The c-map Image of $\mathbb{C}H^n$

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1 Manifolds Related to the PSK Manifolds $\mathbb{C}H^n$

Over the course of several papers, Cortés and collaborators have studied a number of physics-derived geometric constructions relating several types of so-called *special* geometries [1–5]. As part of their work, they have worked out the relations between special

Kähler, hyper-Kähler and quaternionic Kähler manifolds summarized in the following diagram:

$$
\begin{tikzcd}
\mathrm{CASK} \arrow[r, "\text{rigid } c"] \arrow[dd, "\mathbb{C}^*"'] & \mathrm{HK} \arrow[dd, "\mathrm{HK/QK}"'] \arrow[dr, dashed, leftrightarrow, "\text{conify}"] & \\
 & & \widehat{\mathrm{HK}} \arrow[dl, dashed, leftrightarrow, "\text{Swann}"] \\
\mathrm{PSK} \arrow[r, "\text{SUGRA } c"'] & \mathrm{QK} &
\end{tikzcd}
$$

In these notes, we study the properties of the quaternionic Kähler manifold(s) arising from the supergravity (SUGRA) c-map, applied to the projective special Kähler (PSK) manifold $\mathbb{C}H^n$, i.e. complex hyperbolic space, equipped with its symmetric metric. We start by describing the manifolds related to $\mathbb{C}H^n$ through the above diagram, and will end by deriving the quaternionic Kähler (QK) metric on its c-map image; this metric is known as the Ferrara-Sabharwal metric, after the physicists that first showed that it is a QK metric. We will also derive the so-called *one-loop deformation* of this metric.

1.1 The Conical Affine Special Kähler Manifold

To obtain the CASK (conical affine special Kähler) manifold corresponding to $\mathbb{C}H^n$, we simply regard the latter as a subset of $\mathbb{C}\mathrm{P}^n$, and simply take the preimage under the projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$. This means that, as a smooth manifold, the CASK is given by

$$X = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, [z_0 : z_1 : \cdots : z_n] \in \mathbb{C}H^n \subset \mathbb{C}\mathrm{P}^n \right\}$$

or equivalently

$$X = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,\middle|\, -|z_0|^2 + \sum_{i=1}^{n} |z_i|^2 < 0 \right\}$$

Since the metric

$$-\mathrm{d}z_0 \mathrm{d}\bar{z}_0 + \sum_{i=1}^{n} \mathrm{d}z_i \mathrm{d}\bar{z}_i$$

induces the standard symmetric metric on $\mathbb{C}H^n$, and the CASK metric should induces its negative (and should correspondingly be of mostly negative signature), we have

$$g_X = \mathrm{d}z_0 \mathrm{d}\bar{z}_0 - \sum_{i=1}^{n} \mathrm{d}z_i \mathrm{d}\bar{z}_i$$

since this metric is already flat, the special Kähler connection on X is simply the trivial connection d, which is simultaneously the Levi-Cività connection. The Euler vector field on X is simply the restriction of the radial vector field on $\mathbb{C}^2 \setminus \{0\}$ to X.

1.2 The Hyper-Kähler Manifold

Now, we would like to apply the rigid c-map and find the corresponding hyper-Kähler manifold, to which we may apply the HK/QK correspondence. The rigid c-map is explained on page 18 of ACM (Conification paper). Starting from the (pseudo)-CASK manifold X, we consider its cotangent bundle $p : N = T^*X \to X$ and split its tangent bundle into horizontal and vertical subbundles using the special Kähler connection (in this case simply the Levi-Cività connection d). We then have $TN = T^hN \oplus T^vN \cong$

$p^*TX \oplus p^*T^*X$ where the isomorphism comes from the projection on the first summand and is canonical on the second. In our case, X is an open subset of $\mathbb{C}^{n+1}$ and therefore its (co)tangent bundle is trivial; p is simply projection onto the first factor.

With respect to the above decomposition of TN, we can define the hyper-Kähler structure on N as follows:

$$g_N = \begin{pmatrix} g_X & 0 \\ 0 & g_X^{-1} \end{pmatrix} \qquad J_1 = \begin{pmatrix} J_X & 0 \\ 0 & J_X^* \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & -\omega_X^{-1} \\ \omega_X & 0 \end{pmatrix}$$

Here, g_X is the metric on X and g_X^{-1} is the induced metric on T^*X, J_X is the complex structure of X and J_X^* the induced complex structure on the cotangent bundle. Finally, ω_X is the standard symplectic form on $X \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, regarded as an isomorphism $TX \to T^*M$ (and hence as an endomorphism of TN, by identifying one-forms of the coordinates on X with vectors along the fibers via $\mathrm{d}x_j = \partial_{a_j}$ and $\mathrm{d}y_j = -\partial_{b_j}$(!)). Notice that pullbacks are suppressed throughout.

In this case, we have $X = \mathbb{C}H^n \times \mathbb{C}^*$ and $N = \mathbb{C}H^n \times \mathbb{C}^* \times \mathbb{C}^{n+1}$; if $(\vec{z}, \vec{w}) = ((\vec{x}, \vec{y}), (\vec{a}, \vec{b}))$ are (standard) global coordinates on X, we have

$$g_X = 2\left(\mathrm{d}z_0\mathrm{d}\bar{z}_0 - \sum_{i=1}^{n} \mathrm{d}z_i\mathrm{d}\bar{z}_i\right) \qquad J = i \qquad \omega_X = i\left(\mathrm{d}z_0 \wedge \mathrm{d}\bar{z}_0 - \sum_{i=1}^{n} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i\right)$$

which yields explicit expressions for the hyper-Kähler data on N. For easier comparison with ACDM, page 24–25, we have multiplied our previous expression for g_X by two. We set $G_{00} = 1$, $G_{ii} = -1$ for $i > 0$ and $G_{ij} = 0$ for $i \neq j$. We then have

$$g_N = 2\left[\sum_{i=0}^{n} G_{ii}\mathrm{d}z_i\mathrm{d}\bar{z}_i + \sum_{i=0}^{n} G_{ii}\mathrm{d}w_i\mathrm{d}\bar{w}_i\right] \tag{1}$$

$$J_1\partial_{x_j} = \partial_{y_j} \qquad J_1\partial_{y_j} = -\partial_{x_j} \qquad J_1\partial_{a_j} = \partial_{b_j} \qquad J_1\partial_{b_j} = -\partial_{a_j}$$

$$\omega_1 = i\left[\sum_{i=0}^{n} G_{ii}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i + \sum_{i=0}^{n} G_{ii}\mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i\right] \tag{2}$$

$$J_2\partial_{x_j} = -G_{jj}\partial_{b_j} \qquad J_2\partial_{y_j} = -G_{jj}\partial_{a_j} \qquad J_2\partial_{a_j} = G_{jj}\partial_{y_j} \qquad J_2\partial_{b_j} = G_{jj}\partial_{x_j}$$

$$\omega_2 = i\left[\sum_{i=0}^{n} \mathrm{d}z_i \wedge \mathrm{d}w_i - \mathrm{d}\bar{z}_i \wedge \mathrm{d}\bar{w}_i\right] = -2\,\mathrm{Im}\left(\sum_{i=0}^{n} \mathrm{d}z_i \wedge \mathrm{d}w_i\right) \tag{3}$$

$$J_3\partial_{x_j} = G_{jj}\partial_{a_j} \qquad J_3\partial_{y_j} = -G_{jj}\partial_{b_j} \qquad J_3\partial_{a_j} = -G_{jj}\partial_{x_j} \qquad J_3\partial_{b_j} = G_{jj}\partial_{y_j}$$

$$\omega_3 = \sum_{i=0}^{n} \mathrm{d}z_i \wedge \mathrm{d}w_i + \mathrm{d}\bar{z}_i \wedge \mathrm{d}\bar{w}_i = 2\,\mathrm{Re}\left(\sum_{i=0}^{n} \mathrm{d}z_i \wedge \mathrm{d}w_i\right) \tag{4}$$

which summarizes the hyper-Kähler data on N.

1.3 Tensors Involved in the HK/QK Correspondence

Now, we consider the field Z generating the S^1-action on N: The action comes from the units inside $\mathbb{C}^*$, which acts diagonally on the base manifold X; the normalization of Z is derived from the requirement $L_Z J_2 = -2J_3$. The diagonal action of $S^1 \subset \mathbb{C}^*$ on the base X implies that Z is proportional to the standard angular coordinate vector field in every copy of $\mathbb{C}$, i.e.

$$Z = iC\sum_{i=0}^{n}(z_i\partial_{z_i} - \bar{z}_i\partial_{\bar{z}_i}) \qquad C \in \mathbb{R}$$

To compute the Lie derivative of J_2, we use

$$(L_Z J_2)(X) = [Z, J_2(X)] - J_2([Z, X])$$

We use that

$$[Z, \partial_{x_j}] = -C\partial_{y_j} \qquad [Z, \partial_{y_j}] = C\partial_{x_j} \qquad [Z, \partial_{a_j}] = [Z, \partial_{b_j}] = 0$$

to find

$$\begin{aligned} (L_Z J_2)(\partial_{x_j}) &= -CG_{jj}\partial_{a_j} \qquad (L_Z J_2)(\partial_{y_j}) = CG_{jj}\partial_{b_j} \\ (L_Z J_2)(\partial_{a_j}) &= CG_{jj}\partial_{x_j} \qquad (L_Z J_2)(\partial_{b_j}) = -CG_{jj}\partial_{y_j} \end{aligned}$$

comparing with the expressions for J_3, we see that $C = 2$ is the correct normalization. It is easily computed that

$$g(Z, Z) = 8\sum_{i=0}^{n} G_{ii}|z_i|^2$$

From the fact that Z is an angular vector field and the form of the metric, it is clear that Z is Killing. To see that it is symplectic with respect to ω_1, one checks $L_Z J_1 = 0$, which is another short computation. In fact, it is a Hamiltonian vector field:

$$\omega_1(Z, -) = -2\sum_{i=0}^{n} G_{ii} z_i \mathrm{d}\bar{z}_i + \bar{z}_i \mathrm{d}\bar{z}_i = -\mathrm{d}\left(2\sum_{i=0}^{n} G_{ii}|z_i|^2\right) =: -\mathrm{d}f \tag{5}$$

where $f =: r^2$ is the standard choice of Hamiltonian (shifts in f will correspond to the one-loop deformation). Now we set

$$f_1 := f - \frac{1}{2}g(Z, Z) = -2\sum_{i=0}^{n} G_{ii}|z_i|^2 = -r^2$$

Now, we consider the (trivial) principal bundle $P = N \times S^1$, which we equip with a connection η with curvature $\mathrm{d}\eta = \pi^*(\omega_1 - \frac{1}{2}\mathrm{d}(g(Z, -)))$. Since

$$g(Z, -) = 2i\sum_{i=0}^{n} G_{ii}(z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i)$$

a natural choice for η is

$$\begin{aligned} \eta &= \mathrm{d}s + \frac{i}{2}\sum_{i=0}^{n} G_{ii}(z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i + w_i \mathrm{d}\bar{w}_i - \bar{w}_i \mathrm{d}w_i) - \frac{1}{2}g(Z, -) \\ &= \mathrm{d}s - \frac{i}{2}\sum_{i=0}^{n} G_{ii}(z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i) + \frac{i}{2}\sum_{i=0}^{n} G_{ii}(w_i \mathrm{d}\bar{w}_i - \bar{w}_i \mathrm{d}w_i) \\ &=: \mathrm{d}s - \frac{r^2}{4}\tilde{\eta} + \eta_{\mathrm{can}} =: \mathrm{d}s + \eta_N \end{aligned} \tag{6}$$

where s is the standard (angular) coordinate on the circle $S^1 = \{e^{is} \mid s \in \mathbb{R}\}$, and η_{can} is universal, while $\tilde{\eta} := \frac{1}{r^2}g(Z, -)$ is given by

$$\tilde{\eta} = 2i\frac{\sum_{i=0}^{n} G_{ii}(z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i)}{r^2} = \frac{i\sum_{i=0}^{n} G_{ii}(z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i)}{\sum_{j=0}^{n} G_{jj}|z_j|^2}$$

The fundamental vector field of the principal action $S^1 \curvearrowright P$ is simply ∂_s, and the η-horizontal lift $\tilde{Z}$ of Z (which satisfies $\eta(\tilde{Z}) = 0$) is of course simply $Z - \eta_N(Z)\partial_s$. Now if we set $Z_1 = \tilde{Z} + f_1\partial_s$, the fact that $\eta_N(Z) = -r^2$ implies that $Z_1 = Z$. We define the one-forms θ_i^P, $i = 0, 1, 2, 3$ on P via

$$\theta_0^P = -\frac{1}{2}\mathrm{d}f = \frac{1}{2}\omega_1(Z, -) \qquad \theta_1^P = \eta + \frac{1}{2}g(Z, -)$$
$$\theta_2^P = \frac{1}{2}\omega_3(Z, -) \qquad \theta_3^P = -\frac{1}{2}\omega_2(Z, -)$$

which are given by

$$\theta_0^P = -2\sum_{i=0}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i + \bar{z}_i\mathrm{d}z_i) \tag{7}$$

$$\theta_1^P = \mathrm{d}s + \frac{i}{2}\sum_{i=0}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i - \bar{z}_i\mathrm{d}z_i + w_i\mathrm{d}\bar{w}_i - \bar{w}_i\mathrm{d}w_i) \tag{8}$$

$$\theta_2^P = i\sum_{i=0}^{n} z_i\mathrm{d}w_i - \bar{z}_i\mathrm{d}\bar{w}_i \tag{9}$$

$$\theta_3^P = -\sum_{i=0}^{n} z_i\mathrm{d}w_i + \bar{z}_i\mathrm{d}\bar{w}_i \tag{10}$$

Together with

$$g_P := \frac{2}{f_1}\eta^2 + p^*g_N$$

they yield the tensor field

$$\tilde{g}_P := g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2$$

which will yield the quaternionic Kähler metric on a hypersurface $M' \subset P$ transversal to Z, via the expression

$$g_{QK} = \frac{1}{2|f|}\tilde{g}_P\big|_{M'} = \frac{1}{|f|f_1}\eta^2 + \frac{p^*g_N}{2|f|} - \frac{1}{f|f|}\sum_{a=0}^{3}(\theta_a^P)^2 \tag{11}$$

which, in our case, simplifies to

$$g_{QK} = -\frac{1}{r^4}\Big(\eta^2 + \sum_{a=0}^{3}(\theta_a^P)^2\Big)^2 + \frac{p^*g_N}{2r^2} \tag{12}$$

In fact, this will not quite reproduce the Ferrara-Sabharwal metric g_{FS}, which is related via $g_{FS} = -2g_{QK}$. In particular, g_{QK} will be negative-definite.

1.4 Deriving the QK Metric Through the HK/QK Correspondence; Undeformed Case

Following ACDM, we pick $M' = \{\arg z_0 = 0\} \subset P$. We set $\varphi = \arg z_0$; the coordinate differential $\mathrm{d}\varphi$ can be written as

$$\mathrm{d}\varphi = \frac{1}{2i}\left(\frac{\mathrm{d}z_0}{z_0} - \frac{\mathrm{d}\bar{z}_0}{z_0}\right) = \frac{1}{2i|z_0|^2}(\bar{z}_0\mathrm{d}z_0 - z_0\mathrm{d}\bar{z}_0) = \frac{i}{2|z_0|^2}(z_0\mathrm{d}\bar{z}_0 - \bar{z}_0\mathrm{d}z_0)$$

and therefore we have

$$g(Z, -)\big|_{M'} = 2i\sum_{i=1}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i - \bar{z}_i\mathrm{d}z_i)$$

since the first summand vanishes. Similarly

$$\eta_N\big|_{M'} = -\frac{i}{2}\sum_{i=1}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i - \bar{z}_i\mathrm{d}z_i) + \frac{i}{2}\sum_{i=0}^{n} G_{ii}(w_i\mathrm{d}\bar{w}_i - \bar{w}_i\mathrm{d}w_i)$$

Following [1, 4], we define $\rho = r^2$, $w_j =: \frac{1}{2}(\tilde{\zeta}_j + iG_{jj}\zeta^j)$ and $X_{i>0} = z_i/z_0$. Then $\{X_i, \rho, \tilde{\zeta}_i, \zeta^i, s\} \in \mathbb{C}^n \times \mathbb{R}_{>0} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}$ are local coordinates around any point in M' (recall that $\vec{X}$ lies in the unit ball in $\mathbb{C}^n$). We now proceed to write the prospective metric on M' in terms of these coordinates. the replacements $w_i \mapsto (\tilde{\zeta}_i, \zeta^i)$ are rather simple:

$$\eta_{\mathrm{can}} = \frac{1}{4}\sum_{i=0}^{n} \tilde{\zeta}_i\mathrm{d}\zeta^i - \zeta^i\mathrm{d}\tilde{\zeta}_i \tag{13}$$

$$\sum_{i=0}^{n} G_{ii}\mathrm{d}w_i\mathrm{d}\bar{w}_i = \frac{1}{2}\sum_{i=0}^{n}\left((\mathrm{d}\tilde{\zeta}_i)^2 + (\mathrm{d}\zeta^i)^2\right) \tag{14}$$

Now we want to replace the z_i-expressions by equations featuring ρ, X_i. On M', the following identity holds:

$$\frac{1}{|z_0|^2}\sum_{i=1}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i - \bar{z}_i\mathrm{d}z_i) = \sum_{i=1}^{n} G_{ii}(X_i\mathrm{d}\bar{X}_i - \bar{X}_i\mathrm{d}X_i)$$

This implies

$$\tilde{\eta}\big|_{M'} = \frac{i\sum_{i=1}^{n} G_{ii}(z_i\mathrm{d}\bar{z}_i - \bar{z}_i\mathrm{d}z_i)}{\sum_{j=0}^{n} G_{jj}|z_j|^2} = \frac{i\sum_{i=1}^{n}(X_i\mathrm{d}\bar{X}_i - \bar{X}_i\mathrm{d}X_i)}{1-\sum_{i=1}^{n}|X_i|^2} \tag{15}$$

Furthermore, since $\rho = 2|z_0|^2(1-\sum_i |X_i|^2)$ and $\varphi = 0$ on M', we set $|X|^2 = \sum_i |X_i|^2$ and find

$$\sqrt{2}z_0|_{M'} = \sqrt{\frac{\rho}{1-|X|^2}}$$

and consequently

$$\frac{\mathrm{d}z_0}{z_0}\bigg|_{M'} = \frac{1}{2}\left(\frac{\mathrm{d}\rho}{\rho} + \frac{\sum_i X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right) \tag{16}$$

Thus, we may write

$$\begin{aligned}
\frac{1}{|z_0|^2}\sum_{i=0}^{n} G_{ii}\mathrm{d}z_i\mathrm{d}\bar{z}_i\Big|_{M'} &= \frac{\mathrm{d}z_0}{z_0}\frac{\mathrm{d}\bar{z}_0}{\bar{z}_0} - \sum_{i=1}^{n}\frac{\mathrm{d}z_i}{z_0}\frac{\mathrm{d}\bar{z}_i}{\bar{z}_0} \\
=&\frac{1}{4}\left(1-|X|^2\right)\left(\frac{\mathrm{d}\rho}{\rho} + \frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right)^2 \\
&- \sum \mathrm{d}X_i\mathrm{d}\bar{X}_i - \frac{1}{2}\sum(X_k\mathrm{d}\bar{X}_k + \bar{X}_k\mathrm{d}X_k)\left(\frac{\mathrm{d}\rho}{\rho} + \frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right) \\
=&\frac{1}{4}\left(1-|X|^2\right)\left(\frac{\mathrm{d}\rho^2}{\rho^2} + \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right]^2\right) \\
&- \sum \mathrm{d}X_i\mathrm{d}\bar{X}_i - \frac{1}{2}\frac{\left(\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i\right)^2}{1-|X|^2} \\
=&\frac{1}{4}\left(1-|X|^2\right)\left(\frac{\mathrm{d}\rho^2}{\rho^2} - \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right]^2\right) - \sum \mathrm{d}X_i\mathrm{d}\bar{X}_i
\end{aligned}$$

where we used $\mathrm{d}z_i/z_0 = \mathrm{d}X_i + z_i/z_0^2 \mathrm{d}z_0 = \mathrm{d}X_i + X_i \mathrm{d}z_0/z_0$, and canceled some terms in each further step. We conclude (using $\rho = 2|z_0|^2(1-|X|^2)$) that

$$\frac{1}{2\rho}\sum_{i=0}^{n} G_{i\bar{i}}\mathrm{d}z_i\mathrm{d}\bar{z}_i\Big|_{M'} = \frac{1}{16}\left(\frac{\mathrm{d}\rho^2}{\rho^2} - \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right]^2\right) - \frac{1}{4}\frac{\sum \mathrm{d}X_i\mathrm{d}\bar{X}_i}{1-|X|^2} \tag{17}$$

and correspondingly

$$\begin{aligned} \frac{1}{2\rho}p^*g_N\big|_{M'} = \frac{1}{8}\left(\frac{\mathrm{d}\rho^2}{\rho^2} - \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right]^2\right) \\ - \frac{1}{2}\frac{\sum \mathrm{d}X_i\mathrm{d}\bar{X}_i}{1-|X|^2} + \frac{1}{2\rho}\sum_{i=0}^{n}\left((\mathrm{d}\tilde{\zeta}_i)^2 + (\mathrm{d}\zeta^i)^2\right) \end{aligned} \tag{18}$$

This is the first piece of the quaternionic Kähler metric g_{QK} (cf. equation (12)). The next piece is

$$-\frac{\eta^2}{\rho^2} = -\frac{1}{\rho^2}\left(\mathrm{d}s - \frac{\rho\tilde{\eta}}{4} + \eta_{\mathrm{can}}\right)^2 = -\frac{1}{\rho^2}(\mathrm{d}s+\eta_{\mathrm{can}})^2 - \frac{\tilde{\eta}^2}{16} + \frac{1}{2\rho}\tilde{\eta}(\mathrm{d}s+\eta_{\mathrm{can}})$$

where we separated out the universal part. We compute its terms one-by-one (implicitly restricting to M' throughout, from here on):

$$\begin{aligned} -\frac{1}{\rho^2}(\mathrm{d}s+\eta_{\mathrm{can}})^2 &= -\frac{1}{16\rho^2}\left(4\mathrm{d}s + \sum_{i=0}^{n}\tilde{\zeta}_i\mathrm{d}\zeta^i - \zeta^i\mathrm{d}\tilde{\zeta}_i\right)^2 \\ -\frac{\tilde{\eta}^2}{16} &= -\frac{1}{16}\left[\frac{\sum_{i=1}^{n}(X_i\mathrm{d}\bar{X}_i - \bar{X}_i\mathrm{d}X_i)}{1-|X|^2}\right]^2 \end{aligned}$$

We need not bother with the final (cross-)term, because we will soon see it cancels out against second term in equation (12), i.e. against $-\frac{1}{\rho^2}\sum_{i=0}^{3}(\theta_i^P)^2$. These squares can be rather easily computed: Firstly, we have $\theta_0^P = -r\mathrm{d}r = -\frac{1}{2}\mathrm{d}\rho$, so $-(\theta_0^P)^2/\rho^2 = -\mathrm{d}\rho^2/4\rho^2$. Secondly, we find

$$\begin{aligned} \theta_1^P &= \mathrm{d}s + \eta_{\mathrm{can}} + \frac{\rho\tilde{\eta}}{4} \\ -\frac{1}{\rho^2}(\theta_1^P)^2 &= -\frac{1}{\rho^2}(\mathrm{d}s+\eta_{\mathrm{can}})^2 - \frac{\tilde{\eta}^2}{16} - \frac{1}{2\rho}\tilde{\eta}(\mathrm{d}s+\eta_{\mathrm{can}}) \end{aligned}$$

which yields the promised cancellation. Finally, we have

$$(\theta_2^P)^2 + (\theta_3^P)^2 = 4\sum_{i,j=0}^{n} z_i\bar{z}_j\mathrm{d}w_i\mathrm{d}\bar{w}_j$$

hence

$$\begin{aligned} -\frac{1}{\rho^2}\left((\theta_2^P)^2 + (\theta_3^P)^2\right) &= -\frac{2}{\rho}\frac{\sum_{i,j=0}^{n} z_i\bar{z}_j\mathrm{d}w_i\mathrm{d}\bar{w}_j}{|z_0|^2\left(1-|X|^2\right)} \\ &= -\frac{2}{\rho}\frac{\mathrm{d}w_0\mathrm{d}\bar{w}_0 + \sum_{i=1}^{n} X_i\mathrm{d}w_i\mathrm{d}\bar{w}_0 + \bar{X}_i\mathrm{d}w_0\mathrm{d}\bar{w}_i + \sum_{i,j=1}^{n} X_i\bar{X}_j\mathrm{d}w_i\mathrm{d}\bar{w}_j}{1-|X|^2} \\ &= -\frac{2}{\rho}\frac{1}{1-|X|^2}\left|\mathrm{d}w_0 + \sum_{i=1}^{n} X_i\mathrm{d}w_i\right|^2 \end{aligned}$$

Now, all that is left to do is add up all the pieces to obtain the (undeformed) Ferrara-Sabharwal metric on the corresponding quaternionic Kähler manifold. We will use the notation of [4], where the metric is explicitly written out in Corollary 15. We have

$$\begin{aligned}
-\frac{\eta^2}{\rho^2} - \frac{1}{\rho^2}\sum_{i=0}^{3}(\theta_i^P)^2 &= -\frac{2}{\rho^2}(\mathrm{d}s + \eta_{\mathrm{can}})^2 - \frac{\tilde{\eta}^2}{8} - \frac{\mathrm{d}\rho^2}{4\rho^2} - \frac{2}{\rho}\frac{\left|\mathrm{d}w_0 + \sum_{i=1}^n X_i \mathrm{d}w_i\right|^2}{1-|X|^2} \\
&= -\frac{1}{8\rho^2}\left(\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\sum_{i=0}^{n} G_{ii}\bar{w}_i \mathrm{d}w_i\right)^2 - \frac{\mathrm{d}\rho^2}{4\rho^2} - \frac{2}{\rho}\frac{\left|\mathrm{d}w_0 + \sum_{i=1}^n X_i \mathrm{d}w_i\right|^2}{1-|X|^2} \\
&\quad - \frac{1}{8}\left[\frac{\sum_{i=1}^n (X_i \mathrm{d}\bar{X}_i - \bar{X}_i \mathrm{d}X_i)}{1-|X|^2}\right]^2
\end{aligned}$$

where we set $\tilde{\phi} = -4s$ and pulled out the sign of the first squared term. To this, we must add

$$\begin{aligned}
\frac{1}{2\rho}p^* g_N\big|_{M'} = \frac{1}{8}&\left(\frac{\mathrm{d}\rho^2}{\rho^2} - \left[\frac{\sum X_i \mathrm{d}\bar{X}_i + \bar{X}_i \mathrm{d}X_i}{1-|X|^2}\right]^2\right) \\
&- \frac{1}{2}\frac{\sum \mathrm{d}X_i \mathrm{d}\bar{X}_i}{1-|X|^2} + \frac{1}{\rho}\sum_{i=0}^{n} G_{ii}\mathrm{d}w_i \mathrm{d}\bar{w}_i
\end{aligned}$$

which yields

$$\begin{aligned}
g' = &-\frac{\mathrm{d}\rho^2}{\rho^2} + \frac{1}{\rho}\sum_{i=0}^{n} G_{ii}\mathrm{d}w_i \mathrm{d}\bar{w}_i - \frac{2}{\rho}\frac{\left|\mathrm{d}w_0 + \sum_{i=1}^n X_i \mathrm{d}w_i\right|^2}{1-|X|^2} \\
&- \frac{1}{8\rho^2}\left(\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\sum_{i=0}^{n} G_{ii}\bar{w}_i \mathrm{d}w_i\right)^2 \\
&- \frac{1}{2}\frac{1}{1-|X|^2}\left(\sum \mathrm{d}X_i \mathrm{d}\bar{X}_i + \frac{\sum |X_i \mathrm{d}\bar{X}_i|^2}{1-|X|^2}\right)
\end{aligned}$$

This is precisely $-\frac{1}{2}g_{FS}$, where g_{FS} is the undeformed Ferrara-Sabharwal metric (cf. [4], Corollary 15).

1.5 The One-Loop Deformation

From a physical point of view, g_{FS} is a *classical* object which will receive corrections from quantum effects. Because of certain supersymmetric non-renormalization theorems, the only perturbative corrections arise at one-loop order, i.e. the metric is perturbatively one-loop exact. The one-loop corrections lead to a one-parameter family of complete quaternionic-Kähler metrics, parametrized by a real, positive constant c. For $c = 0$, we recover the standard Ferrara-Sabharwal metric, while any two metrics corresponding to $c, c' > 0$ are isometric (cf. [4], proposition 10).

One of the reasons why the HK/QK correspondence is so useful is that this so-called *one-loop deformation* of the quaternionic Kähler metric corresponds to a more-or-less trivial modification on the hyper-Kähler side. Indeed, recall that the function f is only defined up to a constant $c \in \mathbb{R}$ (by $\omega(Z, -) = -\mathrm{d}f$), and hence we may take

$$f = r^2 - c = 2\sum_{i=0}^{n} G_{ii}|z_i|^2 - c =: \rho$$

and correspondingly $f_1 = -2\sum_{i=0}^n G_{ii}|z_i|^2 - c$. This will change the expression we obtain for the QK metric $-2g'$; the corresponding metrics are called one-loop deformed. We will now compute them, assuming $c > 0$ throughout. We have

$$\rho + c = 2|z_0|^2\Big(1 - |X|^2\Big) \implies \sqrt{2}z_0\big|_{M'} = \sqrt{\frac{\rho + c}{1 - |X|^2}}$$

and correspondingly (always restricting to M')

$$\frac{\mathrm{d}z_0}{z_0} = \frac{1}{2}\left(\frac{\mathrm{d}\rho}{\rho + c} + \frac{\sum_i X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1 - |X|^2}\right)$$

This means that

$$\frac{1}{|z_0|^2}\sum_i G_{ii}\mathrm{d}z_i\mathrm{d}\bar{z}_i = \frac{1}{4}(1 - |X|^2)\left(\frac{\mathrm{d}\rho^2}{(\rho + c)^2} - \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1 - |X|^2}\right]^2\right) - \sum \mathrm{d}X_i\mathrm{d}\bar{X}_i$$

and correspondingly

$$\begin{aligned}
\frac{p^*g_N}{2\rho} &= \frac{p^*g_N}{2(2|z_0|^2(1 - |X|^2) - c)} = \frac{p^*g_N}{4|z_0|^2(1 - |X|^2)}\frac{\rho + c}{\rho} \\
&= \frac{\rho + c}{\rho}\left[\frac{1}{8}\left(\frac{\mathrm{d}\rho^2}{(\rho + c)^2} - \left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1 - |X|^2}\right]^2\right) - \frac{1}{2}\frac{\sum \mathrm{d}X_i\mathrm{d}\bar{X}_i}{1 - |X|^2}\right] \\
&\quad + \frac{1}{\rho}\sum_{i=0}^n G_{ii}\mathrm{d}w_i\mathrm{d}\bar{w}_i
\end{aligned} \tag{19}$$

The expressions for $\tilde{\eta}$ and η_{can} remain unchanged, but we now have

$$\eta = \mathrm{d}s + \eta_{\mathrm{can}} - \frac{\rho + c}{4}\tilde{\eta}$$

and thus there are extra terms:

$$\frac{\eta^2}{f_1|f|} = -\frac{\eta^2}{\rho(\rho + 2c)} = \frac{1}{\rho(\rho + 2c)}\Big(\mathrm{d}s + \eta_{\mathrm{can}} - \frac{c}{4}\tilde{\eta}\Big)^2 - \frac{\rho}{\rho + 2c}\frac{\tilde{\eta}^2}{16} + \frac{1}{2(\rho + 2c)}\tilde{\eta}\Big(\mathrm{d}s + \eta_{\mathrm{can}} - \frac{c}{4}\tilde{\eta}\Big)$$

The θ_a^P-terms are also slightly modified:

$$\begin{aligned}
-\frac{1}{\rho^2}(\theta_0^P)^2 &= -\frac{\mathrm{d}\rho^2}{4\rho^2} \\
-\frac{1}{\rho^2}(\theta_1^P)^2 &= -\frac{1}{\rho^2}\Big(\mathrm{d}s + \eta_{\mathrm{can}} + \frac{\rho + c}{4}\tilde{\eta}\Big)^2 \\
&= -\frac{1}{\rho^2}\Big(\mathrm{d}s + \eta_{\mathrm{can}} + \frac{c\tilde{\eta}}{4}\Big)^2 - \frac{\tilde{\eta}^2}{16} - \frac{1}{2\rho}\tilde{\eta}\Big(\mathrm{d}s + \eta_{\mathrm{can}} + \frac{c}{4}\tilde{\eta}\Big) \\
-\frac{1}{\rho^2}\big((\theta_2^P)^2 + (\theta_3^P)^2\big) &= -\frac{\rho + c}{\rho^2}\frac{2}{1 - |X|^2}\Big|\mathrm{d}w_0 + \sum_{i=1}^n X_i\mathrm{d}w_i\Big|^2
\end{aligned} \tag{20}$$

Hence, we have:

$$\begin{aligned}
&-\frac{\eta^2}{\rho^2} - \frac{1}{\rho^2}\sum_{a=0}^3(\theta_a^P)^2 = \\
&-\frac{1}{\rho^2}\left[\frac{\rho}{\rho + 2c}\Big(\mathrm{d}s + \eta_{\mathrm{can}} - \frac{c\tilde{\eta}}{4}\Big)^2 + \Big(\mathrm{d}s + \eta_{\mathrm{can}} + \frac{c\tilde{\eta}}{4}\Big)^2\right] + \frac{\tilde{\eta}}{2}(\mathrm{d}s + \eta_{\mathrm{can}})\Big(\frac{1}{\rho + 2c} - \frac{1}{\rho}\Big) \\
&-\frac{\tilde{\eta}^2}{8}\Big(1 + \frac{c}{\rho}\Big) - \frac{\rho + c}{\rho^2}\frac{2}{1 - |X|^2}\Big|\mathrm{d}w_0 + \sum_{i=1}^n X_i\mathrm{d}w_i\Big|^2 - \frac{\mathrm{d}\rho^2}{4\rho^2}
\end{aligned}$$

Summing it all up, we see that the $\mathrm{d}w\mathrm{d}\bar{w}$-part is already in a nice form, and so is the $\mathrm{d}X_i\mathrm{d}\bar{X}_i$-term, as well as the $|\mathrm{d}w_0 + X_i\mathrm{d}w_i|$-term. For the terms proportional to $\mathrm{d}\rho^2$, we have

$$-\frac{\mathrm{d}\rho^2}{8\rho^2}\Big(2 - \frac{\rho+c}{\rho}\frac{1}{(1+\frac{c}{\rho})^2}\Big) = -\frac{\mathrm{d}\rho^2}{8\rho^2}\frac{2(\rho+c)-\rho}{\rho+c} = -\frac{\rho+2c}{\rho+c}\frac{\mathrm{d}\rho^2}{8\rho^2} \tag{21}$$

There are only a few terms left to rewrite, namely

$$\begin{aligned}&-\frac{1}{\rho^2}\left[\frac{\rho}{\rho+2c}\Big(\mathrm{d}s+\eta_{\mathrm{can}}-\frac{c\tilde{\eta}}{4}\Big)^2 + \Big(\mathrm{d}s+\eta_{\mathrm{can}}+\frac{c\tilde{\eta}}{4}\Big)^2\right] + \frac{\tilde{\eta}}{2}(\mathrm{d}s+\eta_{\mathrm{can}})\Big(\frac{1}{\rho+2c}-\frac{1}{\rho}\Big)\\ &-\frac{\tilde{\eta}^2}{8}\Big(1+\frac{c}{\rho}\Big) - \frac{1}{8}\frac{\rho+c}{\rho}\left[\frac{\sum X_i\mathrm{d}\bar{X}_i + \bar{X}_i\mathrm{d}X_i}{1-|X|^2}\right]^2\end{aligned}$$

The second line amounts exactly to

$$-\frac{1}{8}\frac{\rho+c}{\rho}\frac{1}{\big(1-|X|^2\big)^2}\Big|\sum_i \bar{X}_i\mathrm{d}X_i\Big| \tag{22}$$

The first can be simplified by noting that the term in square brackets is

$$\begin{aligned}&-\frac{1}{\rho^2}\left[\left((\mathrm{d}s+\eta_{\mathrm{can}})^2 + \Big[\frac{c\tilde{\eta}}{4}\Big]^2\right)\Big(1+\frac{\rho}{\rho+2c}\Big) + 2\frac{c\tilde{\eta}}{4}(\mathrm{d}s+\eta_{\mathrm{can}})\Big(1-\frac{\rho}{\rho+2c}\Big)\right]\\ =&-\frac{2}{\rho^2}\left[\frac{\rho+c}{\rho+2c}\left((\mathrm{d}s+\eta_{\mathrm{can}})^2 + \Big[\frac{c\tilde{\eta}}{4}\Big]^2\right) + 2(\mathrm{d}s+\eta_{\mathrm{can}})\frac{c\tilde{\eta}}{4}\frac{c}{\rho+2c}\right]\end{aligned}$$

while the term outside the brackets is

$$-\frac{1}{\rho^2}\left[2(\mathrm{d}s+\eta_{\mathrm{can}})\frac{c\tilde{\eta}}{4}\Big(\frac{\rho^2}{\rho+2c}-\rho\Big)\right] = -\frac{2}{\rho^2}\left[2(\mathrm{d}s+\eta_{\mathrm{can}})\frac{\rho}{\rho+2c}\right]$$

so that adding them up leads to the neat expression

$$\begin{aligned}&-\frac{2}{\rho^2}\frac{\rho+c}{\rho+2c}\left[\mathrm{d}s+\eta_{\mathrm{can}}+\frac{c\tilde{\eta}}{4}\right]^2\\ =&-\frac{1}{8\rho^2}\frac{\rho+c}{\rho+2c}\left[\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\Big(\sum_i G_{ii}\bar{w}_i\mathrm{d}w_i\Big) + \frac{2c}{1-|X|^2}\mathrm{Im}\Big(\sum_i \bar{X}_i\mathrm{d}X_i\Big)\right]^2\end{aligned} \tag{23}$$

We can now finally sum this equation with (the relevant terms of) equations (21), (22), (19) and (20) and obtain the expression in full after multiplying by -2. This is the one-loop deformed Ferarra-Sabharwal metric:

$$\begin{aligned}g^c_{\mathrm{FS}} =& \frac{\rho+c}{\rho}\frac{1}{1-|X|^2}\left[\sum_{i=1}^n \mathrm{d}X_i\mathrm{d}\bar{X}_i + \frac{1}{1-|X|^2}\Big|\sum_{i=1}^n \bar{X}_i\mathrm{d}X_i\Big|^2\right]\\ &+\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}\mathrm{d}\rho^2 - \frac{2}{\rho}\sum_{i=0}^n G_{ii}\mathrm{d}w_i\mathrm{d}\bar{w}_i\\ &+\frac{\rho+c}{\rho^2}\frac{4}{1-|X|^2}\Big|\mathrm{d}w_0 + \sum_{i=1}^n X_i\mathrm{d}w_i\Big|^2\\ &+\frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\left[\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\Big(\sum_i G_{ii}\bar{w}_i\mathrm{d}w_i\Big) + \frac{2c}{1-|X|^2}\mathrm{Im}\Big(\sum_i \bar{X}_i\mathrm{d}X_i\Big)\right]^2\end{aligned}$$

2 The Supergravity c-Map

The above method gives us one explicit method of deriving the metric, but there is another method to arrive at this expression. This method will be helpful to us in determining the isometry group of the c-map image; it is a more direct description, which immediately gives us the quaternionic Kähler metric from the PSK metric on $\mathbb{C}H^n$; it is known as the *supergravity (SUGRA) c-map.*

2.1 General Setup

If (M, g_M) is a projective special Kähler (PSK) manifold of complex dimension n, the supergravity c-map associates a quaternionic Kähler (QK) manifold (N, g_N) of dimension $4n+4$ to it. N is diffeomorphic to $M \times \mathbb{R}^{2n+3} \times \mathbb{R}_+ \cong M \times \mathbb{R}^{2n+4}$ and its metric is given by the following expression:

$$
\begin{aligned}
g_N &= g_M + g_G \\
g_G &= \frac{1}{4\phi^2}\left[\mathrm{d}\phi^2 + \left(\mathrm{d}\tilde{\phi} + \sum_I \zeta^I \mathrm{d}\tilde{\zeta}_I - \tilde{\zeta}_I \mathrm{d}\zeta^I\right)^2\right] \\
&+ \frac{1}{2\phi}\left[\sum_{IJ} \mathcal{J}_{IJ}(p)\mathrm{d}\zeta^I \mathrm{d}\zeta^I + \sum_{IJ} \mathcal{J}^{IJ}\left(\mathrm{d}\tilde{\zeta}_I + \sum_K \mathcal{R}_{IK}(p)\mathrm{d}\zeta^K\right)\left(\mathrm{d}\tilde{\zeta}_J + \sum_L \mathcal{R}_{JL}(p)\mathrm{d}\zeta^L\right)\right]
\end{aligned}
$$

where $(\phi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I)$, $I = 0, \dots, n$ are coordinates on $\mathbb{R}_+ \times \mathbb{R}^{1+(n+1)+(n+1)}$ and $\phi > 0$. The matrix $\mathcal{J}_{IJ}$ is symmetric and positive-definite, with inverse $\mathcal{J}^{IJ}$.

2.2 Basic Expressions for the Fiber G

A more compact notation can be adopted. Define the column vector $p_a = (\tilde{\zeta}_I, \zeta^I)$, where $a = 1, \dots, 2n+2$. Furthermore consider the positive-definite matrix

$$
\tilde{H}^{ab} := \begin{pmatrix} \mathcal{J}^{-1} & \mathcal{J}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{J}^{-1} & \mathcal{J} + \mathcal{R}\mathcal{J}^{-1}\mathcal{R} \end{pmatrix}
$$

with inverse matrix

$$
(\tilde{H}^{-1})_{ab} := \begin{pmatrix} \mathcal{J} + \mathcal{R}\mathcal{J}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{J}^{-1} \\ -\mathcal{J}^{-1}\mathcal{R} & \mathcal{J}^{-1} \end{pmatrix}
$$

Then we have

$$
g_G = \frac{1}{4\phi^2}\left[\mathrm{d}\phi^2 + \left(\mathrm{d}\tilde{\phi} + \sum_I \zeta^I \mathrm{d}\tilde{\zeta}_I - \tilde{\zeta}_I \mathrm{d}\zeta^I\right)^2\right] + \sum_{a,b} \frac{1}{2\phi}\mathrm{d}p_a \tilde{H}^{ab}\mathrm{d}p_b
$$

Remark 1. The matrix $\tilde{H}^{ab}$ corresponds to H^{ab} in the paper *Completeness in Supergravity Constructions* by Cortés, Han and Mohaupt. They think of H^{ab} as the inverse of H_{ab}, but we will think of $\tilde{H}^{ab}$ as the "original" one, so translating to their notation comes with an inversion of H (though the position of the indices does match).

This can be further simplified by noting that

$$
\sum_I \zeta^I \mathrm{d}\tilde{\zeta}_I - \tilde{\zeta}_I \mathrm{d}\zeta^I = (\tilde{\zeta}_I, \zeta^I)\begin{pmatrix} 0_{n+1} & -\mathbb{1}_{n+1} \\ \mathbb{1}_{n+1} & 0_{n+1} \end{pmatrix}\begin{pmatrix} \mathrm{d}\tilde{\zeta}_I \\ \mathrm{d}\zeta^I \end{pmatrix} =: p_a K^{ab}\mathrm{d}p_b
$$

Thus, the metric takes the form

$$g_G = \frac{1}{4\phi^2}\left[\mathrm{d}\phi^2 + \mathrm{d}\tilde{\phi}^2 + \sum_{a,b} p_a K^{ab}(\mathrm{d}p_b \mathrm{d}\tilde{\phi} + \mathrm{d}\tilde{\phi}\mathrm{d}p_b)\right] + \sum_{a,b}\left[\left(\frac{1}{2\phi}p_a K^{ab}\mathrm{d}p_b\right)^2 + \frac{1}{2\phi}\mathrm{d}p_a \tilde{H}^{ab}\mathrm{d}p_b\right]$$

Alternatively, we may take $P_a = (-\tilde{\zeta}_I, \zeta^I)$ and find

$$g_G = \frac{1}{4\phi^2}\left[\mathrm{d}\phi^2 + \mathrm{d}\tilde{\phi}^2 + \sum_{a,b} P_a K^{ab}(\mathrm{d}P_b \mathrm{d}\tilde{\phi} + \mathrm{d}\tilde{\phi}\mathrm{d}P_b)\right] + \sum_{a,b}\left[\frac{1}{4\phi^2}\mathrm{d}P_a \mathrm{d}P_b + \frac{1}{2\phi}\mathrm{d}P_a \bar{H}^{ab}\mathrm{d}P_b\right]$$

Note that some entries of $\bar{H}$ have different sign.

We can write the metric in matrix form as follows:

$$g_G = \begin{pmatrix} \frac{1}{4\phi^2} & 0 & 0 \\ 0 & \frac{1}{4\phi^2} & \frac{1}{4\phi^2}\sum_a p_a K^{ab} \\ 0 & \frac{1}{4\phi^2}\sum_a p_a K^{ab} & \frac{1}{4\phi^2}p_a K^{ab} p_c K^{cf} + \frac{1}{2\phi}\tilde{H}^{bf} \end{pmatrix}$$

The inverse is given by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

which is valid for any block-matrix. The inverse metric is easy to determine:

$$\begin{aligned} g_G^{-1} &= \begin{pmatrix} 4\phi^2 & 0 & 0 \\ 0 & 4\phi^2 + 2\phi p_a K^{ab}(\tilde{H}^{-1})_{bc} p_f K^{fc} & -2\phi p_c K^{cb}(\tilde{H}^{-1})_{ba} \\ 0 & -2\phi(\tilde{H}^{-1})_{ab} p_f K^{fb} & 2\phi(\tilde{H}^{-1})_{ab} \end{pmatrix} \\ &= \begin{pmatrix} 4\phi^2 & 0 & 0 \\ 0 & 4\phi^2 + 2\phi p_a \tilde{H}^{ab} p_b & -2\phi p_c K^{cb}(\tilde{H}^{-1})_{ba} \\ 0 & -2\phi(\tilde{H}^{-1})_{ab} p_f K^{fb} & 2\phi(\tilde{H}^{-1})_{ab} \end{pmatrix} \end{aligned}$$

where we used the fact that $K^{ab}(\tilde{H}^{-1})_{bc}K^{fc} = \tilde{H}^{af}$.

3 Isometries of the Undeformed Ferrara-Sabharwal Metric

3.1 The Unit Ball as a Homogeneous Space

To understand the symmetries of the c-map image of $\mathbb{C}H^n$, we first study the base PSK in a bit more depth. $\mathbb{C}H^n$ is the unit ball in $\mathbb{C}^n$, equipped with a symmetric metric of constant negative curvature. Its standard realization is as a homogeneous space under $SU(n,1)$, which is most easily understood by viewing the unit ball as a subset of $\mathbb{C}\mathrm{P}^n$. Let $[z_0 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{C}\mathrm{P}^n$, and consider the open set $U = \{z_n \neq 0\} \subset \mathbb{C}\mathrm{P}^n$. Then the standard affine chart

$$\begin{aligned} U &\longrightarrow (\mathbb{C}^n, 1) \\ [z_1 : \cdots : z_n] &\longmapsto \left(\tfrac{z_1}{z_{n+1}}, \ldots, \tfrac{z_n}{z_{n+1}}, 1\right) \end{aligned}$$

shows us that we can realize the unit ball in $\mathbb{C}^n$ as

$$B = \left\{ [z_1 : \cdots : z_{n+1}] \in \mathbb{C}\mathrm{P}^n \;\middle|\; \sum_{j=1}^{n} |z_j|^2 - |z_{n+1}|^2 = \langle z, z \rangle < 0 \right\}$$

Consider $\mathbb{C}^{n+1}$, equipped with the indefinite Hermitian form

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j - z_{n+1} \bar{w}_{n+1}$$

Now $SU(n,1)$ is defined as the set of special linear transformations which preserve this form. Its action on $\mathbb{C}^{n+1}$ descends to $\mathbb{C}\mathrm{P}^n$, and preserves the unit ball B since B is defined by the sign of $\langle z, z \rangle$.

To prove transitivity of this action, we will show that any point can be mapped to the origin. Recall that $U(n)$, essentially by definition, acts simply transitively on the space of orthonormal bases of $\mathbb{C}^n$. This shows that $SU(n)$ acts transitively on the sphere of radius r, so we have reduced the claim to showing that we can map $(r, 0, \ldots, 0)$ to the origin, where $r \in \mathbb{R}$. There is a copy of $SU(1,1)$ sitting inside $SU(n,1)$, given by matrices of the following form:

$$\begin{pmatrix} a & 0 & b \\ 0 & \mathbb{1}_{n-1} & 0 \\ c & 0 & d \end{pmatrix}$$

Acting on projective space, it sends

$$[r : 0 : \cdots : 0 : 1] \mapsto [ar + b : 0 : \cdots : 0 : cr + d] = \left[\frac{ar+b}{cr+d} : 0 : \cdots : 0 : 1 \right]$$

and correspondingly it sends

$$(r, 0, \ldots, 0) \longmapsto \left(\tfrac{ar+b}{cr+d}, 0 \ldots, 0 \right)$$

The condition that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies inside $SU(1,1)$ means that $ad - bc = 1$ while also $|a|^2 - |c|^2 = 1 = |b|^2 - |d|^2$ and $a\bar{b} - c\bar{d} = 0$. If $b = 0$, we easily get that $c = 0$ and $d = a^*$, while if $b \neq 0$ we find

$$a = \frac{c\bar{d}}{\bar{b}} \implies \frac{c\bar{d}}{\bar{b}} - bc = 1 \implies \frac{c}{\bar{b}}(|d|^2 - |b|^2) = 1 \implies c = \bar{b} \implies a = \bar{d}$$

and we see that the matrix takes the form $\left(\begin{smallmatrix} a & b \\ b^* & a^* \end{smallmatrix}\right)$. Now, we want to satisfy $\frac{ar+b}{b^* r + a^*} = 0$ as well as $|a|^2 - |b|^2 = 1$. If we force $ar + b = 0$, we only have the condition $|b|^2 = (r^{-2} - 1)^{-1}$, which obviously admits a solution.

Now, we determine the isotropy subgroup. Let $\left(\begin{smallmatrix} A & \vec{b} \\ \vec{c}^T & d \end{smallmatrix}\right)$ be an $(n+1) \times (n+1)$ complex matrix. It fixes $[0 : \cdots : 0 : 1] \in \mathbb{C}\mathrm{P}^n$, or correspondingly $0 \in \mathbb{C}^n$, if and only if $\vec{b} = 0$. It then lies in $U(n,1)$ precisely if

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ \vec{c}^T & d \end{pmatrix}^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & 0 \\ \vec{c}^T & d \end{pmatrix} = \begin{pmatrix} A^\dagger A - \bar{\vec{c}} \vec{c}^T & -d\bar{\vec{c}} \\ -\bar{d}\vec{c}^T & -|d|^2 \end{pmatrix}$$

For it to lie in $SU(n,1)$, we must have $d \neq 0$ and it follows that $\vec{c} = 0$ as well as $A \in U(n)$ and finally $d \in U(1)$ such that $d \det A = 1$. This means that the stabilizer is $S(U(n) \times U(1)) \cong U(n)$ and we conclude

$$\mathbb{C}H^n \cong \frac{SU(n,1)}{U(n)}$$

3.2 The Unit Ball as a Lie Group

There is a general decomposition, called the Iwasawa decomposition, which casts a non-compact semisimple Lie group G uniquely (up to conjugation) as a product (as a manifold, but not a direct product of Lie groups!) $G = KAN$, where K is the maximal compact subgroup (unique up to conjugation), A is maximal Abelian (its dimension gives the rank) and N is nilpotent. The subgroup $AN \subset G$ is called the *Iwasawa subgroup* of G; it is always simply connected.

It is a fact, which we assume known, that $U(n)$ is the maximal compact subgroup of $SU(n,1)$, hence we can identify $\mathbb{C}H^n$ with the Iwasawa subgroup $\mathrm{Iwa}(SU(n,1))$, which therefore acts simply transitively on the unit ball. As such, $\mathbb{C}H^n$ admits a group structure. We will denote this group by $G(n)$ This group is of course diffeomorphic to $\mathbb{R}^{2n}$, and its group structure not very hard to describe explicitly, at least on the level of Lie algebras.

As described in CHM (page 199), $G(n) = \mathrm{Iwa}(SU(n,1))$ is a rank one extension of the Heisenberg group of dimension $2n-1$. On the level of Lie algebras, we therefore have

$$\mathfrak{g} = \mathfrak{D} + \mathfrak{heis}$$

where $+$ denotes a direct sum of vector spaces, but not as Lie algebras. $\mathfrak{D}$ is one-dimensional, spanned by D. The Lie algebra of the Heisenberg group is constructed as follows: Consider the standard symplectic vector space $(\mathbb{R}^{2n-2}, \omega_{\mathrm{std}})$. Then $\mathfrak{heis}$ is defined as follows: $\mathfrak{heis} = \mathfrak{Z} + \mathbb{R}^{2n-2}$, where $\mathfrak{Z}$ is one-dimensional and spanned by Z, and the Lie bracket is defined as follows: For $X, Y \in \mathbb{R}^{2n-2}$, $[X,Y] = \omega(X,Y)Z$, while $[Z,X] = 0$. Clearly, Z spans the (one-dimensional) center, and the Lie algebra structure only depends on the symplectic structure on $\mathbb{R}^{2n-2}$. Now that we have the Lie algebra structure of $\mathfrak{heis}$, we only need to know the action of ad_D. It is given by $[D,Z] = Z$ and $[D,X] = \frac{1}{2}X$ for $X \in \mathbb{R}^{2n-2} \subset \mathfrak{heis}$: The factor $\frac{1}{2}$ is what makes the Jacobi identity work out.

The group multiplication can be explicitly written out: Let $(\tilde{\zeta}, \zeta, \tilde{\phi}, \phi), (\tilde{\zeta}', \zeta', \tilde{\phi}', \phi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^{2n}$; then their product is

$$(\tilde{\zeta} + e^{\phi/2}\tilde{\zeta}', \zeta + e^{\phi/2}\zeta', \tilde{\phi} + e^{\phi}\tilde{\phi}' + e^{\phi/2}(\zeta^T\tilde{\zeta}' - \zeta'^T\tilde{\zeta}), \phi + \phi')$$

In CHM, $G(n)$ is described as $\mathbb{R}^{2n-1} \times \mathbb{R}_+$, an identification achieved by the diffeomorphism $(\tilde{\zeta}, \zeta, \tilde{\phi}, \phi) \mapsto (\tilde{\zeta}, \zeta, \tilde{\phi}, e^{\phi})$.

3.3 The Group Structure of the c-map Image

Recall from section 1.4 that the undeformed Ferrara-Sabharwal metric on the c-map image of $\mathbb{C}H^n$ is

$$\begin{aligned} g_{FS} =& \frac{1}{1-|X|^2}\left(\sum \mathrm{d}X^\mu \mathrm{d}\bar{X}^\mu + \frac{1}{1-|X|^2}\left|\sum \bar{X}^\mu \mathrm{d}X^\mu\right|^2\right) \\ &+ \frac{1}{4\phi^2}\mathrm{d}\phi^2 + \frac{1}{4\phi^2}\left(\mathrm{d}\tilde{\phi} - 4\,\mathrm{Im}\left[\bar{w}_0 \mathrm{d}w_0 - \sum \bar{w}_\mu \mathrm{d}w_\mu\right]\right)^2 \\ &- \frac{2}{\phi}\left(\mathrm{d}w_0 \mathrm{d}\bar{w}_0 - \sum \mathrm{d}w_\mu \mathrm{d}\bar{w}_\mu\right) + \frac{1}{\phi}\frac{4}{1-|X|^2}\left|\mathrm{d}w_0 + \sum X^\mu \mathrm{d}w_\mu\right|^2 \end{aligned}$$

where the coordinates $(X, \phi, \tilde{\phi}, w)$ take values in $\mathbb{C}^n \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{C}^{n+1}$ and $|X| < 1$.

It is known (see CDS, example 14) that this metric turns the c-map image N into the symmetric space $SU(n+1,2)/S(U(n+1)\times U(2)) \cong \mathrm{Iwa}(SU(n+1,2))$. In particular, it should be left-invariant with respect to a group structure described above. The base PSK manifold and the fiber each have a group structure or equivalently admit a simply transitive group action, but the group structure on N cannot be simply a direct product structure, since the metric on the fibers depends on the point in the base. Thus, we have to try to understand the group structure on N, which is amounts to finding a simply transitive group action, which furthermore is isometric.

The action of $G(n+2)$ on the fiber is of course simply transitive, and since the metric on the base does not depend on the fiber this group, one may expect that $G(n+2)$ is a normal subgroup of the group we are looking for: Indeed, if we can extend the action of $G(n)$ on the base to an isometric action on all of N, then we will have realized N as a semidirect product of $G(n)$ with $G(n+2)$, where the latter is the normal subgroup since the base metric does not depend on the fiber. This also amounts to presenting $\mathrm{Iwa}(SU(n+1,2))$ as $G(n)\ltimes G(n+2)$.

Thus, we are looking to define an action $G(n) \curvearrowright G(n+2)$ with respect to which the c-map metric (Ferrara-Sabharwal metric) is left-invariant. Since $G(n) = \mathrm{Iwa}(SU(n,1))$, we have a natural inclusion $G(n) \subset SU(n,1) \subset Sp(2n+2)$. This allows us to act on the w-coordinates by symplectic transformations, which is natural from a physical perspective because string-theoretic arguments dictate that there should be some kind of $Sp(2n+2)$-duality symmetry—we will check whether/that these transformations are isometries of the c-map metric.

To do this, we need two ingredients:

(i) Explicit expressions for the isometric action $G(n) \curvearrowright \mathbb{C}H^n$.

(ii) An explicit embedding $G(n) = \mathrm{Iwa}(SU(n,1)) \subset Sp(2n+2)$.

For simplicity, we will start by considering the case $n=1$. In this case, we know that

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \middle| |a|^2 - |b|^2 = 1 \right\}$$

Our discussion of the Lie group model of $\mathbb{C}H^n$ shows that, in the case $n=1$, the quotient $SU(1,1)/U(1)$ is given by considering a modulo $U(1)$, which means simply that we may realize the Iwasawa subgroup as the subgroup for which a is positive and real—in fact $a > 1$ because $|b|^2 = a^2 - 1$.

We start with the second step. The embedding $SU(1,1) \subset Sp(4)$ is given by first embedding it into $Sp(2,2)$: This map is induced by the identification $\mathbb{C}^2 = \mathbb{R}^4$. Composing this with the isomorphism $Sp(2,2) \cong Sp(4)$ induced by the permutation $(x_1,y_1,x_2,y_2) \mapsto (x_1,y_1,y_2,x_2)$, we find that the image of $G(1) = \mathrm{Iwa}(SU(1,1))$ is:

$$A = \begin{pmatrix} a & 0 & -\operatorname{Im} b & \operatorname{Re} b \\ 0 & a & \operatorname{Re} b & \operatorname{Im} b \\ -\operatorname{Im} b & \operatorname{Re} b & a & 0 \\ \operatorname{Re} b & \operatorname{Im} b & 0 & a \end{pmatrix} \qquad \mathbb{R} \ni a \geq 1 \qquad |b|^2 = a^2 - 1$$

For the first step, we have the unit disk in $\mathbb{C}$, acted upon by $SU(1,1)$. The most obvious action is by fractional linear transformations: $X \mapsto \frac{aX+b}{b^*X+a^*}$. It is not hard to check that the PSK metric (which is also induced by g_{FS}) on the disk is left-invariant with respect to this action of $SU(1,1)$. In the $n = 1$ case, the metric on the base simplifies to

$$\frac{\mathrm{d}X\mathrm{d}\bar{X}}{(1-|X|^2)^2}$$

As $X \mapsto \frac{aX+b}{b^*X+a^*}$, we have

$$\mathrm{d}X \longmapsto \frac{1}{(b^*X+a^*)^2}\big(a(b^*X+a^*)-b^*(aX+b)\big)\mathrm{d}X = \frac{\mathrm{d}X}{(b^*X+a^*)^2}$$

and so we obtain

$$\frac{\mathrm{d}X\mathrm{d}\bar{X}}{\left[|b^*X+a^*|^2\big(1-|\frac{aX+b}{b^*X+a^*}|^2\big)\right]^2} = \frac{\mathrm{d}X\mathrm{d}\bar{X}}{(|b^*X+a^*|^2-|aX+b|^2)^2} = \frac{\mathrm{d}X\mathrm{d}\bar{X}}{(1-|X|^2)^2}$$

where the final step uses $|a|^2 - |b|^2 = 1$.

This choice of action, however, is not at all unique. Other actions can be obtained by precomposing the standard action by an automorphism of $SU(1,1)$, but also by conjugating with a biholomorphism φ of the unit disk which is compatible with the PSK structure in the sense that $\varphi(X)$ still gives a *special* (i.e. preferred) coordinate system corresponding to a PSK structure on the disk. Later, we will make use of the example $\varphi(X) = iX$. The metric on the base remains invariant.

3.4 Homogeneity of the Metric ($n = 1$)

Now, we want to show that g_{FS} is invariant under the action of $G(1)$ which is the standard action on the base, but simultaneously acts in the above form (as a symplectic transformation) on the copy of $\mathbb{R}^4$ spanned by $(\tilde{\zeta}_0, \zeta^0, \tilde{\zeta}_1, \zeta^1)$ in the fiber. To do this, we first write the metric in terms of the ζ's, using $w_0 = \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$ and $w_\mu = \frac{1}{2}(\tilde{\zeta}_\mu - i\zeta^\mu)$.

$$\begin{aligned}
g_{FS} =& \frac{\mathrm{d}X\mathrm{d}\bar{X}}{(1-|X|^2)^2} + \frac{1}{4\phi^2}\mathrm{d}\phi^2 + \frac{1}{4\phi^2}\left(\mathrm{d}\tilde{\phi} + \zeta^0\mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0\mathrm{d}\zeta^0 + \zeta^1\mathrm{d}\tilde{\zeta}_1 - \tilde{\zeta}_1\mathrm{d}\zeta^1\right)^2 \\
& - \frac{1}{2\phi}\big(\mathrm{d}\tilde{\zeta}_0^2 + (\mathrm{d}\zeta^0)^2 - \mathrm{d}\tilde{\zeta}_1^2 - (\mathrm{d}\zeta^1)^2\big) \\
& + \frac{1}{\phi}\frac{1}{1-|X|^2}\Big(\mathrm{d}\tilde{\zeta}_0^2 + (\mathrm{d}\zeta^0)^2 + |X|^2\big(\mathrm{d}\tilde{\zeta}_1^2 + (\mathrm{d}\zeta^1)^2\big) \\
& \qquad + 2\operatorname{Re}X\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\tilde{\zeta}_1 - \mathrm{d}\zeta^0\mathrm{d}\zeta^1\big) + 2\operatorname{Im}X\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\zeta^1 + \mathrm{d}\zeta^0\mathrm{d}\tilde{\zeta}_1\big)\Big)
\end{aligned}$$

We already checked invariance of the first term, and for the second term it is trivial since $\phi, \tilde{\phi}$ are invariant. Our method to check invariance for the remaining terms is to treat them one-by-one as follows: Write $g_{FS} = (\mathrm{d}\vec{x})^T G(X, \vec{x})\mathrm{d}\vec{x}$, where $\vec{x} = (\tilde{\zeta}_0, \zeta^0, \tilde{\zeta}_1, \zeta^1)$ and G is the matrix of metric coefficients. Now, invariance of the metric means that

$$A^T G\left(\frac{aX+b}{b^*X+a}, A\vec{x}\right)A = G(X, \vec{x})$$

We should check three terms: The last term on the first line, the second line, and the third plus fourth lines. Using `Mathematica`, it is not hard to check invariance of the first and second line terms. For the last part, we put the components of this piece of the metric in a matrix (with respect to the basis $(\tilde{\zeta}_0, \zeta^0, \tilde{\zeta}_1, \zeta^1)$):

$$B(X) = \frac{1}{1-|X|^2}\begin{pmatrix} 1 & 0 & \operatorname{Re} X & \operatorname{Im} X \\ 0 & 1 & \operatorname{Im} X & -\operatorname{Re} X \\ \operatorname{Re} X & \operatorname{Im} X & |X|^2 & 0 \\ \operatorname{Im} X & -\operatorname{Re} X & 0 & |X|^2 \end{pmatrix}$$

Invariance then translates to

$$A^T B(X')A = B(X) \qquad\qquad X' = A \cdot X$$

where $A \cdot X$ denotes the action of $\operatorname{Iwa}(SU(1,1))$ on X. We first compute $A^T G(X)A$ and only then perform the transformation on X.

To do the matrix multiplication by hand, one needs the following facts:

$$\begin{pmatrix} -\operatorname{Im} b & \operatorname{Re} b \\ \operatorname{Re} b & \operatorname{Im} b \end{pmatrix}^2 = \begin{pmatrix} |b|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \qquad \begin{pmatrix} \operatorname{Re} X & \operatorname{Im} X \\ \operatorname{Im} X & -\operatorname{Re} X \end{pmatrix}^2 = \begin{pmatrix} |X|^2 & 0 \\ 0 & |X|^2 \end{pmatrix}$$

$$\begin{pmatrix} -\operatorname{Im} b & \operatorname{Re} b \\ \operatorname{Re} b & \operatorname{Im} b \end{pmatrix}\begin{pmatrix} \operatorname{Re} X & \operatorname{Im} X \\ \operatorname{Im} X & -\operatorname{Re} X \end{pmatrix} = \begin{pmatrix} -\operatorname{Im}(\bar{b}X) & -\operatorname{Re}(\bar{b}X) \\ \operatorname{Re}(\bar{b}X) & \operatorname{Im}(\bar{b}X) \end{pmatrix}$$

$$\begin{pmatrix} \operatorname{Re} X & \operatorname{Im} X \\ \operatorname{Im} X & -\operatorname{Re} X \end{pmatrix}\begin{pmatrix} -\operatorname{Im} b & \operatorname{Re} b \\ \operatorname{Re} b & \operatorname{Im} b \end{pmatrix} = \begin{pmatrix} \operatorname{Im}(\bar{b}X) & \operatorname{Re}(\bar{b}X) \\ -\operatorname{Re}(\bar{b}X) & \operatorname{Im}(\bar{b}X) \end{pmatrix}$$

$$\begin{pmatrix} -\operatorname{Im} b & \operatorname{Re} b \\ \operatorname{Re} b & \operatorname{Im} b \end{pmatrix}\begin{pmatrix} \operatorname{Im}(\bar{b}X) & \operatorname{Re}(\bar{b}X) \\ -\operatorname{Re}(\bar{b}X) & \operatorname{Im}(\bar{b}X) \end{pmatrix} = \begin{pmatrix} -\operatorname{Re}(\bar{b}^2X) & \operatorname{Im}(\bar{b}^2X) \\ \operatorname{Im}(\bar{b}^2X) & \operatorname{Re}(\bar{b}^2X) \end{pmatrix}$$

This allows us to calculate $A^T(GA)$ with relative ease (note, by the way, that $A^T = A$). The result is

$$A^T B(X)A = \frac{1}{1-|X|^2}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

where

$$\begin{aligned}
\alpha &= \begin{pmatrix} a^2 + 2a\operatorname{Im}(\bar{b}X) + |bX|^2 & 0 \\ 0 & a^2 + 2a\operatorname{Im}(\bar{b}X) + |bX|^2 \end{pmatrix} \\
&= \begin{pmatrix} |a - i\bar{b}X|^2 & 0 \\ 0 & |a - i\bar{b}X|^2 \end{pmatrix} \\
\gamma &= \begin{pmatrix} |b^2| + 2a\operatorname{Im}(\bar{b}X) + a^2|X|^2 & 0 \\ 0 & |b|^2 + 2a\operatorname{Im}(\bar{b}X) + a^2|X|^2 \end{pmatrix} \\
&= \begin{pmatrix} |b - iaX|^2 & 0 \\ 0 & |b - iaX|^2 \end{pmatrix} \\
\beta &= \begin{pmatrix} a(1+|X|^2)\operatorname{Im}\bar{b} - \operatorname{Re}(\bar{b}^2X) + a^2\operatorname{Re} X & \beta_{12} = \beta_{21} \\ a(1+|X|^2)\operatorname{Re}\bar{b} + \operatorname{Im}(\bar{b}^2X) + a^2\operatorname{Im} X & \beta_{22} = -\beta_{11} \end{pmatrix}
\end{aligned}$$

This result is confirmed by a computation in `Mathematica`. Now, we have to carry out the $\operatorname{Iwa}(SU(1,1))$-transformation on the base manifold.

3.4.1 Choosing the Correct Base Transformation

As discussed above, perhaps the most natural choice for the base action is the correspondence $\left(\begin{smallmatrix} a & b \\ b^* & a \end{smallmatrix}\right) \longleftrightarrow \left(X \mapsto \frac{aX+b}{b^*X+a}\right)$. It is easy to check that the prefactor transforms as

$$\frac{1}{1-|X|^2} \longmapsto \frac{|\bar{b}X + a|^2}{1-|X|^2}$$

Hence, invariance is equivalent to the statement that the components of α, β, γ are transformed simply by a multiplicative factor $|\bar{b}X + a|^{-2}$. However, this does not seem to be the case. The simplest way to see this is to consider the non-zero part of α. It is sent to

$$\frac{|a(\bar{b}X + a) - i\bar{b}(aX + b)|^2}{|\bar{b}X + a|^2}$$

which gives us what we need if the factor i is replaced by a 1, as the X-dependent terms cancel. As is, however, this does not work.

The factor i suggests that one should try to modify the action to get an extra factor $-i$ on the second term. Precomposing the action by an automorphism of $SU(1,1)$ does not seem like a real possibility, since it would mean that the transformation essentially gets multiplied by i, which can never be the result of an automorphism (since $i^2 = -1$).

However, we *can* conjugate our action by the biholomorphic map $\varphi(X) = iX$. This is allowed since if $(\mathbb{C}H^1, g, i, \nabla)$ is the standard PSK structure on $\mathbb{C}H^1$, for which X is the special coordinate, then iX is the special coordinate for $(\mathbb{C}H^1, g, i, \nabla')$, where $\nabla' = -i \circ \nabla \circ i$. In general, a PSK manifold can be equipped with such a *conjugate* PSK structure, and the new special coordinates are obtained from the old ones by multiplication by i. The corresponding action is:

$$X \longmapsto -i\frac{a(iX) + b}{b^*(iX) + a} = \frac{aX - ib}{ib^*X + a}$$

This action clearly leads to the correct results for α, γ. The fact that it does for β as well was done by computer, though it should be feasible by hand as well. This concludes the proof that g_{FS} is a left-invariant metric on the Lie group $\mathrm{Iwa}(SU(1,1)) \ltimes \mathrm{Iwa}(SU(3,1))$, with the semidirect product structure described above.

3.4.2 The Extension to All of $SU(1,1)$

Since the metric g_{FS} is in fact the symmetric metric on the non-compact symmetric space

$$N = \frac{SU(n+1,2)}{S(U(n+1) \times U(2))}$$

we know that it should even be $SU(n+1,2)$-invariant. In particular, the action of *all* of $SU(n,1)$ should extend symplectically to an isometric action on the fibers.

We will now continue with the case $n = 1$; we already checked the metric on the base, so we only need to check it on the fiber. This is done in `Mathematica`: Again, the last lines of the metric are the only ones that are difficult to check. `Mathematica` is able to sufficiently simplify the off-diagonal blocks to show that their entries do reduce to the correct expressions, but the diagonal blocks resist its attempts. However, starting from `Mathematica`'s result after applying the symplectic matrix, it is not hard to show by hand that they give the correct result. To demonstrate this, we explicitly carry out the computation for the $(4,4)$-entry, for which `Mathematica` gives:

$$(X\bar{a} - \operatorname{Im} b)(\bar{X}a - \operatorname{Im} b) + \operatorname{Re} b(\operatorname{Re} b - 2X \operatorname{Im} a + 2a \operatorname{Im} X)$$

We rewrite this as follows:

$$\begin{aligned}
&|b|^2 + |aX|^2 - \operatorname{Im} b(\bar{a}X + a\bar{X}) + 2 \operatorname{Re} b(a \operatorname{Im} X - X \operatorname{Im} a) \\
=&|b|^2 + |aX|^2 - 2 \operatorname{Im} b \operatorname{Re}(\bar{a}X) - i \operatorname{Re} b(\bar{a}X - a\bar{X}) \\
=&|b|^2 + |aX|^2 - 2 \operatorname{Im} b \operatorname{Re}(\bar{a}X) + 2 \operatorname{Re} b \operatorname{Im}(\bar{a}X) \\
=&|b|^2 + |aX|^2 + 2 \operatorname{Im}(\bar{b}\bar{a}X) \\
=&|b - i\bar{a}X|^2
\end{aligned}$$

As we transform X in the same fashion as before, this turns into $|X|^2$, which is what we wanted to show. Thus, the metric is invariant under all of $SU(1,1)$ (though this action is of course not free).

4 Isometries of the One-Loop Deformed Metric

Recall the one-loop deformed metric from section 1.5:

$$\begin{aligned}
g^c_{FS} = &\frac{\phi + c}{\phi} \frac{1}{1 - |X|^2} \left(\sum \mathrm{d}X^\mu \mathrm{d}\bar{X}^\mu + \frac{1}{1 - |X|^2} \left| \sum \bar{X}^\mu \mathrm{d}X^\mu \right|^2 \right) \\
&+ \frac{\phi + 2c}{\phi + c} \frac{1}{4\phi^2} \mathrm{d}\phi^2 - \frac{2}{\phi} \left(\mathrm{d}w_0 \mathrm{d}\bar{w}_0 - \sum \mathrm{d}w_\mu \mathrm{d}\bar{w}_\mu \right) \\
&+ \frac{\phi + c}{\phi^2} \frac{4}{1 - |X|^2} \left| \mathrm{d}w_0 + \sum X^\mu \mathrm{d}w_\mu \right|^2 \\
&+ \frac{\phi + c}{\phi + 2c} \frac{1}{4\phi^2} \left(\mathrm{d}\tilde{\phi} - 4 \operatorname{Im} \left[\bar{w}_0 \mathrm{d}w_0 - \sum \bar{w}_\mu \mathrm{d}w_\mu \right] + \frac{2c}{1 - |X|^2} \operatorname{Im} \left[\sum \bar{X}^\mu \mathrm{d}X^\mu \right] \right)^2
\end{aligned}$$

In the case $n = 1$, we have

$$\begin{aligned}
g^c_{FS} = &\frac{\phi + c}{\phi} \frac{\mathrm{d}X \mathrm{d}\bar{X}}{(1 - |X|^2)^2} + \frac{\phi + 2c}{\phi + c} \frac{1}{4\phi^2} \mathrm{d}\phi^2 - \frac{2}{\phi} \left(\mathrm{d}w_0 \mathrm{d}\bar{w}_0 - \mathrm{d}w_1 \mathrm{d}\bar{w}_1 \right) \\
&+ \frac{\phi + c}{\phi^2} \frac{4}{1 - |X|^2} \left| \mathrm{d}w_0 + X \mathrm{d}w_1 \right|^2 \\
&+ \frac{\phi + c}{\phi + 2c} \frac{1}{4\phi^2} \left(\mathrm{d}\tilde{\phi} - 4 \operatorname{Im} \left[\bar{w}_0 \mathrm{d}w_0 - \bar{w}_1 \mathrm{d}w_1 \right] + \frac{2c}{1 - |X|^2} \operatorname{Im} \left[\bar{X} \mathrm{d}X \right] \right)^2
\end{aligned}$$

Decomposing the w's into ζ-coordinates and setting $X = x + iy$, this is:

$$
\begin{aligned}
g^c_{FS} =& \frac{\phi + c}{\phi}\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 - x^2 - y^2)^2} + \frac{\phi + 2c}{\phi + c}\frac{1}{4\phi^2}\mathrm{d}\phi^2 - \frac{1}{2\phi}\big(\mathrm{d}\tilde{\zeta}_0^2 + (\mathrm{d}\zeta^0)^2 - \mathrm{d}\tilde{\zeta}_1^2 - (\mathrm{d}\zeta^1)^2\big) \\
&+ \frac{\phi + c}{\phi^2}\frac{1}{1 - x^2 - y^2}\Big(\mathrm{d}\tilde{\zeta}_0^2 + (\mathrm{d}\zeta^0)^2 + (x^2 + y^2)\big(\mathrm{d}\tilde{\zeta}_1^2 + (\mathrm{d}\zeta^1)^2\big) \\
&\qquad\qquad + 2x\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\tilde{\zeta}_1 - \mathrm{d}\zeta^0\mathrm{d}\zeta^1\big) + 2y\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\zeta^1 + \mathrm{d}\zeta^0\mathrm{d}\tilde{\zeta}_1\big)\Big) \\
&+ \frac{\phi + c}{\phi + 2c}\frac{1}{4\phi^2}\left(\mathrm{d}\tilde{\phi} + \zeta^0\mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0\mathrm{d}\zeta^0 + \zeta^1\mathrm{d}\tilde{\zeta}_1 - \tilde{\zeta}_1\mathrm{d}\zeta^1 + \frac{2c}{1 - x^2 - y^2}(x\mathrm{d}y - y\mathrm{d}x)\right)^2 \\
=& \frac{\phi + c}{\phi}\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 - x^2 - y^2)^2} + \frac{\phi + 2c}{\phi + c}\frac{1}{4\phi^2}\mathrm{d}\phi^2 \\
&+ \left(\frac{\phi + c}{\phi^2}\frac{1}{1 - x^2 - y^2} - \frac{1}{2\phi}\right)(\mathrm{d}\tilde{\zeta}_0^2 + (\mathrm{d}\zeta^0)^2) \\
&+ \left(\frac{\phi + c}{\phi^2}\frac{x^2 + y^2}{1 - x^2 - y^2} + \frac{1}{2\phi}\right)(\mathrm{d}\tilde{\zeta}_1^2 + (\mathrm{d}\zeta^1)^2) \\
&+ \frac{\phi + c}{\phi^2}\frac{1}{1 - x^2 - y^2}\Big(2x\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\tilde{\zeta}_1 - \mathrm{d}\zeta^0\mathrm{d}\zeta^1\big) + 2y\big(\mathrm{d}\tilde{\zeta}_0\mathrm{d}\zeta^1 + \mathrm{d}\zeta^0\mathrm{d}\tilde{\zeta}_1\big)\Big) \\
&+ \frac{\phi + c}{\phi + 2c}\frac{1}{4\phi^2}\left(\mathrm{d}\tilde{\phi} + \zeta^0\mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0\mathrm{d}\zeta^0 + \zeta^1\mathrm{d}\tilde{\zeta}_1 - \tilde{\zeta}_1\mathrm{d}\zeta^1 + \frac{2c}{1 - x^2 - y^2}(x\mathrm{d}y - y\mathrm{d}x)\right)^2
\end{aligned}
$$

4.1 The Unbroken Subgroup of Isometries ($n = 1$)

We expect that many of the isometries of the undeformed FS metric remain isometries in the deformed case, but not all of them can survive. The expectation is that the unbroken subgroup acts by nonzero but low cohomogeneity.

We can do this for the fiber isometries for arbitrary n. The group $G(n+2)$ acts on the fibers as follows (cf. CHM, equation (4.4)):

$$(\tilde{\zeta}, \zeta, \tilde{\phi}, \phi) \longmapsto (e^{\lambda/2}\tilde{\zeta} + \tilde{v}, e^{\lambda/2}\zeta + v, e^{\lambda/2}(\tilde{v}^T\zeta - v^T\tilde{\zeta}) + e^{\lambda}\tilde{\phi} + \alpha, e^{\lambda}\phi)$$

where $(\tilde{v}, v, \alpha, e^{\lambda}) \in \mathbb{R}^{2n+3} \times \mathbb{R}_+ \cong G(n+2)$. Clearly, the shift-invariance of $\tilde{\phi}$ is not spoiled by the one-loop deformation, but scaling ϕ is now no longer a symmetry. The shifts in $\tilde{\zeta}$ and ζ, which are compensated (in the last term) by the shift in $\tilde{\phi}$, also remain symmetries. Thus, $G(n+2)$ acts with cohomogeneity one; every orbit corresponds to a fixed value of $\phi \in \mathbb{R}_+$.

We now restrict to $n = 1$, where we check the isometries coming from the base, i.e. the the (symplectically extended) $SU(1,1)$ transformations. All terms except for the last line are the same (up to a factor) as the old ones, so we only need to check invariance of the last line. Inspecting the final term, we see that the base metric will get an additional term

$$\frac{2c}{1 - |X|^2}(\bar{X}^2\mathrm{d}X^2 + X^2\mathrm{d}\bar{X}^2 - 2|X|^2\mathrm{d}X\mathrm{d}\bar{X})$$

Imposing invariance of the base metric, it is already clear that the transformation

$$X \longmapsto \frac{aX - ib}{i\bar{b}X + \bar{a}}$$

can only be an isometry if the prefactor $(1 - |X|^2)^{-1}$ is invariant: If it transforms by a non-trivial factor, then this factor simply cannot be canceled. This is equivalent to requiring $|i\bar{b}X + \bar{a}|^2 = 1$, which in turn is equivalent to $b = 0$, $a = e^{i\theta}$, where $\theta \in \mathbb{R}$ is a phase; at the same time, $X \mapsto e^{2i\theta}X$. Thus, the subgroup of isometries corresponding to $SU(1,1)$ is broken to a circle subgroup, *at best*. We will now check that the full metric is indeed invariant under this subgroup.

The easiest way to proceed is to translate the transformation of $(\tilde{\zeta}_0, \zeta^0, \tilde{\zeta}_1, \zeta^1)$ to one of (w_0, w_1): Recall that $w_0 = \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$ while $w_1 = \frac{1}{2}(\tilde{\zeta}_1 - i\zeta^1)$; then it is not hard to see that $w_0 \mapsto e^{i\theta}w_0$ while $w_1 \mapsto e^{-i\theta}w_1$. By inspection, one immediately sees that all terms except perhaps $|\mathrm{d}w_0 - X\mathrm{d}w_1|^2$ are invariant. But this term is invariant too, since $X \mapsto e^{2i\theta}X$. Thus, we have an unbroken S^1-subgroup of isometries of the deformed metric, i.e. the base is acted on with cohomogeneity one. The orbits are standard circles in $\mathbb{C}$, rotating X while leaving $|X|$ constant.

Thus, the deformed metric is acted on by a group of isometries of cohomogeneity at most 2. We would like to go further and compute the full isometry group of the deformed space. Our method to try to prove this is to start with arguing that every isometry must in fact preserve every orbit of the group remaining subgroup of $G(n) \ltimes G(n+2)$ which acts by isometries, i.e. fix the pair $(\phi, |X|)$. To prove this, we proceed in analogy with the $n = 0$ case, namely we will look for (real) functions which are invariant under isometries by construction and distinguish all orbits. Since the action is cohomogeneity two, we must find two functions: The most natural thing to try is to take functions built from the curvature tensor, i.e. certain norms of it and its covariant derivative. Computing these is the next step.

We start by computing the so-called *Kretschmann scalar* κ, which (in a local frame) is given by the coordinate-expression $R^{ijkl}R_{ijkl}$, where R is the curvature tensor. Viewing R as an endomorphism $\mathcal{R} : \bigwedge^2 T^*M \to \bigwedge^2 T^*M$, the Kretschmann scalar is proportional to its Hilbert-Schmidt norm $\|\mathcal{R}\|^2 = \sum \lambda^2$, where $\lambda \in \mathbb{R}$ are its eigenvalues (recall that $\mathcal{R}$ is self-adjoint, hence there exists an orthonormal eigenbasis). A computation using `Mathematica` shows that

$$\kappa = \frac{128(176c^6 + 528c^5\phi + 672c^4\phi^2 + 464c^3\phi^3 + 186c^2\phi^4 + 42c\phi^5 + 5\phi^6)}{(2c + \phi)^6}$$

However, proceeding with the computation of further polynomial invariants in the curvature and its covariant derivatives, one finds that it is hard to find any that depend on $|X|$. Thus, we are lead to suspect that there is another (continuous) isometry changing the value of $|X|$.

4.2 Determining the Cohomogeneity

There are two possible directions we may want to explore next, in order to determine the full deformed isometry group.

4.2.1 Pointwise methods

We are interested in finding potential isometries changing the value of $|X|$, but in fact we can already draw some interesting conclusions if we are able to find the other isometries: Since M is of cohomogeneity at most 2, there exist (many) points $p \in M$ such that the orbit $\mathcal{L} \cdot p$ under the remainder of the undeformed isometry group $\mathcal{L}$ is six-dimensional. In fact, this is the case whenever $X(p) \neq 0$ Assume that we can determine the stabilizer $K = G_p$ (G denotes the full isometry group) and denote its Lie algebra by $\mathfrak{k}$. If $\mathfrak{l}$ denotes the Lie algebra of $\mathcal{L}$, we know that either $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$ or $\mathfrak{k} + \mathfrak{l}$ defines a codimension one subspace (since the cohomogeneity is at least 1). In many cases, knowledge of $\mathfrak{k}$ lets us shed some light on this question.

Consider $W := T_p(\mathcal{L} \cdot p) \subset T_pM$, and the action of $\mathfrak{k}$ on this subspace. Then we have the following cases:

(i) $\mathfrak{k} \cdot W \not\subset W$. Then there must be some non-stabilizing elements of (the identity component of) G which lie *outside* $\mathcal{L}$, since if $g \cdot p \in \mathcal{L} \cdot p$ for every $g \in G$ then $\mathfrak{g} \cdot W \subset W$. Thus, we know that $\mathfrak{g}$ is strictly *larger* than $\mathfrak{k} + \mathfrak{l}$.

(ii) $\mathfrak{k} \cdot W \subset W$ and $\mathfrak{k} \cdot W^{\perp} \neq 0$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$.

(iii) $\mathfrak{k} \cdot W \subset W$ and $\mathfrak{k} \cdot W^{\perp} = 0$. Then we cannot decide whether $\mathfrak{g}$ is larger than $\mathfrak{k} + \mathfrak{l}$ or not. However, in any case, we can look at the Lie algebra $\mathfrak{k} + \mathfrak{l}$, which forms a Lie subalgebra of $\mathfrak{g}$, and consider all of its one-dimensional extensions. There might not be many, and so we may only have a few candidate isometry groups.

To find K or $\mathfrak{k}$, we can try to solve the equations

$$A \cdot R = 0 \qquad\qquad A \cdot g = 0$$

where we can consider $A \in SO(T_pM)$ to determine K or its linearization $A \in \mathfrak{so}(T_pM)$ to find $\mathfrak{k}$.

4.2.2 Using the HK/QK Correspondence

As mentioned in section 1.5, the one-loop deformation is rather trivial on the hyper-Kähler side of the HK/QK correspondence. Therefore, if we can find a way to carry over isometries from from the hyper-Kähler to the quaternionic Kähler side, there is a good chance that this procedure will be insensitive to the one-loop deformation, and therefore will give us a way to deform isometries to remain such on the quaternionic Kähler side. The next section is dedicated to this approach.

5 Twists and the HK/QK Correspondence

Inspired by certain physical constructions derived from T-duality, Swann [8] invented a general method to associate to a manifold M with a torus action (subject to some conditions) another manifold, called the *twist* of M. Moreover, tensors invariant under the action can be carried over from M to its twist in unique fashion. The HK/QK correspondence, as described in section 1, can be viewed as one instance of this general construction [6, 7]. Demonstrating this is the aim of this section.

6 The Twist Construction

For our purposes, it will suffice to describe the twist construction for *circle* actions; the interested reader is referred to [8] for twists of arbitrary-dimensional torus actions.

Consider a manifold M equipped with an S^1-action generated by a vector field $Z \in \mathfrak{X}(M)$. To apply the twist construction, we must first construct an associated manifold, namely the total space P of an S^1-principal bundle $\pi_M : P \to M$. We furthermore equip this principal bundle with a principal S^1-connection $\eta \in \Omega^1(P)$ with curvature F. Let $X \in \mathfrak{X}(P)$ denote the vector field generating the principal action. We wish to lift the given action $S^1 \curvearrowright M$ to an action on P which preserves the connection θ and commutes with the principal circle action.

Proposition 2. *A lift as above exists if and only if* $[\iota_Z F] = 0 \in H^1(M;\mathbb{Z})$.

Proof. Such a lift is specified by a vector field $Z_1 \in \mathfrak{X}(M)$, which will be the generator of our action. We then require $L_{Z_1}\theta = 0$. Writing $Z_1 = \tilde{Z} + f_1 X$, where $\tilde{Z}$ denotes the θ-horizontal lift, this translates to

$$0 = \iota_{Z_1}\mathrm{d}\theta + \mathrm{d}\iota_{Z_1}\theta = \iota_{Z_1}\pi^*F + \mathrm{d}(\iota_{f_1X}\theta) = \iota_{\tilde{Z}}\pi^*F + \mathrm{d}f_1\theta(X) = \pi^*(\iota_Z F) + \mathrm{d}f_1$$

and therefore $\mathrm{d}f_1 = -\pi^*(\iota_Z F)$. Such an f_1 is constant along fibers of π, since $X(f_1) = \iota_X \mathrm{d}f_1 = -\iota_X\pi^*(\iota_Z F) = 0$. This means that $f_1 = \pi^* f$ for some $f \in C^\infty(M)$, and $\pi^*\mathrm{d}f = \mathrm{d}f_1 = -\pi^*(\iota_Z F)$ and therefore $\iota_Z F = -\mathrm{d}f$ and in particular, $[\iota_Z F] = 0 \in H^1(M;\mathbb{Z})$.

Since Lie(S^1) is one-dimensional, Z_1 trivially defines a Lie algebra. Choosing f_1 as above*, we obtain a circle actions $S^1_{Z_1} \curvearrowright P$; we check that it commutes with the principal S^1-action. The horizontal component of $[Z_1, X]$, which is measured by its projection to M, vanishes because of naturality of the Lie bracket and verticality of X. Thus, we only need to investigate the vertical component. We can instead check the vertical component of $[\tilde{Z}, X]$, since $[f_1X, X] = 0$ by constancy of f_1 on the fibers. Thus, we compute

$$\theta([\tilde{Z}, X]) = \tilde{Z}(\theta(X)) - (L_{\tilde{Z}}\theta)(X) = -\mathrm{d}\theta(\tilde{Z}, X) = -\pi^*F(\tilde{Z}, X) = 0$$

where the final step uses verticality of X. Finally, if $\iota_Z F = -\mathrm{d}f$ for some $f \in C^\infty(M)$, then $\pi^*(\iota_Z F) = \mathrm{d}f_1$ where $f_1 = \pi^* f$, and hence $L_{Z_1}\theta = 0$. This establishes the converse statement. □

Definition 3. Let F be a closed 2-form on a manifold M. A circle action $S^1 \curvearrowright M$ is called *F-Hamiltonian* if $\iota_Z F = -\mathrm{d}f$ for some $f \in C^\infty(M)$.

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*Notice that f_1 is only determined up to a real constant; this corresponds to the one-loop deformation parameter in the HK/QK correspondence.

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