of course, the cubic form is (1.26).

Remark 3.7. The preceding discussion applies to the pseudo-Kähler case with one modification: the polarization classes $[\rho_m]$ are no longer positive definite. So X_m is an affine torus with an indefinite polarization. We term this an *indefinite algebraic integrable system.*

The discussion in Sect. 2 applies directly to the quotient T^*M/Λ, and so Theorem 2.1 yields the following.

Theorem 3.8. *Let $(X \to M, \eta, [\rho_m])$ be an algebraic integrable system. Then X carries a canonical hyperkähler structure.*

4. Projective Special Kähler Manifolds

We term the triple (M, L, ω) a *Hodge manifold* if (M, ω) is Kähler and $L \to M$ is a holomorphic hermitian line bundle with curvature[11] $-2\pi i\omega$. This implies $[\omega] \in H^2(M; \mathbb{R})$ is an integral class. We begin with a geometric lemma about the principal $\mathbb{C}^\times$bundle $\pi : \tilde{M} \to M$ obtained by deleting the zero section from $L \to M$. First, the hermitian connection on L is also a connection on $\pi : \tilde{M} \to M$, that is, a $\mathbb{C}^\times$-invariant distribution of horizontal subspaces. Also, the bundle $\pi^*L \to \tilde{M}$ has a canonical nonzero holomorphic section s.

Lemma 4.1. *Let $\tilde{\omega} \in \Omega^{1,1}(\tilde{M})$ denote the form which equals $|s|^2\pi^*\omega$ on pairs of horizontal vectors, vanishes on a horizontal vector paired with a vertical vector, and is $-1/\pi$ times the canonical Kähler form on pairs of vertical vectors. Then*

$$\tilde{\omega} = \frac{i}{2\pi}\bar{\partial}\partial|s|^2. \tag{4.2}$$

Thus $d\tilde{\omega} = 0$, which implies that $\tilde{\omega}$ is a pseudo-Kähler metric on $\tilde{M}$ of Lorentz type. Finally,

$$\pi^*\omega = \frac{i}{2\pi}\bar{\partial}\partial \log |s|^2. \tag{4.3}$$

The canonical Kähler form on a hermitian line L is $\frac{i}{2}\partial\bar{\partial}|s|^2,\ s \in L$. The metric $\tilde{\omega}$ is negative definite on fibers and positive definite on horizontal subspaces. It has signature $(n, 1)$, where $n = \dim M$.

Proof. Let t be a nonzero holomorphic section of $L\big|_U \to U$ for an open set $U \in M$, and set $h(z) = |t(z)|^2,\ z \in U$. We use local coordinates $\langle z, \lambda\rangle \mapsto \lambda\, t(z) \in \pi^{-1}U \subset \tilde{M}$, where $\lambda \in \mathbb{C}^\times$. Now $s(z, \lambda) = \lambda t(z)$, and so $|s(z, \lambda)|^2 = |\lambda|^2 h(z)$. Compute the right-hand side of (4.2). To verify the description of $\tilde{\omega}$ given before (4.2), note that $\xi - \lambda h^{-1}\partial h(\xi)\frac{\partial}{\partial\lambda}$ is the horizontal lift of a tangent vector ξ in U. Formula (4.3) is the standard curvature formula for the hermitian connection. □

The usual definition for what we call a projective special Kähler structure is a particular type of *variation of Hodge structure*, which was considered specifically in a paper of Bryant and Griffiths [BG]. We discuss this first and defer our description to Proposition 4.6(b). Our version of the usual definition emphasizes the fact that the parameter space is a Hodge manifold, but it is equivalent to the definition in [BG] (cf., [C] for the relationship to [St]).

[11] Since we do not use so many indices in this section, we revert to the standard notation $i = \sqrt{-1}$.

Definition 4.4. (i) *A* **projective special Kähler structure** *on an n dimensional Hodge manifold (M, L, ω) is a triple (V, ∇, Q) where*

(a) *$V \to M$ is a holomorphic vector bundle of rank $n+1$ with a given holomorphic inclusion $L \hookrightarrow V$;*

(b) *∇ is a flat connection on the underlying real bundle $V_{\mathbb{R}} \to M$ such that $\nabla(L) \subset V$ and the section*

$$\begin{aligned} M &\longrightarrow \mathbb{P}\big[(V_{\mathbb{R}})_{\mathbb{C}}\big] \\ m &\longmapsto L_m \end{aligned} \tag{4.5}$$

is an immersion with respect to ∇;

(c) *Q is a nondegenerate skew form on $V_{\mathbb{R}}$ which has type (1,1) with respect to the complex structure and satisfies $\nabla Q = 0$. Furthermore, we assume that $Q\big|_{L\times \overline{L}}$ is $i/2\pi$ times the hermitian metric on L.*

(ii) *An* **integral projective special Kähler structure** *is a quadruple (Λ, V, ∇, Q) with (V, ∇, Q) as in (*i) and $\Lambda \subset V_{\mathbb{R}}$ a flat submanifold which intersects each fiber in a full lattice such that $Q\big|_{\Lambda\times\Lambda}$ has integral values.

In this definition ∇ and Q are extended to the complexification $(V_{\mathbb{R}})_{\mathbb{C}}$ of $V_{\mathbb{R}}$. The flat connection gives a local identification of $V_{\mathbb{R}}$ – hence also of its complexification $(V_{\mathbb{R}})_{\mathbb{C}}$ and the projectivization $\mathbb{P}\big[(V_{\mathbb{R}})_{\mathbb{C}}\big]$ – with any fiber. The immersion condition in (b) states that $m \mapsto L_m$ is an immersion into the $2n+1$ dimensional projective space of a local trivialization of $\mathbb{P}\big[(V_{\mathbb{R}})_{\mathbb{C}}\big]$.

The data in (ii) define a variation of polarized Hodge structures of weight 3 with Hodge numbers $h^{3,0} = 1,\ h^{2,1} = n$ with an extra immersion condition. This is the form of the definition in [BG]. (See [CGGH] for the basic definitions related to variations of Hodge structures.) We recover the Hodge filtration $\{F^p\}$ by setting $F^3 = L$, $F^2 = V$, $F^1 = F^{3\perp}$, and $F^0 = (V_{\mathbb{R}})_{\mathbb{C}}$. (Here "$^\perp$" is with respect to Q.) The Griffiths transversality condition $\nabla(F^3) \subset F^2$ is given in (b) above; the condition $\nabla(F^2) \subset F^1$ follows from this and the immersion condition [BG, pp.82–83]. Proposition 4.6 below implies that $iQ\big|_{H^{2,1}\times\overline{H^{2,1}}}$ is positive definite, where $H^{2,1} = F^2 \cap \overline{F^1}$. Variations of Hodge structure without the lattice, as in (i), were considered in [S].

Our main observation in this section is the following. We prefer to take the structure in (b) as the definition of projective special Kähler.

Proposition 4.6. *Let (M, L, ω) be a Hodge manifold with associated pseudo-Kähler manifold $(\tilde{M}, \tilde{\omega})$ and canonical section s.*

(a) *A projective special Kähler structure on (M, L, ω) induces a $\mathbb{C}^\times$-invariant special pseudo-Kähler structure $\tilde{\nabla}$ on $(\tilde{M}, \tilde{\omega})$ with $\tilde{\nabla} s = \pi^{(1,0)}$.*

(b) *Conversely, a $\mathbb{C}^\times$-invariant special pseudo-Kähler structure $\tilde{\nabla}$ on $(\tilde{M}, \tilde{\omega})$ which satisfies $\tilde{\nabla} s = \pi^{(1,0)}$ induces a projective special Kähler structure on (M, L, ω).*

Recall that $\tilde{\omega}$ is defined in Lemma 4.1. The canonical section s defined there can be viewed as the holomorphic vertical vector field on $\tilde{M}$ induced by the $\mathbb{C}^\times$ action.

Proof. (a) Let $\tilde{\nabla} = \pi^*\nabla$ be the lifted flat connection on π^*V. Using the inclusion $L \hookrightarrow V$ we view s as a section of π^*V. The immersion condition (4.5) implies that

$$\tilde{\nabla} s : T\tilde{M} \longrightarrow \pi^* V \tag{4.7}$$

is an isomorphism. (Note that $\widetilde{\nabla} s \subset \pi^* V$ by the Griffiths transversality in (b).) Using the real isomorphism underlying (4.7) we obtain a real flat connection on $\tilde{M}$; we also denote it by '$\widetilde{\nabla}$'. Furthermore, under (4.7) the form $\tilde{Q} = \pi^* Q$ pulls back to $-\tilde{\omega}$. This follows by differentiating the equation

$$\frac{i}{2\pi}|s|^2 = \tilde{Q}(s, \bar{s}),$$

assumed in (c), to obtain

$$\tilde{\omega} = \frac{i}{2\pi}\bar{\partial}\partial|s|^2 = -\tilde{Q}(\widetilde{\nabla} s, \overline{\widetilde{\nabla} s}).$$

Thus $\widetilde{\nabla}\tilde{\omega} = 0$. Now under (4.7) the section s corresponds to a holomorphic vector field ζ which satisfies (1.8). This proves that $\widetilde{\nabla}$ satisfies the special Kähler condition (1.7), and by Remark 1.19 $\widetilde{\nabla}$ is also torsionfree.

(b) We simply indicate the construction of (V, ∇, Q). First, let V be the quotient of $T\tilde{M}$ by the $\mathbb{C}^\times$ action. Then V is a holomorphic bundle over M, and the inclusion of vertical vectors in $T\tilde{M}$ induces an inclusion $L \hookrightarrow V$. The connection $\widetilde{\nabla}$ on $(T\tilde{M})_{\mathbb{R}}$ induces a connection ∇ on $V_{\mathbb{R}}$; the immersion condition in Definition 4.4(i)(b) follows from the hypothesis $\widetilde{\nabla} s = \pi^{(1,0)}$. The form $-\tilde{\omega}$ on $\tilde{M}$ induces a skew form Q on V. $\square$

Notice as a consequence of (c) and the description of $\tilde{\omega}$ in Lemma 4.1 that $iQ\,\big|_{H^{2,1}\times\overline{H^{2,1}}}$ is positive definite.

Now the discussion of special coordinates, holomorphic prepotential, etc. from Sect. 1 applies to $(\tilde{M}, \tilde{\omega}, \widetilde{\nabla})$. We make $\mathbb{C}^\times$-equivariant choices on $\tilde{M}$ and consider the induced tensors on M. We work on $\pi^{-1}(U)$ for $U \subset M$ a sufficiently small open set. We do not choose Darboux coordinates, but only a flat local symplectic framing[12] of $T\tilde{M}$, which we require to be $\mathbb{C}^\times$-invariant. We say that a complex tensor field on $\tilde{M}$ has *degree n* if it transforms under $\lambda \in \mathbb{C}^\times$ by multiplication by λ^n. The vector field ζ (which corresponds to s under (4.7)) has degree 1. So from (1.9) we see that a special coordinate function z^i also has degree 1. In other words, z^i is a local holomorphic section of $L \to M$. Thus a special coordinate system $\{z^i\}$ on $\tilde{M}$ gives rise to local *projective* coordinates on M (which transform as sections of L). From (1.13) we see that the period matrix (τ_{ij}) is a scalar, and from (1.14) that the holomorphic prepotential $\mathfrak{F}$ has degree 2, i.e., $\mathfrak{F}$ is a local holomorphic section of $L^{\otimes 2}$. Because of the $\mathbb{C}^\times$-invariance there is less flexibility in choosing $\{z^i\}$ and $\mathfrak{F}$ than in the nonprojective case – different choices differ by a *homogeneous* function.

Proposition 4.8. *Let $(M, L, \omega, \widetilde{\nabla})$ be a projective special Kähler manifold.*

(a) *Given a special projective coordinate system $\{z^i\}$ the holomorphic prepotential $\mathfrak{F}$ is determined up to a change*

$$\mathfrak{F} \longrightarrow \mathfrak{F} + \frac{1}{2}A_{ij}z^i z^j,$$

where $A = (A_{ij})$ is a real symmetric matrix. Hence the conjugate special projective coordinate system $\{w_j\}$ is determined up to a change

$$w_j \longrightarrow w_j + A_{jk}z^k.$$

[12] It is denoted $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_j}\}$ in Sect. 1, but here we do not consider coordinate functions x^i and y_j.

(b) *If* $\{z^i\}, \{w_j\}$ *are conjugate special projective coordinate systems, then any other pair* $\{\tilde{z}^i\}, \{\tilde{w}_j\}$ *of conjugate special projective coordinate systems are related by*

$$\begin{pmatrix} z \\ w \end{pmatrix} = P \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix}, \quad P \in Sp(2n;\mathbb{R}).$$

From (4.3) we see that the lift of the metric ω to $\tilde{M}$ has a global "Kähler potential" $\tilde{K}$, which we write in special coordinates as

$$\begin{aligned} \tilde{K} &\doteq \frac{-1}{\pi} \log |s|^2 \\ &\doteq \frac{-1}{\pi} \log \tilde{Q}(s,\bar{s}) \\ &\doteq \frac{-1}{\pi} \log\bigl(-\tilde{\omega}(\zeta,\bar{\zeta})\bigr) \\ &\doteq \frac{-1}{\pi} \log \operatorname{Im}(z^i \bar{w}_i) \\ &\doteq \frac{-1}{\pi} \log \operatorname{Im}\bigl(z^i \frac{\partial \overline{\mathfrak{F}}}{\partial \overline{z^i}}\bigr). \end{aligned}$$

Here "$\doteq$" means "equals up to an additive constant". $\tilde{K}$ pulls down to a local Kähler potential on M via a local holomorphic section of $\pi : \tilde{M} \to M$.

The cubic form $\tilde{\Xi}$ of the special Kähler structure on $\tilde{M}$ (see (1.26)) is a holomorphic section

$$\Xi \in H^0(M, Sym^3 T^*M \otimes L^{\otimes 2}),$$

as follows easily from (1.28). It is a basic ingredient in the analysis of [BG], where it is derived from an *infinitesimal* variation of Hodge structure. Since ζ is holomorphic of type (1, 0), we have $\omega(\zeta, \widetilde{\nabla}\zeta) = 0$, and by differentiating $\omega(\zeta, \widetilde{\nabla}^2\zeta) = 0$. Differentiating once more we conclude from (1.27) that the cubic form in this case is

$$\Xi = \omega(\zeta, \widetilde{\nabla}^3\zeta) = \tilde{Q}(\nabla^3 s, s).$$

We can use $\tilde{\Xi} \in H^0(\tilde{M}, Sym^3 T^*\tilde{M})$ and the associated $\tilde{A} \in \Omega^{1,0}\bigl(\operatorname{Hom}(T\tilde{M}, \overline{T\tilde{M}})\bigr)$ to introduce an algebra structure on $T\tilde{M} \otimes_{\mathbb{R}} \mathbb{C}$. Fix $\tilde{m} \in \tilde{M}$ and denote $V = T_{\tilde{m}}\tilde{M}$. It is easy to see that $\tilde{A}$ vanishes on ζ, and it is a well-defined map $W \otimes W \to \overline{W}$, where W is the orthogonal complement to ζ. (Under the projection π we can identify $W \cong T_m M$, where $m = \pi(\tilde{m})$.) We now obtain a graded algebra C: Set $C_0 = \mathbb{C} \cdot \zeta$, $C_1 = W$, $C_2 = \overline{W}$, and $C_3 = \mathbb{C} \cdot \bar{\zeta}$; then ζ acts as the identity, the multiplication $C_1 \otimes C_1 \to C_2$ is given by $\tilde{A}$, and the multiplication $C_1 \otimes C_2 \to C_3$ is $\alpha \otimes \bar{\beta} \mapsto \omega(\alpha, \bar{\beta})\bar{\zeta}$. Associativity is trivial to verify.

Now we consider the implications of the lattice $\Lambda \subset V_{\mathbb{R}}$ in an *integral* projective special Kähler structure on M. Under the isomorphism (4.7) the lift $\pi^*\Lambda \subset \pi^* V_{\mathbb{R}}$ induces a lattice $\tilde{\Lambda} \subset T\tilde{M}$. Now $\tilde{\Lambda}$ is $\widetilde{\nabla}$-flat by hypothesis, so by Proposition 1.22 it is a complex submanifold. Locally $\tilde{\Lambda}$ is the graph of a $\widetilde{\nabla}$-flat vector field on $\tilde{M}$, so the dual $\tilde{\Lambda}^*$ is locally the graph of a $\widetilde{\nabla}$-flat 1-form. Since $\widetilde{\nabla}$ is torsionfree, this 1-form is also holomorphic and so $\tilde{\Lambda}^* \subset T^*\tilde{M}$ is complex lagrangian. Thus Theorem 3.4(b) and Theorem 3.8 apply to give the following conclusion.

Proposition 4.9. *Suppose* $(M, L, \omega, \widetilde{\nabla}, \Lambda)$ *is an integral projective special Kähler manifold of dimension* n*. Then there is an associated indefinite algebraic integrable system* $X \to \tilde{M}$*, where* $\tilde{M}$ *is* L *with the zero section removed. The total space* X *carries a "pseudo-hyperkähler" structure of real signature* $(4n, 4)$.

The fibers of this integrable system are the intermediate Jacobians associated to the underlying variation of Hodge structure. The symplectic form on this family of intermediate Jacobians was constructed by Donagi and Markman [DM2] (for the case of a family of Calabi-Yau manifolds). The pseudo-hyperkähler structure was also given by Cortés [C]. As in the nonprojective case we restrict our local Darboux framings to lie in the lattice, and so the matrices A, P in Proposition 4.8 must be integral.

5. Remarks on $N = 2$ Gauge Theories in Four Dimensions

We make some brief remarks on the role of special Kähler manifolds in global supersymmetric theories. We do not comment on their role in supergravity. References for the quantum physics are [SW1] and [SW2]. For a mathematical development of the relevant classical supersymmetry, see [DF]. The quantum aspects of our discussion have no pretension to rigor.

We first recall the origin of the local formula (1.15) for the Kähler potential. It arises from the lagrangian for the complex scalars in the four dimensional $N = 2$ *vector multiplets*. There is a superspace description in terms of the superspace $N^{4|8}$, which is an extension of ordinary four dimensional Minkowski space with eight odd dimensions. The complexification of the odd distribution splits into two pieces, and there is a corresponding notion of a *chiral* map $\Sigma : N^{4|8} \to \mathbb{C}$. Such a map describes an (abelian) $N = 2$ vector multiplet. (More precisely, it is a component of the curvature of a constrained connection on superspace.) The most general supersymmetric lagrangian for n such multiplets is specified by a holomorphic function $\mathfrak{F} : \mathbb{C}^n \to \mathbb{C}$. The theory is free if $\mathfrak{F}$ is quadratic. Upon reduction to $N = 1$ superspace $N^{4|4}$ each multiplet Σ decomposes into an $N = 1$ chiral multiplet Φ and an $N = 1$ vector multiplet $\mathcal{A}$. The lagrangian for the chiral multiplets is determined from the Kähler potential K, and a computation gives the formula (1.15) for K in terms of $\mathfrak{F}$.

Next, we emphasize that a special Kähler manifold does *not* define a *classical* field theory for $N = 2$ vector multiplets. We do obtain a classical lagrangian from a special coordinate system, as explained in the previous paragraph. Furthermore, any Kähler manifold M does determine a well-defined $N = 1$ supersymmetric field theory for a chiral field $\Phi : N^{4|4} \to M$. However, the change of special coordinates (1.39) must be accompanied by a duality[13] transformation on the gauge field in the vector multiplet $\mathcal{A}$, and this only makes sense in the *quantum* theory. Moreover, this duality transformation only makes sense when the holonomy of ∇ is contained in the *integral* symplectic group. Thus a special Kähler manifold M with a lattice as in Theorem 3.4 determines[14] a *quantum* field theory which locally has a semiclassical description in terms of $N = 2$ vector multiplets. The manifold M is the moduli space of quantum vacua. According to Theorem 3.4 such a theory is always specified by an algebraic integrable system.

These abelian theories describe the low energy behavior of the Coulomb branch of nonabelian $N = 2$ supersymmetric gauge theories, with or without matter. The

[13] That is, electromagnetic duality.

[14] Since typically M is incomplete this is not yet a full description of a theory. Also, an abelian gauge theory, which has a positive β-function, only makes sense as an effective field theory, not as a fundamental theory.

simplest example [SW1] has gauge group $SU(2)$ and no matter. Then M is the universal curve $M(2)$ for the modular group $\Gamma(2) \subset SL(2;\mathbb{Z})$, which we can identify as $\mathbb{CP}^1$ with 3 points omitted, say $M(2) = \mathbb{CP}^1 - \{-1, 1, \infty\}$. The universal curve $X(2) \to M(2)$ is the algebraic integrable system which defines the model. Many more examples have been found, all of course involving integrable systems. (See [D] for a review.)

So far we have taken M to be smooth. As stated above, a nonflat M is not complete and an honest physical theory is formulated on some completion of M. For example, for the pure $SU(2)$ gauge theory the special Kähler metric on the moduli space $\mathbb{CP}^1 - \{-1, 1, \infty\}$ is complete near ∞, but the singular points $-1, 1$ are at finite distance. At these points other fields are massless and must be added to the low energy description.

We now remark further on the physical origin of the lattice Λ. It is a feature of four dimensional abelian gauge theories; supersymmetry is irrelevant. (See [AgZ, Sect. 3] for a recent discussion.) Consider a four dimensional gauge theory with gauge group $G = \mathbb{T}^n$, where $\mathbb{T} \cong U(1)$ is the circle group. The theory is specified by a complex bilinear form τ on the Lie algebra $\mathfrak{g}$ whose imaginary part $\operatorname{Im}\tau$ is an inner product.[15] The lagrangian density in Minkowski space is

$$L = \left\{ -\frac{1}{8\pi} \operatorname{Im}\tau(F_A, *F_A) + \frac{1}{8\pi} \operatorname{Re}\tau(F_A, F_A) \right\} |d^4x|, \tag{5.1}$$

where A is a connection and $|d^4x|$ the standard density. There is a lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ whose elements exponentiate to the identity in G, and each basis of this lattice produces a matrix $(\tau_{ij}) \in \mathbb{H}_n$ which represents the form τ. The group $GL(n;\mathbb{Z})$ permutes these bases. The larger *duality group* $Sp(2n;\mathbb{Z})$ is generated by this group together with the electromagnetic duality transformation. The latter expresses the theory in terms of a "dual" connection $\tilde{A}$ and the bilinear form $-\tau^{-1}$. The lagrangian has the same form as (5.1), and the operator F_A in the original theory corresponds to $*F_{\tilde{A}}$ in the dual theory. The action of $Sp(2n;\mathbb{Z})$ which is generated acts on τ by (1.40).

Fix a basis of $\mathfrak{g}_{\mathbb{Z}}$ and so write the curvature as $F_A = (F_A^i)_{i=1,\dots,n}$. There are n electric charges q^i and n magnetic charges g_i for charged matter we might put into the theory. Classically, the electric charge in a spatial region bounded by a surface Σ is defined to be

$$q^i = \int_\Sigma \frac{\sqrt{-1}}{2\pi} * F_A^i,$$

and the enclosed magnetic charge is

$$g^i = (n_m)^i = \int_\Sigma \frac{\sqrt{-1}}{2\pi} F_A^i.$$

The electric and magnetic charges of a *quantum* state are computed from the corresponding operators in the quantum theory. Now $(n_m)^i$ is an integer by Chern-Weil theory for the *compact* gauge group $\mathbb{T}^n$. In the classical theory $(n_m)^i$ is an integer-valued function on the space of classical solutions; in the quantum theory it assigns an integer to each quantum state. There are other integers $(n_e)_i$ attached to quantum states from the Noether charges associated to global infinitesimal gauge transformations. Here the integrality is from the fact that certain exponentials of these infinitesimal transformations

[15] For $n = 1$ in the standard basis the form τ is usually written $\tau = \frac{\theta}{\pi} + \frac{8\pi\sqrt{-1}}{e^2}$, where e is the coupling constant.

are the identity operator. These integers are related to the electric charge of states via the formula

$$q^i = \big((\operatorname{Im}\tau)^{-1}\big)^{ij}\big((\operatorname{Re}\tau)_{jk}(n_m)^k + (n_e)_j\big).$$

It is convenient to consider the complex quantity

$$q^i + \sqrt{-1}g^i = \big((\operatorname{Im}\tau)^{-1}\big)^{ij}\big(\tau_{jk}(n_m)^k + (n_e)_j\big).$$

As the n_m, n_e run over all integers, this runs over the points of the *(electromagnetic) charge lattice* Λ^* in $\mathbb{C}^n$. There is an integral symplectic form ω on Λ^* defined by

$$\begin{aligned}\omega\Big(\big(\begin{smallmatrix} g\\ q\end{smallmatrix}\big), \big(\begin{smallmatrix} \tilde g\\ \tilde q\end{smallmatrix}\big)\Big) &= g^i(\operatorname{Im}\frac{\tau}{2})_{ij}\tilde q^j - \tilde g^i(\operatorname{Im}\frac{\tau}{2})_{ij}q^j \\ &= (n_m)^i(\tilde n_e)_i - (\tilde n_m)^i(n_e)_i.\end{aligned} \tag{5.2}$$

It is preserved by the duality group. Equation (5.2) is the form of charge quantization usually referred to as the "Dirac-Schwinger-Zwanziger condition".

Returning to an $N = 2$ supersymmetric abelian gauge theory, we have the moduli space M of the complex scalars which carries the special geometry we have been discussing. There is a distinguished set of conjugate special coordinate systems related by integral coordinate transformations. In each such coordinate system we have a lagrangian description as a gauge theory with gauge group $G = \mathbb{T}^n$ (with distinguished basis for the Lie algebra). We should regard the $\mathbb{C}^n$ where the coordinates live as the complexified Lie algebra with its distinguished basis. The electromagnetic charge lattice Λ^* discussed in the previous paragraph defines a global lattice in the complex conjugate cotangent bundle $\overline{T^*M} \cong TM$. Note in the notation of Sect. 1 that $\left(\begin{smallmatrix} (n_m)^i\\ (n_e)_j\end{smallmatrix}\right)$ transforms analogously to $\left(\begin{smallmatrix} dx^i\\ dy_j\end{smallmatrix}\right)$, and $q^i + \sqrt{-1}g^i$ transforms analogously to $\overline{dz^i}$. (See formulas (1.6) and (1.12).)

There is a further geometric input from the *BPS mass formula*. Namely, in the classical theory the central charge Z in the $N = 2$ supersymmetry algebra is a complex-valued locally constant function on the space of solutions to the classical field equations. In the quantum theory (at a point $m \in M$) it is an operator whose eigenvalues are complex numbers. Let $\{z^i\}, \{w_j\}$ be distinguished conjugate special coordinate systems. This means that there is a lagrangian description for the $N = 2$ theory in terms of a prepotential $\mathfrak{F}(z^1, \dots, z^n)$ with $w_j = \partial\mathfrak{F}/\partial z^j$. The BPS formula involves possible additional $\mathbb{T}$ charges S^α which may appear in the theory. These have integer eigenvalues. The BPS formula asserts that the eigenvalue of Z is

$$z^i(n_e)_i + w_i(n_m)^i + s^\alpha\frac{m_\alpha}{\sqrt{2}},$$

where s^α is the eigenvalue of S^α and $(n_e)_i, (n_m)^i$ are the integers defined above. Let $\Gamma \subset \mathbb{C}$ denote the set of points so described. If there are no S^α, then the fact that Γ is intrinsic and the transformation law for $\left(\begin{smallmatrix} (n_m)^i\\ (n_e)_j\end{smallmatrix}\right)$ implies that there is no translation in the coordinate change (1.39) between different sets of distinguished conjugate coordinate systems. However, in the presence of charges S^α there may be a nonzero translational component.

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