A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one

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Abstract

We classify all complete projective special real manifolds with reducible cubic potential, obtaining four series. For two of the series the manifolds are homogeneous, for the two others the automorphism group acts with co-homogeneity one. We show that, for each dimension $n \geq 3$, each of the two homogeneous examples can be deformed by a family depending on $n-2$ parameters of complete projective special real manifolds that are pairwise inequivalent. The two homogeneous examples are the boundary points of a curve which lies in the union of the two families. Complete projective special real manifolds give rise to complete quaternionic Kähler manifolds via the supergravity q-map which is the composition of the supergravity c-map and r-map. We develop curvature formulas for manifolds in the image of the q-map. Applying the q-map to one of the above series of projective special real manifolds we obtain a series of complete quaternionic Kähler manifolds, which are shown to be inhomogeneous (of co-homogeneity one) based on our curvature formulas.

Keywords: projective special real manifolds, projective special Kähler manifolds, quaternionic Kähler manifolds, co-homogeneity one

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Contents

Introduction

In this paper we are concerned with hypersurfaces $\mathcal{H} \subset \mathbb{R}^{n+1}$ contained in the level set $\{h = 1\}$ of a homogeneous cubic polynomial h. The hypersurface is equipped with the symmetric tensor field $g_{\mathcal{H}}$ on $\mathcal{H}$ induced by $-\frac{1}{3}\partial^2 h$. We require that $g_{\mathcal{H}}$ is a Riemannian metric. Then $(\mathcal{H}, g_{\mathcal{H}})$ is called a *projective special real manifold*, see Definition 5, h is called its *cubic potential* and $g_{\mathcal{H}}$ is called the *projective special real metric*. The polynomials h which admit such a hypersurface are called *hyperbolic*, cf. Definition 4. Projective special real manifolds occur in the physics literature as the scalar manifolds of 5-dimensional supergravity coupled to vector multiplets, see [GST]. These manifolds are related to *projective special Kähler manifolds* by a construction known as the *r-map* [DV], which is induced by the dimensional reduction of the supergravity theory from 5 to 4 space-time dimensions. Similarly projective special Kähler manifolds are related to *quaternionic Kähler manifolds* of negative scalar curvature by the *c-map*, which is induced by dimensional reduction to 3 dimensions [FS]. It is known [CHM] that the r- and c-map preserve the completeness of the underlying Riemannian metrics. It follows that the same is true for their composition, the *q-map*. In this way the study of the completeness of quaternionic Kähler manifolds obtained by the q-map is reduced to the study of the completeness of the initial projective special real manifold. Complete projective special real manifolds are characterized as follows, see [CNS, Thm. 2.5].

Theorem 1. *A projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ is complete with respect to the metric $g_{\mathcal{H}}$ if and only if $\mathcal{H}$ is closed as a subset of $\mathbb{R}^{n+1}$.*

It follows from Theorem 1 that the classification of complete projective special real manifolds is equivalent to the solution of the following two problems:

(i) Classification of all hyperbolic homogeneous cubic polynomials h, up to linear transformations.

(ii) For each such polynomial determine all locally strictly convex components of the level set $\{h = 1\}$, up to linear transformations.

While it is certainly possible to solve these problems in low dimensions, see [CDL] for the solution up to polynomials in 3 variables, we do not expect a simple solution valid in all dimensions. A very rough idea about problem (i) is obtained by observing that the dimension of the space of homogeneous cubic polynomials grows cubically whereas the dimension of the general linear group grows only quadratically with the number of variables. Notice that the hyperbolic polynomials form an open subset in the space of homogeneous cubic polynomials in a given number of variables. An interesting class of projective special real manifolds is provided by considering those with reducible cubic potentials h, that is h is a product of polynomials of lower degree. Applying the q-map to the complete manifolds in this class we obtain a class of complete quaternionic Kähler manifolds, as follows from the general result [CHM, Thm. 6]. In this way one obtains, in particular, the series of symmetric spaces

$$\frac{\mathrm{SO}_0(4,m)}{\mathrm{SO}(4)\times\mathrm{SO}(m)}, \quad m \geq 3, \tag{0.1}$$

as well as the series of homogeneous non-symmetric spaces $\mathcal{T}(p)$, $p \geq 1$, of rank 3, see [DV, C]. One of the results of this paper is that one also obtains a series of complete quaternionic Kähler manifolds that are not locally homogeneous, see Theorem 24. In fact, we show that there are precisely 4 series of complete projective special real manifolds with reducible cubic potential, three of which correspond to the 3 series of quaternionic Kähler manifolds mentioned above. More precisely, by solving the above problems (i) and (ii) under the assumption that h is reducible we will obtain the following result.

Theorem 2. *Every complete projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ of dimension $n \geq 2$ for which h is reducible is linearly equivalent to exactly one of the following complete projective special real manifolds:*

a) $\{x_{n+1}(\sum_{i=1}^{n-1} x_i^2 - x_n^2) = 1, \quad x_{n+1} < 0, x_n > 0\}$,

b) $\{(x_1 + x_{n+1})(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 + x_{n+1} < 0\}$,

c) $\{x_1(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 < 0, x_{n+1} > 0\}$,

d) $\{x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}$.

Notice that in the case $n = 2$ the result follows from [CDL, Thm. 1] and that the above list is also valid in the case $n = 1$ but then the curves a) and b) are linearly equivalent, as well as c) and d), see [CHM, Cor. 4].

Under the q-map the series a) with $n \geq 1$ corresponds to the series (0.1) of symmetric quaternionic Kähler manifolds with $m = n + 2$. Similarly, b) corresponds to the series $\mathcal{T}(p)$ of homogeneous quaternionic Kähler manifolds with $p = n - 1 \geq 0$, where only the first member $\mathcal{T}(0) = \frac{SO_0(4,3)}{SO(4) \times SO(3)}$ of the series is symmetric. The quaternionic Kähler manifolds obtained from the series c) and d) admit a Lie group acting isometrically with co-homogeneity one. For d) we will prove the following stronger result.

Theorem 3. *The quaternionic Kähler manifolds associated with the projective special real manifolds $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}$, $n \geq 1$, are complete of negative scalar curvature and the isometry group acts with co-homogeneity one.*

The claim that the quaternionic Kähler manifolds in Theorem 3, and similarly the ones obtained from the series c) in Theorem 2, admit a *subgroup* of the isometry group acting with an orbit of codimension one follows from the fact that the automorphism group of the initial projective special real manifolds acts with an orbit of codimension one. In fact, every automorphism of a projective special real manifold extends to an isometry of the corresponding quaternionic Kähler manifold under the q-map and the r-map as well as the c-map each produce a freely acting additional solvable Lie group of automorphisms, see [DV, CHM]. The dimensions of the latter groups coincide with the number of extra dimensions created by the r- and c-map, respectively. Therefore the co-homogeneity does not increase under these constructions.

The main difficulty is to prove that the quaternionic Kähler manifolds of Theorem 3 are not of co-homogeneity zero, this is the content of Theorem 24. The proof proceeds by computing the point-wise norm of the curvature tensor and showing that for each of these manifolds it is a non-constant rational function depending only on one coordinate x out of a system of $4n+8$ global coordinates. It relies on general curvature formulas for quaternionic Kähler manifolds obtained by the q-map, which constitute another important result of this paper, see Theorem 22 and Corollary 23. Incidentally, we expect that the isometry groups of the quaternionic Kähler manifolds corresponding to the remaining series c) in Theorem 2 do likewise have co-homogeneity precisely one. The corresponding curvature calculations are more involved in that case.

Another result of this paper is the construction of two multi-parameter families, depending on $(n-2)$ parameters, of n-dimensional complete projective special real manifolds that are pairwise inequivalent, see Theorem 9. Until now, only a one-parameter family of pairwise inequivalent projective special real surfaces was known, where the corresponding cubic potentials are the (homogenised) Weierstraß cubics with positive discriminant, cf. [CDL, Thm. 1]. We use these multi-parameter families to define a curve (parametrised over a compact interval) in the vector space of homogeneous cubic polynomials, such that each interior point of the curve is contained in one of the two multi-parameter families and the endpoints are, up to equivalence, the potentials corresponding to the symmetric spaces (0.1) and the homogeneous non-symmetric spaces $\mathcal{T}(p)$, respectively. Furthermore, we determine the automorphism group of each element in the two multi-parameter families, see Corollary 13.

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1 Classification of complete projective special real manifolds with reducible cubic potential

In this section we will classify all complete projective special real manifolds with reducible cubic potential up to linear transformations. After giving some basic definitions we will first classify up to equivalence all non-degenerate reducible homogeneous cubic polynomials in Section 1.1 and among these all hyperbolic ones in Section 1.2. In the same section we determine, for each of the resulting hyperbolic polynomials h, those connected components (up to linear transformations) of the level sets $\{h = 1\}$ which contain a hyperbolic point. In particular we determine all such components which are locally strictly convex or, equivalently, consist solely of hyperbolic points. As a consequence of Theorem 1 these components give precisely all complete projective special real manifolds with reducible cubic potential (up to linear transformations).

Definition 4. *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a homogeneous cubic polynomial. We will call h* non-degenerate *if there exists $p \in \mathbb{R}^{n+1}$, such that* $\det \partial^2 h_p \neq 0$. *It is called* hyperbolic *if there exists a* hyperbolic point *$p \in \mathbb{R}^{n+1}$, that is a point such that $h(p) > 0$ and $\partial^2 h_p$ is of signature $(1, n)$. Two homogeneous cubic polynomials are called* equivalent *if they are related by a linear transformation.*

Notice that the notions of non-degeneracy and hyperbolicity are invariant under linear transformations and that $\det \partial^2 h_p \neq 0$ implies $h(p) \neq 0$.

Definition 5. *A hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is called a* projective special real manifold *if there exists a homogeneous cubic polynomial $h : \mathbb{R}^{n+1} \to \mathbb{R}$, such that*

(i) $\mathcal{H} \subset \{x \in \mathbb{R}^{n+1} \mid h(x) = 1\}$ *and*

(ii) $g_{\mathcal{H}} := -\frac{1}{3}\partial^2 h|_{T\mathcal{H}\times T\mathcal{H}} > 0.$

The hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ is endowed with the Riemannian metric $g_{\mathcal{H}}$ which is called the projective special real metric[1] *or* centroaffine metric*, see [CNS] for an explanation of this terminology. Two projective special real manifolds are called* isomorphic *if there is a linear transformation inducing a bijection between them.*

Remark 6. It is easy to see that for every projective special real manifold $\mathcal{H}$ the symmetric tensor $\partial^2 h_p$ is of signature $(1, n)$ for all $p \in \mathcal{H}$ and that $\mathcal{H}$ is perpendicular to the position vector p with respect to $\partial^2 h_p$. In particular, h is hyperbolic. Notice also that a linear transformation mapping a projective special real manifold $\mathcal{H} \subset \mathbb{R}^{n+1}$ to another projective special real manifold $\mathcal{H}' \subset \mathbb{R}^{n+1}$ is automatically an isometry with respect to the centroaffine metrics. In particular, isomorphic projective special real manifolds are isometric.

In order to avoid special cases in low dimensions, and since the case $n \leq 2$ has already been studied [CDL], we will always assume that $n \geq 3$ in the following classifications.

[1] For practical reasons, we prefer to work with $-\frac{1}{2}\partial^2 h$ instead of $-\frac{1}{3}\partial^2 h$ below.

1.1 Classification of non-degenerate reducible homogeneous polynomials

For $m \in \mathbb{N}$ and $k \in \{0, \dots, m\}$, we introduce the following quadratic polynomials on $\mathbb{R}^m$:

$$Q_k^m := \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{m} x_i^2.$$

Proposition 7. *Any non-degenerate reducible homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$, $n \geq 3$, is equivalent to precisely one of the following:*

I) $x_{n+1} Q_k^n$, $\frac{n}{2} \leq k \leq n$,

II) $x_1 Q_k^{n+1}$, $1 \leq k \leq n+1$,

III) $(x_1 + x_{n+1}) Q_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n$.

Proof. Let $h = LQ$ be a non-zero reducible cubic polynomial on $\mathbb{R}^{n+1}$, where L is a linear and Q a quadratic factor. Up to a linear transformation, we can assume that $Q = Q_k^m$, $1 \leq m \leq n+1$, $\frac{m}{2} \leq k \leq m$. In the following, let

$$L := \sum_{j=1}^{n+1} a_j x_j.$$

Next we examine for which choices of Q_k^m and L the polynomial $h = LQ_k^m$ is non-degenerate.

Notice that $m = n$ or $m = n+1$, since otherwise $0 \neq \ker dL \cap \ker \partial^2 Q \subset \ker \partial^2 h_p$ for all $p \in \mathbb{R}^{n+1}$. In the case $m = n$ the non-degeneracy of h clearly implies that $a_{n+1} \neq 0$ and without loss of generality we can assume that $L = x^{n+1}$. We compute

$$\partial^2 h = 2 \begin{pmatrix} x_{n+1} & & & & & & x_1 \\ & \ddots & & & & & \vdots \\ & & x_{n+1} & & & & x_k \\ & & & -x_{n+1} & & & -x_{k+1} \\ & & & & \ddots & & \vdots \\ & & & & & -x_{n+1} & -x_n \\ x_1 & \dots & x_k & -x_{k+1} & \dots & -x_n & 0 \end{pmatrix},$$

where the remaining entries are zero. The determinant is given by

$$\det \partial^2 h = 2^{n+1} (-1)^{n-k+1} x_{n+1}^{n-2} h,$$

which shows that $h = x_{n+1} Q_k^n$ is non-degenerate for all $\frac{n}{2} \leq k \leq n$. These are precisely the polynomials listed in I).

It remains to check the case $m = n+1$, that is, $h = LQ_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n+1$. Using the transitive action of the pseudo-orthogonal group of the quadratic form Q_k^{n+1} on each pseudo-sphere and on the cone of non-zero light-like vectors we can assume up to a positive rescaling that $L = x_1$ (L space-like), $L = x_{n+1}$ (L time-like), or $L = x_1 + x_{n+1}$ (L light-like), where the latter two cases need only to be considered for $k \leq n$. Since x_{n+1} is space-like with respect to $-Q_k^{n+1}$ for $\frac{n+1}{2} \leq k \leq n$ and $-Q_k^{n+1}$ is equivalent to Q_{n+1-k}^{n+1}, $1 \leq n+1-k \leq \frac{n+1}{2}$, we are left with the two cases II) and III).

In case II), $h = x_1 Q_k^{n+1}$ with $1 \leq k \leq n+1$ and

$$\partial^2 h = 2 \begin{pmatrix} 3x_1 & x_2 & \dots & x_k & -x_{k+1} & \dots & -x_{n+1} \\ x_2 & x_1 & & & & & \\ \vdots & & \ddots & & & & \\ x_k & & & x_1 & & & \\ -x_{k+1} & & & & -x_1 & & \\ \vdots & & & & & \ddots & \\ -x_{n+1} & & & & & & -x_1 \end{pmatrix}.$$

We obtain

$$\det \partial^2 h = (-1)^{n+1-k} 2^{n+1} x_1^{n-2} (4x_1^3 - h),$$

which, for all $1 \leq k \leq n+1$, is not the zero polynomial. Hence, all polynomials listed in II) are non-degenerate.

In case III), that is $h = (x_1 + x_{n+1}) Q_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n$, it is convenient to change the coordinates the following way:

$$x_1 + x_{n+1} = \xi,$$
$$x_1 - x_{n+1} = \eta.$$

h is now of the form

$$h = \xi \left(\xi\eta + \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2 \right).$$

In the coordinates $(\xi, \eta, x_2, \dots, x_n)$ we have

$$\partial^2 h = 2 \begin{pmatrix} \eta & \xi & x_2 & \dots & x_k & -x_{k+1} & \dots & -x_n \\ \xi & 0 & & & & & & \\ x_2 & & \xi & & & & & \\ \vdots & & & \ddots & & & & \\ x_k & & & & \xi & & & \\ -x_{k+1} & & & & & -\xi & & \\ \vdots & & & & & & \ddots & \\ -x_n & & & & & & & -\xi \end{pmatrix}.$$

It is now easy to see that

$$\det \partial^2 h = (-1)^{n+1-k} \xi^{n+1}.$$

We conclude that all polynomials considered in III) are non-degenerate. $\square$

1.2 Classification of hyperbolic reducible homogeneous polynomials and complete projective special real manifolds

Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a hyperbolic homogeneous cubic polynomial. We consider the open subset $\mathcal{H}(h)$ of the hypersurface $\{h = 1\}$ consisting of the hyperbolic points of h:

$$\mathcal{H}(h) = \{p \in \mathbb{R}^{n+1} \mid h(p) = 1, -\partial^2 h_p \text{ has Lorentzian signature } (n, 1)\}.$$

Proposition 8. *Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$, $n \geq 3$, be a reducible hyperbolic homogeneous cubic polynomial and let $(x_1, \ldots, x_{n+1})$ denote the standard coordinates of $\mathbb{R}^{n+1}$. Then h is equivalent to one of the following polynomials and the corresponding hypersurface $\mathcal{H}(h)$ endowed with the Riemannian metric $-\frac{1}{2}\partial^2 h|_{T\mathcal{H}(h)\times T\mathcal{H}(h)}$ has the following properties:*

a) $h = x_1 \left(x_1^2 - \sum_{i=2}^{n+1} x_i^2 \right)$, $\mathcal{H}(h) = \{h = 1,\ x_1 > 0\}$ *has one connected component and it is closed.*

b) $h = x_1 \left(x_1^2 + x_2^2 - \sum_{i=3}^{n+1} x_i^2 \right)$, $\mathcal{H}(h) = \{h = 1\} \cap \{\frac{1}{\sqrt[3]{4}} > x_1 > 0\}$ *has two connected components. They are isomorphic and not closed.*

c) $h = x_1 \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right)$, $\mathcal{H}(h) = \{h = 1,\ x_1 < 0\}$ *has two connected components, both closed and isomorphic.*

d) $h = x_{n+1} \left(\sum_{i=1}^{n-1} x_i^2 - x_n^2 \right)$, $\mathcal{H}(h) = \{h = 1,\ x_{n+1} < 0\}$ *has two connected components, both closed and isomorphic.*

e) $h = (x_1 + x_{n+1}) \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right)$, $\mathcal{H}(h) = \{h = 1,\ x_1 + x_{n+1} < 0\}$ *has one connected component and it is closed.*

In particular, the closed connected components of the respective $\mathcal{H}(h)$ are complete projective special real manifolds.

Proof. In Proposition 7 we have listed all non-degenerate cubic homogeneous polynomials up to equivalence. It remains to determine which ones are hyperbolic and to analyse the properties of the connected components of $\mathcal{H}(h)$. In the following we treat each of the cases I-III) of Proposition 7.

I) Recall that the family I) of Proposition 7 contains the polynomials $h = x_{n+1}Q_k^n$, $\frac{n}{2} \leq k \leq n$, with

$$-\frac{1}{2}\partial^2 h = -x_{n+1}\left(\sum_{i=1}^{k} dx_i^2 - \sum_{i=k+1}^{n} dx_i^2\right) - 2\left(\sum_{i=1}^{k} x_i dx_i - \sum_{i=k+1}^{n} x_i dx_i\right) dx_{n+1}.$$

To check that a point $p \in \mathbb{R}^{n+1}$ is hyperbolic it suffices to construct an orthogonal basis of $T_p\mathbb{R}^{n+1}$ with respect to $-\frac{1}{2}\partial^2 h$ and to check that the Gram matrix has Lorentzian signature. Note that the vectors $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ are orthogonal at each point:

$$-\frac{1}{2}\partial^2 h(\partial_{x_i}, \partial_{x_j}) = \begin{cases} -\delta_i^j x_{n+1}, & 1 \leq i, j \leq k, \\ \delta_i^j x_{n+1}, & k+1 \leq i, j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Now the restrictions $n \geq 3$, $k \geq \frac{n}{2}$, allow us to limit the possibility of hyperbolic points to the cases $k = n-1$ and $k = n$ and we obtain the requirement $x_{n+1} < 0$. Otherwise we would have at least two time-like vectors in an orthogonal basis of the form $(v, \partial_{x_1}, \ldots, \partial_{x_n})$. For $v = \sum_{i=1}^{n+1} v_i \partial_{x_i}$ to be orthogonal to ∂_{x_i} for all $1 \leq i \leq n$ it has to fulfil

$$x_{n+1}v_i + x_i v_{n+1} = 0 \quad \forall 1 \leq i \leq n.$$

Hence, $v_i = -\frac{x_i v_{n+1}}{x_{n+1}}$ for $1 \leq i \leq n$ and $v = v_{n+1}\left(-\sum_{i=1}^{n} \frac{x_i}{x_{n+1}}\partial_{x_i} + \partial_{x_{n+1}}\right)$. Since $x_{n+1} < 0$, we might choose $v = \sum_{i=1}^{n} x_i \partial_{x_i} - x_{n+1}\partial_{x_{n+1}}$ and obtain

$$-\frac{1}{2}\partial^2 h(v,v) = x_{n+1}\left(\sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right) = h.$$

Hyperbolic points need to fulfil $h(p) > 0$ by definition, which implies $-\frac{1}{2}\partial^2 h(v,v) > 0$. Hence, $h = x_{n+1}Q_k^n$, $\frac{n}{2} \leq k \leq n$, is hyperbolic if and only if $k = n-1$, that is $h = x_{n+1}\left(\sum\limits_{i=1}^{n-1} x_i^2 - x_n^2\right)$ is the polynomial $d)$ of this proposition. The hypersurface $\mathcal{H}(h)$ consists of the connected components

$$\mathcal{H}_1 := \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid h(x_1, \ldots, x_{n+1}) = 1,\ x_n < 0, x_{n+1} < 0\right\}$$

and

$$\mathcal{H}_2 := \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid h(x_1, \ldots, x_{n+1}) = 1,\ x_n > 0, x_{n+1} < 0\right\}.$$

One can easily verify that $\mathcal{H}_1$ and $\mathcal{H}_2$ are both closed in $\mathbb{R}^{n+1}$ and related by the involution $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, -x_n, x_{n+1})$.

II) The family II) of Proposition 7 contains polynomials of the form $h = x_1 Q_k^{n+1}$, $1 \leq k \leq n+1$. We will construct an orthogonal basis for each $p \in \{h > 0\}$, $p = (x_1, \ldots, x_{n+1})$, with respect to

$$-\frac{1}{2}\partial^2 h = x_1 \left(-3dx_1^2 - \sum_{i=2}^{k} dx_i^2 + \sum_{i=k+1}^{n+1} dx_i^2 \right) - 2dx_1 \left(\sum_{i=2}^{k} x_i dx_i - \sum_{i=k+1}^{n+1} x_i dx_i \right).$$

We define

$$v = x_1 \partial_{x_1} - \sum_{i=2}^{n+1} x_i \partial_{x_i}.$$

Then one can check, for $x_1 \neq 0$, that $(v, \partial_{x_2}, \ldots, \partial_{x_{n+1}})$ is an orthogonal basis with respect to $-\frac{1}{2}\partial^2 h$ and that

$$-\frac{1}{2}\partial^2 h(v,v) = -4x_1^3 + h.$$

Thus, the possible values for k that do not exclude the possibility for h to be hyperbolic, the respective requirements for the possibly hyperbolic points, and the corresponding polynomials are (recall $n \geq 3$):

$A)$ $\quad k = 1, \quad x_1 > 0, \quad h < 4x_1^3 \ \left(-\frac{1}{2}\partial^2 h(v,v) < 0\right); \quad h = x_1 \left(x_1^2 - \sum_{i=2}^{n+1} x_i^2\right),$

$B)$ $\quad k = 2, \quad x_1 > 0, \quad h > 4x_1^3 \ \left(-\frac{1}{2}\partial^2 h(v,v) > 0\right); \quad h = x_1 \left(x_1^2 + x_2^2 - \sum_{i=3}^{n+1} x_i^2\right),$

$C)$ $\quad k = n, \quad x_1 < 0, \quad h > 4x_1^3 \ \left(-\frac{1}{2}\partial^2 h(v,v) > 0\right); \quad h = x_1 \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2\right),$

$D)$ $\quad k = n+1, \quad x_1 < 0, \quad h < 4x_1^3 \ \left(-\frac{1}{2}\partial^2 h(v,v) < 0\right); \quad h = x_1 \left(\sum_{i=1}^{n+1} x_i^2\right).$

The polynomials in A), B), and C) are, in fact, hyperbolic, as seen by specifying a hyperbolic point:

$$A) \quad p_A = (1, 0, \ldots, 0), \qquad h(p_A) = 1,$$

$$B) \quad p_B = (1, 2, 0, \ldots, 0), \qquad h(p_B) = 5,$$

$$C) \quad p_C = (-1, 0, \ldots, 0, 2), \quad h(p_C) = 3.$$

These three series of polynomials are, in the same order, the first three cases a), b), and c) of this proposition. The polynomials in D) are not hyperbolic, since the specified conditions are not compatible with $h > 0$. We will now describe the sets $\mathcal{H}(h)$.

In case A), the set of hyperbolic points of $\mathbb{R}^{n+1}$ with respect to h was described by the inequalities $x_1 > 0$ and $h < 4x_1^3$. The second inequality follows from the first since $Q_1^{n+1} \leq x_1^2$. This shows that $\mathcal{H}(h) = \{h = 1,\ x_1 > 0\}$, which has one connected component. To see this consider for fixed $u = (x_2, \ldots, x_{n+1}) \in \mathbb{R}^n$ the function

$$(\rho, \infty) \to \mathbb{R}, \ x_1 \mapsto h(x_1, u),$$

where $\rho = |u|$ and notice that it is a strictly monotonously increasing diffeomorphism onto $(0,\infty)$. In particular, for all $u \in \mathbb{R}^n$ there is a unique $x_1(u) \in (\rho,\infty)$ such that $h(x_1(u),u) = 1$. We obtain a bijection

$$\mathbb{R}^n \to \mathcal{H}(h), \quad u \mapsto (x_1(u),u),$$

which is a diffeomorphism by the implicit function theorem. In particular, $\mathcal{H}(h)$ is connected. This implies that it is a connected component of $\{h=1\}$ and, thus, closed in $\mathbb{R}^{n+1}$.

In case B), the requirement for hyperbolicity on $\{h = x_1\left(x_1^2 + x_2^2 - \sum_{i=3}^{n+1} x_i^2\right) = 1\}$ is $\frac{1}{\sqrt[3]{4}} > x_1 > 0$, which implies $x_2 \neq 0$. Observe that

$$h = 1 \;\Leftrightarrow\; x_2^2 = \frac{1}{x_1}(1 - x_1^3) + \sum_{i=3}^{n+1} x_i^2.$$

Hence, $\mathcal{H}(h) = \{h=1\} \cap \{\frac{1}{\sqrt[3]{4}} > x_1 > 0\}$ has two connected components, namely $\{h=1\} \cap \{\frac{1}{\sqrt[3]{4}} > x_1 > 0\} \cap \{x_2 > 0\}$ and $\{h=1\} \cap \{\frac{1}{\sqrt[3]{4}} > x_1 > 0\} \cap \{x_2 < 0\}$. They are related by the involution $x_2 \mapsto -x_2$, which preserves the polynomial h. The two components of $\mathcal{H}(h)$ are not closed in $\mathbb{R}^{n+1}$, since its boundary is given by

$$\partial\mathcal{H}(h) = \left\{h = 1,\, x_1 = \frac{1}{\sqrt[3]{4}},\, \det \partial^2 h = 0\right\} = \left\{h = 1,\, x_1 = \frac{1}{\sqrt[3]{4}}\right\} = \left\{x_2^2 - \sum_{i=3}^{n+1} x_i^2 = \frac{3}{4^{\frac{2}{3}}}\right\}.$$

In case C), the requirement $x_1 < 0$ automatically implies the second requirement $h > 4x_1^3$ on $\{h = x_1\left(\sum_{i=1}^n x_i^2 - x_{n+1}^2\right) = 1\}$ and, hence, $\mathcal{H}(h) = \{h = 1,\ x_1 < 0\}$. Note that $\{h=1\} \cap \{x_1 = 0\} = \emptyset$ implies that the connected components of $\mathcal{H}(h)$ are also connected components of $\{h=1\}$, and thus are closed. $x_1 < 0$ and $h = x_1\left(\sum_{i=1}^n x_i^2 - x_{n+1}^2\right) = 1$ implies $\sum_{i=1}^n x_i^2 - x_{n+1}^2 < 0$, which implies $x_{n+1} \neq 0$. Hence, the connected components of $\mathcal{H}(h)$ are given by the two graphs $\{h = 1,\ x_1 < 0,\ x_{n+1} > 0\}$ and $\{h = 1,\ x_1 < 0,\ x_{n+1} < 0\}$. They are related by the involution $x_{n+1} \mapsto -x_{n+1}$.

III) Recall that each $h = (x_1 + x_{n+1})Q_k^{n+1}$ contained in family III) of Proposition 7 is equivalent to $h = \xi\left(\xi\eta + \sum_{i=2}^k x_i^2 - \sum_{i=k+1}^n x_i^2\right)$. In these coordinates

$$\begin{aligned} -\frac{1}{2}\partial^2 h = &-\eta d\xi^2 - 2\xi d\eta d\xi + \left(-2\sum_{i=2}^k x_i dx_i + 2\sum_{i=k+1}^n x_i dx_i\right) d\xi \\ &+ \xi\left(-\sum_{i=2}^k dx_i^2 + \sum_{i=k+1}^n dx_i^2\right). \end{aligned}$$

The set $\left\{h = \xi\left(\xi\eta + \sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right) = 1\right\}$ consists of exactly two connected components:

$$\mathcal{H}_1 := \left\{(\xi, \eta, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \eta = \frac{1 - \xi\left(\sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right)}{\xi^2},\ \xi > 0\right\}$$

and

$$\mathcal{H}_2 := \left\{(\xi, \eta, x_2, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \eta = \frac{1 - \xi\left(\sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right)}{\xi^2},\ \xi < 0\right\}.$$

In order to determine which of the polynomials in this family are hyperbolic, we will pull back $-\frac{1}{2}\partial^2 h$ to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We will use that h is hyperbolic if and only if the pullback is Riemannian at least at one point contained in $\{h = 1\}$. We first determine the differential of $\eta = \eta(\xi, x_2, \ldots, x_n)$:

$$d\eta = \frac{-2 + \xi\left(\sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right)}{\xi^3} d\xi + \frac{-2\sum_{i=2}^{k} x_i dx_i + 2\sum_{i=k+1}^{n} x_i dx_i}{\xi}.$$

Hence, the pullback of $-\frac{1}{2}\partial^2 h$ to $\mathcal{H}_j$ which we denote by g_j, $j \in \{1, 2\}$, is of the following form:

$$\begin{aligned} g_j = {} & \frac{3 - \xi\left(\sum_{i=2}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2\right)}{\xi^2} d\xi^2 + 2\left(\sum_{i=2}^{k} x_i dx_i - \sum_{i=k+1}^{n} x_i dx_i\right) d\xi \\ & + \xi\left(-\sum_{i=2}^{k} dx_i^2 + \sum_{i=k+1}^{n} dx_i^2\right). \end{aligned}$$

For each $\frac{n+1}{2} \leq k \leq n$ there exists exactly one $\widetilde{k}$ with $1 \leq \widetilde{k} \leq \frac{n+1}{2}$, such that $\mathcal{H}_1$ corresponding to $h = (x_1 + x_{n+1})Q_k^{n+1}$ is isometric to $\mathcal{H}_2$ corresponding to $\widetilde{h} = (x_1 + x_{n+1})Q_{\widetilde{k}}^{n+1}$, namely $\widetilde{k} = n - (k-1)$. In the coordinates $(\xi, \eta, x_2, \ldots, x_n)$ the corresponding isometry is given by $\xi \mapsto -\xi$, $x_\ell \mapsto x_{n-(\ell-2)}$ for $2 \leq \ell \leq n$. Hence, we can reduce our analysis to $\mathcal{H}_1$, that is $\xi > 0$, but need to increase the range for k to $1 \leq k \leq n$.

Returning to the study of g_1, we obtain

$$g_1(\partial_{x_i}, \partial_{x_j}) = \begin{cases} -\delta_i^j \xi, & 2 \leq i, j \leq k, \\ \delta_i^j \xi, & k+1 \leq i, j \leq n. \end{cases}$$

For g_1 to be Riemannian, this implies that $k = 1$. Hence, the only possibly hyperbolic polynomial is $h = \xi\left(\xi\eta - \sum_{k=2}^{n} x_i^2\right)$ and the corresponding metric g_1 reads

$$g_1 = \frac{3}{\xi^2} d\xi^2 + \frac{1}{\xi} \sum_{i=2}^{n} (x_i d\xi - \xi dx_i)^2,$$

which is indeed Riemannian at all points of $\mathcal{H}_1$. Hence, the only hyperbolic polynomial of the form $h = (x_1 + x_{n+1})Q_k^{n+1}$, $\frac{n+1}{2} \leq k \leq n$, is given by

$$h = (x_1 + x_{n+1}) \left(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2 \right).$$

The corresponding $\mathcal{H}(h) = \{h = 1,\ x_1 + x_{n+1} < 0\}$ has a single connected component. It is closed in $\mathbb{R}^{n+1}$, since $\{h = 1\} \cap \{x_1 + x_{n+1} = 0\} = \emptyset$ implies that $\mathcal{H}(h)$ is also a connected component of $\{h = 1\}$. This polynomial is the polynomial $e)$ of this proposition. $\square$

2 Two multi-parameter families of complete projective special real manifolds

Let $n \geq 3,\ n \in \mathbb{N}$. We will give two examples of $(n-2)$-parameter families in $S^3(\mathbb{R}^{n+1})^*$, each consisting of pairwise inequivalent hyperbolic cubic polynomials which define a complete projective special real manifold of dimension n. We will use this result to find a curve in $S^3(\mathbb{R}^{n+1})^*$, such that each point in the curve is a hyperbolic polynomial which defines a complete projective special real manifold and that the endpoints of that curve are linearly equivalent to the polynomials $a)$ and $b)$ in Theorem 2.

In the following we will denote $z = (z_1, \ldots, z_{n-1})^T$ and by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product on $\mathbb{R}^{n-1} \subset \mathbb{R}^{n+1} = \left\{ \left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix} \right) \middle| \, z \in \mathbb{R}^{n-1}, w, x \in \mathbb{R} \right\}$.

Theorem 9. *The $(n-2)$-parameter families*

$$\mathcal{F} := \left\{ h = x(-w^2 + \langle z, z \rangle) + w \sum_{i=1}^{n-1} b_i z_i^2 \;\middle|\; 1 = b_1 \geq \ldots \geq b_{n-1} \geq 0 \right\}$$

and

$$\mathcal{G} := \left\{ h = x \left(-w^2 + \sum_{i=1}^{n-1} b_i z_i^2 \right) + w \langle z, z \rangle \;\middle|\; 1 = b_1 \geq \ldots \geq b_{n-1} \geq 0 \right\}$$

consist of pairwise inequivalent hyperbolic cubic polynomials. The corresponding projective special real manifolds

$$\mathcal{H}(h) = \left\{ h = 1 \;\middle|\; h \in \mathcal{F},\ x < 0,\ w < 0,\ w^2 > \langle z, z \rangle \right\}$$

and

$$\mathcal{H}(h) = \left\{ h = 1 \;\middle|\; h \in \mathcal{G},\ x < 0,\ w < 0,\ w^2 > \sum_{i=1}^{n-1} b_i z_i^2 \right\},$$

respectively, are complete.

Proof. Let $M, N \in \mathrm{Mat}((n-1)\times(n-1), \mathbb{R})$ be symmetric positive semi-definite matrices, such that $\mathrm{rk}(M) = (n-1)$ or $\mathrm{rk}(N) = (n-1)$, and denote by $M(z,z) = z^T M z,\ N(z,z) = z^T N z$. We will show that

$$h = x\left(-w^2 + N(z,z)\right) + wM(z,z)$$

is hyperbolic for any such M and N on the set $\mathcal{H} := \{h = 1 \mid x < 0,\ w < 0,\ w^2 > N(z,z)\}$. Consider the vector fields ∂_w and $w\partial_w - x\partial_x$, which are both non-vanishing along $\mathcal{H}$. One can check that they are orthogonal to each other with respect to

$$\begin{aligned} g &= -\frac{1}{2}\partial^2 h \\ &= -xN(dz,dz) - wM(dz,dz) + xdw^2 - 2M(z,dz)dw - 2N(z,dz)dx + 2wdwdx, \end{aligned}$$

and that $g(\partial_w, \partial_w) = x < 0$, $g(w\partial_w - x\partial_x, w\partial_w - x\partial_x) = -xw^2 > 0$ along $\mathcal{H}$. In the above formula dz is considered as column vector with components dz_i. We will now show that g is positive definite on the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\{\partial_w, w\partial_w - x\partial_x\}$ along $\mathcal{H}$ with respect to g and thereby prove our claim. One can easily verify that every vector field Y along $\mathcal{H}$ which is perpendicular to $\mathrm{span}_{\mathbb{R}}\{\partial_w, w\partial_w - x\partial_x\}$ can be written as

$$Y = X + \frac{N(z,X)}{w}\partial_w + \frac{wM(z,X) - xN(z,X)}{w^2}\partial_x,$$

where $X = \sum_{i=1}^{n-1} X^i \partial_{z_i}$. Note that $Y = 0$ if and only if $X = 0$. We obtain

$$g(Y,Y) = \frac{1}{w^2}\left(-xw^2 N(X,X) - w^3 M(X,X) - 2wM(z,X)N(z,X) + xN(z,X)^2\right).$$

If $0 \neq X \in \ker N$ it follows by assumption that $M > 0$ and, hence, $g(Y,Y) > 0$ along $\mathcal{H}$. Assume now that $N(X,X) \neq 0$. Observe that $h = 1$ is equivalent to $-xw^2 = 1 - wM(z,z) - xN(z,z)$. Hence, along $\mathcal{H}$ we have

$$\begin{aligned} & -xw^2 N(X,X) + xN(z,X)^2 \\ = \ & \underbrace{N(X,X)}_{>0} - x(\underbrace{N(X,X)N(z,z) - N(z,X)^2}_{\geq 0}) - wM(z,z)N(X,X) \\ > \ & -wM(z,z)N(X,X). \end{aligned}$$

Using this estimate and $w^2 > N(z,z)$, we obtain

$$g(Y,Y) > \frac{1}{-w}(M(z,z)N(X,X) + 2M(z,X)N(z,X) + M(X,X)N(z,z))$$

along $\mathcal{H}$. If $z \in \ker N$, it follows that $g(Y,Y) > 0$. Assume that $z \notin \ker N$. Consider

$$Q(z,X,\widetilde{z},\widetilde{X}) := M(\widetilde{z},\widetilde{z})N(X,X) + 2M(\widetilde{z},\widetilde{X})N(z,X) + M(\widetilde{X},\widetilde{X})N(z,z).$$

One observes that $Q(z, X, \widetilde{z}, \widetilde{X}) \geq 0$ for all $z, X, \widetilde{z}, \widetilde{X} \in \mathbb{R}^{n-1}$ if $M(\widetilde{z}, \widetilde{z})M(\widetilde{X}, \widetilde{X}) \geq M(\widetilde{z}, \widetilde{X})^2$ for all $\widetilde{z}, \widetilde{X} \in \mathbb{R}^{n-1}$. The latter estimate is true since M is positive semi-definite. Hence, $Q(z, X, z, X) \geq 0$ for all $z, X \in \mathbb{R}^{n-1}$, which shows that $g(Y, Y) > 0$ for $Y \neq 0$. This proves that the pullback of g to $\mathcal{H}$ is a Riemannian metric, so that $\mathcal{H}$ is a projective special real manifold.

We will now show that $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed in the subspace topology. Notice that $\mathcal{H}$ can be written as a graph over $U := \{w < 0,\ w^2 > N(z, z)\} \subset \mathbb{R}^n$ by rewriting the equation $h = 1$ as $x = \frac{1 - wM(z,z)}{-w^2 + N(z,z)}$. We need to check that $x \to -\infty$ for $(w, z) \to \partial U$. Observe that $\partial U = \{w \leq 0,\ -w^2 + N(z, z) = 0\}$. For $(z, w) \in U$ we have

$$x = \frac{1 - wM(z, z)}{-w^2 + N(z, z)} \leq \frac{1}{-w^2 + N(z, z)}$$

and the right-hand side goes to $-\infty$ for all sequences in $\{(z(j), w(j)),\ j \in \mathbb{N}\} \subset U$ with the property $\lim_{j \to \infty} (-w(j)^2 + N(z(j), z(j))) = 0$. This shows that $\partial\mathcal{H}$ is empty and, hence, that $\mathcal{H}$ is closed in $\mathbb{R}^{n+1}$. By [CNS, Thm. 2.5] this implies that the projective special real manifold $\mathcal{H}$ is complete.

Summarizing, we have shown that $\mathcal{H}(h)$ is a complete projective special real manifold for all $h \in \mathcal{F}$ and all $h \in \mathcal{G}$. It remains to show that $\mathcal{F}$ and $\mathcal{G}$ each consist of pairwise inequivalent polynomials.

We will start with the family $\mathcal{F}$. We define

$$K := \{x(-w^2 + \langle z, z\rangle) + wM(z, z) \mid 0 \neq M \geq 0\}$$

and see that for all $h \in K$, $\mathcal{H}(h) = \{h = 1 \mid h \in K,\ x < 0,\ w < 0,\ w^2 > \langle z, z\rangle\}$ is a complete special real manifold. This follows from setting $N(\cdot, \cdot) = \langle \cdot, \cdot \rangle$. Furthermore, $\mathcal{F} \subset K$. In order to study equivalence classes of elements of K, it turns out that we have to study the cases (i) $\dim \ker M \neq 1$ and (ii) $\dim \ker M = 1$ separately. In both cases we will make use of properties of the singularity set $\{dh = 0\}$. For a given $h \in K$ we will determine all possible $A \in \mathrm{GL}(n+1)$, such that $h \circ A \in K$. In case (i) we will see that this set of transformations is independent of the chosen h. In case (ii) it will turn out that this set of transformations will depend on the chosen h. We will then use the results to show that $\mathcal{F} \subset K$ consists of pairwise inequivalent polynomials and that for each polynomial $h \in K$ there is a unique representative in $\mathcal{F}$ of the $\mathrm{GL}(n+1)$-orbit of h.

For case (i) we will employ the following lemma.

Lemma 10. *Let $h \in K$ and M the corresponding positive semi-definite bilinear form, such that* $\dim \ker M \neq 1$. *Then for $A \in \mathrm{GL}(n+1)$, $h \circ A \in K$ if and only if A is of the* *form*

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right), \ r > 0, \ E \in \mathrm{O}(n-1).$$

Proof. (of Lemma 10) Observe that for all $A \in \mathrm{GL}(n+1)$, $dh_p = 0$ if and only if $d(h \circ A)_{A^{-1}p} = 0$, i.e. $\{d(h \circ A) = 0\}$ is precisely the image of $\{dh = 0\}$ under A^{-1}. First we describe $\{dh = 0\}$ explicitly. We have

$$dh = 2x\langle z, dz\rangle + 2wM(z, dz) + (-2xw + M(z, z))dw + (-w^2 + \langle z, z\rangle)dx.$$

To determine the points $p = (z, w, x)$ such that $dh_p = 0$ we distinguish the cases $w = 0$ and $w \neq 0$. If $w = 0$ then $dh_p = 0$ if and only if $z = 0$. If $w \neq 0$ then $dh_p = 0$ if and only if $w^2 = \langle z, z\rangle$, $z \in \ker M$, and $x = 0$. To see this it suffices to substitute $2xw = M(z, z)$ and $w^2 = \langle z, z\rangle$ into $2xw\langle z, dz\rangle + 2w^2M(z, dz) = 0$ and insert the position vector z on the left hand side of the latter equation. We have thus determined the set $\{dh = 0\}$ and see that the cone $\{dh = 0\} \setminus \{0\}$ has the following components :

$$\begin{aligned} \{dh = 0\} \setminus \{0\} = \{z = 0, \ w = 0, \ x > 0\} &\dot\cup \{z = 0, \ w = 0, \ x < 0\} \\ &\dot\cup \{z \in \ker M \setminus \{0\}, \ w = \sqrt{\langle z, z\rangle}, \ x = 0\} \\ &\dot\cup \{z \in \ker M \setminus \{0\}, \ w = -\sqrt{\langle z, z\rangle}, \ x = 0\}. \end{aligned}$$

The latter two sets are either smooth manifolds of dimension $\dim\ker M$ in the case that $\dim\ker M \neq 0$, or empty if $M > 0$. By assumption they are not of dimension 1 and, hence, connected. Since A^{-1} maps connected components of $\{dh = 0\} \setminus \{0\}$ to connected components of $\{d(h \circ A) = 0\} \setminus \{0\}$, we see that if $\bar{h} = h \circ A$ is contained in K and, hence, associated with some $\overline{M} \geq 0$, then M and $\overline{M}$ have the same rank and A maps the line $\{z = 0, \ w = 0, \ x \in \mathbb{R}\}$ to itself. Note that it is precisely at this point that we have used the condition $\dim\ker M \neq 1$. This means that A has the following form:

$$A = \left(\begin{array}{c|c} B & \\ \hline (\alpha^T, \beta) & r \end{array}\right), \ B \in \mathrm{Mat}(n \times n, \ \mathbb{R}), \ \alpha \in \mathbb{R}^{n-1}, \ \beta \in \mathbb{R}, \ r \in \mathbb{R} \setminus \{0\}.$$

By writing down $(h \circ A)(z, w, x)$, one can easily verify that $r > 0$ and $B = r^{-\frac{1}{2}}C$, $C \in \mathrm{O}(n-1, 1)$, are necessary for $h \circ A$ to be contained in K. Here $\mathrm{O}(n-1, 1)$ is the automorphism group of the quadratic form $-w^2 + \langle z, z\rangle$ on $\mathbb{R}^n$. Using the notation $C\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right)$ we obtain

$$(h \circ A)(z, w, x) = x(-w^2 + \langle z, z\rangle) + r^{-\frac{3}{2}}((\langle r^{\frac{1}{2}}\alpha, z\rangle + r^{\frac{1}{2}}\beta w)(-w^2 + \langle z, z\rangle) + \widetilde{w}M(\widetilde{z}, \widetilde{z})).$$

C is of the form

$$C = \left(\begin{array}{c|c} E & \xi \\ \hline \eta^T & \mu \end{array}\right), \ E \in \mathrm{Mat}((n-1) \times (n-1), \ \mathbb{R}), \ \eta, \xi \in \mathbb{R}^{n-1}, \ \mu \in \mathbb{R},$$

and fulfils

$$C^T \left(\begin{array}{c|c} \mathbf{1} & \\ \hline & -1 \end{array}\right) C = \left(\begin{array}{c|c} \mathbf{1} & \\ \hline & -1 \end{array}\right).$$

The left hand side of the above equation equals

$$\left(\begin{array}{c|c} E^T E - \eta \otimes \langle \eta, \cdot \rangle & E^T \xi - \mu\eta \\ \hline \xi^T E - \mu\eta^T & \langle \xi, \xi \rangle - \mu^2 \end{array}\right),$$

which in particular implies that $\mu \neq 0$ and $\mathrm{rk}\, E = n-1$. To see the latter, assume that there exists a $0 \neq v \in \ker E$. Since $E^T\xi - \mu\eta = 0$, it follows that $\eta = \mu^{-1}E^T\xi$. Hence,

$$\begin{aligned}(E^T E - \eta \otimes \langle \eta, \cdot \rangle)v &= E^T E v - \mu^{-2} E^T \xi \langle E^T \xi, v\rangle \\ &= -\mu^{-2} E^T \xi \langle \xi, Ev \rangle = 0,\end{aligned}$$

which contradicts the assumption that $E^T E - \eta \otimes \langle \eta, \cdot \rangle = \mathbf{1}$. With $\kappa := r^{\frac{1}{2}}\alpha$ and $\rho := r^{\frac{1}{2}}\beta$,

$$\begin{aligned}(h \circ A)(z, w, x) = x(&-w^2 + \langle z, z\rangle) \\ &+ r^{-\frac{3}{2}}(w^3(\mu M(\xi,\xi) - \rho) \qquad (1) \\ &+ w^2(2\mu M(Ez,\xi) + \langle \eta, z\rangle M(\xi,\xi) - \langle \kappa, z\rangle) \qquad (2) \\ &+ w(\mu M(Ez,Ez) + 2\langle \eta, z\rangle M(Ez,\xi) + \rho\langle z,z\rangle) \\ &+ \langle \eta, z\rangle M(Ez,Ez) + \langle \kappa, z\rangle\langle z,z\rangle). \qquad (3)\end{aligned}$$

The requirements for $h \circ A$ to be contained in K are (1) = (2) = (3) = 0 and

$$\mu M(Ez,Ez) + 2\langle \eta, z\rangle M(Ez,\xi) + \rho\langle z,z\rangle \geq 0 \ \forall z \in \mathbb{R}^{n-1}. \qquad (4)$$

We will show that this implies $\kappa = 0$ and $\rho = 0$ and, consequently, $\alpha = 0$ and $\beta = 0$. Firstly, we will show that $\rho = 0$ implies $\kappa = 0$, and secondly that a transformation with $\rho \neq 0$ contradicts the requirement $C \in \mathrm{O}(n-1,1)$.

Assume $\rho = 0$. Then (1) is equivalent to $M(\xi,\xi) = 0$. Since $M \geq 0$, this implies $\xi \in \ker M$. Equation (2) is thus equivalent to $\langle \kappa, z\rangle = 0$ for all $z \in \mathbb{R}^{n-1}$. This shows $\kappa = 0$.

Now assume that $\rho \neq 0$. Then by equation (1)

$$M(\xi,\xi) = \mu^{-1}\rho.$$

Note that this implies $\mu^{-1}\rho > 0$ and in particular $\xi \notin \ker M$. Inserting the above equation in (2) yields

$$2\mu M(Ez,\xi) + \langle \eta, z\rangle \mu^{-1}\rho = \langle \kappa, z\rangle.$$

Using that, (3) becomes

$$\langle \eta, z\rangle(M(Ez,Ez) + \mu^{-1}\rho\langle z,z\rangle) + 2\mu M(Ez,\xi)\langle z,z\rangle = 0.$$

Since $C \in \mathrm{O}(n-1,1)$, we have $\eta = \mu^{-1}E^T\xi$ and, hence,

$$\langle z, E^T\xi\rangle\big(\underbrace{M(Ez,Ez) + \mu^{-1}\rho\langle z,z\rangle}_{>0\ \forall z\neq 0}\big) + \langle z, E^T M\xi\rangle\, \underbrace{2\mu^2\langle z,z\rangle}_{>0\ \forall z\neq 0} = 0.$$

An immediate consequence is that $E^T\xi$ and $E^TM\xi$ are linearly dependent. Since $\ker E^T = \{0\}$ and $\xi \notin \ker M$ this is equivalent to $E^TM\xi = sE^T\xi$ for some $s \in \mathbb{R}\setminus\{0\}$, which shows that $M\xi = s\xi$, that is ξ needs to be an eigenvector of M. This also shows $s > 0$. Hence,

$$\langle z, E^T\xi\rangle\big(\underbrace{M(Ez,Ez) + (\mu^{-1}\rho + 2\mu^2 s)\langle z,z\rangle}_{>0\ \forall z\neq 0}\big) = 0.$$

This shows that $E^T\xi = 0$ which contradicts $\ker E = \{0\}$. This proves $\rho = 0$, $\kappa = 0$, and $\xi \in \ker M$.

Summarizing, we have shown that A needs to be of the form

$$A = \left(\begin{array}{c|c} r^{-\frac{1}{2}}C & \\ \hline & r \end{array}\right),\ C \in \mathrm{O}(n-1,1),\ r > 0.$$

For such A, equations (1) and (2) are automatically fulfilled, and equation (3) becomes

$$\langle \eta, z\rangle M(Ez,Ez) = 0. \tag{3}$$

Since rk $E = n-1$ we know that $M(Ez,Ez)$ is a non-vanishing quadratic polynomial. Hence, (3) is true if and only if $\eta = 0$. As we have seen before, $\eta = 0$ implies $\xi = 0$ since $C \in \mathrm{O}(n-1,1)$. Observe that $\xi = 0$ and $C \in \mathrm{O}(n-1,1)$ also imply $-\mu^2 = -1$. The inequality (4) becomes $\mu M(Ez,Ez) \geq 0$, from which we deduce that $\mu = 1$. Hence, all possible transformations such that $h \circ A \in K$ with

$$h = x(-w^2 + \langle z,z\rangle) + wM(z,z),\ M \geq 0,\ M \neq 0,\ \dim\ker M \neq 1,$$

can be written as

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right),\ E \in \mathrm{O}(n-1),\ r > 0, \tag{2.1}$$

independent of the choice of $h \in K$. $\square$

Next, we will deal with case (ii).

Lemma 11. *Let $A \in \mathrm{GL}(n+1)$, $h \in K$ and M the corresponding positive semi-definite bilinear form, such that* $\dim\ker M = 1$. *Then $h \circ A \in K$ if and only if M has at least* 2 *distinct positive eigenvalues and A is of the form (2.1) or, if M has precisely* 1 *positive eigenvalue, A can be written as a product of transformations of the form (2.1) and*

$$\left(\begin{array}{c|c|c|c} \mathbf{1} & & & \\ \hline & \frac{1}{2} & \frac{-1}{2} & 1 \\ \hline & \frac{-1}{2} & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right).$$

Furthermore, in the case when M has precisely 1 *positive eigenvalue the sets* $\{h \circ A \mid A \in \mathrm{GL}(n+1),\ h \circ A \in K\}$ *and* $\{h \circ A \mid A$ *is of the form* $(2.1)\}$ *coincide.*

Proof. (of Lemma 11) In case (ii), that is $\dim \ker M = 1$, $\{dh = 0\}$ consists of 3 distinct lines that intersect at $0 \in \mathbb{R}^{n+1}$,

$$\begin{aligned} \{dh = 0\} = &\{z = 0,\ w = 0,\ x \in \mathbb{R}\} \\ &\cup \{z \in \ker M,\ w = \sqrt{\langle z, z\rangle},\ x = 0\} \\ &\cup \{z \in \ker M,\ w = -\sqrt{\langle z, z\rangle},\ x = 0\}. \end{aligned}$$

Note that each of the latter two sets is not a line, but their union is a union of two distinct lines. Contrary to case (i) we can no longer assume that a transformation mapping $h = x(-w^2 + \langle z, z\rangle) + wM(z, z) \in K$ to $\overline{h} = x(-w^2 + \langle z, z\rangle) + w\overline{M}(z, z) \in K$ preserves the line $\{z = 0,\ w = 0,\ x \in \mathbb{R}\}$, since all connected components of $\{dh = 0\} \setminus \{0\}$ are of dimension one. Note that we can, after a possible orthogonal transformation of the z-coordinates, assume that

$$M = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_{n-2} & \\ & & & 0 \end{pmatrix}, \quad \overline{M} = \begin{pmatrix} \overline{\lambda}_1 & & & \\ & \ddots & & \\ & & \overline{\lambda}_{n-2} & \\ & & & 0 \end{pmatrix},$$

which in particular implies $\ker M = \ker \overline{M}$. Thus in addition to the transformations (2.1), considered in case (i), we need to consider transformations of the form

$$A = \left(\begin{array}{c|c|c} E & \xi & v \\ \hline \eta^T & \mu & \pm\|v\| \\ \hline \alpha^T & \beta & 0 \end{array}\right),\ v \in \ker M \setminus \{0\},$$

which map $\{z = 0,\ w = 0,\ x \in \mathbb{R}\}$ to either $\{z = rv,\ w = r\|v\|,\ x = 0 \mid r \in \mathbb{R}\}$ or $\{z = rv,\ w = -r\|v\|,\ x = 0 \mid r \in \mathbb{R}\}$, and are required to preserve $\{dh = 0\} = \{d\overline{h} = 0\}$. By calculating $(h \circ A)(z, w, x)$, we obtain the following system of equations, which is equivalent to $h \circ A = \bar{h}$:

$$\begin{aligned}
\mp 2\|v\|\beta\langle \eta, z\rangle + 2\beta\langle Ez, v\rangle \pm 2\|v\|M(Ez,\xi) \mp 2\|v\|\mu\langle \alpha, z\rangle + 2\langle \xi, v\rangle\langle \alpha, z\rangle &= 0 && (1)\\
\beta(-\mu^2 + \langle \xi,\xi\rangle) + \mu M(\xi,\xi) &= 0 && (2)\\
\langle \alpha, z\rangle(-\mu^2 + \langle \xi,\xi\rangle) + \langle \eta, z\rangle(-2\beta\mu + M(\xi,\xi)) + 2\beta\langle Ez,\xi\rangle + 2\mu M(Ez,\xi) &= 0 && (3)\\
-\langle \alpha, z\rangle\langle \eta, z\rangle^2 + \langle \alpha, z\rangle\langle Ez, Ez\rangle + \langle \eta, z\rangle M(Ez,Ez) &= 0 && (4)\\
\mp 2\|v\|\beta\mu + 2\beta\langle \xi, v\rangle \pm \|v\|M(\xi,\xi) &= -1 && (5)\\
\langle \alpha, z\rangle(\mp 2\|v\|\langle \eta, z\rangle + 2\langle Ez, v\rangle) \pm \|v\|M(Ez,Ez) &= \langle z, z\rangle && (6)\\
-2\mu\langle \alpha, z\rangle\langle \eta, z\rangle + 2\langle \alpha, z\rangle\langle Ez,\xi\rangle - \beta\langle \eta, z\rangle^2 & \\
+\beta\langle Ez, Ez\rangle + 2\langle \eta, z\rangle M(Ez,\xi) + \mu M(Ez,Ez) &= \overline{M}(z,z) && (7)
\end{aligned}$$

We will show that such a transformation exists if and only if $\lambda_1 = \ldots = \lambda_{n-2}$.

Claim 1: $\dim\ker E \leq 1$.

Proof. In general, $\dim\ker\langle \alpha, \cdot\rangle \geq n-2$. Assume $\dim\ker E > 1$. Then there exists $Y \in \mathbb{R}^{n-1} \setminus \{0\}$, such that $Y \in \ker\langle \alpha, \cdot\rangle \cap \ker E$. Hence, by equation (6), $0 = \langle Y, Y\rangle$, which is a contradiction to $Y \neq 0$. □

Claim 2: $\dim\ker E = 1 \Rightarrow \ker E \not\subset \ker\langle \alpha, \cdot\rangle$.

Proof. Assume $\dim\ker E = 1$ and $\ker E \subset \ker\langle \alpha, \cdot\rangle$, and let $0 \neq Y \in \ker E$. Again, equation (6) implies $0 = \langle Y, Y\rangle$ and, hence, contradicts $Y \neq 0$. □

Claim 3: $\dim\ker E = 1 \Rightarrow \ker E \subset \ker\langle \eta, \cdot\rangle$.

Proof. Let $0 \neq Y \in \ker E$. Equation (4) reads

$$-\underbrace{\langle \alpha, Y\rangle}_{\neq 0}\langle \eta, Y\rangle^2 = 0,$$

which shows that $Y \in \ker\langle \eta, \cdot\rangle$. □

Claim 4: $\dim\ker E = 0$.

Proof. Assume that $\dim\ker E \neq 0$. We have shown that the only other possible case would be $\dim\ker E = 1$. For $0 \neq Y \in \ker E$, we have also shown that $Y \in \ker\langle \eta, \cdot\rangle$. Now equation (6) implies $0 = \langle Y, Y\rangle$, which, again, contradicts $Y \neq 0$. Hence, we have shown that $\ker E = \{0\}$, i.e. $E \in \mathrm{GL}(n-1)$. □

Claim 5: $\alpha \neq 0$**.**

Proof. Assume $\alpha = 0$. Equation (6) is now equivalent to

$$\pm\|v\|E^T ME = \mathbf{1}.$$

Since $E \in \mathrm{GL}(n-1)$, this implies that M is invertible, which contradicts the assumption $\dim\ker M = 1$. $\square$

Claim 6: $\eta = s\alpha,\ s \neq 0$**.**

Proof. If $\eta \notin \mathbb{R}\alpha \setminus \{0\}$ then there exists $Y \in \ker\langle\eta, \cdot\rangle$, such that $\langle\alpha, Y\rangle \neq 0$. Together with $E \in \mathrm{GL}(n-1)$ this implies $\langle\alpha, Y\rangle\langle EY, EY\rangle \neq 0$, which contradicts equation (4). $\square$

Claim 7: $A\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}v\\\|v\|\\0\end{smallmatrix}\right)$**.**

Proof. Assume on the contrary that $A\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}v\\-\|v\|\\0\end{smallmatrix}\right)$. Then for all $Y \in \ker\langle\alpha, \cdot\rangle$ equation (6) implies $-\|v\|M(EY, EY) = \langle Y, Y\rangle$. But M is positive semi-definite, hence this is a contradiction. Note that this means that in equations (1)-(7), every "$\pm$" needs to be "$+$", and every "$\mp$" needs to be "$-$". $\square$

Claim 8: $\xi \in \ker M$**.**

Proof. By construction, A is required to map the set $\{dh = 0\} = \{d\overline{h} = 0\}$ onto itself, that is induces a permutation of the three lines $\mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\mathbb{R}\begin{pmatrix}v\\\|v\|\\0\end{pmatrix}$, and $\mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}$. We already know that the first line is mapped to the second. Therefore, either

$$A\begin{pmatrix}v\\\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}, \tag{a}$$

or

$$A\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}v\\\|v\|\\0\end{pmatrix} \in \mathbb{R}\begin{pmatrix}v\\-\|v\|\\0\end{pmatrix}. \tag{b}$$

In case (a), $Ev + \|v\|\xi = 0$, and, hence, using the second equation in (a), $Ev - \|v\|\xi = -2\|v\|\xi \in \mathbb{R}v = \ker M$. Similarly, in case (b) we have $Ev - \|v\|\xi = 0$, showing that $Ev + \|v\|\xi = 2\|v\|\xi \in \mathbb{R}v$. $\square$

In the following we will write $\xi = kv$, $k \in \mathbb{R}$.

Claim 9: $\beta \neq 0$**.**

Proof. This follows from the previous claim and equation (5). $\square$

Claim 10: $\xi = -\frac{1}{4\beta\langle v,v\rangle}v$**,** $\mu = \frac{1}{4\beta\|v\|}$**,** $s = -\frac{1}{4\beta^2\|v\|}$**,** $\alpha = 4\beta^2 E^T v$**.**

Proof. We have shown that $\beta \neq 0$ and $\xi = kv \in \ker M$. Hence, (2) implies $\mu = \pm k\|v\|$. Furthermore the previous results imply that (5) is equivalent to $-2\|v\|\beta\mu + 2\beta k\langle v, v\rangle = -1$. This shows that $\mu = -k\|v\|$ and, hence,

$$k = -\frac{1}{4\beta\langle v, v\rangle}, \quad \mu = \frac{1}{4\beta\|v\|}.$$

One can easily check that equation (3) is equivalent to $\langle \alpha, z\rangle(-2\beta\mu s) + 2\beta\langle Ez, \xi\rangle = 0$, which shows that

$$\langle \alpha, z\rangle = -\frac{1}{s\|v\|}\langle Ez, v\rangle.$$

Using this, equation (1) is equivalent to

$$s = \frac{k\|v\|}{\beta} = -\frac{1}{4\beta^2\|v\|}.$$

Hence, $\langle \alpha, z\rangle = 4\beta^2\langle Ez, v\rangle$. $\square$

The restrictions derived from the equations (1)–(7) in the above series of claims already imply the equations (1), (2), (3), and (5). With the above results, one can show that the remaining equations (4), (6), and (7) are equivalent to

$$-\frac{1}{\langle v, v\rangle}\langle Ez, v\rangle^2 + \langle Ez, Ez\rangle - \frac{1}{4\beta^2\|v\|}M(Ez, Ez) = 0, \tag{4'}$$

$$16\beta^2\langle Ez, v\rangle^2 + \|v\|M(Ez, Ez) = \langle z, z\rangle, \tag{6'}$$

$$-\frac{\beta}{\langle v, v\rangle}\langle Ez, v\rangle^2 + \beta\langle Ez, Ez\rangle + \frac{1}{4\beta\|v\|}M(Ez, Ez) = \overline{M}(z, z), \tag{7'}$$

respectively.

Claim 11: $\overline{M}(z, z) = \frac{1}{2\beta\langle v,v\rangle}\langle z, z\rangle - \frac{8\beta}{\langle v,v\rangle}\langle Ez, v\rangle^2$**.**

Proof. By multiplying both sides of equation (4′) with $-\beta$ and adding them to (7′) we obtain

$$\frac{1}{2\beta\|v\|}M(Ez, Ez) = \overline{M}(z, z).$$

By considering equation (6′) we see that $\frac{1}{2\beta\|v\|}M(Ez, Ez) = \frac{1}{2\beta\langle v,v\rangle}\langle z, z\rangle - \frac{8\beta}{\langle v,v\rangle}\langle Ez, v\rangle^2$, which proves the claim. $\square$

Claim 12: E is of the form $E = \left(\begin{array}{c|c} B & \\ \hline & \pm\frac{1}{4\beta\|v\|} \end{array}\right)$, $B \in \mathrm{GL}(n-2)$.

Proof. By the assumption $\overline{M}(z,z) = \sum_{i=1}^{n-2} \overline{\lambda}_i z_i^2$, it follows that either $v = \|v\|\partial_{z_{n-1}}$, or $v = -\|v\|\partial_{z_{n-1}}$. Note that the sign does not depend on the cases (a) and (b) described in Claim 8. Using this, one can easily check that Claim 11 restricts E to be of the form

$$E = \left(\begin{array}{c|c} * & * \\ \hline 0 & \pm\frac{1}{4\beta\|v\|} \end{array}\right).$$

Recall that by Claim 8, $Ev = -\|v\|\xi = \frac{1}{4\beta\|v\|}v$ in case (a), or $Ev = \|v\|\xi = -\frac{1}{4\beta\|v\|}v$ in case (b). This shows that E needs to be of the form

$$E = \left(\begin{array}{c|c} * & 0 \\ \hline * & \pm\frac{1}{4\beta\|v\|} \end{array}\right),$$

where "+" corresponds to case (a) and "−" to case (b). This and the requirement $E \in \mathrm{GL}(n-1)$ show that E is of the claimed form. $\square$

This shows that under our assumptions the equations (1)-(7) can only be satisfied if $\overline{M}$ has precisely one positive eigenvalue, i.e.

$$\overline{M}(z,z) = \frac{1}{2\beta\langle v,v\rangle}\sum_{i=1}^{n-2} z_i^2.$$

This also shows that $\beta > 0$ is a necessary requirement.

Claim 13: E is of the form $E = \frac{1}{2\beta\|v\|}\left(\begin{array}{c|c} C & \\ \hline & \pm\frac{1}{2} \end{array}\right)$, $C \in \mathrm{O}(n-2)$.

Proof. Observe that Claim 12 shows $E^T v = Ev$, which implies $\langle Ez, v\rangle^2 = \frac{z_{n-1}^2}{16\beta^2}$. Hence, equation $(6')$ is equivalent to

$$\|v\|M(Ez,Ez) = \sum_{i=1}^{n-2} z_i^2, \tag{6''}$$

and equation $(4')$ is equivalent to

$$\|v\|M(Ez,Ez) = 4\beta^2\langle v,v\rangle \left\langle B\begin{pmatrix} z_1 \\ \vdots \\ z_{n-2}\end{pmatrix}, B\begin{pmatrix} z_1 \\ \vdots \\ z_{n-2}\end{pmatrix}\right\rangle. \tag{4''}$$

On the right-hand side of $(4'')$, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^{n-2}$. Note that, since E is invertible, $(4'')$ shows that

$$M(z,z) = 4\beta^2\|v\|\sum_{i=1}^{n-2} z_i^2,$$

so M also has exactly one positive eigenvalue. By comparing $(4'')$ and $(6'')$ we see that $B = \frac{1}{2\beta\|v\|}C$ for some $C \in \mathrm{O}(n-2)$. This proves that $E = \frac{1}{2\beta\|v\|}\begin{pmatrix} C & \\ & \pm\frac{1}{2}\end{pmatrix}$, $C \in \mathrm{O}(n-2)$. $\square$

Since $M(z,z)$ is a positive scalar multiple of $\sum_{i=1}^{n-2} z_i^2$, h is invariant under transformations of the form

$$\widehat{C} = \begin{pmatrix} C^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad C \in \mathrm{O}(n-2).$$

Replacing A by the matrix $\widehat{C}A$, we can assume without restriction of generality that $E = \frac{1}{2\beta\|v\|}\begin{pmatrix} \mathbf{1} & \\ & \pm\frac{1}{2} \end{pmatrix}$. Summarizing, we have shown that in case (a), depending on the choice of the sign of $v = \pm\|v\|\partial_{z_{n-1}}$,

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbf{1}}{2\beta\|v\|} & & & \\ \hline & \frac{1}{4\beta\|v\|} & \frac{\mp 1}{4\beta\|v\|} & \pm\|v\| \\ \hline & \frac{\mp 1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \pm\beta & \beta & 0 \end{array}\right),$$

and in case (b)

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbf{1}}{2\beta\|v\|} & & & \\ \hline & \frac{-1}{4\beta\|v\|} & \frac{\mp 1}{4\beta\|v\|} & \pm\|v\| \\ \hline & \frac{\pm 1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \mp\beta & \beta & 0 \end{array}\right),$$

which again depends on the sign of $v = \pm\|v\|\partial_{z_{n-1}}$.

Since both h and $\overline{h}$ are invariant under the transformation

$$K := \left(\begin{array}{c|c|c|c} \mathbf{1} & & & \\ \hline & -1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right),$$

we see that, up to automorphisms of h and $\bar{h}$, in each of the 4 possible cases we only need to consider

$$A = \left(\begin{array}{c|c|c|c} \frac{\mathbf{1}}{2\beta\|v\|} & & & \\ \hline & \frac{1}{4\beta\|v\|} & \frac{-1}{4\beta\|v\|} & \|v\| \\ \hline & \frac{-1}{4\beta\|v\|} & \frac{1}{4\beta\|v\|} & \|v\| \\ \hline & \beta & \beta & 0 \end{array}\right).$$

We set $\lambda := 4\beta^2\|v\|$, so that $M(z,z) = \lambda \sum_{i=1}^{n-2} z_i^2$, $\overline{M}(z,z) = \frac{8\beta^3}{\lambda^2} \sum_{i=1}^{n-2} z_i^2$, and

$$A = \left(\begin{array}{c|c|c|c} \frac{2\beta}{\lambda}\mathbf{1} & & & \\ \hline & \frac{\beta}{\lambda} & \frac{-\beta}{\lambda} & \frac{\lambda}{4\beta^2} \\ \hline & \frac{-\beta}{\lambda} & \frac{\beta}{\lambda} & \frac{\lambda}{4\beta^2} \\ \hline & \beta & \beta & 0 \end{array}\right).$$

We define

$$R_r := \begin{pmatrix} r\mathbf{1} & & & \\ & r & & \\ & & r & \\ & & & \frac{1}{r^2} \end{pmatrix}, \quad \widehat{A} := \begin{pmatrix} \mathbf{1} & & & \\ & \frac{1}{2} & \frac{-1}{2} & 1 \\ & \frac{-1}{2} & \frac{1}{2} & 1 \\ & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

One can now verify that $A = R_{\lambda^{-\frac{1}{3}}} \widehat{A} R_{\frac{2\beta}{\lambda^{\frac{2}{3}}}}$. Note that $\widehat{A}^2 = \mathbf{1}$ and that $\widehat{A}$ is an automorphism of the polynomial $h_1 := x(-w^2 + \langle z, z\rangle) + w \sum\limits_{i=1}^{n-2} z_i^2$.

Claim 13 shows that the additional transformations obtained in the special case that h is equivalent to h_1 when compared to the other considered cases are all conjugated to a composition of the additional automorphism $\widehat{A}$ of h_1 and transformations of the form (2.1). This shows that

$$\{h \circ A \mid A \in \mathrm{GL}(n+1),\ h \circ A \in K\} = \{h \circ A \mid A \text{ is of the form (2.1)}\}.$$

Hence, for choosing a representative of an h in $\mathcal{F}$ when h has the property that the corresponding M has exactly one positive eigenvalue and $\dim \ker M = 1$, it suffices to consider transformations of the form (2.1). This finishes the proof of Lemma 11. $\square$

With the help of Lemma 10 and Lemma 11 we will now choose a unique representative in $\mathcal{F}$ for the $\mathrm{GL}(n+1)$-orbit of an element $h \in K$. For a given positive semi-definite bilinear form M there is a unique bilinear form

$$\widehat{M} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-1} \end{pmatrix},\ \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0,$$

such that there exists $E \in \mathrm{O}(n-1)$ with the property that $E^T M E = \widehat{M}$. The λ_i are the eigenvalues of M. $M \neq 0$ implies that M has at least one positive eigenvalue $\lambda_1 > 0$. Applying the corresponding transformation (2.1) with $r = \lambda_1^{\frac{2}{3}}$, we see that $h = x(-w^2 + \langle z, z\rangle) + wM(z, z)$ is equivalent to

$$\widehat{h} \in \mathcal{F},\ \widehat{h} = x(-w^2 + \langle z, z\rangle) + w \sum_{i=1}^{n-1} b_i z_i^2,\ b_1 = 1,\ b_1 \geq \ldots \geq b_{n-1} \geq 0,$$

and the b_i's thus uniquely determined by M. Summarizing up to this point, we have shown that the $(n-2)$-parameter family $\mathcal{F}$ consists of pairwise inequivalent hyperbolic homogeneous polynomials, all of which define a complete projective special real manifold of dimension n.

We will now consider the family $\mathcal{G}$ and proceed similarly as for the family $\mathcal{F}$. Consider the set of homogeneous cubic polynomials

$$L := \{x(-w^2 + N(z, z)) + w\langle z, z\rangle \mid 0 \neq N \geq 0\}.$$

It is clear that $\mathcal{G} \subset L$ and that any element in L is contained in the $\mathrm{GL}(n+1)$-orbit of some element in $\mathcal{G}$. For a given $h = x(-w^2 + N(z,z)) + w\langle z, z\rangle$ we want to determine all possible $A \in \mathrm{GL}(n+1)$, such that $(h \circ A)(z,w,x) \in L$. We will see that the answer is independent of the chosen h.

For $\dim\ker N = 0$, h is equivalent to some $\widetilde{h} = x(-w^2 + \langle z, z\rangle) + wM(z,z) \in K$ with the property $M > 0$. In this case we know that there is a unique representative of $\widetilde{h}$ under the $\mathrm{GL}(n+1)$-action in $\mathcal{F}$ of the form

$$\widehat{h} = x(-w^2 + \langle z, z\rangle) + w\sum_{i=1}^{n-1} b_i z_i^2,\ b_1 = 1,\ b_1 \geq \ldots \geq b_{n-1} > 0,$$

which can easily be checked to be equivalent to

$$\check{h} = x\left(-w^2 + \sum_{i=1}^{n-1} \frac{b_{n-1}}{b_{n-i}} z_i^2\right) + w\langle z, z\rangle,\ 1 = \frac{b_{n-1}}{b_{n-1}} \geq \ldots \geq \frac{b_{n-1}}{b_1} > 0.$$

Hence, $\check{h} \in \mathcal{G}$. The uniqueness property can be shown the following way. Assume that $h = x\left(-w^2 + \sum_{i=1}^{n-1} c_i z_i^2\right) + w\langle z, z\rangle \in \mathcal{G}$, $c_1 = 1$, $c_1 \geq \ldots \geq c_{n-1} > 0$, and $\overline{h} = x\left(-w^2 + \sum_{i=1}^{n-1} \overline{c}_i z_i^2\right) + w\langle z, z\rangle \in \mathcal{G}$, $\overline{c}_1 = 1$, $\overline{c}_1 \geq \ldots \geq \overline{c}_{n-1} > 0$, are equivalent. h and $\overline{h}$ are equivalent to

$$h' = x(-w^2 + \langle z, z\rangle) + w\sum_{i=1}^{n-1} \frac{c_{n-1}}{c_{n-i}} z_i^2 \in \mathcal{F}$$

and

$$\overline{h}' = x(-w^2 + \langle z, z\rangle) + w\sum_{i=1}^{n-1} \frac{\overline{c}_{n-1}}{\overline{c}_{n-i}} z_i^2 \in \mathcal{F},$$

respectively. We have shown that h' and $\overline{h}'$ are equivalent if and only if $\frac{c_{n-1}}{c_{n-i}} = \frac{\overline{c}_{n-1}}{\overline{c}_{n-i}}$ for all $1 \leq i \leq n-1$. Since $c_1 = \overline{c}_1 = 1$, this shows that $c_{n-1} = \overline{c}_{n-1}$. Hence, $c_i = \overline{c}_i$ must hold for all $1 \leq i \leq n-1$.

Thus, we can reduce this question and assume that the $h \in L$ we are starting with has the property that $N \geq 0$, $N \neq 0$, and $\dim\ker N \neq 0$.

Lemma 12. *Let $h \in L \setminus \{x(-w^2 + N(z,z)) + w\langle z, z\rangle \mid N > 0\}$. Then $h \circ A \in L$, $A \in \mathrm{GL}(n+1)$, if and only if*

$$A = \left(\begin{array}{c|c|c} r^{\frac{1}{4}}F & & \\ \hline & r^{-\frac{1}{2}} & \\ \hline & & r \end{array}\right),\ F \in \mathrm{O}(n-1),\ r > 0.$$

In particular the possible choices for A do not depend on h.

Proof. Let $h = x(-w^2 + N(z,z)) + w\langle z, z\rangle$. We obtain

$$dh = 2xN(z,dz) + 2w\langle z, dz\rangle + (-2wx + \langle z, z\rangle)dw + (-w^2 + N(z,z))dx.$$

We will determine the set $\{dh = 0\}$. Observe that for $w = 0$ it follows that $\langle z, z\rangle = 0$ and, hence, $z = 0$. Then all entries of dh are 0 for all $x \in \mathbb{R}$. For $w \neq 0$, substitute the equations $2wx = \langle z, z\rangle$ and $w^2 = N(z,z)$ into $2wxN(z,\cdot) + 2w^2\langle z, \cdot\rangle = 0$, which is the first equation in $dh = 0$ multiplied by w. We obtain $\langle z, z\rangle N(z,\cdot) + 2\langle z, \cdot\rangle N(z,z) = 0$, which in particular implies $3\langle z, z\rangle N(z,z) = 0$. This shows that $z \in \ker N$. But then $w^2 = N(z,z) = 0$, which is a contradiction to the assumption $w \neq 0$. Summarizing, we have shown that for all $N \geq 0$

$$\{dh = 0\} = \{z = 0,\ w = 0,\ x \in \mathbb{R}\}.$$

Hence, A needs to be of the form

$$A = \left(\begin{array}{c|c} B & \\ \hline (\alpha^T, \beta) & r \end{array}\right),\ B \in \mathrm{Mat}(n \times n,\ \mathbb{R}),\ \alpha \in \mathbb{R}^{n-1},\ \beta \in \mathbb{R},\ r \in \mathbb{R} \setminus \{0\}.$$

Let $\overline{h} = x(-w^2 + \overline{N}(z,z)) + w\langle z, z\rangle$ and assume that $h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) = \overline{h}\left(\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right)$. Denote by $\left(\begin{smallmatrix} \widetilde{z} \\ \widetilde{w} \\ \widetilde{x} \end{smallmatrix}\right) = A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)$. We obtain

$$h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) = (\langle \alpha, z\rangle + \beta w + rx)(-\widetilde{w}^2 + N(\widetilde{z}, \widetilde{z})) + \widetilde{w}\langle \widetilde{z}, \widetilde{z}\rangle.$$

Since $\widetilde{w}\langle \widetilde{z}, \widetilde{z}\rangle$ does not depend on the variable x, this shows that $-\widetilde{w}^2 + N(\widetilde{z}, \widetilde{z}) = r^{-1}(-w^2 + \overline{N}(z,z))$. Hence, $B = r^{-\frac{1}{2}}C$ with

$$C^T\left(\begin{array}{c|c} N & \\ \hline & -1 \end{array}\right)C = \left(\begin{array}{c|c} \overline{N} & \\ \hline & -1 \end{array}\right),\ C \in \mathrm{GL}(n).$$

For $C = \left(\begin{array}{c|c} E & \xi \\ \hline \eta^T & \mu \end{array}\right)$ the above equation is equivalent to

$$\left(\begin{array}{c|c} E^TNE - \eta \otimes \langle \eta, \cdot\rangle & E^TN\xi - \mu\eta \\ \hline \xi^TNE - \mu\eta^T & N(\xi,\xi) - \mu^2 \end{array}\right) = \left(\begin{array}{c|c} \overline{N} & \\ \hline & -1 \end{array}\right).$$

Note that this shows $\mu \neq 0$. This is equivalent to

$$\mu^2 = 1 + N(\xi,\xi), \tag{I}$$

$$E^TN\xi = \mu\eta, \tag{II}$$

$$E^TNE - \eta \otimes \langle \eta, \cdot\rangle = \overline{N}. \tag{III}$$

In particular $\mu \neq 0$. Up to this point, we have shown that

$$A = \left(\begin{array}{c|c|c} r^{-\frac{1}{2}}E & r^{-\frac{1}{2}}\xi & \\ \hline r^{-\frac{1}{2}}\eta^T & r^{-\frac{1}{2}}\mu & \\ \hline \alpha & \beta & r \end{array}\right).$$

We calculate

$$
\begin{aligned}
h\left(A\left(\begin{smallmatrix} z \\ w \\ x \end{smallmatrix}\right)\right) &= x(-w^2+\overline{N}(z,z)) \\
&+ w^3\left(-\beta r^{-1} + r^{-\frac{3}{2}}\mu\langle \xi,\xi\rangle\right) \\
&+ w^2\left(-r^{-1}\langle \alpha, z\rangle + r^{-\frac{3}{2}}\langle \eta, z\rangle\langle \xi,\xi\rangle + 2r^{-\frac{3}{2}}\mu\langle Ez,\xi\rangle\right) \\
&+ w\left(\beta r^{-1}\overline{N}(z,z) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle\right) \\
&+ r^{-1}\langle \alpha,z\rangle\overline{N}(z,z) + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,Ez\rangle.
\end{aligned}
$$

By assumption, the entries of A need to fulfil the equations

$$-\beta r^{-1} + r^{-\frac{3}{2}}\mu\langle \xi,\xi\rangle = 0, \tag{1}$$

$$-r^{-1}\langle \alpha,z\rangle + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle \xi,\xi\rangle + 2r^{-\frac{3}{2}}\mu\langle Ez,\xi\rangle = 0, \tag{2}$$

$$\beta r^{-1}\overline{N}(z,z) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle = \langle z,z\rangle, \tag{3}$$

$$r^{-1}\langle \alpha,z\rangle\overline{N}(z,z) + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,Ez\rangle = 0. \tag{4}$$

Claim 1: $E \in \mathrm{GL}(n-1)$**.**

Proof. Substituting (III) into (3) yields

$$\beta r^{-1}(N(Ez,Ez) - \langle \eta,z\rangle^2) + r^{-\frac{3}{2}}\mu\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,\xi\rangle = \langle z,z\rangle. \tag{3$'$}$$

We multiply both sides of (3′) by μ^2 and substitute (II) to obtain

$$\beta r^{-1}(\mu^2 N(Ez,Ez) - N(Ez,\xi)^2) + r^{-\frac{3}{2}}\mu^3\langle Ez,Ez\rangle + 2r^{-\frac{3}{2}}\mu N(Ez,\xi)\langle Ez,\xi\rangle = \mu^2\langle z,z\rangle. \tag{3$''$}$$

Assume $y \in \ker E$. Then (3″) implies $0 = \mu^2\langle y,y\rangle$. Since $\mu \neq 0$ this implies $y = 0$. This proves our claim. □

Claim 2: $\alpha = 0$**.**

Proof. Assume $\alpha \neq 0$. Substituting (III) into (4), we obtain

$$r^{-1}\langle \alpha,z\rangle(N(Ez,Ez) - \langle \eta,z\rangle^2) + r^{-\frac{3}{2}}\langle \eta,z\rangle\langle Ez,Ez\rangle = 0. \tag{4$'$}$$

Multiply both sides of (4′) by $r\mu^2$ and substitute (II) to obtain

$$\langle \alpha,z\rangle(\mu^2 N(Ez,Ez) - N(Ez,\xi)^2) + r^{-\frac{1}{2}}\mu N(Ez,\xi)\langle Ez,Ez\rangle = 0. \tag{4$''$}$$

Claim 2.1: $\alpha \neq 0 \;\Rightarrow\; E^T N\xi = s\alpha$**.**

Proof. Equation $(4'')$ and $E \in \mathrm{GL}(n-1)$ show that $y \in \ker\langle \alpha, \cdot\rangle$ implies $N(Ey,\xi) = 0$. Hence, $N(E\cdot,\xi) = s\langle \alpha, \cdot\rangle$. □

Claim 2.2: $\alpha \neq 0 \;\Rightarrow\; s \neq 0,\; \xi \notin \ker N$**.**

Proof. Assume that $s = 0$. Then $(4'')$ becomes $\langle \alpha, z\rangle N(Ez, Ez) = 0$ for all $z \in \mathbb{R}^{n-1}$. But $E \in \mathrm{GL}(n-1)$, $N \neq 0$, and $\alpha \neq 0$, so this is a contradiction. Since $E^T N\xi = s\alpha \neq 0$, it immediately follows that $\xi \notin \ker N$. □

Claim 2.3: $E^T\xi = t\alpha$**,** $t \neq 0$**.**

Proof. Equation (2) implies that α, η, and $E^T\xi$ are linearly dependent. Since $\eta = \mu^{-1}E^T N\xi = \mu^{-1}s\alpha$, it follows that $E^T\xi = t\alpha$. Then $t \neq 0$ follows from $E^T \in \mathrm{GL}(n-1)$ and $\xi \neq 0$. □

Claim 2.4: $\mathrm{sgn}(\mu) = \mathrm{sgn}(-s)$ **and** $\dim\ker N = 1$**.**

Proof. Observe that Claim 2.1-2.3 and $\alpha \neq 0$ show that $(4'')$ is equivalent to

$$\mu^2 N(Ez, Ez) - s^2\langle \alpha, z\rangle^2 + r^{-\frac{1}{2}}\mu s\langle Ez, Ez\rangle = 0.$$

Thus, for all $y \in \ker\langle \alpha, \cdot\rangle$ we have

$$\mu^2 N(Ey, Ey) + r^{-\frac{1}{2}}\mu s\langle Ey, Ey\rangle = 0.$$

$N \geq 0$ and $E \in \mathrm{GL}(n-1)$ imply that $\mu s < 0$, which shows $\mathrm{sgn}(\mu) = \mathrm{sgn}(-s)$. Since $\langle E\cdot, E\cdot\rangle|_{\ker\langle\alpha,\cdot\rangle} > 0$ it follows that $N(E\cdot, E\cdot)|_{\ker\langle\alpha,\cdot\rangle} > 0$. Hence, N is of rank $n-2$ or $n-1$, the latter being excluded by the assumption that $N \geq 0$ but not $N > 0$. □

Claim 2.5: $\mathrm{sgn}(s) = \mathrm{sgn}(t)$**.**

Proof. We have $\alpha = s^{-1}E^T N\xi$ and $\alpha = t^{-1}E^T\xi$. The invertibility of E shows $N\xi = st^{-1}\xi$. Since $\xi \notin \ker N$ and $N \geq 0$, it follows that $\mathrm{sgn}(st^{-1}) = 1$. □

To conclude the proof of Claim 2, multiply both sides of equation (2) by $r\mu$ and substitute (II) to obtain

$$-\mu\langle \alpha, z\rangle + r^{-\frac{1}{2}}\langle \xi, \xi\rangle N(Ez, \xi) + 2r^{-\frac{1}{2}}\mu^2\langle Ez, \xi\rangle = 0. \tag{$2'$}$$

Claim 2.1-2.3 and $\alpha \neq 0$ show that (2′) is equivalent to

$$-\mu + r^{-\frac{1}{2}}\langle \xi, \xi \rangle s + 2r^{-\frac{1}{2}}\mu^2 t = 0. \tag{2''}$$

We have shown that all terms are non-vanishing and, by Claim 2.4-2.5, have the same sign. Hence, (2″) cannot be true. This completes the proof of Claim 2, that is $\alpha = 0$. $\square$

Claim 3: $\xi = \eta = 0$**.**

Proof. Since $\alpha = 0$, using (I) and (II) shows that equation (4) is equivalent to $N(E\cdot, \xi) = 0$. But $E \in \mathrm{GL}(n-1)$, thus it follows that $\xi \in \ker N$ and $\eta = 0$. Equation (2) and $E \in \mathrm{GL}(n-1)$ now show that $\xi = 0$. $\square$

Claim 4: $\beta = 0$**,** $\mu = 1$**, and** $E = r^{\frac{3}{4}}F$**,** $F \in \mathrm{O}(n-1)$**.**

Proof. Equation (1), $\xi = 0$, and $r > 0$ imply $\beta = 0$. Using $\xi = 0$ we see that equation (3″) is equivalent to

$$r^{-\frac{3}{2}}\mu\langle Ez, Ez \rangle = \langle z, z \rangle. \tag{3'''}$$

Equations (I) and (3‴) are satisfied if and only if $\mu = 1$ and $r^{-\frac{3}{4}}E \in O(n-1)$, that is $E = r^{\frac{3}{4}}F$ with $F \in \mathrm{O}(n-1)$. $\square$

This finishes the proof of Lemma 12. $\square$

Now one can show in the exact same way as for the family $\mathcal{F}$ that each element of L has a unique representative in $\mathcal{G}$. Hence, the $(n-2)$-parameter family $\mathcal{G}$ consists of pairwise inequivalent hyperbolic homogeneous cubic polynomials, each defining a complete projective special real manifold of dimension n. This concludes the proof of Theorem 9. $\square$

A consequence of the Lemmata 10, 11, and 12 is the following corollary.

Corollary 13. *The automorphism groups of elements $h \in \mathcal{G}$ and $h \in \mathcal{F}$, $h \neq h_1 := x(-w^2 + \langle z, z \rangle) + w \sum_{i=1}^{n-2} z_i^2$, are of the form*

$$\mathrm{Aut}(h) = \mathrm{O}(m_1) \times \ldots \times \mathrm{O}(m_k),\ 1 \leq k \leq n-1,\ \sum_{j=1}^{k} m_k = n-1.$$

The automorphism group of h_1 is generated by $\mathrm{O}(n-2)$ *and $\widehat{A}$ defined as*

$$\widehat{A} := \left(\begin{array}{c|c|c|c} \mathbf{1} & & & \\ \hline & \frac{1}{2} & \frac{-1}{2} & 1 \\ \hline & \frac{-1}{2} & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right),$$

i.e.
$$\mathrm{Aut}(h_1) \cong \mathrm{O}(n-2) \rtimes \mathbb{Z}_2.$$

Recall that the projective special real manifolds associated to the polynomials $a)$ and $b)$ in Theorem 2 are homogeneous. Theorem 9 now implies the following.

Corollary 14. *For $n \geq 3$ there exists a smooth curve $\gamma : [0,1] \to S^3\left(\mathbb{R}^{n+1}\right)^*$, such that $\gamma(0) = x(-w^2 + \langle z,z\rangle)$, that is the polynomial $a)$ in Theorem 2, and $\gamma(1) = x(-w^2) + w\langle z,z\rangle$, which is equivalent to the polynomial $b)$ in Theorem 2, with the property that for each $t \in (0,1)$, the level set $\{\gamma(t) = 1\}$ contains a complete projective special real manifold.*

Note that the above corollary is also true for $n = 1$ and $n = 2$. For $n = 1$, the polynomials $a)$ and $b)$ in Theorem 2 are equivalent, cf. [CHM, Cor. 4]. For $n = 2$, one choice for γ is
$$\gamma(t) = x(-w^2 + (1-t)z^2) + twz^2.$$
If we compare these polynomials with [CDL, Thm. 1], we see that $\gamma(0)$ is equivalent to $a)$, that is xyz, $\gamma(1)$ is equivalent to $b)$, that is $x(xy - z^2)$, and $\gamma(t)$ for all $t \in (0,1)$ is equivalent to $e)$, that is $x(y^2 - z^2) + y^3$.

3 Curvature formulas for the q-map

In this section, we introduce the supergravity r- and c-map and derive curvature formulas for their composition, the q-map. Note that compared to the last section, the dimension n is shifted by one: In this section, the projective special real manifold $\mathcal{H}$ is defined by a cubic polynomial h in n variables and has dimension $\dim \mathcal{H} = n-1$. The corresponding projective special Kähler manifold $\bar{M}$ in the image of the supergravity r-map has real dimension $2n$ and the quaternionic Kähler manifold $\bar{N}$ in the image of the q-map has real dimension $4m = 4(n+1)$.

3.1 Conical affine and projective special Kähler geometry

First, we recall the definitions of conical affine and projective special Kähler manifolds [ACD, CM]:

Definition 15. *A* conical affine special Kähler manifold (M, g_M, J, ∇, ξ) *is a pseudo-Kähler manifold (M, g_M, J) endowed with a flat torsionfree connection ∇ and a vector field ξ such that* *i)* $\nabla\omega_M = 0$, *where* $\omega_M := g_M(J\cdot,\cdot)$ *is the Kähler form,*

ii) $(\nabla_X J)Y = (\nabla_Y J)X$ *for all* $X, Y \in \Gamma(TM)$,

iii) $\nabla\xi = D\xi = \mathrm{Id}$, *where* D *is the Levi-Civita connection,*

iv) g_M *is positive definite on* $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ *and negative definite on* $\mathcal{D}^\perp$.

Let (M, J, g_M, ∇, ξ) be a conical affine special Kähler manifold of complex dimension $n+1$. Then ξ and $J\xi$ are commuting holomorphic vector fields that are homothetic and Killing respectively [CM]. We assume that the holomorphic Killing vector field $J\xi$ induces a free S^1-action and that the holomorphic homothety ξ induces a free $\mathbb{R}^{>0}$-action on M. Then (M, g_M) is a metric cone over (S, g_S), where $S := \{p \in M | g_M(\xi(p), \xi(p)) = 1\}$, $g_S := g_M|_S$; and $-g_S$ induces a Riemannian metric $g_{\bar{M}}$ on $\bar{M} := S/S^1_{J\xi}$. $(\bar{M}, -g_{\bar{M}})$ is obtained from (M, J, g) via a Kähler reduction with respect to $J\xi$ and, hence, $g_{\bar{M}}$ is a Kähler metric (see e.g. [CHM]). The corresponding Kähler form $\omega_{\bar{M}}$ is obtained from ω_M by symplectic reduction. This determines the complex structure $J_{\bar{M}}$.

Definition 16. *The Kähler manifold* $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ *is called a* projective special Kähler manifold.

Locally, there exist so-called *conical special holomorphic coordinates* $z = (z^I) = (z^0, \dots, z^n) : U \xrightarrow{\sim} \tilde{U} \subset \mathbb{C}^{n+1}$ such that the geometric data on the domain $U \subset M$ is encoded in a holomorphic function $F : \tilde{U} \to \mathbb{C}$ that is homogeneous of degree 2 [ACD, CM]. Namely, we have [CM]

$$g_M|_U = \sum_{I,J} N_{IJ} dz^I d\bar{z}^J, \quad N_{IJ}(z, \bar{z}) := 2\mathrm{Im}\, F_{IJ}(z) := 2\mathrm{Im}\, \frac{\partial^2 F(z)}{\partial z^I \partial z^J} \quad (I, J = 0, \dots, n)$$

and $\xi|_U = \sum z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I}$. The Kähler potential for $g_M|_U$ is given by $r^2|_U = g_M(\xi, \xi)|_U = \sum z^I N_{IJ} \bar{z}^J$.

The $\mathbb{C}^*$-invariant functions $X^\mu := \frac{z^\mu}{z^0}$, $\mu = 1, \dots, n$, define a local holomorphic coordinate system on $\bar{M}$. The Kähler potential for $g_{\bar{M}}$ is $\mathcal{K} := -\log \sum_{I,J=0}^n X^I N_{IJ}(X) \bar{X}^J$, where $X := (X^0, \dots, X^n)$ with $X^0 := 1$. Note that for every function $f_U(z)$ on U, we define a function $f_{\bar{U}}(X)$ on the corresponding subset $\bar{U} \subset \bar{M}$ by $f_{\bar{U}}(X) := f_U(1, X^1, \dots, X^n)$. In most cases, we will suppress the subscripts $_U$ and $_{\bar{U}}$ and use the same notation for corresponding functions on U and $\bar{U}$.

3.2 The supergravity c-map

Let $(\bar{M}, g_{\bar{M}})$ be a projective special Kähler manifold of complex dimension n which is globally defined by a single holomorphic function F. The *supergravity c-map* [FS] associates with $(\bar{M}, g_{\bar{M}})$ a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}})$ of dimension $4n+4$. Following the conventions of [CHM], we have $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ and

$$\begin{aligned} g_{\bar{N}} &= g_{\bar{M}} + g_G, \\ g_G &= \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(m) d\zeta^I d\zeta^J \\ &\quad + \frac{1}{2\rho}\sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(m)d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m)d\zeta^L), \end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on $m \in \bar{M}$ and $\mathcal{I}(m)$ is invertible with the inverse $\mathcal{I}^{-1}(m) =: (\mathcal{I}^{IJ}(m))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK} z^K \sum_L N_{JL} z^L}{\sum_{IJ} N_{IJ} z^I z^J}, \quad N_{IJ} := 2\,\mathrm{Im}\, F_{IJ}, \tag{3.1}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates z^I on the underlying conical special Kähler manifold $M \to \bar{M}$. Notice that the expressions are homogeneous of degree zero and, hence, well defined functions on $\bar{M}$. It is shown in [CHM, Cor. 5] that the matrix $\mathcal{I}(m)$ is positive definite and hence invertible and that the metric $g_{\bar{N}}$ does not depend on the choice of special coordinates [CHM, Thm. 9]. It is also shown that $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, g_{\bar{M}})$ is complete [CHM, Thm. 5].

Using $(p_a)_{a=1,\ldots,2n+2} := (\tilde{\zeta}_I, \zeta^J)_{IJ=0,\ldots,n}$ and $(\hat{H}^{ab}) := \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix}$, we can combine the last two terms of g_G into $\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b$, i.e. the quaternionic Kähler metric is given by

$$g_{FS} := g_{\bar{N}} = g_{\bar{M}} + \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b. \tag{3.2}$$

3.3 The supergravity r-map

Let $(\mathcal{H} := \{x \in U \mid h(x) = 1\}, g_{\mathcal{H}} := -\partial^2 h\big|_{\mathcal{H}})$ be a projective special real manifold defined by a real homogeneous cubic polynomial h and an $\mathbb{R}^{>0}$-invariant domain $U \subset \mathbb{R}^n \backslash \{0\}$. Let $\bar{M} := \mathbb{R}^n + iU \subset \mathbb{C}^n$ be endowed with the standard complex structure $J_{\bar{M}}$ induced from $\mathbb{C}^n$ and with holomorpic coordinates $(X^\mu = y^\mu + ix^\mu)_{\mu=1,\ldots,n} \in \mathbb{R}^n + iU$. We define a Kähler metric

$$g_{\bar{M}} = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\mu} dX^\mu d\bar{X}^\nu \tag{3.3}$$

on $\bar{M}$ with Kähler potential

$$\mathcal{K}(X, \bar{X}) := -\log 8h(x), \tag{3.4}$$

where $x = (\operatorname{Im} X^1, \ldots, \operatorname{Im} X^n) \in U$.

Definition 17. *The correspondence* $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ *is called the* **supergravity r-map**.

Remark 1. Note that any manifold $(\bar{M},\, g_{\bar{M}},\, J_{\bar{M}})$ in the image of the supergravity r-map is a projective special Kähler manifold (see Section 3.1). The corresponding conical affine special Kähler manifold is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1,\, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*,\ X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}$$

endowed with the standard complex structure J and the metric g_M defined by the holomorphic function

$$F : M \to \mathbb{C}, \quad F(z^0,\, \ldots,\, z^n) = \frac{h(z^1,\, \cdots,\, z^n)}{z^0}.$$

Note that in general, the flat connection[2] ∇ on M is not the standard one induced from $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. The homothetic vector field ξ is given by $\xi = \sum_{I=0}^n (z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I})$. To check that $g_{\bar{M}}$ is the corresponding projective special Kähler metric, one uses the fact that

$$8|z^0|^2 h(x) = \sum_{I,\, J=0}^{n} z^I N_{IJ}(z,\, \bar{z}) \bar{z}^J, \tag{3.5}$$

where as above, $x = (\operatorname{Im} X^1,\, \ldots,\, \operatorname{Im} X^n) = (\operatorname{Im} \frac{z^1}{z^0},\, \ldots,\, \operatorname{Im} \frac{z^n}{z^0}) \in U$ (see [CHM]).

3.4 Curvature formulas for the supergravity r-map

Under the assumptions of Section 3.3, let $(e^a_\mu)_{a,\, \mu=1,\, \ldots,\, n}$ be a real $n \times n$ matrix-valued function on some open subset in $\bar{M}$ such that $\sum_{a=1}^n e^a_\mu \bar{e}^a_\nu = \sum_{a=1}^n e^a_\mu e^a_\nu = \mathcal{K}_{\mu\bar{\nu}}$, where

$$\mathcal{K}_{\mu\bar{\nu}} = -\frac{\partial^2 \log\, h(x)}{\partial X^\mu \partial \bar{X}^\nu} = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)}. \tag{3.6}$$

Here, subscripts of the cubic polynomial h denote derivatives with respect to the standard coordinates on U, e.g. $h_\mu(x) = \frac{\partial h(x)}{\partial x^\mu}$. The holomorphic one-forms

$$\sigma^a := \sum_{\mu=1}^n e^a_\mu dX^\mu \tag{3.7}$$

constitute a *unitary coframe* $(\sigma^a)_{a=1,\, \ldots,\, n}$, i.e. the metric can locally be written as

$$g_{\bar{M}} = \sum_{a=1}^n \sigma^a \bar{\sigma}^a = \frac{1}{2} \sum_{a=1}^n (\sigma^a \otimes \bar{\sigma}^a + \bar{\sigma}^a \otimes \sigma^a). \tag{3.8}$$

[2] ∇ is defined by $\mathbf{x}^I = \operatorname{Re} z^I$ and $\mathbf{y}_I = \operatorname{Re} F_I(z)$ being flat for $I = 0,\, \ldots,\, n$ (see [ACD]).

Let $(\sigma_a := \sum_{\mu=1}^n e_a^\mu \frac{\partial}{\partial X^\mu})_{a=1,\ldots,n}$ denote the corresponding local frame in $T^{1,0}\bar{M}$ dual to $(\sigma^a)_{a=1,\ldots,n}$, i.e. $(e_a^\mu) = (e_\mu^a)^{-1}$. Then $\sigma^a = 2g_{\bar{M}}(\bar{\sigma}_a, \cdot)$ and $\sigma^a(\sigma_b) = \bar{\sigma}^a(\bar{\sigma}_b) = \delta^a{}_b$, $\sigma^a(\bar{\sigma}_b) = \bar{\sigma}^a(\sigma_b) = 0$.

Note that the inverse of the matrix-valued function $(\mathcal{K}_{\mu\bar{\nu}})_{\mu,\nu=1,\ldots,n}$ (see Eq. (3.6)) is given by

$$\mathcal{K}^{\bar{\nu}\rho} = -4h(x)h^{\nu\rho}(x) + 2x^\nu x^\rho, \tag{3.9}$$

where $(h^{\mu\nu})_{\mu,\nu=1,\ldots,n} = (h_{\mu\nu})^{-1}_{\mu,\nu=1,\ldots,n}$.

Note that in this section, ∇ denotes the Levi-Civita connection of the projective special Kähler metric $g_{\bar{M}}$. The expressions for the Christoffel symbols

$$\begin{aligned}\Gamma^\rho_{\sigma\mu} &:= dX^\rho(\nabla_{\partial_{X^\sigma}}\partial_{X^\mu}) = \sum_{\kappa=1}^n \mathcal{K}^{\rho\bar{\kappa}}\partial_{X^\sigma}\mathcal{K}_{\mu\bar{\kappa}} \\ &= -\frac{i}{2h}\left(h\sum_{\kappa=1}^n h^{\rho\kappa}h_{\kappa\mu\sigma} - h_\sigma\delta^\rho_\mu - h_\mu\delta^\rho_\sigma + \frac{1}{2}x^\rho h_{\mu\sigma}\right)\end{aligned} \tag{3.10}$$

and the coefficients

$$\begin{aligned}R^\rho{}_{\sigma\mu\bar{\nu}} &:= dX^\rho\left(R(\partial_{X^\mu}, \partial_{\bar{X}^\nu})\partial_{X^\sigma}\right) = -\partial_{\bar{X}^\nu}\Gamma^\rho_{\sigma\mu} = -\frac{i}{2}\partial_{x^\nu}\Gamma^\rho_{\sigma\mu} \\ &= -\frac{1}{4h^2}\Bigg[\frac{1}{2}x^\rho(hh_{\mu\sigma\nu} - h_{\mu\sigma}h_\nu) + h_\mu h_\nu\delta^\rho_\sigma + h_\sigma h_\nu\delta^\rho_\mu \\ &\qquad - h\left(h_{\sigma\nu}\delta^\rho_\mu + h_{\mu\nu}\delta^\rho_\sigma - \frac{1}{2}h_{\mu\sigma}\delta^\rho_\nu\right) - h^2\sum_{\alpha,\beta,\gamma=1}^n h^{\rho\alpha}h_{\nu\alpha\beta}h^{\beta\gamma}h_{\gamma\mu\sigma}\Bigg] \\ &= -\delta^\rho_\sigma\mathcal{K}_{\mu\bar{\nu}} - \delta^\rho_\mu\mathcal{K}_{\sigma\bar{\nu}} + e^{2\mathcal{K}}\sum_{\alpha,\beta,\gamma=1}^n \mathcal{K}^{\rho\bar{\alpha}}h_{\alpha\nu\beta}\mathcal{K}^{\beta\bar{\gamma}}h_{\gamma\mu\sigma}\end{aligned} \tag{3.11}$$

of the Riemann curvature tensor

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \qquad (X,Y,Z \in \mathfrak{X}(\bar{M}))$$

have been calculated for instance in [CDL, Theorem 3].

We denote the coefficients of the local Levi-Civita connection one-form associated to the unitary local coframe $(\sigma^a)_{a=1,\ldots,n}$ by $\omega^a{}_b$, i.e. $\nabla_\cdot\sigma^a = \sum_{b=1}^n \omega^a{}_b(\cdot)\sigma^b$. Compatibility with the metric and torsion-freeness translate into the conditions that the complex one-form valued matrix $(\omega^a{}_b)_{a,b=1,\ldots,n}$ is anti-Hermitian and satisfies $d\sigma^a + \sum_{b=1}^n \omega^a{}_b \wedge \sigma^b = 0$ for $a = 1, \ldots, n$. These are fulfilled by the following general formula that holds for all Kähler manifolds[3]:

$$\omega^a{}_b = \sum_{\mu=1}^n (e^a_\mu \bar{\partial} e^\mu_b - \bar{e}^b_\mu \partial \bar{e}^\mu_a). \tag{3.12}$$

[3]Note that for arbitrary Kähler manifolds, the functions e^a_μ can in general not be chosen to be real.

In terms of the local connection one-form, the curvature tensor of a Kähler manifold is given by

$$R(X,\,Y)\sigma_c = \sum_{d=1}^{n}(d\omega^d{}_c + \sum_{c'=1}^{n}\omega^d{}_{c'} \wedge \omega^{c'}{}_c)(X,\,Y)\sigma_d =: \sum_{d=1}^{n}\tilde{R}^d{}_c(X,\,Y)\sigma_d. \tag{3.13}$$

Using Eq. (3.11) and $\mathcal{K}^{\mu\bar{\nu}} = \sum_{c=1}^{n} e^{\mu}_c \bar{e}^{\nu}_c$, one gets the following proposition (see [D, Prop. 7.2.1]):

Proposition 18. *In terms of the unitary local coframe $(\sigma^a)_{a=1,\dots,n}$, the Riemann curvature tensor of a projective special Kähler manifold in the image of the supergravity r-map reads*

$$\tilde{R}^a{}_b = -\delta^a_b \sum_{c=1}^{n} \sigma^c \wedge \bar{\sigma}^c - \sigma^a \wedge \bar{\sigma}^b + e^{2\mathcal{K}} \sum_{c,e,d=1}^{n} \tilde{h}_{adc}\tilde{h}_{ceb}\sigma^e \wedge \bar{\sigma}^d, \tag{3.14}$$

where $\tilde{h}_{abc} := \sum_{\mu,\nu,\sigma=1}^{n} e^{\mu}_a e^{\nu}_b e^{\sigma}_c h_{\mu\nu\sigma}$ for $a, b, c = 1, \dots, n$.

3.5 Levi-Civita connection for quaternionic Kähler manifolds in the image of the q-map

In this and the following section, we will introduce the *quaternionic vielbein formalism*, which was used in [FS] to determine the Levi-Civita connection and the Riemann curvature tensor of manifolds in the image of the supergravity c-map. The formulas in this formalism arise from well-known formulas in the differential geometry literature expressed in terms of local frames in the complex vector bundles E and H whose tensor product is identified with the complexified tangent bundle of a quaternionic Kähler manifold in Salamon's E-H formalism [S] (see e.g. [D, Ch. 7] for detailed explanations of the relation between the formulas used in the physics, respectively mathematics literature). The *q-map* is the composition of the supergravity r- and c-map. It assigns a quaternionic Kähler manifold of dimension $4m = 4(n+1)$ to any projective special real manifold of dimension $n-1$. We apply the quaternionic vielbein formalism to quaternionic Kähler manifolds in the image of the q-map and derive formulas for the Levi-Civita connection and the Riemann curvature tensor of these manifolds, expressed in terms of the cubic polynomial h, which defines the initial projective special real manifold. Up to changing conventions and fixing inaccuracies, these results can also be obtained by restricting the formulas in [FS] for the c-map to the case of the q-map. The Riemann curvature tensor of a quaternionic Kähler manifold is determined by its trace-free part, the *quaternionic Weyl tensor*. The latter can be expressed in terms of a certain symmetric quartic tensor field $\Omega \in \Gamma(S^4E^*)$ in the complex vector bundle E. In addition to the above-mentioned results, we derive a formula expressing this quartic tensor field in terms of the cubic polynomial

h for manifolds in the image of the q-map. This result is used in Subsection 3.7 to give a general formula for the squared pointwise norm of the Riemann curvature tensor of any quaternionic Kähler manifold in the image of the q-map.

We will restrict ourselves to manifolds in the image of the *q-map*, which is the composition of the supergravity r- and c-map, i.e. we consider the Ferrara-Sabharwal metric (3.2) defined on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ for a projective special Kähler manifold $(\bar{M} = \mathbb{R}^n + iU, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map, which is defined by a real homogeneous cubic polynomial h. On $\bar{N}$, we define the following complex-valued one-forms:

$$\beta^0 := ie^{\mathcal{K}/2}\frac{1}{\sqrt{\rho}}\sum_{I=0}^{n} X^I A_I, \qquad \beta^a := \sum_{I=0}^{n} P^a_I dX^I = \sigma^a, \tag{3.15}$$
$$\alpha^0 := -\frac{1}{2\rho}\left(d\rho - i(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))\right), \qquad \alpha^a := \frac{i}{\sqrt{\rho}}e^{-\mathcal{K}/2}\sum_{I,J=0}^{n}\overline{P}^a_I N^{IJ} A_J$$

for $a = 1, \ldots, n$, where $(P^a_I)_{I=0,\ldots,n} = (P^a_0,\, P^a_\mu)_{\mu=1,\ldots,n} = (-\sum_{\nu=1}^n X^\nu e^a_\nu,\, e^a_\mu)_{\mu=1,\ldots,n}$ and $A_I = d\tilde{\zeta}_I + \sum_{J=0}^n F_{IJ}(X)d\zeta^J$ for $I = 0, \ldots, n$. In terms of these one-forms, the Ferrara-Sabharwal metric reads (see e.g. [D, Lemma 7.3.1])

$$g_{FS} = \sum_{A=0}^{n}(\beta^A\bar{\beta}^A + \alpha^A\bar{\alpha}^A). \tag{3.16}$$

The equations $J_1^*\alpha^A = i\alpha^A$, $J_1^*\beta^A = i\beta^A$, $J_2^*\alpha^A = \bar{\beta}^A$ for $A = 0, \ldots, n$ and $J_1J_2 = J_3$ define an almost hyper-complex structure (J_1, J_2, J_3) on $\bar{N}$. J_1, J_2 and J_3 span a quaternionic structure Q on $\bar{N}$ that is compatible with the quaternionic Kähler metric g_{FS}. Note that J_1 defines an integrable[4] complex structure on $\bar{N}$.

Direct calculation gives the following expressions for the exterior derivatives of the above one-forms (see [D, Prop. 7.3.3]):

Proposition 19.

$$d\beta^0 = \frac{1}{2}\left(\alpha^0 + \bar{\alpha}^0 - id^c\mathcal{K}\right)\wedge\beta^0 + \sum_{b=1}^{n}\alpha^b\wedge\beta^b,$$
$$d\beta^a = -\sum_{b=1}^{n}\omega^a{}_b\wedge\beta^b,$$
$$d\alpha^0 = -\alpha^0\wedge\bar{\alpha}^0 + \beta^0\wedge\bar{\beta}^0 - \sum_{b=1}^{n}\alpha^b\wedge\bar{\alpha}^b,$$

[4]This can either be shown by direct calculation (see [CLST]) or deduced from the fact that all quaternionic Kähler manifolds obtained from the HK/QK correspondence admit a globally defined compatible integrable complex structure (see [D, Rem. 5.5.5]).

$$d\alpha^a = \frac{1}{2}(\alpha^0 + \bar{\alpha}^0 - id^c\mathcal{K}) \wedge \alpha^a + \beta^0 \wedge \bar{\beta}^a - \sum_{b=1}^{n} \overline{\omega^a{}_b} \wedge \alpha^b - ie^{\mathcal{K}} \sum_{b,c=1}^{n} \tilde{h}_{abc}\bar{\alpha}^b \wedge \beta^c,$$

where $\tilde{h}_{abc} = \sum_{\mu,\nu,\sigma=1}^{n} e^\mu_a e^\nu_b e^\sigma_c h_{\mu\nu\sigma}$ *for* $a, b, c = 1, \ldots, n$ *and* $(\omega^a{}_b)_{a,b=1,\ldots,n}$ *is the (pullback to* $\bar{N}$ *of the) local connection one-form of the Levi-Civita connection on* $\bar{M}$ *with respect to the local unitary coframe* $(\sigma^a)_{a=1,\ldots,n}$ *on* $\bar{M}$*.*

Note that the above proposition uses the following explicit formula for the local Levi-Civita connection one-form of a projective special Kähler manifold:

$$\omega^a{}_b = e^{-\mathcal{K}}\big((\bar{\partial}P^a_I)N^{IJ}\bar{P}^b_J - P^a_I N^{IJ}(\partial\bar{P}^b_J)\big) \tag{3.17}$$

$$= \delta^a_b\partial\mathcal{K} + e^{-\mathcal{K}}d(P^a_I N^{IJ})\bar{P}^b_J + ie^{-\mathcal{K}}P^a_I N^{IK}\,\overline{dF_{KL}(X)}\,N^{LJ}\bar{P}^b_J, . \tag{3.18}$$

The components $\bar{\theta}_\alpha$ of the local Sp(1)-connection one-form of a quaternionic Kähler manifold (with Levi-Civita connection ∇) with respect to a local oriented orthonormal frame (J_1, J_2, J_3) in the quaternionic structure are defined by

$$\nabla.J_\alpha = 2(\bar{\theta}_\beta(\cdot)J_\gamma - \bar{\theta}_\gamma(\cdot)J_\beta) \tag{3.19}$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$. The local fundamental two-forms $\omega_\alpha = g(J_\alpha\cdot, \cdot)$ are then given by

$$\frac{\nu}{2}\omega_\alpha = d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma, \tag{3.20}$$

where $\nu := \frac{scal}{4m(m+2)}$ ($\dim_{\mathbb{R}}\bar{N} = 4m = 4(n+1)$) is the *reduced scalar curvature*. For manifolds in the image of the supergravity c-map, we have (see [D, Rem. 5.5.3 and 5.5.4]) $\nu = -2$ and

$$\bar{\theta}_1 = -\frac{1}{4\rho}\Big(d\tilde{\phi} + \rho\, d^c\mathcal{K} - \sum_{I=0}^{n}(\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I)\Big) = -\frac{1}{2}\mathrm{Im}\,\alpha^0 - \frac{1}{4}d^c\mathcal{K},$$

$$\bar{\theta}_2 + i\bar{\theta}_3 = i\frac{1}{\sqrt{\rho}}e^{\mathcal{K}/2}\sum_{I=0}^{n} X^I A_I = \beta^0. \tag{3.21}$$

We combine the one-forms defined in Eq. (3.15) into the following *quaternionic vielbein*, which is a $(4n+4) \times (4n+4)$ matrix of complex-valued one-forms:

$$(f^{\alpha\Gamma})_{\alpha=1,2;\Gamma=1,\ldots,2n+2} = \begin{pmatrix} f^{1A} & f^{1\tilde{A}} \\ f^{2A} & f^{2\tilde{A}} \end{pmatrix}_{A=0,\ldots,n} := \begin{pmatrix} \beta^A & \alpha^A \\ -\bar{\alpha}^A & \bar{\beta}^A \end{pmatrix}_{A=0,\ldots,n}. \tag{3.22}$$

Let β_A, α_A be complex-valued vector fields on $\bar{N}$ such that $\beta^A = 2\overline{g(\beta_A, \cdot)}$ and $\alpha^A = 2\overline{g(\alpha_A, \cdot)}$ for $A = 0, \ldots, n$. These vector-fields are combined into the following local frame in $T^{\mathbb{C}}\bar{N}$, which is dual to $(f_{\alpha\Gamma})$:

$$(f_{\alpha\Gamma})_{\alpha=1,2;\Gamma=1,\ldots,2n+2} = \begin{pmatrix} f_{1A} & f_{1\tilde{A}} \\ f_{2A} & f_{2\tilde{A}} \end{pmatrix}_{A=0,\ldots,n} := \begin{pmatrix} \beta_A & \alpha_A \\ -\bar{\alpha}_A & \bar{\beta}_A \end{pmatrix}_{A=0,\,\ldots,\,n}. \tag{3.23}$$

With respect to the local frame $(f_{\alpha\Gamma})$, the components of the Levi-Civita connection one-form are given by

$$f^{\alpha\Gamma}(\nabla_X f_{\beta\Delta}) = p^\alpha{}_\beta(X)\delta^\Gamma{}_\Delta + \delta^\alpha{}_\beta\Theta^\Gamma{}_\Delta(X) \tag{3.24}$$

for $\alpha, \beta = 1, 2$ and $\Gamma, \Delta = 1, \ldots, 2n + 2$, where

$$p = (p^\alpha{}_\beta) = \begin{pmatrix} p^1{}_1 & p^1{}_2 \\ p^2{}_1 & p^2{}_2 \end{pmatrix} = \begin{pmatrix} -i\bar{\theta}_1 & -\bar{\theta}_2 - i\bar{\theta}_3 \\ \bar{\theta}_2 - i\bar{\theta}_3 & i\bar{\theta}_1 \end{pmatrix} \tag{3.25}$$

and

$$\Theta = (\Theta^\Gamma{}_\Delta)_{\Gamma=1,\ldots,2n+2} = \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix}, \tag{3.26}$$

where q, t are complex 1-form-valued $(n + 1) \times (n + 1)$ matrices that are anti-Hermitian, respectively symmetric ($q^\dagger := \bar{q}^t = -q$, $t^t = t$) and fulfill

$$0 = d\beta^A + p^1{}_1 \wedge \beta^A - p^1{}_2 \wedge \bar{\alpha}^A + \sum_{B=0}^{n}(q^A{}_B \wedge \beta^B + t^A{}_B \wedge \alpha^B), \tag{3.27}$$

$$0 = d\alpha^A + p^1{}_1 \wedge \alpha^A + p^1{}_2 \wedge \bar{\beta}^A + \sum_{B=0}^{n}(-\bar{t}^A{}_B \wedge \beta^B + \bar{q}^A{}_B \wedge \alpha^B) \tag{3.28}$$

for $A = 0, \ldots, n$. The following is a straightforward corollary of Proposition 19:

Corollary 20. *The* $\mathrm{Sp}(n)$*-part of the Levi-Civita connection of a quaternionic Kähler manifold in the image of the q-map is given by* $(\Theta^\Gamma{}_\Delta)_{\Gamma=1,\ldots,2n+2} = \begin{pmatrix} q^A{}_B & t^A{}_{\tilde{B}} \\ -\bar{t}^{\tilde{A}}{}_B & \bar{q}^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix}_{A,B=0,\ldots,n}$*, where*

$$q = (q^A{}_B)_{A,B=0,\ldots,n} = \begin{pmatrix} \frac{i}{4}d^c\mathcal{K} + \frac{3}{4}(\bar{\alpha}^0 - \alpha^0) & -\alpha^b \\ \bar{\alpha}^a & \omega^a{}_b + \frac{1}{4}(-id^c\mathcal{K} + (\bar{\alpha}^0 - \alpha^0))\delta^a{}_b \end{pmatrix}_{a,b=1,\ldots,n}$$

and

$$t = (t^A{}_{\tilde{B}})_{A,B=0,\ldots,n} = \begin{pmatrix} 0 & 0 \\ 0 & ie^{\mathcal{K}}\sum_{c=1}^n \tilde{h}_{abc}\,\alpha^c \end{pmatrix}_{a,b=1,\ldots,n}.$$

3.6 Riemann curvature tensor for quaternionic Kähler manifolds in the image of the q-map

We consider a manifold in the image of the q-map and use the notation introduced in the last section. In terms of the local frame (3.23), the Riemann curvature tensor of a quaternionic Kähler manifold reads

$$f^{\alpha\Gamma}(R(X, Y)f_{\beta\Delta}) = \tilde{R}_H{}^\alpha{}_\beta(X, Y)\delta^\Gamma{}_\Delta + \delta^\alpha{}_\beta\tilde{R}_E{}^\Gamma{}_\Delta(X, Y), \tag{3.29}$$

where

$$
\begin{aligned}
\tilde{R}_H &= dp + p \wedge p \\
&= \begin{pmatrix} -id\bar{\theta}_1 + 2i\bar{\theta}_2 \wedge \bar{\theta}_3 & -(d\bar{\theta}_2 + id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 + i\bar{\theta}_3) \\ (d\bar{\theta}_2 - id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 - i\bar{\theta}_3) & id\bar{\theta}_1 - 2i\bar{\theta}_2 \wedge \bar{\theta}_3 \end{pmatrix} \\
&\overset{(3.20)}{=} \frac{\nu}{2} \begin{pmatrix} -i\omega_1 & -\omega_2 - i\omega_3 \\ \omega_2 - i\omega_3 & i\omega_1 \end{pmatrix}
\end{aligned}
\tag{3.30}
$$

and

$$
\tilde{R}_E = d\Theta + \Theta \wedge \Theta. \tag{3.31}
$$

We write the $\mathrm{Sp}(n)$-part of the curvature tensor as

$$
\tilde{R}_E = \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix}, \tag{3.32}
$$

where r, s are complex two-form valued $(n+1) \times (n+1)$ matrices that fulfill $r^\dagger = -r$, $s^t = s$. In terms of this splitting, Eqs. (3.26) and (3.31) read

$$
r^A{}_B = dq^A{}_B + \sum_{C=0}^{n} (q^A{}_C \wedge q^C{}_B - t^A{}_C \wedge \bar{t}^C{}_B) \tag{3.33}
$$

$$
s^A{}_B = dt^A{}_B + \sum_{C=0}^{n} (q^A{}_C \wedge t^C{}_B + t^A{}_C \wedge \bar{q}^C{}_B), \tag{3.34}
$$

for $A, B = 0, \ldots, n$.

Since the quaternionic Weyl tensor of a quaternionic Kähler manifold can be expressed in terms of a quartic symmetric tensor field, the $\mathrm{Sp}(n)$-part (3.31) of the curvature tensor can be expressed as follows (see e.g. [D, Cor. 7.1.6]):

$$
\tilde{R}_E{}^\Lambda{}_\Xi = \sum_{\alpha,\,\beta=1}^{2} \sum_{\Delta=1}^{2n+2} \frac{\nu}{4} \epsilon_{\alpha\beta} C_{\Xi\Delta} f^{\alpha\Lambda} \wedge f^{\beta\Delta} + \sum_{\alpha,\,\beta=1}^{2} \sum_{\Lambda',\Gamma,\Delta=1}^{2n+2} C^{\Lambda\Lambda'} \Omega_{\Lambda'\Xi\Gamma\Delta} \epsilon_{\alpha\beta} f^{\alpha\Gamma} \wedge f^{\beta\Delta}, \tag{3.35}
$$

where $(C_{\Gamma\Delta})_{\Gamma,\Delta=1,\ldots,2n+2} = -(C^{\Gamma\Delta})_{\Gamma,\Delta=1,\ldots,2n+2}$ and $(\epsilon_{\alpha\beta})_{\alpha,\beta=1,2}$ are constant real-valued matrices defined by $C_{A\tilde{B}} = -C_{\tilde{A}B} = \delta_{AB}$, $C_{AB} = C_{\tilde{A}\tilde{B}} = 0$ $(A,\, B = 0,\, \ldots,\, n)$ and $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$, and $\Omega_{\Lambda'\Xi\Gamma\Delta}$ are complex-valued functions on $\bar{N}$, that are totally symmetric in all four indices.

Using the expressions for the local Levi-Civita connection one-form given in Corollary 20, one obtains the following result (see [D, Prop. 7.3.5]):

Proposition 21. *The* $\mathrm{Sp}(n)$*-part of the curvature two-form for any quaternionic Kähler* *manifold in the image of the q-map is given by* $(\tilde{R}_E{}^\Gamma{}_\Delta) = \begin{pmatrix} r^A{}_B & s^A{}_{\tilde{B}} \\ -\bar{s}^{\tilde{A}}{}_B & \bar{r}^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix}_{A,B=0,\dots,n}$ *with*[5]

$$
\begin{aligned}
r &= (r^A{}_B) \\
&= \begin{pmatrix}
\begin{array}{l}\frac{1}{2}\big(\alpha^0\wedge\bar\alpha^0 - \beta^0\wedge\bar\beta^0 \\ \quad + \sum_{C=0}^{n}\alpha^C\wedge\bar\alpha^C - \beta^C\wedge\bar\beta^C\big)\end{array} & \alpha^b\wedge\bar\alpha^0 + \bar\beta^b\wedge\beta^0 + ie^{\mathcal{K}}\tilde{h}_{bcd}\bar\alpha^c\wedge\beta^d \\
\begin{array}{c}\alpha^0\wedge\bar\alpha^a + \bar\beta^0\wedge\beta^a \\ +ie^{\mathcal{K}}\tilde{h}_{acd}\alpha^c\wedge\bar\beta^d\end{array} & \begin{array}{l}\frac{1}{2}\delta^a{}_b\sum_{C=0}^{n}(\alpha^C\wedge\bar\alpha^C - \beta^C\wedge\bar\beta^C) \\ \quad -(\beta^a\wedge\bar\beta^b + \bar\alpha^a\wedge\alpha^b) \\ \quad - e^{2\mathcal{K}}\tilde{h}_{adc}\tilde{h}_{ceb}(\alpha^d\wedge\bar\alpha^e + \bar\beta^d\wedge\beta^e)\end{array}
\end{pmatrix}_{a,b=1,\dots,n}
\end{aligned}
$$

and

$$
\begin{aligned}
s &= (s^A{}_{\tilde{B}}) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & ie^{\mathcal{K}}\tilde{h}_{abc}(\beta^0\wedge\bar\beta^c + \bar\alpha^0\wedge\alpha^c) + e^{2\mathcal{K}}\tilde{h}_{abf}\tilde{h}_{fde}\bar\alpha^d\wedge\beta^e - 2S_{abcd}\alpha^c\wedge\bar\beta^d \end{pmatrix}_{a,b=1,\dots,n},
\end{aligned}
$$

where

$$
\begin{aligned}
S_{abcd} := -\frac{1}{2}e^{2\mathcal{K}}\Big(&(\tilde{h}_{bcf}\tilde{h}_{fad} - 4\tilde{h}_{bc}\tilde{h}_{ad}) + (\tilde{h}_{acf}\tilde{h}_{fbd} - 4\tilde{h}_{ac}\tilde{h}_{bd}) + (\tilde{h}_{abf}\tilde{h}_{fcd} - 4\tilde{h}_{ab}\tilde{h}_{cd}) \\
&+ 4\tilde{h}_a\tilde{h}_{bcd} + 4\tilde{h}_b\tilde{h}_{cda} + 4\tilde{h}_c\tilde{h}_{dab} + 4\tilde{h}_d\tilde{h}_{abc}\Big). \qquad (3.36)
\end{aligned}
$$

Remark 2. Note that the vanishing of the symmetric quartic tensor field[6]

$$
\begin{aligned}
&S_{abcd}\,\sigma^a\otimes\sigma^b\otimes\sigma^c\otimes\sigma^d \\
&= -\frac{1}{2}\frac{1}{4^3h^2}\left(3h_{\tau(\mu\nu}\mathcal{K}^{\tau\tau'}h_{\sigma\rho)\tau'} - 12h_{(\mu\nu}h_{\sigma\rho)} + 16h_{(\mu}h_{\nu\sigma\rho)}\right)dX^\mu\otimes dX^\nu\otimes dX^\sigma\otimes dX^\rho \\
&= -\frac{1}{2}\frac{1}{4^3h^2}\left(-12h_{\tau(\mu\nu}h^{\tau\tau'}h_{\sigma\rho)\tau'} - 6h_{(\mu\nu}h_{\sigma\rho)} + 16h_{(\mu}h_{\nu\sigma\rho)}\right)dX^\mu\otimes dX^\nu\otimes dX^\sigma\otimes dX^\rho \\
&=: S_{\mu\nu\sigma\rho}\,dX^\mu\otimes dX^\nu\otimes dX^\sigma\otimes dX^\rho \qquad (3.37)
\end{aligned}
$$

on the projective special Kähler manifold $(\bar{M},\, g_{\bar{M}},\, J_{\bar{M}})$ is a necessary and sufficient condition for $(\bar{M},\, g_{\bar{M}})$ to be symmetric [CV].

Careful comparison of the expressions given in the above proposition with Eq. (3.35) leads to the following expression for the quartic symmetric tensor field determining the Riemann curvature tensor of a quaternionic Kähler manifold:

[5] All repeated lower case indices are summed over $1,\dots,n$.

[6] All repeated indices are summed over $1,\dots,n$. Note that the symmetrization denoted by $(\dots)$ over four indices includes a factor of $\frac{1}{4!}$.

Theorem 22. [D, Th. 7.3.7]
For manifolds in the image of the q-map, the non-vanishing components of the quartic symmetric tensor field defined in Eq. (3.35) *are given by*

$$\Omega_{00\tilde{0}\tilde{0}} = \frac{1}{2}, \quad \Omega_{0b\tilde{0}\tilde{d}} = \frac{1}{4}\delta_{bd}, \quad \Omega_{ab\tilde{c}\tilde{d}} = \frac{1}{4}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) - \frac{1}{2}e^{2\mathcal{K}}\sum_{f=1}^{n}\tilde{h}_{abf}\tilde{h}_{fcd},$$

$$\Omega_{\tilde{0}bcd} = \Omega_{0\tilde{b}\tilde{c}\tilde{d}} = -\frac{i}{2}e^{\mathcal{K}}\tilde{h}_{bcd}, \quad \Omega_{abcd} = \Omega_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = S_{abcd}$$

and symmetrization thereof, where $a, b, c, d = 1, \ldots, n$.

3.7 Pointwise norm of the Riemann curvature tensor for quaternionic Kähler manifolds in the image of the q-map

In this section, we give a general formula for a certain curvature invariant $\mathcal{S}_{\mathcal{W}}$ for all quaternionic Kähler manifolds $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ in the image of the q-map. $\mathcal{S}_{\mathcal{W}}$ is a real-valued scalar function, that (up to a factor of 64) coincides with the squared pointwise norm of the quaternionic Weyl tensor. We express $\mathcal{S}_{\mathcal{W}}$ as the linear combination of three curvature invariants on the corresponding projective special Kähler manifold $\bar{M}$. Its relation to the squared pointwise norm of the Riemann curvature tensor $\mathcal{R}$ is given by (see [D, Rem. 7.4.2])

$$\|\mathcal{R}\|^2 = 80(n+1)^2 + 16(n+1) + 64\,\mathcal{S}_{\mathcal{W}}. \tag{3.38}$$

The scalar curvature of a projective special Kähler manifold $\bar{M}$ in the image of the supergravity r-map can be calculated to be (see Theorem 3 in[7] [CDL] for the general formula)

$$\begin{aligned} scal_{\bar{M}} &= -2n^2 + n - 2h \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{\alpha',\beta',\gamma'=1}^{n} h_{\alpha\beta\gamma} h^{\alpha\alpha'} h^{\beta\beta'} h^{\gamma\gamma'} h_{\alpha'\beta'\gamma'} \\ &= -2n(n+1) + \frac{1}{32h^2} \sum_{\alpha,\beta,\gamma=1}^{n} \sum_{\alpha',\beta',\gamma'=1}^{n} h_{\alpha\beta\gamma} \mathcal{K}^{\alpha\alpha'} \mathcal{K}^{\beta\beta'} \mathcal{K}^{\gamma\gamma'} h_{\alpha'\beta'\gamma'}. \end{aligned} \tag{3.39}$$

The squared pointwise norm of the Riemann tensor of a projective special Kähler

[7] Note that compared to [CDL] we scaled the projective special Kähler metric $g_{\bar{M}}$ by a factor of $\frac{1}{2}$, which leads to a scaling of the scalar curvature $scal_{\bar{M}}$ by a factor of 2.

manifold $\bar{M}$ in the image of the r-map is

$$\|R_{\bar{M}}\|^2 = 16 \sum_{\mu,\nu,\rho,\sigma=1}^{n} \sum_{\mu',\nu',\rho',\sigma'=1}^{n} R_{\bar{\mu}\nu\sigma\bar{\rho}} \mathcal{K}^{\mu\mu'} \mathcal{K}^{\nu\nu'} \mathcal{K}^{\sigma\sigma'} \mathcal{K}^{\rho\rho'} R_{\mu'\bar{\nu}'\bar{\sigma}'\rho'}, \tag{3.40}$$

$$= -32\, scal_{\bar{M}} - 32n(n+1) \tag{3.41}$$

$$+ \frac{1}{4^4 h^4} \sum_{\mu,\nu,\sigma,\rho=1}^{n} \sum_{\mu',\nu',\sigma',\rho'=1}^{n} B_{\rho\sigma\mu\nu} \mathcal{K}^{\rho\rho'} \mathcal{K}^{\sigma\sigma'} \mathcal{K}^{\mu\mu'} \mathcal{K}^{\nu\nu'} B_{\rho'\sigma'\mu'\nu'}$$

where

$$R_{\bar{\mu}\nu\sigma\bar{\rho}} = \sum_{\alpha=1}^{n} \mathcal{K}_{\bar{\mu}\alpha} R^{\alpha}{}_{\nu\sigma\bar{\rho}} = -\mathcal{K}_{\bar{\mu}\nu}\mathcal{K}_{\sigma\bar{\rho}} - \mathcal{K}_{\bar{\mu}\sigma}\mathcal{K}_{\nu\bar{\rho}} + e^{2\mathcal{K}} \sum_{\beta,\gamma=1}^{n} h_{\mu\rho\beta} \mathcal{K}^{\beta\gamma} h_{\gamma\sigma\nu} \tag{3.42}$$

and

$$B_{\mu\nu\sigma\rho} := \sum_{\kappa,\kappa'=1}^{n} h_{\mu\nu\kappa} \mathcal{K}^{\kappa\kappa'} h_{\kappa'\sigma\rho}. \tag{3.43}$$

The third real-valued function on $\bar{M}$ relevant for this discussion is

$$\sum_{a,b,c,d=1}^{n} (S_{abcd})^2 = \sum_{\mu,\nu,\sigma,\rho=1}^{n} \sum_{\mu',\nu',\sigma',\rho'=1}^{n} S_{\mu\nu\sigma\rho} \mathcal{K}^{\mu\mu'} \mathcal{K}^{\nu\nu'} \mathcal{K}^{\sigma\sigma'} \mathcal{K}^{\rho\rho'} S_{\mu'\nu'\sigma'\rho'}, \tag{3.44}$$

where the respective components are defined in Eqs. (3.36) and (3.37).

Using the quartic tensor field introduced in (3.35), we define the following function on $\bar{N}$:

$$\mathcal{S}_{\mathcal{W}} := \sum_{\Gamma,\Gamma',\Gamma'',\Gamma'''=1}^{2n+2} \sum_{\Delta,\Delta',\Delta'',\Delta'''=1}^{2n+2} \Omega_{\Gamma\Gamma'\Gamma''\Gamma'''} C^{\Gamma\Delta} C^{\Gamma'\Delta'} C^{\Gamma''\Delta''} C^{\Gamma'''\Delta'''} \Omega_{\Delta\Delta'\Delta''\Delta'''}. \tag{3.45}$$

Using the formulas for Ω given in Theorem 22, we find the following expression for $\mathcal{S}_{\mathcal{W}}$:

$$\begin{aligned}
\mathcal{S}_{\mathcal{W}} &= 2\Omega_{ABCD}\Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} - 8\Omega_{ABC\tilde{D}}\Omega_{\tilde{A}\tilde{B}\tilde{C}D} + 6\Omega_{AB\tilde{C}\tilde{D}}\Omega_{\tilde{A}\tilde{B}CD} \\
&= 2\Omega_{abcd}\Omega_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} - 8\Omega_{abc\tilde{0}}\Omega_{\tilde{a}\tilde{b}\tilde{c}0} + 6(\Omega_{00\tilde{0}\tilde{0}})^2 + 24\Omega_{0b\tilde{0}\tilde{d}}\Omega_{\tilde{0}\tilde{b}0d} + 6\Omega_{ab\tilde{c}\tilde{d}}\Omega_{\tilde{a}\tilde{b}cd} \\
&= 2S_{abcd}S_{abcd} + 2n(n+1) + scal_{\bar{M}} + \frac{3}{2}(n+1) + 6\left(\frac{1}{4^3}\|R_{\bar{M}}\|^2 + \frac{1}{4}scal_{\bar{M}} + \frac{n^2+n}{8}\right) \\
&= 2S_{abcd}S_{abcd} + \frac{1}{4}(11n+6)(n+1) + \frac{3}{32}\|R_{\bar{M}}\|^2 + \frac{5}{2}scal_{\bar{M}}.
\end{aligned} \tag{3.46}$$

Together with Eq. (3.38), we obtain the following corollary:

Corollary 23. *The squared pointwise norm of the Riemann curvature tensor for any quaternionic Kähler manifold in the image of the q-map, defined by a cubic polynomial h in n variables, is*

$$\|\mathcal{R}\|^2 = 64(n+1)(4n+3) + 160\, scal_{\bar{M}} + 6\|\mathcal{R}_{\bar{M}}\|^2 + 128 \sum_{a,b,c,d=1}^{n} (S_{abcd})^2. \tag{3.47}$$

3.8 Example: A series of inhomogeneous complete quaternionic Kähler manifolds

For $n \in \mathbb{N}$, we consider the following series of projective special real manifolds:

$$\mathcal{H} = \{h = 1,\ x > 0\} \subset \mathbb{R}^n,\ h := x\left(x^2 - \sum_{i=1}^{n-1} y_i^2\right). \tag{3.48}$$

The scalar curvature of the corresponding projective special Kähler manifold $\bar{M}$ in the image of the supergravity r-map can be calculated using Eq. (3.39) and reads

$$scal_{\bar{M}} = -n \cdot (2n-1) + 3h \cdot \frac{n-2}{h-4x^3} + \frac{36x^3h^2}{(h-4x^3)^3}. \tag{3.49}$$

Furthermore, we find

$$\begin{aligned}\|R_{\bar{M}}\|^2 = \frac{16}{(h-4x^3)^6}\Big(&h^6(n(3n-8)+9) - 4h^5(n(17n-46)+57)x^3 \\ &+ 4h^4(n(161n-382)+537)x^6 - 64h^3(n(51n-97)+99)x^9 \\ &+ 128h^2(n(73n-107)+78)x^{12} - 2048h(n(7n-8)+3)x^{15} \\ &+ 1024n(9n-8)x^{18}\Big)\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}\sum_{a,b,c,d=1}^{n} (S_{abcd})^2 &= \sum_{\mu,\nu,\sigma,\rho=1}^{n} \sum_{\mu',\nu',\sigma',\rho'=1}^{n} S_{\mu\nu\sigma\rho}\mathcal{K}^{\mu\mu'}\mathcal{K}^{\nu\nu'}\mathcal{K}^{\sigma\sigma'}\mathcal{K}^{\rho\rho'}S_{\mu'\nu'\sigma'\rho'} \\ &= \frac{3x^6}{(h-4x^3)^6}\Big(h^4(n(n+16)+207) - 16h^3(n-2)(n+9)x^3 \\ &\qquad + 96h^2\left(n^2+n-6\right)x^6 - 256h(n-2)nx^9 \\ &\qquad + 256(n-2)nx^{12}\Big).\end{aligned} \tag{3.51}$$

Using Eq. (3.46), the function $\mathcal{S}_{\mathcal{W}}$ is calculated to be

$$\begin{aligned}\mathcal{S}_{\mathcal{W}} = \frac{3}{2\,(h-4x^3)^6}\Big(&h^6n(n+1) - 4h^5(n+1)(5n-2)x^3 + 8h^4(n(21n+37)+112)x^6 \\ &- 256h^3(n(3n+10)-11)x^9 + 256h^2(n(8n+33)-20)x^{12} \\ &- 1024h(n(3n+11)+2)x^{15} + 2048(n+1)(n+2)x^{18}\Big) + \frac{3n}{4}(n+1).\end{aligned} \tag{3.52}$$

By evaluating the above function in different points, one can check that it is non-constant for $n > 1$. Due to Eq. (3.38), this also applies to the squared pointwise norm of the Riemann curvature tensor, which manifestly is a curvature invariant. This shows that the quaternionic Kähler metrics obtained from the series of polynomials in Eq. (3.48) are not locally-homogeneous for $n > 1$. In total, we have the following:

Theorem 24. *For $n > 1$, the series of manifolds obtained from the complete projective special real manifolds in Eq.* (3.48) *via the q-map consists of complete quaternionic Kähler manifolds that are not locally homogeneous.*

Remark 3. Using computer algebra software, we have calculated the squared pointwise norm $\|\mathcal{R}\|^2$ of the Riemann tensor for $n = 2$ and $n = 3$ and have checked that it agrees with Eqs. (3.52) and (3.38).

References

[C] V. Cortés, *Alekseevskian spaces*, Differential Geom. Appl. **6** (1996), no. 2, 129–168

[DV] B. de Wit, A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Comm. Math. Phys. **149** (1992), no. 2, 307–333.

[GST] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $N = 2$ Maxwell–Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984), 244–268.

[ACD] D.V. Alekseevsky, V. Cortés and C. Devchand, *Special complex manifolds*, J. Geom. Phys. **42** (2002), no. 1–2, 85–105.

[ACDM] D.V. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. **92** (2015), 271–287.

[CDL] V. Cortés, M. Dyckmanns and D. Lindemann, *Classification of complete projective special real surfaces*, Proc. London Math. Soc. **109** (2014), no. 2, 423–445.

[CHM] V. Cortés, X. Han and T. Mohaupt, *Completeness in supergravity constructions*, Commun. Math. Phys. **311** (2012), no. 1, 191–213.

[CLST] V. Cortés, J. Louis, P. Smyth and H. Triendl, *On certain Kähler quotients of quaternionic Kähler manifolds*, Commun. Math. Phys. **317** (2013), no. 3, 787–816

[CM] V. Cortés and T. Mohaupt, *Special Geometry of Euclidean Supersymmetry III: the local r-map, instantons and black holes*, JHEP **0907** 066 (2009).

[CNS] V. Cortés, M. Nardmann and S. Suhr, *Completeness of hyperbolic centroaffine hypersurfaces*, to appear in Comm. Anal. Geom., arXiv:1407.3251.

[CV] E. Cremmer and A. Van Proeyen, *Classification Of Kähler Manifolds In N=2 Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2** (1985), no. 4, 445–454.

[D] M. Dyckmanns, *The hyper-Kähler/quaternionic Kähler correspondence and the geometry of the c-map*, PhD thesis, University of Hamburg, 2015.

[FS] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nucl. Phys. **B332** (1990), no. 2, 317–332.

[S] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171.