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Quaternionic Kähler metrics associated with special Kähler manifolds

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ABSTRACT

We give an explicit formula for the quaternionic Kähler metrics obtained by the HK/QK correspondence. As an application, we give a new proof of the fact that the Ferrara–Sabharwal metric as well as its one-loop deformation is quaternionic Kähler. A similar explicit formula is given for the analogous (K/K) correspondence between Kähler manifolds endowed with a Hamiltonian Killing vector field. As an example, we apply this formula in the case of an arbitrary conical Kähler manifold.

0. Introduction

Extending results by Haydys [1], it was proven in [2] that any pseudo-hyper-Kähler manifold (M, g, J_1, J_2, J_3) of dimension $4n$ endowed with a space-like or time-like ω_1-Hamiltonian Killing vector field Z which acts as a rotation in the plane spanned by J_2 and J_3 gives rise to a one-parameter family of conical[1] pseudo-hyper-Kähler manifolds of dimension $4n+4$ and finally to a one-parameter family of possibly indefinite quaternionic Kähler manifolds of dimension $4n$. Here $\omega_\alpha := gJ_\alpha := g \circ J_\alpha := g(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$, are the three symplectic forms associated with the pseudo-hyper-Kähler structure and the parameter in the above one-parameter families is related to the choice of a Hamiltonian function for Z. Under the assumptions on the Hamiltonian specified in [2], the resulting quaternionic Kähler metrics are positive definite.

Following [3,4] (but allowing indefinite metrics) we will call the above relation between hyper-Kähler and quaternionic Kähler manifolds of the same dimension the HK/QK correspondence. The analogous construction relating (possibly indefinite) Kähler manifolds of the same dimension, which follows from the Kähler conification in [2], will be called the K/K correspondence.

It was also proven in [2] that the cotangent bundle of any conical special Kähler manifold admits a canonical vector field Z which satisfies the above assumptions with respect to the pseudo-hyper-Kähler structure (g, J_1, J_2, J_3) provided by the (rigid) c-map [5] (see Section 4.2). Using techniques from supergravity and twistor theory, Alexandrov, Persson and Pioline

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[1] See Definition 1.

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[3] show that the Ferrara–Sabharwal metric [6] (also known as the supergravity c-map metric, see Section 4.3) and its one-loop deformation are related to the c-map pseudo-hyper-Kähler metric g under the HK/QK correspondence. It was shown in [2] that the above vector field Z has a canonical Hamiltonian function conjecturing that the quaternionic Kähler metric associated with this particular choice of the parameter is precisely the Ferrara–Sabharwal metric. It was checked that the sign of the scalar curvature is negative and thus consistent with the latter conjecture. Finally, the precise relation between the parameter in the choice of the Hamiltonian and the one-loop quantum deformation parameter occurring in [7,3] was left for future investigation.

In this paper we verify the above conjecture and determine the precise relation between the Hamiltonian parameter and the one-loop parameter. In fact, we apply the HK/QK correspondence to the pseudo-hyper-Kähler manifolds obtained from the rigid c-map starting with a conical affine special Kähler manifold. The final result is the formula (4.11) for the quaternionic Kähler metric, see Theorem 5. This is precisely the one-loop deformed Ferrara–Sabharwal metric as described in [7,3]. As a corollary this implies:

Corollary 1. *The Ferrara–Sabharwal metric and its one-loop deformation* (4.11) *are quaternionic Kähler.*

Notice that this generalizes the result that the Ferrara–Sabharwal metric is quaternionic Kähler [6,8].

Our proof is based on a new explicit formula for the quaternionic Kähler metric in the HK/QK correspondence, see Theorem 2. A similar result is obtained in the Kähler case, that is for the K/K correspondence, see Theorem 3. To obtain the explicit formula for the quaternionic Kähler metric we start by reviewing the Swann bundle construction and the moment map of a tri-holomorphic Killing vector field on the Swann bundle in Section 1. Our approach allows to control the signature of the resulting metrics. In particular, we specify for any given value of the one-loop parameter c the maximal domain on which the deformed Ferrara–Sabharwal metric is positive definite. For $c \geq 0$, this domain coincides with the manifold on which the Ferrara–Sabharwal metric is defined. These results generalize those of Antoniadis, Minasian, Theisen and Vanhove [9] in four dimensions (for the universal hypermultiplet).

We have included the Appendix, in which we discuss the simplest case of the HK/QK correspondence in which the initial hyper-Kähler manifold is (flat) four-dimensional, for the reader's convenience. The resulting quaternionic Kähler manifold is the complex hyperbolic plane (universal hypermultiplet).

For the K/K correspondence we apply our formula for the resulting metric in the case when the initial pseudo-Kähler manifold is conical, see Theorem 4. In particular, for a conical affine special Kähler manifold (M, J, g, ∇, ξ) we obtain (up to a cyclic covering) the product $\mathbb{C}H^1 \times \bar{M}$ of the projective special Kähler manifold $\bar{M}$ underlying M and the complex hyperbolic line, see Remark 5. Notice that this is a maximal totally geodesic Kähler submanifold $\mathbb{C}H^1 \times \bar{M} \subset \bar{N}$ of the Ferrara–Sabharwal manifold $\bar{N}$, which is related to $\bar{M}$ by the supergravity c-map.

1. The Swann bundle revisited

In this section we derive explicit formulas relating the metric of a quaternionic Kähler manifold to the pseudo-hyper-Kähler metric of its Swann bundle [10]. This will be used in Section 2 to obtain an explicit formula for the quaternionic Kähler metric in the HK/QK correspondence from the conical pseudo-hyper-Kähler metric constructed in [2].

1.1. The pseudo-hyper-Kähler structure

Let (M, g, Q) be a (possibly indefinite) quaternionic Kähler manifold of nonzero scalar curvature, where $Q \subset \mathfrak{so}(TM)$ denotes its quaternionic structure. Let us denote by $\pi : S \to M$ the principal SO(3)-bundle of frames (J_1, J_2, J_3) in Q such that $J_3 = J_1 J_2$ and $J_\alpha^2 = -\mathrm{Id}$, $\alpha = 1, 2, 3$. The principal action of an element $A \in \mathrm{SO}(3)$ is given by

$$s = (J_1, J_2, J_3) \mapsto \tau(A, s) := (J_1, J_2, J_3)A^\epsilon,$$

where $\epsilon = 1$ if we consider S as a right-principal bundle and $\epsilon = -1$ if we prefer a left-principal bundle. Let us denote by Z_α the fundamental vector fields associated with some basis (e_α) of $\mathfrak{so}(3)$:

$$Z_\alpha(s) = \left.\frac{\partial}{\partial t}\right|_{t=0} \tau(\exp(te_\alpha), s).$$

We may choose the basis corresponding to the standard basis of $\mathfrak{sp}(1) = \mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$ under the canonical isomorphism $\mathfrak{sp}(1) \cong ad(\mathfrak{sp}(1)) = \mathfrak{so}(3)$. Then

$$[e_\alpha, e_\beta] = 2e_\gamma, \qquad [Z_\alpha, Z_\beta] = 2\epsilon Z_\gamma, \tag{1.1}$$

for every cyclic permutation (α, β, γ) of $(1, 2, 3)$. In the following, (α, β, γ) will be always a cyclic permutation, whenever the three letters appear in an expression.

The Levi-Civita connection ∇ of (M, g) induces a principal connection

$$\theta = \sum \theta_\alpha e_\alpha : TS \to \mathfrak{so}(3)$$

on S. Its curvature is defined by

$$\Omega := d\theta + \epsilon \frac{1}{2}[\theta \wedge \theta],$$

where

$$\frac{1}{2}[\theta \wedge \theta](X, Y) := [\theta(X), \theta(Y)], \quad X, Y \in T_sS,\ s \in S.$$

Writing $\Omega = \sum \Omega_\alpha e_\alpha$ and using (1.1) we have

$$\Omega_\alpha = d\theta_\alpha + 2\epsilon\theta_\beta \wedge \theta_\gamma. \tag{1.2}$$

From the definition of the connection and curvature forms we get the following lemma.

Lemma 1.

$$\mathcal{L}_{Z_\alpha}\theta_\alpha = \mathcal{L}_{Z_\alpha}\Omega_\alpha = 0, \qquad \mathcal{L}_{Z_\alpha}\theta_\beta = 2\epsilon\theta_\gamma, \qquad \mathcal{L}_{Z_\alpha}\Omega_\beta = 2\epsilon\Omega_\gamma.$$

Given a local section $\boldsymbol{\sigma} = (J_1, J_2, J_3) \in \Gamma(U, S)$, defined over some open subset $U \subset M$, we can also define a vector-valued 1-form

$$\bar{\theta} = \sum \bar{\theta}_\alpha e_\alpha$$

on U by

$$\nabla J_\alpha = -2\epsilon(\bar{\theta}_\beta \otimes J_\gamma - \bar{\theta}_\gamma \otimes J_\beta).$$

The coefficient is chosen such that

$$\nabla(J_1, J_2, J_3) = (J_1, J_2, J_3)\epsilon\bar{\theta}.$$

Notice that then

$$\nabla B = dB + \epsilon \sum \bar{\theta}_\alpha \otimes [J_\alpha, B], \tag{1.3}$$

for every section $B = \sum b_\alpha J_\alpha$ of Q, where $d = d_{\boldsymbol{\sigma}}$ is defined by $dB := \sum db_\alpha \otimes J_\alpha$. The vector-valued 1-forms $\bar{\theta}$ on $U \subset M$ and θ on S are related by

$$\bar{\theta} = \boldsymbol{\sigma}^*\theta.$$

In the local trivialization $\pi^{-1}(U) \cong U \times \mathrm{SO}(3)$ of S given by σ we can write

$$\theta = \pi^*\bar{\theta} + \varphi,$$

where $\varphi = \sum \varphi_\alpha e_\alpha$ is the Maurer–Cartan form on SO(3) defined by $\varphi_\alpha(Z_\beta) = \delta_{\alpha\beta}$. From (1.3) we compute the curvature $R^Q \in \Gamma(\wedge^2 T^*M \otimes Q)$, $Q \cong ad(Q) \subset \mathrm{End}\, Q$, of the vector bundle Q, which is

$$R^Q = \sum \bar{\Omega}_\alpha J_\alpha, \quad \bar{\Omega}_\alpha = \epsilon d\bar{\theta}_\alpha + 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma.$$

It is a well-known result by Alekseevsky [11] that

$$\bar{\Omega}_\alpha = -\frac{\nu}{2}\omega_\alpha,$$

where $\omega_\alpha = gJ_\alpha$ and

$$\nu := \frac{scal}{4n(n+2)} \quad (\dim M = 4n)$$

is the reduced scalar curvature. Since the curvature form of a principal connection is horizontal, this implies that

$$\epsilon\Omega_\alpha|_{\sigma(U)} = \pi^*\bar{\Omega}_\alpha|_{\sigma(U)} = -\frac{\nu}{2}\pi^*\omega_\alpha|_{\sigma(U)}. \tag{1.4}$$

We endow the manifold S with the pseudo-Riemannian metric

$$g_S = \sum \theta_\alpha^2 + \frac{\nu}{4}\pi^* g.$$

Now we consider the cone $\hat{M} = S \times \mathbb{R}^{>0}$ over S with the Euler vector field $\xi := Z_0 := r\partial_r$ and the following exact 2-forms

$$\hat{\omega}_\alpha := d\hat{\theta}_\alpha, \quad \hat{\theta}_\alpha := -\epsilon\frac{r^2}{2}\theta_\alpha.$$

For later use we state the following lemma, which follows from Lemma 1 and the fact that $Z_0 = \xi$ preserves θ_α.

Lemma 2. *The Lie algebra* $\mathrm{span}\{Z_i | i = 0, \ldots, 3\} \cong \mathfrak{co}(3)$ *acts on* $\mathrm{span}\{\hat{\theta}_\alpha | \alpha = 1, 2, 3\}$ *by the standard representation:*

$$\mathcal{L}_{Z_0}\hat{\theta}_\alpha = 2\hat{\theta}_\alpha, \qquad \mathcal{L}_{Z_\alpha}\hat{\theta}_\beta = 2\epsilon\hat{\theta}_\gamma.$$

Using the above data we recover Swann's hyper-Kähler structure on $\hat{M}$:

Theorem 1. *The cone metric $\hat{g} = dr^2 + r^2 g_S$ is a pseudo-hyper-Kähler metric on $\hat{M}$ with the Kähler forms $\hat{\omega}_\alpha$. The signature of $\hat{g}$ is $(4+4k, 4l)$ if $\nu > 0$ and $(4+4l, 4k)$ if $\nu < 0$, where $(4k, 4l)$ is the signature of the quaternionic Kähler metric g on M.*

Proof. Let us denote by $T^v\hat{M} \subset T\hat{M}$ the vertical distribution with respect to the projection $\hat{\pi} := \pi \circ \mathrm{pr}_S : \hat{M} \to M$, $\mathrm{pr}_S : \hat{M} = S \times \mathbb{R}^{>0} \to S$ and by $T^h\hat{M}$ the horizontal distribution defined by its $\hat{g}$-orthogonal complement. Let $\hat{J}_\alpha$ be the uniquely determined 3 almost complex structures on $\hat{M}$ which preserve the horizontal distribution and satisfy

$$\hat{J}_\alpha Z_0 = -\epsilon Z_\alpha, \qquad \hat{J}_\alpha Z_\alpha = \epsilon Z_0, \qquad \hat{J}_\alpha Z_\beta = Z_\gamma, \qquad \hat{J}_\alpha Z_\gamma = -Z_\beta, \qquad \hat{\pi}_* \circ \hat{J}_\alpha|_{(s,r)} = J_\alpha \circ \hat{\pi}_*,$$

where $s = (J_1, J_2, J_3)$. We see that these structures satisfy $\hat{J}_1\hat{J}_2 = \hat{J}_3$ and pairwise anti-commute. Then, using (1.2) and (1.4), one can easily check $\hat{g}\hat{J}_\alpha = \hat{\omega}_\alpha$. This proves that the 2-forms $\hat{\omega}_\alpha$ are not only closed but also non-degenerate and that $\hat{J}_\alpha = -\hat{\omega}_\beta^{-1}\hat{\omega}_\gamma$ are three anti-commuting skew-symmetric almost complex structures on $(\hat{M}, \hat{g})$. By the Hitchin Lemma [12, Lemma 6.8], this shows that $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ is a pseudo-hyper-Kähler structure on $\hat{M}$. □

1.2. The moment map of an infinitesimal automorphism

Let $\hat{M}$ be the Swann bundle over a (possibly indefinite) quaternionic Kähler manifold (M, g, Q). We will follow the conventions in Section 1.1 with $\epsilon = -1$. We endow $\hat{M}$ with the hyper-Kähler structure $(g_{\hat{M}} := \sigma\hat{g}, (\hat{J}_\alpha))$, where $\sigma = \pm 1$. The corresponding Kähler forms are $\sigma\hat{\omega}_\alpha = d(\sigma\hat{\theta}_\alpha)$.

Let X be a tri-holomorphic space-like or time-like Killing vector field on $\hat{M}$, which commutes with the Euler vector field $\xi = r\partial_r = Z_0$.

Proposition 1. *The vector field X is tri-Hamiltonian with moment map $-\mu$, where*

$$\mu : \hat{M} \to \mathbb{R}^3, \quad x \mapsto (\mu_1(x), \mu_2(x), \mu_3(x)), \quad \mu_\alpha := \hat{\theta}_\alpha(X).$$

In fact, the functions μ_α satisfy

$$d\mu_\alpha = -\iota_X\hat{\omega}_\alpha. \tag{1.5}$$

Proof. Notice first that, since X is tri-holomorphic, it commutes not only with ξ but also with $Z_\alpha = \hat{J}_\alpha\xi$. This implies already that the Killing field X preserves the horizontal distribution $T^h\hat{M} = (T^v\hat{M})^\perp$ and hence the three one-forms θ_α. Furthermore, $\mathcal{L}_X(r^2) = \mathcal{L}_X g_{\hat{M}}(\xi, \xi) = 0$, since X is Killing and commutes with ξ. This implies that

$$\mathcal{L}_X\hat{\theta}_\alpha = \mathcal{L}_X\left(\frac{r^2}{2}\theta_\alpha\right) = 0.$$

Using this equation, we have

$$d\mu_\alpha = d\iota_X\hat{\theta}_\alpha = \mathcal{L}_X\hat{\theta}_\alpha - \iota_X d\hat{\theta}_\alpha = -\iota_X\hat{\omega}_\alpha. \quad \square$$

We will now explain how to recover the quaternionic Kähler metric on M from the geometric data on the level set of the moment map μ

$$P = \{\mu_1 = 1,\ \mu_2 = \mu_3 = 0\} \subset \hat{M}.$$

Since the group $\mathbb{R}^{>0} \times \mathrm{SO}(3)$ generated by ξ, Z_1, Z_2, Z_3 acts as the standard conformal linear group CO(3) on the three-dimensional vector space spanned by the functions μ_α, i.e.

$$\mathcal{L}_{Z_0}\mu_\alpha = 2\mu_\alpha, \qquad \mathcal{L}_{Z_\alpha}\mu_\beta = -2\mu_\gamma,$$

we see that

$$\hat{M} \setminus \{\mu = 0\} = \bigcup_{a \in \mathbb{R}^{>0} \times \mathrm{SO}(3)} aP.$$

In particular, P is nonempty. Then (1.5) shows that $P \subset \hat{M}$ is a smooth submanifold of codimension 3. On P we have the following data:

$$g_P := g_{\hat{M}}|_P = \sigma\hat{g}|_P \in \Gamma(\mathrm{Sym}^2 T^*P)$$
$$\theta_\alpha^P := \sigma\hat{\theta}_\alpha|_P \in \Omega^1(P) \quad (\alpha = 1, 2, 3)$$
$$f := \sigma\left.\frac{r^2}{2}\right|_P \in C^\infty(P)$$

$$\theta_0^P := -\frac{1}{2}df \in \Omega^1(P)$$
$$X_P := \sigma X|_P \in \mathfrak{X}(P)$$
$$Z_1^P := Z_1|_P \in \mathfrak{X}(P).$$

The fact that Z_1 is tangent to P follows from

$$d\mu_\alpha Z_1 = \iota_{Z_1} d\mu_\alpha = \mathcal{L}_{Z_1}\mu_\alpha = -2\delta_{2\alpha}\mu_3 + 2\delta_{3\alpha}\mu_2,$$

since $\mu_2 = \mu_3 = 0$ on P.

With these definitions, the formula

$$\hat{g} = dr^2 + r^2\left(\sum_{\alpha=1}^{3}\theta_\alpha^2 + \frac{\nu}{4}\pi^* g\right) \tag{1.6}$$

implies:

Proposition 2. *The quaternionic Kähler metric g on M is related as follows to the geometric data on the level set $P \subset \hat{M}$ of the moment map:*

$$\nu\pi^* g|_P = \frac{2}{f}\left(g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\right). \tag{1.7}$$

Proof. Solving Eq. (1.6) for $\nu\pi^* g$ yields:

$$\nu\pi^* g = \frac{4}{r^2}\left(\hat{g} - dr^2 - r^2\sum\theta_\alpha^2\right) = \frac{4}{\sigma r^2}\left(\sigma\hat{g} - \sigma dr^2 - \sigma r^2\sum\theta_\alpha^2\right).$$

Restricting to P we first obtain:

$$\nu\pi^* g|_P = \frac{2}{f}\left(g_P - \sigma dr^2|_P - 2f\sum\theta_\alpha^2|_P\right). \tag{1.8}$$

The above definitions imply $\theta_\alpha|_P = f^{-1}\theta_\alpha^P$. Therefore, $f\theta_\alpha^2|_P = f^{-1}(\theta_\alpha^P)^2$. Similarly, $\sigma dr^2|_P = 2f^{-1}(\theta_0^P)^2$. This shows that (1.8) implies (1.7). $\square$

Corollary 2. *The tensor field on the right-hand side of (1.7) is invariant under Z_1^P and has one-dimensional kernel $\mathbb{R}Z_1^P$.*

Proof. The Z_1^P-invariance follows from the Z_1-invariance of $\pi^* g$. The statement about the kernel follows from

$$T^v\hat{M} \cap TP = \mathbb{R}Z_1, \tag{1.9}$$

which is a consequence of

$$d\mu_\alpha\xi = \mathcal{L}_\xi\mu_\alpha = 2\mu_\alpha$$
$$d\mu_\alpha Z_2 = \mathcal{L}_{Z_2}\mu_\alpha = 2\delta_{1\alpha}\mu_3 - 2\delta_{3\alpha}\mu_1$$
$$d\mu_\alpha Z_3 = \mathcal{L}_{Z_3}\mu_\alpha = -2\delta_{1\alpha}\mu_2 + 2\delta_{2\alpha}\mu_1.$$

In fact, we have already shown that the vertical vector field Z_1 is tangent to P and these equations show now that the three vector fields ξ, Z_2, Z_3 are mapped to (constant) linearly independent vectors under the vector-valued one-form $d\mu = (d\mu_\alpha) : T\hat{M} \to \mathbb{R}^3$. This implies (1.9), since $TP = \ker d\mu$. $\square$

In the next section we apply the above results to the case when $\hat{M}$ is obtained by conification of a hyper-Kähler manifold, in the sense of [2].

2. Explicit formula for the HK/QK correspondence

Let (M, g, J_1, J_2, J_3) be a possibly indefinite hyper-Kähler manifold with the Kähler forms $\omega_\alpha = gJ_\alpha$, $\alpha = 1, 2, 3$, and a time-like or space-like ω_1-Hamiltonian Killing vector field Z such that $\mathcal{L}_Z J_2 = -2J_3$. According to [2], with any choice of function $f \in C^\infty(M)$ such that $df = -\omega_1 Z$ and such that $f_1 = f - \frac{g(Z,Z)}{2}$ is not zero, one can, at least locally, associate a quaternionic Kähler metric g' on a manifold M' of dimension $\dim M$. (One has to assume, in particular, that the functions f and f_1 are nowhere zero, which may require to restrict the manifold M.)

Following [2], let $P \to M$ be an S^1-principal bundle with a principal connection η with the curvature $d\eta = \omega_1 - \frac{1}{2}dgZ$. We endow P with the pseudo-Riemannian metric

$$g_P := \frac{2}{f_1}\eta^2 + \pi^* g \tag{2.1}$$

and with the vector field

$$Z_1^P := \tilde{Z} + f_1 X_P, \tag{2.2}$$

where $\tilde{Z}$ denotes the horizontal lift of Z and X_P the fundamental vector field of the principal action. Furthermore, we endow P with the following one-forms:

$$\begin{aligned} \theta_0^P &:= -\frac{1}{2}df \\ \theta_1^P &:= \eta + \frac{1}{2}gZ \\ \theta_2^P &:= \frac{1}{2}\omega_3 Z \\ \theta_3^P &:= -\frac{1}{2}\omega_2 Z. \end{aligned} \tag{2.3}$$

Theorem 2. *The tensor field*

$$\tilde{g}_P := g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2 \tag{2.4}$$

on P is invariant under Z_1^P and has one-dimensional kernel $\mathbb{R}Z_1^P$. Let M' be a codimension 1 submanifold of P which is transversal to the vector field Z_1^P. Then

$$g' := \frac{1}{2|f|}\tilde{g}_P|_{M'}$$

is a possibly indefinite quaternionic Kähler metric on M'.

Proof. Analysing the constructions of [1,2], we see that the data g_P, θ_α^P, f, X_P, Z_1^P are obtained by restriction from data $\sigma\hat{g}$, $\sigma\hat{\theta}_\alpha$, $\sigma r^2/2$, X and Z_1 on the conical pseudo-hyper-Kähler manifold $\hat{M}$, as in Section 1.2. Therefore, the theorem follows from Proposition 2 and Corollary 2. The tensor field $\frac{1}{2|f|}\tilde{g}_P$ corresponds to $\frac{\sigma\nu}{4}\pi^*\bar{g}|_P$, where $(\bar{M}, \bar{g})$ denotes the underlying quaternionic Kähler manifold, when $\hat{M}$ is represented locally as a Swann bundle $\hat{\pi} : \hat{M} \to \bar{M}$. (Recall that $\sigma = \operatorname{sgn} f$.) □

The metric g' is the quaternionic Kähler metric which corresponds under the HK/QK correspondence to the hyper-Kähler manifold $(M, g, (J_\alpha))$ with the data (Z, f). Notice that the principal projection $\pi : (S, g_{\hat{M}}|_S = \sigma g_S) \to (\bar{M}, \bar{g})$ is a pseudo-Riemannian submersion if and only if $\frac{\sigma\nu}{4} = 1$. This is why we normalized the metric g' such that its reduced scalar curvature is $\nu' = 4\sigma$.

Remark 1. If Z_1^P generates a free and proper action of a one-dimensional Lie group A ($\cong S^1$ or $\mathbb{R}$) and if M' is a global section for the A-action, then we can identify M' with the orbit space P/A, which inherits the quaternionic Kähler metric g'.

In the next section we present a similar result for the K/K correspondence.

3. Explicit formula for the K/K correspondence

Let (M, g, J) be a possibly indefinite Kähler manifold endowed with a time-like or space-like Killing vector field Z, which is Hamiltonian with respect to the Kähler form $\omega = gJ$. According to [2], with any choice of function $f \in C^\infty(M)$ such that $df = -\omega Z$ and such that $f_1 = f - \frac{g(Z,Z)}{2}$ is not zero, one can, at least locally, associate a conical pseudo-Kähler manifold $\hat{M}$ of (real) dimension $\dim M + 2$ and, hence, a pseudo-Kähler manifold M' of dimension $\dim M$. In fact, $\hat{M}$ is a metric cone over a pseudo-Sasaki manifold S which has a pseudo-Kähler structure transversal to the Reeb foliation. Therefore, any codimension 1 submanifold of S transversal to the Reeb foliation inherits a pseudo-Kähler structure (J', g'). Now we give an explicit formula for the metric g' in terms of the initial data.

Following [2], let $\pi : P \to M$ be an S^1-principal bundle with a principal connection η with the curvature $d\eta = \omega - \frac{1}{2}dgZ$. We endow P with the pseudo-Riemannian metric

$$g_P := \frac{2}{f_1}\eta^2 + \pi^* g$$

and with the vector field

$$Z^P := \tilde{Z} + f_1 X_P,$$

where $\tilde{Z}$ denotes the horizontal lift of Z and X_P the fundamental vector field of the principal action. Furthermore, we endow P with the following one-forms:

$$\theta_0^P := -\frac{1}{2}df$$

$$\theta_1^P := \eta + \frac{1}{2}gZ.$$

Then $\hat{M} = \mathbb{R} \times P$ is endowed with a conical pseudo-Kähler structure described explicitly in terms of the above data on P, see [2]. In particular, the Euler vector field is given by $\xi = \partial_t$, where t is the coordinate on the $\mathbb{R}$-factor. It is related to the radial variable $r > 0$ of the metric cone over the pseudo-Sasaki manifold S by $e^{2t} = \frac{r^2}{2|f|}$. This implies that $S = \{p \in \hat{M} | r(p) = 1\}$ is a circle bundle over M diffeomorphic to P.

Theorem 3. *The tensor field*

$$\tilde{g}_P := g_P - \frac{2}{f}\sum_{a=0}^{1}(\theta_a^P)^2$$

on P is invariant under Z^P and has one-dimensional kernel $\mathbb{R}Z^P$. Let M' be a codimension 1 submanifold of P which is transversal to the vector field Z^P. Then

$$g' := \frac{1}{2|f|}\tilde{g}_P|_{M'}$$

is a possibly indefinite Kähler metric on M'.

Proof. The proof is similar to that of Theorem 2. It relies on the representation of the pseudo-Kähler manifold $(\hat{M}, g_{\hat{M}})$ as a metric cone over a pseudo-Sasaki manifold S. $\hat{M} = \mathbb{R}^{>0} \times S$ is equipped with the metric $g_{\hat{M}} = \sigma\hat{g} = \sigma(dr^2 + r^2 g_S)$, where $\sigma = \operatorname{sgn} f \in \{-1, 1\}$. One can (locally) assume that $S = I \times \bar{M} \subset \mathbb{R} \times \bar{M}$ is contained in a trivial principal bundle with structure group $\mathbb{R}$ over a pseudo-Kähler manifold $(\bar{M}, \bar{g})$, where $I \subset \mathbb{R}$ is an interval. Let us denote by $\bar{\omega}$ the Kähler form of $(\bar{M}, \bar{g})$. The pseudo-Sasaki metric takes the form $g_S = \theta^2 + \bar{g}$, where θ is a principal connection with curvature given by $2\bar{\omega}$. Analysing the construction of [2], we see that the tensor field $\frac{1}{2|f|}\tilde{g}_P$ corresponds to $\sigma\pi^*\bar{g}|_P$, where $\pi : \hat{M} \to \bar{M}$ is the composition of the two projections $\hat{M} \to S$ and $S \to \bar{M}$. Here $P = \{t = 0\} \times P \subset \hat{M} = \mathbb{R} \times P$ is the level set $\{\mu = 1\}$ of the moment map $\mu = e^{2t}$ associated with the holomorphic Killing vector field X on $\hat{M}$ which canonically extends the vector field X_P on P. □

3.1. K/K correspondence for conical Kähler manifolds

As an example, we apply the K/K correspondence to an arbitrary conical pseudo-Kähler manifold (M, J, g, ξ) endowed with the holomorphic Killing field $Z := 2J\xi$. Recall the following definition:

Definition 1. A pseudo-Riemannian manifold (M, g) is called **conical** if it is endowed with a space-like or time-like vector field ξ (called the **Euler vector field**) such that $D\xi = \mathrm{Id}$, where D is the Levi–Civita connection.

Geometrically this means that M is locally isometric to a (space-like or time-like, respectively) metric cone $C_\pm(S) = (\mathbb{R}^{>0} \times S, \pm dr^2 + r^2 g_S)$ over a pseudo-Riemannian manifold (S, g_S). Notice that in this local representation the Euler vector field ξ is given by $r\partial_r$. If g happens to be a pseudo-Kähler metric for some complex structure J on M, then (M, J, g, ξ) is called a **conical pseudo-Kähler manifold**. In this case M is locally isometric to a **pseudo-Kähler cone**, that is a metric cone $C_\pm(S)$ over a pseudo-Sasaki manifold (S, g_S) with Reeb vector field $J\xi|_S$, see e.g. [13,14].

From now on we assume that (M, J, g, ξ) is a conical pseudo-Kähler manifold. Using $r^2 := |g(\xi, \xi)|$, $\lambda := \operatorname{sgn} g(\xi, \xi)$, $\tilde{\eta} := \frac{\lambda}{r^2}g(J\xi, \cdot)$, we can write the metric as

$$g = \frac{(g(\xi, \cdot))^2}{g(\xi, \xi)} + \frac{(g(J\xi, \cdot))^2}{g(\xi, \xi)} + |g(\xi, \xi)|\breve{g} = \lambda(dr^2 + r^2(\tilde{\eta}^2 + \lambda\breve{g})). \tag{3.1}$$

This equation defines the tensor $\breve{g}$ on M, which has $\ker \breve{g} = \operatorname{span}\{\xi, J\xi\}$ and fulfills $\mathcal{L}_\xi\breve{g} = \mathcal{L}_{J\xi}\breve{g} = 0$. Assume that $S := \{r = 1\} \subset M$ is non-empty and let $\breve{M} \subset S$ be a codimension 1 submanifold that is transversal to the Reeb vector field $J\xi|_S \in \Gamma(TS)$. Then $\breve{M}$ inherits a complex structure $\breve{J}$ from J such that $(\breve{M}, \breve{J}, \breve{g}|_{\breve{M}})$ is pseudo-Kähler. For simplicity (and without restriction of generality), we assume in the following theorem that $M = \mathbb{R}^{>0} \times S$ is globally a cone.

Theorem 4. *The K/K correspondence assigns to any pseudo-Kähler cone $(M = \mathbb{R}^{>0} \times S, J, g, \xi)$ endowed with the holomorphic Killing field $Z = 2J\xi$ the manifolds $M'_\pm := I_\pm \times S^1 \times \breve{M}$,*

$$I_+ := \begin{cases} (\max\{0, -2c\}, \infty) & \text{for } \lambda = 1 \\ (\min\{-2c, 0\}, -c) & \text{for } \lambda = -1, \end{cases} \qquad I_- := \begin{cases} (-c, \max\{0, -2c\}) & \text{for } \lambda = 1 \\ (-\infty, \min\{-2c, 0\}) & \text{for } \lambda = -1; \end{cases}$$

endowed with the metric

$$g' = \frac{1}{2|\rho|}\left[\lambda(\rho+c)\breve{g} - \frac{1}{4\rho}\frac{\rho+2c}{\rho+c}d\rho^2 - \frac{1}{4\rho}\frac{\rho+c}{\rho+2c}(d\tilde{\phi} - 2c\tilde{\eta}|_{\breve{M}})^2\right] \tag{3.2}$$

for each $c \in \mathbb{R}$. *Here,* $\tilde{\phi}$ *is a local coordinate on the* S^1*-factor* $S^1 = \{e^{-\frac{i}{4}\tilde{\phi}}|\tilde{\phi} \in \mathbb{R}\}$, $\rho \in I_\pm$ *and* $\lambda = \operatorname{sgn} g(\xi, \xi)$. *The signature of* (M'_+, g') *is* $(2k, 2l+2)$ *and that of* (M'_-, g') *is* $(2k+2, 2l)$, *where* $(2k, 2l)$ *is the signature of* $\breve{g}$.

For $c = 0$, *we get*

$$(M'_\pm, 2g') = (\pm\mathbb{R}^{>0} \times S^1 \times \breve{M}, \mp g_{\mathbb{C}H^1} + \breve{g}) \cong (\mathbb{R}^{>0} \times S^1 \times \breve{M}, \mp g_{\mathbb{C}H^1} + \breve{g}),$$

where $\pm$ *corresponds to* $\lambda = \pm 1$, *respectively, and* $g_{\mathbb{C}H^1} := \frac{1}{4\rho^2}(d\rho^2 + d\tilde{\phi}^2)$.

Proof. $f := \lambda r^2 - c$ fulfills $\omega(Z, \cdot) = -2g(\xi, \cdot) = -2\lambda r\, dr = -df$, where $r^2 = |g(\xi, \xi)|$. The Kähler form of (M, J, g) is given by $\omega = \lambda r dr \wedge \tilde{\eta} + r^2\breve{\omega}$, where $\breve{\omega} := \breve{g}(J\cdot, \cdot)$. One can check that $\breve{\omega} = \frac{1}{2}\lambda d\tilde{\eta}$. Using this, one finds $d\beta = 4\omega$, where $\beta := g(Z, \cdot) = 2\lambda r^2\tilde{\eta}$.

We endow the trivial S^1-principal bundle $P := M \times S^1 \to M$ with the principal connection

$$\eta := ds - \frac{1}{4}\beta = ds - \frac{\lambda}{2}r^2\tilde{\eta},$$

which has curvature $d\eta = \omega - \frac{1}{2}d\beta = -\omega$. Here, s is the natural coordinate on $S^1 = \{e^{is}|s \in \mathbb{R}\}$. The metric and one-forms on P are given by

$$g_P = \frac{2}{f_1}\eta^2 + g$$
$$\theta_0^P = -\frac{1}{2}df = -\lambda r\, dr$$
$$\theta_1^P = \eta + \frac{1}{2}\beta = ds + \frac{\lambda}{2}r^2\tilde{\eta},$$

where $f_1 = f - \frac{1}{2}g(Z, Z) = -\lambda r^2 - c$.

We compute the degenerate tensor field $\tilde{g}_P$:

$$\begin{aligned}\tilde{g}_P = g_P - \frac{2}{f}((\theta_0^P)^2 + (\theta_1^P)^2) &= \frac{2}{f_1}\left(ds + \frac{f_1+c}{2}\tilde{\eta}\right)^2 + g - \frac{2}{f}\left(r^2\, dr^2 + \left(ds + \frac{f+c}{2}\tilde{\eta}\right)^2\right)\\ &\overset{(3.1)}{=} \left(\frac{2}{f_1} - \frac{2}{f}\right)\left(ds + \frac{c}{2}\tilde{\eta}\right)^2 + \left(\lambda - \frac{2r^2}{f}\right)dr^2 + r^2\breve{g}\\ &= -\frac{4}{f}\frac{f+c}{f+2c}\left(ds + \frac{c}{2}\tilde{\eta}\right)^2 - \frac{1}{4f}\frac{f+2c}{f+c}df^2 + \lambda(f+c)\breve{g}.\end{aligned}$$

Since $\mathbb{R}^{>0} \times \breve{M} \subset \mathbb{R}^{>0} \times S = M$ is transversal to $J\xi \in \Gamma(TM)$, $\tilde{M}' := \mathbb{R}^{>0} \times \breve{M} \times S^1 \subset P$ is transversal to $Z^P := \tilde{Z} + f_1\partial_s = Z - (\eta(Z) - f_1)\partial_s = 2J\xi - c\partial_s \in \Gamma(TP)$. Replacing the coordinates r and s by $\rho := f$ and $\tilde{\phi} := -4s$, we obtain the Kähler metric $g' = \frac{1}{2|\rho|}\tilde{g}_P|_{M'}$ obtained from the K/K correspondence (Theorem 3) as given in Eq. (3.2). Here,

$$M' := \begin{cases}(-c, \infty) \times S^1 \times \breve{M} & \text{for } \lambda = 1\\ (-\infty, -c) \times S^1 \times \breve{M} & \text{for } \lambda = -1\end{cases}$$

is obtained from $\tilde{M}'$ via the coordinate change $r \mapsto \rho = \lambda r^2 - c$. For the metric g' to be defined, we need to restrict to $\{f = \rho \neq 0,\ -f_1 = \rho + 2c \neq 0\} \subset M'$.

The signature of g is given by $(2k+2, 2l)$ if $\lambda = 1$ and $(2k, 2l+2)$ if $\lambda = -1$, where $(2k, 2l)$ is the signature of $\breve{g}$. The signature of g' is related to the one of g by

$$\operatorname{sign} g' = \begin{cases}(+2, -2) + \operatorname{sign} g & \text{for } f_1 > 0,\ f < 0\\ \operatorname{sign} g & \text{for } f f_1 > 0\\ (-2, +2) + \operatorname{sign} g & \text{for } f_1 < 0, f > 0.\end{cases}$$

Using $f = \rho, f_1 = -(\rho + 2c)$ and taking into account $r^2 = \lambda(\rho + c) > 0$, one finds that on the subsets $M_\pm = \{\rho \in I_\pm\} \subset M'$ given in the Theorem, g' has signature $(2k, 2l+2)$, $(2k+2, 2l)$ respectively.

For the last statement, one just has to notice that for $c = 0$, $\operatorname{sgn}\rho = \lambda$. □

4. HK/QK correspondence for the *c*-map

In this section, we use the explicit formula given in Theorem 2 to show that the pseudo-hyper-Kähler structure on the cotangent bundle of a conical affine special Kähler manifold given by the rigid *c*-map is related to the quaternionic Kähler metric obtained from the supergravity *c*-map via the HK/QK correspondence. In fact, we get a one-parameter family of positive definite quaternionic Kähler metrics, which corresponds to one-loop corrections of the hypermultiplet moduli space in string theory compactifications on Calabi–Yau 3-folds (if the corresponding model is realized in string theory). As a corollary, this proves that the Ferrara–Sabharwal metric and its one-loop deformation are indeed quaternionic Kähler.

4.1. Conical affine and projective special Kähler geometry

First, we recall the definitions of conical affine and projective special Kähler manifolds [15,16]:

Definition 2. A **conical affine special Kähler manifold** (M, J, g_M, ∇, ξ) is a pseudo-Kähler manifold (M, J, g_M) endowed with a flat torsionfree connection ∇ and a vector field ξ such that

(i) $\nabla\omega_M = 0$, where $\omega_M := g_M(J\cdot, \cdot)$ is the Kähler form,
(ii) $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y \in \Gamma(TM)$,
(iii) $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection,
(iv) g_M is positive definite on $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^\perp$.

Let (M, J, g_M, ∇, ξ) be a conical affine special Kähler manifold of complex dimension $n+1$. Then ξ and $J\xi$ are commuting holomorphic vector fields that are homothetic and Killing respectively [16]. We assume that the holomorphic Killing vector field $J\xi$ induces a free S^1-action and that the holomorphic homothety ξ induces a free $\mathbb{R}^{>0}$-action on M. Then (M, g_M) is a metric cone over (S, g_S), where $S := \{p \in M | g_M(\xi(p), \xi(p)) = 1\}$, $g_S := g_M|_S$; and $-g_S$ induces a Riemannian metric $g_{\bar{M}}$ on $\bar{M} := S/S^1_{J\xi}$. $(\bar{M}, -g_{\bar{M}})$ is obtained from (M, J, g) via a Kähler reduction with respect to $J\xi$ and, hence, $g_{\bar{M}}$ is a Kähler metric (see e.g. [17]). The corresponding Kähler form $\omega_{\bar{M}}$ is obtained from ω_M by symplectic reduction. This determines the complex structure $J_{\bar{M}}$.

Definition 3. The Kähler manifold $(\bar{M}, J_{\bar{M}}, g_{\bar{M}})$ is called a **projective special Kähler manifold**.

More precisely, S is a (Lorentzian) Sasakian manifold and introducing the radial coordinate $r := \sqrt{g(\xi, \xi)}$, we can write the metric on M as [13,14]

$$g_M = dr^2 + r^2\pi^* g_S, \quad g_S = g_M|_S = \tilde{\eta} \otimes \tilde{\eta}|_S - \bar{\pi}^* g_{\bar{M}}, \tag{4.1}$$

where

$$\tilde{\eta} := \frac{1}{r^2} g_M(J\xi, \cdot) = d^c \log r = i(\bar{\partial} - \partial)\log r \tag{4.2}$$

is the contact one-form form when restricted to S and $\pi : M \to S = M/\mathbb{R}^{>0}_\xi$, $\bar{\pi} : S \to \bar{M} = S/S^1_{J\xi}$ are the canonical projection maps. From now on, we will drop π^* and $\bar{\pi}^*$ and identify, e.g., $g_{\bar{M}}$ with a $(0, 2)$ tensor field on M that has the distribution $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ as its kernel.

Locally, there exist so-called *conical special holomorphic coordinates* $z = (z^I) = (z^0, \ldots, z^n) : U \xrightarrow{\sim} \tilde{U} \subset \mathbb{C}^{n+1}$ such that the geometric data on the domain $U \subset M$ is encoded in a holomorphic function $F : \tilde{U} \to \mathbb{C}$ that is homogeneous of degree 2 [15,16]. Namely, we have [16]

$$g_M|_U = \sum_{I,J} N_{IJ} dz^I d\bar{z}^J, \qquad N_{IJ}(z, \bar{z}) := 2\mathrm{Im}\, F_{IJ}(z) := 2\mathrm{Im}\, \frac{\partial^2 F(z)}{\partial z^I \partial z^J} \quad (I, J = 0, \ldots, n)$$

and $\xi|_U = \sum z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I}$. The Kähler potential for $g_M|_U$ is given by $r^2|_U = g_M(\xi, \xi)|_U = \sum z^I N_{IJ} \bar{z}^J$.

The $\mathbb{C}^*$-invariant functions $X^\mu := \frac{z^\mu}{z^0}$, $\mu = 1, \ldots, n$, define a local holomorphic coordinate system on $\bar{M}$. The Kähler potential for $g_{\bar{M}}$ is $\mathcal{K} := -\log \sum_{I,J=0}^n X^I N_{IJ}(X) \bar{X}^J$, where $X := (X^0, \ldots, X^n)$ with $X^0 := 1$.

4.2. The rigid c-map

Now, we introduce the *rigid c-map*, which assigns to each affine special (pseudo-)Kähler manifold (M, J, g_M, ∇) and in particular to any conical affine special Kähler manifold (M, J, g_M, ∇, ξ) of real dimension $2n+2$ a (pseudo-)hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$ of dimension $4n+4$ [5,15].

From now on, we assume for simplicity that $(M \subset \mathbb{C}^{n+1}, J = J_{can}, g_M, \nabla, \xi)$ is a conical affine special Kähler manifold that is globally described by a homogeneous holomorphic function F of degree 2 defined on a $\mathbb{C}^*$-invariant domain M in standard holomorphic coordinates $z = (z^I) = (z^0, \ldots, z^n)$ induced from $\mathbb{C}^{n+1}$. Here, J_{can} denotes the standard complex structure induced from $\mathbb{C}^{n+1}$.

The real coordinates $(q^a)_{a=1,\dots,2n+2} := (x^I, y_J)_{I,J=0,\dots,n} := (\mathrm{Re}\, z^I,\ \mathrm{Re}\, F_J(z) := \mathrm{Re}\, \frac{\partial F(z)}{\partial z^J})$ on M are ∇-affine and fulfill $\omega_M = -2\sum dx^I \wedge dy_I$, where $\omega_M = g(J\cdot,\cdot)$ is the Kähler form on M [16]. We consider the cotangent bundle $\pi_N : N := T^*M \to M$ and introduce real functions $(p_a) := (\tilde{\zeta}_I, \zeta^J)$ on N such that together with $(\pi_N^* q^a)$, they form a system of canonical coordinates.

Proposition 3. *In the above coordinates (z^I, p_a), the hyper-Kähler structure on $N = T^*M$ obtained from the rigid c-map is given by*

$$g_N = \sum dz^I N_{IJ} d\bar{z}^J + \sum A_I N^{IJ} \bar{A}_J, \tag{4.3}$$

$$\omega_1 = \frac{i}{2}\sum N_{IJ} dz^I \wedge d\bar{z}^J + \frac{i}{2}\sum N^{IJ} A_I \wedge \bar{A}_J, \tag{4.4}$$

$$\omega_2 = -\frac{i}{2}\sum (d\bar{z}^I \wedge \bar{A}_I - dz^I \wedge A_I), \tag{4.5}$$

$$\omega_3 = \frac{1}{2}\sum (dz^I \wedge A_I + d\bar{z}^I \wedge \bar{A}_I), \tag{4.6}$$

where $A_I := d\tilde{\zeta}_I + \sum_J F_{IJ}(z) d\zeta^J$ $(I = 0, \dots, n)$ are complex-valued one-forms on N and $\omega_\alpha = g_N(J_\alpha\cdot,\cdot)$. (Here and in the following, we identify functions and one-forms on M with their pullbacks to N.)

Proof. One can check by a direct calculation that the metric and Kähler forms, (4.3)–(4.6) agree with the geometric data for the rigid c-map given in Section 3 of [15] (see also Section 3 of [2]), up to a conventional sign in the definition of the Kähler forms $\omega_\alpha = g_N(J_\alpha\cdot,\cdot) = -g_N(\cdot, J_\alpha\cdot)$ in [15]. For instance, we can write ω_1 and ω_3 as

$$\omega_1 = -2\sum dx^I \wedge dy_I + \frac{1}{2}\sum d\tilde{\zeta}_I \wedge d\zeta^I, \tag{4.7}$$

$$\omega_3 = \sum dx^I \wedge d\tilde{\zeta}_I + \sum dy_I \wedge d\zeta^I = \sum dq^a \wedge dp_a. \quad \square$$

Remark 2. It follows from the intrinsic geometric description in [15] that the pseudo-hyper-Kähler structure is independent of the particular description of the special Kähler structure in terms of a holomorphic function F.

Remark 3. We introduce holomorphic functions w_I, $I = 0, \dots, n$, on (N, J_1) that together with the holomorphic coordinates $z = (z^I)$ on (M, J) form a system of canonical holomorphic coordinates on $(N = T^*M, J_1)$. Then (w_I) and $(\tilde{\zeta}_I, \zeta^J)$ are related by

$$\begin{aligned}\sum_I w_I dz^I + \bar{w}_I d\bar{z}^I &\overset{!}{=} \sum_I \tilde{\zeta}_I dx^I + \zeta^I dy_I \\ &= \sum_I \frac{\tilde{\zeta}_I}{2}(dz^I + d\bar{z}^I) + \frac{\zeta^I}{2}\left(\sum_J F_{IJ}(z) dz^J + \overline{F_{IJ}(z)} d\bar{z}^J\right),\end{aligned}$$

which is equivalent to

$$w_I = \frac{1}{2}\left(\tilde{\zeta}_I + \sum_J F_{IJ}(z)\zeta^J\right) \quad (I = 0, \dots, n). \tag{4.8}$$

With the identification (4.8), (4.3)–(4.6) also agree, up to conventional factors, with the rigid c-map as given in Appendix B of [5] and throughout the physics literature.

4.3. The supergravity c-map

Let $(\bar{M}, g_{\bar{M}})$ be a projective special Kähler manifold of complex dimension n which is globally defined by a single holomorphic function F. The *supergravity c-map* [6] associates with $(\bar{M}, g_{\bar{M}})$ a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}})$ of dimension $4n+4$. Following the conventions of [17], we have $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ and

$$\begin{aligned} g_{\bar{N}} &= g_{\bar{M}} + g_G, \\ g_G &= \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)\right)^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(m) d\zeta^I d\zeta^J \\ &\quad + \frac{1}{2\rho}\sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(m) d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m) d\zeta^L), \end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on $m \in \bar{M}$ and $\mathcal{I}(m)$ is invertible with the inverse $\mathcal{I}^{-1}(m) =: (\mathcal{I}^{IJ}(m))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum\limits_K N_{IK}z^K \sum\limits_L N_{JL}z^L}{\sum\limits_{IJ} N_{IJ}z^I z^J}, \quad N_{IJ} := 2\mathrm{Im}F_{IJ}, \tag{4.9}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates z^I on the underlying conical special Kähler manifold $M \to \bar{M}$. Notice that the expressions are homogeneous of degree zero and, hence, well defined functions on $\bar{M}$. It is shown in [17, Cor. 5] that the matrix $\mathcal{I}(m)$ is positive definite and hence invertible and that the metric $g_{\bar{N}}$ does not depend on the choice of special coordinates [17, Thm. 9]. It is also shown that $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, g_{\bar{M}})$ is complete [17, Thm. 5].

Using $(p_a)_{a=1,\ldots,2n+2} := (\tilde{\zeta}_I, \zeta^J)_{IJ=0,\ldots,n}$ and $(\hat{H}^{ab}) := \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix}$, we can combine the last two terms of g_G into $\frac{1}{2\rho}\sum dp_a\hat{H}^{ab}dp_b$, i.e. the quaternionic Kähler metric is given by

$$g_{FS} := g_{\bar{N}} = g_{\bar{M}} + \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}\left(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)\right)^2 + \frac{1}{2\rho}\sum dp_a\hat{H}^{ab}dp_b. \tag{4.10}$$

4.4. HK/QK correspondence for the c-map

Again, we assume that $(M \subset \mathbb{C}^{n+1}, J = J_{can}, g_M, \nabla, \xi)$ is a conical affine special Kähler manifold that is globally described by a homogeneous holomorphic function F of degree 2 in standard holomorphic coordinates $z = (z^I) = (z^0, \ldots, z^n)$ induced from $\mathbb{C}^{n+1}$. We want to apply the HK/QK correspondence to the hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$ of signature $(4, 4n)$ obtained from the rigid c-map (see Section 4.2). In [2], it was shown that the vector field $Z := 2(J\xi)^h = 2J_1\xi^h$ on N fulfills the assumptions of the HK/QK correspondence, i.e. it is a space-like ω_1-Hamiltonian Killing vector field with $\mathcal{L}_Z J_2 = -2J_3$. Here, $X^h \in \Gamma(TN)$ is defined for any vector field $X \in \Gamma(TM)$ by $X^h(\pi_N^* q^a) = \pi_N^* X(q^a)$ and $X^h(p_a) = 0$ for all $a = 1, \ldots, 2n+2$. (X^h is the horizontal lift with respect to the flat connection ∇.)

Theorem 5. *Applying the HK/QK correspondence to (N, g_N, J_1, J_2, J_3) endowed with the ω_1-Hamiltonian Killing vector field Z gives (up to a constant conventional factor) the one-parameter family g^c_{FS} (4.11) of quaternionic pseudo-Kähler metrics, which includes the Ferrara–Sabharwal metric g_{FS} (4.10). The metric g^c_{FS} is positive definite and of negative scalar curvature on the domain $\{\rho > -2c\} \subset \bar{N}$ (which coincides with $\bar{N}$ if $c \geq 0$, see Section 4.3). If $c < 0$ the metric g^c_{FS} is of signature $(4n, 4)$ on the domain $\{-c < \rho < -2c\} \subset \bar{N}$. Furthermore, if $c > 0$ the metric g^c_{FS} is of signature $(4, 4n)$ on the domain $\bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3}$.*

Proof. We start from the hyper-Kähler structure on $N = T^*M$ given in Eqs. (4.3)–(4.6). As in Section 4.2, we identify functions and differential forms on M with their pullbacks to $\pi_N : N \to M$. We first compute the geometric data involved in the HK/QK correspondence, cf. Section 2. The moment map for $-\omega_1$ w.r.t. $Z = 2(J\xi)^h$ is given by $f := r^2 - c$, where $r := \|\xi\|_{g_M} = \sqrt{\sum z^I N_{IJ}\bar{z}^J}$ and $c \in \mathbb{R}$:

$$\omega_1(Z, \cdot) = -g_M(2\xi, \cdot) = -\sum(z^I N_{IJ}d\bar{z}^J + N_{IJ}\bar{z}^J dz^I) = -d(r^2) = -df,$$

since $\sum_I z^I \frac{\partial F_{IJ}(z)}{\partial z^K} = 0$. With $g_N(Z, Z) = 4g_M(\xi, \xi) = 4r^2$, we get

$$f_1 := f - \frac{1}{2}g_N(Z, Z) = -r^2 - c.$$

For the functions f and f_1 nowhere to vanish, we have to restrict N to $\{r^2 \neq |c|\} \subset N$. Using the contact one form $\tilde{\eta} := \frac{1}{r^2}g_M(J\xi, \cdot)$ on M (see (4.2)), we get

$$\beta := g_N(Z, \cdot) = 2g_M(J\xi, \cdot) = 2r^2\tilde{\eta}.$$

We consider the trivial S^1-principal bundle

$$P := N \times S^1, \quad S^1 = \{e^{is} | s \in \mathbb{R}\},$$

with the connection form

$$\eta = ds + \eta_N,$$

where η_N is the following one-form on N:

$$\eta_N := -\frac{1}{2}r^2\tilde{\eta} + \eta_{can} = \frac{f_1 + c}{2}\tilde{\eta} + \eta_{can}, \quad \eta_{can} := \frac{1}{4}\sum(\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I).$$

Then

$$d\eta = d\eta_N = -\frac{1}{4}d\beta + d\eta_{can} = \omega_1 - \frac{1}{2}d\beta,$$

where we used that ω_1 can be written as

$$\omega_1 \overset{(4.7)}{=} \pi_N^*\omega_M + \frac{1}{2}\sum d\tilde{\zeta}_I \wedge d\zeta^I = \frac{1}{4}d\beta + d\eta_{can},$$

since $\pi_N^*\omega_M = \frac{1}{4}\pi_N^* dd^c(r^2)$ and $\pi_N^* d^c(r^2) = \pi_N^*(2r^2 d^c \log r) \overset{(4.2)}{=} \pi_N^*(2r^2\tilde{\eta}) = \beta$, see Section 4.1.

Now we compute the one-forms $\theta_j^P, j = 0, 1, 2, 3$ on P, introduced in (2.3):

$$\theta_0^P = -\frac{1}{2}df = -rdr,$$
$$\theta_1^P = \eta + \frac{1}{2}\beta = ds + \frac{1}{2}r^2\tilde{\eta} + \eta_{can} = ds + \frac{f+c}{2}\tilde{\eta} + \eta_{can},$$
$$\theta_2^P = \frac{1}{2}\omega_3(Z, \cdot) = -\frac{i}{2}\sum(\bar{z}^I\bar{A}_I - z^I A_I) = -\mathrm{Im}\sum z^I A_I,$$
$$\theta_3^P = -\frac{1}{2}\omega_2(Z, \cdot) = \frac{1}{2}\sum(z^I A_I + \bar{z}^I\bar{A}_I) = \mathrm{Re}\sum z^I A_I.$$

For the calculation of θ_2^P and θ_3^P, we used $Z = 2i\sum(z^I\frac{\partial}{\partial z^I} - \bar{z}^I\frac{\partial}{\partial \bar{z}^I})^h$ and (4.5)–(4.6).

We compute the pseudo-Riemannian metric

$$g_P = \frac{2}{f_1}\eta^2 + \pi^* g_N \overset{(4.3)}{=} \frac{2}{f_1}\left(ds + \frac{c}{2}\tilde{\eta} + \eta_{can} + \frac{f_1}{2}\tilde{\eta}\right)^2 + g_M + \sum A_I N^{IJ}\bar{A}_J$$

and the degenerate tensor field

$$\begin{aligned}
\tilde{g}_P &= g_P - \frac{2}{f}\sum_{j=0}^{3}(\theta_j^P)^2 \\
&= g_P - \frac{2}{f}\left(r^2dr^2 + \left(ds + \frac{c}{2}\tilde{\eta} + \eta_{can} + \frac{f}{2}\tilde{\eta}\right)^2 + \left(\sum z^I A_I\right)\left(\sum \bar{z}^J\bar{A}_J\right)\right) \\
&= \left(\frac{2}{f_1} - \frac{2}{f}\right)\left(ds + \frac{c}{2}\tilde{\eta} + \eta_{can}\right)^2 + \left(\frac{f_1}{2} - \frac{f}{2}\right)\tilde{\eta}^2 - \frac{2}{f}r^2dr^2 + g_M + \sum A_I N^{IJ}\bar{A}_J - \frac{2}{f}\left(\sum z^I A_I\right)\left(\sum \bar{z}^J\bar{A}_J\right),
\end{aligned}$$

see (2.1) and (2.4). As always, pullbacks from M and N to P are implied where necessary. Using $\frac{f_1}{2} - \frac{f}{2} = -r^2 = -(f+c)$, $\frac{2}{f_1} - \frac{2}{f} = -\frac{4}{f}\frac{f+c}{f+2c}$, $\frac{2}{f} = \frac{2}{r^2} + \frac{2c}{f(f+c)}$ and $g_M \overset{(4.1)}{=} dr^2 + r^2(\tilde{\eta}^2 - g_{\bar{M}})$, we get

$$\begin{aligned}
\tilde{g}_P &= -r^2 g_{\bar{M}} - \frac{f+2c}{f}dr^2 - \frac{4}{f}\frac{f+c}{f+2c}\left(ds + \frac{c}{2}\tilde{\eta} + \eta_{can}\right)^2 - \frac{2c}{f(f+c)}\left(\sum z^I A_I\right)\left(\sum \bar{z}^J\bar{A}_J\right) \\
&\quad + \sum A_I N^{IJ}\bar{A}_J - \frac{2}{r^2}\left(\sum z^I A_I\right)\left(\sum \bar{z}^J\bar{A}_J\right).
\end{aligned}$$

We claim that the last two terms can be combined into $-\frac{1}{2}\sum dp_a\hat{H}^{ab}dp_b$, which appeared in the Ferrara–Sabharwal metric (4.10). This will be proven in the lemma below, see (4.12).

We use the local coordinates

$$r = \sqrt{\sum z^I N_{IJ}\bar{z}^J}, \qquad \phi := \arg z^0, \qquad X^\mu = \frac{z^\mu}{z^0}$$

on the conical affine special Kähler base M and choose the submanifold $N' = \{\phi = 0\} \subset P = N \times S^1$, which is transversal to

$$Z_1^P = (Z - \eta(Z)X_P) + f_1 X_P = Z + (r^2 + f_1)X_P = 2\partial_\phi - c\partial_s,$$

where $X_P = \partial_s$ is the fundamental vector field on P, cf. (2.2).

In these coordinates, we have

$$|z^0|^2 = r^2 e^{\mathcal{K}}$$

and, hence,

$$\tilde{\eta} = \frac{1}{2}d^c\log r^2 = \frac{1}{2}d^c\log|z^0|^2 - \frac{1}{2}d^c\mathcal{K} = d\phi - \frac{1}{2}d^c\mathcal{K} = d\phi + \sum\frac{iN_{IJ}(X)}{2X^tN\bar{X}}(X^I d\bar{X}^J - \bar{X}^J dX^I)$$

and

$$\sum (z^I A_I) \sum (\bar{z}^J \bar{A}_J) = |z^0|^2 \sum (X^I A_I) \sum (\bar{X}^J \bar{A}_J) = r^2 e^{\mathcal{K}} \left| \sum (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I) \right|^2 ,$$

where $\mathcal{K} = -\log X^t N \bar{X}$, $X^t N \bar{X} := \sum X^I N_{IJ} \bar{X}^J$, is the Kähler potential for the projective special Kähler metric $g_{\bar{M}}$. Replacing the coordinates r and s by $\rho := f$ and $\tilde{\phi} := -4s$ and recalling that $\sigma = \operatorname{sgn} f$, we obtain the quaternionic Kähler metric $g' = \frac{1}{2|f|} \tilde{g}_P|_{N'}$ from the HK/QK correspondence (Theorem 2) such that $g^c_{FS} := -2\sigma g'$ is given by

$$\begin{aligned} g^c_{FS} = {} & \frac{\rho + c}{\rho} g_{\bar{M}} + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \left(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c \mathcal{K} \right)^2 \\ & + \frac{1}{2\rho} \sum dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2} e^{\mathcal{K}} \left| \sum (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I) \right|^2 . \end{aligned} \tag{4.11}$$

For $c = 0$, g^c_{FS} reduces to the Ferrara–Sabharwal metric (4.10).

Notice that the above metric g^c_{FS} obtained from the HK/QK correspondence is defined on a subset of $\bar{M} \times \mathbb{R}^* \times S^1 \times \mathbb{R}^{2n+2}$, where the $\mathbb{R}^*$-factor corresponds to the coordinate ρ (which may now take negative values) and the S^1-factor is parametrized by the coordinate $\tilde{\phi} = -4s$ considered modulo $8\pi\mathbb{Z}$. Replacing the above subset by its universal covering (that is replacing S^1 by $\mathbb{R}$) we obtain a subset of $\bar{M} \times \mathbb{R}^* \times \mathbb{R}^{2n+3}$. In particular, $g_{FS} = g^0_{FS}$ is defined on $\tilde{N}$ as well as on the cyclic quotient $\tilde{N}/\mathbb{Z} = \bar{M} \times \mathbb{R}^{>0} \times S^1 \times \mathbb{R}^{2n+2}$.

The pseudo-hyper-Kähler metric g_N has signature $(4, 4n)$ and Z is space-like. Hence, g' is negative definite if $f > 0$ and $f_1 < 0$, it has signature $(4, 4n)$ if $f_1 f > 0$ and it has signature $(8, 4(n-1))$ if $f < 0$ and $f_1 > 0$ (see Corollary 1 in [2]). Using $f = \rho$ and $f_1 = -\rho - 2c$, we get

$$\operatorname{sign} g' = \begin{cases} (0, 4n+4) & \text{for } \rho > \max\{0, -2c\} \\ (4, 4n) & \text{for } 0 < \rho < -2c, \; c < 0 \\ (4, 4n) & \text{for } -2c < \rho < 0, \; c > 0 \\ (8, 4(n-1)) & \text{for } \rho < \min\{0, -2c\}. \end{cases}$$

Taking into account that by definition $r^2 = g_M(\xi, \xi) > 0$, i.e. $\rho > -c$, we get

$$\operatorname{sign} g' = \begin{cases} (0, 4n+4) & \text{for } \rho > \max\{0, -2c\} \; (\Leftrightarrow r^2 > |c|) \\ (4, 4n) & \text{for } -c < \rho < \max\{0, -2c\} \; (\Leftrightarrow 0 < r^2 < |c|). \end{cases}$$

It remains to prove.

Lemma 3.

$$\sum dp_a \hat{H}^{ab} dp_b = -2 \sum A_I N^{IJ} \bar{A}_J + \frac{4}{r^2} \left(\sum z^I A_I \right) \left(\sum \bar{z}^J \bar{A}_J \right), \tag{4.12}$$

where, as in the last section, $(p_a) = (\tilde{\zeta}_I, \zeta^J)$ *and* $(\hat{H}^{ab}) = \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix}$.

Proof. Recall that $A_I = d\tilde{\zeta}_I + \sum_J F_{IJ} d\zeta^J$, $I = 0, \ldots, n$. We write $A = (A_I) = d\tilde{\zeta} + \mathrm{F} d\zeta$, where $d\tilde{\zeta} = (d\tilde{\zeta}_I)$, $d\zeta = (d\zeta^I)$ are form-valued column vectors and $\mathrm{F} := (F_{IJ})$.

First, we show that $\sum A_I N^{IJ} \bar{A}_J = \sum dp_a H^{ab} dp_b$ with

$$(H^{ab}) := \begin{pmatrix} N^{-1} & \frac{1}{2} N^{-1} R \\ \frac{1}{2} R N^{-1} & \frac{1}{4}(N + R N^{-1} R) \end{pmatrix},$$

where $R := 2\mathrm{Re}\,\mathrm{F}$:

$$\begin{aligned} \sum A_I N^{IJ} \bar{A}_J &= (d\tilde{\zeta}^t + d\zeta^t \mathrm{F}) N^{-1} (d\tilde{\zeta} + \bar{\mathrm{F}} d\zeta) \\ &= \left(d\tilde{\zeta}^t + d\zeta^t \frac{1}{2}(R + iN) \right) N^{-1} \left(d\tilde{\zeta} + \frac{1}{2}(R - iN) d\zeta \right) \\ &= d\tilde{\zeta}^t N^{-1} d\zeta + d\tilde{\zeta}^t \frac{1}{2} N^{-1} R \, d\zeta + d\zeta^t \frac{1}{2} R N^{-1} d\tilde{\zeta} + d\zeta^t \frac{1}{4}(N + R N^{-1} R) d\zeta . \end{aligned}$$

Now, we show that $(\sum z^I A_I)(\sum \bar{A}_J \bar{z}^J) = \sum dp_a \breve{H}^{ab} dp_b$ with

$$(\breve{H}^{ab}) := \frac{1}{2} \begin{pmatrix} z\bar{z}^t + \bar{z}z^t & z\bar{z}^t \bar{\mathrm{F}} + \bar{z}z^t \mathrm{F} \\ \bar{\mathrm{F}}\bar{z}z^t + \mathrm{F}z\bar{z}^t & \mathrm{F}z\bar{z}^t\bar{\mathrm{F}} + \bar{\mathrm{F}}\bar{z}z^t\mathrm{F} \end{pmatrix} :$$

$$\begin{aligned}\left(\sum z^I A_I\right)\left(\sum \bar{z}^J \bar{A}_J\right) &= (d\tilde{\zeta}^t z + d\zeta^t \mathrm{F} z)(\bar{z}^t d\tilde{\zeta} + \bar{z}^t \bar{\mathrm{F}} d\zeta) \\ &= d\tilde{\zeta}^t z\bar{z}^t d\tilde{\zeta} + d\tilde{\zeta}^t z\bar{z}^t \bar{\mathrm{F}} d\zeta + d\zeta^t \mathrm{F} z\bar{z}^t d\tilde{\zeta} + d\zeta^t \mathrm{F} z\bar{z}^t \bar{\mathrm{F}} d\zeta \\ &= d\tilde{\zeta}^t \frac{1}{2}(z\bar{z}^t + \bar{z}z^t) d\tilde{\zeta} + d\tilde{\zeta}^t \frac{1}{2}(z\bar{z}^t \bar{\mathrm{F}} + \bar{z}z^t \mathrm{F}) d\zeta \\ &\quad + d\zeta^t \frac{1}{2}(\mathrm{F} z\bar{z}^t + \bar{\mathrm{F}}\bar{z}z^t) d\tilde{\zeta} + d\zeta^t \frac{1}{2}(\mathrm{F} z\bar{z}^t \bar{\mathrm{F}} + \bar{\mathrm{F}}\bar{z}z^t \mathrm{F}) d\zeta.\end{aligned}$$

Hence, the right side of Eq. (4.12) is given by $\sum dp_a(-2H^{ab} + \frac{4}{r^2}\breve{H}^{ab})dp_b$.

To rewrite the left side of (4.12), we need to invert $\mathcal{I} = \mathrm{Im}\,\mathcal{N} = -\frac{1}{2}N + \frac{Nzz^tN}{2z^tNz} + \frac{N\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}$. It is easy to check that the inverse of $\mathcal{I}$ is given by [18]

$$\mathcal{I}^{-1} = -2N^{-1} + \frac{2}{z^tN\bar{z}}(z\bar{z}^t + \bar{z}z^t).$$

Using $\mathcal{R} = \mathrm{Re}\,\mathcal{N} = \frac{1}{2}R + \frac{iNzz^tN}{2z^tNz} - \frac{iN\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}$, we obtain

$$\mathcal{I}^{-1}\mathcal{R} = -N^{-1}R + \frac{1}{z^tN\bar{z}}(z\bar{z}^t(R - iN) + \bar{z}z^t(R + iN)) = -N^{-1}R + \frac{2}{r^2}(z\bar{z}^t\bar{\mathrm{F}} + \bar{z}z^t\mathrm{F})$$

and hence

$$\mathcal{R}\mathcal{I}^{-1} = (\mathcal{I}^{-1}\mathcal{R})^t = -RN^{-1} + \frac{2}{r^2}(\bar{\mathrm{F}}\bar{z}z^t + \mathrm{F}z\bar{z}^t).$$

For the lower right block in $(\hat{H}^{ab})$, we calculate

$$\begin{aligned}\mathcal{R}\mathcal{I}^{-1}\mathcal{R} &= -\frac{1}{2}RN^{-1}R + \frac{1}{z^tN\bar{z}}(\bar{\mathrm{F}}\bar{z}z^t(R + iN) + \mathrm{F}z\bar{z}^t(R - iN)) \\ &\quad + \frac{i}{z^tNz}\left(-\frac{1}{2}R + F\right) zz^tN - \frac{i}{\bar{z}^tN\bar{z}}\left(-\frac{1}{2}R + \bar{\mathrm{F}}\right)\bar{z}\bar{z}^tN \\ &= -\frac{1}{2}RN^{-1}R + \frac{2}{r^2}(\bar{\mathrm{F}}\bar{z}z^t\mathrm{F} + \mathrm{F}z\bar{z}^t\bar{\mathrm{F}}) - \frac{Nzz^tN}{2z^tNz} - \frac{N\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}\end{aligned}$$

and hence

$$\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} = -\frac{1}{2}(N + RN^{-1}R) + \frac{2}{r^2}(\bar{\mathrm{F}}\bar{z}z^t\mathrm{F} + \mathrm{F}z\bar{z}^t\bar{\mathrm{F}}).$$

This shows that $(\hat{H}^{ab}) = -2(H^{ab}) + \frac{4}{r^2}(\breve{H}^{ab})$ and thus proves Eq. (4.12). □

This proves Theorem 5. □

Remark 4. Note that the quaternionic Kähler metric g^c_{FS} given in (4.11) agrees with the one-loop deformed Ferrara–Sabharwal metric first obtained in [7] (see also [3], eq. (2.93)).

Remark 5. One can check that the restriction of the metric $g' = -\frac{\sigma}{2}g^c_{FS}$ given in (4.11) to $M' := \{\zeta = \tilde{\zeta} = 0\} \subset N'$ is the metric that one obtains when applying the K/K correspondence to the original conical affine special Kähler manifold (M, J, g_M, ∇, ξ) with respect to the holomorphic Killing field $Z = 2J\xi$. This is a special case of Theorem 4. For $c = 0$, we have

$$(M', g_{FS}|_{M'}) = (\mathbb{R}^{>0} \times S^1 \times \bar{M}, g_{\mathbb{C}H^1} + g_{\bar{M}}),$$

where $(\bar{M}, J_{\bar{M}}, g_{\bar{M}})$ is the underlying projective special Kähler manifold and $(\mathbb{R}^{>0} \times S^1, g_{\mathbb{C}H^1})$ is a $\mathbb{Z}$-quotient of the complex hyperbolic line. The hyperbolic metric $g_{\mathbb{C}H^1} = \frac{1}{4\rho^2}(d\rho^2 + d\tilde{\phi}^2)$ is normalized such that its scalar curvature is -8.

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Appendix. A simple example of the HK/QK correspondence

Here we consider

$$M = \{(z, w) \in \mathbb{C}^2 | z \neq 0\} \subset \mathbb{C}^2$$

with its standard flat hyper-Kähler structure[2] (g, J_1, J_2, J_3). This manifold is in the image of the rigid c-map and therefore admits a vector field Z verifying the above assumptions, see [2, Proposition 2]. The canonical choice of function f which leads to a definite quaternionic Kähler metric g' is $f = \frac{1}{4}g(Z, Z)$, see [2, Corollary 4]. The metric g' is in fact negative definite if we take g positive definite and vice versa.

Let us first compute all the relevant geometric data on M in terms of the standard J_1-holomorphic coordinates (z, w) of $\mathbb{C}^2$, which satisfy $J_3^* dw = d\bar{z}$. The metric and Kähler forms are given by:

$$\begin{aligned}
g &= 2(|dz|^2 + |dw|^2),\\
\omega_1 &= i(dz \wedge d\bar{z} + dw \wedge d\bar{w}),\\
\omega_2 &= i(dz \wedge dw - d\bar{z} \wedge d\bar{w}),\\
\omega_3 &= dz \wedge dw + d\bar{z} \wedge d\bar{w}.
\end{aligned}$$

The vector field Z is given by

$$Z = 2\left(iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}}\right),$$
$$g(Z, Z) = 8|z|^2,$$

with the canonical choice of Hamiltonian given by

$$f = 2|z|^2,$$

such that $df = -\omega_1 Z$ and

$$f_1 = f - \frac{g(Z, Z)}{2} = -2|z|^2.$$

Notice that the functions f and f_1 are nowhere vanishing on M. Then we consider the trivial S^1-principal bundle

$$P = M \times S^1, \quad S^1 = \{e^{is} | s \in \mathbb{R}\}$$

with the connection form

$$\eta = ds + \eta_M,$$

where s is the natural coordinate on $S^1 = \{e^{is} | s \in \mathbb{R}\}$ and η_M is the following one-form on M

$$\eta_M = \frac{i}{2}(zd\bar{z} - \bar{z}dz + wd\bar{w} - \bar{w}dw) - \frac{1}{2}gZ.$$

Computing

$$gZ = 2i(zd\bar{z} - \bar{z}dz), \tag{A.1}$$

we get

$$\eta_M = \frac{i}{2}(-zd\bar{z} + \bar{z}dz + wd\bar{w} - \bar{w}dw).$$

Remark 6. Notice that, in the trivialization of P, $X_P = \partial_s$, $\tilde{Z} = Z - \eta_M(Z)\partial_s$ and $Z_1 = Z + (-\eta_M(Z) + f_1)\partial_s$. The above formula for η_M implies that $\eta_M(Z) = -2|z|^2 = f_1$ and, thus, $Z_1 = Z$, in the given trivialization.

We define M' as the submanifold of $P = M \times S^1$ defined by $\operatorname{Im} z = 0$. We will use $w, r = \sqrt{2}|z|$ and s as local coordinates on M'. M' intersects each orbit of the S^1-action $S^1_{Z_1}$ generated by Z_1 in exactly one point such that we can identify M' with the orbit space $P/S^1_{Z_1}$. Now we compute the one-forms gZ and θ_a, $a = 0, 1, 2, 3$, on M'. Writing $z = \frac{r}{\sqrt{2}}e^{i \arg z}$, from (A.1) we get

$$gZ = 2r^2 d \arg z,$$

which shows that gZ vanishes on M' and that

$$\eta_M|_{M'} = \frac{i}{2}(wd\bar{w} - \bar{w}dw) = \frac{1}{4}(\tilde{\zeta}d\zeta - \zeta d\tilde{\zeta}) = \eta_{can},$$

if we write $w = \frac{1}{2}(\tilde{\zeta} + i\zeta)$ and define $\eta_{can} = \frac{1}{4}(\tilde{\zeta}d\zeta - \zeta d\tilde{\zeta})$. Using that $gZ|_{M'} = 0$, we have:

$$\begin{aligned}
\theta_0|_{M'} &= -(zd\bar{z} + \bar{z}dz)|_{M'} = -rdr,\\
\theta_1|_{M'} &= \eta|_{M'},
\end{aligned}$$

[2] The flat hyper-Kähler manifold $\mathbb{C}^2$ is also considered in [4] but the corresponding quaternionic Kähler metric is not computed there.

$$\theta_2|_{M'} = \frac{i}{\sqrt{2}} r(dw - d\bar{w}),$$
$$\theta_3|_{M'} = \frac{1}{\sqrt{2}} r(dw + d\bar{w}),$$

which implies

$$\sum_a (\theta_a^P)^2 = (\eta|_{M'})^2 + r^2(dr^2 + 2|dw|^2).$$

So

$$\tilde{g}_P|_{M'} = \left(\frac{2}{f_1} - \frac{2}{f}\right)\eta^2|_{M'} + (dr^2 + 2|dw|^2) - 2(dr^2 + 2|dw|^2) = \left(\frac{2}{f_1} - \frac{2}{f}\right)\eta^2|_{M'} - (dr^2 + 2|dw|^2).$$

Now

$$\frac{2}{f_1} - \frac{2}{f} = \frac{2(f - f_1)}{ff_1} = \frac{g(Z,Z)}{ff_1} = -\frac{4}{r^2}$$

and

$$\eta|_{M'} = ds + \eta_{can}.$$

Therefore, we can rewrite

$$\tilde{g}_P|_{M'} = -\frac{4}{r^2}(ds + \eta_{can})^2 - (dr^2 + 2|dw|^2)$$

and

$$-2g' = -\frac{1}{f}\tilde{g}_P|_{M'} = \frac{4}{r^4}(ds + \eta_{can})^2 + \frac{dr^2}{r^2} + 2\frac{|dw|^2}{r^2}.$$

Putting $\rho = r^2$ and $\tilde{\phi} = -4s$, we can rewrite this as

$$-2g' = \frac{1}{4\rho^2}(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta} d\zeta)^2 + \frac{d\rho^2}{4\rho^2} + \frac{d\tilde{\zeta}^2 + d\zeta^2}{2\rho}.$$

The Riemannian metric $-2g'$ is precisely the Ferrara–Sabharwal metric (cf. [6,17]), which in the present case coincides with the complex hyperbolic metric.

Remark 7. The manifold M' is a cyclic quotient of the complex hyperbolic plane. The complex hyperbolic plane is parametrized by the global coordinates $(w, r > 0, s)$ and M' is obtained as the quotient by $2\pi\mathbb{Z}$ acting by translations in s.

Remark 8. Carrying out the above calculation with the Hamiltonian function f replaced by $f - c$, one obtains the deformed Ferrara–Sabharwal metric (4.11) for the special case where the underlying projective special Kähler manifold is a point (i.e. for holomorphic prepotential $F = \frac{i}{2}(z^0)^2$):

$$g^c_{UH} := \frac{1}{4\rho^2}\left(\frac{\rho + 2c}{\rho + c}d\rho^2 + \frac{\rho + c}{\rho + 2c}(d\tilde{\phi} + \zeta^0 d\tilde{\zeta}_0 - \tilde{\zeta}_0 d\zeta^0)^2 + 2(\rho + 2c)((d\tilde{\zeta}_0)^2 + (d\zeta^0)^2)\right).$$

This is known to physicists as the one-loop corrected universal hypermultiplet metric and was derived in [9]. As was already noticed in [9], this metric admits an isometric action of the three-dimensional Heisenberg group generated by the Killing vector fields

$$\frac{\partial}{\partial\tilde{\phi}}, \qquad \frac{\partial}{\partial\tilde{\zeta}_0} + \zeta^0\frac{\partial}{\partial\tilde{\phi}}, \qquad \frac{\partial}{\partial\zeta^0} - \tilde{\zeta}_0\frac{\partial}{\partial\tilde{\phi}};$$

and hence falls under the classification of 4-dimensional self-dual Einstein metrics with non-zero scalar curvature admitting two commuting Killing vector fields by Calderbank and Pedersen[3] [19].

For $c > 0$, g^c_{UH} is positive definite and of negative scalar curvature on the domains $\{-c < \rho < 0\}$ and $\{\rho > 0\}$ in $\mathbb{R}^4$. For $c < 0$, g^c_{UH} is positive definite and of negative scalar curvature on $\{\rho > -2c\} \subset \mathbb{R}^4$ and $-g^c_{UH}$ is positive definite and of positive scalar curvature on $\{-c < \rho < -2c\} \subset \mathbb{R}^4$, cf. Theorem 5.

[3] Calderbank and Pedersen express such a metric in terms of an eigenfunction $\mathcal{F}(r,\eta)$ of the Laplacian on the hyperbolic plane $\{(r,\eta) \in \mathbb{R}^{>0} \times \mathbb{R}\}$. In their formalism, the metric $-2g^c_{UH}$ corresponds to the function $\mathcal{F} = \frac{r^2 - c}{\sqrt{r}}$ and their coordinates (r, η, ϕ, ψ) are related to our coordinates by $\rho = r^2 - c$, $\tilde{\phi} = 2\psi + \phi\eta$, $\tilde{\zeta}_0 = \frac{1}{\sqrt{2}}\phi$, $\zeta^0 = \sqrt{2}\eta$.

Notice that the complex hyperbolic metric g^0_{UH} is symmetric and hence complete. Using this, one can show that g^c_{UH} is complete on the domain $\{\rho > 0\}$ for $c > 0$. In fact, $g^c_{UH} > \frac{1}{2}g^0_{UH}$. On the other domains mentioned above, however, the positive or negative definite metric g^c_{UH} is incomplete, as stated in the proposition below. This is in agreement with the result of A. Haydys, who studied $-g^c_{UH}$ on $\{-c < \rho < -2c\} \subset \mathbb{R}^4$ for the special case $c = -1$ (see [1, Example 9] (resp. 3.2 in the arXiv version)).

Proposition 4. (i) *For $c > 0$ the quaternionic Kähler metric g^c_{UH} of negative scalar curvature is complete on the domain $\{\rho > 0\}$ and incomplete on $\{-c < \rho < 0\}$.*
(ii) *For $c < 0$ the quaternionic Kähler metric g^c_{UH} is incomplete and of negative scalar curvature on $\{\rho > -2c\}$.*
(iii) *For $c < 0$ the quaternionic Kähler metric $-g^c_{UH}$ is incomplete and of positive scalar curvature on $\{-c < \rho < -2c\}$.*

Proof. It remains to prove the incompleteness in the corresponding cases. In cases (i) and (iii) we consider the curve

$$\rho = t - c, \qquad \tilde{\phi} = \tilde{\zeta}_0 = \zeta^0 = 0, \quad 0 < t < \frac{|c|}{2},$$

which approaches the boundary of the respective domain for $t \to 0$. Its length is given by

$$\frac{1}{2}\int_0^{\frac{|c|}{2}} \frac{1}{|t-c|}\sqrt{\frac{|t+c|}{t}}dt \geq C\int_0^{\frac{|c|}{2}} \frac{dt}{2\sqrt{t}} < \infty,$$

where $C > 0$ is a lower bound for the continuous function $\frac{\sqrt{|t+c|}}{|t-c|}$ on the compact interval $[0, \frac{|c|}{2}]$. In case (ii) we consider instead the curve

$$\rho = t - 2c, \qquad \tilde{\phi} = \tilde{\zeta}_0 = \zeta^0 = 0, \quad 0 < t < 1,$$

which approaches the boundary for $t \to 0$. Its length is the integral of the continuous function $\frac{1}{2|t-2c|}\sqrt{\frac{t}{t-c}}$ on the compact interval $[0, 1]$ and, hence, finite. □

Remark 9. Notice that the above proof for the incompleteness in case (ii) is still valid in higher dimensions for the positive definite quaternionic Kähler metric g^c_{FS}, $c < 0$, on the domain $\{\rho > -2c\}$, see Theorem 5 for a description of the domain of positivity of the one-loop deformed Ferrara–Sabharwal metric (4.11) depending on the sign of c. On the contrary, the proof of the completeness in case (i), given in Remark 8, does not extend in a straightforward way to higher dimensions.[4]

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[4] Note added in proof: In the meantime it has been proven that the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} for $c > 0$ on the domain $\rho > 0$ is complete for every complete projective special Kähler manifold in the image of the supergravity r-map, see M. Dyckmanns, Ph.D. dissertation, University of Hamburg, to appear in 2015.