# Twist geometry of the c-map

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#### **Abstract**

We discuss the geometry of the c-map from projective special Kähler to quaternionic Kähler manifolds using the twist construction to provide a global approach to Hitchin's description. As found by Alexandrov et al. and Alekseevsky et al. this is related to the quaternionic flip of Haydys. We prove uniqueness statements for several steps of the construction. In particular, we show that given a hyperKähler manifold with a rotating symmetry, there is essentially only a one parameter degree of freedom in constructing a quaternionic Kähler manifold of the same dimension. We demonstrate how examples on group manifolds arise from this picture.

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2010 Mathematics Subject Classification: Primary 53C26; Secondary 57S25 53C80

Keywords: special Kähler, hyperKähler, quaternionic Kähler, twist.

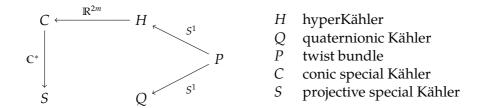
## 1 Introduction

The c-map was introduced in the physics literature by Cecotti, Ferrara and Girardello [9] and explicit local expressions for the metrics were provided by Ferrara and Sabharwal [15]. It is a remarkable construction that for certain Kähler manifolds S of dimension 2n creates a quaternionic Kähler manifold Q of dimension 4n+4. The manifolds Q are Einstein with negative scalar curvature and have holonomy contained in the group Sp(n+1)Sp(1). The map is derived from a duality of moduli spaces for type IIA and IIB string theories which takes a product  $S \times Q' \times \mathbb{C}H(1)$  to a product  $S' \times Q \times \mathbb{C}H(1)$ , with Q the c-map of S and S are S and S an

Historically the c-map construction has had particular importance for the study of homogeneous quaternionic Kähler manifolds. In [9], Alekseevsky's method [1] for classifying completely solvable Lie groups admitting a quaternionic Kähler metric was used to elucidate the geometric properties of the c-map for homogeneous spaces. Later the c-map was used by de Wit and Van Proeyen [13] to show that the above homogeneous classification was missing one family of spaces, a derivation of the correct result using Alekseevsky's construction was then provided by Cortés [10]. The significance of these spaces is that they include the only known examples of homogeneous quaternionic Kähler manifolds that are not symmetric.

Given the importance of these homogeneous results, it is surprising that the first mathematical description of the c-map was published by Hitchin [20] in 2009, building on the description of the relevant Kähler geometries in Freed [16]. As was well-known, an intermediate step is the construction of a hyperKähler manifold H, which is the rigid c-map of a cone C on S. It then became apparent that the passage from the hyperKähler manifold H to the quaternionic Kähler Q, was a pseudo-Riemannian version of a much more general construction of Haydys [18]. This has now been dubbed the hK/qK correspondence and has been studied both in the physics literature, for example [4], and mathematically, for example [22, 21]. In the particular context of the c-map, Cortés et al. [2, 3, 11] have shown that this correspondence reproduces the explicit local expressions for the c-map and its one-loop deformation, first seen from a twistor viewpoint by Alexandrov et al. [4], and used it to construct new examples of complete inhomogeneous quaternionic Kähler manifolds.

Given that the c-map is a manifestation of a general string theory duality, it is useful to put the above constructions into a wider context. In [28, 29] a twist construction was introduced as a geometric interpretation of a broad class of T-duality constructions. It was successfully exploited to produce geometries with torsion and particular examples of hypercomplex structures. However, it was not apparent how it could produce Riemannian metrics of special holonomy. The central purpose of this paper is to show how to adapt the twist construction so that hyperKähler manifolds with



**Figure 1.** Spaces in the c-map

a rotating circle symmetry may be used to produce quaternionic Kähler manifolds. The method to do this involves considering a combination of a conformal change in the hyperKähler metric together with a different conformal scaling along quaternionic directions associated to the symmetry. We prove that there is essentially only one degree of freedom if the result is to be quaternionic Kähler. It immediately follows that the descriptions of the hK/qK correspondence in Haydys [18], Hitchin [22] and Alekseevsky, Cortés and Mohaupt [2] agree and that the twist construction may be used to describe the c-map. We will therefore present the construction of the c-map in this context, with emphasis on proving uniqueness of the constructions involved. We take a uniformly global approach, with the aim of elucidating the geometric basis for the c-map and illustrating how information may be extracted from the twist construction. We consider metrics of arbitrary signature when appropriate. In [30] similar constructions are considered for tri-holomorphic actions on hyperKähler manifolds; in contrast to the single parameter in the present paper it turns that there is much more freedom in the way of obtaining twists that are again hyperKähler.

In outline the constructions are as follows, see Figure 1. A projective special Kähler manifold S may be defined via a complex reduction of a regular conic special Kähler manifold C. For supergravity the latter has complex Lorentzian signature. The rigid c-map defines an indefinite hyperKähler metric on  $H = T^*C$ . This carries a circle action and the aim is to produce a quaternionic Kähler manifold *Q* of the same dimension as *H*. Hitchin provides a local description, using the flat special Kähler connection to write  $T^*C = C \times \mathbb{R}^{2m}$ , putting  $Q = S \times CH(m+1)$  and adjusting the metric along quaternionic directions related to the symmetry. Patching of Hitchin's local construction is described by Cortés, Han and Mohaupt [11]. However, we will show that a global picture may be obtained by considering Haydys' quaternionic flip construction in indefinite signature, as also found by Alexandrov et al. [4] and Alekseevsky et al. [2], and that in addition the constructions may be directly described via the twist construction of [29]. The twist construction lifts the circle action on H to a principal circle bundle P and constructs Q as the circle quotient of P by the lifted action. The main conclusions are that the c-map for projective special Kähler

manifolds is obtained from the rigid c-map, via general constructions for hyperKähler manifolds with a rotating circle symmetry, and that these latter constructions are essentially unique.

Acknowledgements We thank Thomas Bruun Madsen, Nigel Hitchin, Vicente Cortés, Christoph Böhm, Paul Gauduchon and Henrik Pedersen for useful conversations, and Maxim Kontsevich for an instigating question. A preliminary version of this material was presented at the Srní Winter School, 2011, we thank the organisers for that opportunity. This work is partially supported by the Danish Council for Independent Research, Natural Sciences and by the Spanish Agency for Scientific and Technical Research (DGICT) and FEDER project MTM2010-15444.

# 2 The rigid c-map

The rigid c-map constructs hyperKähler manifolds of dimension 4n from so-called special Kähler manifolds of dimension 2n. It was described mathematically by Freed [16]. Let us demonstrate this construction using the language of structure bundles and show how the special Kähler condition arises naturally. For later use we will need the case of indefinite metrics.

**Definition 2.1.** A *special Kähler* manifold is a Kähler manifold  $(M, g, I, \omega)$ , with g a metric of signature (2p, 2q), together with a flat, torsion-free, symplectic connection  $\nabla$  satisfying  $d^{\nabla}I = 0$ , meaning

$$(\nabla_X I)Y = (\nabla_Y I)X,\tag{2.1}$$

for all  $X, Y \in TM$ .

Our conventions are such that  $\omega_I(\cdot,\cdot)=g(I\cdot,\cdot)$ . When M is a Kähler manifold, there are sophisticated constructions of hyperKähler metrics on neighbourhoods of the zero section of  $T^*M$  due to Feix [14] and Kaledin [23]. We will show how the extra conditions of a special Kähler geometry arise naturally from the desire to have a simply defined hyperKähler metric on the whole cotangent bundle.

#### 2.1 HyperKähler structures on flat vector spaces

Let V denote  $\mathbb{C}^{p,q}$  regarded as  $(\mathbb{R}^{2m}, \mathbf{G}, \mathbf{i})$ , m = p + q, so that the (indefinite) inner product is  $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{G} \mathbf{w}$  and  $\mathbf{i}$  is the complex structure  $\mathbf{i} \mathbf{G} = \mathbf{G} \mathbf{i}$ ,  $\mathbf{i}^T = -\mathbf{i}$ . Concretely, take  $\mathbf{G} = \operatorname{diag}(\operatorname{Id}_{2p}, -\operatorname{Id}_{2q})$  and  $\mathbf{i} = \operatorname{diag}(\mathbf{i}_2, \dots, \mathbf{i}_2)$ , where  $\mathbf{i}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The vector space  $H = V + V^*$  is isomorphic to  $\mathbb{H}^{p,q}$  and the flat (indefinite) hyperKähler metric may be described as follows.

Let  $\theta \in \Omega^1(V, \mathbb{R}^{2m})$  be the tautological one-form from the identification  $T_x V \cong V = \mathbb{R}^{2m}$ . In other words,

$$\theta = (dx_1, dy_1, dx_2, dy_2, \dots, dx_m, dy_m)^T,$$

where  $(x_1, y_1, \dots)$  are the standard coordinates on  $\mathbb{R}^{2m}$  and the flat metric on V is

$$g = \sum_{i=1}^{p} dx_i^2 + dy_i^2 - \sum_{j=p+1}^{m} dx_j^2 + dy_j^2.$$

The Kähler form on *V* is

$$\omega = \sum_{i=1}^{p} dx_i \wedge dy_i - \sum_{j=p+1}^{m} dx_j \wedge dy_j = -\frac{1}{2}\theta^T \wedge \mathbf{s}\theta,$$

with

$$s = Gi$$
.

Similarly on  $V^*$ , let  $\alpha \in \Omega^1(V^*, \mathbb{R}^{2m^*})$  be the tautological form:

$$\alpha = (du_1, dv_1, du_2, dv_2, \dots, du_m, dv_m),$$

with  $(u_1, v_1, \dots)$  dual coordinates to  $(x_1, y_1, \dots)$ . An induced Kähler form on  $V^*$  is

$$\omega^* = \sum_{i=1}^p du_i \wedge dv_i - \sum_{j=p+1}^m du_j \wedge dv_j = -\frac{1}{2}\alpha \wedge \mathbf{s}\alpha^T.$$

Now regarding  $H = V + V^*$  as  $T^*V$  there is a canonical symplectic form

$$\omega_J = \alpha \wedge \theta = \sum_{i=1}^m du_i \wedge dx_i + dv_i \wedge dy_i.$$

In addition, using i we may construct

$$\omega_K = -\alpha \wedge \mathbf{i}\theta = \sum_{i=1}^m du_i \wedge dy_i - dv_i \wedge dx_i.$$

Together with

$$\omega_I = \omega - \omega^* = \frac{1}{2}(\alpha \wedge \mathbf{s}\alpha^T - \theta^T \wedge \mathbf{s}\theta)$$

we obtain a flat hyperKähler structure with metric  $\sum_{i=1}^{m} \varepsilon_i (dx_i^2 + dy_i^2 + du_i^2 + dv_i^2)$ , where  $\varepsilon_i = \mathbf{G}_{ii} = \pm 1$ , and complex structures:

$$Idx_i = dy_i$$
,  $Idu_i = -dv_i$ ,  $Jdv_i = dy_i$ ,  $Jdx_i = -du_i$ ,  $Kdu_i = dy_i$ ,  $Kdv_i = -dx_i$ .

## 2.2 The cotangent bundle

Let us start with a general manifold M of dimension n. The bundle GL(M) of frames, i.e., linear isomorphisms  $u\colon\mathbb{R}^n\to T_aM$ , is a principal  $GL(n,\mathbb{R})$ -bundle with action  $(R_gu)(v)=(u\cdot g)(v)=u(gv)$ , for  $g\in GL(n,\mathbb{R})$  and  $v\in\mathbb{R}^n$ . It carries a canonical one-form  $\theta\in\Omega^1(GL(M),\mathbb{R}^n)$  given by  $\theta_u(X)=u^{-1}(\pi_*X)$ , where  $\pi\colon GL(M)\to M$  is the projection. This satisfies  $R_g^*\theta=g^{-1}\theta$ . A connection one-form  $\omega_\nabla\in\Omega^1(GL(M),\operatorname{End}(\mathbb{R}^n))$  is by definition a form such that  $R_g^*\omega_\nabla=g^{-1}\omega_\nabla g$  and  $\omega_\nabla(\xi^*)=\xi$ , where  $\xi^*$  is the vector field on GL(M) generated by the infinitesimal action of  $\xi\in\operatorname{End}(\mathbb{R}^n)$ , the Lie algebra of  $GL(n,\mathbb{R})$ . The connection is torsion-free if and only if

$$d\theta = -\omega_{\nabla} \wedge \theta$$
.

The cotangent bundle may be constructed as the associated bundle

$$GL(M) \times_{GL(n,\mathbb{R})} (\mathbb{R}^n)^* = T^*M,$$
  
 $R_g(u,v) = (u \cdot g, vg) \mapsto v \circ u^{-1}.$ 

Writing  $x: (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$  for the identity map, we have  $R_g^* x = xg$ . The form

$$\alpha = dx - x\omega_{\nabla}$$

on  $GL(M) \times (\mathbb{R}^n)^*$  agrees with dx on  $(\mathbb{R}^n)^*$  and is zero on vectors tangent to the  $GL(n,\mathbb{R})$ -action. It satisfies  $R_g^*\alpha = \alpha g$ . As the kernels of the forms  $\alpha$  and  $\theta$  are preserved by the group action, these kernels descend to provide a splitting

$$T(T^*M) = \mathcal{V} \oplus \mathcal{H},\tag{2.2}$$

with  $V = \ker \pi_*$  and  $\mathcal{H}_{[u,v]} \cong T_{\pi(u)}M$ .

The canonical symplectic form on  $T^*M$  is now

$$\omega_I = d(x\theta),$$

since  $x\theta$  is the tautological one-form. Expanding the right-hand side gives  $\omega_I = dx \wedge \theta + xd\theta$ . This gives

**Lemma 2.2.** 
$$\omega_I = \alpha \wedge \theta$$
 if and only if  $\omega_{\nabla}$  is torsion-free.

Now suppose that M carries an almost complex structure I and that n=2m. Consider the bundle  $GL(\mathbb{C},M)$  of  $\mathbb{C}$ -linear frames  $u:\mathbb{C}^m=\mathbb{R}^{2m}\to T_aM$ ,  $u\circ \mathbf{i}=I\circ u$ . If we identify the cotangent bundle with  $\Lambda^{1,0}M=GL(\mathbb{C},M)\times_{GL(m,\mathbb{C})}(\mathbb{C}^m)^*$ , then we obtain a canonical non-degenerate closed (2,0)-form. Its real part is the canonical symplectic structure  $\omega_I$  given above, where  $\theta$  is pull-backed to  $GL(\mathbb{C},M)$ . The imaginary part is

$$\omega_K = -d(x\mathbf{i}\theta).$$

This expands to  $\omega_K = -dx\mathbf{i} \wedge \theta - x\mathbf{i}d\theta = -(dx\mathbf{i} - x\mathbf{i}\omega_\nabla) \wedge \theta$ . This gives

**Lemma 2.3.** Suppose  $\omega_{\nabla}$  is torsion free. We have  $\omega_K = -\alpha \wedge \mathbf{i}\theta$  if and only if the complex structure I on M is integrable and the pair  $(\nabla, I)$  satisfies the special condition (2.1).

*Proof.* The expansion of  $\omega_K$  above shows that  $\omega_K = -\alpha \wedge \mathbf{i}\theta$  if and only if

$$(\mathbf{i}\omega_{\nabla} - \omega_{\nabla}\mathbf{i}) \wedge \theta = 0, \tag{2.3}$$

where  $\omega_{\nabla}$  is pulled-back to  $GL(\mathbb{C}, M)$ . Writing  $\omega_{\nabla} = \omega_{\mathbb{C}} + \eta_{\mathbb{A}}$ , where  $\omega_{\mathbb{C}} = \frac{1}{2}(\omega_{\nabla} - \mathbf{i}\omega_{\nabla}\mathbf{i})$  and  $\eta_{\mathbb{A}} = \frac{1}{2}(\omega_{\nabla} + \mathbf{i}\omega_{\nabla}\mathbf{i})$  are the complex and anti-complex parts of  $\omega_{\nabla}$ , we have that  $\omega_{\mathbb{C}}$  takes values in  $\mathfrak{gl}(m,\mathbb{C})$ . Equation (2.3) is equivalent to

$$\eta_{\rm A} \wedge \theta = 0. \tag{2.4}$$

As  $\omega_{\nabla}$  is torsion-free, we have  $d\theta = -\omega_{\mathbb{C}} \wedge \theta$  and so  $\omega_{\mathbb{C}}$  is a torsion-free  $GL(m,\mathbb{C})$  connection. Thus we have a torsion-free connection  $\nabla^{\mathbb{C}}$  such that  $\nabla^{\mathbb{C}}I = 0$ . By the Newlander-Nirenberg Theorem, such a connection exists if and only if I is integrable.

Now equation (2.4) is equivalent to  $0 = \mathbf{i}\eta_A \wedge \theta = -\eta_A \wedge \mathbf{i}\theta$ . This is zero on vertical vectors, and for any  $X, Y \in TM$  taking any lifts X', Y', (IX)', (IY)' to TU(M) we have

$$\mathbf{i}\theta(X') = \theta((IX)')$$
 and  $0 = \eta_A(X')\theta((IY)') - \eta_A(Y')\theta((IX)')$ ,

giving  $(\nabla^{C} - \nabla)_{X}(IY) = (\nabla^{C} - \nabla)_{Y}(IX)$ . As  $\nabla^{C}I = 0$  and the two connections  $\nabla^{C}$  and  $\nabla$  are torsion-free, this is equivalent to (2.1).

Let us now assume that (M,I) has a Hermitian metric g, possibly of indefinite signature. Then the structure group reduces to U(p,q). Let U(M) denote the bundle of unitary frames. Pulling  $\theta$  and  $\omega_{\nabla}$  back to U(M), these forms satisfy the identities given on GL(M), in particular  $\omega_{\nabla}(\xi^*) = \xi$  for each  $\xi \in \mathfrak{u}(p,q) \subset \mathfrak{o}(2p,2q) \subset \operatorname{End}(\mathbb{R}^n)$ .

Considering the flat model we see that the Hermitian form on *M* pullsback to

$$-\frac{1}{2}\theta^T\wedge\mathbf{s}\theta.$$

Indeed the identity  $\pi^*g = \theta^T \mathbf{G} \theta$  implies  $\pi^*\omega(X,Y) = \pi^*g(IX,Y) = \theta(IX)^T \mathbf{G} \theta(Y) = (\mathbf{i}\theta(X))^T \mathbf{G} \theta(Y) = -\frac{1}{2}(\theta^T \wedge \mathbf{s}\theta)(X,Y).$ 

On  $T^*M$ , it is natural to look for a hyperKähler metric whose Kähler form for I is

$$\omega_I = \frac{1}{2} (\alpha \wedge \mathbf{s} \alpha^T - \theta^T \wedge \mathbf{s} \theta). \tag{2.5}$$

Since such a hyperKähler metric pulls-back to the zero section as the given Hermitian structure on M, we see that M is necessarily Kähler. In particular, the connection one-form  $\omega_{\rm LC}$  for the Levi-Civita connection on M is a

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U(p,q)-connection and torsion-free, so

$$\omega_{\mathrm{LC}}^{T}\mathbf{G} = -\mathbf{G}\omega_{\mathrm{LC}}, \quad \omega_{\mathrm{LC}}\mathbf{i} = \mathbf{i}\omega_{\mathrm{LC}} \quad \text{and} \quad d\theta = -\omega_{\mathrm{LC}} \wedge \theta.$$

**Proposition 2.4.** The two-forms  $\omega_I$  of (2.5),  $\omega_J = \alpha \wedge \theta$ ,  $\omega_K = -\alpha \wedge \mathbf{i}\theta$  on  $T^*M$  give a hyperKähler structure compatible with the standard complex symplectic structure if and only if  $(M, I, g, \nabla)$  is special Kähler.

The passage from  $(M, I, g, \nabla)$  to the above hyperKähler structure on  $T^*M$  is known as the *rigid c-map*.

*Proof.* It remains to show that for a torsion-free connection  $\nabla$ , closure of  $\omega_I$  corresponds to  $\nabla$  being flat and symplectic. We compute

$$2d\omega_I = -d\alpha \wedge \mathbf{s}\alpha^T + \alpha \mathbf{s} \wedge d\alpha^T.$$

We have  $d\alpha = -dx \wedge \omega_{\nabla} - xd\omega_{\nabla} = -\alpha \wedge \omega_{\nabla} - x\Omega_{\nabla}$ , where  $\Omega_{\nabla} = d\omega_{\nabla} + \omega_{\nabla} \wedge \omega_{\nabla}$  is the curvature of  $\nabla$ . This gives

$$2d\omega_{I} = \alpha \wedge \omega_{\nabla} \mathbf{s} \wedge \alpha^{T} + x\Omega_{\nabla} \mathbf{s} \wedge \alpha^{T} + \alpha \wedge \mathbf{s}\omega_{\nabla}^{T} \wedge \alpha^{T} - \alpha \wedge \mathbf{s}\Omega_{\nabla}^{T} x^{T} = 2(\alpha \wedge \omega_{\nabla} \mathbf{s} \wedge \alpha^{T} + x\Omega_{\nabla} \mathbf{s} \wedge \alpha^{T}),$$
(2.6)

where we have used  $(\beta \wedge \gamma)^T = (-1)^{|\beta||\gamma|} \gamma^T \wedge \beta^T$ ,  $\mathbf{s}^T = -\mathbf{s}$  and noted that each summand  $\sigma$  of  $d\omega_I$  takes values in scalars, so satisfies  $\sigma^T = \sigma$ . Evaluating (2.6) on X,Y,Z with  $X,Y \in TU(M)$ , tangent to the principal frame bundle U(M), and  $Z \in T(\mathbb{R}^{2m^*})$ , we see that  $d\omega_I = 0$  implies  $\Omega_{\nabla} = 0$ , i.e.,  $\nabla$  is flat. Evaluation on  $X \in TU(M)$  and  $Y,Z \in T(\mathbb{R}^{2m^*})$ , gives that for  $\nabla$  flat  $d\omega_I = 0$  is equivalent to

$$\omega_{\nabla} \mathbf{s} + \mathbf{s} \omega_{\nabla}^{T} = 0. \tag{2.7}$$

This says that  $\omega_{\nabla}$  is symplectic, since

$$\mathfrak{sp}(2m,\mathbb{R})\cong\{A\in M_n(\mathbb{R}):A^T\mathbf{j}+\mathbf{j}A=0\},$$

for any **j** with  $j^2 = -1$ , and in particular for j = s.

For future use we note that  $\omega_{\nabla}$  satisfies

$$(\mathbf{G}\omega_{\nabla} + \omega_{\nabla}^{T}\mathbf{G}) \wedge \theta = 0, \tag{2.8}$$

which follows from (2.3) and (2.7).

Now that we have a flat symplectic connection  $\nabla$ , it is reasonable to write out the above structures in adapted coordinates. Suppose s is a flat

symplectic frame over on open subset  $M_0$  of M, i.e., a section  $M_0 \to Sp(M_0) \subset GL(M_0)$  of the bundle of symplectic frames. We have  $s^*\omega_{\nabla} = 0$ ,  $\sigma_a := (s^*\theta)_a = s(a)^{-1} \colon T_aM \to \mathbb{R}^{2m}$ . Thus writing  $\tilde{s}$  for the map

$$\tilde{s} := s \times \mathrm{Id} \colon M_0 \times (\mathbb{R}^{2m})^* \to GL(M_0) \times (\mathbb{R}^{2m})^*,$$
 (2.9)

gives

$$\tilde{s}^*(\omega_I) = dx \wedge \sigma, \quad \tilde{s}^*(\omega_K) = -dx \wedge h^{-1} \mathbf{i} h \, \sigma, \tilde{s}^*(\omega_I) = \frac{1}{2} (dx \wedge \mathbf{s} dx^T - \sigma^T \wedge \mathbf{s} \sigma),$$
 (2.10)

where  $s = R_h u$  for any local unitary frame u. Note that h is a function on  $M_0$ , so x only enters the above expressions through its differential. Thus translations  $T_v(x) = x + v$ , for each  $v \in (\mathbb{R}^{2m})^*$ , are triholomorphic isometries of the hyperKähler structure.

Remark 2.5. There is a natural circle action on the fibres given by  $x \mapsto xe^{\mathbf{i}t}$ . This rotates the pair  $\omega_I$  and  $\omega_K$ . However, the infinitesimal action on  $\alpha$  is  $\alpha \mapsto dx\mathbf{i} - x\mathbf{i}\omega_\nabla = dx\mathbf{i} - x\mathbf{i}(\omega_C + \eta_A) = \alpha\mathbf{i} + 2x\eta_A\mathbf{i}$ . It follows that under the infinitesimal action  $\omega_I \mapsto -2x\eta_A\mathbf{G} \wedge \alpha^T$ . Thus the action preserves  $\omega_I$  if and only if  $\omega_\nabla = \omega_{LC}$ , so M is a flat Kähler manifold. Thus in general the hyperKähler metric obtained from the rigid c-map is different from the hyperKähler metrics on cotangent bundles constructed by Feix [14] and Kaledin [23].

# 3 Conic special Kähler manifolds

For the local c-map the central starting object is a 'projective special Kähler manifold'. We will adopt the usual strategy [16, 11, 24] of defining these in terms of conic special Kähler manifolds.

**Definition 3.1.** A special Kähler manifold  $(M, g, I, \omega, \nabla)$  is *conic* if it admits a vector field X such that

- (i) g(X, X) is nowhere vanishing, and
- (ii)  $\nabla X = -I = \nabla^{LC} X$ .

We say that a conic structure is *periodic* or *quasi-regular* if *X* exponentiates to a circle action, *regular* if that circle action is free. We call *X* a *conic isometry* of *C*.

Note that if  $(C_1, X_1)$  and  $(C_2, X_2)$  conic special Kähler, then  $(C_1 \times C_2, X_1 + X_2)$  is too. In this way, by considering regular examples with different periods one gets examples that are (i) quasi-regular, but not regular, or (ii) non-periodic depending on whether the periods are rationally related or not.

Lemma 3.2. On a conic special Kähler manifold

(i) the vector field X is an isometry preserving I, and

(ii) the vector field IX is a homothety that preserves both I and the special connection  $\nabla$ .

*Proof.* The results for *X* follow purely from  $\nabla^{LC}X = -I$ : we have

$$\begin{split} (L_X g)(U,V) &= X g(U,V) - g([X,U],V) - g(U,[X,V]) \\ &= (g(\nabla_X^{LC} U,V) + g(U,\nabla_X^{LC} V)) - g(\nabla_X^{LC} U - \nabla_U^{LC} X,V) \\ &- g(U,\nabla_X^{LC} V - \nabla_V^{LC} X) \\ &= g(\nabla_U^{LC} X,V) + g(U,\nabla_V^{LC} X) = -g(IU,V) - g(U,IV) = 0 \end{split}$$

and

$$(L_X I)U = [X, IU] - I[X, U] = (\nabla_X^{LC} I)U - \nabla_{IU}^{LC} X + I\nabla_U^{LC} X$$
  
= 0 + U - U = 0.

For IX we have  $\nabla^{LC}(IX) = I\nabla^{LC}X = \text{Id}$  and simple modifications of the above arguments show that  $L_{IX}g = 2g$ ,  $L_{IX}I = 0$ . Its infinitesimal action on  $\nabla$  is given by

$$L_{IX}\nabla_{U}V - \nabla_{[IX,U]}V - \nabla_{U}[IX,V]$$

$$= \nabla_{IX}(\nabla_{U}V) - \nabla_{\nabla_{U}V}(IX) - \nabla_{[IX,U]}V - \nabla_{U}(\nabla_{IX}V - \nabla_{V}(IX)) \quad (3.1)$$

$$= R_{IX,U}^{\nabla}V - \nabla_{\nabla_{U}V}(IX) + \nabla_{U}(\nabla_{V}(IX)).$$

The first term vanishes since  $\nabla$  is flat. For the other terms we need to determine  $\nabla(IX)$ . Write  $\nabla = \nabla^{LC} + \eta$ . Putting  $\eta(A,B,C) = g(\eta_A B,C)$  the special Kähler conditions imply  $\eta$  is type  $\{3,0\}$  and totally symmetric. In particular,  $\nabla X = \nabla^{LC} X$  implies

$$X \perp \eta = 0$$
 and hence  $IX \perp \eta = 0$ . (3.2)

This gives  $\nabla(IX) = \nabla^{LC}(IX) = \text{Id}$ . The remaining part of (3.1) is thus equal to  $-\nabla_U V + \nabla_U (V) = 0$ , show that IX preserves  $\nabla$ .

Note that *X* itself does not preserve  $\nabla$ :

$$(L_X \nabla_U V) - \nabla_{[X,U]} V - \nabla_U [X,V] = -\nabla_{\nabla_U V} X + \nabla_U (\nabla_V X)$$
  
=  $I \nabla_U V - \nabla_U (IV) = -(\nabla_U I) V$ ,

which is only symmetric, not zero.

**Lemma 3.3.** The function  $\mu = \frac{1}{2}g(X,X)$  is both a moment map for the conic isometry X and a Kähler potential for g.

*Proof.* To be a moment map we need  $\mu$  satisfy  $d\mu = X \bot \omega$ . We have

$$(d\mu)(Y) = \frac{1}{2}Y(g(X,X)) = g(\nabla_Y^{LC}X, X) = -g(IY,X) = g(IX,Y)$$
  
=  $(X \cup \omega)(Y)$ .

It follows that

$$(dId\mu)(Y,Z) = (dI(X_{} \bot \omega))(Y,Z) = -Yg(X,Z) + Zg(X,Y) + g(X,[Y,Z])$$
  
=  $-g(\nabla_{Y}^{LC}X,Z) + g(\nabla_{Z}^{LC}X,Y) = 2\omega(Y,Z)$ 

so  $\omega = \frac{1}{2}dId\mu$  and  $\mu$  is a Kähler potential.

**Definition 3.4.** A *projective special Kähler* manifold is a Kähler quotient  $S = C/\!\!/_c X = \mu^{-1}(c)/X$  of conic special Kähler manifold C by a conic isometry X at some level  $c \in \mathbb{R}$ , together with the data necessary to reconstruct C up to equivalence.

We will not dwell on the extra data needed on S to specify C, as we will not need it at this stage. However, we do note that, for  $c \neq 0$  the projection  $\mu^{-1}(c) \to S$  is a (pseudo-) Riemannian submersion, and that the Kähler form  $\omega_S$  on S pulls-back to  $\mu^{-1}(c)$  as  $\iota^*\omega$ , where  $\iota\colon \mu^{-1}(c) \to C$  is the inclusion. The Kähler structure on S is of Hodge type, since the connection form  $\varphi = \iota^* X^{\flat}/g(X,X) = 2\iota^* X^{\flat}/c$  has curvature

$$d\varphi = 2\iota^* dX^{\flat} / c = -4\iota^* \omega / c. \tag{3.3}$$

Let  $\widetilde{X}$  be the horizontal lift of a conic isometry X to the cotangent bundle  $H=T^*C$ . This is the vector field in  $TT^*C=\mathcal{V}\oplus\mathcal{H}=\ker\pi_*\oplus\ker\alpha$ , see (2.2), defined by  $\alpha(\widetilde{X})=0$ ,  $\pi_*(\widetilde{X})=X$ . Equip H with the hyperKähler geometry of section 2.

**Proposition 3.5.** The horizontal lift  $\widetilde{X}$  is an isometry of H preserving  $\omega_I$  and with

$$L_{\widetilde{X}}\omega_I = \omega_K, \qquad L_{\widetilde{X}}\omega_K = -\omega_I.$$

*Proof.* Let  $\widetilde{X}$  also denote any choice of lift of  $\widetilde{X}$  to a vector field on  $U(C) \times (\mathbb{R}^{2m})^*$  or  $GL(C) \times (\mathbb{R}^{2m})^*$ . The essential point is to compute the quantity  $d\chi$ , where  $\chi_{(u,v)} = \theta_u(\widetilde{X}) = u^{-1}(X)$ .

On U(C), we claim that

$$d\chi = -\mathbf{i}\theta - \omega_{\nabla}\chi. \tag{3.4}$$

To see this, note that  $\chi \colon GL(C) \to \mathbb{R}^{2m}$  is the equivariant map representing the section X of TC. It follows that  $\nabla X$  is represented by the form  $d\chi + \omega_{\nabla} \chi \in \Omega^1(GL(C), \mathbb{R}^{2m})$ . But  $\nabla X = -I$ , so  $\nabla_A X$  is represented by  $u \mapsto$ 

 $u(\theta_u(-(IA)'))$ , where (IA)' is any vector on GL(C) projecting to  $A \in TC$ . On U(C), we have  $\theta_u(-(IA)') = u^{-1}(-IA) = -\mathbf{i}u^{-1}(A) = -\mathbf{i}\theta_u(A')$  and this gives the claimed formula (3.4).

We now find

$$L_{\widetilde{X}}\theta = d(\widetilde{X} \cup \theta) + \widetilde{X} \cup d\theta = d\chi - \omega_{\nabla}(\widetilde{X})\theta + \omega_{\nabla}\chi = -\mathbf{i}\theta - \omega_{\nabla}(\widetilde{X})\theta.$$

Since  $L_{\widetilde{X}}x = dx(\widetilde{X})$ , we get

$$L_{\widetilde{X}}\omega_I = d(L_{\widetilde{X}}(x\theta)) = d(\alpha(\widetilde{X})\theta - x\mathbf{i}\theta) = -d(x\mathbf{i}\theta) = \omega_K.$$

Similarly  $L_{\widetilde{X}}\omega_K=-\omega_J$ . On the other hand,

$$L_{\widetilde{X}}\omega_I = d(\widetilde{X} \lrcorner \omega_I) = -\frac{1}{2}d(\widetilde{X} \lrcorner \theta^T \wedge \mathbf{s}\theta) = \pi^*d(X \lrcorner \omega) = 0.$$

As the hyperKähler metric is specified by  $\omega_I$ ,  $\omega_J$  and  $\omega_K$ , it follows that  $\widetilde{X}$  is an isometry.  $\square$ 

For reference, flatness of  $\nabla$  implies

$$L_{\widetilde{X}}\alpha = \widetilde{X} \rfloor d\alpha = \widetilde{X} \rfloor (-\alpha \wedge \omega_{\nabla} - x\Omega_{\nabla}) = \alpha \omega_{\nabla}(\widetilde{X}).$$

# 4 Twisting hyperKähler manifolds by a rotating circle symmetry

Let (M, g, I, J, K) be a hyperKähler manifold. Suppose that X generates a circle action that is isometric, preserves I but rotates J and K. More precisely assume that

$$L_X g = 0$$
,  $L_X I = 0$  and  $L_X J = K$ . (4.1)

We write  $\omega_I(\cdot, \cdot) = g(I \cdot, \cdot)$ , etc., for the Kähler forms. and put

$$\alpha_0 = X^{\flat} = g(X, \cdot), \quad \alpha_A = (AX)^{\flat} = A\alpha_0 \quad \text{for } A = I, J, K.$$

The question we wish to address is when can this circle action be used to twist (M, g) to a quaternionic Kähler metric. As the twist construction [29] does not preserve closed forms, we expect to have to adjust our original structures before twisting. Therefore consider the metric

$$g_N = fg + h\left(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2\right)$$

for some unknown functions  $f,h \in C^{\infty}(M)$ . It will be convenient to allow this metric to be indefinite. Twisting the geometry by an unknown curvature form

$$F \in \Omega^2_{\mathbb{Z}}(M)$$

via a twisting function  $a \in C^{\infty}(M)$ , requires

$$da = -X \,\lrcorner\, F. \tag{4.2}$$

Let W be the twist of M with respect to X, F and a. Topologically  $W = P/\langle X' \rangle$ , where  $P \to M$  is a principal circle bundle with connection one-form  $\theta_P$  whose is curvature F. If the principal action on P is generated by Y and  $\hat{X}$  is the horizontal lift of X to P, then  $X' = \hat{X} + aY$ . The geometry on W is induced from that on M by pulling invariant tensors (metrics, complex structures, etc.) back to the horizontal distribution  $\mathscr{H} = \ker \theta_P$  and then pushing them down to the quotient W; we say that such tensor are  $\mathscr{H}$ -related and write  $\sim_{\mathscr{H}}$  for this relation. For this to work we need X' to be transverse to  $\mathscr{H}$ , which is equivalent to the non-vanishing of a.

Note that the original hyperKähler metric has fundamental four form

$$\Omega = \omega_I^2 + \omega_I^2 + \omega_K^2,$$

which is invariant under the action of X. We write  $\omega_I^N$  for the 2-form defined by  $(g_N, I)$ , etc., and  $\Omega_N$  for the corresponding four-form. We will look for twists W where  $\Omega_N$  is  $\mathscr{H}$ -related to a quaternionic Kähler four-form  $\Omega_W$ . Pointwise  $\Omega_N$  and  $\Omega_W$  agree when pulled back to  $\mathscr{H}$ , which is isomorphic to both  $T_xM$  and  $T_yW$ . In dimensions at least 12, when  $g_N$  is non-degenerate, the only condition now required for  $\Omega_W$  to define a quaternionic Kähler metric is that this four-form be closed [26]. The result we will prove is:

**Theorem 4.1.** Suppose X is a non-null vector field satisfying (4.1) on a hyper-Kähler manifold of dimension at least 8. Then the only twists of  $g_N$  that are quaternionic Kähler are given by the data

$$F = k(dX^{\flat} + \omega_I), \qquad a = k(||X||^2 - \mu + c),$$

with

$$f = \frac{B}{\mu - c}$$
 and  $h = -\frac{B}{(\mu - c)^2}$ ,

where  $\mu$  is a Kähler moment map for the action of X on (M,g,I) and c,k,B are constants.

We will start by concentrating on the case when  $\dim M$  is at least 12. However, first let us note that it is a simple consequence that this twist data is usable in all dimensions.

**Corollary 4.2.** In all dimensions, the data F, a, f and h of Theorem 4.1 define a twist that is quaternionic Kähler when  $g_N$  is non-degenerate.

*Proof.* Consider  $M \times \mathbb{H}^2$ , with circle action on  $\mathbb{H}^2$  given by  $q = z + jw \mapsto z + je^{-i\theta}w = e^{i\theta/2}qe^{-i\theta/2}$ . Then the twist W is a quaternionic submanifold of the twist of  $M \times \mathbb{H}^2$ , so totally geodesic and hence quaternionic Kähler by Gray [17].

The constants c, k, B in Theorem 4.1 have the following significance. Firstly B is just an overall scaling of the metric, so only adjusts the result by a homothety. The scalar k changes the curvature form, and thus affects the topology of the twist. Note for global constructions the curvature form F must have integral periods. Scaling of k can help to achieve this. Different choices of k then correspond to coverings of the twist manifold. Finally c is the only constant which affects the local properties of the quaternionic Kähler metric, but it also affects the global picture by changing the lift of X to the twist bundle.

It follows that the twist construction above agrees with the constructions of Haydys [18, via eqn. (18)], Hitchin [22] and Alekseevsky, Cortés and Mohaupt [2, via eqn. (2.4)] and that those constructions do not admit further variants of the type above. It also follows that this construction is inverted by the quaternionic flip of Haydys [18]. In particular, given an isometry Y of a quaternionic Kähler manifold Q, by [27] one may lift Y to a tri-holomorphic isometry  $Y_U$  if the associated bundle  $\mathcal{U}(Q)$ . Then the one-dimensional family of corresponding hyperKähler manifolds with rotating X are provided by the hyperKähler quotients of  $\mathcal{U}(Q)$  by  $Y_U$  at different levels.

## 4.1 Quaternionic Kähler twists in high dimensions

This section will be devoted to proving Theorem 4.1 in most dimensions. In particular, for dim  $M \geqslant 12$  we will verify that this twist data always leads to a quaternionic Kähler manifold and we will prove that for this is the only such data that suffices.

First, to determined  $d\Omega_W$  for a general twist, we need a little more notation, part of which is contained in the next result.

**Lemma 4.3.** The exterior derivatives of  $\alpha_I$ ,  $\alpha_I$ ,  $\alpha_K$  and  $\alpha_0 = X^{\flat}$  are

$$d\alpha_I = 0, \quad d\alpha_I = \omega_K, \quad d\alpha_K = -\omega_I$$
 (4.3)

and

$$d\alpha_0 = G - \omega_I, \tag{4.4}$$

for some  $G \in S^2E = \bigcap_{A=I,I,K} \Lambda_A^{1,1}$ .

*Proof.* The first three assertions follow from  $L_X\omega_I=0$ ,  $L_X\omega_J=\omega_K$  via Cartan's formula  $L_X\omega_A=X\lrcorner d\omega_A+d(X\lrcorner \omega_A)=0+d\alpha_A$ . For the final relation, start by noting that the Killing vector field X preserves both the Sp(n)Sp(1)-structure, where Sp(n)Sp(1) is the normaliser of Sp(n) in SO(4n), and the Kähler structure (g,I), which has structure group  $U(2n)_I$ . It follows that  $\nabla^{LC}X\in (\mathfrak{sp}(n)+\mathfrak{sp}(1))\cap\mathfrak{u}(2n)_I\subset \Lambda^2T^*M$ . Under the action of Sp(n)Sp(1), we have  $\Lambda^2T^*M=S^2E+S^2H+\Lambda_0^2ES^2H$ , where

 $E \cong \mathbb{C}^{2n}$  is the fundamental representation of Sp(n) and  $H \cong \mathbb{C}^2$  is that of Sp(1). Now the three-dimensional subspace  $\mathfrak{sp}(1) = S^2H \subset \Lambda^2T^*M$  is spanned by the Kähler forms  $\omega_I$ ,  $\omega_J$  and  $\omega_K$ . With respect to I we have  $\Lambda_I^{1,1}M = S^2E + \mathbb{R}\omega_I + \Lambda_0^2E\mathbb{R}_I$ , with  $S^2E + \mathbb{R}\omega_I = \mathfrak{sp}(n) + \mathfrak{u}(1)_I$ . So we conclude that  $\nabla^{\mathrm{LC}}X \in S^2E + \mathbb{R}\omega_I$ . To compute the coefficient of  $\omega_I$ , we note that this component is the  $\{2,0\}_I$  part of  $d\omega_I$ . From  $d\alpha_I = \omega_K$ , we find

$$g(KA, B) = \omega_K(A, B) = d(JX)^{\flat}(A, B) = g(\nabla_A^{LC}(JX), B) - g(\nabla_B^{LC}(JX), A)$$
  
=  $-g(\nabla_A^{LC}X, JB) + g(\nabla_B^{LC}X, JA) = -\frac{1}{2}dX^{\flat}(A, JB) + \frac{1}{2}dX^{\flat}(B, JA)$   
=  $-\frac{1}{2}(d\alpha_0(JA, B) + d\alpha_0(A, JB)).$ 

This implies that  $d\alpha_0^{\{2,0\}_J} = \frac{1}{2}(1-J)d\alpha_0 = -g(KJ\cdot,\cdot) = -\omega_I$ , as claimed.  $\Box$ 

On the quaternionic span of X, the forms  $\alpha_i$ , i = 0, I, J, K, give a volume element

$$vol_{\alpha} = \alpha_{0IJK} = \alpha_0 \wedge \alpha_I \wedge \alpha_I \wedge \alpha_K$$

and 2-forms

$$\omega_I^{\alpha} = \alpha_{0I} + \alpha_{IK}, \quad \omega_I^{\alpha} = \alpha_{0I} + \alpha_{KI} \quad \text{and} \quad \omega_K^{\alpha} = \alpha_{0K} + \alpha_{II}.$$
 (4.5)

With this notation the Hermitian forms of  $g_N$  are

$$\omega_I^N = f\omega_I + h\omega_I^{\alpha}$$
, etc

**Proposition 4.4.** The four-form  $\Omega_W$   $\mathcal{H}$ -related to  $\Omega_N$  satisfies

$$\begin{split} d\Omega_{W} \sim_{\mathcal{H}} d(f^{2}) \wedge \Omega + 6d(h^{2}) \wedge \operatorname{vol}_{\alpha} - 2fh\alpha_{I} \wedge \Omega \\ &+ 2(d(fh) - 3h^{2}\alpha_{I}) \wedge \sum_{A=I,J,K} \omega_{A}^{\alpha} \wedge \omega_{A} \\ &+ 2fH \wedge \sum_{A=I,J,K} \alpha_{A} \wedge \omega_{A} + 6hH \wedge \alpha_{IJK}, \end{split}$$

where

$$H = hG - \frac{1}{a}(f + h||X||^2)F.$$
 (4.6)

*Proof.* If  $\gamma \in \Omega^p(M)$  is X-invariant, then the  $\mathscr{H}$ -related form  $\gamma_W$  satisfies

$$d\gamma_W \sim_{\mathcal{H}} d_W \gamma := d\gamma - \frac{1}{a} F \wedge X \rfloor \gamma, \tag{4.7}$$

see [29]. Note that  $d_W$  is a derivation on the graded algebra  $(\Omega^*(M), \wedge)$  of all forms, so we have

$$d\Omega_W \sim_{\mathcal{H}} d_W \Omega_N = 2(d_W \omega_I^N \wedge \omega_I^N + d_W \omega_I^N \wedge \omega_I^N + d_W \omega_K^N \wedge \omega_K^N), \quad (4.8)$$

even though  $\omega_I$  and  $\omega_K$  are not invariant. We now compute

$$d_{W}\omega_{I}^{N} = d(f\omega_{I} + h\omega_{I}^{\alpha}) - \frac{1}{a}F \wedge X \rfloor (f\omega_{I} + h\omega_{I}^{\alpha})$$

$$= df \wedge \omega_{I} + dh \wedge \omega_{I}^{\alpha} + h(d\alpha_{0} \wedge \alpha_{I} + \alpha_{J} \wedge \omega_{J} + \alpha_{K} \wedge \omega_{K})$$

$$- \frac{1}{a}F \wedge (f\alpha_{I} + h\|X\|^{2}\alpha_{I})$$

$$= (df - h\alpha_{I}) \wedge \omega_{I} + dh \wedge \omega_{I}^{\alpha} + h(\alpha_{I} \wedge \omega_{I} + \alpha_{K} \wedge \omega_{K}) + H \wedge \alpha_{I}$$

with H as in (4.6). Similar computations lead to

$$d_{W}\begin{pmatrix} \omega_{I}^{N} \\ \omega_{I}^{N} \\ \omega_{K}^{N} \end{pmatrix} = \begin{pmatrix} df - h\alpha_{I} & h\alpha_{J} & h\alpha_{K} \\ -h\alpha_{J} & df - h\alpha_{I} & -h\alpha_{0} \\ -h\alpha_{K} & h\alpha_{0} & df - h\alpha_{I} \end{pmatrix} \wedge \begin{pmatrix} \omega_{I} \\ \omega_{J} \\ \omega_{K} \end{pmatrix} + dh \wedge \begin{pmatrix} \omega_{I}^{\alpha} \\ \omega_{K}^{\alpha} \\ \omega_{K}^{\alpha} \end{pmatrix} + H \wedge \begin{pmatrix} \alpha_{I} \\ \alpha_{J} \\ \alpha_{K} \end{pmatrix}.$$

$$(4.9)$$

Combining these formulae with (4.8) gives the desired result.

We may now confirm that Theorem 4.1 does indeed give quaternionic Kähler twists.

**Lemma 4.5.** The data of Theorem 4.1 gives  $d_W \Omega_N = 0$ .

*Proof.* Note that  $-\mu + c = -B/f$  and that  $f' = -f^2/B$ . Also we have F = kG, so equation (4.6) becomes

$$H = f'G - \frac{f + f'\|X\|^2}{\|X\|^2 - \mu + c}G$$

$$= \frac{f'(\|X\|^2 - B/f) - f - f'\|X\|^2}{\|X\|^2 - \mu + c}F = 0.$$

Now  $d(h^2) \wedge \operatorname{vol}_{\alpha} = 2hh'\alpha_I \wedge \operatorname{vol}_{\alpha} = 0$ ,  $d(f^2) = 2ff'\alpha_I = 2fh\alpha_I$  and  $d(fh) = -d(B^2/(\mu-c)^3) = 3B^2/(\mu-c)^4\alpha_I = 3h^2\alpha_I$ , so  $d_W\Omega_N$  is indeed zero.

The proof of the uniqueness part of Theorem 4.1 now proceeds by decomposing  $d_W\Omega_N=0$  in to type components corresponding to the splitting  $TM=\mathbb{H}X+\mathcal{B}$  with  $\mathcal{B}=(\mathbb{H}X)^{\perp}$ . We call elements of  $\langle \alpha_0,\alpha_I,\alpha_J,\alpha_K\rangle\subset T^*M$  'type (1,0)', and elements in the orthogonal complement  $T^*\mathcal{B}$  'type (0,1)'. This induces a type decomposition of each  $\Lambda^kT^*M$ . In particular,  $d_W\Omega_N\in\Omega^5(M)$  splits into five components. Note that  $\omega$  is type (2,0)+(0,2), and that  $\Omega$  is type (4,0)+(2,2)+(0,4).

Firstly the type (0,5) part gives

$$d(f^2)^{(0,1)} \wedge \Omega^{(0,4)} = 0.$$

The form  $\Omega^{(0,4)}$  is a quaternionic form on  $\mathscr{B}$ , and so the Lapage like map  $\Omega^{(0,4)}\wedge\cdot\colon \Lambda^k\mathscr{B}^*\to \Lambda^{k+4}\mathscr{B}^*$  is injective for each  $k\leqslant \frac{1}{2}\dim\mathscr{B}-2$  by Bonan [7]. We conclude that  $d(f^2)^{(0,1)}=0$ , so  $d(f^2)$  is type (1,0).

Now the type (1,4) part of  $d_W \Omega_N = 0$  is

$$(d(f^2) - 2fh\alpha_I) \wedge \Omega^{(0,4)} + 2fH^{(0,2)} \wedge \sum_{A=I,I,K} \alpha_A \wedge \omega_A^{(0,2)} = 0.$$
 (4.10)

This gives directly that  $d(f^2) = \sum_{A=I,J,K} f_A \alpha_A$  with no  $\alpha_0$ -component. Now considering the (0,2)-component of

$$0 = d^{2}(f^{2}) = \sum_{A=I,I,K} df_{A} \wedge \alpha_{A} + f_{J}\omega_{K} - f_{K}\omega_{J}$$

we see that  $f_I = 0 = f_K$  and so  $d(f^2) = f_I \alpha_I$  with  $df_I \wedge \alpha_I = 0$ . Since  $\mu$  is a moment map for the action of X with respect to  $\omega_I$ , we have  $\alpha_I = X \,\lrcorner\, \omega_I = d\mu$ . We conclude that  $f = f(\mu)$  and so  $d(f^2) = 2ff'\alpha_I$ , where ' denotes the derivative with respect to  $\mu$ .

Considering the coefficient of  $\alpha_J$  in (4.10) we have  $fH^{(0,2)} \wedge \omega_J^{(0,2)} = 0$ , implying that  $H^{(0,2)} = 0$ . Now the  $\alpha_I$ -component of (4.10) reads  $2f(f' - h) \wedge \Omega^{(0,4)} = 0$ , giving h = f'. In particular  $d(h^2) \wedge \operatorname{vol}_{\alpha} = 2f''f'\alpha_I \wedge \operatorname{vol}_{\alpha} = 0$ .

Using this information, we have

$$0 = d_{W}\Omega_{N} = ((f^{2})'' - 6f'^{2})\alpha_{I} \wedge \sum_{A=I,J,K} \omega_{A}^{\alpha} \wedge \omega_{A}$$

$$+ 2fH \wedge \sum_{A=I,J,K} \alpha_{A} \wedge \omega_{A} + 6f'H \wedge \alpha_{IJK},$$

$$(4.11)$$

with  $H^{(0,2)} = 0$ . Taking the (2, 3)-part of this equation gives

$$2fH^{(1,1)} \wedge (\alpha_I \wedge \omega_I^{(0,2)} + \alpha_J \wedge \omega_J^{(0,2)} + \alpha_K \wedge \omega_K^{(0,2)})$$
 (4.12)

Writing  $H^{(1,1)} = \sum_{i=0,I,J,K} \alpha_i \wedge H_i$ , with  $H_i$  of type (0,1), equation (4.12) becomes

$$H_0 \wedge \omega_A^{(0,2)} = 0$$
 and  $H_A \wedge \omega_B^{(0,2)} = H_B \wedge \omega_A^{(0,2)}$ ,

for A, B = I, J, K. As dim  $\mathscr{B} \geqslant 8$ , we conclude that  $H^{(1,1)} = 0$ .

We now have that H is type (2,0), so  $H \wedge \alpha_{IJK} = 0$  and the remaining terms in  $d_W \Omega_N$  are all of type (3,2). Write  $H = \sum_{0 \le i < j \le K} H_{ij} \alpha_i \wedge \alpha_j$ . The coefficient of  $\omega_I^{(0,2)}$  in (4.11) is

$$((f^2)'' - 6f'^2)\alpha_{IJK} + 2f(H_{JK}\alpha_{IJK} + H_{0J}\alpha_{0JI} + H_{0K}\alpha_{0KI}) = 0,$$

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so  $H_{0J}=0=H_{0K}$  and  $H_{JK}=(6f'^2-(f^2)'')/2f$ . On the other hand, the coefficient of  $\omega_J^{(0,2)}$  is

$$-((f^2)'' - 6f'^2)\alpha_{0II} + 2f(H_{IK}\alpha_{IKI} + H_{0I}\alpha_{0II}) = 0,$$

giving  $H_{IK}=0$  and  $H_{0I}=-H_{JK}$ . Finally, the coefficient of  $\omega_K^{(0,2)}$  leads to  $H_{II}=0$ . Thus

$$H = \frac{1}{2f}((f^2)'' - 6f'^2)\overline{\omega_I^{\alpha}},\tag{4.13}$$

where  $\overline{\omega_I^{\alpha}} := \alpha_{0I} - \alpha_{JK}$ . Using the definition (4.6) of H shows that

$$F = \frac{a}{f + f' \|X\|^2} \left( f'G - \frac{1}{2f} ((f^2)'' - 6f'^2) \overline{\omega_I^{\alpha}} \right). \tag{4.14}$$

Note that non-degeneracy of  $g_N$  ensures that  $(f + f' \|X\|^2) = (f + h \|X\|^2)$  and f are non-zero.

The two-form F in (4.14) needs to be closed and to satisfy (4.2). To evaluate these conditions, introduce the function R = f'/f. Then  $R' = f''/f - f'^2/f^2$ ,  $dR = R'\alpha_I$  and

$$F = \frac{a}{1 + R||X||^2} (RG + (R^2 - R')\overline{\omega_I^{\alpha}}). \tag{4.15}$$

Note that  $G = dX^{\flat} + \omega_I$  and that  $d\|X\|^2 = -X \rfloor dX^{\flat}$ . Equation (4.15) then implies that

Thus equation (4.2) gives  $d \log a = d \log((1 + R||X||^2)/f)$ , since the twist construction requires a to be non-zero. We conclude that

$$a = \frac{k_0}{f} (1 + R \|X\|^2)$$

for some non-zero constant  $k_0$ .

It remains to determine when F is closed. When  $k_0 \neq 0$ , we substitute

the expression for a into (4.14). This gives

$$\begin{split} 0 &= \frac{1}{k_0} dF = d \left( \frac{1}{f} (RG + (R^2 - R') \overline{\omega_I^{\alpha}}) \right) \\ &= -\frac{R}{f} \alpha_I \wedge (RG + (R^2 - R') \overline{\omega_I^{\alpha}}) + \frac{1}{f} (R' \alpha_I \wedge G - (R^2 - R')' \alpha_{IJK}) \\ &+ \frac{1}{f} (R^2 - R') ((G - \omega_I) \wedge \alpha_I - \omega_K \wedge \alpha_K - \alpha_J \wedge \omega_J) \\ &= \frac{1}{f} (R(R^2 - R') - (R^2 - R')') \alpha_{IJK} - \frac{1}{f} (R^2 - R') \sum_{A = I,I,K} \alpha_A \wedge \omega_A, \end{split}$$

since the coefficient of  $G \wedge \alpha_I$  sums to zero. Taking the (1,2) component, we see that  $R^2 - R' = 0$  and that this is the only equation that needs to be satisfied. But this gives either R = 0, so f' = 0, F = 0 and the twist is trivial, or  $R = 1/(-\mu + c)$  and  $f = B/(\mu - c)$  for constants c and d. This latter case then has  $d = f' = -B/(\mu - c)^2$  and  $d = d(||X||^2 - \mu + c)$ , with  $d = d(||X||^2 - \mu + c)$ , with  $d(||X||^2 - \mu + c)$  and we have the claimed result.

## 4.2 Uniqueness in dimension eight

Here we extend Theorem 4.1 to manifolds of dimension 8. By Corollary 4.2, we know that the given twist data does lead to a quaternionic Kähler metric. It thus remains to prove that this twist data is unique.

Recall [27] that an almost quaternion Hermitian structure  $(W, g_W, \Omega_W)$  in dimension 8 is quaternionic Kähler if and only if its fundamental fourform is closed and  $d(S^2H) \subset T^*W \wedge S^2H \subset \Omega^3(W)$ . The latter condition is the requirement that locally the Hermitian forms  $\omega_I^W, \omega_J^W, \omega_K^W$  associated to the compatible local almost complex structures generate a differential ideal. With respect to such a local triple of Hermitian forms the quaternionic Kähler condition in all dimensions at least 8 is equivalent to the existence of a local one-form  $\sigma = (\sigma_{AB}) \in \Omega^1(U,\mathfrak{so}(3))$  such that  $\omega^W = (\omega_I^W, \omega_I^W, \omega_K^W)^T$ 

$$d\omega^{W} = \sigma \wedge \omega^{W}. \tag{4.16}$$

Our problem with computing the derivatives of  $\omega_A^W$  is that they do not correspond to invariant forms on M, and so we can not directly use the formulae of [29]. To resolve this first note that it is sufficient to work away from the fixed point set of X: this is a totally geodesic submanifold of codimension at least two, so its complement is open and dense and the complement corresponds to an open dense subset of the twist.

Let  $\theta(t,q) = \theta_t(q)$ , for  $t \in \mathbb{R}$  and  $q \in M$ , be the one-parameter group generated by X. On M, we have  $\theta_t^* \omega_I = \cos(t)\omega_I + \sin(t)\omega_K$  and  $\theta_t^* \omega_K = -\sin(t)\omega_I + \cos(t)\omega_K$ .

In a neighbourhood of a point p outside of the fixed-point set, we may choose a slice  $\mathscr{S}_p$  transverse to the orbits of X and an open neighbourhood  $U_p$  of  $\{0\} \times \mathscr{S}_p$  in  $\mathbb{R} \times \mathscr{S}_p$  such that  $\theta \colon U_p \to M$  is an embedding with image an open set  $B_p$ . On  $B_p$ , we define invariant forms  $\omega_{\widetilde{I}}$ ,  $\omega_{\widetilde{K}}$  as the translates under  $\theta_t$  of  $\omega_I$  and  $\omega_K$  over  $\mathscr{S}_p$ . We have

$$\omega_{\tilde{I}} = u\omega_{\tilde{I}} + v\omega_{K}, \quad \omega_{\tilde{K}} = -v\omega_{\tilde{I}} + u\omega_{K}$$

for some functions  $u, v \in C^{\infty}(B_p)$  satisfying  $u^2 + v^2 = 1$ . Putting

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & v \\ 0 & -v & u \end{pmatrix}$$

and  $\tilde{\omega} = (\omega_I, \omega_{\tilde{I}}, \omega_{\tilde{K}})^T$ , we write  $\tilde{\omega}^N = \gamma \omega^N$  and define  $\tilde{\omega}^W$  by  $\tilde{\omega}^W \sim_{\mathscr{H}} \tilde{\omega}^N$ . Now  $d_W \tilde{\omega}^N = d\gamma \wedge \omega^N + \gamma d_W \omega^N$  and it follows that  $\tilde{\omega}^W$  satisfies the quaternionic Kähler condition (4.16)  $d\tilde{\omega}^W = \tilde{\sigma} \wedge \tilde{\omega}^W$  if and only if

$$d_W \omega^N = \sigma^N \wedge \omega^N \tag{4.17}$$

for a  $\sigma^N \in \Omega^1(B_p,\mathfrak{so}(3))$ . The  $\mathfrak{so}(3)$ -connections  $\sigma^N$  and  $\tilde{\sigma}^W$  satisfy the gauge type relation  $\tilde{\sigma}^W \sim_{\mathscr{H}} \gamma \sigma^N \gamma^{-1} - (d\gamma) \gamma^{-1}$ .

Now to solve (4.17), we use (4.9). As in the proof of Theorem 4.1, we consider the components of (4.16) according to their types with respect to the splitting of  $\Lambda^k T^*M$  induced by  $\langle \alpha_0, \dots, \alpha_K \rangle \subset T^*M$  and its orthogonal complement.

Write  $V = ||X||^{-2}$ , so that  $\omega^N = f\omega^\beta + (Vf + h)\omega^\alpha$ . Then, the (2,1)-component of (4.9) gives

$$(Vd^{(0,1)}f + d^{(0,1)}h) \wedge \omega_A^{\alpha} + H^{(1,1)} \wedge \alpha_A = \sum_{B=I,I,K} (\sigma_{AB}^N)^{(0,1)} \wedge (Vf + h)\omega_B^{\alpha}$$

for each A=I,J,K. As each non-zero element in the span of  $\omega_I^{\alpha}$ ,  $\omega_J^{\alpha}$  and  $\omega_K^{\alpha}$  is non-degenerate, it follows, that

$$H^{(1,1)} = 0. (4.18)$$

Now the fact that  $\sigma^N$  is skew-symmetric yields

$$(\sigma^N)^{(0,1)} = 0$$
 and  $Vd^{(0,1)}f + d^{(0,1)}h = 0.$  (4.19)

Now consider the (1, 2)-component of (4.17). We have immediately that  $H^{(0,2)}=H_I^{02}\wedge\omega_I^\beta+H_J^{02}\wedge\omega_J^\beta+H_K^{02}\wedge\omega_K^\beta$  for some  $H_A^{02}$ . Considering the coefficients of  $\omega_A^\beta$  and putting  $\delta=d^{(1,0)}f-h\alpha_I$ , the equations then give

$$\begin{pmatrix} \delta + H_I^{02} \alpha_I & h \alpha_J + H_J^{02} \alpha_I & h \alpha_K + H_K^{02} \alpha_I \\ -h \alpha_J + H_I^{02} \alpha_J & \delta + H_J^{02} \alpha_J & -h \alpha_0 + H_K^{02} \alpha_J \\ -h \alpha_K + H_I^{02} \alpha_K & h \alpha_0 + H_J^{02} \alpha_K & \delta + H_K^{02} \alpha_K \end{pmatrix} = f \sigma^N.$$

The off-diagonal terms and the skew-symmetry of  $\sigma^N$  imply that  $H^{(0,2)}=0$ . It then follows that  $\delta=0$ , i.e.,

$$d^{(1,0)}f = h\alpha_I, (4.20)$$

and that

$$f\sigma_{IJ}^{N} = h\alpha_{J}, \quad f\sigma_{IK}^{N} = h\alpha_{K} \quad \text{and} \quad f\sigma_{JK}^{N} = -h\alpha_{0}.$$
 (4.21)

Using  $H^{(0,2)}=0$  and  $(\sigma^N)^{(0,1)}=0$ , the (0,3)-component of (4.17) implies  $d^{(0,1)}f=0$ . Together with (4.20), we thus have  $df=h\alpha_I$ , so  $f=f(\mu)$  and h=f'.

Finally, all that remains of (4.17) is the (3,0)-component. This reduces to

$$h'\alpha_I \wedge \omega^{\alpha} + H \wedge \alpha = h\sigma^N \wedge \omega^{\alpha}.$$

Multiplying through by f, we use (4.21) to get

$$fH \wedge \alpha_I = (2h^2 - fh')\alpha_{IJK}, \quad fH \wedge \alpha_J = -(2h^2 - fh')\alpha_{0IJ},$$
  
 $fH \wedge \alpha_K = -(2h^2 - fh')\alpha_{0IK}.$ 

The first of these equations implies that  $fH=(2h^2-fh')\alpha_{JK}+\alpha_I\wedge(\lambda_0\alpha_0+\lambda_J\alpha_J+\lambda_K\alpha_K)$ , the second and third equations then give  $\lambda_0=2h^2-fh'$ ,  $\lambda_J=0=\lambda_K$ . We conclude that  $fH=-(2h^2-fh')(\alpha_{0I}-\alpha_{JK})$ . Substituting h=f', we see that H is given by equation (4.13) as in the higher-dimensional case. The arguments following (4.13), then provide the claimed uniqueness in dimension 8.

# 5 Geometry of the twist

Let us start by specialising the above results when the hyperKähler manifold is the image of the rigid c-map for *C* conic special Kähler. By Alekseevsky et al. [2] and the remarks after Corollary 4.2, we now know this gives the c-map and its one loop deformations via Figure 1. We wish to show how the properties listed by Ferrara and Sabharwal [15] may be obtained from the twist picture and describe some global aspects.

Let X be the conic isometry, and put  $H = T^*C$ . By Proposition 3.5, we have that the horizontal lift  $\tilde{X}$  is a hyperKähler isometry of H rotating J and K.

**Lemma 5.1.** The twist data for the symmetry  $\widetilde{X}$  of  $H = T^*C$  is

$$F = \frac{1}{2}k(\alpha \wedge \mathbf{s}\alpha^T + \theta^T \wedge \mathbf{s}\theta), \qquad a = k(\mu + c), \qquad \mu = \frac{1}{2}\|\widetilde{X}\|^2.$$

Moreover, F is exact.

*Proof.* We first compute  $d\widetilde{X}^{\flat}$ . In the notation of Proposition 3.5, we have  $\alpha_0 = \widetilde{X}^{\flat} = \widetilde{X} \,\lrcorner\, \theta^T \mathbf{G} \theta = \chi^T \mathbf{G} \theta$ . Thus

$$d\alpha_0 = (d\chi)^T \wedge \mathbf{G}\theta + \chi^T \mathbf{G}d\theta = (-\mathbf{i}\theta - \omega_{\nabla}\chi)^T \wedge \mathbf{G}\theta - \chi^T \mathbf{G}\omega_{\nabla} \wedge \theta$$
$$= \theta^T \wedge \mathbf{i}\mathbf{G}\theta - \chi^T (\omega_{\nabla}^T \mathbf{G} + \mathbf{G}\omega_{\nabla}) \wedge \theta$$
$$= \theta^T \wedge \mathbf{s}\theta$$
 (5.1)

by Proposition 3.5 and (2.8). As  $F = k(d\widetilde{X}^{\flat} + \omega_I)$ , the claimed expression for F follows from (2.5):  $\omega_I = \frac{1}{2}(\alpha \wedge \mathbf{s}\alpha^T - \theta^T \wedge \mathbf{s}\theta)$ . To check the formula for  $\mu$ , we wish to show that  $d\mu = \widetilde{X} \, \omega_I$ . We compute

$$d\|\widetilde{X}\|^{2} = d\{(\theta^{T}\mathbf{G}\theta)(\widetilde{X}, \widetilde{X})\} = d\{\chi^{T}\mathbf{G}\chi\}$$

$$= (-\mathbf{i}\theta - \omega_{\nabla}\chi)^{T}\mathbf{G}\chi + \chi^{T}\mathbf{G}(-\mathbf{i}\theta - \omega_{\nabla}\chi)$$

$$= \widetilde{X}_{\perp}(-\theta^{T} \wedge \mathbf{s}\theta) + \widetilde{X}_{\perp}\chi^{T}\{(\omega_{\nabla}^{T}\mathbf{G} + \mathbf{G}\omega_{\nabla}) \wedge \theta\}$$

$$= 2\widetilde{X}_{\perp}\omega_{I},$$

where we again have used (2.8). It follows that  $\mu = \frac{1}{2} \|\widetilde{X}\|^2$ , as claimed, and the expression for a is now obtained from  $a = k(\|\widetilde{X}\|^2 - \mu + c)$ .

To show that F is exact, it is enough to show  $\alpha \wedge \mathbf{s} \alpha^T$  is exact, since  $\theta^T \wedge \mathbf{s} \theta = d\alpha_0$ . But  $d(\alpha \mathbf{s} x^T) = -\alpha \wedge \omega_{\nabla} \mathbf{s} x^T - \alpha \mathbf{s} dx^T = \alpha \wedge \mathbf{s} (\omega_{\nabla}^T x^T - dx^T) = \alpha \wedge \mathbf{s} \alpha^T$ , by (2.7), giving  $F = d(\frac{1}{2}k(\alpha \mathbf{s} x^T + \alpha_0))$  as desired.

We now wish to describe the geometric properties of the quaternionic Kähler manifolds  $Q_c$  that we obtain from the above twist construction. We put  $Q = Q_0$ .

**Lemma 5.2.** For general c, the metric  $g_N$  is

$$g_N = \frac{B}{\mu + c} \left( g_{\perp} - \frac{\mu - c}{\mu + c} g_{HX} \right)$$

where B is an arbitrary constant,  $g_{\mathbb{H}X}$  is the restriction of g to  $\mathbb{H}X \subset TH$ , and  $g_{\perp}$  is the restriction to  $(\mathbb{H}X)^{\perp}$ .

In particular, for c = 0, the metric  $g_N$  is

$$g_N = \frac{B}{\mu}(g_{\perp} - g_{\mathbb{H}X}).$$

*Proof.* By Theorem 4.1,  $g_N = fg + hg_\alpha$ , where  $g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$ . As  $g = g_\perp + g_{\rm HX} = g_\perp + \frac{1}{2\mu}g_\alpha$ , the relations  $f = B/(\mu + c)$  and  $h = -B/(\mu + c)^2$  imply the claimed result.

Since  $g_{HX}/\mu$  is positive definite, it is natural to take

$$B = -1$$

when c=0. In all cases, we may take k=1, since F and a only occur as the combination  $\frac{1}{a}F$ , and so the k's cancel. In fact, this choice is good topologically.

**Proposition 5.3.** Let k=1 in Lemma 5.1. For c=0, the twist of  $H=T^*C$  is diffeomorphic to the product  $(H/\langle \widetilde{X} \rangle) \times S^1$  as diffeological spaces.

In particular, when  $H/\langle \widetilde{X} \rangle$  is smooth, the diffeomorphism is as manifolds. For X regular, this is the case when C has a global flat symplectic frame; such a frame always exists on some discrete cover of C.

*Proof.* From Lemma 5.1,  $F = d\beta$  with  $\beta = \frac{1}{2}(\alpha \mathbf{s} x^T + \alpha_0)$ . This has the property that  $\widetilde{X} \,\lrcorner\, \beta = \frac{1}{2}\alpha_0(\widetilde{X}) = \frac{1}{2}\|\widetilde{X}\|^2 = \mu$  which is the twisting function a. As F is exact, the twist bundle P is trivial,  $P = H \times S^1$ , with connection one-form  $\phi = \beta + d\tau$ , where  $\tau$  is the parameter on the  $S^1$ -factor. The twist W is the quotient  $P/\langle X' \rangle$ , where  $X' = \widetilde{X}^\phi + a \frac{\partial}{\partial \tau}$  with  $\widetilde{X}^\phi$  the  $\phi$ -horizontal lift of  $\widetilde{X}$  to P. This means that in the product structure  $P = H \times S^1$ , we have  $\widetilde{X}^\phi = \widetilde{X} + \lambda \frac{\partial}{\partial \tau}$  and  $0 = \phi(\widetilde{X}^\phi) = \beta(\widetilde{X}) + \lambda = a + \lambda$ . Thus we have  $X' = \widetilde{X}$  and  $W = (H/\langle \widetilde{X} \rangle) \times S^1$ .

Note that for general c, we get  $X' = \widetilde{X} - c \frac{\partial}{\partial \tau}$  above and the twist is  $(H \times S^1)/\langle \widetilde{X} - c \frac{\partial}{\partial \tau} \rangle$ .

Now in the general twist construction, if  $N \subset M$  is an X-invariant submanifold, it inherits natural twist data from M: writing  $\iota \colon N \to M$  for the inclusion, the circle bundle is  $P_N = \iota^* P$  with curvature  $\iota^* F$  preserved by X, and the twist function is simply  $\iota^* a$ , as  $d\iota^* a = \iota^* da = -\iota^* X \lrcorner F = -X \lrcorner \iota^* F$ . This implies that the lift X' of X to P its tangent to the submanifold  $P_N$ . In particular, if X' is regular, then the twist  $P_N/\langle X'\rangle$  of N is a submanifold of the twist  $P/\langle X'\rangle$  of M.

We first consider the image of the cone *C* under the twist.

**Proposition 5.4.** If X is regular, the image  $C_Q$  in the twist Q of the conic manifold C at c=0 is a Kähler product of the projective special Kähler manifold S and the quotient  $\mathbb{CH}(1)/\mathbb{Z}$  of a one-dimensional complex hyperbolic space.

*Proof.* Let  $\iota_C \colon C \to H$  be the inclusion, where C lies in  $H = T^*C$  as the zero-section. The twist data is  $\iota^*F = -\iota^*\omega$ ,  $\iota^*a = \iota^*\mu = s$  and

$$\iota^* g_N = -\frac{1}{\mu} g_{C,\perp} + \frac{1}{2\mu^2} (\alpha_0^2 + \alpha_I^2).$$

Let *S* be the Kähler quotient  $\mu^{-1}(s)/X$  of *C* at some regular value *s*. As *C* is conic with *X* regular, we have projections  $\pi_S \colon C \to S$  and  $\overline{\pi}_S \colon H \to S$ ;

furthermore,  $\pi_S^* g_S = g_{C,\perp}$  on  $\mu^{-1}(s)$ . Since  $L_{IX}g = 2g$ , we have  $L_{IX}(g/\mu) = 0$  and so  $\pi_S^* g_S = sg_{C,\perp}/\mu$  on all of C.

Suppose  $\gamma \in \Omega^*(S)$ . Then  $\overline{\pi}_S^* \gamma$  is X-invariant, and there is an  $\mathscr{H}$ -related differential form  $\gamma_Q$  on Q. We have  $d\gamma_Q \sim_{\mathscr{H}} \overline{\pi}_S^* d\gamma - \frac{1}{a} F \wedge X \,\lrcorner\, \pi_S^* \gamma = \overline{\pi}_S^* d\gamma$ . Thus the set

$$\Omega_S^* = \{ \gamma_O \in \Omega^*(Q) \mid \exists \gamma \in \Omega^*(S) \text{ such that } \gamma_O \sim_{\mathscr{H}} \overline{\pi}_S^* \gamma \}$$
 (5.2)

is a differential subalgebra and pulls back to a subalgebra of  $\Omega^*(C_Q)$ . Putting  $\tilde{\alpha}_0 \sim_{\mathscr{H}} \alpha_0/\mu$  and  $\tilde{\alpha}_I \sim_{\mathscr{H}} \alpha_I/\mu$ , we use (5.1) to compute

$$d\tilde{\alpha}_{0} \sim_{\mathcal{H}} -\frac{1}{\mu^{2}} d\mu \wedge \alpha_{0} + \frac{1}{\mu} d\alpha_{0} - \frac{1}{\mu^{2}} F \wedge \alpha_{0}(X)$$

$$= -\frac{1}{\mu^{2}} \alpha_{I} \wedge \alpha_{0} + \frac{1}{\mu} d\alpha_{0} - \frac{2}{\mu} (d\alpha_{0} + \omega_{I})$$

$$= \frac{1}{\mu^{2}} \alpha_{0} \wedge \alpha_{I} - \frac{1}{\mu} (\alpha \wedge \mathbf{s} \alpha^{T})$$
(5.3)

and

$$d\tilde{\alpha}_I \sim_{\mathcal{H}} d(1/\mu) \wedge \alpha_I = 0.$$

Pulling back to  $C_Q$ , we have that  $d\iota^*\tilde{\alpha}_0 = \iota^*\tilde{\alpha}_0 \wedge \iota^*\tilde{\alpha}_I$  and so  $\mathcal{S}_1 = \ker(\iota^*\alpha_0) \cap \ker(\iota^*\alpha_I)$  is an integrable foliation of  $C_Q$ . Similarly the kernel of  $\iota^*(\Omega_S^1)$  is a complementary two-dimensional integrable distribution  $\mathcal{S}_2$  of  $C_Q$ . The leaves of  $\mathcal{S}_1$  have the same ring of functions as S, so are diffeomorphic to S. Moreover the two distributions  $S_1$  and  $S_2$  are orthogonal in the induced metric. The metric on  $S_1$  is  $\mathscr{H}$ -related to  $-g_{C,\perp}/\mu = g_S/s$ , that on  $S_2$  is  $\frac{1}{2}\iota^*(\tilde{\alpha}_0^2 + \tilde{\alpha}_I^2)$ . Each distribution inherits a complex structure from I and we obtain a Kähler product. The differential relations, together with the completeness of the metric on  $S_2$ , show that the universal cover of  $S_2$  is  $\mathrm{CH}(1)$  as a solvable group. By Proposition 5.3,  $S_2 = \mathbb{R}_{>0} \times S^1 = \mathrm{CH}(1)/\mathbb{Z}$ .  $\square$ 

**Proposition 5.5.** Let X be regular and c = 0. The differential algebra  $\Omega_S^*$  in (5.2) gives a distribution

$$\mathcal{D} = \ker \Omega^1_S$$

on Q that is integrable. Its leaves are discrete quotients of complex Heisenberg groups  $\mathbb{C}H(m+1)$ ,  $\dim_{\mathbb{C}}C=m$ , with left-invariant Riemannian structures homothetic to the standard Kähler metric.

*Proof.* Integrability of  $\mathcal{D}$  follows directly from that fact that  $\Omega_S^*$  is closed under the exterior differential. Each leaf  $L_Q$  of D is a twist of a leaf L of  $\overline{\pi}_S^*\Omega^1(S)$  on  $H = T^*C$ . Now  $C_L := L \cap C$  is an orbit of X and IX, so by regularity is a copy of  $\mathbb{C}^*$ .

Write  $\iota\colon C_L\to C$  for the inclusion of this orbit. The leaf L is the restriction  $\iota^*T^*C=T^*C|_{C_L}$ . The relations in (3.2), show that  $\iota^*\nabla$  and  $\iota^*\nabla^{LC}$  agree as connections on the bundle  $\iota^*TC$ . In particular, this bundle is flat, and there is a discrete cover  $C'_L$  over which it is trivial. We thus have that the pull-back  $\iota^*Sp(C)$  of the bundle of symplectic frames admits a flat symplectic section s over  $C'_L$ . In our case s may be chosen to be unitary with span of the negative definite directions equal to the span of X and IX. As in (2.10) each  $v\in (\mathbb{R}^{2m})^*$  gives rise to a tri-holomorphic isometry of H' around  $C'_L$ . Write  $U_v$  for the corresponding vector field.

Write  $\mu = \varepsilon v^2$ , for some  $\varepsilon \in \{\pm 1\}$  and a smooth function  $v \colon L' \to \mathbb{R}$ . Put  $V_v = v U_v$ . We claim that  $X, IX, V_v, v \in (\mathbb{R}^{2m})^*$ , are  $\mathscr{H}$ -related to vector fields  $X_Q, IX_Q, (V_v)_Q$  on  $L'_O$  generating a complex Heisenberg algebra.

If A and B are vector fields on L' that are  $\mathcal{H}$ -related to  $A_Q$  and  $B_Q$  on  $L'_{O'}$ , then the Lie brackets are related by

$$[A_Q, B_Q] \sim_{\mathscr{H}} [A, B] + \frac{1}{\iota^* a} (\iota^* F)(A, B) X,$$

see [29, Lemma 3.7]. In our case, from Lemma 5.1 we have  $a = \mu = \varepsilon v^2$  and  $F = \frac{1}{2}(\alpha \wedge \mathbf{s}\alpha^T + \theta^T \wedge \mathbf{s}\theta) = \omega^* - \omega$ .

Firstly, 
$$\iota^* F(IX, X) = -i^* \omega(IX, X) = g(X, X) = 2\mu$$
. Hence,

$$[(IX)_Q, X_Q] = \frac{1}{\mu} \iota^* F(IX, X) X_Q = 2X_Q.$$

Considering the Lie brackets of vertical vector fields, we find

$$[(V_v)_Q, (V_w)_Q] = \frac{1}{\mu} \nu^2 \omega^*(U_v, U_w) X_Q = \frac{\varepsilon}{2} (v \mathbf{s} w^T) X_Q.$$

For mixed brackets, note that X and IX commute with the fibre translations  $U_v$ . Now  $d\nu$  is given by  $X \,\lrcorner\, \omega_I = d\mu = 2\varepsilon \,\nu d\nu$ , so  $X\nu = 0$  and  $IX\nu = \omega_I(X,IX)/2\varepsilon\nu = \nu$ . We thus have

$$[X_Q,(V_v)_Q]=0$$

and

$$[(IX)_Q, (V_v)_Q] = [IX, (V_v)]_Q = (IX\nu)(U_v)_Q = \nu(U_v)_Q = (V_v)_Q.$$

We thus see that  $(IX)_Q$ ,  $X_Q$ ,  $(V_v)_Q$ ,  $v \in (\mathbb{R}^{2m})^*$ , span a Lie algebra  $\mathfrak{g}$ ;  $(IX)_Q$  acts with weight 2 on  $X_Q$  and weight 1 on  $(V_v)_Q$ ; furthermore  $X_Q$ ,  $(V_v)_Q$  span a Heisenberg algebra with derived algebra generated by  $X_Q$ . Thus  $\mathfrak{g}$  is the solvable algebra of  $\mathbb{C}H(m+1)$  as shown by Hitchin in [20]. This is a discrete cover of the leaf  $L_Q$ .

For the metric  $g_N$  and our choice of section s, the above Lie algebra generators, where v runs over the standard unitary basis for G on  $(\mathbb{R}^{2m})^*$ ,

has constant coefficients and so is left-invariant. Indeed this homothetic to the standard orthonormal basis of CH(m+1) when with a scaling factor of  $\sqrt{2}$ .

Note that the form  $\omega_I^N$  does not induce a Kähler form on  $\mathbb{C}H(m+1)$ . Indeed direct computation gives  $d_W \iota^* \omega_I^N = \alpha_{IJK}/\mu^2$ , which is non-zero.

# 6 The hyperbolic plane

To provide more concrete examples of the c-map, we consider indefinite projective special Kähler structures on open subsets S of the hyperbolic plane  $\mathbb{R}H(2)$  with constant curvature Kähler metric. The space  $\mathbb{R}H(2)$  is a solvable group of dimension 2 with one-dimensional derived algebra. Choosing any unit vector B in the derived algebra and putting A = -IB, we obtain an orthonormal basis for the Lie algebra satisfying

$$[A, B] = \lambda B$$

for some non-zero constant  $\lambda \in \mathbb{R}$ .

Write *a*, *b* for the dual basis to *A*, *B*. Then these one-forms satisfy

$$da = 0$$
 and  $db = -\lambda a \wedge b$ .

The metric is  $g_S = a^2 + b^2$  and the corresponding Kähler form is  $\omega_S = a \wedge b$ . We note that Ia = b.

Locally any special Kähler cone C over  $S \subset \mathbb{R}H(2)$  is of the form  $C = \mathbb{R}_{>0} \times C_0$ . Here  $C_0$  is a bundle over S, with connection one-form  $\varphi$  satisfying  $d\varphi = 2\pi^*\omega_S = 2\tilde{a} \wedge \tilde{b}$ , where  $\tilde{a} = \pi^*a$ , etc., corresponding to reduction at level g(X,X) = -1 in (3.3).

**Lemma 6.1.** The metric and Kähler form of  $C = \mathbb{R}_{>0} \times C_0$  are

$$g_C = \hat{a}^2 + \hat{b}^2 - \hat{\psi}^2 - \hat{\varphi}^2, \qquad \omega_C = \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi},$$

where  $\hat{a} = t\tilde{a}$ ,  $\hat{b} = t\tilde{b}$ ,  $\hat{\varphi} = t\varphi$  and  $\hat{\psi} = dt$ , with t denoting the standard coordinate on  $\mathbb{R}_{>0}$ . The conic symmetry X satisfies

$$IX = t \frac{\partial}{\partial t}.$$

*Proof.* In general the metric is as claimed with  $\hat{\psi} = k dt$  for some  $k \in \mathbb{R}_{>0}$  and  $\omega_C = \hat{a} \wedge \hat{b} - \varepsilon \hat{\phi} \wedge \hat{\psi}$ , for some  $\varepsilon = \pm 1$ . For this structure to be Kähler we need  $d\omega_C = 0$ . However,

$$d\hat{\psi} = 0, \quad d\hat{\varphi} = \frac{1}{t} (dt \wedge \hat{\varphi} + 2\hat{a} \wedge \hat{b}),$$

$$d\hat{a} = \frac{1}{t} dt \wedge \hat{a}, \quad d\hat{b} = \frac{1}{t} (dt \wedge \hat{b} - \lambda \hat{a} \wedge \hat{b}),$$
(6.1)

so  $d\omega_C = 2\hat{a} \wedge \hat{b} \wedge (dt - \varepsilon \hat{\psi})/t$ , implying that  $\varepsilon = +1$  and k = 1. Noting that  $I\hat{a} = \hat{b}$  and  $I\hat{\varphi} = \hat{\psi}$ , we have from (6.1) that  $\Lambda^{1,0} = \operatorname{Span}\{\hat{a} - i\hat{b}, \hat{\varphi} - i\hat{\psi}\}$  satisfies  $d\Lambda^{1,0} \subset \Lambda^{2,0} + \Lambda^{1,1}$ , and so I is integrable.

To obtain the conic symmetry, write  $IX = f\partial/\partial t$ . Then  $L_{IX}\hat{a} = IX \, d\hat{a} + d(IX \, \hat{a}) = f\hat{a}/t$ , and similarly for  $\hat{b}$  and  $\hat{\phi}$ . On the other hand  $L_{IX}I = 0$  and  $\hat{\psi} = I\hat{\phi}$  give  $L_{IX}\hat{\psi} = IL_{IX}\hat{\phi} = f\hat{\psi}/t$ . Thus the condition  $L_{IX}g = 2g$  implies that f = t.

**Lemma 6.2.** The pseudo-Riemannian metric  $g_C$  is flat if and only if  $\lambda^2 = 4$ .

*Proof.* With respect to the unitary coframe  $s^*\theta = (\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi})$  the connection one-form  $\omega_{LC}$  is uniquely determined by  $ds^*\theta = -s^*\omega_{LC} \wedge s^*\theta$ , with  $\omega_{LC}^T \mathbf{G} + \mathbf{G}\omega_{LC} = 0$  and  $\mathbf{i}\omega_{LC} = \omega_{LC}\mathbf{i}$ . It follows from (6.1) that

$$s^*\omega_{\mathrm{LC}} = rac{1}{t} egin{pmatrix} 0 & \hat{\phi} + \lambda \hat{b} & \hat{b} & \hat{a} \\ -\hat{\phi} - \lambda \hat{b} & 0 & -\hat{a} & \hat{b} \\ \hat{b} & -\hat{a} & 0 & \hat{\phi} \\ \hat{a} & \hat{b} & -\hat{\phi} & 0 \end{pmatrix}.$$

The curvature is then given by

and the flatness result follows.

**Proposition 6.3.** The cone  $(C, g_C, \omega_C)$  over  $S \subset \mathbb{R}H(2)$  is conic special Kähler if and only if  $\lambda^2 = 4/3$  or 4.

*Proof.* We need to determine when the geometry admits a flat symplectic connection. In the notation of the proof of Lemma 6.2, we need to determine a matrix-valued one-form  $\eta$  such that  $s^*\omega_{\nabla}=s^*\omega_{LC}+\eta$  is flat, with (i)  $\eta \wedge s^*\theta=0$ , (ii)  $\mathbf{i}\eta=-\eta\mathbf{i}$  and (iii)  $\eta^T\mathbf{s}=-\mathbf{s}\eta$ . Furthermore the analysis of Lemma 3.2, shows that we must have  $X \perp \eta=0$  and  $IX \perp \eta=0$ . The latter implies that the entries of  $\eta$  lie in the span of  $\hat{a}$  and  $\hat{b}$ . Now (ii) and (iii) imply

$$\eta = \begin{pmatrix} u & v & p & q \\ v & -u & q & -p \\ -p & -q & x & y \\ -q & p & y & -x \end{pmatrix}$$

with all entries in the span of  $\hat{a}$  and  $\hat{b}$ . Using (i), and considering the coefficients of  $\hat{\varphi}$  and  $\hat{\psi}$  gives first p=0=q and then x=0=y. We then have

$$u \wedge \hat{a} + v \wedge \hat{b} = 0$$
 and  $v \wedge \hat{a} - u \wedge \hat{b} = 0$ . (6.2)

TWIST GEOMETRY OF THE C-MAP

The curvature of  $\nabla$  is  $\Omega_{\nabla} = d\omega_{\nabla} + \omega_{\nabla} \wedge \omega_{\nabla}$  which pulls back to

where

$$U = du + \frac{2}{t}(\hat{\varphi} + \lambda \hat{b}) \wedge v, \quad V = dv - \frac{2}{t}(\hat{\varphi} + \lambda \hat{b}) \wedge u,$$
$$W = \frac{4 - \lambda^2}{t^2} \hat{a} \wedge \hat{b} + 2u \wedge v.$$

Equations (6.2) are solved in general by writing  $u = s_+ \tilde{a} + s_- \tilde{b}$ ,  $v = s_- \tilde{a} - s_+ \tilde{b}$ , since  $\hat{a} = t\tilde{a}$  etc. Then W = 0 is equivalent to

$$s_{+}^{2} + s_{-}^{2} = \frac{1}{2}(4 - \lambda^{2}) \tag{6.3}$$

which is constant. We may thus write  $s_+ = r \cos z$ ,  $s_- = r \sin z$  for some local function z and constant r > 0 satisfying  $2r^2 = (4 - \lambda^2)$ .

Computing du and dv, the equations U = 0 = V become

$$(dz - 2\varphi - 3\lambda \tilde{b}) \wedge v = 0 = (dz - 2\varphi - 3\lambda \tilde{b}) \wedge u$$

If r=0, we have u=v=0,  $\nabla=\nabla^{C}$  and  $\lambda^{2}=4$ . For  $r\neq 0$ , we see that  $dz=2\varphi+3\lambda\tilde{b}$ . In particular the right-hand side must be closed. But  $d(2\varphi+3\lambda\tilde{b})=(4-3\lambda^{2})\tilde{a}\wedge\tilde{b}$ , so  $\lambda^{2}=4/3$ , as claimed.

Note that this result does not require the conic symmetry to be periodic or even quasi-regular.

#### 6.1 The flat case

In the case  $\lambda=2$ , we have that the Levi-Civita and special Kähler connections coincide. Let  $C_0$  be the principal  $S^1$ -bundle over all of  $\mathbb{R}H(2)$ , with connection one form  $\varphi$  satisfying  $d\varphi=2\pi^*(a\wedge b)=2\tilde{a}\wedge\tilde{b}$ . Write the principal action as  $p\mapsto e^{i\tau}\cdot p$ . The proof of Lemma 6.2 provides us with the derivative of the unitary coframe  $s^*\theta=(\hat{a},\hat{b},\hat{\varphi},\hat{\psi})$ . The hyperKähler structure on  $H=T^*C$  of the rigid c-map is

$$\begin{split} g_H &= \hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 - \hat{\Phi}^2 - \hat{\Psi}^2, \\ \omega_I &= \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}, \\ \omega_J &= \hat{A} \wedge \hat{a} + \hat{B} \wedge \hat{b} + \hat{\Phi} \wedge \hat{\varphi} + \hat{\Psi} \wedge \hat{\psi}, \\ \omega_K &= \hat{A} \wedge \hat{b} - \hat{B} \wedge \hat{a} + \hat{\Phi} \wedge \hat{\psi} - \hat{\Psi} \wedge \hat{\varphi}, \end{split}$$

where  $(\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi}) := s^*\alpha = dx - xs^*\omega_{LC}$ . We have  $ds^*\alpha = -s^*\alpha \wedge s^*\omega_{LC}$  and  $\widetilde{X} \lrcorner s^*\alpha = 0$  by definition. Now  $\alpha_I = I\widetilde{X} \lrcorner g_H = IX \lrcorner (-\hat{\psi}^2) = -t\hat{\psi}$ , so  $\alpha_0 = -I\alpha_I = -t\hat{\varphi}$  and  $\mu = \frac{1}{2} ||\widetilde{X}||^2 = -t^2/2$ . Similarly  $\alpha_J = I\widetilde{X} \lrcorner \omega_K = -t\hat{\Phi}$  and  $\alpha_K = -I\widetilde{X} \lrcorner \omega_I = -t\hat{\Psi} = I\alpha_I$ .

Now the metric we twist is

$$g_N = -\frac{1}{\mu}g_H + \frac{1}{\mu^2}g(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2)$$
  
=  $\frac{2}{t^2}(\hat{a}^2 + \hat{b}^2 + \hat{\varphi}^2 + \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 + \hat{\Phi}^2 + \hat{\Psi}^2),$ 

which is a complete metric on (t > 0). Thus  $g_N$  has constant coefficients with respect to the coframe  $(\tilde{a}, \tilde{b}, \varphi, \tilde{\psi}, \tilde{A}, \dots, \tilde{\Psi})$ , where the last five terms are given by  $\tilde{\psi} = \hat{\psi}/t$ , etc.

The twist data is

$$F = -\hat{a} \wedge \hat{b} + \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

with twist function a equal to  $\mu = -t^2/2$ .

The coframe  $\gamma := s^*\theta/t = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$  on C is invariant under X, so we can compute these twisted differentials immediately:

$$d_{W}\tilde{a} = d\tilde{a} = 0, \quad d_{W}\tilde{b} = d\tilde{b} = 2\tilde{b} \wedge \tilde{a},$$

$$d_{W}\varphi = d\varphi + \frac{2}{t^{2}}F \wedge X \cup \varphi = 2\tilde{a} \wedge \tilde{b} + \frac{2}{t^{2}}F = 2(\varphi \wedge \tilde{\psi} - \tilde{A} \wedge \tilde{B} + \tilde{\Phi} \wedge \tilde{\Psi}),$$

$$d_{W}\tilde{\psi} = d(dt/t) = 0.$$
(6.4)

On the other hand  $\tilde{\delta}:=(s^*\alpha)/t=(\tilde{A},\tilde{B},\tilde{\Phi},\tilde{\Psi})$ , has  $L_{\widetilde{X}}\tilde{\delta}=(L_{\widetilde{X}}s^*\alpha)/t=\widetilde{X}\lrcorner ds^*\alpha/t=\widetilde{\delta}(\widetilde{X}\lrcorner s^*\omega_{LC})=-\tilde{\delta}\mathbf{i}$ , so these forms are not invariant. However the bundle  $C_0\to\mathbb{R}H(2)$  is trivial; indeed  $d\varphi=2\tilde{a}\wedge\tilde{b}=-d\tilde{b}$ , and we may write  $\varphi=d\tau-\tilde{b}$  with  $\exp(\mathbf{i}\tau)$  globally defined. We have  $X\tau=1$ , and  $L_{\widetilde{X}}(\tilde{\delta}e^{\mathbf{i}\tau})=0$ . Putting  $\delta=\tilde{\delta}e^{\mathbf{i}\tau}$ , we find that

$$d_{W}\delta = d(\frac{1}{t}s^{*}\alpha e^{\mathbf{i}\tau})$$

$$= -\frac{1}{t^{2}}\hat{\psi}\wedge s^{*}\alpha e^{\mathbf{i}\tau} - \frac{1}{t}s^{*}\alpha \wedge s^{*}\omega_{LC}e^{\mathbf{i}\tau} - \frac{1}{t}s^{*}\alpha \wedge \mathbf{i}e^{\mathbf{i}\tau}d\tau$$

$$= -\tilde{\psi}\wedge\delta - \delta\wedge s^{*}\omega_{LC} - \delta\wedge(\varphi + \tilde{b})\mathbf{i}$$

$$= \delta\wedge\begin{pmatrix}\tilde{\psi} & -\tilde{b} & -\tilde{b} & -\tilde{a}\\\tilde{b} & \tilde{\psi} & \tilde{a} & -\tilde{b}\\ -\tilde{b} & \tilde{a} & \tilde{\psi} & \tilde{b}\\ -\tilde{a} & -\tilde{b} & -\tilde{b} & \tilde{\psi}\end{pmatrix},$$
(6.5)

which has constant coefficients with respect to the coframe  $(\gamma, \delta)$ , as do the relations (6.4).

As  $g_N$  is complete, it follows that the universal cover of the twist gives a left-invariant metric on a Lie group G. Writing  $\epsilon = \frac{1}{\sqrt{2}}(\delta_1 + \delta_4, \delta_2 - \delta_3, -\delta_2 - \delta_3, -\delta_1 + \delta_4)$ , equation (6.5) gives

$$d_W \epsilon = \epsilon \wedge egin{pmatrix} ilde{\psi} - ilde{a} & 0 & 2 ilde{b} & 0 \ 0 & ilde{\psi} - ilde{a} & 0 & -2 ilde{b} \ 0 & 0 & ilde{\psi} + ilde{a} & 0 \ 0 & 0 & ilde{\psi} + ilde{a} \end{pmatrix}$$

and we have  $d_W \varphi = 2(\varphi \wedge \tilde{\psi} + \epsilon_1 \wedge \epsilon_3 + \epsilon_4 \wedge \epsilon_2)$ .

The Lie algebra  $\mathfrak g$  is completely solvable, with derived algebra  $\mathfrak n$  of codimension 2, while  $\mathfrak n$  is 3-step nilpotent, with  $\mathfrak n'=[\mathfrak n,\mathfrak n]$  of dimension 3 and  $\mathfrak n^{(2)}=[\mathfrak n,\mathfrak n']$  of dimension 1. We see that the Lie algebra is isomorphic to the solvable algebra corresponding to the non-compact symmetric space  $\mathrm{Gr}_2^+(\mathbb C^{2,2})=U(2,2)/(U(2)\times U(2))$ . This solvable algebra may be identified with the Lie algebra of matrices

$$\begin{pmatrix} x & u & v & iw \\ 0 & y & iz & -\overline{v} \\ 0 & 0 & -y & -\overline{u} \\ 0 & 0 & 0 & -x \end{pmatrix}, \quad x, y, z, w \in \mathbb{R}, u, v \in \mathbb{C},$$

when one realises  $\mathfrak{u}(2,2)$  as matrices preserving  $\mathbf{i}$  and the inner product given by a matrix with non-zero entries 1 in each anti-diagonal position (i,4-i). Dually  $(\tilde{a},\tilde{b},\varphi,\tilde{\psi},\epsilon_1,\ldots,\epsilon_4)$  correspond to  $((y=1),(z=2),(w=1),(x=1),(\mathrm{Re}\,u=1),(\mathrm{Im}\,u=1),(\mathrm{Im}\,v=1),(\mathrm{Re}\,v=1))$ , where for example (y=1) means the matrix with y=1 and all other variables zero.

### 6.2 The non-flat case

In this case, we take  $\lambda=2/\sqrt{3}$ . The difference  $\eta$  between  $s^*\omega_{\nabla}$  and  $s^*\omega_{LC}$  is described in the proof of Proposition 6.3. Note that the local function z appearing there satisfies  $dz=2(\varphi+\sqrt{3}\,\tilde{b})$ , so  $\varphi=\frac{1}{2}dz-\sqrt{3}\,\tilde{b}$ . But  $d\varphi=2\tilde{a}\wedge\tilde{b}=-\sqrt{3}d\tilde{b}$ , so corresponding to the flat case we write  $\tau=z/2$ . This again has  $X\tau=1$ .

If we write  $(\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi}) = s^*\alpha = dx - xs^*\omega_{\nabla} = dx - xs^*\omega_{LC} - x\eta$ , the description of  $g_H$  and  $g_N$  is as in the flat case, and the twist data is the same. Putting  $\gamma = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$ , we have

$$d_{W}\tilde{a} = d\tilde{a} = 0, \quad d_{W}\tilde{b} = d\tilde{b} = -\frac{2}{\sqrt{3}}\tilde{a} \wedge \tilde{b},$$

$$d_{W}\varphi = 2(\varphi \wedge \tilde{\psi} - \tilde{A} \wedge \tilde{B} + \tilde{\Phi} \wedge \tilde{\Psi}), \quad d_{W}\tilde{\psi} = d(dt/t) = 0.$$
(6.6)

Once again  $\tilde{\delta} \coloneqq (s^*\alpha)/t = (\tilde{A}, \tilde{B}, \tilde{\Phi}, \tilde{\Psi})$ , has  $L_{\widetilde{X}}\tilde{\delta} = (L_{\widetilde{X}}s^*\alpha)/t = \widetilde{X} \lrcorner ds^*\alpha/t =$ 

 $\tilde{\delta}(\widetilde{X} \, | \, s^* \omega_{\rm LC}) = -\tilde{\delta} i$ , so we consider  $\delta \coloneqq \tilde{\delta} e^{i\tau}$ . This satisfies

$$d_{W}\delta = d(\frac{1}{t}s^{*}\alpha e^{i\tau})$$

$$= -\frac{1}{t^{2}}\hat{\psi} \wedge s^{*}\alpha e^{i\tau} - \frac{1}{t}s^{*}\alpha \wedge (s^{*}\omega_{LC} + \eta) e^{i\tau} - \frac{1}{t}s^{*}\alpha \wedge e^{i\tau} i d\tau$$

$$= -\tilde{\psi} \wedge \delta - \delta \wedge (s^{*}\omega_{LC} + e^{-i\tau}\eta e^{i\tau}) - \delta \wedge (\varphi + \sqrt{3}\tilde{b}) i$$

$$= -\tilde{\psi} \wedge \delta - \delta \wedge (s^{*}\omega_{LC} + \eta e^{2i\tau}) - \delta \wedge (\varphi + \sqrt{3}\tilde{b}) i$$

$$= \delta \wedge \left\{ \tilde{\psi}1_{4} + \tilde{a} \begin{pmatrix} -2/\sqrt{3} & 0 & 0 & -1\\ 0 & 2/\sqrt{3} & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix} \right.$$

$$\left. + \tilde{b} \begin{pmatrix} 0 & \sqrt{3} & -1 & 0\\ 1/\sqrt{3} & 0 & 0 & -1\\ -1 & 0 & 0 & \sqrt{3}\\ 0 & -1 & -\sqrt{3} & 0 \end{pmatrix} \right\},$$

$$\left. (6.7)$$

which has constant coefficients with respect to the coframe  $(\gamma, \delta)$ . As  $g_N$  is complete, we conclude that the universal cover of the twist is a Lie group G, with left-invariant quaternionic Kähler metric.

Let  $\zeta = \sqrt[4]{3}$  and put  $\epsilon = \frac{1}{\sqrt{2}} (\zeta(\delta_2 \zeta + \delta_3 / \zeta), \frac{1}{\zeta} (\delta_1 / \zeta - \delta_4 \zeta), \frac{1}{\zeta} (\delta_2 / \zeta - \delta_3 \zeta), \zeta(\delta_1 \zeta + \delta_4 / \zeta))$ . We have

$$d_W \epsilon = \epsilon \wedge \left\{ \tilde{\psi} 1_4 + \frac{1}{\sqrt{3}} \tilde{a} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} + \frac{2}{\sqrt{3}} \tilde{b} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

Moreover,  $d_W \varphi = 2\varphi \wedge \tilde{\psi} - 3\epsilon_2 \wedge \epsilon_3 + \epsilon_1 \wedge \epsilon_4$ . The Lie algebra is completely solvable with derived algebra n of codimension 2, which is 4-step nilpotent. With  $a' = \tilde{a}/\sqrt{3}$  and  $b' = \frac{2}{\sqrt{3}}\tilde{b}$ , we have  $d_W a' = 0$ ,  $d_W b' = 2b' \wedge a'$ . In this coframe  $(a',b',\varphi,\tilde{\psi},\epsilon_1,\ldots,\epsilon_4)$  we see the structure of solvable algebra associated to  $G_2^*/SO(4)$ : an explicit matrix representation of this solvable algebra is provided by Castrillón López, Gadea and Oubiña [8]; in their notation the dual to our coframe is  $(A_2,U_3,X_3,2A_1+A_2,-X_2,U_1,U_2,X_1)$ .

# 7 Deformations and related geometries

Our description of the c-map via the twist construction has the advantage that there are obvious changes that can be made that still yield quaternionic Kähler twists.

- (i) The most obvious is that a constant c can be subtracted from the  $\mu$  given in §5, as in Theorem 4.1. This change has been identified by Alexandrov et al. [4] and Alekseevsky et al. [2] as the one-loop deformation of the Ferrara-Sabharwal metric, originally found by Cecotti, Ferrara and Girardello [9] and also studied by Robles Llana, Saueressig and Vandoren [25].
- (ii) The rigid c-map produces hyperKähler metrics with many tri-holomorphic isometries. If Y is one of these, we may consider twists by  $Z = \widetilde{X} + \lambda Y$  for any real constant  $\lambda$ . Since Z satisfies  $L_Z \omega_J = \omega_K$  and is a Kähler isometry for  $(g, \omega_I)$ , Theorem 4.1 shows that the twists of  $H = T^*C$  by Z are still quaternionic Kähler.
- (iii) If Y is a tri-Hamiltonian isometry acting freely and properly on H and commuting with  $\widetilde{X}$ , the one can create a new hyperKähler manifold with rotating circle action as follows. Recall that the hyperKähler modification [12] of  $H = T^*C$  by Y is  $H_{\text{mod}} = (H \times \mathbb{H}) /\!/\!/ \mathbb{R}$ , the hyperKähler quotient of  $H \times \mathbb{H}$  by the action  $(p,q) \mapsto (t \cdot p, e^{-it}q)$ , where Y generates the action  $p \mapsto t \cdot p$ . On the product  $H \times \mathbb{H}$  we have a rotating circle action given by the action of  $\widetilde{X}$  on the first factor and that of  $q \mapsto e^{it/2}qe^{-it/2}$  on  $\mathbb{H}$ . This action descends to  $H_{\text{mod}}$  such that the hypotheses of Theorem 4.1 are satisfied. It is therefore possible to twist  $H_{\text{mod}}$  to produce quaternionic Kähler metrics.
- (iv) The construction of item (iii) may be generalised further. Firstly one may replace the factor  $\mathbb H$  by any hyperKähler four-manifold that admits a tri-Hamiltonian action that commutes with a circle action rotation satisfying (4.1). In general this is specified by  $S^1$ -invariant monopoles on open subsets U of  $\mathbb R^3$ , cf. Hitchin [19]. Such monopoles are specified by a harmonic function on U that is required to be positive. Secondly, in [30] it shown how such generalised modifications may be interpreted as twists via tri-holomorphic isometries, and there form part of a wider class of hyper-Kähler twists constructions, corresponding to a relaxation of the positivity condition on the harmonic function on  $U \subset \mathbb R^3$ .

In Alexandrov et al. [5, 6] various deformations of hyperKähler and quaternionic Kähler metrics are discussed. It seems likely that some of the constructions above under items (ii) to (iv) describe their deformations, and that some simply provide isometric deformations of the quaternionic Kähler metric. However, at this stage the correspondence between these constructions in not clear. We believe items (ii) to (iv) describe a wider class of quaternionic Kähler metrics.

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