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HyperKähler and quaternionic Kähler manifolds with S^1-symmetries

Andriy Haydys*

Fakultät für Mathematik, Universität Bielefeld, P.O. Box 100 131, D-33501 Bielefeld, Germany

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Abstract

We study relations between quaternionic Riemannian manifolds admitting different types of symmetries. We show that any hyperKähler manifold admitting hyperKähler potential and triholomorphic action of S^1 can be constructed from another hyperKähler manifold (of lower dimension) with an action of S^1 that fixes one complex structure and rotates the other two and vice versa. We also study the corresponding quaternionic Kähler manifolds equipped with a quaternionic Kähler action of the circle. In particular we show that any positive quaternionic Kähler manifolds with S^1-symmetry admits a Kähler metric on an open everywhere dense subset.

MSC: 53C26

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1. Introduction

In a short Berger's list [3] of groups that can occur as holonomy group of locally irreducible Riemannian manifold are in particular $Sp(n)$ and $Sp(n)Sp(1)$, the first being a holonomy group of hyperKähler manifold while the second a quaternionic Kähler one. In other words, a Riemannian manifold (M, g) is hyperKähler if it admits three covariantly constant complex structures $I_r,\ r = 1, 2, 3$ with quaternionic relations

$$I_r^2 = -id, \qquad I_1 I_2 = -I_2 I_1 = I_3,$$

compatible with the Riemannian structure: $g(I_r\cdot, I_r\cdot) = g(\cdot,\cdot)$. Hitchin [13] proved the following criteria of integrability of complex structures: I_r are covariantly constant if and only if 2-forms $\omega_r(\cdot,\cdot) = g(\cdot, I_r\cdot)$ are symplectic (i.e. closed). It is convenient to consider all three symplectic forms as a single 2-form with values in imaginary quaternions:

$$\omega = \omega_1 i + \omega_2 j + \omega_3 k.$$

In contrast to hyperKähler manifolds, a quaternionic Kähler manifold N admits almost complex structures (and correspondingly 2-forms ω_r) only locally. Nevertheless the 4-form $\Omega = w_1 \wedge \omega_1 + w_2 \wedge \omega_2 + w_3 \wedge \omega_3$, called the

* Tel.: +49 5211064966.
E-mail address: haydys@math.uni-bielefeld.de.

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fundamental 4-form, exists globally and determines the quaternionic Kähler structure. In this case the integrability means that the fundamental 4-form is covariantly constant and this is equivalent to $d\Omega = 0$ provided $\dim N \geq 12$. In dimension 4 quaternionic Kähler by definition means Einstein and self-dual.

An important link between hyperKähler and quaternionic Kähler geometries provides the Swann construction [18]. Suppose that the group $SU(2) \cong Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ acts on M isometrically and *permutes complex structures:*

$$\mathcal{L}_\zeta \omega = [\zeta, \omega] \quad \Leftrightarrow \quad \left(L_q\right)^* \omega = q\omega\bar{q}, \tag{1}$$

where $\zeta \in \operatorname{Im}\mathbb{H} \cong \mathfrak{sp}(1)$. We also say, that the action of $Sp(1)$ with the above property is *permuting*. Swann shows that such an action can be extended to homothetic action of the whole $\mathbb{H}^* = \mathbb{R}^*_+ \times Sp(1)$ if the vector field IY_I is independent of a complex structure I, where Y_I is a Killing vector field of $S^1 \subset Sp(1)$ which preserves I. In particular

$$I_1 Y_1 = I_2 Y_2 = I_3 Y_3 = -Y_0, \tag{2}$$

where we put $Y_r = Y_{I_r}$ for short and a vector field Y_0 generates *homothetic* action of $\mathbb{R}^*_+ \subset \mathbb{H}^* : \ (L_r)^* g = r^2 g$. We will also call such an $\mathbb{H}^*$-action *permuting*. Under these circumstances $N = M/\mathbb{H}^*$ has positive scalar curvature and carries a quaternionic Kähler structure. On the other hand, for any quaternionic Kähler manifold N with positive scalar curvature Swann constructs a hyperKähler manifold $\mathcal{U}(N)$ which enjoys the permuting action of $\mathbb{H}^*$. Such manifolds are also distinguished by the property of carrying a hyperKähler potential, i.e. function $\rho : M \to \mathbb{R}$ which is the Kähler potential simultaneously for each complex structure.

In this paper we study the influence of S^1-symmetry on relations between different types of quaternionic Riemannian geometries. We consider the hyperKähler manifolds M with the hyperKähler potential and additional triholomorphic and isometric action of S^1 and show that such manifolds can be reconstructed from their hyperKähler reductions $\tilde{M}$ with respect to a *nonzero* value of momentum map. The main result of this paper is Theorem 3 which describes M as the total space of a certain fibre bundle with the fibre $\mathbb{H}^*$. Moreover, the hyperKähler structure is described quite explicitly which allows to obtain not only existence results but metric and symplectic forms themselves. Dividing such manifolds by $\mathbb{H}^*$ as described in [18] we obtain quaternionic Kähler manifolds with S^1-symmetry (Theorem 7). In Section 4 we describe new examples of hyper- and quaternionic-Kahler manifolds making use of a certain freedom in choice of the parameters of construction.

We also show in Theorem 14 that positive quaternionic Kähler manifold N with S^1-symmetry admits a Kähler structure on an open everywhere dense submanifold. The complex structure is a section of the structure bundle of N, however the Kähler metric is different from the quaternionic Kähler one.

2. S^1-symmetry

Let M be a hyperKähler manifold with $\mathbb{H}^*$-action permuting complex structures. Suppose also that M admits a hyperKähler action of S^1 (which we prefer to denote as S^1_0 in order to distinguish this group from another one, which will appear later and is also isomorphic to S^1) with momentum map $\mu : M \to \operatorname{Im}\mathbb{H}$,

$$\mathrm{d}\mu = -\iota_{K_0}\omega,$$

where K_0 is the Killing vector field of S^1_0. We also assume that these two actions commute and that μ is $\mathbb{H}^*$-equivariant:

$$\mu \circ L_x = x\mu\bar{x}, \quad x \in \mathbb{H}^*. \tag{3}$$

Fix an imaginary quaternion, say i, and consider the corresponding level set $P = \mu^{-1}(i)$. Since $xm = L_x m \in \mu^{-1}(xi\bar{x})$ and $\mathbb{H}^*$ acts transitively on $\operatorname{Im}\mathbb{H} \setminus \{0\}$ the map

$$f : \mathbb{H}^* \times P \to M \setminus \mu^{-1}(0), \quad (x, m) \mapsto xm, \tag{4}$$

is surjective. Notice that

$$M_0 = M \setminus \mu^{-1}(0)$$

is open and everywhere dense submanifold of M.

However the map f is not injective. Indeed $\mu^{-1}(i) = P$ inherits action of $\mathrm{Stab}_i = S^1 \subset \mathbb{H}^*$ (to which we now give a label S^1_r) and it follows that points (x, m) and $(xz, \bar{z}m)$, $z \in S^1_r$ are mapped into the same point xm in M_0. Thus the manifold M_0 can be described as $\mathbb{H}^* \times_{S^1_r} P$. Now the challenge is to express the hyperKähler structure of M_0 in terms of its "components" $\mathbb{H}^*$ and P.

While the first "component" $\mathbb{H}^*$ is very simple, the second one needs to be understood more deeply for the future purposes.

2.1. Induced structure on P

It follows from $\mathbb{H}^*$-equivariancy of μ that each nonzero imaginary quaternion is a regular value of μ. Assuming that S^1_0 acts freely on P, we see that $\tilde{M} := P/S^1_0$ is just a hyperKähler reduction of M and therefore is itself a hyperKähler manifold. Thus P can be thought of as S^1_0-principal bundle over $\tilde{M}$. Moreover it gets equipped with a connection,[1] namely

$$\xi(\cdot) = vg(K_0, \cdot) \in \Omega^1(P),$$

where $v^{-1} = g(K_0, K_0)$, $v : \tilde{M} \to \mathbb{R}_{>0}$. Notice that the induced metric $\tilde{g}$ on $\tilde{M}$, the connection ξ and the function v together determine the metric on P since $T_\cdot P \cong \mathbb{R}K_0 \oplus T_\cdot \tilde{M}$:

$$g_{\mathrm{P}} = \tilde{g} + v^{-1}\xi^2. \tag{5}$$

The connection ξ defines a horizontal lift $\hat{\mathrm{u}} \in TP$ of a tangent vector $\mathrm{u} \in T\tilde{M}$.

As we have already remarked P inherits the action of S^1_r, which descends to $\tilde{M}$. The latter action has a nice property (inherited from M) of fixing complex structure I_1 and rotating the plane spanned by I_2 and I_3. Denote by K_r a Killing vector field of S^1_r-action on $\tilde{M}$ and by w the squared norm of K_r:

$$w : \tilde{M} \to \mathbb{R}_{>0}, \quad w = \|K_r\|^2.$$

Below we will also use a quaternion-valued 1-form η generated by K_r:

$$\eta = \iota_{K_r}\tilde{g} + \iota_{K_r}\tilde{\omega} \in \Omega^1(\tilde{M}; \mathbb{H}). \tag{6}$$

Further, recall that Y_1 is the Killing vector field of the S^1_r-action on P. Then Y_1 and K_r are related as follows. First observe that $T_\cdot M = T_\cdot P \oplus \mathbb{R}I_1K_0 \oplus \mathbb{R}I_2K_0 \oplus \mathbb{R}I_3K_0$ and one also has

$$T_\cdot P = \mathrm{Ker}\,\mu_*, \qquad \mu_* I_1 K_0 = v^{-1}i, \qquad \mu_* I_2 K_0 = v^{-1}j \quad \text{and} \quad \mu_* I_3 K_0 = v^{-1}k.$$

Now taking $x = \exp(it)$ in formula (3) and differentiating with respect to t one obtains that the formula $Y_1 = \hat{K}_r + aK_0$ holds on P. The same argument gives that $\mu_* Y_0 = 2i$ or in other words $Y_0 = \hat{Y}' + bK_0 + 2vI_1K_0$. It follows from the equation $I_1Y_0 = Y_1$ that $Y' = -I_1K_r$, $b = 0$, $a = -2v$. Summing up we obtain

$$\begin{aligned} Y_0 &= -I_1\hat{K}_r - 2vI_1K_0, \qquad Y_1 = \hat{K}_r + 2vK_0, \\ Y_2 &= I_3\hat{K}_r + 2vI_3K_0, \qquad Y_3 = -I_2\hat{K}_r - 2vI_2K_0. \end{aligned} \tag{7}$$

Remark 1. Since actions of S^1_0 and $S^1_r \subset \mathbb{H}^*$ commute, it follows that the connection ξ enjoys additional property of being S^1_r invariant. On infinitesimal level this means that $0 = \mathcal{L}_{Y_1}\xi = \iota_{Y_1}d\xi + d\,\iota_{Y_1}\xi = \iota_{K_r}F_\xi + 2dv$, where $F_\xi \in \Omega^1(\tilde{M})$ denotes the curvature form of ξ. Thus, invariance of ξ with respect to action of S^1_r on P is equivalent to

$$\iota_{K_r}F_\xi + 2dv = 0. \tag{8}$$

Note also that the function v is S^1_r-invariant by the same reason.

[1] Notice that we identify the Lie algebra of S^1 with $\mathbb{R}$, not with $i\mathbb{R}$.

2.2. Metric

Since M is a Riemannian manifold the map f, defined by (4), induces a metric f^*g on $\mathbb{H}^* \times P$. Notice that since f is not injective, this metric degenerates on tangent vectors to fibres. Our next aim is to calculate f^*g explicitly in terms of tensors on $\mathbb{H}^*$ and $\tilde{M}$ as well as connection ξ and function v.

Let $(x, m) \in \mathbb{H}^* \times P$ and $(h_1, \mathrm{v}_1),\ (h_2, \mathrm{v}_2) \in T_x\mathbb{H}^* \times T_m P$. Put also $\alpha = x^{-1}h_1$, $\beta = x^{-1}h_2 \in T_1\mathbb{H}^*$ and denote by Y_α and Y_β the Killing vector fields of $\mathbb{H}^*$-action at the point m corresponding to the Lie algebra elements α and β. Obviously $(\mathsf{Y}_1, \mathsf{Y}_i, \mathsf{Y}_j, \mathsf{Y}_k) = (Y_0, Y_1, Y_2, Y_3)$. Further, one has

$$\begin{aligned} f^*g\left((h_1, \mathrm{v}_1), (h_2, \mathrm{v}_2)\right) &= g\left((L_x)_* \left(\mathsf{Y}_\alpha + \mathrm{v}_1\right), (L_x)_* \left(\mathsf{Y}_\beta + \mathrm{v}_2\right)\right) \\ &= |x|^2 g\left(\mathsf{Y}_\alpha + \mathrm{v}_1, \mathsf{Y}_\beta + \mathrm{v}_2\right). \end{aligned}$$

Thus we see that essentially the following three terms have to be computed: $g\left(\mathsf{Y}_\alpha, \mathsf{Y}_\beta\right)$, $g\left(\mathsf{Y}_\alpha, \mathrm{v}\right)$ and $g\left(\mathrm{v}_1, \mathrm{v}_2\right)$.

The first term. Since relation (2) holds, we get

$$g\left(\mathsf{Y}_\alpha, \mathsf{Y}_\beta\right) = g\left(\sum_{r=0}^{3} \alpha_r Y_r\, , \sum_{r=0}^{3} \beta_r Y_r\right) = g\left(Y_0, Y_0\right) \operatorname{Re}\left(\alpha\bar{\beta}\right).$$

Recall that w denotes the squared norm of K_r and therefore it follows from (7) that $g(Y_0, Y_0) = w + 4v^2v^{-1} = 4v + w$. So finally we have

$$g\left(\mathsf{Y}_\alpha, \mathsf{Y}_\beta\right) = (4v + w) \operatorname{Re}\left(\alpha\bar{\beta}\right).$$

The second term. First decompose v into horizontal and vertical parts: $\mathrm{v} = \hat{\mathrm{v}}' + \xi(\mathrm{v}) K_0$. Taking into account formulae (7) again, one obtains

$$\begin{aligned} g(\mathsf{Y}_\beta, \mathrm{v}) &= g\left(\mathsf{Y}_\beta, \mathrm{v}'\right) + 2\beta_1\xi(\mathrm{v}) \\ &= \tilde{g}\left(-\beta_0 I_1 K_r + \beta_1 K_r + \beta_2 I_3 K_r - \beta_3 I_2 K_r, \mathrm{v}'\right) + 2\beta_1\xi(\mathrm{v}) \\ &= \beta_0\tilde{\omega}_1(K_r, \mathrm{v}') + \beta_1\tilde{g}(K_r, \mathrm{v}') - \beta_2\tilde{\omega}_3(K_r, \mathrm{v}') + \beta_3\tilde{\omega}_2(K_r, \mathrm{v}') + 2\beta_1\xi(\mathrm{v}). \end{aligned}$$

Slightly abusing notations, we also use the letter η for the pull-back of the form (6) to P. Then the above formula can be written in a more compact form:

$$g\left((R_m)_*\, \beta, \mathrm{v}\right) = -\operatorname{Re}\left(2\beta i\xi(\mathrm{v}) + \beta i\eta(\mathrm{v})\right).$$

The third term. This has been already computed and is given by (5).

Remark 2. Below we follow conventions of [11]. In particular, if ζ_1 and ζ_2 are (quaternion-valued) 1-forms, then

$$\begin{aligned} &\left(\zeta_1 \odot \zeta_2\right)(v_1, v_2) = \zeta_1(v_1)\zeta_2(v_2) + \zeta_1(v_2)\zeta_2(v_1), \\ &\left(\zeta_1 \wedge \zeta_2\right)(v_1, v_2) = \zeta_1(v_1)\zeta_2(v_2) - \zeta_1(v_2)\zeta_2(v_1). \end{aligned}$$

Now, recalling that α and β contain shift by $x^{-1} = \bar{x}/|x|^2$ we obtain a final form of the metric:

$$f^*g = (4v + w)\operatorname{Re} dx \otimes d\bar{x} - \operatorname{Re}\left(\bar{x}dxi \odot (2\xi + \eta)\right) + |x|^2\left(\tilde{g} + v^{-1}\xi^2\right). \tag{9}$$

2.3. Symplectic forms

In this section we will describe symplectic forms in a similar manner as we did with the metric above.

The pull-back of ω can be written as

$$\begin{aligned} f^*\omega((h_1, \mathrm{v}_1), (h_2, \mathrm{v}_2)) &= \omega\left((L_x)_* \left(\mathsf{Y}_\alpha + \mathrm{v}_1\right), (L_x)_* \left(\mathsf{Y}_\beta + \mathrm{v}_2\right)\right) \\ &= x\omega\left(\mathsf{Y}_\alpha + \mathrm{v}_1, \mathsf{Y}_\beta + \mathrm{v}_2\right)\bar{x}, \end{aligned}$$

where α and β are the same as in Section 2.2. Therefore we have to compute three terms analogous to those, which appear in the metric computation.

The first term. The computation is similar to the one above:

$$\begin{aligned}\omega(\mathsf{Y}_\alpha, \mathsf{Y}_\beta) &= ig\,(\alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3,\, \beta_0 Y_1 - \beta_1 Y_0 + \beta_2 Y_3 - \beta_3 Y_2)\\ &\quad + jg\,(\alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3,\, \beta_0 Y_2 - \beta_1 Y_3 - \beta_2 Y_0 + \beta_3 Y_1)\\ &\quad + kg\,(\alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3,\, \beta_0 Y_3 + \beta_1 Y_2 - \beta_2 Y_1 - \beta_3 Y_0)\\ &= g(Y_0, Y_0)\,(i\,(-\alpha_0\beta_1 + \alpha_1\beta_0 - \alpha_2\beta_3 + \alpha_3\beta_2)\\ &\quad + \ j\,(-\alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_1\beta_3 - \alpha_3\beta_1) + k\,(-\alpha_0\beta_3 + \alpha_3\beta_0 - \alpha_1\beta_2 + \alpha_2\beta_1))\\ &= (4v + w)\mathrm{Im}\,\left(\alpha\bar{\beta}\right).\end{aligned}$$

The second term. Decomposing v into horizontal $\hat{\mathrm{v}}'$ and vertical $\xi(\mathrm{v})K_0$ parts one obtains:

$$\begin{aligned}\omega\,(\mathsf{Y}_\alpha, \mathrm{v}) &= i\left(-2\alpha_0\xi(\mathrm{v}) + \omega_1\left((R_m)_*\,\alpha, \hat{\mathrm{v}}'\right)\right)\\ &\quad + j\left(-2\alpha_3\xi(\mathrm{v}) + \omega_2\left((R_m)_*\,\alpha, \hat{\mathrm{v}}'\right)\right) + k\left(-2\alpha_2\xi(\mathrm{v}) + \omega_3\left((R_m)_*\,\alpha, \hat{\mathrm{v}}'\right)\right)\\ &= i\left(-2\alpha_0\xi(\mathrm{v}) - \alpha_0\tilde{g}(K_r, \mathrm{v}') + \alpha_1\tilde{\omega}_1(K_r, \mathrm{v}') - \alpha_2\tilde{\omega}_2(K_r, \mathrm{v}') - \alpha_3\tilde{\omega}_3(K_r, \mathrm{v}')\right)\\ &\quad + j\left(-2\alpha_3\xi(\mathrm{v}) + \alpha_0\tilde{\omega}_3(K_r, \mathrm{v}') + \alpha_1\tilde{\omega}_2(K_r, \mathrm{v}') + \alpha_2\tilde{\omega}_1(K_r, \mathrm{v}') - \alpha_3\tilde{g}(K_r, \mathrm{v}')\right)\\ &\quad + k\left(-2\alpha_2\xi(\mathrm{v}) - \alpha_0\tilde{\omega}_2(K_r, \mathrm{v}') + \alpha_1\tilde{\omega}_3(K_r, \mathrm{v}') + \alpha_2\tilde{g}(K_r, \mathrm{v}') + \alpha_3\tilde{\omega}_1(K_r, \mathrm{v}')\right)\\ &= -2\,\mathrm{Im}\,(\alpha i)\,\xi(\mathrm{v}) - \mathrm{Im}\,(\alpha i\eta(\mathrm{v}))\,.\end{aligned}$$

The third term. It is easy to see that

$$\omega(\mathrm{v}_1, \mathrm{v}_2) = \omega(\mathrm{v}_1', \mathrm{v}_2') = \tilde{\omega}(\mathrm{v}_1, \mathrm{v}_2),$$

where the pull-back is also implied.

Thus, recalling that $\alpha = x^{-1}h_1 = |x|^{-2}\bar{x}h_1$, the $\mathrm{Im}\,\mathbb{H}$-valued form $\varphi = f^*\omega$ can be written as

$$\varphi = \frac{4v + w}{2}dx \wedge d\bar{x} + x\tilde{\omega}\bar{x} - 2\,\mathrm{Im}\,(dxi\bar{x}) \wedge \xi - \mathrm{Im}\,(dxi \wedge \eta\bar{x})\,. \tag{10}$$

2.4. Inverse construction

Now we can look on the above considerations in reverse order in the following sense. Suppose $\tilde{M}$ is a hyperKähler manifold with metric $\tilde{g}$ and hyperKähler structure $\tilde{\omega}$. Further, a group S^1_r acts on $\tilde{M}$ preserving complex structure I_1 and rotating I_2 and I_3 in the sense $(L_z)^*\,\tilde{\omega} = z\tilde{\omega}\bar{z},\ z \in S^1_r$. Pick an S^1_0-principal bundle P with a connection ξ and extend the action of S^1_r to P such that it commutes with S^1_0 (at least locally such extension always exists).

Consider further a manifold $M_0 = \mathbb{H}^* \times_{S^1_r} P$. We would like to define a metric g and hyperKähler structure ω on M_0 such that their pull-backs to $\mathbb{H}^* \times P$ are given by formulae (9) and (10) respectively. The first thing to show is that these expressions define invariant and basic tensors on $\mathbb{H}^* \times P$. One can easily check that both tensors are invariant provided ξ is S^1_r-invariant (see also Remark 5). Let χ be a Killing vector field of the S^1_r-action on $\mathbb{H}^* \times P$. It follows that $\chi = K^* - Y_1$, where K^* is a Killing vector field of the S^1_r-action on $\mathbb{H}^*$ by right multiplication, i.e. $dx(K^*) = xi$. Then the equalities $\iota_\chi\, g = 0,\ \ \iota_\chi\, \varphi = 0$ can be checked directly. For example, the last one follows from the following computation:

$$\begin{aligned}\left(\iota_\chi\,\varphi\right)(\alpha, \mathrm{v}) &= \frac{1}{2}(4v + w)\left(xi\bar{\alpha} - \alpha\overline{xi}\right) - x\tilde{\omega}\,(K_r, \mathrm{v})\,\bar{x}\\ &\quad - 2\,\mathrm{Im}\,(xii\bar{x}\xi\,(\mathrm{v}) + \alpha i\bar{x}\,2v) - \mathrm{Im}\,(xii\eta\,(\mathrm{v})\,\bar{x} + \alpha i\eta\,(K_r)\,\bar{x})\\ &= (4v + w)\mathrm{Im}\,(xi\bar{\alpha}) - x\tilde{\omega}\,(K_r, \mathrm{v})\,\bar{x} - 2\cdot 0\\ &\quad - 4v\mathrm{Im}\,(xi\bar{\alpha}) + x\tilde{\omega}\,(K_r, \mathrm{v})\,\bar{x} - w\mathrm{Im}\,xi\bar{\alpha}\\ &= 0.\end{aligned}$$

The next question is whether the 2-form $\omega \in \Omega^2(M_0; \operatorname{Im}\mathbb{H})$ is closed. As we have seen, the pull-back φ of ω to $\mathbb{H}^* \times P$ is basic and therefore this is equivalent to φ being closed. Now $d\varphi$ is a quaternion-valued 3-form on $\mathbb{H}^* \times P$ and by the Künneth formula $\Omega^3(\mathbb{H}^* \times P; \operatorname{Im}\mathbb{H}) \cong \bigoplus_{l=0}^{3} \Omega^l(\mathbb{H}^*; \operatorname{Im}\mathbb{H}) \otimes \Omega^{3-l}(P; \operatorname{Im}\mathbb{H})$. Thus $d\varphi$ decomposes in 4 components: $d\varphi = \sum_{l=0}^{3} (d\varphi)_{(l,3-l)}$, $(d\varphi)_{(l,3-l)} \in \Omega^l(\mathbb{H}^*; \operatorname{Im}\mathbb{H}) \otimes \Omega^{3-l}(P; \operatorname{Im}\mathbb{H})$. It is easy to see that $(d\varphi)_{(0,3)}$ and $(d\varphi)_{(3,0)}$ vanish identically and it remains to compute the remaining two components of $d\varphi$.

It follows directly from the expression for φ that

$$\begin{aligned}(d\varphi)_{(1,2)} &= dx \wedge \tilde{\omega}\bar{x} + x\tilde{\omega} \wedge d\bar{x} + 2\operatorname{Im}(dxi\bar{x}) \wedge F_\xi + \operatorname{Im}(dxi \wedge d\eta\bar{x}) \\ &= \operatorname{Im}\left(dx \wedge \left(2\tilde{\omega} + 2iF_\xi + id\eta\right)\bar{x}\right)\end{aligned}$$

and this vanishes iff

$$-2i\tilde{\omega} + 2F_\xi + d\eta = 0.$$

By the Cartan formula $[i, \tilde{\omega}] = \mathcal{L}_{K_r}\tilde{\omega} = d\left(\iota_{K_r}\tilde{\omega}\right)$. But then the above equation can be rewritten as $2F_\xi = -d\left(\iota_{K_r}\tilde{g}\right) - d\left(\iota_{K_r}\tilde{\omega}\right) + 2i\tilde{\omega} = -d\left(\iota_{K_r}\tilde{g}\right) - 2\tilde{\omega}_1$. Thus the vanishing of $(d\varphi)_{(1,2)}$ is equivalent to

$$F_\xi = -\frac{1}{2}d\left(\iota_{K_r}\tilde{g}\right) - \tilde{\omega}_1. \tag{11}$$

For the other nontrivial component of $d\varphi$ one obtains

$$\begin{aligned}(d\varphi)_{(2,1)} &= \frac{1}{2}(4dv + dw) \wedge dx \wedge d\bar{x} + 2\operatorname{Im}(dxi \wedge d\bar{x}) \wedge \xi - \operatorname{Im}(dxi \wedge \eta \wedge d\bar{x}) \\ &= \frac{1}{2}\operatorname{Im}\left(dx \wedge \left(-(4dv + dw) - 4i\xi - 2i\eta\right) \wedge d\bar{x}\right).\end{aligned}$$

Suppose θ is a quaternion-valued 1-form on $\tilde{M}$ and consider the equation $\operatorname{Im}(dx \wedge \theta \wedge d\bar{x}) = 0$ on $\mathbb{H}^* \times \tilde{M}$, which turns out to be equivalent to $\operatorname{Re}\theta = 0$. Indeed, $dx \wedge \theta \wedge d\bar{x} = -(\operatorname{Re}\theta) \wedge dx \wedge d\bar{x} + dx \wedge \operatorname{Im}\theta \wedge d\bar{x}$ and the last summand is real-valued: $\overline{dx \wedge \operatorname{Im}\theta \wedge d\bar{x}} = (-1)dx \wedge \overline{\operatorname{Im}\theta} \wedge d\bar{x} = dx \wedge \operatorname{Im}\theta \wedge d\bar{x}$.

Therefore $(d\varphi)_{(2,1)}$ vanishes iff

$$4dv + dw = 2\,\iota_{K_r}\tilde{\omega}_1. \tag{12}$$

Thus, the 2-form φ descends to a closed form on $M_0 = \mathbb{H}^* \times_{S^1_r} P$ if and only if the three equations are satisfied: (8), (11) and (12). But the last equation follows from the first two. Indeed, since S^1_r acts isometrically we have $0 = \mathcal{L}_{K_r}\left(\iota_{K_r}\tilde{g}\right) = \iota_{K_r}\, d\left(\iota_{K_r}\tilde{g}\right) + d\left(\iota_{K_r}\iota_{K_r}\tilde{g}\right)$ which means $\iota_{K_r}\, d\left(\iota_{K_r}\tilde{g}\right) = -dw$. Now taking the operator ι_{K_r} of both sides of Eq. (11) and using (8) we obtain Eq. (12).

It was first remarked in [14] that a hyperKähler manifold with an S^1-action which preserves one complex structure and permutes the other two has a *Kähler* potential. Since our conventions slightly differ we reproduce this simple computation.

Let $\tilde{\rho} : \tilde{M} \to \mathbb{R}$ be a momentum map of S^1_r, i.e. a solution of the equation

$$d\tilde{\rho} = -\,\iota_{K_r}\tilde{\omega}_1.$$

On the one hand we have $d\left(I_2^* d\tilde{\rho}\right) = d\,i\left(\partial_2 - \bar{\partial}_2\right)\tilde{\rho} = -2i\partial_2\bar{\partial}_2\tilde{\rho}$. But on the other hand $I_2^*\,\iota_{K_r}\tilde{\omega}_1 = \iota_{K_r}\tilde{\omega}_3$ and therefore $-d\left(I_2^* d\tilde{\rho}\right) = d\left(I_2^*\,\iota_{K_r}\tilde{\omega}_1\right) = d\left(\iota_{K_r}\tilde{\omega}_3\right) = \mathcal{L}_{K_r}\tilde{\omega}_3 = 2\tilde{\omega}_2$. Putting this together we obtain that $\tilde{\rho}$ satisfies

$$i\partial_2\bar{\partial}_2\tilde{\rho} = \tilde{\omega}_2$$

or, in other words, $\tilde{\rho}$ is a *Kähler* potential for $\tilde{\omega}_2$. It is clear that $\tilde{\rho}$ is also a Kähler potential for $\tilde{\omega}_3$ since these forms are not distinguished by the S^1_r-action. However $\tilde{\rho}$ need not be a Kähler potential for $\tilde{\omega}_1$.

Now if we remark that $I_1^* d\tilde{\rho} = \iota_{K_r}\tilde{g}$ and consequently

$$-2i\partial_1\bar{\partial}_1\tilde{\rho} = d\left(\iota_{K_r}\tilde{g}\right),$$

then Eq. (11) can be written in a particularly nice form: $F_\xi = i\partial_1\bar{\partial}_1\tilde{\rho} - \tilde{\omega}_1$, i.e. the function $\tilde{\rho}$ is a hyperKähler potential iff $F_\xi = 0$.

Now we can find the function v (up to a constant) from Eq. (12) (or, equivalently, from (8)):

$$v = -\frac{w + 2\tilde{\rho}}{4}. \tag{13}$$

Therefore the following theorem is essentially proven.

Theorem 3. *Assume that the group S^1_r act isometrically on a hyperKähler manifold $\tilde{M}$ such that $(L_z)^*\tilde{\omega} = z\tilde{\omega}\bar{z}$. Let $P \to \tilde{M}$ be an S^1_0-principal bundle with a connection $\xi \in \Omega^1(P)$. Suppose also, that the function v defined by formula (13) is everywhere positive, where w denotes squared norm of the Killing vector field K_r of S^1_r, while $\tilde{\rho}$ is its momentum map. Extend the action of S^1_r to P such that it commutes with the action of S^1_0. Then (9) and (10) define a hyperKähler structure on $M_0 = \mathcal{H}(\tilde{M}) = \mathbb{H}^* \times_{S^1_r} P$ if and only if*

$$F_\xi = i\,\partial_1\bar{\partial}_1\tilde{\rho} - \tilde{\omega}_1. \tag{14}$$

Furthermore the left action of $\mathbb{H}^*$ induces a transitive action on the 2-sphere of complex structures and therefore $\mathcal{H}(\tilde{M})$ has a hyperKähler potential

$$\rho = -\frac{4v + w}{2}|x|^2. \tag{15}$$

Finally, for any hyperKähler manifold M with permuting action of $\mathbb{H}^*$ and triholomorphic one of S^1, the open everywhere dense submanifold $M_0 = M \setminus \mu^{-1}(0)$ can be obtained as $\mathcal{H}(\tilde{M})$, where $\tilde{M}$ is as above.

Proof. It remains to show that the symmetric tensor given by formula (9) provides a non-negative bilinear form at any point of the tangent space to $\mathbb{H}^* \times P$ as well as to prove formula (15) for the hyperKähler potential.

First we have a decomposition $T(\mathbb{H}^* \times P) = T\mathbb{H} \oplus TP = T\mathbb{H} \oplus \mathbb{R}K_0 \oplus \pi^* T\tilde{M}$. Further decompose $T\tilde{M}$ as $\mathtt{span}(K_r, I_1K_r, I_2K_r, I_3K_r) \oplus E$, where E denotes the orthogonal complement. Thus we have

$$T\left(\mathbb{H}^* \times P\right) = T\mathbb{H} \oplus \mathbb{R}K_0 \oplus \pi^*\mathtt{span}\left(K_r, I_1K_r, I_2K_r, I_3K_r\right) \oplus \pi^*E,$$

and we can write a tangent vector as $\mathrm{w} = \mathrm{w}^* + aK_0 + \beta K_r + \mathrm{v}$, where a is a real number and β is a quaternion.[2] If $dx(\mathrm{w}^*) = \alpha \in \mathbb{H}$, then

$$\begin{aligned} g(\mathrm{w},\mathrm{w}) &= (4v + w)|\alpha|^2 + |x|^2\left(|\beta|^2 + \|\mathrm{v}\|^2 + \frac{a^2}{v}\right) - 2\mathrm{Re}\,(\bar{x}\alpha i\,(2a + w\beta)) \\ &= 4v|\alpha|^2 - 4a\mathrm{Re}\,(\bar{x}\alpha i) + |x|^2 v^{-1}a^2 + w\left(|\alpha|^2 - 2\mathrm{Re}\,(\bar{x}\alpha i\beta) + |\beta x|^2\right) + |x|^2\|\mathrm{v}\|^2 \\ &= \left|2\sqrt{v}\alpha i - \frac{ax}{\sqrt{v}}\right|^2 + w\,|\alpha i - \beta\bar{x}|^2 + |x|^2\|\mathrm{v}\|^2 \geq 0. \end{aligned}$$

Further, it was shown in [5] that if a hyperKähler manifold M admits a permuting $\mathbb{H}^*$-action, then the squared norm of any Killing vector field generating this action is a hyperKähler potential (up to a constant[3] -2). Now the permuting action of $\mathbb{H}^*$ on $\mathcal{H}(\tilde{M})$ is induced by the left multiplication on the first component of $\mathbb{H}^* \times P$. In particular, the Killing vector field of $\mathbb{R}^* \subset \mathbb{H}^*$ is the vector field w^* s.t. $dx(\mathrm{w}^*) = x$. Its squared norm (multiplied by $-1/2$) with respect to metric (9) is exactly the right-hand side of (15). □

Remark 4. It is easy to see that the hyperKähler reduction of $\mathcal{H}(\tilde{M})$ by S^1_0 is $\tilde{M}$ (certainly not a surprise in view of Section 2.1). Thus the construction $\mathcal{H}(\cdot)$ may be regarded as a kind of "hyperKähler induction", i.e. an inverse construction to the hyperKähler reduction.

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[2] Any tangent space of a hyperKähler manifold carries an action of $\mathbb{H}$. In particular if $\beta = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k \in \mathbb{H}$, we write βK_r instead of $\beta_0 K_r + \sum_{l=1}^{3}\beta_l I_l K_r$ for the sake of brevity.

[3] The minus sign appears because of different sign convention in the definition of hyperKähler potential.

Remark 5. From Eq. (14) we have that the Chern class of P is $-\frac{1}{2\pi}[\tilde{\omega}_1]$. It follows that $\frac{1}{2\pi}[\tilde{\omega}_1]$ must represent an integral cohomology class and in this case Eq. (14) does have a solution. Furthermore, any solution ξ is automatically S^1_r-invariant and the Killing vector field Y_1 of S^1_r-action on P satisfies $Y_1 = \hat{K}_r + 2vK_0$. Indeed, as we have seen the right-hand side of Eq. (14) may be written in the form $-\frac{1}{2}d\left(\iota_{K_r}\tilde{g}\right) - \tilde{\omega}_1$ and this immediately implies S^1_r-invariancy of ξ. Further, we may decompose Y_1 on the horizontal and vertical parts: $Y_1 = \hat{K}_r + aK_0$. Then by Remark 1 we have $\iota_{K_r} F_\xi + da = 0$. On the other hand Eq. (14) implies $\iota_{K_r} F_\xi = -2dv$ and the statement follows.

It is worth pointing out that equality $a = 2v$ holds only up to a constant. This phenomenon will be discussed in details in Section 4. At this point we will ignore this subtlety implying that a constant is chosen properly, such that the equation $Y_1 = \hat{K}_r + 2vK_0$ holds.

Remark 6. Suppose that the S^1_r-action is induced by a permuting $\mathbb{H}^*$-action (and standard inclusion $S^1_r \subset \mathbb{H}^*$), or equivalently, the momentum map $\tilde{\rho}$ of the S^1_r-action is not only Kähler potential but also hyperKähler [18]. It follows from Eq. (14) that the bundle P is flat and we can take it to be trivial so that topologically $\mathcal{H}(\tilde{M}) = \mathbb{H}^* \times \tilde{M}$. Moreover, it follows from the proof of the theorem that v is constant so that we may put $v = 1$. This determines a metric and symplectic forms.

Further, it turns out that in this case $\mathcal{H}(\tilde{M})$ is *isometric* to $\mathbb{H}^* \times \tilde{M}$ with its product metric. Indeed, direct computation shows that the map $\mathcal{H}(\tilde{M}) \to \mathbb{H}^* \times \tilde{M}$, $(x, m) \mapsto (x, xm)$ is an isometry.

2.5. *Quaternionic flip*

In the previous section for any hyperKähler manifold $\tilde{M}$ with a certain S^1-symmetry we have constructed another hyperKähler manifold $M_0 = \mathcal{H}(\tilde{M})$ with hyperKähler potential. Then Swann's results [18] imply that the manifold $N_0 = M_0/\mathbb{H}^* = P/S^1_r$ is quaternionic Kähler. In this section we will describe its quaternionic Kähler structure.

First notice that in order to obtain quaternionic Kähler structure on N_0 we have to consider a *Riemannian* version of the quotient $M_0/\mathbb{H}^*$, that is to pick a level set of a hyperKähler potential and divide it by the group $Sp(1)$; in this case we may view complex structures of N_0 as induced by those of M_0 on $\mathtt{span}(Y_0, Y_1, Y_2, Y_3)^\perp \subset TM_0$.

Let us again return to the viewpoint of Section 2.1, i.e. $P = \mu^{-1}(i) \subset M$ and let $\lambda = (4v + w)^{-1/2}$. Since the restriction of the hyperKähler potential ρ to P equals $-(4v + w)/2$, a map

$$l \colon p \mapsto \lambda(p) \cdot p = L_{\lambda(p)} p, \quad p \in P, \tag{16}$$

is a diffeomorphism between P and $Q = \rho^{-1}(-1/2) \cap \mu_c^{-1}(0) \cap \{\mu_1 > 0\}$, where $\mu_c = \mu_2 + i\mu_3$. Thus our next aim is to compute the tensors $g(pr \circ l_* \cdot, pr \circ l_* \cdot)$ and $\omega(pr \circ l_* \cdot, pr \circ l_* \cdot)$, where pr means a projection onto $\mathtt{span}(Y_0, Y_1, Y_2, Y_3)^\perp$.

First we may decompose a vector $\mathrm{u} \in T_pP \subset T_pM$ as $\mathrm{u}' + \sum_{l=0}^{3} a_l Y_l$. The coefficients a_l can be found from the following relations:

$$\begin{aligned}
a_0 g(Y_0, Y_0) &= g(\mathrm{u}, Y_0) = g(\mathrm{u}, -I_1\hat{K}_r - 2vI_1K_0) = \tilde{\omega}_1(K_r, \mathrm{u}),\\
a_1 g(Y_1, Y_1) &= g(\mathrm{u}, Y_1) = g(\mathrm{u}, \hat{K}_r + 2vK_0) = 2\xi(\mathrm{u}) + \tilde{g}(K_r, \mathrm{u}),\\
a_2 g(Y_2, Y_2) &= g(\mathrm{u}, Y_2) = g(\mathrm{u}, I_3\hat{K}_r + 2vI_3K_0) = -\tilde{\omega}_3(K_r, u),\\
a_3 g(Y_3, Y_3) &= g(\mathrm{u}, Y_3) = g(\mathrm{u}, -I_2\hat{K}_r - 2vI_2K_0) = \tilde{\omega}_2(K_r, u).
\end{aligned}$$

The expressions for a_l become more compact in quaternionic notations. Indeed, if we put $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ and recall the definition (6) of 1-form η, then

$$a = \frac{1}{4v + w}\,(2\xi(\mathrm{u}) + \bar{\eta}(\mathrm{u}))\,i.$$

Since $l_* = \left(L_{\lambda(p)}\right)_* + d\lambda\, Y_0(l(p))$, we have $pr\, l_*\mathrm{v} = \left(L_{\lambda(p)}\right)_* \mathrm{v}'$ and therefore

$$\begin{aligned}
g\left(pr\, l_*\mathrm{u}, pr\, l_*\mathrm{v}\right) &= g\left(\left(L_{\lambda(p)}\right)_* \mathrm{u}', \left(L_{\lambda(p)}\right)_* \mathrm{v}'\right)\\
&= \lambda^2 g\left(\mathrm{u}', \mathrm{v}'\right)\\
&= \lambda^2 g\left(\mathrm{u} - aY_0, \mathrm{v} - bY_0\right)\\
&= \lambda^2\left(g(\mathrm{u}, \mathrm{v}) - g(aY_0, \mathrm{v}) - g(\mathrm{u}, bY_0) + g(aY_0, bY_0)\right),
\end{aligned}$$

where we also put $\mathrm{v} = \mathrm{v}' + bY_0$. Now it is easy to compute every single summand in the last expression. Indeed, the first summand is given by formula (5). Taking into account the decompositions (7) one obtains $g(aY_0, \mathrm{v}) = -\mathrm{Re}\,(ai\,(2\xi+\eta)\,(\mathrm{v})) = (4v+w)^{-1}\mathrm{Re}\,((2\xi+\bar\eta)\,(\mathrm{u})\,(2\xi+\eta)\,(\mathrm{v}))$. Since vectors Y_l are pairwise orthogonal we get: $g(aY_0, bY_0) = (4v+w)\mathrm{Re}\,\left(a\bar b\right) = (4v+w)^{-1}\mathrm{Re}\,((2\xi+\bar\eta)\,(\mathrm{u})\,(2\xi+\eta)\,(\mathrm{v}))$. Finally, gathering all terms together, one has after a simplification:

$$g_{\mathrm{N}} = \frac{1}{4v+w}\left(\tilde g + \frac{1}{v}\xi^2 - \frac{1}{2(4v+w)}\,(2\xi+\bar\eta)\odot(2\xi+\eta)\right).$$

It is convenient to introduce a 1-form

$$\psi = \frac{1}{g(Y_1,Y_1)}g(Y_1,\cdot) = \frac{1}{4v+w}(2\xi + \imath_{K_r}\tilde g), \tag{17}$$

which is a connection on the principal fibre bundle $P \to N_0$ (assuming that S^1_r acts freely on P), i.e. it is S^1_r-invariant and $\psi(Y_1) = 1$. Then the expression for the metric takes the following form:

$$g_{\mathrm{N}} = \frac{1}{4v+w}\left(\tilde g + \frac{1}{v}\xi^2 - \frac{1}{2}\psi^2\right) - \frac{1}{2(4v+w)^2}\sum_{l=1}^{3}\left(\imath_{K_r}\tilde\omega_l\right)^2. \tag{18}$$

Arguments similar to those at the beginning of Section 2.4 show that formula (18) defines a metric on N_0.

The fundamental 4-form Ω can be obtained in the similar manner. Indeed,

$$\begin{aligned}\chi(\mathrm{u},\mathrm{v}) &= \omega(pr\,l_*\mathrm{u}, pr\,l_*\mathrm{v})\lambda^2\omega(\mathrm{u}-aY_0, \mathrm{v}-bY_0)\\ &= \lambda^2\left(\omega\,(\mathrm{u},\mathrm{v}) - \omega\,(aY_0,\mathrm{v}) - \omega(\mathrm{u}, bY_0) + \omega(aY_0, bY_0)\right).\end{aligned}$$

Arguing similarly as we did when computing the metric, we obtain finally that

$$\chi = \frac{1}{4v+w}\tilde\omega - \frac{1}{2(4v+w)^2}\,(2\xi+\bar\eta)\wedge(2\xi+\eta)\,.$$

Componentwise spelling of this formula is

$$\begin{aligned}\chi_1 &= \frac{1}{4v+w}\left(\tilde\omega_1 - \psi\wedge\imath_{K_r}\tilde\omega_1\right) + \frac{1}{(4v+w)^2}\,\imath_{K_r}\tilde\omega_2\wedge\imath_{K_r}\tilde\omega_3,\\ \chi_2 &= \frac{1}{4v+w}\left(\tilde\omega_2 - \psi\wedge\imath_{K_r}\tilde\omega_2\right) - \frac{1}{(4v+w)^2}\,\imath_{K_r}\tilde\omega_1\wedge\imath_{K_r}\tilde\omega_3,\\ \chi_3 &= \frac{1}{4v+w}\left(\tilde\omega_3 - \psi\wedge\imath_{K_r}\tilde\omega_3\right) + \frac{1}{(4v+w)^2}\,\imath_{K_r}\tilde\omega_1\wedge\imath_{K_r}\tilde\omega_2.\end{aligned} \tag{19}$$

With respect to the action of S^1_r on P all three forms $\chi_l \in \Omega^2(P)$ are basic, however only χ_1 is invariant:

$$\begin{aligned}\mathcal{L}_{Y_1}\chi_1 &= \frac{1}{(4v+w)^2}\left(-2\,\imath_{K_r}\tilde\omega_3\wedge\imath_{K_r}\tilde\omega_3 + \imath_{K_r}\tilde\omega_2\wedge 2\,\imath_{K_r}\tilde\omega_2\right) = 0,\\ \mathcal{L}_{Y_1}\chi_2 &= \frac{1}{4v+w}\left(-2\tilde\omega_3 + \psi\wedge 2\,\imath_{K_r}\tilde\omega_3\right) + \frac{1}{(4v+w)^2}\,\imath_{K_r}\tilde\omega_1\wedge 2\,\imath_{K_r}\tilde\omega_2 = -2\chi_3,\\ \mathcal{L}_{Y_1}\chi_3 &= \frac{1}{4v+w}\left(2\tilde\omega_2 - \psi\wedge 2\,\imath_{K_r}\tilde\omega_2\right) - \frac{1}{(4v+w)^2}\,\imath_{K_r}\tilde\omega_1\wedge 2\,\imath_{K_r}\tilde\omega_3 = 2\chi_2.\end{aligned}$$

It follows that a 4-form

$$\Omega = \chi_1\wedge\chi_1 + \chi_2\wedge\chi_2 + \chi_3\wedge\chi_3 \tag{20}$$

is basic and invariant and therefore descends to N_0. Integrability of such defined quaternionic Kähler structure follows from the integrability of the hyperKähler structure on $M_0 = \mathcal{H}(\tilde M) = \mathcal{U}(N_0)$ [18].

Theorem 7. *Let the assumptions of Theorem 3 be satisfied. Then the metric (18) and the fundamental 4-form (20) define a quaternionic Kähler structure on $N_0 = \mathcal{Q}(\tilde{M}) = P/S^1_r$, where χ_l and ψ are defined by (19) and (17) respectively. Moreover $\mathcal{Q}(\tilde{M})$ admits a quaternionic Kähler action of S^1 and its Swann bundle $\mathcal{U}(N_0)$ is $\mathcal{H}(\tilde{M})$.* □

3. Examples

Example 8 (*$T^*\mathbb{CP}^n$ with the Calabi Metric*). The hyperKähler quotient of $\mathbb{H}^{n+1}$ by S^1 acting by multiplication on the left with respect to nonzero value of the momentum map is topologically $T^*\mathbb{CP}^n$. Hitchin [12] showed that the metric coincides with the one defined by Calabi [6]. Therefore $\mathcal{H}(T^*\mathbb{CP}^n) = \mathbb{H}^{n+1}$ with its flat metric and $\mathcal{Q}(T^*\mathbb{CP}^n) = \mathbb{HP}^n$ (in both cases with zero level set of corresponding momentum map being removed).

Example 9 (*Flat Manifold, Adjoint Action*). Let us take a copy of quaternions $\mathbb{H}_y$ as a manifold $\tilde{M}$ with the following action of $S^1_r : (z, y) \mapsto zy\bar{z}$ (one can also regard $\mathbb{H}$ as $T^*\mathbb{C}$ with fibrewise action of S^1_r; see also Remark 10). In this case $1/4(w + 2\tilde{\rho}) = 1/2(y_2^2 + y_3^2)$, where $y = y_0 + y_1 i + y_2 j + y_3 k$. Adding 1/2 we may write function v in the form

$$v = \frac{1}{2}(1 - y_2^2 - y_3^2)$$

and it is positive on $\mathbb{R}^2_{y_0\,y_1} \times \mathrm{D}^2_{y_2\,y_3}$, where $\mathrm{D}^2 \subset \mathbb{R}^2$ is an open disc of radius 1. The principal bundle P is trivial and therefore $\mathcal{Q}(\mathbb{R}^2 \times \mathrm{D}^2) = \mathbb{R}^2 \times \mathrm{D}^2$ with the following metric:

$$g_{\mathrm{N}} = \frac{1}{2(1+d)}\left(\frac{1-d}{1+d}\,\mathrm{Re}\,dy \otimes d\bar{y} + \frac{4d}{1-d^2}\,(y_0 dy_1 + y_3 dy_2)^2 \right.$$
$$\left. - \frac{1}{1+d}\,(y_0 dy_1 + y_3 dy_2) \odot (y_2 dy_3 - y_3 dy_2)\right),$$

where $d = y_2^2 + y_3^2$. Therefore the above metric is Einstein and self-dual. However it is incomplete.

Similarly, one can compute the metric and symplectic forms on $\mathcal{H}(\mathbb{R}^2 \times \mathrm{D}^2) = \mathbb{H}^* \times \mathbb{R}^2 \times \mathrm{D}^2$ but the metric is also incomplete.

Remark 10. The above manifolds are examples of a large class of hyperKähler manifolds admitting S^1_r-action. Namely Kaledin [15] and independently Feix [7] proved the existence of a hyperKähler metric on a (neighborhood of the zero section of) cotangent bundle $T^*\mathcal{M}$ to a real-analytic Kähler manifold $\mathcal{M}$. The above examples show that the function v can be both positive everywhere and only on a proper open subset of $T^*\mathcal{M}$.

Example 11 (*The Gibbons–Hawking Spaces*). All hyperKähler four-manifolds with S^1-symmetry were described by Gibbons and Hawking [10] and their construction is as follows. If Z^4 is hyperKähler and admits S^1-symmetry with a Killing vector field K_0, then its hyperKähler momentum map $\mu = \mu_1 i + \mu_2 j + \mu_3 k$ represents Z as a fibration over $\mathbb{R}^3$ with generic fibre S^1, so that, excluding the critical points of the momentum map, one can write the metric as

$$g_{\mathrm{GH}} = \nu\left(dx_1^2 + dx_2^2 + dx_3^2\right) + \nu^{-1}\xi^2, \qquad x_l = \mu_l,\, l = 1, 2, 3, \tag{21}$$

where $\nu : \mathbb{R}^3 \to \mathbb{R}_{>0}$, $\nu^{-1} = \|K_0\|^2$, and ξ is a connection form. It is then an easy exercise to write down 2-forms which are closed provided

$$F_\xi = - * d\nu. \tag{22}$$

It follows from the Bianchi identity that ν is harmonic. We would like to point out that Z^4 is determined by the function ν (harmonic and positive) since the connection ξ can be found from Eq. (22). The Gibbons–Hawking ansatz is a choice of a particular function ν:

$$\nu(x) = \sum_{i=1}^{n} \frac{1}{|x - y_i|}, \qquad y_i \in \mathbb{R}^3.$$

In general the above four-manifold does not admit an S^1_r-action. However when all the poles y_i of the function ν lie on one line (say x_1-axis), then such action does exist; its projection to $\mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H}$ is then $(z, x) \mapsto zx\bar{z}$, $x \in \mathrm{Im}\,\mathbb{H}$. A direct (and tedious) computation shows that the function v is positive everywhere and therefore the construction $\mathcal{H}(Z)$ is defined on the whole Gibbons–Hawking space Z. Alternatively, one can observe [8] that the Gibbons–Hawking spaces can be obtained as hyperKähler reductions of a flat space acted upon by a torus with respect to a nonzero value of the corresponding momentum map, i.e. Gibbons–Hawking spaces are examples of toric hyperKähler manifolds [4]. Therefore, if the value of the momentum map is chosen properly, the action of S^1_r can be obtained from the corresponding action on the flat space. In this case one can also show that the manifold $\mathcal{H}(Z)$ is also toric.

4. Indeterminacy of function v: Further examples

As we have seen in Section 2.4 the function v is defined by formula (13) only up to a constant and this has strong consequences as we will see below. Recall that the action of S^1_r should be lifted from $\tilde{M}$ to P such that its Killing vector field Y_1 equals $\hat{K}_r + 2vK_0$ (see (7)). This implies that we are free to take $\tilde{v} = v + m/2$ instead of v, where m is an integer, whenever $v + m/2$ remains everywhere positive. However in this case one needs to modify the lifting of the S^1_r-action to P to get that the Killing vector field is given by $\hat{K}_r + 2\tilde{v}K_0$. Therefore we get that the manifolds $\mathbb{H}^* \times_{S^1_r} P$ and P/S^1_r again carry hyperKähler and quaternionic Kähler structures correspondingly, where the modified action of S^1_r is implied. It turns out that the modification of the S^1_r-action can change the topology of the $\mathcal{H}$ and $\mathcal{Q}$ constructions.

In the rest of this section we carry out the above observation in detail for the case of cotangent bundle of a complex Grassmannian. HyperKähler metrics on cotangent bundles of Grassmannians with required symmetries were obtained long ago (see [8] and references therein). More generally, Nakajima [16] constructed such metrics on cotangent bundles of quiver varieties, however we shall consider only Grassmannians for the sake of simplicity. First we review construction of hyperKähler structure on $T^*Gr_k(\mathbb{C}^n)$ and then illustrate the impact of the modification of S^1_r-action.

Choose the left quaternionic structure on the flat space $M_{n,k}(\mathbb{H})$ consisting of matrices with n rows and k columns. We have

$$g(A_1, A_2) = \operatorname{Re}\operatorname{tr}(A_1\bar{A}_2^t), \qquad \omega(A_1, A_2) = \operatorname{Im}\operatorname{tr}(A_1\bar{A}_2^t).$$

The group $U(k)$ acts on $M_{n,k}(\mathbb{H})$ by multiplication on the right. Write $A = B + C^t j$, where $B \in M_{n,k}(\mathbb{C})$ and $C \in M_{k,n}(\mathbb{C})$. Then the hyperKähler momentum map $\mu = \mu_{\mathbb{R}} i + \mu_{\mathbb{C}} j$ is given by

$$\mu_{\mathbb{R}}(B, C) = \frac{1}{2}\left(\bar{B}^t B - C\bar{C}^t\right), \quad \mu_{\mathbb{C}}(B, C) = -CB.$$

Probably the easiest way to see that the hyperKähler reduction of $M_{n,k}(\mathbb{H})$ is isomorphic to $T^*Gr_k(\mathbb{C}^n)$ is to observe [16] that

$$\mu^{-1}(i)\ /U(k) \cong \{(B, C) \in M_{n,k}(\mathbb{C}) \times M_{k,n}(\mathbb{C}) \mid \operatorname{rk} B = k,\ BC = 0\}/GL_k(\mathbb{C}), \tag{23}$$

where $GL_k(\mathbb{C})$ acts on pairs of matrices as follows: $(B, C)\cdot g = (Bg, g^{-1}C)$. Think about B as a k-frame in $\mathbb{C}^n$. Then $\{B|\operatorname{rk}B = k\}/GL_k(\mathbb{C}) = Gr_k(\mathbb{C}^n)$ and it follows that the right-hand side of (23) is $S \otimes Q^\vee$, where S and Q denote the tautological and quotient vector bundles over $Gr_k(\mathbb{C}^n)$ respectively. Recalling that $TGr_k(\mathbb{C}^n) \cong S^\vee \otimes Q$ we get the result.

The action of S^1_r, inherited from the permuting action of $\mathbb{H}^*$ on $M_{n,k}(\mathbb{H})$, is the following one[4]: $z\cdot[B, C] = [zB, zC]$. For such action the function v defined by (13) must be positive everywhere on $T^*Gr_k(\mathbb{C}^n)$. Indeed, this follows from the following observation. The space $M_{n,k}/\!\!/\!\!/_{\mu=i}U(k)$ can be obtained in two steps: first consider $M_{n,k}/\!\!/\!\!/_{\mu=0}SU(k)$ and then take its hyperKähler reduction with respect to $S^1 \subset U(k)$. Since the space $M_{n,k}/\!\!/\!\!/_{\mu=0}SU(k)$ inherits permuting action of $\mathbb{H}^*$, the statement follows. Notice also that the corresponding S^1-principal bundle $P = \mu^{-1}_{S^1}(i) \subset M_{n,k}/\!\!/\!\!/_{\mu=0}SU(k)$ is pull-back of the principal bundle of $\Lambda^{top}S$.

Take $\tilde{v} = v + m/2$ instead of v, where m is a positive integer. Then the modified action of S^1_r on $\mu^{-1}(i)$ is given by

$$z\cdot(B, C) = (Bz^{m+1}, z^{1-m}C). \tag{24}$$

Before proceeding we need the following lemma.

[4] Notice that this action is different from the one considered by Nakajima in [16].

Lemma 12. *Let $P \to X$ be an S^1-principal bundle and $L \to X$ be the corresponding line bundle. Consider the following action of S^1 on $P \times \mathbb{C}$: $z \cdot (p, w) = (pz^r, z^s w)$, where r and s are integers and r is positive. Then*

$$(P \times \mathbb{C})/S^1 \cong L^{-s}.$$

Proof. Let $Q_{r,s}$ denote the space $P \times \mathbb{C}$ with the action of S^1 as in the statement of the lemma. Then we have an equivariant map $Q_{r,s} \to Q_{r,rs}$, $(p, w) \mapsto (p, w^r)$. Clearly it is surjective; although it is not injective, it descends to a bijective map of quotients $Q_{r,s}/S^1 \to Q_{r,rs}/S^1$. But the last quotient is exactly L^{-s}. □

Therefore the action (24) can be replaced by the following one

$$z \cdot (B, C) = (Bz, z^{-1}Cz^{2-m}).$$

This action is induced by the inclusion $S^1 \subset U(k)$ followed by the action of $U(k)$

$$(B, C) \cdot g = (Bg, g^{-1}C(\det g)^{-s}),$$

provided $m = ks + 2$, $s \in \mathbb{Z}$. If L denotes the top exterior power of the tautological bundle of $Gr_k(\mathbb{C}^n)$, then we get

$$\mu_{S^1}^{-1}(i)/S^1_r \cong \mu^{-1}(i)/U(k) \cong L^s \otimes T^*Gr_k(\mathbb{C}^n).$$

Thus, summing up we get the following result.

Theorem 13. *The total space of $L^s \otimes T^*Gr_k(\mathbb{C}^n)$, $s \in \mathbb{Z}$, $ks + 2 \geq 0$ carries a quaternionic Kähler structure with positive scalar curvature. Its Swann bundle is the total space of $\left(L \oplus L^{-1}\right)_0 \oplus L^s \otimes T^*Gr_k(\mathbb{C}^n)$, where the index 0 indicates that the image of the zero section is removed.* □

In the case of $k = 1$, $n = 2$ one obtains quaternionic Kähler structures on total spaces of $\mathcal{O}_{\mathbb{P}^1}(-s-2) = \mathcal{O}_{\mathbb{P}^1}(-m)$ for $m \geq 0$, which is certainly diffeomorphic to $\mathcal{O}_{\mathbb{P}^1}(m)$. These spaces were obtained by Galicki and Lawson [9] as open subsets of four-dimensional quaternionic Kähler orbifolds.

5. Kähler structure on N_0

In contrast to a hyperKähler manifold, almost complex structures of a quaternionic Kähler manifold N are defined only locally, i.e. we have a distinguished rank 3 subbundle $\mathcal{I} \subset End(TN)$ called a *structure bundle*, which locally admits a basis consisting of three almost complex structures with quaternionic relations. Since the metric induces an isomorphism $TN \cong T^*N$, one gets an embedding of $\mathcal{I}$ in Λ^2T^*N. Locally this is given by passing from an almost complex structure I to the associated 2-form $\omega_I(\cdot,\cdot) = g(\cdot, I\cdot)$. We will not distinguish between $\mathcal{I}$ and its image in Λ^2T^*N. An analogue of a momentum map can be defined in the quaternionic Kähler context, but now it will be a section of a structure bundle (see [8] for details).

Theorem 14. *Let N be a quaternionic Kähler manifold of positive scalar curvature. Suppose also that N admits a quaternionic Kähler action of S^1 with momentum section $\mu_N \in \Gamma(\mathcal{I})$. Then $N_0 = N \backslash \{\mu_N = 0\}$ is a Kähler manifold.*

Proof. Let M be the Swann bundle of N. Then M admits a hyperKähler action of S^1 [18]. Let $\mu = \mu_1 i + \mu_c j$: $M \to \operatorname{Im}\mathbb{H}$ be its momentum map. Since the function $\mu_c : M \to \mathbb{C}$ is I_1-holomorphic, the subvariety $M_c = \{m \in M : \mu_c(m) = 0\}$ has an induced Kähler structure. Further, $M_c^+ = \{m \in M_c : \mu_1(m) > 0\}$ is an open submanifold of M_c. The group S^1_r preserves I_1 and one may consider the Kähler reduction of M_c^+ with respect to a nonzero value of the momentum map: $M_c^+//S^1_r \cong M_c^+/\mathbb{C}^*_r$. It remains to observe that $M_c^+ = \mu^{-1}(i) \times \mathbb{R}_{>0} = P \times \mathbb{R}_{>0}$ and therefore $M_c^+/\mathbb{C}^*_r \cong P/S^1_r \cong N_0$. □

When a quaternionic Kähler manifold N admits an action of S^1, one can normalize the momentum section $\mu_N \in \Gamma(\mathcal{I})$ and consider it as an almost complex structure $\hat{I}$ over N_0. It turns out that $\hat{I}$ is *integrable* [2,17] and it is easy to see from the proof that it coincides with the complex structure implied by Theorem 14 (our proof of the above theorem itself represents an alternative proof of the integrability of $\hat{I}$ in case when N has positive scalar curvature). Although the complex structure $\hat{I}$ is a section of the structure bundle $\mathcal{I}$, the *Kähler* metric of N_0 must not coincide with the quaternionic Kähler one as we will see in what follows. Note also that $\mathcal{I}$ does not admit a section which defines an integrable complex structure on the whole manifold N (see [1] for extensive discussion of this phenomenon). Taking this into account, one may consider N_0 as "the largest" open submanifold of N where it is still possible to choose an integrable complex structure. Our next aim is to express the Kähler structure of N_0 similarly to the quaternionic Kähler one (see Section 2.5).

Recall that $N_0 \cong P/S^1_r = M_c^+/\mathbb{C}^*_r$. In order to get a metric and Kähler form on N_0 we have to express N_0 as a Kähler reduction, i.e. we have to fix a level set of momentum map and divide it by $S^1_r \subset \mathbb{C}^*_r$. In our case the momentum map of the S^1_r-action is nothing else but the hyperKähler potential ρ (restricted to M_c^+). Recall also that we have an isomorphism (16) between P and $Q = \rho^{-1}(-1/2) \cap M_c^+$. Further, one has $T_. M_c = \mathtt{span}(I_2K_0, I_3K_0)^\perp \subset T_. M$ and $T_. \rho^{-1}(-1/2) = \mathtt{span}(Y_0)^\perp$. It follows that $T_. Q = \mathtt{span}(I_2K_0, I_3K_0, Y_0)^\perp$ because Y_0 is perpendicular to both I_2K_0 and I_3K_0 (see (7)). In particular $Y_1 \in TQ$; this also follows from the fact that S^1_r preserves Q. Further, the Kähler reduction procedure implies that $T_. N_0$ is identified with $\mathtt{span}(Y_1)^\perp \subset T_. Q$ and the Kähler form and metric are obtained as a restriction of the corresponding tensors to $\mathtt{span}(Y_1)^\perp$. Remark that the quaternionic Kähler metric was obtained as the one induced on a different subbundle, namely on $\mathtt{span}(Y_1, Y_2, Y_3)^\perp \subset TQ$.

Let $\mathrm{u} \in T_pP$. Then we may decompose $\mathrm{u} = \mathrm{u}' + \psi(\mathrm{u})Y_1$, where u' is orthogonal to Y_1. Now denote by Π an orthogonal projector on $\mathtt{span}(Y_1)^\perp$ in TQ. Then for the Kähler metric $\hat{g}_{\mathrm{N}}$ we have:

$$\begin{aligned}\hat{g}_{\mathrm{N}}(\mathrm{u},\mathrm{v}) &= g\left(\Pi\, l_*\mathrm{u}, \Pi\, l_*\mathrm{v}\right)\\ &= g\left(\left(L_{\lambda(p)}\right)_* \mathrm{u}' + d\lambda(\mathrm{u})Y_0(\lambda(p)p), \left(L_{\lambda(p)}\right)_* \mathrm{v}' + d\lambda(\mathrm{v})Y_0(\lambda(p)p)\right)\\ &= \lambda^2 g\left(\mathrm{u} - \psi(\mathrm{u})Y_1 + d\lambda(\mathrm{u})Y_0,\ \mathrm{v} - \psi(\mathrm{v})Y_1 + d\lambda(\mathrm{v})Y_0\right)\\ &= \lambda^2\left(g(\mathrm{u},\mathrm{v}) - \psi(\mathrm{u})g(Y_1,\mathrm{v}) - \psi(\mathrm{v})g(Y_1,\mathrm{u}) + d\lambda(\mathrm{u})\,g(Y_0,\mathrm{v})\,d\lambda(\mathrm{v})\,g(Y_0,\mathrm{u})\right).\end{aligned}$$

As we already know $g(\mathrm{u},\mathrm{v}) = \left(\tilde{g} + v^{-1}\xi^2\right)(\mathrm{u},\mathrm{v})$. By the definition of ψ one has $g(Y_1,\mathrm{v}) = (4v + w)\psi(\mathrm{v})$. Further, $g(Y_0,\mathrm{v}) = g\left(-I_1K_r - 2vI_1K_0, \hat{\mathrm{v}} + \xi(\mathrm{v})K_0\right) = \iota_{K_r}\tilde{\omega}_1(\mathrm{v})$. Therefore we obtain $\hat{g}_{\mathrm{N}} = \lambda^2\left(\tilde{g} + v^{-1}\xi^2 - (4v+w)\psi^2 + d\lambda \odot \iota_{K_r}\tilde{\omega}_1\right)$. Since $d\lambda = (4v+w)^{-3/2}\,\iota_{K_r}\tilde{\omega}_1$ we may finally write

$$\hat{g}_{\mathrm{N}} = \frac{1}{4v+w}\,\tilde{g} + \frac{1}{v(4v+w)}\,\xi^2 - \psi^2 + \frac{1}{(4v+w)^{5/2}}\,\left(\iota_{K_r}\tilde{\omega}_1\right)^2.$$

The Kähler form $\hat{\omega}_N$ may be obtained in a similar manner. Indeed,

$$\begin{aligned}\hat{\omega}_N(\mathrm{u},\mathrm{v}) &= \omega_1\left(\Pi\, l_*\mathrm{u}, \Pi\, l_*\mathrm{v}\right)\\ &= \omega_1\left(\left(L_{\lambda(p)}\right)_* \mathrm{u}' + d\lambda(\mathrm{u})Y_0\left(l(p)\right), \left(L_{\lambda(p)}\right)_* \mathrm{v}' + d\lambda(\mathrm{v})Y_0\left(l(p)\right)\right)\\ &= \omega_1\left(\left(L_{\lambda(p)}\right)_* \mathrm{u}', \left(L_{\lambda(p)}\right)_* \mathrm{v}'\right)\\ &= \lambda^2\omega_1\left(\mathrm{u} - \psi(\mathrm{u})Y_1,\ \mathrm{v} - \psi(\mathrm{v})Y_1\right).\end{aligned}$$

Since $\omega_1(\mathrm{u},\mathrm{v}) = \tilde{\omega}_1(\mathrm{u},\mathrm{v})$ and $\omega_1(Y_1,\mathrm{u}) = g(\hat{K}_r + 2vK_0, I_1\hat{\mathrm{u}} + \xi(\mathrm{u})I_1K_0) = \iota_{K_r}\tilde{\omega}_1(\mathrm{u})$, we obtain the Kähler form as

$$\hat{\omega}_N = \frac{1}{4v+w}\left(\tilde{\omega}_1 - \psi \wedge \iota_{K_r}\tilde{\omega}_1\right).$$

Remark 15. As we have already remarked, we may regard the form ψ as a connection on the S^1_r-principal bundle $P \to N_0$. Let us compute its curvature. We have

$$F_\psi = -\frac{1}{(4v+w)^2}(4dv + dw) \wedge (2\xi + \iota_{K_r}\tilde{g}) + \frac{1}{4v+w}\left(2d\xi + d\,\iota_{K_r}\tilde{g}\right).$$

It follows from Eqs. (11) and (12) that

$$F_\psi = -\frac{2}{4v+w}\left(\iota_{K_r}\tilde{\omega}_1 \wedge \psi + \tilde{\omega}_1\right) = -2\hat{\omega}_N.$$

This observation provides an "intrinsic" interpretation of the Kähler form $\hat{\omega}_N$ in the following sense. Let N be a quaternionic Kähler manifold with positive scalar curvature and $F \to N$ be the principal $SO(3)$ bundle associated to the structure bundle $\mathcal{I}$. Observe that F is equipped with the natural connection induced by the Levi-Civita one.

Suppose also that N admits a quaternionic Kähler action of the circle and denote by μ_N its momentum section. As it was explained above one can think about μ_N on $N_0 = N \setminus \{\mu_N = 0\}$ as a section of F (restricted to N_0). This means that we get S^1-subbundle Q of F. The curvature of the induced connection ψ is a $-1/2$-multiple of the Kähler form $\hat{\omega}_N$.

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