Introduction to Geometric Quantization

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1 Some Notions from Symplectic Geometry

Recall that the phase space of a classical mechanical system (such as n particles in $\mathbb{R}^3$) can be modeled by a symplectic manifold (M, ω), where ω is a closed and non-degenerate two-form. Observables of the system are given by functions on M.

Classical mechanics	
Phase space	Symplectic manifold (M, ω)
Observables	Functions

Definition 1.1. To each observable $f \in C^\infty(M)$, we canonically associate a *Hamiltonian vector field* X_f such that $\omega(X_f, -) = -\mathrm{d}f$ (the sign is convention-dependent). For $f, g \in C^\infty(M)$, we define the *Poisson bracket* $\{f, g\}$ by $\{f, g\} := \omega(X_f, X_g)$.

Remark 1.2. Note that $\{f, g\} = -\omega(X_g, X_f) = \mathrm{d}g(X_f) = X_f(g) = -X_g(f)$.

With some small computations which are carried out in full in my notes, one can verify:

Lemma 1.3. *$(C^\infty(M), \{-,-\})$ is a Lie algebra, and the map $(C^\infty(M), \{-,-\}) \to (\mathfrak{X}(M), [-,-])$ which sends $f \mapsto X_f$ is a Lie algebra homomorphism.*

Of course, there is a second algebraic structure induced by point-wise multiplication, which turns $C^\infty(M)$ into a commutative and associative algebra. The two algebraic structures are compatible in the sense that the Poisson bracket is a derivation of the algebra $(C^\infty(M), \cdot)$. These compatible algebraic structures turn $C^\infty(M)$ into a commutative *Poisson algebra.*

2 Prequantization

2.1 Basic Idea of Prequantization

One of the main goals of modern physics is to *quantize* classical systems. Thus, we are faced with the problem of how to "quantize" a symplectic manifold. Roughly, this should mean the following: The manifold is replaced by a (quantum) *Hilbert space* of states $\mathcal{H}$, and observables are promoted to (symmetric) *operators*, which are supposed to act on (a dense subspace of) $\mathcal{H}$. This map should ideally be compatible with all the algebraic structures. In particular, one requires that the map $f \mapsto \hat{f}$ is a Lie algebra representation:

$\widehat{\{f,g\}} = \frac{i}{\hbar}[\hat{f},\hat{g}]$. We also want $\hat{1} = \mathbb{1}$ and more generally constant functions should map to the corresponding multiplication operators.

Ideally, we would even like to have compatibility with the full Poisson structure of $C^\infty(M)$. This would mean that composite observables $h = f \cdot g$ get mapped to composite operators $\hat{h} = \hat{f} \cdot \hat{g}$. However, this leads to all kinds of problems (e.g. ordering ambiguities) and we will not attempt this now.

It turns out that this is always possible (under one technical assumption). This procedure , which takes a symplectic manifold and its Lie algebra of functions as input, and spits out a Hilbert space which carries a Lie algebra representation of $C^\infty(M)$, is called *prequantization* in mathematics. The technical assumption that needs to be made is that the cohomology class of the symplectic form ω is *integral*. Some authors, including Hitchin, prefer to assume $\frac{1}{2\pi}[\omega]$ is integral, but this is just a convention. To understand this integrality condition, and why we need to impose it, we have to introduce sheaves and Čech cohomology.

2.2 Interlude: Sheaves and Čech Cohomology

I will be extremely brief, only giving definitions without extensive motivation or examples.

Definition 2.1. Given a topological space $(X, \mathcal{T})$, a presheaf of Abelian groups (sets, R-modules, etc.) is a pair $(\mathcal{F}, \rho)$ consisting of

(i) A family $\mathcal{F} = (\mathcal{F}(U))_{U \in \mathcal{T}}$ of Abelian groups ($\mathcal{F}(U)$ is called the *space of sections* of $\mathcal{F}$ over U);

(ii) For every $U, V \in \mathcal{T}$ such that $V \subset U$, a *restriction* homomorphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ and if $W \subset V \subset U$, then $\rho^U_W = \rho^U_V \circ \rho^V_W$.

A sheaf is a presheaf which satisfies some additional axioms:

Definition 2.2. A sheaf on X is a presheaf $\mathcal{F}$ on X such that for any open set $U \subset X$ and any family $U_i \subset U$ with $U = \bigcup U_i$, the following holds:

(i) **Locality**: If two sections $f, g \in \mathcal{F}(U)$ satisfy $\rho^U_{U_i} f =: f|_{U_i} = g|_{U_i}$ for every U_i, then $f = g$.

(ii) **Gluing**: Given a family $f_i \in \mathcal{F}(U_i)$ such that for $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for any i, j, then there exists a section $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

One of the most useful technical tools of sheaf theory is *Čech cohomology*, which we will now define.

Definition 2.3. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X indexed by I, and let $\mathcal{F}$ be a (pre)sheaf of Abelian groups on X. Then the *Čech cochain groups* with respect to the open covering $\mathcal{U}$ are defined as follows:

$$C^k(\mathcal{U}, \mathcal{F}) \coloneqq \prod_{(i_0,\ldots,i_k) \in I^{k+1}} \mathcal{F}(U_{i_0}) \cap \cdots \cap \mathcal{F}(U_{i_k})$$

Hence, a k-cochain is a family $(f_{i_0,\ldots,i_k})_{(i_0,\ldots,i_k) \in I^{k+1}}$ of sections: Each $f_{i_0,\ldots,i_k}$ is a section over $U_{i_0 \ldots i_k} \coloneqq U_{i_0} \cap \cdots \cap U_{i_k}$.

Now we want to turn this sequence of groups into a complex; this is done by introducing the Čech coboundary operator δ, defined through

$$(\delta f)_{i_0,\ldots,i_{k+1}} = \sum_{j=0}^{k} (-1)^j f_{i_0\ldots\hat{i}_j\ldots i_{k+1}} \qquad \text{on } U_{i_0\ldots i_k}$$

Here, $f \in C^k(\mathcal{U}, \mathcal{F})$ and the hat denotes omission of an index. It is easily checked that indeed $\delta^2 = 0$; we have constructed the Čech cochain complex. Now one defines the cohomology as usual:

Definition 2.4. We call $(f_{i_0\ldots i_k}) \in C^k(\mathcal{U}, \mathcal{F})$ a *cocycle* if $\delta f = 0$, and a *coboundary* if $f = \delta g$ for some $(g_{i_0\ldots i_{k-1}}) \in C^{k-1}(\mathcal{U}, \mathcal{F})$. The space of k-cocycles is denoted by $Z^k(\mathcal{U}, \mathcal{F})$ and the space of k-coboundaries by $B^k(\mathcal{U}, \mathcal{F})$. Then the *Čech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$ are defined as

$$\check{H}^k(\mathcal{U}; \mathcal{F}) := \frac{Z^k(\mathcal{U}, \mathcal{F})}{B^k(\mathcal{U}, \mathcal{F})}$$

A priori, the cohomology depends on our choice of covering. An independent theory is obtained only after taking a direct limit over (arbitrarily fine) open coverings of X. However, we can circumvent this discussion by considering *good* coverings:

Definition 2.5. Let X be a topological space. We call a covering $\mathcal{U} = \{U_i\}$ of X *good* if all finite intersections $U_{i_1\ldots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ are contractible.

Any smooth manifold admits a good covering; they can be constructed by for instance picking a Riemannian metric and using so-called geodesically convex neighborhoods around every point. Good coverings are useful because of the following fundamental theorem:

Theorem 2.6 (Leray)**.** *Let $\mathcal{U}$ be a good covering of X, and let $\mathcal{F}$ be any sheaf on X. Then $\check{H}^k(\mathcal{U}; \mathcal{F}) = \check{H}^k(X; \mathcal{F})$.*

Now we related the sheaf cohomology of the sheaf whose sections are locally constant real-valued functions to de Rham cohomology

Proposition 2.7. *There is a canonical isomorphism $H^2_{\mathrm{dR}}(M; \mathbb{R}) \cong \check{H}^2(M; \mathbb{R})$, where $\mathbb{R}$ denotes the sheaf of locally constant real functions.*

Proof. Let $\mathcal{U} = \{U_i\}$ be a good covering for M and $[\alpha] \in H^2_{\mathrm{dR}}(M; \mathbb{R})$. Since U_i is contractible, there exist one-forms β_i such that $\alpha|_{U_i} = \mathrm{d}\beta_i$. On U_{ij}, $\mathrm{d}(\beta_j - \beta_i) = 0$ hence $\beta_j - \beta_i = \mathrm{d}f_{ij}$ for some smooth function f_{ij}. On U_{ijk}, set $c_{ijk} = f_{jk} - f_{ik} + f_{ij}$. These functions are locally constant by construction, i.e. $(c_{ijk}) \in C^2(\mathcal{U}, \mathbb{R})$. Furthermore $(\delta c)_{ijkl} = 0$, hence it defines a Čech cohomology class. This class is independent of the choices we made. See the notes for the details.

This yields a homomorphism $H^2_{\mathrm{dR}}(M; \mathbb{R}) \to \check{H}^2(M; \mathbb{R})$. We construct its inverse to show that it is an isomorphism. Given $(c_{ijk}) \in Z^2(\mathcal{U}, \mathbb{R})$, we have

$$(\delta c)_{ijkl} = c_{jkl} - c_{ikl} + c_{ijl} - c_{ijk} = 0$$

Let $\{\varphi_j\}$ be a partition of unity subordinate to $\mathcal{U}$, i.e. $\operatorname{supp} \varphi_j \subset U_j$. Then $\varphi_k c_{ijk}$ vanishes outside U_i and can be regarded as a section of the sheaf of smooth functions over U_{jk} after extending it by zero. More generally, setting $f_{ij} := \sum_k \varphi_k c_{ijk} \in$ yields a 1-cochain which satisfies

$$\begin{aligned}(\delta f)_{ijk} = f_{jk} - f_{ik} + f_{ij} &= \sum_i \varphi_l(c_{jkl} - c_{ikl} + c_{ijl}) \\ &= \sum_l \varphi_l c_{ijk} = c_{ijk}\end{aligned}$$

where, in going to the second line, we used the cocycle condition. Note that (c_{ijk}) is *not* necessarily a coboundary when regarded as an element of $C^2(\mathcal{U};\mathbb{R})$, since (f_{ij}) does not generally consist of locally constant functions.

Since $0 = \mathrm{d}(c_{ijk}) = \mathrm{d}f_{jk} - \mathrm{d}f_{ik} + \mathrm{d}f_{ij} = (\delta(\mathrm{d}f))_{ijk}$, $(\mathrm{d}f_{ij})$ is a cocycle of the sheaf of smooth one-forms. Proceeding as before, one shows that $(\mathrm{d}f_{ij})$ is a coboundary, i.e. we have a 0-cochain of one-forms (β_i) such that $(\delta\beta)_{ij} = \beta_j - \beta_i = \mathrm{d}f_{ij}$. Then clearly $\mathrm{d}\beta_i = \mathrm{d}\beta_j$ on U_{ij} and by the gluing axiom for the sheaf of smooth 2-forms, there exists a global 2-form α such that $\alpha|_{U_i} = \beta_i$. This construction inverts our homomorphism. □

2.3 Prequantization Line Bundles

In $\check{H}^2(M;\mathbb{R})$, there is a subset of classes for which the cocycle (c_{ijk}) can be chosen to take integer values; the corresponding classes in $H^2_{\mathrm{dR}}(M;\mathbb{R})$ are called *integral*. Integral classes have a geometric interpretation in terms of curvature:

Proposition 2.8. *Let M be a smooth manifold and ω a closed two-form on M. Then $[\omega] \in H^2_{\mathrm{dR}}(M;\mathbb{R})$ is an integral class if and only if ω can be realized as the curvature of a connection on a $T^1 = \mathbb{R}/\mathbb{Z}$-principal bundle P.*

Proof. Pick a good cover $\mathcal{U} = \{U_i\}$ of M. Since ω is integral, the corresponding Čech cohomology class can be represented by an integer-valued cocycle (c_{ijk}). As above, we construct smooth functions f_{ij} on U_{ij} such that for every U_{ijk} we have

$$f_{jk} - f_{ik} + f_{ij} = c_{ijk} \equiv 0 \mod \mathbb{Z}$$

This means that they descend to a one-cocycle $(\tilde{f}_{ij})$ of the sheaf of smooth functions into T^1 by simply considering the functions modulo $\mathbb{Z}$. This data defines a T^1-principal bundle $\pi : P \to M$, viewing the $\tilde{f}_{ij}$ as transition functions: Glue the trivial bundles $U_i \times T^1$ together via $(x,t) \sim (x, t + \tilde{f}_{ij}(x))$ for every $x \in U_{ij}$. The consistency of this prescription is equivalent to the cocycle condition.

The forms β_i define local connection one-forms; the rule $\beta_j = \beta_i + \mathrm{d}f_{ij}$ ensures that the local forms glue together to a globally defined connection θ on P. The curvature $F^\theta \in \Omega^2(M)$ is characterized by $\pi^* F^\theta = \mathrm{d}\theta \implies F^\theta|_{U_i} = s_i^*\mathrm{d}\theta = \omega|_{U_i}$ for trivializations s_i on U_i. By locality, $F^\theta = \omega$.

Conversely, if F^θ is the curvature of a connection θ on a T^1-principal bundle P over M, pick local sections $s_i : U_i \to P$ and set $\beta_i = s_i^*\theta$. They satisfy $\mathrm{d}\beta_i = F^\theta|_{U_i}$. Since U_{ij} is contractible, $s_j - s_i = \tilde{f}_{ij}$ lifts to a function $f_{ij} : U_{ij} \to \mathbb{R}$ such that $s_j - s_i \equiv f_{ij} \mod \mathbb{Z}$. A computation using the properties of a connection shows $\beta_j - \beta_i = \mathrm{d}f_{ij}$. Thus, if we set

$$c_{ijk} = f_{jk} - f_{ik} + f_{ij} \equiv 0 \mod \mathbb{Z}$$

we see that F^θ corresponds to the integral class $[(c)_{ijk}] \in \check{H}^2(M;\mathbb{R})$. □

Remark 2.9.

(i) Regarding uniqueness, the space of T^1-principal bundles with fixed curvature ω is $H^1(M, T^1) \cong \mathrm{Hom}(H_1(M, \mathbb{Z}), T^1) \cong \mathrm{Hom}(\pi_1(M), T^1$. In particular, if M is simply connected then P is unique.

(ii) Since the standard symplectic structure on a cotangent bundle is exact, its class is trivially integral.

We can henceforth think of an integral symplectic form as the curvature of a T^1-principal bundle P.

Representations of T^1 yield *associated bundles* of P. Since T^1 is Abelian, the only irreducible representations are one-dimensional, of the form $\rho_k : T^1 \to S^1 \subset \mathbb{C}^*$, $t \mapsto e^{2\pi ikt}$. These are just k-fold tensor powers of the fundamental representation ρ_1 (or its dual, for negative k). To each of them, They give rise to complex line bundles

$$L_k = P \times_{\rho_k} \mathbb{C}$$

where $P \times_{\rho_k} \mathbb{C}$ is defined as $P \times \mathbb{C}/\sim$, with $(p, z) \sim (p', z')$ when there exists some $t \in T^1$ such that $(p - t, e^{2\pi ikt}z) = (p', z')$. Note that $L_k \cong L_1^{\otimes k}$. These bundles have natural Hermitian metrics: Let u be a local section of P, and let $s_1 = [(u, f_1)]$ and $s_2 = [(u, f_2)]$ be local sections of L_k. Then we define $\langle s_1, s_2\rangle = f_1\bar{f}_2$; this is well-defined since $f_1\bar{f}_2 = e^{2\pi ikt}f_1\overline{e^{2\pi ikt}f_2}$.

Proposition 2.10. *A principal T^1-connection on P induces a Hermitian connection ∇ on $(L_k, \langle -, -\rangle)$ for every $k \in \mathbb{Z}$.*

Proof. We give a local description: Let $u : U \to P$ and $s = [(u, f)]$ be local sections of P and L_k, where f is a complex function. We set

$$\nabla_X s = [(u, X(f) + 2\pi iku^*\theta(X)f)] =: [(u, \nabla^u_X f)]$$

This seems to depend on our choice of u, but in fact it does not.

We check that ∇ is Hermitian. Let s_1, s_2 be local sections and f_1, f_2 the corresponding complex functions. Then

$$\begin{aligned} X(\langle s_1, s_2\rangle) &= X(f_1\bar{f}_2) = X(f_1)\bar{f}_2 + f_1\overline{X(f_2)} \\ &= X(f_1)\bar{f}_2 + 2\pi iku^*\theta(X)f_1\bar{f}_2 + f_1\overline{Xf_2)} + f_1\overline{2\pi iku^*\theta(X)f_2} \\ &= \nabla^u_X(f_1)\bar{f}_2 + f_1\overline{\nabla^u_X f_2} \\ &= \langle\nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle \end{aligned}$$

which is to say that ∇ is Hermitian. □

Remark 2.11. The curvature of the induced connections is given by

$$F^\nabla(X, Y)f = [\nabla^u_X, \nabla^u_Y]f - \nabla^u_{[X,Y]}f$$

A short computation shows that

$$F^\nabla(X, Y)f = 2\pi iku^*(\mathrm{d}\theta)(X, Y)f = 2\pi ikF^\theta$$

In particular,

$$c_1(L_k) = -k[\omega]$$

where the sign is convention-dependent. In particular, $c_1(L_{-1}) = [\omega]$ and this is the line bundle L that Hitchin discusses in his paper.

2.4 The Prequantization Map

Now we are in a position to define a Lie algebra representation of $C^\infty(M)$ on a (family of) Hilbert space(s) naturally associated to the symplectic structure. This "prequantization map" is the final main result of this lecture. To each L_k, we can associate its space of smooth sections with compact support, $\Gamma_c(L_k)$. There is a natural L^2 inner product, induced by the Hermitian structure and the symplectic volume form on M: For $s_1, s_2 \in \Gamma_c(L_k)$, set

$$(s_1, s_2) = \int_M \langle s_1, s_2 \rangle \frac{\omega^n}{n!}$$

$\Gamma_c(L_k)$ is of course a complex vector space, but it is not yet a Hilbert space since it may fail to be complete. To remedy this, we take the closure with respect to the L^2-inner product and set $\mathcal{H}_k := \overline{\Gamma_c(L_k)}^{L^2}$.

On the dense subset $\Gamma_c(L_k)$ we can define differential operators and in particular we have a notion of *first order* differential operators [1], which are of the form $\nabla_X + A$ for a vector field X an endomorphism field A. We define the commutator $[D_1, D_2]_\hbar s = \frac{i}{\hbar}(D_1 D_2 s - D_2 D_1 s)$. Thanks to the curvature formula $F^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, the space $\mathcal{D}_1(\Gamma_c(L_k))$ of (at most) first-order differential operators forms a Lie (sub)algebra.

Proposition 2.12. *Let (M, ω) be a symplectic manifold with integral symplectic form and let $(L_k, \langle -,- \rangle, \nabla)$ be as above. Then*

$$f \longmapsto \hat{f} := -i\hbar(\nabla_{X_f} - 2\pi i k f) = -i\hbar\nabla_{X_f} - 2\pi\hbar k f$$

defines a Lie algebra homomorphism $(C^\infty(M), \{-,-\}) \to (\mathcal{D}_1(\Gamma_c(L_k)), [-,-])$. Moreover, these densely defined operators are symmetric: $(\hat{f}s_1, s_2) = (s_1, \hat{f}s_2)$ for every $s_1, s_2 \in \Gamma_c(L_k)$.

Proof. To establish the homomorphism property, we simply compute

$$\begin{aligned}
\frac{i}{\hbar}[\hat{f}, \hat{g}] &= -i\hbar[\nabla_{X_f}, \nabla_{X_g}] - 2\pi k\hbar(X_f(g) - X_g(f)) \\
&= -i\hbar(F^\nabla(X_f, X_g) + \nabla_{[X,Y]}) - 4\pi k\hbar\{f,g\} \\
&= 2\pi k\hbar\omega(X_f, X_g) - i\hbar\nabla_{X_{\{f,g\}}} - 4\pi k\hbar\{f,g\} = -i\hbar\nabla_{X_{\{f,g\}}} - 2\pi k\hbar\{f,g\} \\
&= \widehat{\{f,g\}}
\end{aligned}$$

Now we check symmetry. Since ∇ is compatible with $\langle -,- \rangle$, we have

$$\begin{aligned}
((-i\hbar\nabla_{X_f} - 2\pi\hbar k f)s_1, s_2) &= \int_M \langle (-i\hbar\nabla_{X_f} - 2\pi\hbar k f)s_1, s_2 \rangle \frac{\omega^n}{n!} \\
&= \int_M \Big(\langle s_1, (-i\hbar\nabla_{X_f} - 2\pi\hbar k f)s_2 \rangle - i\hbar X_f\langle s_1, s_2 \rangle \Big) \frac{\omega^n}{n!}
\end{aligned}$$

Hence, it suffices to show that

$$\int_M X_f\langle s_1, s_2 \rangle \omega^n = 0$$

$L_{X_f}\omega^n = 0$, hence the integrand equals $L_{X_f}(\langle s_1, s_2 \rangle \omega^n) = d\iota_{X_f}(\langle s_1, s_2 \rangle \omega^n)$. But both sections have compact support, hence the integrand does too and we may invoke Stokes' theorem to conclude that the integral vanishes. □

[1] Given a differential operator P between vector bundles E and F, let $e \in \Gamma(E)$ be arbitrary and choose $f \in C^\infty(M)$ such that $f(p) = 0$ and $D_p f = \xi$ for some $\xi \in T_p^* M$. We say that P is of order at most k if $(Pf^{k+1}e)_p = 0$ for any e.

Remark 2.13. In order to have the desired property $\hat{1} = \mathbb{1}$, we must set $\hbar = -(2\pi k)^{-1}$. In particular, for Hitchin's line bundle L_{-1} we have $\hbar = 1/2\pi$, which seems rather reasonable from a physical perspective.

Example 2.14. To get a feeling for what the result of prequantization looks like, we carry the procedure out in the case where M is the cotangent bundle $\pi : T^*N \to N$ of a smooth manifold, equipped with its standard symplectic structure $\omega = -\mathrm{d}\lambda$, where λ is the tautological one-form. Since ω is exact, the corresponding principal T^1-bundle $p : P \to M$ is trivial. The curvature of $\theta = \mathrm{d}t - p^*\lambda$ satisfies $F^\theta = -\mathrm{d}\lambda = \omega$.

Using the canonical section $u : M \to M \times T^1$ which sends $x \mapsto (x, [0])$, we induce connections on the L_k. Since $u^*\mathrm{d}t = 0$, we have $u^*\theta = -\lambda$ and

$$\nabla_X s = [(u, X(f) + 2\pi i k \lambda(X) f)]$$

In local coordinates $q^j = x^j \circ \pi$ and $p_j(\xi) = \xi(\partial_{x^j})$ associated to local coordinates $\{x^j\}$ on N, $\omega = \sum \mathrm{d}q^j \wedge \mathrm{d}p_j$. Thus $X_{q^j} = -\partial_{p_j}$ and $X_{p_j} = \partial_{q^j}$ and hence

$$\nabla_{X_{q^j}} s = [(u, -\partial_{p_j} f)] \qquad \nabla_{X_{p_j}} s = [(u, \partial_{q^j} f + 2\pi i k p_j f)]$$

We conclude:

$$\widehat{q^j}[(u, f)] = [(u, i\hbar\partial_{p_j} f - 2\pi k\hbar q^j f)] \qquad \widehat{p_j}[(u, f)] = [(u, -i\hbar\partial_{q^j} f)]$$

We conclude:

$$\widehat{q^j} = i\hbar\frac{\partial}{\partial p_j} - 2\pi k\hbar q^j$$
$$\widehat{p_j} = -i\hbar\frac{\partial}{\partial q^j}$$

The expressions for $\widehat{p_j}$ and the second term of $\widehat{q^j}$ are in accordance with the usual prescription of canonical quantization if we impose the above-mentioned equation $\hbar = -(2\pi k)^{-1}$:

$$\widehat{q^j} = i\hbar\frac{\partial}{\partial p_j} + \hbar q^j$$

However, the first term is completely off! There is one obvious remedy: Restricting to $\pi^* C^\infty(N) \subset C^\infty(M)$, or equivalently imposing that the corresponding section $s = [(u, f)]$ satisfies $\nabla_X s = 0$ for every $X \in \Gamma(T^V N)$, where $T^V N = \ker D\pi \subset TM$ is the vertical tangent bundle. This will lead to the notion of *polarization*.

It turns out that this failure of prequantization is a general feature: The restriction to certain types of functions and sections which are *parallel* along a (Lagrangian) distribution leads to the notion of what is called *quantization* without the pre-. This will be discussed in the following lecture.