A GEOMETRIC INTERPRETATION OF THE C-MAP

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Universiteit Utrecht

Master's Thesis
Mathematical Institute
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Supervisors: dr. J. Stienstra
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Mathematical and physical aspects of a bundle of intermediate Jacobians over the complex structure deformation space of a family of Calabi-Yau$_3$ manifolds

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Abstract

T-duality is a duality between type IIA and type IIB superstring theory. After compactification on a Calabi-Yau$_3$ manifold this duality induces a relation between the vector multiplet moduli space of type IIB (IIA) and the hypermultiplet moduli space of type IIA (IIB), which is called the c-map. We have investigated this relation from the geometric point of view of the internal Calabi-Yau$_3$ manifold, with and without coupling to gravity. In the $N = 2$ rigid supersymmetry situation it is known that the c-map constructs a bundle of Griffiths intermediate Jacobians on the vector multiplet moduli space, while a bundle of Weil intermediate Jacobians is found in the $N = 2$ supergravity situation. In the latter case an additional gravitational dilaton-axion system is connected with the Weil intermediate Jacobians through a dilatationally extended Heisenberg group structure that amounts from symplectic invariance, dilatational invariance and the Peccei-Quinn isometries of the Ramond fields and Neveu-Schwarz axion. The total hypermultiplet moduli space gets therefore the interpretation of a principal-like fibre bundle whose fibres are identified with a semi-direct product of a subgroup of the symplectic group and a dilatationally extended Heisenberg group modulo their integer subgroups. The invariant metric on the fibre bundle is given by a Wess-Zumino-Witten model consisting of the Killing bilinear form acting on the structure group's Maurer-Cartan form.

Contents

Voorwoord

Met de afronding van mijn scriptie komt ook voor mij het moment dat ik mag afstuderen. Het vormt het einde van een enerverende en vormende studietijd, maar ook de opstap naar een opnieuw leerzame en verder verdiepende periode waarin ik in Leiden verder mag "wetenschappen". Bij deze wil ik een klein woordje richten aan eenieder die mij geholpen heeft dit schakelpunt te bereiken.

Allereerst wil ik mijn afstudeerbegeleiders, Jan Stienstra en Stefan Vandoren, bedanken. Ze hebben mij gevormd als junior-onderzoeker gedurende de laatste helft van mijn Master-opleiding en ik heb veel geleerd en genoten van onze samenwerking. Hoewel ik denk dat natuur- en wiskundigen niet zonder elkaar kunnen, blijken communicatieproblemen er in de praktijk verantwoordelijk voor dat beide wetenschappen elkaar niet optimaal aan kunnen vullen. Jan en Stefan zijn hét voorbeeld hoe deze communicatiedrempel overwonnen kan worden. Hun talent om een uitleg bondig en helder te houden en de wil om elkaars inzichten te begrijpen, hebben er mijns inziens voor gezorgd dat onze gezamenlijke bijeenkomsten interessant waren voor ons allen. Dankzij hun enthousiasme zijn we er met deze scriptie alleraardigst in geslaagd om de schakel tussen natuur- en wiskundige methoden te versterken.

Alle andere hoogleraren en mijn medestudenten, in het bijzonder Aron, Frits, Jan en Ties, verdienen evenwel een welgemeend dankjewel voor het delen van kennis, enthousiasme en een verfrissende kijk van/voor/op natuur- en wiskunde. Het studieplezier is er onmetelijk veel groter door geworden. Ook wil ik eenieder in mijn familie- en vriendenkring bedanken voor het tonen van interesse én voor de benodigde afleiding; het leven bestaat niet alleen uit rekensommetjes.

Drie mensen wil ik in het bijzonder bedanken voor alle steun en toeverlaat, voor het delen van lief en leed. Irene verdient alle lof omdat ze niet is weggerend ondanks al mijn enge praatjes die voor geen normaal weldenkend mens begrijpelijk/interessant zijn en ondanks mijn voorliefde om apathisch voor me uit te staren wanneer ik me heb vastgebeten in een of ander probleem. De stabiliteit die ze me biedt zorgt ervoor dat ik me kan ontspannen als ik thuis ben ongeacht de (sporadische) studiestress van overdag. Tenslotte wil ik 's pap en 's mam bedanken voor alle geboden mogelijkheden. Ik heb mijn best gedaan ze optimaal te benutten, wat ik dankzij de combinatie van hun vertrouwen en stille trots als heel natuurlijk heb ervaren.

Velen die de moeite hebben genomen om mijn scriptie open te slaan en dit voorwoord lezen, zullen zich ongetwijfeld afvragen: "Wat heb je nu eigenlijk gedaan, Ted? Leuk voor je dat het erop zit, maar kun je ook in minder dan 100 pagina's uitleggen wat je onderzoeksonderwerp is geweest (en wat het resultaat is)?" Ongetwijfeld vindt het gros van de mensen die dit voorwoord leest de rest van mijn scriptie te langdradig om door te ploeteren. Daarom benut ik de rest van de hier beschikbare ruimte om in zo makkelijk mogelijke bewoordingen een sluier op te lichten van het onderzoek dat mij anderhalf jaar lang heeft beziggehouden. Als uitgangspunt wil ik daarbij de titel van mijn scriptie nemen: "een meetkundige interpretatie van de c-afbeelding".

Laat me eerst toelichten wat wiskundigen bedoelen met "meetkunde". Het is hopelijk niet onbekend dat de hoeken van een driehoek op een blaadje papier optellen tot 180 graden. De hoeken van een driehoek op een boloppervlak tellen echter op tot *meer* dan 180 graden. Ook rechte lijnen zijn verschillend voor een plat vlak en een boloppervlak. Stel dat we een (rechtstreekse) vlucht van New York naar Amsterdam in de Bosatlas zouden tekenen, dan zouden we zien dat de rechte vlucht een klein boogje maakt, omdat het begrip "rechte lijn" op de sferische aardbol anders is dan op een projectie op het platte vlak. Deze voorbeelden laten zien dat verschillende ruimten (zoals een boloppervlak of plat vlak), in twee maar ook in meer of minder dimensies, verschillende manieren kennen om met hoeken, afstanden en rechte lijnen om te gaan. In de wiskunde is *meetkunde* de verzamelnaam voor alles wat met rechte lijnen, afstanden en hoeken te maken heeft. Aan de hand van meetkundige omschrijvingen van ruimten kunnen wiskundigen meer inzicht verkrijgen in de precieze vorm en structuur van die ruimten.

In mijn onderzoek heb ik met behulp van meetkunde de ruimten bekeken die optreden als domein en bereik van de c-afbeelding (een afbeelding gaat *van* de ene ruimte -het domein- *naar* een andere ruimte -het bereik). Zodoende hebben we een beter (meetkundig) begrip gekregen van de werking van de c-afbeelding. Waarom dit relevant is voor een beschrijving van de Natuur zal ik in het vervolg proberen toe te lichten. Daartoe geef ik nog een ander (fysisch relevant) voorbeeld van een "rechte lijn": pak een zware (bowling)bal en leg deze op een trampoline. De trampoline zal zich tot een kuil vormen door het gewicht van de zware bal. Leg vervolgens een kleiner (pingpong)balletje net onder de rand van de kuil en geef het een zetje in de richting loodrecht op de helling. Zonder wrijving zal het kleinere balletje een eeuwige cirkel beschrijven op een vaste afstand van de zware bal.

Dit voorbeeld is een illustratie van Algemene Relativiteitstheorie: massa kromt de ruimte, waardoor de baan van andere massa's wordt beïnvloed. Zwaartekracht tussen bijvoorbeeld zon en aarde wordt in het voorbeeld meetkundig vertaald als de kuil (een verstoring) in de verder vlakke trampoline. Algemene Relativiteitstheorie beschrijft de Natuur op grote schaal (de schaal van hemellichamen en het heelal). Voor de Natuur op kleine schaal (de schaal van atomen en kleiner) is gedurende de 20e eeuw Kwantummechanica ontwikkeld. Combinatie van Kwantummechanica en Algemene Relativiteitstheorie levert echter geen geschikte theorie op die ook de zwaartekracht op kleine schaal kan beschrijven, hoewel we zo'n theorie wel nodig hebben om zwarte gaten of de oerknal te kunnen begrijpen. Als oplossing daarvoor is Snaartheorie ontwikkeld, een theorie die alle fundamentele deeltjes als *snaren* in plaats van puntdeeltjes beschrijft. Er is echter één probleempje: Snaartheorie vereist een universum van *negen* in plaats van drie ruimtelijke dimensies.

Als oplossing daarvoor hebben natuurkundigen bedacht dat de negen dimensies onderverdeeld zijn in drie "uitgestrekte" dimensies die het zichtbare universum vormen en zes opgerolde, voor ons (nog) niet waarneembare dimensies welke Calabi-Yau ruimten vormen. We zeggen dat er zes dimensies zijn *gecompactificeerd*. De precieze structuur van deze Calabi-Yau ruimten wordt beschreven door *moduli*, net zoals we een hond verder moeten specificeren door te zeggen hoe lang z'n staart is, wat voor snuit hij heeft, hoe lang z'n pootjes zijn, etcetera. De mogelijke waarden die de moduli aan kunnen nemen beschrijven zelf ook een ruimte, de *moduliruimte* genaamd. De moduli kunnen worden onderverdeeld in twee typen: vector multiplet moduli en hypermultiplet moduli (de namen zijn niet zo van belang). Zodoende valt de totale moduliruimte uiteen in twee delen: de *vector multiplet* moduli ruimte en de *hypermultiplet* moduliruimte.

De c-afbeelding is een afbeelding van de vector multiplet moduliruimte naar de hypermultiplet moduliruimte. Met behulp van onze meetkundige interpretatie van de c-afbeelding zijn we erin geslaagd om (een deel van)[1] de hypermultiplet moduliruimte -het bereik van de c-afbeelding- te beschrijven als een bundel van (een specifiek soort) *tori* -fietsbanden- bovenop de vector multiplet moduliruimte: boven elk punt in de vector multiplet moduliruimte bevindt zich een fietsband, zie afbeelding 6.1, en alle fietsbanden bij elkaar (in combinatie met de vector multiplet moduliruimte) vormen de hypermultiplet moduliruimte. Met deze beschrijving is de moduliruimte behorende bij de Calabi-Yau ruimten een stukje beter begrepen en als zodanig is het compactificatieproces dat Snaartheorie moet koppelen aan het waarneembare universum weer een (piep)klein stapje verhelderd.

Tot zover deze "Nederlandse" toelichting bij mijn scriptie. Alle geënthousiasmeerden mogen zeker bij mij aankloppen voor een uitgebreidere uitleg, maar ik wil ook graag *wikipedia* en boeken als *The elegant universe* of *Flatland* onder de aandacht brengen. Volgens mij bieden zulk soort populair-wetenschappelijke bronnen voldoende aanknopingspunten om beter te begrijpen welke ontwikkelingen er gaande zijn in dit rijke en fascinerende onderzoeksgebied.

[1] Zoals gebruikelijk in een wat populairder verhaal bedek ik hier een groot aantal subtiliteiten en verdere (uiterst interessante) ontwikkelingen met de mantel der liefde. Geïnteresseerden verwijs ik naar hoofdstuk 7, waarin alles in detail staat uitgelegd.

Chapter 1

Introduction

Physics needs mathematics. Perhaps not all physicists agree with this observation, since the real interpretation of physical theories and their implications are *not* the equations that describe these theories, but the physicist's understanding of these equations by careful experimental verification. It requires thought and intuition to interpret the language in which the book of Nature is supposedly written. However mathematics encompasses more than just writing down well-defined equations and doing calculus with them. Mathematics is a rigorous instrument which may be used by physicists in order to understand the laws of Nature *very* precisely. When physicists fail to write down their theories in a rigorous mathematical way, it is a sign that Nature is still hiding some of its mysteries, simply because the theory is not described as precisely as it should be. In those situations a better understanding of Nature may come with an analysis of the mathematical description of the particular theory.

Mathematics needs physics. Perhaps not all mathematicians agree with this observation, since the larger part of mathematical ideas does not require the existence of *this* universe anyway. As long as a mathematician has an open mind and ideas in it, he could do well within any universe while developing his own universe of ideas. However physics provides a very useful quality: intuition and motivation. Somehow the wildest ideas and conjectures simply pop up as a necessity in the description of Nature, providing mathematicians with useful relations between mathematical objects that may help to shape their own universe of ideas.

Whatever one may think about these meta-scientific statements, certainly physicists and mathematicians may profit a lot from each other's work. One can be stubborn in insisting that both disciplines could do well without each other, but it is certainly more fruitful when the more difficult problems are attacked in a collaborative effort. One of the obstructions in these collaborations is often the communication between physicists and mathematicians. The rigorous approach of mathematicians as opposed to the more intuitive approach of physicists can be quite incompatible and only when both parties are willing to learn from each other the desired solution may be reached.

This thesis is a combined mathematical and physical report of last year's research that I have performed together with dr. Jan Stienstra and dr. Stefan Vandoren as a part of my Master's education in Mathematical Sciences and Theoretical Physics. In our research we have focussed on one particular problem within type II string theory, hopefully providing the necessary dictionary to translate between physics's and mathematics's results. String theory is a prime example of the possible benefit that is obtained by collaborations between mathematicians and physicists. It has arisen as a simple idea to amalgamate General Relativity (which describes the universe at the large scale) and Quantum Field Theory (which describes the universe sub-microscopically). Both theories seem to be incompatible if one assumes that the fundamental constituents of Nature are *point* particles, which is why the idea of string theory is to replace them with *strings*. Over the past three decades, string theory has evolved from being just a (promising) idea to being a rich new framework of five dual string theories in which old, unsolved problems can be addressed using newfound tools. The theory is not yet fully developed and because of its complexity, highly sophisticated mathematical tools need to be constructed, analyzed and applied to string theory before it can be completely put to work. String theory requires a mathematical approach. In turn the embedding of new mathematical objects into a physical theoretical framework has provided intuition and guidance to the development of the mathematical objects themselves.

A particular example of the mutual benefit is the development of compactification of string theory on Calabi-Yau$_3$ manifolds. String theory has the remarkable property that it is only consistently defined in ten dimensions, making it necessary to compactify six of these dimensions on small manifolds. That is the total ten-dimensional target space of the strings is considered to be a fibre bundle with a particular Calabi-Yau$_3$ fibred at each point in spacetime. The strings move through these extra dimensions, which are too small to be detected by current experiments. Since the geometry of the manifolds puts restrictions on the particular vibrational patterns of the string, it determines (part of) the effective theory in the observable four dimensions. For this reason it is important to fully understand the compactification process.

The process is described by the *moduli* of the Calabi-Yau$_3$, which parameterize the precise shape of the Calabi-Yau$_3$ manifolds above each point in spacetime. These moduli are arranged into two different groups, the vector multiplets and hypermultiplets. In this thesis we have considered the moduli in the vector and hypermultiplets of the type IIA and type IIB string theories, two of the five different appearances of string theory. In the literature the vector multiplet moduli space has been investigated quite thoroughly and its geometry, viz. special Kähler geometry, is understood very well [dWVP84]. It is one of the examples in which physics proposes to consider a certain mathematical object, which is later to be found very interesting as a mathematical object by itself [DM95, Cor98, Fre99]. The hypermultiplet moduli space is less understood, but it may be investigated using our understanding of the vector multiplet moduli space through T-duality. T-duality is the relation that explains that type IIA and type IIB are equivalent ("dual") physical theories. It induces a connection between the vector multiplet moduli of type IIB and the hypermultiplet moduli of type IIA, which is called the c-map [CFG89]. In the easier situation in which gravity is put (momentarily) to zero, the c-map is called the rigid c-map, while the full-fledged connection between the nonzero-gravity effective theories is called the local c-map.

The rigid c-map has been investigated for quite some time and by the work of [Cor98] it has been given a geometric interpretation in terms of the Griffiths intermediate Jacobians of the Calabi-Yau$_3$'s of the compactification process. By a generalization to arbitrary affine special Kähler manifolds, the rigid c-map has led to a construction of a hyperkähler structure on the cotangent bundle of affine special Kähler manifolds, similar to Calabi's result for certain Kähler manifolds [Cal79]. The local c-map is not yet understood as well as the rigid c-map. In this thesis we propose an analogous fibre bundle construction that gives the local c-map a newly developed geometric interpretation in terms of the Weil intermediate Jacobians of the Calabi-Yau$_3$'s, interrelated with the gravitational dilaton-axion system by a dilatationally extended Heisenberg group structure. The resulting fibre bundle on the moduli space of complex structure deformations of the Calabi-Yau$_3$ is the local c-map's analogue of the rigid c-map's construction.

Using this result, future research on the quaternion-Kähler structure of the type IIA hypermultiplet moduli space may be performed using a clear geometric approach. At the same time the construction could be used as a prototype of a quaternion-Kähler construction on a fibre bundle over projective special Kähler manifolds. Furthermore the algebraic description of Weil intermediate Jacobians could lead to a Seiberg-Witten theory on 3-dimensional projective varieties.

This thesis is organized as follows. Chapters 2 and 3 contain mathematical preliminaries. Whereas chapter 2 contains fairly basic material, the content of chapter 3 may be very interesting to anyone not familiar with (nonalgebraic) complex tori. Chapters 4 and 5 contain the physical context in which the rigid and local c-maps are rooted. Chapter 4 is a review of type II string theory compactifications on Calabi-Yau$_3$ manifolds and their moduli spaces, while chapter 5 studies the vector multiplets of $N = 2$ supersymmetry theories more thoroughly. After we have revised the construction of the rigid c-map of [Cor98] in chapter 6, we present our geometric interpretation of the local c-map in chapter 7. We end with conclusions and future perspective in chapter 8.

Chapter 2

Preliminary mathematics

Like any physical theory the detailed description of string theory compactifications requires the language of mathematics. In this chapter we will review some of the familiar or less familiar mathematical objects necessary to understand this thesis. The notions are well explained in great textbooks like [GH78, Bes87, Nak03, Fra04] and we will merely present definitions and properties relevant for the content of this thesis. For more detailed explanations and discussions we refer to standard mathematics literature.

2.1 Fibre bundles and connections

Fibre bundles

Assuming that the concept of a manifold is familiar to the reader, we may construct a bundle of copies of one manifold over another manifold. The resulting construction is a common concept in the compactification process and it will be very important in our geometric understanding of the c-map.

Definition 2.1 (Fibre bundle). Let E, M, F be smooth manifolds, let G be a Lie group that acts on F from the left and let $\pi : E \to M$ be a surjection, called the *projection map*. Then the tuple (E, π, M, F, G) is called a *fibre bundle over* M if

1. there is an open covering $\{U_i\}$ of M with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$ for all $(p, f) \in U_i \times F$. A pair (U_i, ϕ_i) is called a *local trivialization* of the bundle.

2. for every pair $(U_i, \phi_i), (U_j, \phi_j)$ of local trivializations we have a diffeomorphism

$$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j \times F) \to (U_i \cap U_j \times F) : (p, f) \mapsto (p, t_{ij}(p)f),$$

   where $t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ is an element of G, called a *transition function*. Here $\phi_{i,p}(\cdot) = \phi_i(p, \cdot)$ denotes the diffeomorphism $\phi_{i,p} : F \to E_p$, where $E_p = \pi^{-1}(p) \cong F$ is called the *fibre at* p.

Often the fibre bundle is denoted with $\pi : E \to M$, when explicit mentioning of F and G is not necessary. E is called the *total space* of the fibre bundle, F is called the *typical fibre*, M the *base space* and G is called the *structure group*. ⊘

Remark: In an abuse of notation sometimes E itself is called the fibre bundle and the dimension of the typical fibre is then denoted by $\dim(E)$, while actually the dimension of E as a manifold is $\dim(E) = \dim(M) + \dim(F)$. In most cases it should be clear from the context what is meant by $\dim(E)$. ◊

When the typical fibre F is a real vector space of dimension r, the fibre bundle is a *vector bundle of rank r*. Its structure group is $\mathrm{GL}(r,\mathbb{R})$. For $r = 1$ the vector bundle is called a *line bundle*. Another special type of fibre bundle is one for which the fibres are equal to the structure group.

Definition 2.2 (Principal fibre bundle). Let M be a smooth manifold and let G be a Lie group. A *principal fibre bundle* $\pi : P_G \to M$ is a fibre bundle (P_G, π, M, G, G) together with a fibre preserving continuous right action of G on P_G acting freely and transitively on each fibre. A principal fibre bundle P_G is often called a *G-bundle*. $\oslash$

"Functions" from the base manifold with values in the typical fibre of a fibre bundle are called *sections*. For example a tangent vector field $v \in \mathscr{X}(M) = \Gamma(TM)$ is a section of the tangent bundle of a manifold M and an arbitrary (r,q)-tensor field $T \in \mathcal{T}^r_q(M) = \Gamma(\bigotimes^r TM \otimes \bigotimes^q T^*M)$ is a section of the tensor bundle obtained from r copies of the tangent bundle and q copies of the cotangent bundle.

Definition 2.3 (Section of a fibre bundle). Let $\pi : E \to M$ be a fibre bundle and let U be a subset of M, then a *local section of E* is a smooth map $s : U \to E$ satisfying $\pi \circ s = id_U$. The set of local sections on U is denoted by $\Gamma(U, E)$. When $U = M$, a smooth map $s : M \to E$ satisfying $\pi \circ s = id_M$ is simply called a *section of E*. $\oslash$

Connections

Given a vector bundle, we would like to compare data from different fibres. The notion of a *connection* or *covariant derivative* makes it possible to do this in a consistent manner [GH78, Bes87, Nak03, Fra04].

Definition 2.4 (Connection). Let M be an n-dimensional manifold. The *covariant derivative* ∇ is a map $\nabla : (v,T) \mapsto \nabla_v T$, sending a vector field v and an (r,q)-tensor field T to an $(r,q+1)$-tensor field $\nabla_v T$, such that

$$
\begin{aligned}
\nabla_{(fv+gw)}T &= f\nabla_v T + g\nabla_w T,\\
\nabla_v(T+S) &= \nabla_v T + \nabla_v S,\\
\nabla_v(T\otimes S) &= \nabla_v T\otimes S + T\otimes\nabla_v S,\\
\nabla_v f &= v(f),
\end{aligned}
$$

for any vector fields v, w, any tensor fields T, S and any functions f, g.

If $\{e_i\}_{i=1}^n$ is a frame of vectors and $\{f^j\}_{j=1}^n$ its dual frame of 1-forms, then the covariant derivative is completely determined by a *connection*. This is a matrix of *connection 1-forms*, i.e. n^3 coefficients, ${\omega^k}_i = \omega^k_{ji} f^j$ defined by

$$\nabla e_i(e_j) = \nabla_{e_j} e_i = e_k \omega^k_{ji}.$$

The covariant derivative of a vector field, being a vector-valued 1-form, can be given in terms of the connection 1-forms,

$$\nabla e_i = e_k \otimes {\omega^k}_i.$$

Such a connection on the tangent bundle of M can be extended to a connection on a tensor bundle, such that the covariant derivative of an arbitrary (r,q)-tensor field T is given by,

$$
\begin{aligned}
{(\nabla_v T)^{i_1\ldots i_r}}_{j_1\ldots j_q} = v^l\Big[&\partial_l {T^{i_1\ldots i_r}}_{j_1\ldots j_q}\\
&+\omega^{i_1}_{lk}{T^{ki_2\ldots i_r}}_{j_1\ldots j_q} + \omega^{i_2}_{lk}{T^{i_1k\ldots i_r}}_{j_1\ldots j_q} + \ldots + \omega^{i_r}_{lk}{T^{i_1\ldots i_{r-1}k}}_{j_1\ldots j_q}\\
&-\omega^k_{lj_1}{T^{i_1\ldots i_r}}_{kj_2\ldots j_q} - \omega^k_{lj_2}{T^{i_1\ldots i_r}}_{j_1k\ldots j_q} - \ldots - \omega^k_{lj_q}{T^{i_1\ldots i_r}}_{j_1\ldots j_{q-1}k}\Big],
\end{aligned}
$$

where we have used Einstein's summation convention.

A tensor field T for which $\nabla_v T = 0$ for all $v \in \mathscr{X}(M)$ is said to be *parallel* or *covariantly constant* with respect to the connection ∇. $\oslash$

Since covariant differentiation, connection and connection 1-forms all determine the same type of information, viz. how to transport data from one fibre to another, terminology and notation may vary from author to author. Note that different connection 1-forms define different ways to differentiate tensor fields along a vector field. For a Riemannian manifold (M, g) the *Levi-Civita connection* or *metric connection* is uniquely defined by demanding that the metric g is covariantly constant with respect to a symmetric connection ∇ [Car97]. The connection coefficients are then given by the familiar *Christoffel symbols* Γ^k_{ij}, which are used to define the *Riemann curvature tensor*, *Ricci tensor* and *Ricci scalar*.

Similar to the affine connection on vector bundles, the *Maurer-Cartan form* $\Omega \in \mathfrak{g} \otimes \Omega^1(G)$ on a Lie group G may be used to compare adjacent fibres of a principal fibre bundle [Bes87, Nak03, Fra04, Dui04a].

Definition 2.5 (Maurer-Cartan form). The *Maurer-Cartan form* of a Lie group G is the $\mathfrak{g}$-valued 1-form $\Omega : \mathscr{X}(g) \to T_eG = \mathfrak{g}$ defined by

$$\Omega_g(v) = (l_{g^{-1}})_* v, \qquad v \in T_gG,$$

where $l_g : G \to G$ denotes *left multiplication* in G. For a matrix Lie group G, the Maurer-Cartan form Ω is defined at each point $g \in G$ by

$$\Omega_g = g^{-1}\mathrm{d}g. \tag{2.1}$$

⊘

The Maurer-Cartan form is invariant under left multiplication by another (constant) element $h \in G$, $(l_h)^*\Omega_{hg} = \Omega_g$. It defines a connection 1-form on a principal fibre bundle P_G which acts as the identity on $T_eG = \mathfrak{g}$ and which satisfies $\Omega_g = (r_{g^{-1}})^*\Omega$. In chapter 7 we will use it to construct a metric which is invariant under a Lie group action.

2.2 Complex, hermitian and symplectic structure

Complex manifolds

A real differentiable manifold is a topological space which locally looks like an open subset of $\mathbb{R}^n$. Now we will look at an extra structure which we may impose on differentiable manifolds, namely that of a *complex structure*. Differentiable manifolds with a complex structure are topological spaces which locally look like a part of $\mathbb{C}^n$ rather than $\mathbb{R}^n$. In order to define complex manifolds the notion of *holomorphicity* is of inescapable importance.

Definition 2.6 (Cauchy-Riemann equations). Let $f : \mathbb{C}^n \to \mathbb{C}$ be a complex function. We write

$$z = x + iy = (z^j)_{j=1}^n = (x^j + iy^j)_{j=1}^n, \qquad x^j, y^j \in \mathbb{R}^n$$

and $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then f is *holomorphic* if and only if it satisfies the *Cauchy-Riemann equations*,

$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \qquad \frac{\partial v}{\partial x^j} = -\frac{\partial u}{\partial y^j}, \qquad \forall j \in \{1, \ldots, n\}.$$

⊘

Intuitively holomorphicity of a function f is the property of respecting the additional structure present in complex spaces compared with their real counterparts. A holomorphic function respects the constraints, since it only depends on z and not on the complex conjugate $\bar{z}$. From the point of view of $\mathbb{R}^{2n}$, the constraints are exactly the Cauchy-Riemann equations. They ensure that mappings preserve the “complexity” of whatever is defined. For example the definition of a *complex manifold* is similar to that of a real manifold in that it is an object which may locally be viewed as an open subset of $\mathbb{C}^n$. However coordinate transformations are now required to be holomorphic [Gre96, Nak03, Bou07]. Alternatively an n-dimensional complex manifold may be obtained from a $2n$-dimensional real manifold by endowing it with a *complex structure*.

Definition 2.7 (Almost complex structure). Let M be a $2n$-dimensional (real) manifold. A globally defined complex $(1,1)$-tensor field J, which satisfies

$$J_p^2 = -\mathbb{I}_{2n},$$

at each point $p \in M$, is called an *almost complex structure*. The manifold (M, J) is then called an *almost complex manifold*. $\oslash$

Notation: The dimension of a complex manifold M is denoted with $\dim_{\mathbb{C}} M$. Since $\dim_{\mathbb{C}} \mathbb{C}^n = \frac{1}{2}\dim_{\mathbb{R}} \mathbb{R}^{2n}$ the complex dimension of a complex manifold is half the real dimension of the underlying real manifold. We will use subscripts $_{\mathbb{C}}$ and $_{\mathbb{R}}$ to distinguish between both types of dimensions when necessary. $\Diamond$

Locally each $2n$-dimensional (real) manifold admits an almost complex structure, but only on an *almost complex manifold* the tensor field may be smoothly patched together *globally*. Due to the Cauchy-Riemann equations that are obeyed by the coordinate transformations of a complex manifold, every complex manifold is an almost complex manifold by defining the almost complex structure (in local coordinates $\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$),

$$J_p \frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \qquad J_p \frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}. \tag{2.2}$$

Therefore a complex manifold is often given by specifying a real even-dimensional manifold admitting an almost complex structure J, although this glances over the fact that not every almost complex structure defines a proper complex manifold. For this to be the case the almost complex structure has to be *integrable* or equivalently the *Nijenhuis tensor field* $N : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ has to vanish for all $v, w \in \mathscr{X}(M)$ [NN57, Nak03]. In that case the almost complex structure is called a *complex structure* and only then J defines a *complex manifold*. The difference between complex and almost complex structures will be important for the distinction between hyperkähler and quaternion-Kähler manifolds.

When considering tangent and cotangent vectors of an n-dimensional complex manifold it is useful to consider the complexification of the tangent bundle of the underlying $2n$-dimensional real manifold.

Definition 2.8 (Complexification). Let V be a real vector space. We define the *complexification* of V to be the real linear space, $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Multiplication by a complex scalar $\lambda \in \mathbb{C}$ is defined by

$$\lambda \sum_j^n v_j \otimes z_j = \sum_j v_j \otimes (\lambda z_j),$$

and an $\mathbb{R}$-linear map $f : V \to W$ is complexified to a $\mathbb{C}$-linear map $f : V^{\mathbb{C}} \to W^{\mathbb{C}}$ via

$$f\left(\sum_j v_j \otimes z_j\right) = \sum_j f(v_j) \otimes z_j.$$

$\oslash$

Definition 2.9 (Holomorphic tangent bundle). Let (M, J) be an almost complex manifold. Consider the complexification of the tangent bundle TM and the complex extension of J on TM via definition 2.8. We define the *holomorphic* TM^+ and *antiholomorphic tangent bundle* TM^- by

$$T_pM^{\pm} = \{v \in T_pM^{\mathbb{C}} | J_p v = \pm iv\}.$$

Note that $TM^{\mathbb{C}}$ may be divided into these two eigenspaces of J, $TM^{\mathbb{C}} = TM^+ \oplus TM^-$. $\oslash$

Definition 2.10 (Differential forms on complex manifolds). Let M be an n-dimensional complex manifold and let $\omega \in \Omega^k(M)^{\mathbb{C}}$, $k \leq 2n$, be a (complexified) k-form on M. It is called a *differential* (r,s)*-form*, or a differential form of *bidegree* (r,s), if it smoothly assigns at each point $p \in M$ a totally antisymmetric $(0,k)$-tensor, such that $\omega(v_1,\dots,v_k) = 0$ unless r of the v_i are in T_mM^+ and s of them are in T_mM^-. The set of (r,s)-forms over M is denoted with $\Omega^{r,s}(M)$. ⊘

Each complex k-form can be uniquely decomposed into $(r,k-r)$-forms, $1 \leq r \leq k$. We now recall the *exterior derivative* $d : \Omega^k \to \Omega^{k+1}$ and let it (extended to a $\mathbb{C}$-linear map on $(\Omega^k)^{\mathbb{C}}$) act on an (r,s)-form,

$$\mathrm{d} : \Omega^{r,s} \to \Omega^{r+1,s} \oplus \Omega^{r,s+1} : \omega \mapsto \mathrm{d}\omega.$$

In local coordinates,

$$\begin{aligned}
\mathrm{d}\omega \quad = \quad & \frac{1}{r!s!}\frac{\partial}{\partial z^l}\omega_{i_1\dots i_r j_1\dots j_s}\mathrm{d}z^l \wedge \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_r} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{j}_s} \\
& + \frac{1}{r!s!}\frac{\partial}{\partial \bar{z}^{\bar{l}}}\omega_{i_1\dots i_r j_1\dots j_s}\mathrm{d}\bar{z}^{\bar{l}} \wedge \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_r} \wedge \mathrm{d}\bar{z}^{\bar{j}_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{j}_s},
\end{aligned}$$

which shows that the exterior derivative d may be decomposed into two operators, ∂ and $\bar{\partial}$.

Definition 2.11 (Dolbeault operators). We define *Dolbeault operators* $\partial_{r,s} : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$, $\bar{\partial}_{r,s} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$ acting on a complex (r,s)-form $\omega \in \Omega^{r,s}(M)$ by

$$\partial_{r,s} = \pi_{r+1,s} \circ \mathrm{d}, \qquad \bar{\partial}_{r,s} = \pi_{r,s+1} \circ \mathrm{d},$$

where $\pi_{r,s}$ is the projection on $\Omega^{r,s}(M)$. The subscript is suppressed and the actions of ∂ and $\bar{\partial}$ on an arbitrary k-form $\omega = \sum_{r+s=k} \omega^{(r,s)}$, $\omega^{(r,s)} \in \Omega^{r,s}(M)$, is given by

$$\partial\omega = \sum_{r+s=k} \partial\omega^{(r,s)}, \qquad \bar{\partial}\omega = \sum_{r+s=k} \bar{\partial}\omega^{(r,s)}.$$

⊘

Analogously to the concept of a holomorphic function, $\frac{\partial}{\partial \bar{z}} f(z) = 0$, we define the *holomorphic differential form.*

Definition 2.12 (Holomorphic differential form). Let M be a complex manifold. A *holomorphic differential* r*-form* is an $(r,0)$-form $\omega \in \Omega^{r,0}(M)$ which satisfies $\bar{\partial}\omega = 0$. ⊘

Hermitian manifolds

Vital for doing geometry on (complex) manifolds is the notion of a (hermitian) metric.

Definition 2.13 (Riemannian manifold). Let M be an n-dimensional differentiable manifold. A *nondegenerate* symmetric $(0,2)$-tensor field $g \in \mathcal{T}^0_2(M)$ is called a *Riemannian metric* on M. If M is endowed with a Riemannian metric g, we call the pair (M,g) a *Riemannian manifold*. ⊘

Notation: We allow the matrix $g_{ij}(p) = g_p(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j})$ to have both positive as well as negative, but always nonzero, eigenvalues. Strictly speaking the metric is therefore a *pseudo*-metric on a *pseudo*-Riemannian manifold, but we will not use this prefix. The *signature* of the metric is written as (m,k), where m is the number of positive eigenvalues and $k = \mathrm{ind}(g)$ is the number of negative eigenvalues of g. The latter is also called the *index* of g.

The metric $g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ is often denoted with its *line element*

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i)$.

Furthermore in expressions such as $\mathrm{sgn}(g)$ and $\sqrt{g}$, g is understood to be the (absolute value of the) *determinant* of the matrix $g_{ij}(p)$. ♢

The complex analogue of a Riemannian manifold is the *hermitian manifold*. Its metric is defined in terms of a *hermitian bilinear form*.

Definition 2.14 (Hermitian bilinear form). Let V be a complex vector space, then a *hermitian bilinear form* h is a *nondegenerate sesquilinear* map $h : V \times V \to \mathbb{C}$, i.e. $h(\lambda v, w) = \lambda h(v, w) = h(v, \bar{\lambda} w)$, which is *conjugate symmetric*, i.e. $h(v, w) = \overline{h(w, v)}$. $\oslash$

Remark: When we consider h as a matrix acting on V it is a *hermitian matrix*, $\bar{h}^t = h$, which is known to have real eigenvalues. Therefore the *signature* and *index* of a hermitian form are well-defined. $\Diamond$

If we identify the complex n-dimensional vector space V with a vector space V over $\mathbb{R}$ of dimension $\dim_{\mathbb{R}} V = 2n$ by introducing the complex structure J, the conjugate symmetry of h implies that $g = \operatorname{Re} h$ is a nondegenerate (real) *symmetric* bilinear form on V, while $\sigma = \operatorname{Im} h$ is a nondegenerate (real) *antisymmetric* bilinear form on V. Using J they are related via

$$\sigma(v, w) = \operatorname{Im} h(v, w) = -\operatorname{Re} ih(v, w) = -\operatorname{Re} h(Jv, w) = -v^t J^t g w.$$

We see that each of the three objects h, g or σ determines the other two using the complex structure J,

$$\begin{aligned} g =& \operatorname{Re} h, & g =& J^t \sigma, & \sigma =& - J^t g, \\ \sigma =& \operatorname{Im} h, & h =& J^t \sigma + i\sigma, & h =& g - iJ^t g. \end{aligned} \tag{2.3}$$

Due to the following lemma, the conjugate symmetry of h is equivalent with g (and hence σ and h) preserving the complex structure.

Lemma 2.15 (Conjugate symmetry). *Suppose h is a nondegenerate sesquilinear form on a complex vector space V and suppose g is a real inner product on V seen as a vector space over $\mathbb{R}$ by using the complex structure J. If $h(w, v) = \overline{h(v, w)}$, $\forall v, w \in V$ and $g = \operatorname{Re} h$, then $g(Jv, Jw) = g(v, w)$, $\forall v, w \in V$. Conversely if $g(Jv, Jw) = g(v, w)$, $\forall v, w \in V$ and $h = g - iJ^t g$, then $h(w, v) = \overline{h(v, w)}$, $\forall v, w \in V$.*

Proof. Suppose $h^t = \bar{h}$ and $g = \operatorname{Re} h$, then

$$J^t g J = -\sigma J = -(J^t \sigma^t)^t = (J^t \sigma)^t = g^t = g.$$

Conversely suppose $J^t g J = g$ and $h = g - iJ^t g$, then

$$h^t = g^t - ig^t J = g + iJ^t (J^t g J) = g + iJ^t g = \bar{h}.$$ $\square$

The complexification of g is the complex analogue of a metric on the complexified tangent space $T_p M^{\mathbb{C}}$ of a complex manifold. By bundling them we obtain a *hermitian manifold*.

Definition 2.16 (Hermitian manifold). Let M be a complex manifold with smoothly varying hermitian form h_p at each point $p \in M$. The associated real part $g = \operatorname{Re} h$ is then a Riemannian metric on M, satisfying $g(Jv, Jw) = g(v, w)$ $\forall v, w \in V$. The complexification $\tilde{g}$ of g, defined by

$$\tilde{g}(u_1, u_2) = g(v_1, v_2) - g(w_1, w_2) + i \left[g(w_1, v_2) + g(v_1, w_2) \right], \tag{2.4}$$

for any two vector fields $u_1 = v_1 + iw_1, u_2 = v_2 + iw_2 \in \mathscr{X}(M)$, is called a *hermitian metric* and the pair $(M, \tilde{g})$ is called a *hermitian manifold*. $\oslash$

The complexification of the imaginary part of the smoothly varying hermitian bilinear form is called the *hermitian differential form*.

Definition 2.17 (Hermitian differential form). Let $\tilde{g}$ be a hermitian metric on a manifold M, then we define the *hermitian differential form* $\tilde{\sigma}$ by

$$\tilde{\sigma} = J^t \tilde{g}.$$

It is a nondegenerate real-valued antisymmetric differential $(1, 1)$-form. $\oslash$

Remark: The terminology in the literature is not always clear about the notion of a hermitian form or hermitian metric. Obviously the existence of a hermitian (bilinear) form, definition 2.14, and a hermitian (differential) form, definition 2.17, is misleading. Furthermore the difference between the hermitian form h, which is a sesquilinear map acting on TM^+, its real part g and the hermitian metric $\tilde{g}$, which is a bilinear map acting on $TM^{\mathbb{C}}$, is not always made explicitly. Since each of these objects can be obtained from the other it is not always necessary to distinguish them, but it could cause confusion when it is not clear from the context which object is meant precisely.

We will follow the tradition of not exactly distinguishing these objects and will stop writing $\tilde{}$'s on top of the complexified objects. ◊

Symplectic manifolds

At first sight the notion of a *symplectic manifold* is similar to that of Riemannian manifolds. Instead of a *symmetric* bilinear form, *symplectic geometry* focuses on manifolds which allows for an *antisymmetric* bilinear form and studies their properties. It turns out that symplectic geometry is in many cases even richer (and more interesting) than Riemannian geometry. We will only consider a few fundamental concepts of symplectic geometry. Readers interested in this subject could consult [Dui04b] and references therein.

Definition 2.18 (Symplectic vector space). A nondegenerate antisymmetric bilinear form σ on a $2n$-dimensional real vector space V is called a *symplectic form*. The pair (V, σ) is called a *symplectic vector space*. ⊘

Definition 2.19 (Lagrange plane). Let (V, σ) be a symplectic vector space. A linear subspace L is called *isotropic with respect to* σ if $\sigma(u, v) = 0$ for all $u, v \in L$. A maximal isotropic linear subspace L of V is called a *Lagrange plane*. This is equivalent with saying that L is isotropic and $2\dim(L) = \dim(V)$. ⊘

We extend these concepts to the realm of manifolds by demanding that at each point of the manifold the tangent space has a symplectic structure.

Definition 2.20 (Symplectic manifold). Let M be a $2n$-dimensional manifold and let σ be a smooth differential 2-form on M such that

1. σ is closed,
2. the bilinear form σ_p is a symplectic form on T_pM for every $p \in M$.

The pair (M, σ) is called a *symplectic manifold* and σ is called the *symplectic form on* M. ⊘

Remark: The simplest example of a symplectic manifold is $\mathbb{R}^{2n}$ endowed with the 2-form $\sigma = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$, called the *standard* or *canonical symplectic form*. Here $(q^i, p_i)_{i=1}^n$ are coordinate functions of $\mathbb{R}^{2n}$. With respect to a frame $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}\}_{i=1}^n$ of $T\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$, we may write σ as a matrix $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$. *Darboux's lemma* states that any symplectic manifold (M, σ) may locally be brought into canonical form [Dui04b]. For this reason calculations using local coordinates are (almost) always done using the canonical symplectic form, either in its differential form or in its matrix notation. A basis for which the symplectic form is in its canonical matrix form Σ is then called a *symplectic basis*. ◊

Definition 2.21 (Lagrange manifold). Let (M, σ) be a $2n$-dimensional symplectic manifold. An n-dimensional submanifold $\Lambda \subset M$ is called a *Lagrange submanifold* (or simply *Lagrangian*[1]) if for all $p \in \Lambda$, $T_p\Lambda$ is a Lagrange plane in T_pM with respect to the symplectic form σ_p. ⊘

When a complex manifold is even-dimensional we may define the complex generalization of a symplectic manifold, by demanding the symplectic form to be holomorphic.

[1]This notion of "Lagrangian" should not be confused with a Lagrangian $\mathscr{L}$ that determines a physical theory.

Definition 2.22 (Complex symplectic manifold). Let M be a $2n$-dimensional complex manifold and let σ be a smooth closed holomorphic differential 2-form on M that is nondegenerate at each point $p \in M$. The pair (M, σ) is called a *complex symplectic manifold*. ⊘

We see that by demanding suitable conditions it is possible to generalize symplectic geometry to complex manifolds. However complex manifolds already play an important role in symplectic geometry, since they are real even-dimensional by their mere existence. There is a natural way to define a symplectic structure on a hermitian manifold (M, g), in which the complex structure J is compatible with the symplectic structure defined by the hermitian differential form. Comparison of definition 2.20 and 2.17 shows that we can link the two structures by demanding the hermitian differential form to be *closed*. *Kähler manifolds* are a way to combine the complex and symplectic structure of a hermitian manifold seen as a real even-dimensional manifold.

Definition 2.23 (Kähler manifold). Let (M, g) be a hermitian manifold whose hermitian differential form $\sigma_J = J^t g$ is closed, then (M, σ_J) is called a *Kähler manifold*, g is called the *Kähler metric* and σ_J the *Kähler form* on M. Locally there exists a function $K(z, \bar{z})$, such that

$$g_{i\bar{j}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}} \qquad \text{and hence} \qquad \sigma_J = i\partial\bar{\partial}K. \tag{2.5}$$

K is called the *Kähler potential*. ⊘

Proof. Cf. [Nak03] for the existence of a Kähler potential. □

2.3 Homology and cohomology

Dolbeault cohomology groups of Kähler manifolds are important in the study of Calabi-Yau$_3$-compactifications of string theory. In this section we review their most important properties and relations.

Poincaré duality

Definition 2.24 (De Rham cohomology group). Let M be an n-dimensional manifold. The k-th *de Rham cohomology group* is the group of closed differential k-forms modulo exact differential k-forms,

$$H^k(M, R) = \frac{\mathrm{Ker}(\mathrm{d} : \Omega^k \to \Omega^{k+1})}{\mathrm{Im}(\mathrm{d} : \Omega^{k-1} \to \Omega^k)}. \tag{2.6}$$

Here $R = \mathbb{R}, \mathbb{C}$ depending on whether the differential forms are real or complex. ⊘

A similar structure may be defined on *chains* in an n-dimensional manifold M. A k-chain is defined using a formal sum of *singular k-simplices* in the manifold M. A *standard k-simplex σ* in $\mathbb{R}^n$ is the convex set generated by $(k+1)$ points $p_i = (0, \ldots, 1, \ldots, 0)$ where 1 is on the i-th position, $0 \leq i \leq k$. We write $\sigma = (p_0 p_1 \ldots p_k)$. A mapping $f : \sigma \to M$ of the standard k-simplex into the manifold defines the *singular k-simplex* $s = f(\sigma)$. A k-chain c is then a formal sum of such singular k-simplices,

$$c = \sum_l a_l s_l,$$

with coefficients a_l in either $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$. These chains form a group, the *chain group* $C_k(M, R)$.

The *boundary operator* δ maps a k-chain to its oriented boundary, which is a $(k-1)$-chain,

$$\delta c = \sum_l a_l \sum_{j=0}^{k} (-1)^j f_l \left((p_0 p_1 \ldots \widehat{p_j} \ldots p_k) \right).$$

We define the subgroups of *cycles* and *boundaries* respectively by,

$$Z_k(M) = \{c \in C_k | \delta c = 0\}, \qquad B_k(M) = \{c \in C_k | c = \delta b, b \in C_{k+1}\}.$$

Because $\delta^2 = 0$ [Nak03] we may define the *singular homology group*, which measures those k-cycles that are not boundaries of some $(k+1)$-chain.

Definition 2.25 (Singular homology group). Let M be an n-dimensional manifold. The k-th *singular homology group* is the group of k-cycles in M modulo k-boundaries,

$$H_k(M,R) = \frac{\operatorname{Ker}(\delta : C_k \to C_{k-1})}{\operatorname{Im}(\delta : C_{k+1} \to C_k)}. \tag{2.7}$$

Here $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ depending on whether the formal sums are allowed to have integer, real or complex coefficients. ⊘

To give the reader some intuition we give the following example. A triangular piece of a two-dimensional manifold is a 2-chain. Its edges form the boundary of the triangular shape, which is a formal (and oriented) sum of lines, i.e. 1-simplices. In a three-dimensional manifold the triangular shape could be part of the boundary of a tetrahedron, but a two-dimensional manifold is too "small" to contain a tetrahedron. Thus the triangular shape is not the boundary of any 3-chain in two dimensions.

The most important thing to remember about chains, cycles and boundaries is that they may be used to integrate differential forms over. A k-chain may be used as the domain of integration of a differential k-form. Stokes's theorem then ensures that the bilinear map $\int : C_k \times \Omega^k \to R : (c,\omega) \mapsto \int_c \omega$ is well-defined on $H_k(M,R)$ and $H^k(M,R)$. By dimensionality argumentations De Rham was able to show that this induces an isomorphism between vector spaces [GH78, Nak03].

Theorem 2.26 (De Rham's theorem). *Let M be a compact manifold and suppose $H^k(M,R)$, $H_k(M,R)$ are finite dimensional, then the de Rham cohomology group $H^k(M,R)$ and the singular homology group $H_k(M,R)$ are dual vector spaces.*

Another interesting duality is *Poincaré duality*, whose application on the middle singular homology $H_n(M,\mathbb{R})$ of an n-dimensional complex manifold M will be important throughout this thesis. $H_n(M,\mathbb{R})$ has a symplectic basis of cycles $\{\gamma_{A^i}, \gamma_{B_i}\}_{i=1}^m$ (where $m = \frac{1}{2}\dim_{\mathbb{R}} H_n(M,\mathbb{R})$) with respect to the *intersection product* $\gamma_{A^i} \cap \gamma_{B_j} = \delta^i_j$. Because of Poincaré duality the intersection product on $H_n(M,\mathbb{R})$ induces a symplectic form $\int_M \phi \wedge \chi$ on $H^n(M,\mathbb{R})$, called the *intersection form*, for which the basis $\{\alpha_i, \beta^i\}_{i=1}^m \subset H^n(M,\mathbb{R})$ dual to $\{\gamma_{A^i}, \gamma_{B_i}\}_{i=1}^m$ is a symplectic basis[2] [GH78],

$$\int_{A^j} \alpha_i = -\int_{B_i} \beta^j = \int_M \alpha_i \wedge \beta^j = \delta^j_i, \qquad \int_{A^j} \beta^i = \int_{B_i} \alpha_j = 0. \tag{2.8a}$$

An element $\phi \in H^n(M,\mathbb{R})$ may be decomposed with respect to this basis into its A- and *B-periods*,

$$\phi = \sum_{i=1}^m \left(\int_{\gamma_{A^i}} \phi \right) \alpha_i - \left(\int_{\gamma_{B_i}} \phi \right) \beta^i. \tag{2.8b}$$

Proposition 2.27 (Poincaré duality). *Let M be an n-dimensional compact manifold, then for each $k \in \{0, \ldots, n\}$, $H^k(M) \cong (H^{n-k}(M))^*$. Furthermore let X be a closed k-dimensional submanifold of M, then there exists an $(n-k)$-form $\eta_X \in H^{n-k}(M)$ such that*

$$\int_X \omega = \int_M \eta_X \wedge \omega, \qquad \forall\, [\omega] \in H^k(M). \tag{2.9}$$

Proof. Cf. [BT82, Bou07]. □

Corollary 2.28 (Reciprocal law). *Let M be a complex n-dimensional compact manifold and suppose $\{\gamma_{A^i}, \gamma_{B_i}\}_{i=1}^m$ is a basis for $H_n(M,\mathbb{R})$ (where $m = \frac{1}{2}\dim H_n(M,\mathbb{R})$), such that*

$$\gamma_{A^i} \cap \gamma_{B_j} = -\gamma_{B_j} \cap \gamma_{A^i} = \delta^i_j, \qquad \gamma_{A^i} \cap \gamma_{A^j} = \gamma_{B_i} \cap \gamma_{B_j} = 0, \qquad \forall 1 \le i,j \le m.$$

[2] The minus-sign occurring in (2.8) is a choice conventionally made in the physics literature.

Then for any two closed n-forms ϕ, χ,

$$\int_M \phi \wedge \chi = -\sum_{i=1}^{m} \left[\int_{\gamma_{A^i}} \phi \int_{\gamma_{B_i}} \chi - \int_{\gamma_{B_i}} \phi \int_{\gamma_{A^i}} \chi \right]. \tag{2.10}$$

Proof. The formula follows if one applies (2.9) to the submanifolds γ_{A^i} and γ_{B_i}. □

Dolbeault cohomology and harmonic forms

Due to the property $\bar{\partial}^2 = 0$ of the Dolbeault operator $\bar{\partial}$ we may define a refinement of the de Rham cohomology on complex manifolds.

Definition 2.29 (Dolbeault cohomology group). Let M be a complex n-dimensional manifold. The (p,q)-th *Dolbeault cohomology group* is the group of $\bar{\partial}$-closed (complex) differential (p,q)-forms modulo $\bar{\partial}$-exact (complex) differential (p,q)-forms,

$$H^{p,q}(M) = \frac{\mathrm{Ker}(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}. \tag{2.11}$$

⊘

Dolbeault cohomology classes have an interesting relationship with *harmonic forms* on a hermitian manifold through Hodge's theorem. Harmonic forms are those forms that are eigenforms of the *Laplace operator* $\Box$ with eigenvalue 0. This operator is most easily defined using the *Hodge star operator* $\star$.

Definition 2.30 (Hodge star operator). Let (M, g) be an n-dimensional hermitian manifold. The *Hodge star operator* $\star$ is the isomorphism $\star : \Omega^k(M)^{\mathbb{C}} \to \Omega^{2n-k}(M)^{\mathbb{C}}$ which sends a complex k-form ω to the unique $(2n-k)$-form $\star\omega$ satisfying

$$\xi \wedge (\star\omega) = \xi_{i_1 \dots i_k} \overline{\omega^{i_1 \cdots i_k}} \mathrm{d}V_g, \qquad \forall \xi \in \Omega^k(M)^{\mathbb{C}},$$

where $\mathrm{d}V_g$ is the volume form on M, $\mathrm{d}V_g = \sqrt{g}\mathrm{d}^n z \wedge \mathrm{d}^n \bar{z} = \star 1$.

In local coordinates the Hodge star operator acts as

$$(\star\omega)_{i_1 \dots i_{2n-k}} = \frac{1}{k!} \varepsilon_{i_1 \dots i_{2n-k}}{}^{j_1 \dots j_k} \omega_{j_1 \dots j_k},$$

where ε is the *Levi-Civita tensor*. That is ε is obtained from the *Levi-Civita symbol* $\tilde{\varepsilon}$ via $\varepsilon_{i_1 \dots i_{2n}} = \sqrt{g}\tilde{\varepsilon}_{i_1 \dots i_{2n}}$. ⊘

Definition 2.31 (Laplace operators). Let (M, g) be an n-dimensional hermitian manifold. The *Laplace operators* $\Box_{\mathrm{d}}$, $\Box_{\partial}$ and $\Box_{\bar{\partial}}$ are defined via the *adjoint operators* $\mathrm{d}^\dagger$, $\partial^\dagger$ and $\bar{\partial}^\dagger$,

$$\begin{aligned} \mathrm{d}^\dagger &= -\star\mathrm{d}\star, & \partial^\dagger &= -\star\partial\star, & \bar{\partial}^\dagger &= -\star\bar{\partial}\star, \\ \Box_{\mathrm{d}} &= \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}, & \Box_{\partial} &= \partial\partial^\dagger + \partial^\dagger\partial, & \Box_{\bar{\partial}} &= \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \end{aligned}$$

The (complex) *harmonic k-forms* and *harmonic (p,q)-forms* are the eigenforms of the Laplace operators $\Box_{\mathrm{d}}$ and $\Box_{\bar{\partial}}$ with eigenvalue 0,

$$\begin{aligned} \mathcal{H}^k(M) &= \{\omega \in \Omega^k(M)^{\mathbb{C}} | \Box_{\mathrm{d}}\omega = 0\} = \mathrm{Ker}(\Box_{\mathrm{d}} : \Omega^k(M)^{\mathbb{C}} \to \Omega^k(M)^{\mathbb{C}}), \\ \mathcal{H}^{p,q}(M) &= \{\omega \in \Omega^{p,q}(M) | \Box_{\bar{\partial}}\omega = 0\} = \mathrm{Ker}(\Box_{\bar{\partial}} : \Omega^{p,q}(M) \to \Omega^{p,q}). \end{aligned}$$

⊘

Theorem 2.32 (Hodge's theorem). *Let (M, g) be a compact hermitian manifold M, then the space of harmonic (p,q)-forms is isomorphic to the (p,q)-th Dolbeault cohomology group,*

$$\mathcal{H}^{p,q}(M) \cong H^{p,q}(M). \tag{2.12}$$

Proof. Cf. [GH78]. □

Cohomology on Kähler manifolds

On Kähler manifolds the different Laplace operators are related, $\Box_d = 2\Box_{\bar\partial} = 2\Box_\partial$. As a result Hodge's theorem 2.32 induces a decomposition for the de Rham cohomology groups [GH78, Bou07].

Theorem 2.33 (Hodge decomposition). *Let M be an n-dimensional compact Kähler manifold. Then for each $0 \le k \le 2n$ the complex de Rham cohomology has a* Hodge decomposition *into Dolbeault cohomology groups,*

$$H^k(M,\mathbb{C}) = \bigoplus_{\substack{p+q=k\\ 0\le p,q\le k}} H^{p,q}(M).$$

Definition 2.34 (Hodge diamond). Let M be an n-dimensional compact Kähler manifold. We define the *Hodge number* $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$, which are grouped together in the *Hodge diamond*

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & \vdots & & & \ddots & \\
h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n}. \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}$$

$\oslash$

Remark: The values in the Hodge diamond are not all independent. Conjugation $H^{p,q} = \overline{H^{p,q}} \cong H^{q,p}$ gives a symmetry along the vertical line, $h^{p,q} = h^{q,p}$, while Poincaré duality $H^{p,q} = (H^{n-p,n-q})^*$ gives a symmetry about the center of the diamond, $h^{p,q} = h^{n-p,n-q}$. $\Diamond$

Alternatively to the Hodge decomposition we may define the *Hodge filtration* [Gri84].

Definition 2.35 (Hodge filtration). Let M be an n-dimensional compact Kähler manifold and let $k \in \{0,\dots,2n\}$. For $p \in \{0,\dots,k\}$ we define

$$F^pH^k(M) = \bigoplus_{s=p}^{k} H^{s,k-s}(M),$$

which gives the *Hodge filtration*,

$$0 = F^{k+1}H^k(M) \subset F^kH^k(M) \subset \ldots \subset F^1H^k \subset F^0H^k = H^k(M,\mathbb{C}).$$

$\oslash$

Lemma 2.36 (Hodge decomposition vs. filtration). *Let M be an n-dimensional compact Kähler manifold. The Hodge decomposition and filtration are related by the formulas*

$$H^k(M,\mathbb{C}) = F^pH^k(M) \oplus \overline{F^{k-(p-1)}H^k}, \qquad H^{p,q}(M) = F^pH^k(M) \cap \overline{F^qH^k(M)},$$

where $q = k - p$.

The Hodge decomposition and filtration both determine the same structure. However in order to give a decomposition it is necessary to choose representatives for the different parts of the decomposition. This choice involves harmonic analysis (as we need to choose representatives of the harmonic forms), which is not defined in an algebraic manner. Contrarily the Hodge filtration *can* be defined algebraically, which makes this the preferred manner to describe the Hodge structure from an algebraic point of view.

2.4 Hyperkähler, quaternion-Kähler and Calabi-Yau manifolds

We conclude this chapter with definitions of three types of Riemannian manifolds which are assumed familiar when discussing compactifications of type II superstring theory. The first two types of manifolds are the *hyperkähler* and *quaternion-Kähler manifolds*. They arise as target spaces of the hypermultiplet's scalar fields in four-dimensional global and local supersymmetry theories.

Definition 2.37 (Hyperkähler manifold). Let $n \in \mathbb{N}$. A *hyperkähler manifold* (M, g, I, J) is a $4n$-dimensional Riemannian manifold (M, g) admitting two parallel (with respect to the Levi-Civita connection) complex structures I, J such that $IJ = -JI$. ⊘

Note that $K = IJ$ is also a parallel complex structure on a hyperkähler manifold (M, g, I, J). In fact for any $(x, y, z) \in S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$, the complex structure $xI + yJ + zK$ is parallel and we obtain a whole sphere of parallel complex structures on M. For any choice J of parallel complex structure the manifold (M, J, g) is a hermitian manifold, although there is no "natural" choice. The fact that the complex structure is parallel means that (M, J, g) is a Kähler manifold [Nak03].

Definition 2.38 (Quaternion-Kähler manifold). Let $n \in \mathbb{N} \setminus \{1\}$. A *quaternion-Kähler manifold* (M, g, I, J) is a $4n$-dimensional Riemannian manifold (M, g) which admits a covering $\{U_i\}_i$ of open subsets and two *almost* complex structures I, J on each U_i, such that

1. $IJ = -JI$,
2. g is hermitian for I and for J on U_i,
3. the Levi-Civita derivatives of I and J are linear combinations of I, J and $K = IJ$,
4. for any $p \in U_i \cap U_j \subset M$, the vector space of endomorphisms of T_pM generated by I, J, K is the same for i and j.

⊘

Remark: Despite its name a quaternion-Kähler manifold is *not* a Kähler manifold [Bes87]. ◊

The last manifold we will define here is the *Calabi-Yau manifold*, which occurs naturally as the internal manifold on which type II superstring theory is compactified from ten to four dimensions. It may be defined in a number of equivalent ways, which we will give here without proving their equivalences nor explaining all notions. The interested reader could consult [Bes87, Hüb92, Nak03, Bou07].

Definition 2.39 (Calabi-Yau manifold). A *Calabi-Yau$_n$ manifold* is a compact Kähler manifold (M, g) of real dimension $2n$, subject to either one of the following equivalent conditions:

1. a vanishing Ricci curvature,
2. a vanishing first Chern class,
3. a holonomy group contained in $\mathrm{SU}(n)$,
4. a trivial canonical bundle,
5. a globally defined and nowhere vanishing holomorphic n-form.

⊘

Due to the additional restrictions on a Calabi-Yau manifold, a three-dimensional Calabi-Yau manifold has only two independent Hodge numbers [Hüb92, Bou07].

Lemma 2.40 (Hodge diamond of a Calabi-Yau$_3$ manifold). *The Hodge diamond of a Calabi-Yau$_3$ manifold reads*

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1. \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.13}$$

A further interesting duality seems to exist between Hodge numbers of different Calabi-Yau$_3$ manifolds. A conjecture dubbed *mirror symmetry* states that for each Calabi-Yau$_3$ M with Hodge numbers $(h^{1,1}(M), h^{2,1}(M))$ there exists a "mirror"-Calabi-Yau$_3$ W with Hodge numbers $(h^{1,1}(W), h^{2,1}(W)) = (h^{2,1}(M), h^{1,1}(M))$. The conjecture was proposed due to a duality between four-dimensional effective type IIA and type IIB string theory, which states that type IIA compactified on M is equal (as a physical theory) to type IIB compactified on W. The "mirror"-Calabi-Yau$_3$ W is not just any Calabi-Yau$_3$ manifold with Hodge numbers $(h^{2,1}, h^{1,1})$, but it should satisfy requirements obtained from the original Calabi-Yau$_3$ M. This thesis is not about mirror symmetry but its results could pave the way for a better understanding of this phenomenon. For more information about mirror symmetry we refer to [CK99].

Chapter 3

Complex tori

In this thesis we are interested in a fibre bundle construction of intermediate Jacobians over the complex structure moduli of a family of Calabi-Yau$_3$'s. Since intermediate Jacobians may not be familiar to everyone, we will provide an introduction to them in this chapter. The intermediate Jacobian of a Calabi-Yau$_3$ manifold is a specific example of a nondegenerate complex torus, a topological torus endowed with a complex structure and a hermitian form. We start with a general discussion of these complex manifolds, their moduli space and a canonical metric on them. In the rest of the chapter we will turn our attention to the third cohomology class of a Calabi-Yau$_3$ and we will define *two* types of intermediate Jacobians on the Calabi-Yau$_3$. Standard and historical references are [GH78, Gri84, BL92, BL99] and [Wei58, Gri68] respectively.

3.1 Nondegenerate complex tori

Period matrix and complex structure

Complex tori are defined in terms of Lie group formalism.

Definition 3.1 (Complex torus). A *complex torus* of dimension n is a compact connected complex Lie group of dimension n. ⊘

A more practical formulation is provided by lemma 3.3. It describes the torus in terms of its (complex) tangent space and a *discrete lattice*, putting more emphasis on the underlying topology.

Definition 3.2 (Lattice). A *lattice of rank r* is a *free abelian group* of rank r, i.e. it is a finitely generated abelian group isomorphic to $\mathbb{Z}^r$. A lattice in a complex vector space of dimension n, is understood to be a lattice of rank $2n$. ⊘

Lemma 3.3 (Alternative description of a complex torus). *Suppose X is a complex torus, $V = T_0X$ its tangent space at the unit element $0 \in X$ and suppose $\pi : V \to X$ is the exponential map. Then there is an isomorphism*

$$X \cong V/\Lambda,$$

where Λ is a lattice in V, given by $\Lambda = \ker \pi$. Conversely any quotient V/Λ of a finite dimensional complex vector space V and a lattice $\Lambda \subset V$ is a complex torus.

Proof. Cf. [BL92]. □

By choosing a basis $\{e_i\}_{i=1}^{n}$ of V and $\{\lambda_s\}_{s=1}^{2n}$ of Λ, the isomorphism $X \cong V/\Lambda$ can be made explicit through the *period matrix*,

$$\Pi = \begin{pmatrix} e_{1,1} & \cdots & e_{1,n} \\ \vdots & \ddots & \vdots \\ e_{n,1} & \cdots & e_{n,n} \\ e_{n+1,1} & \cdots & e_{n+1,n} \\ \vdots & \ddots & \vdots \\ e_{2n,1} & \cdots & e_{2n,n} \end{pmatrix} \in \mathrm{Mat}(2n \times n, \mathbb{C}), \tag{3.1}$$

where we have found the $e_{s,i} \in \mathbb{C}$ via $e_i = \sum_{s=1}^{2n} e_{s,i}\lambda_s$. Thus the basis of V is expressed with respect to the basis of Λ, which allows us to explicitly perform the quotient,

$$X \cong \Pi\mathbb{C}^n/\mathbb{Z}^{2n}. \tag{3.2}$$

We note that the intrinsic structure of the complex torus X does not depend on the choice of bases of the complex vector space V and the lattice Λ. Different bases will give different isomorphisms, all describing the same complex torus. For this reason we have the freedom to put the period matrix in the following form.

Lemma 3.4 (Special choice of period matrix). *Suppose X is a complex torus of dimension n. By suitable bases choices the period matrix can be put into the form*

$$\Pi = \begin{pmatrix} \mathbb{I}_n \\ \mathrm{T} \end{pmatrix}, \tag{3.3}$$

where $\mathbb{I}_n, \mathrm{T} \in \mathrm{Mat}(n, \mathbb{C})$ are both n-dimensional matrices and $\mathrm{Im}\,\mathrm{T}$ is invertible. Conversely any matrix of the form (3.3) is the period matrix of a complex torus.

Proof. Since the rank of any period matrix Π of an n-dimensional complex torus is n, a suitable basis transformation of the lattice permutes the rows of Π such that the first n rows of Π are invertible. Multiplying (from the right) by the inverse of the first n rows of Π gives (3.3).

The fact that $\mathrm{Im}\,\mathrm{T} \in \mathrm{GL}(n, \mathbb{C})$ follows from the fact that a matrix $\Pi \in \mathrm{Mat}(2n \times n, \mathbb{C})$ is a period matrix if and only if the square matrix $\left(\Pi\ \bar{\Pi} \right) \in \mathrm{Mat}(2n, \mathbb{C})$ is invertible [BL92, BL99]. $\square$

Remark: In mathematics literature the period matrix is often denoted as an $n \times 2n$-matrix instead of a $2n \times n$-matrix, to emphasize the analogy with elliptic curves. Depending on which reference is consulted, the first n or second n columns are chosen to be the unit matrix. We have chosen to use a $2n \times n$-matrix whose first n rows are the unit matrix, in order to emphasize the relation with period matrices in the physics literature, cf. chapter 5-7. $\diamondsuit$

Notation: T is the capital Greek letter τ and should be pronounced as *tau*. It is the matrix-analogue of the period ratio τ in standard 1-dimensional complex algebraic variety theory. We can split T in its complex and imaginary part by writing $\mathrm{T} = \Xi + i\mathrm{H}$, where *xi* is $\Xi = \mathrm{Re}\,\mathrm{T}$ and *eta* is $\mathrm{H} = \mathrm{Im}\,\mathrm{T}$. $\diamondsuit$

The period matrix expresses the directions of the lattice Λ with respect to the complex structure of the complex vector space V. An equivalent description of this competition between lattice directions and complex structure is provided by endowing the real extension of the lattice, from which we may define a *real* torus, with a complex structure. Let J be a complex structure on $\mathbb{R}^{2n}$ and let $\Pi_J : \mathbb{C}^n \xrightarrow{\sim} \mathbb{R}^{2n}$ be an isomorphism between complex vector spaces $\mathbb{C}^n$ and $(\mathbb{R}^{2n}, J)$, that is $\Pi_J(iv) = J\Pi_J(v)$ for all $v \in \mathbb{C}^n$. Π_J is a period matrix for the complex torus $X_J = (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}$. Conversely any period matrix Π of a complex torus $X \cong \Pi\mathbb{C}^n/\mathbb{Z}^{2n}$ determines a complex structure J_Π on the real extension $\mathbb{R}^{2n}$ of the lattice $\mathbb{Z}^{2n}$, as we can see from the following proposition.

Proposition 3.5 (Period matrix and complex structure). *Let $X \cong V/\Lambda$ be an n-dimensional complex torus with period matrix Π, then a complex structure J_Π is automatically induced on the* *real extension of the lattice Λ. Conversely if J is a complex structure on $\mathbb{R}^{2n}$, then each complex isomorphism $\Pi_J : \mathbb{C}^n \tilde{\to} (\mathbb{R}^{2n}, J)$ serves as a period matrix for $X_J = (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}$.*

In this sense complex structures and period matrices are equivalent. The relation between a period matrix and complex structure can be summarized using the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\Pi} & \mathbb{R}^{2n} \\ i\mathbb{I}_n \downarrow & & \downarrow J \\ \mathbb{C}^n & \xrightarrow{\Pi} & \mathbb{R}^{2n}. \end{array}$$

Proof. Suppose Π is of the form (3.3). The map $\Pi : \mathbb{C}^n \to \mathbb{R}^{2n}$ may be interpreted as an $\mathbb{R}$-linear isomorphism between real vector spaces by writing $\Pi = \left(\begin{smallmatrix} \mathbb{I}_n & 0 \\ \Xi & \mathrm{H} \end{smallmatrix}\right)$, where $\mathrm{T} = \Xi + i\mathrm{H}$. The commutative diagram expresses the required $\mathbb{C}$-linearity. Explicitly $J = \Pi i \Pi^{-1}$ is calculated to be

$$J_\Pi = \begin{pmatrix} \mathbb{I}_n & 0 \\ \Xi & \mathrm{H} \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \begin{pmatrix} \mathbb{I}_n & 0 \\ -\mathrm{H}^{-1}\Xi & \mathrm{H}^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{H}^{-1}\Xi & -\mathrm{H}^{-1} \\ \mathrm{H} + \Xi\mathrm{H}^{-1}\Xi & -\Xi\mathrm{H}^{-1} \end{pmatrix}, \tag{3.4}$$

which indeed satisfies $J_\Pi^2 = -\mathbb{I}_{2n}$. □

The moduli space of nondegenerate complex tori

In the previous section we have defined complex tori and we have seen how their period matrices can be linked with a complex structure on the tangent space of the complex torus. In this section we are interested in those complex tori that admit a canonical metric. The moduli space of these *nondegenerate complex tori* is also investigated.

Definition 3.6 (Nondegenerate complex torus). Let $X \cong V/\Lambda$ be a complex torus. A *polarization h of index k* is a hermitian form on $V \cong T_0X$, $h : V \times V \to \mathbb{C}$ with k negative eigenvalues, whose imaginary part takes integer values on the lattice $\Lambda \subset V$, $\operatorname{Im} h(\Lambda \times \Lambda) \subset \mathbb{Z}$. The pair (X, h) is called a *nondegenerate complex torus of index k*. ⊘

Because h is nondegenerate and $\operatorname{Im} h$ takes integral values on Λ, the *elementary divisor theorem* states that the latter can be put into the form $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) \in \mathrm{Mat}(2n, \mathbb{Z})$ for some choice of basis of Λ [GH78, BL99]. Here D is the diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$, whose entries are positive integers $d_1|d_2|\dots|d_n$ that are called the *elementary divisors*. The vector $(d_1, \dots, d_n)$ is called the *type of the polarization*. In this thesis we will be primarily interested in the type of a polarization given by $(1, \dots, 1)$, called the *principal type of a polarization*. Note that in that case the imaginary part of h is the standard symplectic form, i.e. $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$. Furthermore it may be shown that if Σ is the imaginary part of the polarization of a nondegenerate complex torus, the lower part of the period matrix is symmetric [BL99].

Assumption 3.7 (Principal polarization and symmetric T). In this thesis the polarization h of each complex torus X is of principal type. As a result the last n rows of its period matrix $\Pi = \left(\begin{smallmatrix} \mathbb{I}_n \\ \mathrm{T} \end{smallmatrix}\right)$ form a symmetric $n \times n$-matrix T. ⊘

We will now explain how the moduli space of nondegenerate complex tori is identified with a well-defined subset of possible complex structures on these tori. First we consider nondegenerate complex tori with respect to a particular basis.

Definition 3.8 (Nondegenerate complex torus with symplectic basis). Suppose $X \cong V/\Lambda$ is an n-dimensional complex torus, h is a polarization of index k on the torus and suppose $\{\lambda_s\}_{s=1}^{2n}$ is a basis for the lattice Λ. If the alternating bilinear form $\sigma = \operatorname{Im} h$ is the standard symplectic form $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$ with respect to the basis $\{\lambda_s\}_{s=1}^{2n}$, then we call $(X, h, \{\lambda_s\}_{s=1}^{2n})$ a *nondegenerate complex torus of index k with symplectic basis*. ⊘

Definition 3.9 (Symplectic group). For every $n \in \mathbb{N}$ and $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, the *symplectic group* $\mathrm{Sp}(2n, R)$ is the subgroup of $\mathrm{GL}(2n, R)$ defined by

$$\mathrm{Sp}(2n, R) = \left\{ S \in \mathrm{GL}(2n, R) \middle| S^t \Sigma S = \Sigma \right\},$$

where $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$. Equivalently a matrix $S = \left(\begin{smallmatrix} U & Z \\ W & V \end{smallmatrix}\right) \in \mathrm{GL}(2n, R)$ is a symplectic matrix if and only if

$$U^t W = W^t U, \qquad Z^t V = V^t Z, \qquad U^t V - W^t Z = \mathbb{I}_n. \tag{3.5}$$

Note that $\mathrm{Sp}(2, R) = \mathrm{SL}(2, R)$. ⊘

Proposition 3.10 (Moduli space of nondegenerate complex tori with symplectic basis). *Suppose J is an element of the real analytic space*

$$\mathcal{C}_k = \{J \in \mathrm{Sp}(2n, \mathbb{R}) | J^2 = -\mathbb{I}_{2n},\ \mathrm{ind}_{\mathbb{R}}(J^t\Sigma) = 2k\}, \tag{3.6}$$

then $(X_J, h_J, \{\lambda_s\}_{s=1}^{2n})$ is a nondegenerate complex torus of index k with symplectic basis. Here $\{\lambda_s\}_{s=1}^{2n}$ is simply the standard basis of $\mathbb{R}^{2n}$ and

$$X_J = (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}, \qquad h_J = J^t\Sigma + i\Sigma.$$

Conversely let $(X, h, \{\lambda_s\}_{s=1}^{2n})$ be a nondegenerate complex torus of index k with symplectic basis, such that X is described by a complex structure J on $\mathbb{R}^{2n}$, $X \cong (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}$. Then $J \in \mathcal{C}_k$.

Proof. The key ingredient in this proof [BL99] is the fact that if $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$ and J is a complex structure, i.e. $J^2 = (J^t)^2 = -\mathbb{I}_{2n}$, that then $J \in \mathrm{Sp}(2n, \mathbb{R})$ if and only if $J^t\Sigma$ is symmetric. For suppose $J \in \mathrm{Sp}(2n, \mathbb{R})$, then

$$(J^t\Sigma)^t = -\Sigma J = J^t J^t \Sigma J = J^t\Sigma,$$

and if $(J^t\Sigma)^t = J^t\Sigma$, then

$$J^t\Sigma J = \Sigma^t J^2 = (-\Sigma)(-\mathbb{I}_{2n}) = \Sigma.$$

Suppose that $J \in \mathcal{C}_k$ and consider $\mathbb{R}^{2n}$ with standard basis $\{\lambda_s\}_{s=1}^{2n}$. $J \in \mathrm{Sp}(2n, \mathbb{R})$ and hence $J^t\Sigma$ is symmetric. This means that the form

$$h_J = J^t\Sigma + i\Sigma, \tag{3.7}$$

is a hermitian form, which is a polarization of type k since $\mathrm{ind}_{\mathbb{R}}(J^t\Sigma) = 2k$. Furthermore since $J^2 = -\mathbb{I}_{2n}$, J defines a complex structure and thus $X_J = (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}$ is a complex torus.

Conversely suppose that $(X, h, \{\lambda_s\}_{s=1}^{2n})$ is a nondegenerate complex torus of index k with symplectic basis, such that $X \cong (\mathbb{R}^{2n}, J)/\mathbb{Z}^{2n}$ is given by putting a complex structure J on $\mathbb{R}^{2n}$. The hermitian form h and the alternating bilinear form $\sigma = \mathrm{Im}\, h$ are related via $h = J^t\sigma + i\sigma$. We see that $J^t\sigma$ is the real part of a hermitian form and hence it is symmetric. By the "key ingredient" in this proof $J \in \mathrm{Sp}(2n, \mathbb{R})$. Since the hermitian form h is of index k on $\mathbb{C}^n = (\mathbb{R}^{2n}, J)$, its real part satisfies $\mathrm{ind}_{\mathbb{R}}(J^t\sigma) = 2k$. □

Thus as long as a complex structure is in the symplectic group and is of the right signature, it parameterizes a specific nondegenerate complex torus with symplectic basis. Since the nondegenerate complex tori do not depend on their specific basis of the lattice Λ, one should take into account possible integer basis transformations. These transformations should preserve the symplectic structure Σ and therefore the moduli space of nondegenerate complex tori is $\mathcal{C}_k$ modulo the integer symplectic group [BL92, BL99].

Moreover the real symplectic group defines a transitive group action on $\mathcal{C}_k$ given by conjugation. We may pick a particular element $J_0 = \left(\begin{smallmatrix} 0 & \mathbb{I}_{n-k,k} \\ -\mathbb{I}_{n-k,k} & 0 \end{smallmatrix}\right)$ of $\mathcal{C}_k$, where $\mathbb{I}_{p,q} = \left(\begin{smallmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{smallmatrix}\right)$, and calculate its stabilizer group. Since it is isomorphic to $\mathrm{U}(n-k, k)$, where $\mathrm{U}(p, q)$ is the *unitary group* of complex $(p+q) \times (p+q)$-matrices U which satisfy $\bar{U}^t \mathbb{I}_{p,q} U = \mathbb{I}_{p,q}$, the moduli space may be identified as follows [BL99], cf. proposition 7.15.

Proposition 3.11 (Moduli space of nondegenerate complex tori). *The topological space,*

$$\mathscr{M}_{tori} = \mathrm{Sp}(2n, \mathbb{Z}) \backslash \mathcal{C}_k \cong {}_{\mathrm{Sp}(2n,\mathbb{Z})}\diagdown^{\mathrm{Sp}(2n,\mathbb{R})}\diagup_{\mathrm{U}(n-k,k)}, \tag{3.8}$$

parameterizes the nondegenerate complex tori of index k.

Geometry on nondegenerate complex tori

From the preceding section we may construct a canonical metric on the tangent space of a nondegenerate complex torus. A nondegenerate complex torus of index k admits a hermitian form h, whose real part defines a metric on the complex torus of index $2k$. Suppose the torus is described by a period matrix Π as in (3.3) with respect to a symplectic basis $\{\lambda_s\}_{s=1}^{2n}$. Proposition 3.5 and equation (3.7) then automatically give a formula for the hermitian form in terms of the second half $\mathrm{T} = \Xi + i\mathrm{H}$ of the period matrix Π.

Corollary 3.12 (Canonical polarization of a complex torus). *Suppose (X, h) is an n-dimensional nondegenerate complex torus of index k with period matrix $\Pi = \left(\begin{smallmatrix}\mathbb{I}_n \\ \mathrm{T}\end{smallmatrix}\right)$. The canonical metric on the complex torus, seen as a $2n$-dimensional real manifold, is*

$$g_\Pi = J_\Pi^t \Sigma = \begin{pmatrix} -\mathrm{H} - \Xi\mathrm{H}^{-1}\Xi & \Xi\mathrm{H}^{-1} \\ \mathrm{H}^{-1}\Xi & -\mathrm{H}^{-1} \end{pmatrix}, \tag{3.9}$$

where $\mathrm{T} = \Xi + i\mathrm{H}$.

In the rest of this thesis, notably chapters 6 and 7, we will encounter (3.9) a number of times as part of the metrics arising from several physical theories. This will provide the motivation for interpreting the manifolds associated to these physical theories as complex tori. Since most metrics in physical theories are provided in terms of a line element, we will now write the canonical hermitian form as a line element with respect to a symplectic set of coordinates for the tangent space of the complex torus.

Lemma 3.13 (Canonical line element of a complex torus). *Suppose $(X, h, \{\lambda_s\}_{s=1}^{2n})$ is a nondegenerate complex torus with symplectic basis and period matrix $\Pi = \left(\begin{smallmatrix}\mathbb{I}_n \\ \mathrm{T}\end{smallmatrix}\right)$, $\mathrm{T} = \Xi + i\mathrm{H}$. The basis $\{\lambda_s\}_{s=1}^{2n}$ spans the lattice Λ of the torus. Let us denote an arbitrary vector in the real extension of this lattice by*

$$(x, y) = \sum_{i=1}^{n} \left[x^i \lambda_i - y_i \lambda_{n+i} \right],$$

with real coefficients $x^i, y_i \in \mathbb{R}$. The line element of the metric determined by h can be written as

$$\mathrm{d}s^2_{complextorus} = -\bar{Y}^t \mathrm{H}^{-1} Y, \tag{3.10a}$$

where $Y = (Y_i)_{i=1}^n$ are complex differential 1-forms determined by the period matrix via

$$Y_i = \mathrm{d}y_i - \sum_{j=1}^{n} \bar{\mathrm{T}}_{ij} \mathrm{d}x^j. \tag{3.10b}$$

Proof. We just have to verify that the hermitian form determined by

$$h = J_\Pi^t \Sigma + i\Sigma = \begin{pmatrix} -\mathrm{H} - \Xi\mathrm{H}^{-1}\Xi & \Xi\mathrm{H}^{-1} \\ \mathrm{H}^{-1}\Xi & -\mathrm{H}^{-1} \end{pmatrix} + i \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix},$$

is equivalent with our line element (3.10). For this we write (3.10) with respect to the basis $\{\mathrm{d}x^i, \mathrm{d}y_i\}_{i=1}^n$. Note that Y is

$$Y = \bar{\Pi}^t \Sigma \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix},$$

and therefore computing

$$\begin{aligned} \Sigma^t \Pi \mathrm{H}^{-1} \bar{\Pi}^t \Sigma &= \Sigma^t \begin{pmatrix} \mathbb{I}_n \\ \Xi + i\mathrm{H} \end{pmatrix} \mathrm{H}^{-1} \begin{pmatrix} \mathbb{I}_n & \Xi - i\mathrm{H} \end{pmatrix} \Sigma \\ &= \begin{pmatrix} \mathrm{H} + \Xi\mathrm{H}^{-1}\Xi & -\Xi\mathrm{H}^{-1} \\ -\mathrm{H}^{-1}\Xi & \mathrm{H}^{-1} \end{pmatrix} + i \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}, \end{aligned}$$

tells us that with respect to the symplectic basis $\{\mathrm{d}x, \mathrm{d}y\}$, the hermitian form $h = J_\Pi^t \Sigma + i\Sigma$ may indeed be written as the line element (3.10). Note that the antisymmetric symplectic form Σ has no contribution to the line element, since

$$\sum_{i=1}^{n} \left[\mathrm{d}x^i \mathrm{d}y_i - \mathrm{d}y_i \mathrm{d}x^i\right] = \frac{1}{2}\sum_{i=1}^{n} \left[\mathrm{d}x^i \otimes \mathrm{d}y_i + \mathrm{d}y_i \otimes \mathrm{d}x^i - (\mathrm{d}y_i \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}y_i)\right] = 0. \qquad \square$$

3.2 Intermediate Jacobians

Historically the motivation for defining nondegenerate complex tori has come with the analysis of the *Jacobian* of a smooth projective curve of genus g. The Jacobian of a curve is an *abelian variety*, i.e. a complex torus admitting a polarization of index 0 or equivalently a Lie group which is also a *projective algebraic variety*. The generalization to Jacobians of a projective algebraic variety of arbitrary dimension has led to the definition of nondegenerate complex tori of arbitrary index. Weil introduced abelian varieties associated to projective varieties of arbitrary dimension [Wei58]. While being abelian, these varieties have the disadvantage that they do not depend holomorphically on the projective variety with which they are associated. Griffiths fixed this holomorphic variation at the cost of the object being a nonalgebraic complex torus instead of an abelian variety [Gri68].

The Jacobian of a curve

Let C be a smooth projective curve of genus g. By Stokes's theorem integration of holomorphic 1-forms $\omega \in H^0(C, \Omega^1(C))$ over a 1-cycle $\gamma \in H_1(C, \mathbb{Z})$ on C,

$$\int_\gamma : H^0(C, \Omega^1(C)) \to \mathbb{C} : \omega \mapsto \int_\gamma \omega,$$

defines a linear functional $\int_\gamma \in H^0(C, \Omega^1(C))^* = \hom(H^0(C, \Omega^1(C)), \mathbb{C})$ on the vector space $H^0(C, \Omega^1(C))$, which we denote with $\int_\gamma$.

Lemma 3.14 (Jacobian of a curve). *Let C be a smooth projective curve of genus g, then the quotient*

$$\mathcal{J}(C) = \frac{H^0(C, \Omega^1(C))^*}{H_1(C, \mathbb{Z})}$$

is a complex torus of dimension g called the Jacobian *of the curve C.*

Proof. We need to check that $H_1(C, \mathbb{Z})$ is a lattice in $H^0(C, \Omega^1(C))^*$. Consider the canonical map

$$\iota : H_1(C, \mathbb{Z}) \to H^0(C, \Omega^1(C))^*,$$

given by the composition of

$$\int : H_1(C, \mathbb{Z}) \hookrightarrow H^1(C, \mathbb{C})^* : \gamma \mapsto \{\omega \mapsto \int_\gamma \omega\},$$

and the projection map

$$\pi : H^0(C, \Omega^1(C))^* \oplus \overline{H^0(C, \Omega^1(C))^*} \to H^0(C, \Omega^1(C))^*.$$

By the universal coefficient theorem the map $H_1(C, \mathbb{Z}) \hookrightarrow H_1(C, \mathbb{C}) = H^1(C, \mathbb{C})^* : \gamma \mapsto \{\omega \mapsto \int_\gamma \omega\}$ is injective. Now pick an element $\tilde{\gamma} \in H^1(C, \mathbb{Z})^*$. It is obviously invariant under complex conjugation. Therefore using the Hodge decomposition, $\tilde{\gamma}$ is of the form $\tilde{\gamma} = \alpha + \bar{\alpha}$ for some $\alpha \in H^0(C, \Omega^1(C))^*$ [BL92].

We see that the map ι is indeed injective and since

$$\operatorname{rk} \iota(H_1(C, \mathbb{Z})) = \dim_{\mathbb{C}} H^1(C, \mathbb{C})^* = 2 \dim_C H^0(C, \Omega^1(C))^* = 2g,$$

the quotient is a well-defined complex torus of dimension g. [BL99]. $\square$

We can describe $\mathcal{J}(C)$ in terms of a period matrix in the following manner. Pick a basis $\{\gamma_s\}_{s=1}^{2g}$ of $H_1(C,\mathbb{Z})$ and a basis $\{\omega_i\}_{i=1}^{g}$ of $H^0(C,\Omega^1(C))$. We denote with $\{v_i\}_{i=1}^{g}$ the basis of $H^0(C,\Omega^1(C))^*$ dual to $\{\omega_i\}_{i=1}^{g}$. If we consider $\gamma_s \in H_1(C,\mathbb{Z})$ as a linear form on $H^0(C,\Omega^1(C))$ as above, $\gamma_s(\omega_i) = \int_{\gamma_s}\omega_i$, then we may write $\gamma_s = \sum_{j=1}^{g}\left(\int_{\gamma_s}\omega_j\right)v_j$.

Thus we find the period matrix

$$\Psi = \begin{pmatrix} \int_{\gamma_1}\omega_1 & \cdots & \int_{\gamma_g}\omega_1 & \int_{\gamma_{g+1}}\omega_1 & \cdots & \int_{\gamma_{2g}}\omega_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1}\omega_g & \cdots & \int_{\gamma_g}\omega_g & \int_{\gamma_{g+1}}\omega_g & \cdots & \int_{\gamma_{2g}}\omega_g \end{pmatrix} \in \mathrm{Mat}(g\times 2g,\mathbb{C}).$$

The rows of this matrix are called the *periods* of the forms along the different cycles. Their coefficients are the "lengths" of the cycles along the torus. Making identifications $H^0(C,\Omega^1(C))^* \cong H^{0,1}(C)$ and $H_1(C,\mathbb{Z}) \cong H^1(C,\mathbb{Z})$ the Jacobian of C can be written as [Gri84]

$$\mathcal{J}(C) = \frac{H^{0,1}(C)}{H^1(C,\mathbb{Z})}. \tag{3.11}$$

In this case the period matrix is more conveniently defined by expressing the basis $\{\bar\omega_i\}_{i=1}^{g}$ of $H^{0,1}(C)$ in terms of the basis $\{\lambda_s\}_{s=1}^{2g}$ of $H^1(C,\mathbb{Z})$ dual to $\{\gamma_s\}_{s=1}^{2g}$. Since $\gamma_s(\bar\omega_i) = \bar\omega_i(\gamma_s) = \int_{\gamma_s}\bar\omega_i$, we find $\bar\omega_i = \sum_{s=1}^{2g}\left(\int_{\gamma_s}\bar\omega_i\right)\lambda_s$ and thus a period matrix,

$$\Pi = \bar\Psi^t \in \mathrm{Mat}(2g\times g,\mathbb{C}).$$

The Jacobian variety of a compact Riemann surface C is an abelian variety. A canonical principal polarization of index 0 is determined by the *intersection product* in $H_1(C,\mathbb{Z})$, which is Poincaré dual to the cup product in $H^1(C,\mathbb{Z})$, $\gamma_i \cap \gamma_j = \int_C P(\gamma_i)\wedge P(\gamma_j)$ [GH78, BL92]. Here $P : H_1(C,\mathbb{C}) \to H^1(C,\mathbb{C})$ denotes the Poincaré duality isomorphism, cf. proposition 2.27.

The Weil intermediate Jacobian

Now consider a three-dimensional projective algebraic variety M and its odd cohomology groups $H^{2k-1}(M,\mathbb{C})$, $k \in \{1,2,3\}$. On the first and last cohomology group a construction similar to the Jacobian of a curve leads to the definition of the *Picard* and *Albanese variety* respectively [GH78, BL99], which are both abelian varieties. In [Wei58] Weil succeeded in defining an abelian variety on the third cohomology group $H^3(M,\mathbb{C})$ as well. Since it is related to an intermediate cohomology group, it is called the *Weil intermediate Jacobian*. It is inspired by the example of the Jacobian on a curve and the intersection form on it. In the rest of this section we will be interested in Calabi-Yau$_3$ manifolds, which is a special kind of three-dimensional projective algebraic variety. The intersection form on a Calabi-Yau$_3$ is defined as follows, cf. page 21.

Definition 3.15 (Intersection form on $H^3(CY)$). Let CY be a Calabi-Yau$_3$ manifold. For $R = \mathbb{Z},\mathbb{R},\mathbb{C}$ we define the *intersection form* $Q : H^3(CY,R)\times H^3(CY,R) \to R$ by

$$Q(\xi,\omega) = \int_{CY}\xi\wedge\omega. \tag{3.12}$$

⊘

The intersection form satisfies the *Hodge-Riemann bilinear relations* [GH78, Wel80], cf. theorem 3.17. To formulate these concisely we introduce the *Weil operator*.

Definition 3.16 (Weil operator). Suppose CY is a Calabi-Yau$_3$ manifold, then the *Weil operator* is defined as the $\mathbb{C}$-linear operator $C : H^3(CY,\mathbb{C}) \to H^3(CY,\mathbb{C})$ given by

$$C = \sum_{\substack{p+q=3\\ 0\le p,q\le 3}} i^{p-q}\pi_{p,q},$$

where $\pi_{p,q} : H^3(CY,\mathbb{C}) \to H^{p,q}(CY)$ are the natural projections on the Dolbeault cohomology groups. Restricted to the $\mathbb{R}$-linear subvector space $H^3(CY,\mathbb{R})$, C defines a complex structure on it. ⊘

Theorem 3.17 (Hodge-Riemann bilinear relations). *Let CY be a Calabi-Yau$_3$ manifold and let $\xi,\omega \in H^3(CY,\mathbb{C})$, then*

$$Q(\xi,\omega) = -Q(\omega,\xi), \qquad Q(C\xi,C\omega) = Q(\xi,\omega), \qquad Q(\xi,C\omega) = Q(\omega,C\xi).$$

Furthermore for $\xi \neq 0$ and for $\alpha \in H^{p,q}(CY)$, $\beta \in H^{p',q'}(CY)$ we have the Hodge-Riemann bilinear relations*,*

$$\begin{aligned} Q(\alpha,\beta) &= 0 \text{ unless } p' = 3-p \text{ and } q' = 3-q \\ Q(\xi,C\bar{\xi}) &> 0. \end{aligned}$$

We see that the Weil operator acts as a complex structure on $H^3(CY,\mathbb{R})$ such that $Q(\xi,C\bar{\xi}) > 0$. This leads to the definition of the following abelian variety.

Definition 3.18 (Weil intermediate Jacobian). The *Weil intermediate Jacobian* of a Calabi-Yau$_3$ manifold, $(\mathcal{J}_W(CY),h_W)$, is the $(1+h^{2,1})$-dimensional nondegenerate complex torus of index 0 defined by[1]

$$\mathcal{J}_W(CY) = \frac{(H^3(CY,\mathbb{R}),-C)}{H^3(CY,\mathbb{Z})}, \tag{3.13a}$$

$$h_W(\phi,\chi) = Q(-C\phi,\chi) + iQ(\phi,\chi), \ \ \phi,\chi \in H^3(CY,\mathbb{R}). \tag{3.13b}$$

⊘

Proof. Note that Q indeed defines a polarization via 3.13b, since $Q|_{H^3_\mathbb{Z}\times H^3_\mathbb{Z}} \subset \mathbb{Z}$. The index of h_W follows from the Hodge-Riemann bilinear relations, theorem 3.17. By linearity it suffices to consider an element $\phi \in H^3(CY,\mathbb{R})$ that can be written as $\phi = \alpha+\bar{\alpha}$ for a certain $\alpha \in H^{p,q}(CY)$. Note that $C\alpha \in H^{p,q}(CY)$ as well and similarly $\bar{\alpha}, C\bar{\alpha} \in H^{q,p}(CY)$. Then since $Q(\bar{\alpha},C\alpha) = Q(C\bar{\alpha},C^2\alpha) = Q(\alpha,C\bar{\alpha})$, we find that

$$h_W(\phi,\phi) = Q(\phi,C\phi) = Q(\alpha+\bar{\alpha},C\alpha+C\bar{\alpha}) = Q(\alpha,C\bar{\alpha}) + Q(\bar{\alpha},C\alpha) = 2Q(\alpha,C\bar{\alpha}) > 0$$

and hence h_W is a positive definite hermitian form. □

In order to compare the Weil intermediate Jacobian with the Griffiths intermediate Jacobian, we will now describe it with respect to the Dolbeault cohomology groups in $H^3(CY,\mathbb{C})$. The crucial structure, which is also relevant to describe the Griffiths intermediate Jacobian in terms of a complex structure on $H^3(CY,\mathbb{R})$, is given by the following lemma.

Lemma 3.19 (Complex structure on $H^3(CY,\mathbb{R})$). *Let CY be a Calabi-Yau$_3$ manifold and suppose J is a complex structure on $H^3(CY,\mathbb{C})$ such that*

$$J\xi = i\xi \Rightarrow J\bar{\xi} = -i\bar{\xi}, \qquad \forall \xi \in H^3(CY,\mathbb{C}), \tag{3.14a}$$

$$V \oplus \bar{V} = H^3(CY,\mathbb{C}), \tag{3.14b}$$

$$Q(\alpha,\beta) = 0, \qquad \forall \alpha,\beta \in V, \tag{3.14c}$$

where we have defined the vector space $V \subset H^3(CY,\mathbb{C})$ by

$$V = \{\alpha \in H^3(\mathbb{C}) | J\alpha = i\alpha\}.$$

[1]Technically we need to divide by a *free* abelian group, which is why we should consider the torsion free version of the discrete group $H^3(CY,\mathbb{Z})$. Since inclusion of torsion is far from trivial, cf. [Asp00] and references therein, we will assume throughout this thesis torsion free integer cohomology groups for simplicity.

Define the $\mathbb{R}$-linear isomorphism $\Phi : H^3(CY,\mathbb{R}) \hookrightarrow H^3(CY,\mathbb{C}) \xrightarrow{\pi} V$ given by the canonical embedding followed by the natural projection onto $V \subset H^3(CY,\mathbb{C})$ and let h be the hermitian form on $H^3(CY,\mathbb{R})$ associated to the intersection form Q, $h = J^tQ + iQ$.

Then the complex structure J restricts to a complex structure on $H^3(CY,\mathbb{R})$, the linear map $\Phi : (H^3(CY,\mathbb{R}), J) \to V$ is an isomorphism between complex vector spaces and the hermitian form $\tilde{h}$ on V is given by

$$\tilde{h} : V \times V \to \mathbb{C} : (\alpha, \beta) \mapsto 2iQ(\alpha, \bar{\beta}).$$

Proof. We have to verify three statements. First suppose $\phi \in H^3(CY,\mathbb{R}) \subset H^3(CY,\mathbb{C})$ is an arbitrary element in $H^3(CY,\mathbb{R})$. Since it must be real, it is of the form $\phi = \alpha + \bar{\alpha}$ for some $\alpha \in V$. Since

$$J\phi = J\alpha + J\bar{\alpha} = i\alpha - i\bar{\alpha} = i\alpha + \overline{i\alpha},$$

$J\phi \in H^3(CY,\mathbb{R})$ again, which means that J restricts to a complex structure on $H^3(CY,\mathbb{R})$.

Next the $\mathbb{R}$-linear isomorphism $\Phi : H^3(CY,\mathbb{R}) \to V$ is in fact an isomorphism of complex vector spaces when $H^3(CY,\mathbb{R})$ is endowed with the complex structure J, since for an arbitrary element $H^3(CY,\mathbb{R}) \ni \phi = \alpha + \bar{\alpha}$,

$$\Phi(J\phi) = \Phi(J\alpha + J\bar{\alpha}) = \Phi(i\alpha + \overline{i\alpha}) = i\alpha.$$

Finally consider two elements $\phi, \chi \in H^3(CY,\mathbb{R})$. Because any $\gamma + \bar{\gamma} = \psi \in H^3(CY,\mathbb{R})$ satisfies $\Phi(\psi) + \overline{\Phi(\psi)} = \gamma + \bar{\gamma} = \psi$, we find

$$\begin{aligned}
(\Phi^*\tilde{h})(\phi,\chi) &= \tilde{h}(\Phi(\phi),\Phi(\chi)) = 2iQ(\Phi(\phi),\overline{\Phi(\chi)}) + iQ(\overline{\Phi(\phi)},\Phi(\chi)) - iQ(\overline{\Phi(\phi)},\Phi(\chi)) \\
&= Q(i\Phi(\phi) - i\overline{\Phi(\phi)}, \Phi(\chi) + \overline{\Phi(\chi)}) + iQ(\Phi(\phi) + \overline{\Phi(\phi)}, \Phi(\chi) + \overline{\Phi(\chi)}) \\
&= Q(\Phi(J\phi) + \overline{\Phi(J\phi)}, \Phi(\chi) + \overline{\Phi(\chi)}) + iQ(\Phi(\phi) + \overline{\Phi(\phi)}, \Phi(\chi) + \overline{\Phi(\chi)}) \\
&= Q(J\phi,\chi) + iQ(\phi,\chi) = h(\phi,\chi).
\end{aligned}$$

□

Corollary 3.20 (Equivalent description intermediate Jacobians). *The linear map Φ from lemma 3.19 determines an isomorphism between nondegenerate complex tori,*

$$\left(\frac{(H^3(CY,\mathbb{R}), J)}{H^3(CY,\mathbb{Z})}, h\right) \cong \left(\frac{V}{\iota(H^3(CY,\mathbb{Z}))}, \tilde{h}\right),$$

where $\iota : H^3(CY,\mathbb{Z}) \hookrightarrow H^3(CY,\mathbb{R}) \hookrightarrow H^3(CY,\mathbb{C}) \xrightarrow{\pi} V$ is the canonical embedding of $H^3(CY,\mathbb{Z})$ into V.

Proof. The map Φ is a $\mathbb{C}$-linear isomorphism $\Phi\left((H^3(CY,\mathbb{R}), J)\right) = V$ and $\Phi(H^3(CY,\mathbb{Z})) = \iota(H^3(CY,\mathbb{Z}))$. The equivalence between both polarizations is given by $\Phi^*\tilde{h} = h$. □

Proposition 3.21 (Dolbeault description of Weil intermediate Jacobian). *In terms of Dolbeault cohomology groups the Weil intermediate Jacobian $(\mathcal{J}_W(CY), h_W)$ of a Calabi-Yau$_3$ manifold CY is given by*

$$\mathcal{J}_W(CY) = \frac{H^{3,0}(CY) \oplus H^{1,2}(CY)}{\iota(H^3(CY,\mathbb{Z}))}, \tag{3.15a}$$

$$h_W(\alpha,\beta) = 2iQ(\alpha,\bar{\beta}), \tag{3.15b}$$

where $\iota : H^3(CY,\mathbb{Z}) \to H^{3,0}(CY) \oplus H^{1,2}(CY)$ is the canonical embedding.

Proof. The Weil operator C acts as $\pm i$ on the different subvector spaces $H^{p,q}(CY)$ of the Hodge decomposition $H^3(CY,\mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ of the third cohomology group. By definition it is $-i$ on $H^{3,0} \oplus H^{1,2}$. If we put $J = -C$ and $V = H^{3,0} \oplus H^{1,2}$ in the conditions of lemma 3.19, then the result follows from corollary 3.20. □

The Griffiths intermediate Jacobian

In [Gri68] Griffiths proposed an alternative intermediate Jacobian on the third cohomology group of a three-dimensional projective algebraic variety M. His definition was motivated by the fact that the Jacobian variety $\mathcal{J}(C)$ of a curve can be interpreted as dividing the second half $H^{0,1}(C)$ of the first (and only odd) cohomology group of the curve by the lattice $H^1(C,\mathbb{Z})$. Similarly the Picard and Albanese varieties of M can be seen as the second halves $H^{0,1}(M)$ and $H^{2,3}(M)$ respectively of the first and fifth cohomology groups modulo their integer subgroups [GH78, Gri84]. Therefore Griffiths proposed to define the Jacobian on the intermediate (third) cohomology group as the second half of the Hodge decomposition of $H^3(M,\mathbb{C})$ modulo $H^3(M,\mathbb{Z})$. The advantage is that this may be defined using the *Hodge filtration*, cf. definition 2.35, rather than the Hodge decomposition needed for the Weil intermediate Jacobian, cf. proposition 3.21. As a result the Griffiths intermediate Jacobian *does* vary nicely under holomorphic variations of the complex structure of the projective algebraic variety M. The lattice $H^3(M,\mathbb{Z})$ is canonically embedded into $\overline{F^2H^3(M,\mathbb{C})}$,

$$\iota : H^3(M,\mathbb{Z}) \hookrightarrow H^3(M,\mathbb{C}) \xrightarrow{\pi} \frac{H^3(M,\mathbb{C})}{F^2H^3(M,\mathbb{C})} = \overline{F^2H^3(M,\mathbb{C})} \cong H^{1,2}(M) \oplus H^{0,3}(M).$$

Note that the latter isomorphism is only defined when specific representatives for $H^{p,q}(M)$ are chosen, cf. page 23. ι is an injective map by the same argument as used in the proof of lemma 3.14. It has maximal rank in $\overline{F^2H^3(M,\mathbb{C})}$, since

$$\operatorname{rk}\iota(H^3(M,\mathbb{Z})) = \dim_{\mathbb{C}} H^3(M,\mathbb{C}) = 2\dim_{\mathbb{C}} \overline{F^2H^3(M)}.$$

Applied to a Calabi-Yau$_3$, we are led to the definition of the following nonalgebraic complex torus [Gri84, BL99].

Definition 3.22 (Griffiths intermediate Jacobian). The *Griffiths intermediate Jacobian* of a Calabi-Yau$_3$ manifold, $(\mathcal{J}_G(CY), h_G)$, is the $(1+h^{2,1})$-dimensional nondegenerate complex torus of index 1 defined by

$$\mathcal{J}_G(CY) = \frac{H^3(CY,\mathbb{C})}{F^2H^3(CY,\mathbb{C}) \oplus H^3(CY,\mathbb{Z})} = \frac{\overline{F^2H^3(CY,\mathbb{C})}}{\iota(H^3(CY,\mathbb{Z}))}, \tag{3.16a}$$

$$h_G(\alpha,\beta) = 2iQ(\alpha,\bar{\beta}). \tag{3.16b}$$

⊘

Proof. Again the index of h_G follows from the Hodge-Riemann bilinear relations, theorem 3.17. Consider an element $\alpha \in H^{0,3}(CY)$. Since $C\bar{\alpha} = -i\bar{\alpha}$,

$$h_G(\alpha,\alpha) = 2iQ(\alpha,\bar{\alpha}) = 2Q(\alpha,i\bar{\alpha}) = 2Q(\alpha,-C\bar{\alpha}) = -2Q(\alpha,C\bar{\alpha}) < 0.$$

A similar calculation shows that h_G is positive definite on $H^{1,2}(CY)$. □

To understand what it means that these Griffiths intermediate Jacobians vary nicely under holomorphic variations we define the *bundle of Griffiths intermediate Jacobians*.

Definition 3.23 (Bundle of Griffiths intermediate Jacobians). Let $\pi : \mathscr{C}\mathscr{Y} \to S$ be a family of Calabi-Yau$_3$ manifolds, cf. definition 4.10. Thus each fibre $CY_s = \pi^{-1}(s)$ is a Calabi-Yau$_3$ manifold, which has an associated Griffiths intermediate Jacobian $(\mathcal{J}_G(CY_s), h_G)$. We define the *bundle of Griffiths intermediate Jacobians* to be the fibre bundle

$$\pi_{\mathscr{J}_G} : \mathscr{J}_G \to S,$$

whose fibre over the point $s \in S$ is the Griffiths intermediate Jacobian of CY_s, $\pi^{-1}_{\mathscr{J}_G}(s) = \mathcal{J}_G(CY_s)$.

⊘

Similarly we can define the *bundle of Weil intermediate Jacobians*. However the following proposition is only valid for the Griffiths intermediate Jacobian.

Proposition 3.24 (Holomorphic variation of bundle of Griffiths intermediate Jacobians). *The complex torus bundle $\pi_{\mathscr{J}_G} : \mathscr{J}_G \to S$ defined in definition 3.23 is a holomorphic fibre bundle. In particular the total space $\mathscr{J}_G$ carries the structure of a complex manifold and $\pi_{\mathscr{J}_G}$ is a holomorphic map.*

Proof. [Gri68] showed that the Hodge filtration of $H^3(CY, \mathbb{C})$, in contrast to the Hodge decomposition, varies holomorphically under a holomorphic variation of the underlying manifold CY. Since the Griffiths intermediate Jacobian $\mathcal{J}_G(CY)$ can be described completely in terms of the Hodge filtration of $H^3(CY, \mathbb{C})$, it too varies holomorphically with variations of CY. Hence the structure of $\mathscr{J}_G$ is that of a holomorphic fibre bundle over the deformation space S (of complex structures) [Gri68, BL99]. □

To conclude our presentation of the intermediate Jacobians of a Calabi-Yau$_3$, we will define the Griffiths intermediate Jacobian in terms of a complex structure put upon $H^3(CY, \mathbb{R})$. Comparison with lemma 3.19 reveals that we need to find a complex structure J, such that $\overline{F^2H^3(CY, \mathbb{C})} = \{\alpha \in H^3(CY, \mathbb{C}) | J\alpha = i\alpha\}$. Defining an alternative "Weil" operator

$$C' = \sum_{\substack{p+q=3 \\ 0 \leq p,q \leq 3}} i^{\frac{p-q}{|p-q|}} \pi_{p,q}, \tag{3.17}$$

application of corollary 3.20 to $J = -C'$ leads to the following proposition.

Proposition 3.25 (Real cohomology description of Griffiths intermediate Jacobian). *In terms of the real third cohomology group $H^3(CY, \mathbb{R})$ upon which a complex structure $-C'$ is put, the Griffiths intermediate Jacobian $(\mathcal{J}_G(CY), h_G)$ of a Calabi-Yau$_3$ manifold CY is given by*

$$\mathcal{J}_G(CY) = \frac{(H^3(CY, \mathbb{R}), -C')}{H^3(CY, \mathbb{Z})}, \tag{3.18a}$$

$$h_G(\phi, \chi) = Q(-C'\phi, \chi) + iQ(\phi, \chi). \tag{3.18b}$$

In the literature it is not always explicitly mentioned which intermediate Jacobian is meant. Most often the Griffiths intermediate Jacobian is considered for its holomorphic variation property. Nevertheless we have found an application of the Weil intermediate Jacobian in the physics literature. In table 3.1 we have given a summary of the different definitions of the Weil and Griffiths intermediate Jacobian.

Intermediate Jacobians			
		Weil	**Griffiths**
Dolbeault cohom.	Torus	$H^{3,0} \oplus H^{1,2} / \iota(H^3(\mathbb{Z}))$	$H^{1,2} \oplus H^{0,3} / \iota(H^3(\mathbb{Z}))$
	Polarization	$2iQ(\alpha, \bar{\beta})$	$2iQ(\alpha, \bar{\beta})$
Real cohom.	Torus	$(H^3(\mathbb{R}), -C)/H^3(\mathbb{Z})$	$(H^3(\mathbb{R}), -C')/H^3(\mathbb{Z})$
	Polarization	$Q(-C\phi, \chi) + iQ(\phi, \chi)$	$Q(-C'\phi, \chi) + iQ(\phi, \chi)$
	Complex structure	$C = \sum i^{p-q} \pi_{p,q}$	$C' = \sum i^{\frac{p-q}{\lvert p-q \rvert}} \pi_{p,q}$
Index of polarization, $\mathrm{ind}_{\mathbb{C}}$		0	1

Table 3.1: Overview of the differences between the Griffiths and Weil intermediate Jacobian.

Remark: In this section we have only focused on intermediate Jacobians of the third cohomology group of a Calabi-Yau$_3$ manifold. In general for any n-dimensional smooth projective variety M, we may define the k-th Griffiths and Weil intermediate Jacobians on $H^{2k-1}(M, \mathbb{C})$ [BL99]. The

Jacobian of a curve and the Albanese and Picard varieties are the motivating examples of k-th intermediate Jacobians, for which Griffiths and Weil agree in their definition. The first nontrivial example is the second intermediate Jacobian which we consider in this thesis. $\Diamond$

Chapter 4

Type II superstring compactifications on Calabi-Yau$_3$ manifolds

The results in this thesis provide a better understanding of a specific aspect within Calabi-Yau$_3$ compactifications of type II superstring theory, viz. the c-map. In order for the reader to put (the results of) this thesis in a proper physical context, we will present part of the formalism of string theory and Calabi-Yau$_3$ compactifications in this chapter. For a more thorough and complete discussion of this vast subject we refer to any of the standard textbooks [GSW87, LT89, Pol98, BBS07] and lecture notes [Gre96, Asp00]. We will start with a short introduction to the worldsheet action of string theory, type II superstring theory, its massless spectrum and low energy effective action. Then compactification on Calabi-Yau$_3$ manifolds is explained, focusing on the geometry of the corresponding moduli space. Finally we give a physical introduction to T-duality between both type II string theories and the relation it induces on the moduli spaces.

4.1 Type II superstring theory

Superstring theory

String theory may be summarized by saying that it is a quantum field theory describing string particles rather than point particles. A string moves through d-dimensional space and sweeps out a two-dimensional surface, the *worldsheet* of the string, rather than the *worldline* of a point particle. This process is formulated as an embedding of the worldsheet into the *target space* by coordinate functions $(\hat{x}^{\hat{\mu}}(\sigma,\tau))_{\hat{\mu}=0}^{d-1}$. Here σ and τ are local coordinates of the worldsheet. τ is viewed as the proper time of the worldsheet, while $\sigma \in [0,\pi]$ is the parameter along the string itself. In analogy with the point particle situation, classical strings may be realized by a minimization of the area of the worldsheet. The quantum field theory is then given by the variational principle of the *Nambu-Goto action*,

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\sqrt{(\partial_\sigma\hat{x}^{\hat{\mu}}\partial_\tau\hat{x}_{\hat{\mu}})^2 - (\partial_\sigma\hat{x}^{\hat{\mu}})^2(\partial_\tau\hat{x}_{\hat{\mu}})^2}, \tag{4.1}$$

where the *string scale* α' is inversely proportional to the string's tension. It measures the "stringiness" of the string. For small α' the energy scales are so small that the stringlike behavior of the strings is negligible and string theory reduces to a *low energy effective action* of point particles by perturbation theory. When α' is not small perturbative effects and the non-perturbative effects associated with *worldsheet instantons* become relevant for the theory to be meaningful.

The strings in string theory may be open or closed. Open strings are attached to *D-branes* to ensure momentum conservation. If an open string is subject to $(d-p)$ Dirichlet boundary conditions and p Neumann boundary conditions it is attached to a Dp-brane, which is a p-dimensional dynamical object by itself.

By including fermionic degrees of freedom on the worldsheet, a (worldsheet) *supersymmetric* action may be constructed, called *superstring theory*. Supersymmetry is a symmetry between bosons and fermions of a theory, whereas "ordinary" symmetries transform bosonic fields into bosonic fields and fermionic fields into fermionic fields. Since charges belonging to a *super*symmetry are spinors, they relate masses and couplings of bosonic fields with those of fermionic fields, cf. section 4.3.

There are five possible supersymmetric string theories, all connected through an impressive web of dualities. Two of these theories are called *type IIA* and *type IIB superstring theory*. They both consist of closed strings as well as open strings attached to D-branes. Due to worldsheet supersymmetry their low energy effective theories are (spacetime) supersymmetric as well. Although fermions are an essential ingredient in supersymmetric theories, in this thesis we are mainly concerned with the type IIA's and IIB's *bosonic* low energy effective theory for closed strings.

A remarkable restriction for all (super)string theories is that a proper quantization of the theory requires a specific ("critical") dimension for the target space of the strings.

Theorem 4.1 (Critical dimension for superstring theory). *A superstring theory can only be quantized consistently if the target space is ten-dimensional.*

Superstring theory is already a big improvement compared with the purely bosonic Nambu-Goto theory (4.1) for which the critical dimension is 26. Nevertheless its critical dimension is still six dimensions larger than the four-dimensional Lorentzian manifold we believe to live in. In section 4.2 we will answer the question how physicists resolve the inconsistency in the dimension of the target space of superstring theory and the observed spacetime around us.

Type IIA massless spectrum and low energy effective action

Let us present the bosonic low energy effective theory for type IIA closed superstrings. For simplicity we will only consider its massless (lowest order) spectrum.

Theorem 4.2 (Type IIA massless bosonic spectrum of closed strings in ten dimensions). *The massless bosonic spectrum of type IIA closed superstrings is given in table 4.1.*[1]

NS-NS	ϕ	Real scalar field, dilaton
	$B_{\hat{\mu}\hat{\nu}}$	Antisymmetric rank 2 tensor field, Kalb-Ramond field
	$g_{\hat{\mu}\hat{\nu}}$	Traceless symmetric rank 2 tensor field, graviton
R-R	$A_{\hat{\mu}}$	Vector field
	$A_{\hat{\mu}\hat{\nu}\hat{\rho}}$	Antisymmetric rank 3 tensor field

Table 4.1: Type IIA massless bosonic spectrum of closed superstrings.

Remark: Note that physicists are a bit sloppy in their language and call the differential 1-form $A_{\hat{\mu}}$ a vector field instead of a *co*vector field. $\lozenge$

The (NS-NS)-sector is associated with the propagation of the string through target space. Its interactions with the background fields of target space is separated into a trace, a traceless symmetric tensor field and an antisymmetric tensor field. The vacuum expectation value ϕ_0 of the dilaton ϕ is the *string coupling* $g_s = e^{-2\phi_0}$ of the strings and it determines the likelihood of two strings joining or splitting. The symmetric tensor field $g_{\hat{\mu}\hat{\nu}}$ is the *graviton* and its appearance is the reason string theorists claim that string theory provides a quantum field theory of general

[1]The spectrum of table 4.1 is divided into *Ramond* and *Neveu-Schwarz* sectors, which refer to periodic and antiperiodic boundary conditions on the left- and right-moving worldsheet fermions. States in the (R-R)- and (NS-NS)-sectors describe spacetime bosons, while states in the (R-NS)- and (NS-R)-sectors describe spacetime fermions (and are therefore not presented in table 4.1).

relativity. Coupling of the string with the background field $g_{\hat{\mu}\hat{\nu}}$ describes a string moving through *curved* spacetime. In (4.2) the term containing the Ricci scalar becomes the Einstein-Hilbert term of general relativity after a Weyl transformation from the *string frame* to the *Einstein frame* [BBS07], making explicit the incorporation of general relativity into string theory.

The last field in the (NS-NS)-sector, the antisymmetric field $B_{\hat{\mu}\hat{\nu}}$, couples to the string just as the electromagnetic gauge field A^μ couples to a point particle in Quantum Electrodynamics [Ryd99]. For this reason the Kalb-Ramond field B is interpreted as a gauge field coupled to the string and the string is said to be electrically charged under the B-field. Its *field strength* is traditionally denoted with $H = \mathrm{d}B$. The (R-R)-fields are in a similar manner interpreted as "electromagnetic" fields coupled to D-branes; every D$(p-1)$-brane is electrically charged under a (R-R)-p-field. We will denote the field strength of $A^{(p)}$ by $\mathcal{F}^{(p+1)} = \mathrm{d}A^{(p)}$. The analogy with ordinary electromagnetism follows from the equations

$$\mathrm{d}\mathcal{F}^{(p+1)} = 0, \qquad \mathrm{d}\star\mathcal{F}^{(p+1)} = 0,$$

which follow from the exactness (as a differential form) of $\mathcal{F}^{(p+1)}$ and the low energy effective action of type IIA string theory.

Theorem 4.3 (Type IIA low energy effective action). *The low energy effective action for the bosonic part of type IIA superstring theory is given by*

$$\begin{aligned}
S_{IIA} &= S_{NS} + S_R + S_{CS}, && (4.2a)\\
S_{NS} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}\hat{x}\sqrt{g}e^{-2\phi}R + \frac{1}{2\kappa^2}\int e^{-2\phi}\left(4\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi - \frac{1}{2}H\wedge\star\,H\right), && (4.2b)\\
S_R &= -\frac{1}{4\kappa^2}\int\left(\mathcal{F}^{(2)}\wedge\star\,\mathcal{F}^{(2)} + \tilde{\mathcal{F}}^{(4)}\wedge\star\,\tilde{\mathcal{F}}^{(4)}\right), && (4.2c)\\
S_{CS} &= -\frac{1}{4\kappa^2}\int\left(B\wedge\mathcal{F}^{(4)}\wedge\mathcal{F}^{(4)}\right), && (4.2d)
\end{aligned}$$

where $\tilde{\mathcal{F}}^{(4)} = \mathcal{F}^{(4)} - A^{(1)}\wedge H$ *and* $2\kappa^2 = (2\pi)^7\alpha'^4$. *The terms are regrouped with respect to the different sectors.* S_{NS} *contains (NS-NS)-fields,* S_R *contains (R-R)-fields. The term* S_{CS} *contains both and is called the* Chern-Simons *(or* topological*) term.*

Type IIB massless spectrum and low energy effective action

Similarly we present the bosonic low energy effective theory for type IIB closed superstrings.

Theorem 4.4 (Type IIB massless bosonic spectrum of closed strings in ten dimensions). *The massless bosonic spectrum of type IIB closed superstrings is given in table 4.2.*

NS-NS	ϕ	Real scalar field, dilaton
	$B_{\hat{\mu}\hat{\nu}}$	Antisymmetric rank 2 tensor field, Kalb-Ramond field
	$g_{\hat{\mu}\hat{\nu}}$	Traceless symmetric rank 2 tensor field, graviton
R-R	a	Real scalar field, axion
	$C_{\hat{\mu}\hat{\nu}}$	Antisymmetric rank 2 tensor field
	$C_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$	Self dual antisymmetric rank 4 tensor field

Table 4.2: Type IIB massless bosonic spectrum of closed superstrings.

Remark: The name *axion* is used in a number of situations. The Kalb-Ramond field is often called axion because of its Poincaré dual representative in four dimensions. Similarly the 2-form $C^{(2)}$ in type IIB may be dualized to a 0-form which is also called an axion. These two four-dimensional axions and the real scalar a in ten- (or four-)dimensional type IIB string theory should not be confused. ◊

For type IIB string theory the self dual 4 tensor $C^{(4)}$ complicates the construction of a low energy effective action.

Theorem 4.5 (Type IIB low energy effective action). *The low energy effective action for the bosonic part of type IIB superstring theory is given by the action*

$$
\begin{aligned}
S_{IIB} &= S_{NS} + S_R + S_{CS}, && (4.3a)\\
S_{NS} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}\hat{x}\sqrt{g}e^{-2\phi}R + \frac{1}{2\kappa^2}\int e^{-2\phi}\left(4\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi - \frac{1}{2}H\wedge\star\,H\right), && (4.3b)\\
S_R &= -\frac{1}{4\kappa^2}\int\left(\mathcal{F}^{(1)}\wedge\star\,\mathcal{F}^{(1)} + \tilde{\mathcal{F}}^{(3)}\wedge\star\,\tilde{\mathcal{F}}^{(3)} + \frac{1}{2}\tilde{\mathcal{F}}^{(5)}\wedge\star\,\tilde{\mathcal{F}}^{(5)}\right), && (4.3c)\\
S_{CS} &= -\frac{1}{4\kappa^2}\int\left(C^{(4)}\wedge H\wedge\mathcal{F}^{(3)}\right), && (4.3d)
\end{aligned}
$$

where $\tilde{\mathcal{F}}^{(3)} = \mathcal{F}^{(3)} - aH$ and $\tilde{\mathcal{F}}^{(5)} = \mathcal{F}^{(5)} - \frac{1}{2}C^{(2)}\wedge H + \frac{1}{2}B\wedge\mathcal{F}^{(3)}$. As an extra constraint, the solutions of (4.3) should satisfy $\tilde{\mathcal{F}}^{(5)} = \star\,\tilde{\mathcal{F}}^{(5)}$.

Nonlinear sigma models

In this thesis we will analyze the geometry of manifolds associated to physical theories. The relation between geometry and the quantum field theories under consideration is given by the precise shape of the kinetic part of the quantum field theories, called *nonlinear sigma models* [HKLR87]. The four-dimensional type IIA and IIB low energy effective actions are examples of such a model.

Definition 4.6 (Nonlinear sigma model). Let (M,γ) be a d-dimensional Riemannian manifold and (T,g) be an n-dimensional Riemannian manifold. Suppose $\phi^i : M \to T$, $i\in\{1,\ldots,n\}$, is a field on M with values in T, i.e. ϕ is a section of a fibre bundle over M with typical fibre T. T is called *target space* and the fields ϕ^i define an embedding into it. The *base space* M is often called *spacetime*, but this is not always equal to spacetime itself. If the dynamics of the fields ϕ is given by an action S,

$$
\begin{aligned}
S &= \int_M \mathscr{L}[\phi^1,\ldots,\phi^n], && (4.4a)\\
\mathscr{L}[\phi^1,\ldots,\phi^n] &= \sum_{i,j=1}^{n}\sum_{a,b=1}^{d} g_{ij}(\phi)\frac{\partial\phi^i}{\partial\xi^a}\frac{\partial\phi^j}{\partial\xi^b}\sqrt{\gamma}\gamma^{ab}\mathrm{d}^d\xi = \sum_{i,j=1}^{n} g_{ij}(\phi)\mathrm{d}\phi^i\wedge\star\,\mathrm{d}\phi^j, && (4.4b)
\end{aligned}
$$

for which the metric $g_{ij}(\phi)$ on T depends on the fields ϕ, we call the theory a *nonlinear sigma model*. When g_{ij} does not depend on the fields ϕ we call the theory simply a *sigma model.* In both cases the metric g is called the *sigma model metric*. ⊘

String theory itself may be formulated as a nonlinear sigma model for which target space is spacetime and the fields $\phi^{\hat{\mu}} = \hat{x}^{\hat{\mu}}$ are simply the coordinates of the string within the (26- or 10-dimensional) target space. Other examples of nonlinear sigma models are the four-dimensional type IIA and type IIB low energy effective actions (4.15) and (4.16) for which spacetime acts as the base space of the nonlinear sigma model and the target space T is just the space of fields. We associate metrics on the target spaces using the following corollary.

Corollary 4.7 (Sigma model metric). *Let (M,γ), (T,g) be two Riemannian manifolds. Suppose a field theory with fields $\phi^i : M \to T : \xi \mapsto \phi^i(\xi)$, $i\in\{1,\ldots,n\}$ is a nonlinear sigma model given by the Lagrangian*

$$\mathscr{L}[\phi^1,\ldots,\phi^d] = \sum_{i,j=1}^{d} g_{ij}(\phi)\mathrm{d}\phi^i\wedge\star\,\mathrm{d}\phi^j.$$

The fields ϕ^i parameterize the manifold T and in local coordinates x^i, the metric on T is given by

$$\mathrm{d}s^2 = \sum_{i,j=1}^{n} g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j.$$

4.2 The compactified theory

The internal manifold

A consistent quantization of superstring theory requires a ten-dimensional target space, while spacetime around us appears four-dimensional. By *compactification* on an *internal manifold K* the ten-dimensional theory can be reduced to an effective four-dimensional theory. Roughly said by "compactification" we mean that the ten dimensions in superstring theory are divided into two types. Four of them are the extended dimensions we perceive forming (in the simplest case) Minkowski space $\mathbb{R}^{1,3}$. The other six dimensions are small, curled-up and difficult to detect. They constitute the internal manifold K, which is assumed to be compact. Strings may propagate through all of these dimensions and p-forms may live partially in the extended and partially in the curled-up dimensions.

Assumption 4.8 (Quantum fluctuations around the classical compactification limit). The vacuum state (classical solution) obtained from type II superstring theory is defined on a ten-dimensional manifold $X_{II} = \mathbb{R}^{1,3} \times K$, where K is a six-dimensional compact manifold. This Cartesian product is a saddle point around which quantum fluctuations of the background fields (the fields of table 4.1 or 4.2) are performed. The resulting geometry is locally of the form

$$X_{II} = \mathbb{R}^{1,3} \times K.$$

Furthermore there should be an unbroken $N = 1$ supersymmetry in four dimensions, while the gauge group and fermion spectrum should be realistic. $\oslash$

Because of $N = 1$ supersymmetry there exists a Killing spinor on X_{II} which may be locally decomposed into two spinors. The Killing spinor equation belonging to K implies that the holonomy of K is SU(3) [CHSW85, BBS07], cf. definition 2.39.

Theorem 4.9 (Compactification on Calabi-Yau$_3$'s). *Under assumption 4.8 the manifold X_{II} is seen to be a fibre bundle over $\mathbb{R}^{1,3}$. The fibres of this fibre bundle are isomorphic to a Calabi-Yau$_3$ manifold. All fibres in the fibre bundle are topologically equivalent. This is often denoted by a trivialization*

$$X_{II} = \mathbb{R}^{1,3} \times CY.$$

CY is said to be the *Calabi-Yau$_3$ manifold upon which the theory is compactified, although actually the fibres form a family of Calabi-Yau$_3$ manifolds topologically equivalent to CY.*

Definition 4.10 (Family of Calabi-Yau$_3$'s). Let CY be an arbitrary Calabi-Yau$_3$ manifold and consider the moduli space $\mathscr{M}_g$ of its geometric deformations. Note that these do not change the topology of CY. We denote the corresponding *family of Calabi-Yau$_3$ manifolds* with $\mathscr{C}\mathscr{Y} = (CY_{\delta g})_{\delta g \in \mathscr{M}_g} \to \mathscr{M}_g$, $CY_0 = CY$. $\oslash$

At each point of Minkowski space a different member $CY_{\delta g}$ of the family of Calabi-Yau$_3$'s is fibred. Deformations of the metric of CY are expressed by the *geometric moduli* $\delta g \in \mathscr{M}_g$. Later we will see that these may be subdivided into complex structure deformations of CY and deformations of its (complexified) Kähler class. Since all Calabi-Yau$_3$'s in the family are topologically equivalent to CY, it is often said that we compactify on *a* Calabi-Yau$_3$ CY. In this thesis we will consider a generic Calabi-Yau$_3$ manifold, without specifying which one we consider precisely. Furthermore we will apply theorem 4.9 to $N = 2$ supersymmetry (cf. section 4.3), which is an even more restricted situation than assumed in assumption 4.8. Although it is less relevant from a phenomenological point of view, it provides enough restrictions to achieve decent control of the theory for our purposes.

In the compactification assumption the six-dimensional compact Calabi-Yau$_3$ manifolds are assumed to be very small. As a result the only part of the background fields we experience, are the parts that live on the external Minkowski space $\mathbb{R}^{1,3}$. The parts that live on the Calabi-Yau$_3$ manifold are not directly perceived, but may be integrated out to yield a four-dimensional effective theory on Minkowski space. Let us take a look on the type II four-dimensional spectrum obtained from tables 4.1 and 4.2. First we need some notation and an assumption.

Notation: Locally we denote the coordinates $(\hat{x}^{\hat{\mu}})_{\hat{\mu}=0}^{9}$ of X_{II} as $(\hat{x}^{\hat{\mu}})_{\hat{\mu}=0}^{9} = (x^\mu, y^a)$, where $\mu \in \{0,\ldots,3\}$ and $a \in \{1,\ldots,6\}$. In general we will use Greek letters with a hat to denote the indices of X_{II}, Greek letters without a hat to denote the indices of $\mathbb{R}^{1,3}$ and Latin letters to denote the real indices of the Calabi-Yau$_3$ manifold. Since the Calabi-Yau$_3$ manifold is a complex manifold we will also use complex coordinates $(z^i, \bar{z}^{\bar{i}})_{i=1}^{3}$. ♢

Assumption 4.11 (Expansion on harmonic forms). Let ω be a differential form on $X_{II} = \mathbb{R}^{1,3} \times CY$, then we assume ω can be written as

$$\omega(\hat{x}) = \omega(x,y) = \sum_k \omega_4^k(x) \wedge \lambda^k(y),$$

where the summation over k runs over the eigenforms $\lambda_k(y)$ of the Laplace operator $\Box_{CY}$ on CY with eigenvalue zero. ⊘

The rationale behind this assumption is that the equation

$$\Box_{X_{II}} \omega(\hat{x}) = 0 \tag{4.5}$$

is exactly the equation of motion following from the kinetic part of the actions (4.2) and (4.3) for the (fluctuations of the) fields $A^{(p)}$, a, $C^{(p)}$, ϕ and B. Because of the product structure of X_{II}, equation (4.5) applied to an arbitrary p-form $\omega(\hat{x}) = \omega_4(x) \wedge \omega_6(y)$ satisfying $\Box_{CY}\omega_6 = \alpha\omega_6$, yields

$$\Box_{\mathbb{R}^{1,3}}\omega_4(x) + \alpha\omega_4(x) = 0,$$

for the four-dimensional field $\omega_4(x)$. Therefore from the point of view of Minkowski space, separation of variables leads to an effective *mass contribution* for the four-dimensional fields. Since the eigenvalues of $\Box_{CY}$ scale inversely proportional to the volume of the Calabi-Yau$_3$ (which is assumed to be very small), it suffices from a phenomenological point of view to consider only the *massless modes* ω_4. These are the forms wedged with eigenforms of $\Box_{CY}$ having eigenvalue zero, i.e. the *harmonic forms*. By Hodge's theorem 2.32 they coincide with the different Dolbeault cohomology equivalence classes. To fix our thoughts we now choose a set of representatives of nonzero classes.

Notation: Following [BCF91] and considering the Hodge diamond (2.13), we define a basis of nonzero harmonic forms on CY.

$$\begin{aligned}
1 &= 1(y), && (4.6\text{a})\\
V_\Lambda &= V_{\Lambda i\bar{j}}(z,\bar{z})\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}, && (4.6\text{b})\\
vol &= \frac{1}{3!} vol_{ijk}(z)\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k, && (4.6\text{c})\\
\Phi_A &= \frac{1}{2!}\Phi_{Aij\bar{k}}(z,\bar{z})\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{k}}, && (4.6\text{d})
\end{aligned}$$

where $\Lambda \in \{1,\ldots,h^{1,1}\}$ and $A \in \{1,\ldots,h^{2,1}\}$. ♢

Four-dimensional effective type IIA theory

Theorem 4.12 (Type IIA massless bosonic spectrum in four dimensions). *After compactifying the ten-dimensional type IIA bosonic spectrum from table 4.1 on a family of Calabi-Yau$_3$ manifolds, the four-dimensional massless field content is as given in table 4.3. The division into supermultiplets will be further discussed in section 4.3.*

$h^{1,1}$ vector multiplets	d^Λ	Real scalar fields, Kähler moduli
	b^Λ	Real scalar fields
	A^Λ_μ	Vector fields
Tensor multiplet	ϕ	Real scalar field, 4d dilaton
	$B_{\mu\nu} \leftrightarrow \sigma$	Antisymmetric rank 2 tensor field, (Kalb Ramond) axion
	A^0, B_0	Real scalar fields
$h^{2,1}$ hypermultiplets	t^A	Complex scalar fields, complex structure moduli
	A^A, B_A	Real scalar fields
Gravity multiplet	$g_{\mu\nu}$	Traceless symmetric rank 2 tensor field, 4d graviton
	A^0_μ	Vector field, graviphoton
	$A_{\mu\nu\rho}$	Antisymmetric rank 3 tensor field, cosmological constant

Table 4.3: Four-dimensional massless bosonic type IIA spectrum.

Proof. We will show how the fields $B_{\mu\nu}$ and b^Λ in table 4.3 are obtained from the ten-dimensional Kalb-Ramond field $B_{\hat\mu\hat\nu}$ by an expansion on the harmonic forms (4.6) of the Calabi-Yau$_3$. The dilaton and all (R-R)-fields go accordingly. This construction may also be found in [BCF91].

The antisymmetric Kalb-Ramond field $B(\hat x) = \frac{1}{2!}B_{\hat\mu\hat\nu}(\hat x)\mathrm{d}\hat x^{\hat\mu} \wedge \mathrm{d}\hat x^{\hat\nu}$ can be expanded as

$$B_{\hat\mu\hat\nu}(\hat x)\mathrm{d}\hat x^{\hat\mu} \wedge \mathrm{d}\hat x^{\hat\nu} = B_{\mu\nu}(\hat x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + 2B_{\mu a}(\hat x)\mathrm{d}x^\mu \wedge \mathrm{d}y^a + B_{ab}(\hat x)\mathrm{d}y^a \wedge \mathrm{d}y^b.$$

The field $B_{\mu\nu}(\hat x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is a 0-form on the Calabi-Yau$_3$ and can thus be written as $B_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge 1(y)$ or simply as $B_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. This four-dimensional antisymmetric rank 2 tensor field $B_{\mu\nu}(x)$ is often called the *(Kalb-Ramond) axion* because of its Poincaré dual σ in four dimensions. The field $B_{\mu a}(\hat x)\mathrm{d}x^\mu \wedge \mathrm{d}y^a$ is a 1-form on the Calabi-Yau$_3$ and since $h^{1,0} = h^{0,1} = 0$ on the Calabi-Yau$_3$, we see that it has no massless contribution. Finally the field $B_{ab}(\hat x)\mathrm{d}y^a \wedge \mathrm{d}y^b$ is a 2-form on the Calabi-Yau$_3$ and a 0-form on Minkowski space $\mathbb{R}^{1,3}$. Since $h^{2,0} = h^{0,2} = 0$ we conclude that we can write

$$B_{ab}(\hat x)\mathrm{d}y^a \wedge \mathrm{d}y^b = \sum_{\Lambda=1}^{h^{1,1}} b^\Lambda(x)V_{\Lambda i\bar j}(z,\bar z)\mathrm{d}z^i \wedge \mathrm{d}\bar z^{\bar j} = \sum_{\Lambda=1}^{h^{1,1}} b^\Lambda(x)V_\Lambda, \tag{4.7}$$

where the $b^\Lambda(x)$ are $h^{1,1}$ real scalar fields in four dimensions.

The only field of table 4.1 left to compactify is the graviton $g(\hat x) = \frac{1}{2!}g_{\hat\mu\hat\nu}(\hat x)\mathrm{d}\hat x^{\hat\mu} \otimes \mathrm{d}\hat x^{\hat\nu}$, a traceless symmetric rank 2 tensor field. It may be expanded into components

$$g_{\hat\mu\hat\nu}(\hat x)\mathrm{d}\hat x^{\hat\mu} \otimes \mathrm{d}\hat x^{\hat\nu} = g_{\mu\nu}(\hat x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + 2g_{\mu a}(\hat x)\mathrm{d}x^\mu \otimes \mathrm{d}y^a + g_{ab}(\hat x)\mathrm{d}y^a \otimes \mathrm{d}y^b.$$

Using again a separation of variables we find the *four-dimensional graviton* $g_{\mu\nu}(x)$, which is a 0-form on the Calabi-Yau$_3$. The field $g_{\mu a}$ is a 1-form on the Calabi-Yau$_3$ having no massless contribution. To understand the four-dimensional contributions from $g_{ab}(\hat x)$, which is a scalar from the four-dimensional point of view, we follow [Can88] and consider the geometric deformations of the six-dimensional Calabi-Yau$_3$ metric

$$g_{ab} = g^0_{ab} + \delta g_{ab}.$$

As opposed to our assumption of the other background fields, the classical solution g^0_{ab} does not vanish in our assumption but determines the geometry of the specific Calabi-Yau$_3$ manifolds in the fibre bundle. The fluctuations δg_{ab} of the metric should respect the topology of the fibre bundle and hence we demand that

$$\begin{aligned} R_{ab}(g^0) &= 0, & (4.8)\\ R_{ab}(g^0+\delta g) &= 0, & (4.9)\end{aligned}$$

i.e. the Ricci tensor R_{ab} should be zero for both the original metric g^0 as well as for fluctuations around it. Upon linearizing equation (4.9) and using (4.8), one finds the *Lichnerowicz equation*

$$\nabla^c\nabla_c\delta g_{ab} + 2R_{a\ b}^{\ d\ f}\delta g_{df} = 0, \tag{4.10}$$

where ∇ and $R^a{}_{bcd}$ are defined with respect to the original metric g^0 [Can88, BBS07]. Owing to the Kähler properties of the metric and the Riemann tensor, the mixed type indices $\delta g_{i\bar{j}} = \delta\overline{g_{\bar{i}j}}$ do not combine with pure type indices $\delta\overline{g_{ij}} = \delta g_{\bar{i}\bar{j}}$ so both $\delta g_{i\bar{j}}$ and $\delta g_{\bar{i}\bar{j}}$ must satisfy (4.10) independently.

Let us first consider $\delta g_{i\bar{j}}$ by defining the $(1,1)$-form

$$\tilde{g} = i\delta g_{i\bar{j}}\mathrm{d}z^i\wedge\mathrm{d}\bar{z}^{\bar{j}}.$$

It can be shown that because $\delta g_{i\bar{j}}$ satisfies the Lichnerowicz equation (4.10), $\tilde{g}$ is a harmonic form [Can88, BBS07]. Hence $\delta g_{i\bar{j}}$ is uniquely identified with an element of $H^{1,1}(CY)$, which is spanned by the V_Λ's from (4.6). Via this identification we can write

$$i\delta g_{i\bar{j}}(w) = \sum_{\Lambda=1}^{h^{1,1}} d^\Lambda(x)V_{\Lambda i\bar{j}}(z,\bar{z}). \tag{4.11}$$

At each point of Minkowski space $\mathbb{R}^{1,3}$ the $h^{1,1}$ real fields $d^\Lambda(x)$ describe how the mixed type indices are being deformed. Since the mixed type indices determine the Kähler form of the Calabi-Yau$_3$ manifold, the d^Λ deformations are called the *Kähler moduli*. They classify infinitesimal Kähler transformations by slightly deforming the Ricci-flat Kähler form, while preserving Kählerity [Gre96, Bou07].

For $\delta g_{\bar{i}\bar{j}}$ we define a $(0,1)$-form that takes values in the holomorphic tangent bundle TCY

$$\breve{g}^i = -||vol||^2 g^{i\bar{j}}\delta g_{\bar{j}\bar{l}}\mathrm{d}\bar{z}^{\bar{l}}. \tag{4.12}$$

The factor $||vol||^2 = \frac{1}{3!}vol_{ijk}\overline{vol}^{ijk}$ is a conventional factor, which is constant over the Calabi-Yau$_3$ manifold [CdlO91, Hüb92, BBS07]. Again it may be shown that $\delta g_{\bar{i}\bar{j}}$ satisfies the Lichnerowicz equation (4.10) if and only if the $(0,1)$-form $\breve{g}^i$ is harmonic [Can88, BBS07]. Using the holomorphic 3-form vol_{ijk} it is possible to identify this $(0,1)$-form with values in the tangent bundle TCY with a $(2,1)$-form $\omega_{ij\bar{k}}$ on the Calabi-Yau$_3$ via

$$\breve{g}^i = \overline{vol}^{ijk}\omega_{jk\bar{l}}\mathrm{d}\bar{z}^{\bar{l}}. \tag{4.13}$$

Since $\omega_{ij\bar{k}}$ is harmonic if and only if $\breve{g}^i$ is harmonic [Can88], we have found an identification between $H^{2,1}(CY,\mathbb{C})$ and $H^{0,1}(CY,\mathscr{X}(CY))$. Combination of (4.12) and (4.13) tells us that we can write $\delta g_{\bar{i}\bar{j}}$ as a linear combination of the harmonic $(2,1)$-forms

$$\delta g_{\bar{i}\bar{j}}(\hat{x}) = -\frac{1}{||vol||^2}g_{\bar{i}l}(\hat{x})\overline{vol}^{lkn}(\bar{z})\omega_{kn\bar{j}}(z,\bar{z}) = -\frac{1}{||vol||^2}\sum_{A=1}^{h^{2,1}} t^A(x)\overline{vol}_{\bar{i}}^{\ kn}(\bar{z})\Phi_{Akn\bar{j}}(z,\bar{z}). \tag{4.14}$$

At each point of Minkowski space $\mathbb{R}^{1,3}$ the $h^{2,1}$ complex fields $t^A(x)$ describe how the pure type indices are being deformed. These deformations are identified with elements in $H^{2,1}(CY)$, since any two representatives of the same cohomology class yield metric perturbations that can be undone by coordinate transformations. Note that when the pure type indices of a hermitian metric are being deformed, the metric will seize to remain hermitian by definition [Nak03]. A suitable change of variables that puts the metric back in hermitian form will necessarily be not holomorphic, which means that the metric is hermitian with respect to a different complex structure. Therefore the elements of $H^{2,1}(CY)$, which classify pure type deformations of the metric, are called the *complex structure moduli* [Gre96]. □

The real fields d^Λ and b^Λ may be paired up to form complex fields $W^\Lambda = b^\Lambda + id^\Lambda$. From a mathematical point of view this is just a complexification of the Kähler cone, $\sigma_J^{\mathbb{C}} = B + i\sigma_J = W^\Lambda V_\Lambda$. Together d^Λ and b^Λ parameterize the deformations of the complexified Kähler class and the W^Λ are often called the *(complexified) Kähler moduli*. The Kähler moduli $W = (W^\Lambda)_{\Lambda=1}^{h^{1,1}}$ and complex structure moduli $t = (t^A)_{A=1}^{h^{2,1}}$ are the *geometric moduli*, since they are seen to arise from deformations of the Calabi-Yau$_3$ structure itself, cf. definition 4.10. In a string theoretical context physicists also refer to the other fields in table 4.3 as *moduli*, although this may lead to confusion among mathematicians.

Next we will write down the four-dimensional effective action of the compactified type IIA theory. The result is obtained by "simply" integrating out the Calabi-Yau$_3$-degrees of freedom and making necessary field redefinitions to put the action in its required form [BCF91].

Notation: For the description of the moduli fields we use four sets of indices. The indices Λ, Σ and A, B run from 1 to $h^{1,1}$ and $h^{2,1}$ respectively, while the coordinates Γ, Δ and I, J run from 0 to $h^{1,1}$ and $h^{2,1}$ respectively. ♢

Theorem 4.13 (Type IIA effective action in four dimensions). *The four-dimensional effective action for the bosonic part of the type IIA spectrum can be written as*

$$
\begin{aligned}
S^4_{IIA} &= S_{gr} + S_{vm} + S_{hm}, &\text{(4.15a)}\\
S_{gr} &= \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\sqrt{g}R, &\text{(4.15b)}\\
S_{vm} &= \int_{\mathbb{R}^{1,3}} \left[\frac{\partial^2 K^{(1,1)}(W,\bar{W})}{\partial W^\Lambda \partial \bar{W}^{\bar{\Sigma}}}\mathrm{d}W^\Lambda \wedge \star\, \mathrm{d}\bar{W}^{\bar{\Sigma}} + \frac{1}{2}\mathrm{Im}\left(\mathcal{N}^{(1,1)}_{\Gamma\Delta}\mathcal{F}^{+\Gamma}\wedge\star\,\mathcal{F}^{+\Delta}\right)\right], &\text{(4.15c)}\\
S_{hm} &= \int_{\mathbb{R}^{1,3}} \left[\frac{\partial^2 K^{(2,1)}(t,\bar{t})}{\partial t^A \partial \bar{t}^{\bar{B}}}\mathrm{d}t^A \wedge\star\, \mathrm{d}\bar{t}^{\bar{B}}\right.\\
&\quad - \frac{1}{4}\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi - \frac{1}{4}ie^{-\phi}(\mathcal{N}^{(2,1)} - \bar{\mathcal{N}}^{(2,1)})_{IJ}U^I\wedge\star\,\bar{U}^J &\text{(4.15d)}\\
&\quad \left. - \frac{1}{4}e^{-2\phi}\left(\mathrm{d}\sigma - \frac{1}{2}(A^I\mathrm{d}B_I - B_I\mathrm{d}A^I)\right)\wedge\star\left(\mathrm{d}\sigma - \frac{1}{2}(A^I\mathrm{d}B_I - B_I\mathrm{d}A^I)\right)\right],
\end{aligned}
$$

where the two-form field strengths $\mathcal{F}^\Gamma$ are defined by

$$\mathcal{F}^0 = \mathrm{d}A^0, \qquad \mathcal{F}^\Lambda = \mathrm{d}A^\Lambda,$$

and

$$U^I = (\mathcal{N}^{(2,1)} - \bar{\mathcal{N}}^{(2,1)})^{IJ}\left(\mathrm{d}B_J - 2\bar{\mathcal{N}}^{(2,1)}_{JK}\mathrm{d}A^K\right).$$

The objects $\mathcal{N}^{(1,1)}$ and $K^{(1,1)}$ are defined, cf. (5.17) and (5.1), in terms of the holomorphic function $F^{(1,1)}(W)$ that determines the projective special Kähler geometry of the Kähler moduli (cf. chapter 5) and similarly $\mathcal{N}^{(2,1)}$ and $K^{(2,1)}$ follow from the prepotential *$F^{(2,1)}(t)$ of the complex structure moduli space.*

Four-dimensional effective type IIB theory

Theorem 4.14 (Type IIB massless bosonic spectrum in four dimensions). *After compactifying the ten-dimensional type IIB bosonic spectrum from table 4.2 on a family of Calabi-Yau$_3$ manifolds, the four-dimensional massless field content is as given in table 4.4.*

$h^{2,1}$ vector multiplets	t^A	Complex scalar fields, complex structure moduli
	A^A_μ	Vector fields
Tensor multiplet	ϕ	Real scalar field, 4d dilaton
	$B_{\mu\nu} \leftrightarrow \sigma$	Antisymmetric rank 2 tensor field, (Kalb Ramond) axion
	$a = A^0$	Real scalar field, 4d axion
	$C_{\mu\nu} \leftrightarrow B_0$	Antisymmetric rank 2 tensor field
$h^{1,1}$ hypermultiplets	d^Λ	Real scalar fields, Kähler moduli
	b^Λ	Real scalar fields
	A^Λ	Real scalar fields
	B^Λ	Real scalar fields
Gravity multiplet	$g_{\mu\nu}$	Traceless symmetric rank 2 tensor field, 4d graviton
	A^0_μ	Vector field, graviphoton
	$C_{\mu\nu\rho\sigma}$	Antisymmetric rank 4 tensor field

Table 4.4: Four-dimensional massless bosonic type IIB spectrum.

Just as the Kalb-Ramond field may be dualized to a real scalar field σ, the antisymmetric rank 2 tensor field $C_{\mu\nu}$ can be dualized to a real scalar B_0 as well. The axion a will be denoted as A^0 in the next theorem.

Theorem 4.15 (Type IIB bosonic action in four dimensions). *The four-dimensional effective action for the bosonic part of the type IIB spectrum can be written as*

$$
\begin{aligned}
S^4_{IIB} &= S_{gr} + S_{vm} + S_{hm}, && (4.16a)\\
S_{gr} &= \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\sqrt{g}R, && (4.16b)\\
S_{vm} &= \int_{\mathbb{R}^{1,3}} \left[\frac{\partial^2 K^{(2,1)}(t,\bar t)}{\partial t^A \partial \bar t^{\bar B}} \mathrm{d}t^A \wedge \star\, \mathrm{d}\bar t^{\bar B} + \frac{1}{2}\mathrm{Im}\left(\mathcal{N}^{(2,1)}_{IJ}\mathcal{F}^{+I}\wedge\star\,\mathcal{F}^{+J}\right)\right], && (4.16c)\\
S_{hm} &= \int_{\mathbb{R}^{1,3}} \left[\frac{\partial^2 K^{(1,1)}(W,\bar W)}{\partial W^\Lambda \partial \bar W^{\bar\Sigma}} \mathrm{d}W^\Lambda \wedge \star\, \mathrm{d}\bar W^{\bar\Sigma}\right.\\
&\qquad - \frac{1}{4}\mathrm{d}\phi\wedge\star\,\mathrm{d}\phi - \frac{1}{4} i e^{-\phi}(\mathcal{N}^{(1,1)} - \bar{\mathcal{N}}^{(1,1)})_{\Gamma\Delta} U^\Gamma \wedge \star\, \bar U^\Delta && (4.16d)\\
&\qquad \left. - \frac{1}{4}e^{-2\phi}\left(\mathrm{d}\sigma - \frac{1}{2}(A^\Gamma \mathrm{d}B_\Gamma - B_\Gamma \mathrm{d}A^\Gamma)\right)\wedge\star\left(\mathrm{d}\sigma - \frac{1}{2}(A^\Gamma \mathrm{d}B_\Gamma - B_\Gamma\mathrm{d}A^\Gamma)\right)\right],
\end{aligned}
$$

where the two-form field strengths $\mathcal{F}^I$ are defined by

$$\mathcal{F}^0 = \mathrm{d}A^0, \qquad \mathcal{F}^A = \mathrm{d}A^A,$$

and

$$U^\Gamma = (\mathcal{N}^{(1,1)} - \bar{\mathcal{N}}^{(1,1)})^{\Gamma\Delta}\left(\mathrm{d}B_\Delta - 2\bar{\mathcal{N}}^{(1,1)}_{\Delta\Theta}\mathrm{d}A^\Theta\right).$$

The objects $\mathcal{N}^{(1,1)}$ and $K^{(1,1)}$ are defined, cf. (5.17) and (5.1), in terms of the holomorphic function $F^{(1,1)}(W)$ that determines the projective special Kähler geometry of the Kähler moduli (cf. chapter 5) and similarly $\mathcal{N}^{(2,1)}$ and $K^{(2,1)}$ follow from the prepotential $F^{(2,1)}(t)$ of the complex structure moduli space.

4.3 The moduli space of type II/CY compactifications

Multiplets of supersymmetry

Theorems 4.13 and 4.15 state the four-dimensional effective field theories obtained from type IIA and type IIB superstring theory respectively. They are obtained by compactifying the ten-dimensional theories on Calabi-Yau$_3$ manifolds, providing the sought-for connection between ten-dimensional superstring theory and the four-dimensional world around us. Although string theory itself is supposedly unique, the compactification process introduces new fields and new problems on how to describe the dynamics of these new fields. All terms in (4.15) and (4.16) containing scalar fields are terms of a nonlinear sigma model. As such associated to these theories are manifolds $\mathscr{M}^A$ and $\mathscr{M}^B$ parameterized by the scalar moduli. These are called the *moduli spaces* of the IIA and IIB theory respectively and their precise geometry already puts restrictions on the fields. For example we note that since both actions are equal, the moduli spaces must be equal (as manifolds) as well! For this reason we will often just call it $\mathscr{M}$.

A next observation is that both actions describe a metric that is split into two orthogonal parts, which we have denoted with S_{vm} and S_{hm}. Therefore the scalar manifold $\mathscr{M}$ is (locally) a product of two manifolds,

$$\mathscr{M} = \mathscr{M}_{vm} \times \mathscr{M}_{hm}. \tag{4.17}$$

The subscripts refer to *vector multiplets* and *hypermultiplets* respectively, two of the different types of *supermultiplets* with respect to which tables 4.3 and 4.4 are organized as well. This terminology stems from the formalism of $N = 2$ *supersymmetry theories* in which it is a general phenomenon that the vector multiplet and hypermultiplet scalar manifolds decouple [dWLVP85, Asp00]. The four-dimensional effective theories (4.15) and (4.16) are examples of $N = 2$ supersymmetry theories. We do not present an explanation of the distribution of type II moduli into different supermultiplets, which can be found in [VW96, Asp00]. We do hope to provide the reader with some intuition by discussing the different supermultiplets of tables 4.3 and 4.4 a little further.

In a general supersymmetric theory the fields are organized in different supermultiplets, representations of the corresponding Lie superalgebra. For example a vector multiplet consists of a complex scalar X, a real vector A_μ and fermions Ω_i, $1 \leq i \leq N$, which form a closed set under the supersymmetry transformations,

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i, \\
\delta A_\mu &= \tilde{\varepsilon}^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j + \tilde{\varepsilon}_{ij} \bar{\epsilon}^i \gamma_\mu \Omega^j, \\
\delta \Omega_i &= f(X, A_\mu, \epsilon^i),
\end{aligned}$$

where γ_μ are the Dirac matrices and f is a specific function of the bosonic fields X and A_μ and the generator ϵ_i of the supersymmetry transformations [ST83, dWLP$^+$84]. We see that under the supersymmetry transformations bosonic fields are transformed to fermionic fields and vice versa.

The index i runs from 1 to N, where N is the total number of supercharges divided by the smallest possible spinor representation (in a given dimension).[2] When the number of supercharges is equal to the smallest possible spinor representation, the theory obeys the simplest and least restrictive form of supersymmetry ($N = 1$). When there are more supercharges the supersymmetry is *extended* and the theory becomes more restricted ($N > 1$).

A *local* supersymmetric theory, for which the generators ϵ_i depend on variables of spacetime, is called a *supergravity* theory. The four-dimensional effective type IIA and type IIB theories are supergravity theories, although it is often interesting to understand their global supersymmetry before one considers local supersymmetry. Global supersymmetry is often called *rigid supersymmetry* as well.

Apart from the vector multiplet, the four-dimensional type II string spectrum is organized in hypermultiplets, a tensor multiplet and a gravity multiplet. Each *hypermultiplet* consists of four real scalars grouped into a quaternion. Since the four-dimensional Kalb-Ramond field B may be dualized to a scalar field σ, the *tensor multiplet* is also considered a hypermultiplet. It is called the *universal hypermultiplet* since it is always present in a type II string theory compactified on a family of Calabi-Yau$_3$'s, regardless of the values of $h^{2,1}$ and $h^{1,1}$ or the specific type II string theory under consideration. In order to obtain a better understanding of the hypermultiplet sector in general, a lot of study has been made concerning this hypermultiplet [BB99, Ket01a, Ket01b]. The final multiplet in tables 4.3 and 4.4 is the *gravity multiplet* containing the gravitational

[2] The index i is not a spinor index, but an index labelling generators of $\mathrm{SU}(N)_R$-symmetry.

information of the theory. It is a result of the *local* supersymmetry of type II superstring theory. When supersymmetry is only global, gravity decouples and an extra dilatational symmetry breaks the gravity multiplet to an additional (*compensating*) vector multiplet, cf. section 6.1.

Geometry of the moduli space

The analysis of the moduli space can be divided in an analysis of the *vector multiplet moduli space* $\mathscr{M}_{vm}$ and of the *hypermultiplet moduli space* $\mathscr{M}_{hm}$ due to (4.17). The vector multiplet moduli space is understood pretty well and we will recall some results in chapter 5. The result may be summarized by the following theorem.

Theorem 4.16 (Geometry of $\mathscr{M}_{vm}$ in $N = 2$ supersymmetry and $N = 2$ supergravity). *The manifold $\mathscr{M}_{vm}$ of scalars in the vector multiplets of an $N = 2$ supersymmetry theory is affine special Kähler. The manifold $\mathscr{M}_{vm}$ of scalars in the vector multiplets of an $N = 2$ supergravity theory is projective special Kähler.*

Exactly how this comes about and what is meant with affine and projective special Kähler geometry will be postponed to chapter 5. Since in type IIA supergravity the scalars of the vector multiplets are the (complexified) Kähler moduli and in type IIB supergravity the vector multiplets's scalars are the complex structure moduli, we find as a corollary to theorem 4.16,

Corollary 4.17 (Geometry of $\mathscr{M}^{1,1}$ and $\mathscr{M}^{2,1}$). *The moduli space of (complexified) Kähler moduli $\mathscr{M}^{1,1}$ and the moduli space of complex structure moduli $\mathscr{M}^{2,1}$ of a family of Calabi-Yau$_3$'s are both projective special Kähler.*

Corollary 4.17 is an interesting result when considering the geometry of the hypermultiplet moduli space. Observe that in each type II theory the role of the geometric moduli is interchanged. In type IIA the complex structure moduli live in the hypermultiplet sector, while in type IIB a part of the hypermultiplet scalar manifold is parameterized by the Kähler moduli. Since the hypermultiplet sector consists of more than just the geometric moduli, we expect that a special Kähler submanifold is contained within the hypermultiplet scalar manifold $\mathscr{M}_{hm}$. The geometry of the manifold as a whole will also depends on the extra Ramond fields of the hypermultiplet sector. Its structure has been investigated in the past using supergravity arguments leading to the following result [BW83, Sei88, CFG89, dWVVP93].

Theorem 4.18 (Geometry of $\mathscr{M}_{hm}$ in $N = 2$ supergravity). *The manifold $\mathscr{M}_{hm}$ of scalars in the hypermultiplets of an $N = 2$ supergravity theory is quaternion-Kähler.*

Just like we anticipated, it has been found that in type IIA the moduli space of the hypermultiplets contains as a submanifold the special Kähler manifold of the complex structure moduli. Moreover the dilaton and axion, which are elements of the hypermultiplets in both type II theories,[3] have been found to constitute the submanifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$. Thus the type IIA and type IIB quaternion-Kähler manifolds parameterized by the scalars in the hypermultiplets contain the following submanifolds

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \mathscr{M}^{2,1} \subset \mathscr{M}^A_{hm}, \qquad \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \mathscr{M}^{1,1} \subset \mathscr{M}^B_{hm}.$$

The precise structure of the total quaternion-Kähler manifold $\mathscr{M}_{hm}$ is still far from understood. Although the statements about their existence have been a good first step, a real analysis of their structure had to be postponed until specific examples were found. The first to succeed in doing this were Ferrara and Sabharwal [FS90] using a construction called the *local c-map*. They were able to construct a metric which they proved to belong to a quaternion-Kähler space. Based on their result, others [BCF91, BGHL00] were able to construct the metric from a "straightforward" expansion on harmonic forms. Using suitable coordinate substitutions the metric can be seen

[3] Note that the dilaton and axion are elements of the *universal* hypermultiplet.

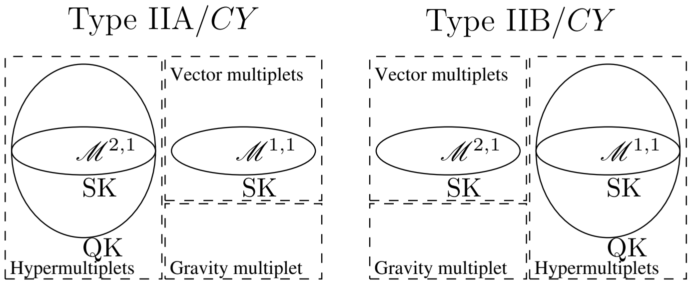

Figure 4.1: Comparison of the multiplet structures of type IIA and type IIB over the same Calabi-Yau$_3$ manifold CY. The scalars of the vector multiplets parameterize a projective special Kähler manifold, the scalars of the hypermultiplets parameterize a quaternion-Kähler manifold which contains a special Kähler submanifold. Comparing one theory with the other we see that the position of the geometric moduli is interchanged.

to incorporate several Kähler manifolds. One is parameterized by the dilaton, axion and the geometric moduli for constant Ramond fields, while the other is parameterized by the dilaton, axion and Ramond fields for constant geometric moduli. The manifold as a whole is *not* a Kähler manifold.

A lot of investigation has been done to obtain a better understanding of the structure of the hypermultiplet moduli space [dWVP90, dWVVP93, DJdWKV98, Ket01b, vG04, RVV06a]. In this thesis we will further analyze the quaternion-Kähler manifold by a geometric analysis of the local c-map. One of the reasons to use this approach is that one may invoke knowledge about the geometry of the *vector multiplet* moduli space, to obtain results about the *hypermultiplet* moduli space. We will further explain this remarkable connection in section 4.4. The discussion of the vector multiplets in chapter 5 will then help us to understand (part of) the hypermultiplet moduli space $\mathscr{M}_{hm}$.

Recall that in many occasions $N = 2$ supersymmetry provides an interesting toy model for problems in $N = 2$ supergravity. We will also consider the hypermultiplet scalar manifold of $N = 2$ supersymmetry theories as a first step in understanding the type IIA's hypermultiplet moduli space in $N = 2$ supergravity. The following result has been obtained by [AGF81].

Theorem 4.19 (Geometry of $\mathscr{M}_{hm}$ in $N = 2$ supersymmetry). *The manifold $\mathscr{M}_{hm}$ of scalars in the hypermultiplets of an $N = 2$ supersymmetry theory is hyperkähler.*

As a summary we provide in figure 4.1 a schematic overview of the moduli spaces of the type IIA and type IIB superstring theories compactified on a Calabi-Yau$_3$ manifold.

4.4 T-duality

In this section we consider *T-duality* to understand an important connection between the vector multiplet moduli space of one type II theory compactified on a Calabi-Yau$_3$ and the hypermultiplet moduli space of the other type II theory compactified on the same Calabi-Yau$_3$. T-duality is one of those fundamental dualities whose combined effect connect all known superstring theories.

Assumption 4.20 (T-duality). Suppose the target space of the strings in type IIA and type IIB string theory contains one spatial dimension that is topological isomorphic to a circle S^1_R of radius R. Then the string states in the two theories are mapped into each other under a transformation of $R \to 1/R$. As such type IIA and type IIB are said to be *T-dual*; they are dual theories describing the same physical states. $\oslash$

The formalism of T-duality is centered around the existence of a circular dimension, with the help of which T-duality transformations can be written down. Nevertheless physicists allow its radius to approach infinity or zero and hence the circular dimension is merely a method of *describing* the workings of T-duality, not a necessary condition of the target space of the strings. As it stands T-duality is a duality between the full-fledged ten-dimensional type II theories. However the duality can be passed over to the compactified four-dimensional theories. An interesting construction is to assume the string background be of the form

$$\mathbb{R}^{1,2} \times S^1_R \times CY.$$

T-duality states that type IIA string theory in the ten-dimensional background $\mathbb{R}^{1,2} \times S^1_R \times CY$ is identical to type IIB string theory in the background $\mathbb{R}^{1,2} \times S^1_{R^{-1}} \times CY$. Blowing up the circle S^1_R by taking $R \to \infty$ we reobtain the original type IIA theory, while blowing up $S^1_{R^{-1}}$ by taking $R \to 0$ we reobtain type IIB. Schematically we may denote this as in figure 4.2.

$$IIA/CY \xleftarrow{\infty \leftarrow R} IIA/(CY \times S^1_R) \cong IIB/(CY \times S^1_{R^{-1}}) \xrightarrow{R^{-1}\to\infty} IIB/CY.$$

Figure 4.2: A schematic overview of T-duality between type IIA and IIB string theory.

The duality is assumed to hold for all aspects of the theory, in particular for the moduli spaces of both theories. We are interested in the relation between the vector multiplet moduli space of one type II theory and the other type II theory's hypermultiplet moduli space. This relation is called the *c-map* and was first considered in [CFG89].

Theorem 4.21 (The c-map). *The hypermultiplet moduli space $\mathscr{M}_{hm}$ of type IIA (IIB) string theory is obtained by a toroidal compactification of the gravity and vector multiplet sector, denoted for the moment with $\mathscr{V}$, of type IIB (IIA) string theory. The map*

$$c : \mathscr{V} \to \mathscr{M}_{hm},$$

providing this relation is called the c-map.

Physicist's proof. For definiteness let us explain the c-map from the type IIA gravitational and vector multiplet sectors to the type IIB hypermultiplet moduli space. The c-map from IIB to IIA is explained in a similar manner.

Under the assumptions of T-duality there is one circular spatial dimension with radius R. If we take the limit $R \to 0$ this compact dimension becomes so small we may perform a compactification of the gravitational and vector multiplet sectors. The Lagrangians describing these sectors are given in (4.15b) and (4.15c). They contain the Ricci scalar R_4, $h^{1,1}$ complex scalar W^Λ and $h^{1,1}+1$ four-dimensional vectors A^Γ_μ, $\mu \in \{0,1,2,3\}$. Note that one of the vectors is interpreted as the graviphoton (living in the gravitational multiplet), which makes it natural to consider the vector multiplet and gravity multiplet simultaneously. Under dimensional reduction a vector field A_μ reduces to a three-dimensional vector field A_m, $m \in \{0,1,2\}$ and a scalar field A. A vector field A_m in three dimensions can be dualized to another scalar field B, so that we have obtained *three-dimensional* scalar fields A and B.

If we decompose the metric

$$g_{\mu\nu} = \begin{pmatrix} g_{lm} & \sigma_l \\ \sigma_m & \phi \end{pmatrix},$$

the compactification of the Ricci scalar, from R_4 to R_3 in (4.15b) contributes terms proportional to R_3, $\partial_m\phi\partial^m\phi$ and $\sigma_m\sigma^m$ [FS90, BCF91, BBS07]. The 1-form σ_m may then be dualized to a scalar σ as before. Thus we may add an extra 2 real scalar fields to the $2h^{1,1}+2$ real scalar fields A^Γ, B_Γ coming from the vectors of the gravity and vector multiplet sectors and the $2h^{1,1}$ real scalar fields W^Λ of the vector multiplets.

We now invoke T-duality. The scalar fields must find their place within the type IIB theory whose background has one circular spatial dimension with radius $1/R$, since type IIA and type IIB are by assumption identical upon an interchange of $R \leftrightarrow 1/R$. Furthermore note that from the point of view of the type IIB theory the radius of the circle is taken to infinity, $1/R \to \infty$,

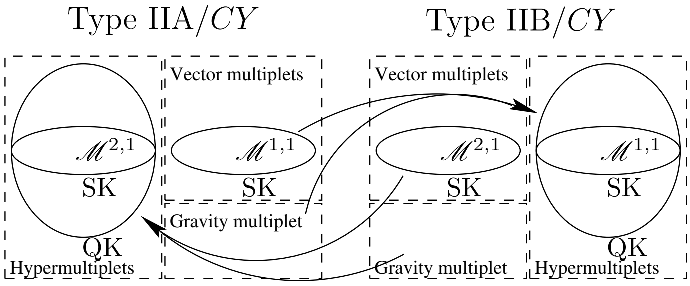

Figure 4.3: The c-map is a map from the gravity and vector multiplet sectors of one type II string theory to the other type II's hypermultiplet sector.

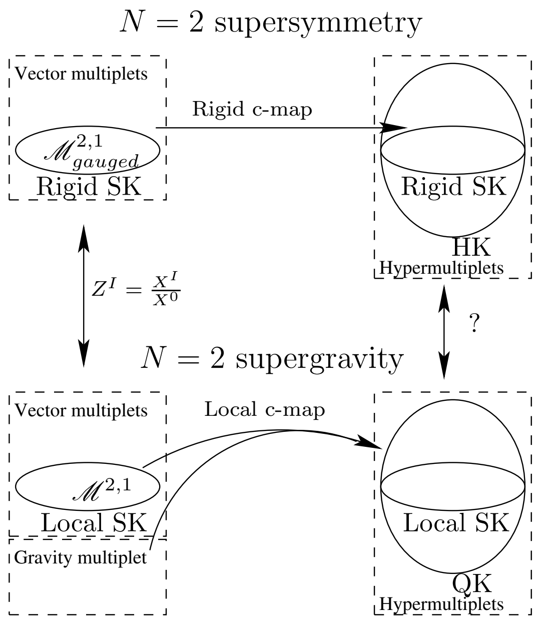

Figure 4.4: The rigid c-map maps the vector multiplets to a hyperkähler hypermultiplet scalar manifold in $N = 2$ rigid supersymmetry. In $N = 2$ supergravity the local c-map maps the vector multiplets and the gravitational multiplet to a quaternion-Kähler hypermultiplet scalar manifold.

decompactifying the fields in this direction. Luckily three-dimensional scalar fields are easily decompactified to four-dimensional scalar fields [CMMS05]. The question is now *which* place is taken within the type IIB/CY theory by the $4h^{1,1} + 4$ *four-dimensional* moduli fields. Since the Kähler moduli are among those fields, it is quite reasonable to predict that the fields must form the $h^{1,1} + 1$ hypermultiplets of the type IIB theory. From the general argument one can even deduce that the universal hypermultiplet must be formed by the scalars coming from the gravitational multiplet, since those are the only fields surviving when $h^{1,1} = 0$. The fields ϕ and σ from the four-dimensional type IIA graviton are in fact identified with the type IIB dilaton and Kalb-Ramond axion respectively. One has to verify these predictions but from general supergravity reasoning they can be explained rigorously [CFG89, FS90]. In figure 4.3 the c-map is schematically depicted. □

Although originally the c-map is a statement for the effective four-dimensional supergravity theories associated to type II superstring theories, it provides interesting results in *arbitrary* $N = 2$ (global or local) supersymmetry theories. In [CFG89, DJdWKV98] the c-map was applied to the simpler problem of obtaining the hypermultiplet scalar manifold from the vector multiplet sector in $N = 2$ global (or rigid) supersymmetry. To distinguish this from the more difficult supergravity problem, the c-map in supersymmetry is called the *rigid c-map* and the one in supergravity is called the *local c-map*. Theorems 4.21, 4.18 and 4.19 are summarized in figure 4.4 for the c-map from IIB to IIA. The relation between the vector multiplet moduli spaces of rigid and local supersymmetry is understood, as taking the homogeneous coordinates of the local supersymmetry's vector multiplet moduli space to be independent yields the vector multiplet moduli space in rigid supersymmetry, cf. chapter 5. The relation between the hypermultiplet scalar manifolds of both supersymmetries is not understood. In fact in this thesis we attempt to find resemblances in the two moduli spaces to get a better grip on the relation between the hypermultiplet scalar manifolds in rigid (global) and local supersymmetry.

In chapters 6 and 7 we will investigate the c-map's construction from a mathematical point of view and thereby try to find a new characterization of the hypermultiplet scalar manifold in $N = 2$ rigid and local supersymmetry.

Chapter 5

Vector multiplets and special Kähler geometry

We have argued that T-duality provides a method for investigating the hypermultiplet moduli space of type II superstring theories compactified on a Calabi-Yau$_3$ manifold by means of the c-map in rigid and local supersymmetry. Since the c-map is a map from the vector multiplet sector to the hypermultiplet sector, we will first need a thorough understanding of the vector multiplet sector. Luckily the vector multiplets in $N = 2$ supersymmetry have been studied a lot and are well understood. We will present some of its features in this chapter. We take a somewhat ahistorical route and first define special geometry mathematically. Then we present the occurrence of special geometry in the moduli space of a family of Calabi-Yau$_3$ manifolds. We proceed by describing the construction of the vector multiplet's Lagrangian within $N = 2$ theories, of which part of the compactified type IIA and IIB theory's Lagrangians are an example. Finally by using electric-magnetic duality we describe the vector multiplet theory in a symplectic setting. Key references are [BG83, Str90, CdlO91, CRTVP97, Fre99, ACD02].

5.1 Special Kähler geometry in mathematics

Special Kähler geometry comes in two flavors. The construction of Lagrangians for the vector multiplets of $N = 2$ rigid supersymmetry [ST83, Gat84] and supergravity [dWVP84] theories in four dimensions led to the extrinsic definition of *rigid* and *local special Kähler manifolds* respectively. When mathematicians succeeded in defining these manifolds intrinsically they preferred to call them *affine* and *projective special Kähler manifolds*. We will refer to the two different types of special Kähler geometry with the mathematical terminology. To clarify the different names used in special Kähler geometry we refer to table 5.1. In the next two sections we will define both special Kähler geometries.

Affine special Kähler geometry

After the discovery of special Kähler geometry in the physics literature [ST83, Gat84, dWVP84], mathematicians have tried to define these manifolds intrinsically, i.e. a definition that uses the tangent bundle of a manifold and other associated bundles rather than bundles which are not constructed directly from the coordinate charts of a manifold. The first to give this intrinsic definition of an *affine special Kähler* manifold was Freed [Fre99].

Definition 5.1 (Affine special Kähler manifold). Let (M, J, σ_J) be a Kähler manifold with Kähler form σ_J and complex structure J, then (M, σ_J, ∇) is an *affine special Kähler manifold* if it has a real connection ∇, cf. definition 2.4, which is

1. *flat*, i.e. $\mathrm{d}_\nabla^2 = 0$,

Physics terminology	
rigid special Kähler appears in $N = 2$ global supersymmetry no coupling to gravity	(local) special Kähler appears in $N = 2$ supergravity coupling to gravity
Mathematics terminology	
(affine) special Kähler	projective special Kähler

Table 5.1: Special Kähler geometry comes in two flavors. Historically rigid special Kähler geometry first appeared in $N = 2$ global supersymmetry, local special Kähler geometry in $N = 2$ supergravity. In most cases a physicist is concerned with a *local* special Kähler structure. Mathematicians have their own terminology of (intrinsically defined) special Kähler geometry. When not otherwise specified they assume special Kähler manifolds to be *affine*. The terminology stems from the fact that in rigid special Kähler geometry all fields are taken to be independent, while in local special Kähler geometry the extra compensator field completes the coordinates to form a set of projective coordinates.

2. *torsion free*, i.e. $\mathrm{d}_\nabla \mathbb{I} = 0$, where $\mathbb{I} \in \Omega^1(TM)$ is the identity endomorphism on TM,
3. and for which the complex structure is ∇-parallel, i.e. $\mathrm{d}_\nabla J = 0 \in \Omega^2(TM)$ or equivalently $(\nabla_X J)Y = (\nabla_Y J)X$ for any two vector fields $X, Y \in \mathscr{X}(M)$.

$\oslash$

In his paper Freed was able to show that this definition leads to the same local formulation of special Kähler geometry in terms of a *prepotential* as was provided by considerations in supersymmetry [Fre99, CRTVP97].

Definition 5.2 (Extrinsic description of an affine special Kähler manifold). Let (M, σ_J) be an n-dimensional Kähler manifold with local coordinates $\{t^i\}_{i=1}^n$. It is an *affine special Kähler manifold*, locally defined, if it satisfies the following conditions.

1. On every chart there exist n independent holomorphic functions $X^I(t)$ on M, where $I \in \{1, \ldots, n\}$, and a holomorphic function $F(X)$, such that the Kähler potential is given by

$$K(t, \bar{t}) = i \sum_{I=1}^{n} X^I \frac{\partial \bar{F}(\bar{X})}{\partial \bar{X}^I} - \bar{X}^I \frac{\partial F(X)}{\partial X^I}. \tag{5.1}$$

2. On overlaps of charts k and l, there are transition functions of the form

$$\begin{pmatrix} X \\ \partial F \end{pmatrix}_{(k)} = e^{ic_{kl}} S_{kl} \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(l)} + b_{kl},$$

where $c_{kl} \in \mathbb{R}$, $S_{kl} \in \mathrm{Sp}(2n, \mathbb{R})$ and $b_{kl} \in \mathbb{C}^{2n}$.

3. The transition functions satisfy the cocycle condition on overlaps of 3 charts.

$\oslash$

It is possible to take the *values* of the fields $\{X^I\}_{I=1}^n$ as complex coordinates $\{t^i\}_{i=1}^n$ of the Kähler manifold. These are then called *special coordinates*. For our purposes it will not always be feasible to identify the coordinates t^i of the special Kähler manifold and the sections X^I of the holomorphic bundle over the special Kähler manifold, which is why we stress their distinction here.

From a mathematical point of view special Kähler manifolds have proven to be very powerful, for example in the subject of integrable systems [Fre99]. Generalizations of the concept such as *special complex manifolds* have been considered and investigated [ACD02]. These may turn out to be valuable in physical theories which do not have a Lagrangian formulation and hence no target space metric [Cor01].

Projective special Kähler geometry

Mathematically projective special Kähler manifolds have long been associated with the article of [BG83], in which a *variation of Hodge structure* of the third cohomology group of a family of Calabi-Yau$_3$ manifolds has been analyzed, cf. section 5.2. The original definition of projective special Kähler structure has been formulated with this variation of Hodge structure in mind [Fre99]. At the same time it is feasible to formulate a definition that emphasizes the connection between the projective special Kähler structure and its affine brother. In [Fre99] such a description of a projective special Kähler structure has been given. It is simultaneously equivalent with the original definition retrieved from [BG83] and has a clear connection with affine special geometry. We will explain this definition using the approach of [ACD02] (which is in its turn consistent with that of [Fre99]).

Definition 5.3 (Local holomorphic $\mathbb{C}^*$-action). A *local holomorphic* $\mathbb{C}^*$*-action* on a complex manifold M is a holomorphic map,

$$s : \mathbb{C}^* \times M \to M : (\lambda, p) \mapsto s_\lambda(p),$$

defined on an open neighborhood W of $\{1\} \times M$ such that $s_1(p) = p$ for all $p \in M$ and $s_\lambda(s_\mu(p)) = s_{\lambda\mu}(p)$ as long as both sides are defined, i.e. if $(\lambda, s_\mu(p)), (\lambda\mu, p) \in W$. ⊘

Definition 5.4 (Conic Kähler manifold). Let (M, J, σ_J, ∇) be a Kähler manifold with Kähler form σ_J, complex structure J and flat, torsion free connection ∇. It is a *conic Kähler manifold* if it admits a local holomorphic $\mathbb{C}^*$-action s such that locally

$$\mathrm{D}s_\lambda.v = re^{\theta J} v,$$

for all ∇-parallel vector fields v. Here $\lambda = re^{i\theta} \in \mathbb{C}^*$ and $re^{\theta J} v = r\cos(\theta)v + r\sin(\theta)Jv$. ⊘

Proposition 5.5 (A conic manifold is special Kähler). *Any conic Kähler manifold is an affine special Kähler manifold.*

Proof. Let (M, J, σ_J, ∇) be a conic Kähler manifold with holomorphic local action s. Comparison of definition 5.1 and 5.4 shows that to prove the statement we only need to verify that the local statement $\mathrm{d}_\nabla J = 0$ holds. This follows nontrivially from the existence of the holomorphic local action s [ACD02]. □

Thus an interesting subclass of all affine special Kähler manifolds are the conic Kähler manifolds. Note that orbits of the action define leaves in such a conic Kähler manifold $\tilde{M}$.

Definition 5.6 (Projective special Kähler manifold). Let $M = \tilde{M}/\mathbb{C}^*$ be the set of orbits of the local holomorphic $\mathbb{C}^*$-action on a conic Kähler manifold $\tilde{M}$ endowed with the quotient topology. If the projection $\tilde{M} \to M = \tilde{M}/\mathbb{C}^*$ is a holomorphic submersion onto a Hausdorff complex manifold, then M is called a *projective special Kähler manifold*. ⊘

The quotient of $\tilde{M}$ by the action $\mathbb{C}^*$ can be considered as a projectivization, which is why mathematicians have coined this terminology. Conversely in [Fre99] the formulation of projective special Kähler geometry is given by constructing a certain $\mathbb{C}^*$-invariant affine special Kähler manifold $(\tilde{M}, \tilde{\sigma}_J, \tilde{\nabla})$ as a subbundle of the holomorphic hermitian line bundle $L \to M$ associated to a Hodge manifold[1] (M, L, σ_J). The manifold (M, L, σ_J) is a projective special Kähler manifold on which the special structure is induced by specific properties of the $\mathbb{C}^*$-invariant affine special Kähler manifold $\tilde{M}$. The advantage of this latter approach is the emphasis on the projective special Kähler manifold being a Hodge manifold. Both definitions show that a projective special Kähler manifold is formulated in terms of a particular class of affine special Kähler manifolds.

[1] A *Hodge manifold* is a Kähler manifold (M, σ_J) with a hermitian line bundle $L \to M$ having curvature $-2\pi i\sigma_J$. According to [Fre99] this implies that $[\sigma_J] \in H^2(M, \mathbb{R})$ is an integral cohomology class.

When formulating projective special Kähler geometry extrinsically with respect to local special coordinates, the reference to affine special Kähler geometry comes about by using projective coordinates on an affine special Kähler manifold. Due to the required invariance under the $\mathbb{C}^*$-action, the holomorphic prepotential is now restricted to be homogeneous of degree 2. The logarithm lifts the Kähler structure on the projective special Kähler manifold to the projective coordinates of the affine special Kähler manifold [Fre99, ACD02].

Definition 5.7 (Extrinsic description of a projective special Kähler manifold). Let (M, L, σ_J) be an n-dimensional Hodge manifold with local coordinates $\{t^a\}_{a=1}^n$. It is a *projective special Kähler manifold*, locally defined, if it satisfies the following conditions.

1. On every chart there exist $n+1$ independent holomorphic functions $X^I(t)$, where $I \in \{0, 1, \ldots, n\}$ and a holomorphic function $F(X)$ that is homogeneous of degree 2 (i.e. $F(\lambda X) = \lambda^2 F(X)$), such that the Kähler potential is given by

$$K(t, \bar{t}) = -\log \left[i \sum_{I \in 0}^{n} \bar{X}^I \frac{\partial F(X)}{\partial X^I} - X^I \frac{\partial \bar{F}(\bar{X})}{\partial \bar{X}^I} \right]. \tag{5.2}$$

2. On overlaps of charts k and l there are transition functions of the form

$$\begin{pmatrix} X \\ \partial F \end{pmatrix}_{(k)} = e^{f_{kl}(t)} S_{kl} \begin{pmatrix} X \\ \partial F \end{pmatrix}_{(l)},$$

where $f_{kl}(t)$ is a holomorphic function and $S_{kl} \in \mathrm{Sp}(2(n+1), \mathbb{R})$.

3. The transition functions satisfy the cocycle condition on overlaps of 3 charts.

⊘

Again the fields $\{X^I\}_{I=0}^n$ are often taken as a (projective) coordinate set of the projective special Kähler manifold. In this context affine coordinates $\{Z^A\}_{A=1}^n$ are defined by $\{Z^I\}_{I=0}^n = \{\frac{X^I}{X^0}\}_{I=0}^n$. Note that these are again special (sets of) coordinates. More generically the manifold may locally be given by complex coordinates $\{t^a\}_{a=1}^n$ on which X^I and Z^A depend holomorphically.

5.2 The geometric moduli space of Calabi-Yau$_3$ deformations

Special Kähler geometry has a natural occurrence in the study of the *geometric moduli space* $\mathscr{M}_g$ of a family of Calabi-Yau$_3$ manifolds. The line element for $\mathscr{M}_g$ is given by

$$\mathrm{d}s^2_{geometric} = \frac{1}{\mathrm{Vol}_{CY}} \int_{CY} \mathrm{d}^3 z \mathrm{d}^3 \bar{z} \sqrt{g} g^{i\bar{j}} g^{k\bar{l}} \left[\delta g_{ik} \delta g_{\bar{j}\bar{l}} + (\delta g_{a\bar{l}} \delta g_{k\bar{j}} + \delta B_{a\bar{l}} \delta B_{\bar{j}k}) \right], \tag{5.3}$$

where δg_{ij} is related to the complex structure moduli t through (4.14) and $\delta g_{i\bar{j}}$ to the complexified Kähler moduli W via (4.7) and (4.11). The line element shows an orthogonal decomposition and therefore we write the geometric moduli space $\mathscr{M}_g$ as a local product structure

$$\mathscr{M}_g = \mathscr{M}^{1,1} \times \mathscr{M}^{2,1}.$$

We will study the moduli space of the (complexified) Kähler moduli $\mathscr{M}^{1,1}$ and that of the complex structure moduli $\mathscr{M}^{2,1}$ individually. The results may also be found in [CdlO91, Hüb92, BBS07].

Complex structure moduli

Definition 5.8 (Metric on complex structure moduli space). Suppose CY is a Calabi-Yau$_3$ manifold and consider deformations of CY by varying the complex structure $t = (t^A)_{A=1}^{h^{2,1}}$, $\mathscr{CY}^{2,1} = (CY_t)_{t\in\mathscr{M}^{2,1}} \to \mathscr{M}^{2,1}$, $CY_0 = CY$. The metric $G^{(2,1)}$ on the moduli space $\mathscr{M}^{2,1}$ of complex structure deformations is defined by

$$\mathrm{d}s^2_{(2,1)} = G^{(2,1)}_{A\bar{B}}\mathrm{d}t^A\mathrm{d}\bar{t}^{\bar{B}} = \frac{1}{4\mathrm{Vol}_{CY}}\int_{CY}\mathrm{d}^3z\mathrm{d}^3\bar{z}\sqrt{g}g^{i\bar{j}}g^{k\bar{l}}\delta g_{ik}\delta g_{\bar{j}\bar{l}}.$$

$\oslash$

Proposition 5.9 (Kähler potential for complex structure moduli space). *The complex structure moduli space $\mathscr{M}^{2,1}$ is a Kähler manifold with Kähler potential*

$$K^{(2,1)} = -\log\left(i\int_{CY} vol\wedge\overline{vol}\right). \tag{5.4}$$

Proof. Using expression (4.14) to relate $\delta g_{\bar{i}\bar{j}}$ and t and using the expression of the volume Vol_{CY_t} of the Calabi-Yau$_3$ in terms of the holomorphic 3-form, $\mathrm{Vol}_{CY_t} = \frac{i}{||vol||^2}\int_{CY_t} vol\wedge\overline{vol}$, we find [Tia87, CdlO91]

$$G^{(2,1)}_{A\bar{B}} = \frac{-i}{\mathrm{Vol}_{CY}||vol||^2}\int_{CY}\Phi_A\wedge\bar{\Phi}_{\bar{B}} = -\frac{\int_{CY}\Phi_A\wedge\bar{\Phi}_{\bar{B}}}{\int_{CY} vol\wedge\overline{vol}}.$$

The basis Φ_A of $H^{2,1}(CY_t)$ can be expressed in terms of a derivative of vol with respect to the complex structure moduli t^A ,

$$\frac{\partial vol}{\partial t^A} = \Phi_A - K_A vol, \tag{5.5}$$

where K_A depends on the moduli t^A but is independent of the coordinates of the Calabi-Yau$_3$. This result is attributed to Kodaira and is not as trivial as it may seem. Indeed under an infinitesimal change in complex structure the holomorphic $(3,0)$-form becomes a linear combination of a $(3,0)$-form and $(2,1)$-forms, since a coordinate 1-form $\mathrm{d}z^i$ becomes partly of bidegree $(1,0)$ and partly of bidegree $(0,1)$ under a change of complex structure. Thus

$$\frac{\partial vol}{\partial t^A}\in H^{3,0}(CY_t)\oplus H^{2,1}(CY_t),$$

which can be seen as an example of *Griffiths transversality*. It is not obvious however that $\frac{\partial vol}{\partial t^A}$ is exactly the given linear combination of vol and Φ_A. Note that although we have *chosen* Φ_A at first, it is related by (4.14) with the complex structure moduli t^A. Hence if we start with complex structure moduli t^A, the inverse of (4.14) expresses Φ_A in terms of t^A,

$$\Phi_{Aij\bar{k}} = -\frac{1}{2}vol_{ij}{}^{\bar{l}}\frac{\partial g_{\bar{k}\bar{l}}}{\partial t^A}.$$

One needs to prove that these Φ_A equal the ones appearing in (5.5). This can be done by explicitly performing the complex structure variation [CdlO91, BBS07]. Having established (5.5) a direct calculation shows that (5.4) is indeed the Kähler potential for $G^{(2,1)}$. $\square$

The scalars K_A in (5.5) can be used to obtain pure $(2,1)$-forms from a *Kähler covariant derivative* $\mathrm{D}^K_A = \partial_A + K_A$ of the volume form $vol\in H^{3,0}(CY_t)$. The contribution in $H^{3,0}(CY_t)$ which is present in differentiating vol with respect to the complex structure is exactly cancelled in the Kähler covariant derivative. What remains is a form in $H^{2,1}(CY_t)$. The following lemma shows that $K_A = \partial_A K^{(2,1)}$ and for this reason the derivative of the Kähler potential gets the interpretation of a connection.

Lemma 5.10 ($\partial_A K^{(2,1)}$ as Kähler connection). *Suppose CY is a Calabi-Yau$_3$ manifold and consider deformations of $CY = CY_0$, $\mathscr{CY}^{2,1} = (CY_t)_{t\in\mathscr{M}^{2,1}} \to \mathscr{M}^{2,1}$, by varying the complex structure $t = (t^A)_{A=1}^{h^{2,1}}$. Let vol_t be the holomorphic $(3,0)$-form in $H^3(CY_t, \mathbb{C})$ and write $\Phi_{Aij\bar{k}}(t) = -\frac{1}{2} vol_{ij}{}^{\bar{l}} \frac{\partial g_{\bar{k}\bar{l}}}{\partial t^A}$ for the $(2,1)$-forms of CY_t. Then the following relation between forms holds,*

$$\Phi_A(t) = \partial_A vol_t + \left(\partial_A K^{(2,1)}(t,\bar{t})\right) vol_t,$$

where $K^{(2,1)}$ is the Kähler potential of the moduli space of complex structure deformations.

Proof. We have already argued that $\Phi_A = \partial_A vol + K_A vol$. By a simple consideration of bidegrees we see that

$$\partial_A K^{(2,1)} = -\frac{i\int_{CY}(\Phi_A - K_A vol)\wedge \bar{vol}}{i\int_{CY} vol \wedge \overline{vol}} = K_A. \qquad \square$$

By definition *Kähler transformations* $\Lambda : t \mapsto \Lambda(t)$ transform the Kähler potential $K^{(2,1)}$ into

$$\tilde{K}^{(2,1)}(t,\bar{t}) = K^{(2,1)}(t,\bar{t}) + \Lambda(t) + \bar{\Lambda}(\bar{t}),$$

and leave the metric unchanged. In our case they follow naturally from a scaling of the holomorphic 3-form

$$\widetilde{vol}_t = e^{-\Lambda(t)} vol_t, \tag{5.6}$$

which expresses the independence of the geometry of the moduli space on the choice of vol_t in $H^{3,0}(CY_t)$. Using the invariance of the theory under a rescaling of vol, the Kähler geometry of the complex structure moduli space turns out to be special. The reason for this is handed to us by Bryant and Griffiths [BG83], who considered deformations of Hodge structure of a Calabi-Yau$_3$. To this end we consider a basis of 3-cycles $\{\gamma_{A^I}, \gamma_{B_I}\}_{I=0}^{h^{2,1}}$ of $H_3(CY, \mathbb{Z})$ with dual basis $\{\alpha_I, \beta^I\}_{I=0}^{h^{2,1}} \subset H^3(CY,\mathbb{Z}) \subset H^3(CY,\mathbb{Z}) \otimes \mathbb{C}$, cf. (2.8). We define the *periods* of the holomorphic 3-form vol as

$$X^I = \int_{\gamma_{A^I}} vol, \qquad F_I = \int_{\gamma_{B_I}} vol. \tag{5.7}$$

The holomorphic 3-form vol is a measure for the complex structure of the Calabi-Yau$_3$. It depends on the complex structure t, but more importantly it specifies the "direction" within $H^3(CY, \mathbb{C})$ of *the* holomorphic part $H^{3,0}(CY_t)$. Picking an arbitrary element of $H^3(CY,\mathbb{C})$ and calling it the holomorphic one, determines the complex structure that has to be taken. Therefore vol and t are equivalent parameters of $\mathscr{M}^{2,1}$. By (2.8) vol may be expanded in terms of the basis $\{\alpha_I,\, \beta^I\}$ via

$$vol_t = X^I(t)\alpha_I - F_I(t)\beta^I,$$

showing explicitly that X^I and F_I depend on the complex structure as well. Note that the basis $\{\alpha_I,\, \beta^I\}$ of $H^3(CY,\mathbb{Z})$ does not know about the complex structure of the Calabi-Yau$_3$, since it is a basis of the *integer* cohomology. Changing the complex structure t "rotates" the position of $vol_t \in H^{3,0}(CY_t)$ within $H^3(CY,\mathbb{Z}) \otimes \mathbb{C}$.

In [BG83] it is noted that the X^I and F_I could not be all independent, since the complex structure moduli space is only $h^{2,1}$-dimensional, while there are $2h^{2,1} + 2$ coordinates X^I and F_I. They were able to formulate this rigorously and this resulted in expressing the periods F_I as functions of X^I,

$$F_I = F_I(X).$$

Moreover since the X^I do not vanish simultaneously and vol (and hence X^I) is only defined up to a scaling, the X^I can be seen as homogeneous coordinates in a projective space. This means that locally the X^I are elements of $\mathbb{CP}^{h^{2,1}}$. The corresponding affine coordinates $\{Z^I\}_{I=0}^n = \{\frac{X^I}{X^0}\}_{I=0}^n$ are special coordinates of $\mathscr{M}^{2,1}$ alternative to the original complex structure moduli $t = (t^A)_{A=1}^{h^{2,1}}$.

We proceed with the homogeneous special coordinates X^I and consider the trivial statement

$$0=\int_{CY}\left(vol\wedge\frac{\partial vol}{\partial X^I}\right)=\int_{CY}\left(-X^K\frac{\partial F_J(X)}{\partial X^I}+F_J\delta_I^K\right)\alpha_K\wedge\beta^J=-X^J\frac{\partial F_J(X)}{\partial X^I}+F_I(X).$$

The first equation follows simply by consideration of bidegree and (5.5). The second equation is the expansion of vol in α- and β-forms, while the third equation is a result of (2.8). From this expression we may conclude that $F_I(X)=\frac{1}{2}\frac{\partial}{\partial X^I}\left(X^KF_K(X)\right)$, which means that

$$\frac{\partial F_I(X)}{\partial X^J}=\frac{1}{2}\frac{\partial^2}{\partial X^I\partial X^J}\left(X^KF_K(X)\right)=\frac{\partial F_J(X)}{\partial X^I}.$$

We conclude that there exists locally a holomorphic function $F^{(2,1)}: X\mapsto F^{(2,1)}(X)$, which satisfies

$$\frac{\partial F^{(2,1)}(X)}{\partial X^I}=F_I(X),\qquad X^IF_I=2F^{(2,1)}.$$

The last equation expresses the fact that $F^{(2,1)}$ is homogeneous of degree 2. In fact $F^{(2,1)}$ is the prepotential of $\mathscr{M}^{2,1}$ since by corollary 2.28,

$$\begin{aligned}K^{(2,1)}&=-\log\left(i\int_{CY}vol\wedge\overline{vol}\right)=-\log -i\left(\int_{\gamma_{A^I}}vol\int_{\gamma_{B_I}}\overline{vol}-\int_{\gamma_{A^I}}\overline{vol}\int_{\gamma_{B_I}}vol\right)\\&=-\log\left(-iX^I\bar{F}_I+i\bar{X}^IF_I\right).\end{aligned}$$

We have established the following theorem.

Theorem 5.11 (Special Kählerity of complex structure moduli space). *The complex structure moduli space $\mathscr{M}^{2,1}$ of a family of Calabi-Yau$_3$ manifolds is projective special Kähler.*

Kähler moduli

Although we are mostly interested in the moduli space of complex structure deformations, we will briefly present the geometry of the moduli space $\mathscr{M}^{1,1}$ of complexified Kähler class deformations $W=(W^\Lambda)_{\Lambda=1}^{h^{1,1}}$ of a family of Calabi-Yau$_3$'s.

Definition 5.12 (Metric on Kähler moduli space). Suppose CY is a Calabi-Yau$_3$ manifold and consider deformations of CY by varying the Kähler structure $W=(W^\Lambda)_{\Lambda=1}^{h^{1,1}}$, $\mathscr{CY}^{1,1}=(CY_W)_{W\in\mathscr{M}^{1,1}}\to\mathscr{M}^{1,1}$, $CY_0=CY$. The metric $G^{(1,1)}$ on the moduli space $\mathscr{M}^{1,1}$ of Kähler deformations is defined by

$$\mathrm{d}s^2_{(1,1)}=G^{(1,1)}_{\Lambda\bar{\Sigma}}\mathrm{d}W^\Lambda\mathrm{d}\bar{W}^{\bar{\Sigma}},$$

$$G^{(1,1)}_{\Lambda\bar{\Sigma}}=\frac{1}{4\mathrm{Vol}_{CY}}\int_{CY}\mathrm{d}^3z\mathrm{d}^3\bar{z}\sqrt{g}g^{i\bar{j}}g^{k\bar{l}}V_{\Lambda i\bar{l}}V_{\Sigma\bar{j}k}=\frac{1}{4\mathrm{Vol}_{CY}}\int_{CY}V_\Lambda\wedge\star V_\Sigma.$$

$\oslash$

Inspired by the structure of the complex structure moduli space, Ferrara and Strominger [FS89] were able to cast the geometry of the Kähler moduli space in a form analogous to the structure of the complex structure moduli space. To make the similarity complete, Candelas and de la Ossa [CdlO91] introduced an auxiliary (complex) coordinate W^0 to make the coordinates $(W^\Lambda)_{\Lambda=1}^{h^{1,1}}$ homogeneous. Upon evaluation, W^0 is set to 1.

Theorem 5.13 (Special Kählerity of Kähler moduli space). *The Kähler moduli space $\mathscr{M}^{1,1}$ of a family of Calabi-Yau$_3$ manifolds is projective special Kähler with holomorphic prepotential*

$$F^{(1,1)}(W)=\frac{1}{6W^0}\int_{CY}\sigma_J^{\mathbb{C}}\wedge\sigma_J^{\mathbb{C}}\wedge\sigma_J^{\mathbb{C}}=\frac{1}{6}\kappa_{\Lambda\Sigma\Pi}\frac{W^\Lambda W^\Sigma W^\Pi}{W^0},$$

which is homogeneous of degree 2. The number $\kappa_{\Lambda\Sigma\Pi}$ is called the triple intersection number $\kappa_{\Lambda\Sigma\Pi}=\int_{CY}V_\Lambda\wedge V_\Sigma\wedge V_\Pi$ *of three basis elements V_Λ,V_Σ,V_Π of $H^{1,1}(CY)$, cf. (4.6b).*

Remarkably the geometry of the complex structure moduli space $\mathscr{M}^{2,1}$ and of the Kähler moduli space $\mathscr{M}^{1,1}$ are the same; a projective special Kähler geometry. From a mathematical point of view this comes rather as a surprise, since no relation between both parts of the geometric moduli space was presupposed. However Calabi-Yau$_3$ manifolds are intimately related with $N = 2$ supergravity theories via the compactification process of ten-dimensional type II string theory on a family of Calabi-Yau$_3$'s. Both parts of the geometric moduli of the Calabi-Yau$_3$ occur as scalar fields in one of the vector multiplet sectors of the resulting four-dimensional effective action. The special geometry that is obeyed by the vector multiplet's scalar manifold of an arbitrary $N = 2$ supergravity theory (cf. section 5.3) is therefore automatically invoked on both parts of the geometric moduli space of the family of Calabi-Yau$_3$'s (cf. corollary 4.17), highlighting an interesting connection between Calabi-Yau$_3$ manifolds and supergravity theory [Asp00].

5.3 $N = 2$ vector multiplet theories

Vector multiplets in $N = 2$ rigid supersymmetry

We have already mentioned that special geometry found its first appearance in the description of vector multiplets in $N = 2$ rigid and local supersymmetry theories. We will now see why special geometry is important to understand vector multiplets. First we consider n_V vector multiplets in a four-dimensional $N = 2$ *rigid* supersymmetry theory, later we will discuss them in a *local* supersymmetry theory. Each vector multiplet consists of a complex scalar X^I and a real vector A^I_μ (and some fermions), where $I \in \{1, \ldots, n_V\}$,

$$(X^I, A^I_\mu, \text{fermionic fields})^{n_V}_{I=1}.$$

A vector field A_μ determines a field strength $\mathcal{F}_{\mu\nu} = (\mathrm{d}A)_{\mu\nu}$, which may be split in a *self dual* and *anti-self dual part*.

Definition 5.14 ((Anti)-self dual field strength in Minkowski space). Consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$ and suppose $\mathcal{F}^{(2)}$ is the field strength of a vector potential $A^{(1)}$, i.e.

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We define the *self dual field strength* $\mathcal{F}^+$ and the *anti-self dual field strength* $\mathcal{F}^-$ by, cf. definition 2.30,

$$\mathcal{F}^\pm_{\mu\nu} = \mathcal{F}_{\mu\nu} \mp i(\star\mathcal{F})_{\mu\nu} = \mathcal{F}_{\mu\nu} \mp \frac{i}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\mathcal{F}_{\rho\sigma}. \tag{5.8}$$

Since $\mathcal{F}$ is real, $\mathcal{F}^+$ and $\mathcal{F}^-$ are related by complex conjugation. ⊘

Using a superfield description, the bosonic part of the vector multiplet theory can be elegantly formulated in terms of a single holomorphic function F, which is called the *prepotential* of the supersymmetry theory [ST83, Gat84, dWVP84, dWLP$^+$84].

Theorem 5.15 (Bosonic Lagrangian of the vector multiplet sector in $N = 2$ rigid supersymmetry). *Let $(X^I, A^I_\mu)^{n_V}_{I=1}$ be vector multiplets in a four-dimensional $N = 2$ rigid supersymmetry theory. The Lagrangian for the vector multiplets is determined by a single holomorphic function $F(X)$. Its bosonic part is given by*

$$\mathscr{L}^{susy}_{vm} = \mathscr{L}_s + \mathscr{L}_v, \tag{5.9a}$$

$$\mathscr{L}_s = i(\partial_\mu F_I \partial^\mu \bar{X}^I - \partial_\mu \bar{F}_I \partial^\mu X^I) = -N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}}, \tag{5.9b}$$

$$\mathscr{L}_v = \frac{1}{2}\mathrm{Im}\,(\mathcal{N}^{susy}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}) = \frac{i}{4}(F_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J\mu\nu} - \bar{F}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}), \tag{5.9c}$$

where

$$\bar{\mathcal{N}}^{susy}_{IJ} = F_{IJ} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}F, \tag{5.10}$$

$$N_{IJ} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial \bar{X}^J}K^{susy}(X,\bar{X}) = i\bar{F}_{IJ} - iF_{IJ} = 2\mathrm{Im}\, F_{IJ}, \tag{5.11}$$

$$K^{susy}(X,\bar{X}) = i\left(X^I\bar{F}_I(\bar{X}) - \bar{X}^I F_I(X)\right). \tag{5.1}$$

Notation: Physicists usually denote the derivatives of F with subscripts, decreasing the length of their expressions using Einstein's summation convention. We will denote the matrix consisting of double derivatives with $\mathrm{Hess}_{IJ} = \frac{\partial^2 F(X)}{\partial X^I \partial X^J}$. $\Diamond$

From $\mathscr{L}^{susy}_{vm}$ it is clear what affine special geometry has to do with the theory of vector multiplets: the scalar Lagrangian describes a nonlinear sigma model whose geometry is given by the prepotential F of an affine special Kähler manifold, cf. definition 5.2.

Vector multiplets in $N = 2$ supergravity

Next we consider the vector and gravity multiplet sectors of a four-dimensional $N = 2$ supergravity theory. Again we denote the number of vector multiplets with n_V. Since the gravity multiplet contains another vector, the *graviphoton*, a supergravity theory is conventionally described using superconformal tensor calculus in which an extra vector multiplet, the *compensating vector multiplet*, is introduced [dWVP84, dWLP+84, VP96]. The procedure amounts to breaking the symmetry of the superconformal group of the resulting system of $n_V + 1$ vector multiplets to that of the super Poincaré group. The *complex* scalar of the compensating vector multiplet disappears as a physical field after gauge fixing the superfluous *complex dilatation symmetry*, which consists of a real dilatation, $\mathbb{R}_{>0}$, and a U(1)-symmetry. The compensating vector multiplet's vector does not disappear; it becomes the physical graviphoton. Together with the *graviton* (and fermionic *gravitino*) it forms the gravity multiplet of the supergravity theory.

Before gauge fixing, the bosonic scalar fields are described by n_V+1 complex scalars $\{L^I\}_{I=0}^{n_V}$. As in the rigid case the Lagrangian of the superconformal theory is elegantly determined by a single holomorphic function $F(L)$. Due to the (unfixed) scaling symmetry, this prepotential must be homogeneous of degree 2 in L [dWVP84], i.e.

$$L^I F_I(L) = 2F(L), \tag{5.12a}$$

$$L^J F_{IJ}(L) = F_I(L), \tag{5.12b}$$

$$L^K F_{IJK}(L) = 0. \tag{5.12c}$$

In order to obtain the Lagrangian for the supergravity theory, we need to gauge fix the dilatation and U(1)-symmetry. For this let us introduce $n_V + 2$ scalar fields a and $\{X^I\}_{I=0}^{n_V}$ via

$$L^I = aX^I, \tag{5.13}$$

together with a *new* gauge invariance,

$$\tilde{a} = ae^{\Lambda_K}, \qquad \tilde{X}^I = X^I e^{-\Lambda_K}, \tag{5.14}$$

where Λ_K is the gauge parameter. For reasons yet to become clear we will call these transformations *Kähler transformations*. The combined action of Kähler transformations, dilatations (generated by the gauge parameter Λ_D) and U(1)-transformations (generated by the gauge parameter $\Lambda_{\mathrm{U}(1)}$) on the fields L^I, X^I and a is

$$\tilde{L}^I = e^{\Lambda_D - i\Lambda_{\mathrm{U}(1)}}L^I, \qquad \tilde{a} = e^{\Lambda_D - i\Lambda_{\mathrm{U}(1)} + \Lambda_K}a, \qquad \tilde{X}^I = e^{-\Lambda_K}X^I.$$

We see that the gauge fixing of the dilatation and U(1)-symmetry can be completely incorporated in the auxiliary field a. The X^I are then fixed by fixing the Kähler transformations associated with Λ_K.

The gauge fixing of the (real) dilatation-invariance and U(1)-invariance is done by setting

$$|a|^2 = e^{K^{sugra}(X,\bar{X})}, \qquad \text{where } K^{sugra} \text{ is defined by} \tag{5.15}$$

$$K^{sugra}(X,\bar{X}) = -\log\left[i\bar{X}^I F_I(X) - iX^I \bar{F}_I(\bar{X})\right], \tag{5.2}$$

and taking a real and positive.[2] The gauge fixed Lagrangian has been constructed in [dWLP$^+$84, dWLVP85]. Its gravitational term, bosonic scalar and vector terms are given by (5.16).

Theorem 5.16 (Bosonic Lagrangian of the vector and gravity multiplet sector in $N = 2$ supergravity). *Let $(L^I, A^I_\mu)^{n_V}_{I=0}$ be $n_V + 1$ superconformal vector multiplets. Let $\{X^I\}^{n_V}_{I=0}$ be given by (5.13). The gauge fixed Lagrangian for the gravity and vector multiplets of the four-dimensional $N = 2$ supergravity theory is determined by a single holomorphic function $F(X)$ that is homogeneous of degree 2. Its bosonic part is given by*

$$\begin{aligned}
\mathscr{L}^{sugra} &= \mathscr{L}_{gr} + \mathscr{L}_{vm} = \mathscr{L}_{gr} + \mathscr{L}_s + \mathscr{L}_v, && (5.16a)\\
\mathscr{L}_{gr} &= i(\bar{L}^I F_I - \bar{F}_I L^I)\sqrt{g}R \;=\; \sqrt{g}R, && (5.16b)\\
\mathscr{L}_s &= \frac{\partial^2 K^{sugra}(X,\bar{X})}{\partial X^I \partial \bar{X}^{\bar{J}}}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}}, && (5.16c)\\
\mathscr{L}_v &= \frac{1}{2}\mathrm{Im}\,(\mathcal{N}^{sugra}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}), && (5.16d)
\end{aligned}$$

where the symmetric matrix $\mathcal{N}^{sugra}$ is given by

$$\begin{aligned}
\mathcal{N}^{sugra}_{IJ} &= \bar{F}_{IJ}(X) + i\frac{(NXX^tN)_{IJ}}{X^tNX}, \qquad I \in \{0,1,\dots,n_V\}, && (5.17a)\\
N_{IJ} &= 2\mathrm{Im}\,F_{IJ}. && (5.17b)
\end{aligned}$$

Clearly the nonlinear sigma model describing the scalars of this theory interprets the fields X^I as special (projective) coordinates on a projective special Kähler manifold, cf. definition 5.7. The arbitrariness of fixing the newly introduced gauge invariance (5.14) is interpreted as choosing local coordinates for the manifold. Different choices are related with each other via a Kähler transformation of the special coordinates X^I. Indeed K^{sugra} transforms under (5.14) as

$$\tilde{K}^{sugra}(X,\bar{X}) = K^{sugra}(X,\bar{X}) + \Lambda_K(X) + \bar{\Lambda}_K(\bar{X}).$$

Comparing (5.6) and (5.7) with (5.14) shows that the scalar fields X^I of the vector multiplets in type IIB theory are A-periods of the holomorphic volume form (note that $n_V = h^{2,1}$ in this case). In type IIA vector multiplet theory ($n_V = h^{1,1}$), the Kähler moduli W^Λ constitute the vector multiplet scalar fields. In this context the auxiliary coordinate W^0 is identified as the scalar of the compensating vector multiplet.

An interesting restriction of the symmetric matrix $\mathcal{N}^{sugra}$ is the negative definiteness of its imaginary part. It has been proven by [CKVP$^+$85] that positive definiteness of the kinetic terms for the scalars and vectors in (5.16) demands a negative definite $\mathrm{Im}\,\mathcal{N}^{sugra}$.

Lemma 5.17 (Negative definiteness of $\mathrm{Im}\,\mathcal{N}^{sugra}$). *The imaginary part of the symmetric matrix $\mathcal{N}^{sugra}$ appearing in the vector multiplet's Lagrangian of an $N = 2$ supergravity theory is negative definite, due to the positive definiteness of the kinetic terms.*

[2] From (5.16b) we see that this is motivated by the fact that a standard Einstein action $\sqrt{g}R$ can be obtained by fixing $i(\bar{L}^I F_I - \bar{F}_I L^I) = 1$. From

$$i(\bar{L}^I F_I(L) - \bar{F}_I(\bar{L})L^I) = |a|^2 i(\bar{X}^I F_I(X) - \bar{F}_I(\bar{X})X^I) = |a|^2 e^{-K^{sugra}(X,\bar{X})},$$

we find that (5.15) is indeed the required gauge.

5.4 Electric-magnetic duality

Although historically the construction of an $N = 2$ supergravity Lagrangian has been the first method of describing special Kähler geometry, other methods have been developed since then. A disadvantage of the sigma model-description using a prepotential is that the fields X^I are singled out as a special set of local coordinates. Via *electric-magnetic duality* considerations, the scalar manifold can more generically be described by a less biased set of coordinates $\{t^a\}_{a=1}^{h^{2,1}}$, without the need of an explicit construction of the Lagrangian in terms of a prepotential [Str90, CDFVP95, dWVP95, CDF96, CRTVP97]. Electric-magnetic duality is a symmetry between electric and magnetic information of the vectors in the vector multiplets. In this section we will see what we mean by this and how symmetry on the *vectors* restricts the geometry of the manifold, parameterized by the *scalars* of the vector multiplets, through the symmetric matrix $\mathcal{N}$ that couples these two parts of the vector multiplets. Electric-magnetic duality puts symplectic transformations at the heart of special geometry, leading to an alternative formulation of special Kähler geometry.

Duality transformations

Consider the vector term of an $N = 2$ supersymmetry theory with n vectors,[3]

$$\mathscr{L}_v = \frac{1}{2}\mathrm{Im}\,(\mathcal{N}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}) = \frac{i}{4}(\bar{\mathcal{N}}_{IJ}\mathcal{F}^{-I}_{\mu\nu}\mathcal{F}^{-J\mu\nu} - \mathcal{N}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}), \tag{5.18}$$

and consider the field equations for the gauge fields A^I_μ that are obtained by an Euler-Lagrange variation of (5.18),

$$-2\partial_\mu\left(\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{+I}_{\mu\nu}} + \frac{\delta\mathscr{L}}{\delta\mathcal{F}^{-I}_{\mu\nu}}\right) = 0.$$

We define $\mathcal{G}^{\pm I}_{\mu\nu}$ by

$$\mathcal{G}^{+}_{I\mu\nu} = 2i\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{+I\mu\nu}} = \mathcal{N}_{IJ}\mathcal{F}^{+J}_{\mu\nu}, \qquad \mathcal{G}^{-}_{I\mu\nu} = -2i\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{-I\mu\nu}} = \bar{\mathcal{N}}_{IJ}\mathcal{F}^{-J}_{\mu\nu}. \tag{5.19}$$

with the help of which we may write the field equations for A^I as,

$$\partial^\mu\mathrm{Im}\,\mathcal{G}^{+}_{I\mu\nu} = 0, \tag{5.20a}$$

Since $\mathcal{F}^I = \mathrm{d}A^I$, $\mathcal{F}^I$ satisfies on top of (5.20a) the *Bianchi identity* $\mathrm{d}\mathcal{F}^I = 0$. This is equivalent to $\partial^\mu(\varepsilon_{\mu\nu}{}^{\rho\sigma}\mathcal{F}^I_{\rho\sigma}) = 0$, which by (5.8) may be written as

$$\partial^\mu\mathrm{Im}\,\mathcal{F}^{+I}_{\mu\nu} = 0. \tag{5.20b}$$

Together the Bianchi identities (5.20b) and field equations (5.20a) describe the total electric-magnetic information of the vectors A^I in the vector multiplets. They are the generalizations of Maxwell's vacuum equations for the gauge vector fields A^I.

Equation (5.20) is invariant under $\mathrm{GL}(2n,\mathbb{R})$-transformations

$$\begin{pmatrix}\tilde{\mathcal{F}}^+\\ \tilde{\mathcal{G}}^+\end{pmatrix} = S\begin{pmatrix}\mathcal{F}^+\\ \mathcal{G}^+\end{pmatrix}, \qquad S = \begin{pmatrix}U & Z\\ W & V\end{pmatrix} \in \mathrm{GL}(2n,\mathbb{R}). \tag{5.21}$$

However since $\mathcal{G}^+_I$ is determined from $\mathcal{F}^{+I}$ via (5.19), the possible transformations are constrained in $\mathrm{GL}(2n,\mathbb{R})$: the *new* self dual and anti-self dual fields $\tilde{\mathcal{G}}^\pm$ should still be obtained from the *new* self dual and anti-self dual field strengths $\tilde{\mathcal{F}}^\pm$ by differentiation of $\tilde{\mathscr{L}}$. For this reason the new matrix $\tilde{\mathcal{N}}$ should be symmetric [CRTVP97]. The transformation of $\mathcal{N}$ induced by (5.21),

$$\begin{pmatrix}\tilde{\mathcal{F}}^+\\ \tilde{\mathcal{G}}^+\end{pmatrix} = \begin{pmatrix}(U+V\mathcal{N})\mathcal{F}^+\\ (W+V\mathcal{N})\mathcal{F}^+\end{pmatrix} = \begin{pmatrix}\tilde{\mathcal{F}}^+\\ (W+V\mathcal{N})(U+Z\mathcal{N})^{-1}\tilde{\mathcal{F}}^+\end{pmatrix},$$

[3] That is $n = n_V$ in rigid supersymmetry and $n = n_V + 1$ in supergravity.

is
$$\tilde{\mathcal{N}} = (W + V\mathcal{N})(U + Z\mathcal{N})^{-1}. \tag{5.22}$$
It can be shown that the matrices $S = \left(\begin{smallmatrix} U & Z \\ W & V \end{smallmatrix}\right) \in \mathrm{GL}(2n, \mathbb{R})$ that map symmetric matrices into symmetric matrices under this transformation are precisely the *symplectic matrices* $S \in \mathrm{Sp}(2n, \mathbb{R})$ [VP95, dW96],
$$\begin{pmatrix} \tilde{\mathcal{F}}^+ \\ \tilde{\mathcal{G}}^+ \end{pmatrix} = S \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix}, \qquad S = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}). \tag{5.23}$$
These rotate the Bianchi identities and field equations in a way that is compatible with the relation (5.19) between $\mathcal{F}^+$ and $\mathcal{G}^+$. They are called *electric-magnetic transformations.* The invariance of Bianchi identities and field equations is called *electric-magnetic duality*, since the electric-magnetic transformations rotate the electric and magnetic degrees of freedom contained in equation (5.20). The concept is related to the magnetic monopole that Dirac introduced to enhance the symmetry in Maxwell's *non*-vacuum equations [Dir31]. When the gauge fields's analogues of these, called *dyonic charges*, are included into the theory, the possible transformations (5.23) become more constrained. By the *Schwinger-Zwanziger quantization condition* these charges are quantized and the rotations are restricted to the lattice of electric and magnetic charges at the nonperturbative level. This restricts the possible duality transformations to lie in the discrete subgroup $\mathrm{Sp}(2n, \mathbb{Z}) \subset \mathrm{Sp}(2n, \mathbb{R})$ [Fre99, CRTVP97, DJdWKV98, Pol98]. We will mostly work at the classical level and we will therefore consider $\mathrm{Sp}(2n, \mathbb{R})$ most often. Nevertheless the effect of $\mathrm{Sp}(2n, \mathbb{Z})$ will already be incorporated into our formalism.

Symplectic vectors

The $2n$-dimensional vector $\left(\begin{smallmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{smallmatrix}\right)$ transforms under the symplectic transformation Λ_S given by (5.23) as a *symplectic vector*, i.e. it is a vector which satisfies
$$\Lambda_S(v) = Sv, \qquad S = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}). \tag{5.24}$$
In the superfield formalism which was also used to construct the supersymmetric Lagrangian, cf. section 5.3, it is natural to consider another $2n$-dimensional vector $v = \left(\begin{smallmatrix} X^I \\ F_I \end{smallmatrix}\right)$. This is because the first derivatives of the holomorphic prepotential, $F_I = \frac{\partial F(X)}{\partial X^I}$, are used as scalars of new vector multiplets. The vectors in the new vector multiplets are associated to the field equations (5.20a) of the Lagrangian, just like the vectors A^I of the original vector multiplets lead to the Bianchi identities of $\mathcal{F}^I$. Thus the Bianchi identities and complex scalars X^I are grouped together in the original vector multiplets, while new vector multiplets are constructed containing complex scalars F_I and the field equations. Since Bianchi identities and field equations form a symplectic vector under the duality transformations, consistency under supersymmetry obliges the scalars X^I and F_I to form a symplectic vector as well [dW96, CRTVP97]. This interplay between supersymmetry (the formation of supermultiplets) and electric-magnetic duality (between Bianchi identities and field equations) makes it possible to understand the vector multiplet Lagrangian and its corresponding scalar manifold in terms of a symplectic vector bundle over a Kähler manifold [Str90, CDFVP95, dWVP95, CDF96, CRTVP97], as we will see in next sections.

Remark: For the vector multiplet theory of compactified type IIB string theory, the symplectic vector $v = \left(\begin{smallmatrix} X^I \\ F_I \end{smallmatrix}\right)$ equals the periods of *vol* over the A- and B-cycles, cf. (5.7). The symplectic transformations rotate the Lagrange plane associated to the choice of these A- and B-cycles. ◊

Symplectic formulation of the vector multiplets in $N = 2$ rigid supersymmetry

Let us first consider $N = 2$ rigid supersymmetry with n_V vector multiplets, cf. section 5.3. The fact that X^I and F_I form a symplectic vector under the duality transformations puts a restriction on the scalar manifold parameterized by the X^I. An alternative definition of the affine special Kähler manifold parameterized by these scalars can therefore be given in terms of symplectically invariant objects.

Definition 5.18 (Symplectic inner product). Let $E \to M$ be a vector bundle of rank $2n$ over a manifold M, then we define the *symplectic inner product* $\langle\cdot,\cdot\rangle : \Gamma(M,E)\times\Gamma(M,E) \to C^\infty(M)$ by

$$\langle v, w\rangle = v^t \Sigma w,$$

where $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{smallmatrix}\right)$ and $v, w \in E$. ⊘

Definition 5.19 (Alternative extrinsic description of an affine special Kähler manifold). Let (M, σ_J) be an n-dimensional Kähler manifold. It is an *affine special Kähler manifold*, locally defined, if it satisfies the following conditions.

1. There exists a $\mathrm{U}(1)\times\mathrm{ISp}(2n,\mathbb{R})$ vector bundle[4] over M with constant transition functions, in the sense that on overlap of charts k and l,

   $$w_{(k)} = e^{ic_{kl}} S_{kl} w_{(l)} + b_{kl},$$

   where w is a holomorphic section, $c_{kl}\in\mathbb{R}$, $S_{kl}\in\mathrm{Sp}(2n,\mathbb{R})$ and $b_{kl}\in\mathbb{C}^{2n}$.

2. This bundle should have a holomorphic section v such that the Kähler form σ_J is given by

   $$\sigma_J = -\partial\bar\partial\langle v, \bar v\rangle, \tag{5.25}$$

   and such that it satisfies in local coordinates $\{t^i\}_{i=1}^n$,

   $$\langle \partial_i v, \partial_j v\rangle = 0. \tag{5.26}$$

⊘

The equivalence of this definition and definition 5.2 is carefully proven in [CRTVP97]. It follows from the discussion on page 66 that definition 5.2 leads to definition 5.19. Conversely one should prove that the existence of a prepotential follows from the conditions in definition 5.19. In [CRTVP97] it is shown that this can be done by virtue of (5.26). We will see a similar reasoning in another description of affine special Kähler geometry in terms of Lagrangian Kähler submanifolds of a complex symplectic vector space in chapter 6. Once the existence of a prepotential is proven the (symplectically invariant) Kähler potential $K = i\langle v,\bar v\rangle = iX^I\bar F_I - i\bar X^I F_I$ follows from formula (5.25) which is obviously consistent with (5.1).

Thus affine special Kähler geometry can equally well be described by a symplectic bundle on a Kähler manifold as it can by a prepotential that determines a supersymmetric nonlinear sigma model. Therefore the scalar term of (5.9) has gotten a description purely in terms of symplectic objects. To complete the equivalence between a prepotential or symplectic description of the vector multiplets, we will proceed by constructing the kinetic term of the vectors (5.9c) in terms of the holomorphic symplectic vector v. That is we want to define the central object $\mathcal{N}^{susy}$ in terms of v. To this end we define [CRTVP97, CDF96]

$$e^I{}_i = \frac{\partial X^I(t)}{\partial t^i}, \qquad h_{Ii} = \frac{\partial F_I(t)}{\partial t^i},$$

and a $2n_V \times n_V$-matrix $u = \left(\begin{smallmatrix} u_1 & \cdots & u_n \end{smallmatrix}\right)$ via

$$u_i = \frac{\partial v}{\partial t^i} = \begin{pmatrix} e^I{}_i \\ h_{Ii} \end{pmatrix}.$$

The columns of u are labelled by $i\in\{1,\ldots,n_V\}$ and its rows are labelled by $I\in\{1,\ldots,n_V\}$. Each column u_i is a symplectic vector.

[4] The semi-direct product $\mathrm{ISp}(2n,\mathbb{R})$ is the group of real symplectic matrices extended with *complex* inhomogeneous shifts.

Proposition 5.20 (Symmetric matrix $\mathcal{N}^{susy}$ from symplectic vector). *The symmetric matrix $\mathcal{N}^{susy}$ appearing in the vector term of the $N=2$ supersymmetric vector multiplet Lagrangian is defined by*

$$\mathcal{N}^{susy}_{IJ} = \bar{h}_{I\bar{i}} \bar{e}^{-1\bar{i}}{}_{J}, \tag{5.27}$$

in terms of the symplectic vector v of definition 5.19.

Proof. We need to verify that (5.27) equals (5.10) when the symplectic vector v is written as $\begin{pmatrix} X^I \\ F_I \end{pmatrix}$, with $F_I = \frac{\partial F}{\partial X^I}$. Using the chain rule,

$$\bar{h}_{I\bar{i}} \bar{e}^{-1\bar{i}}{}_{J} = \frac{\partial \bar{F}_I}{\partial \bar{t}^{\bar{i}}} \frac{\partial \bar{t}^{\bar{i}}}{\partial \bar{X}^J} = \bar{F}_{IJ}. \qquad \square$$

Symplectic formulation of the vector multiplets in $N=2$ supergravity

Next we consider the n_V scalars and n_V+1 vectors in an $N=2$ supergravity theory with n_V vector multiplets. In section 5.3 we have given a local description of the theory and its underlying scalar manifold using a nonlinear sigma model constructed with the help of a prepotential. However in some coordinates a prepotential does not exist, while the intrinsic notion of projective special Kähler geometry does not depend on a coordinate choice. The symplectic formulation inspired by the electric-magnetic duality transformations has been proven more successful, as it succeeds in defining projective special Kähler geometry in a coordinate independent fashion [CRTVP97].

Definition 5.21 (Alternative extrinsic description of a projective special Kähler manifold). Let (M, L, σ_J) be an n-dimensional Hodge manifold. It is a *projective special Kähler manifold*, locally defined, if it satisfies the following conditions.

1. There exists a holomorphic $\mathrm{Sp}(2n+2,\mathbb{R})$ vector bundle E over M with constant transition functions and a holomorphic section v of $L \otimes E$, such that the Kähler form σ_J is given by

$$\sigma_J = -i\partial\bar{\partial} \log\left[i\langle \bar{v}, v\rangle\right]. \tag{5.28}$$

Here L denotes the holomorphic line bundle over M whose transition functions are

$$v_{(k)} = e^{-(\Lambda_K)_{kl}} v_{(l)}$$

and whose first Chern class equals the cohomology class of the Kähler form, $c_1(L) = [\sigma_J]$.

2. The section v satisfies

$$\langle v, \mathrm{D}^K_a v\rangle = 0, \tag{5.29}$$

where D^K_a is a *Kähler covariant derivative* with respect to the local coordinates $\{t^a\}_{a=1}^n$ of M.

⊘

The Kähler covariant derivative, cf. lemma 5.10, acts upon v as $\mathrm{D}^K_a v = \partial_a v + (\partial_a K^{sugra})v$. It defines a proper differentiation of the holomorphic section v, because the Kähler potential's first derivative $\partial_a K$ is an abelian connection on the holomorphic line bundle L [Str90, FL92].

Using (5.29) it is shown in [CRTVP97] that there exists a coordinate system in which a prepotential exists, i.e. definition 5.21 implies definition 5.7. Note that (5.28) is the symplectic formulation of the Kähler form corresponding to (5.2). The converse (5.7 implies 5.21) is automatic by the construction of the symplectic vector v on page 66. As in the rigid case the scalar manifold of the supergravity vector multiplet theory can be described in a symplectic fashion. We conclude our discussion of the supergravity vector multiplet theory (5.16) with a symplectic definition of the matrix $\mathcal{N}^{sugra}$ appearing in the kinetic term of the vectors. It completes the vector multiplet's description in terms of symplectic objects.

Proposition 5.22 (Symmetric matrix $\mathcal{N}^{sugra}$ from symplectic vector). *The symmetric matrix $\mathcal{N}^{sugra}$ appearing in the vector term of the $N=2$ supergravity vector multiplet Lagrangian is defined by*

$$\mathcal{N}^{sugra} = \begin{pmatrix} W & \bar{h} \end{pmatrix} \begin{pmatrix} X & \bar{f} \end{pmatrix}^{-1}, \tag{5.30}$$

in terms of the symplectic vector v of definition 5.21. Here X and W are (n_V+1)-dimensional column vectors defined by $v(t) = \begin{pmatrix} X^I(t) \\ W_I(t) \end{pmatrix}$, and f and h are $(n_V+1)\times n_V$-matrices defined by

$$u_a(t) = \mathrm{D}^K_a v = \begin{pmatrix} f^I{}_a(t) \\ h_{Ia}(t) \end{pmatrix}, \qquad I \in \{0,\ldots,n_V\},\ a \in \{1,\ldots,n_V\}.$$

Proof. We need to verify that (5.30) equals (5.17) when a prepotential F exists. Suppose a prepotential exists, then the symplectic vector v can be written as $\begin{pmatrix} X^I \\ W_I \end{pmatrix}$ with $W_I = \frac{\partial F}{\partial X^I}$. The homogeneity (5.12) of F can be expressed as a matrix multiplication $W = \mathrm{Hess}X$, where Hess is the Hessian of F with respect to X. Equation (5.30) is equivalent with finding a matrix $\mathcal{N}^{sugra}$ which satisfies,

$$\begin{aligned} \bar{h} &= \mathcal{N}^{sugra}\bar{f}, & (5.31a)\\ W &= \mathcal{N}^{sugra}X. & (5.31b) \end{aligned}$$

As in the rigid case, the first equation is solved by $\mathcal{N}^{sugra} = \overline{\mathrm{Hess}}$,

$$\left(\overline{\mathrm{Hess}}\bar{f}\right)_{\bar{a}} = \overline{\mathrm{Hess}}\left(\partial_{\bar{a}}\bar{X} + (\partial_{\bar{a}}K)\bar{X}\right) = \overline{\mathrm{Hess}}\partial_{\bar{a}}\bar{X} + (\partial_{\bar{a}}K)\overline{\mathrm{Hess}}\bar{X} = \partial_{\bar{a}}\bar{W} + (\partial_{\bar{a}}K)\bar{W} = \bar{h}_{\bar{a}}.$$

However equation (5.31b) is *not* solved by $\mathcal{N} = \overline{\mathrm{Hess}}$, since

$$W = \mathrm{Hess}X = \left(\overline{\mathrm{Hess}} + (\mathrm{Hess} - \overline{\mathrm{Hess}})\right)X = \left(\overline{\mathrm{Hess}} + iN\right)X.$$

To correct this we define a matrix P which acts as a projector on the space spanned by X and $\bar{f}_{\bar{a}}$,

$$P = \frac{XX^tN}{X^tNX}. \tag{5.32}$$

Indeed we find

$$\begin{aligned} PX &= \frac{X(X^tNX)}{X^tNX} = X,\\ P\bar{f}_{\bar{a}} &= P\left(\partial_{\bar{a}}\bar{X} + \bar{X}(\partial_{\bar{a}}K)\right) = \frac{XX^tN}{X^tNX}\left(\partial_{\bar{a}}\bar{X} - \bar{X}\frac{X^tN\partial_{\bar{a}}\bar{X}}{X^tN\bar{X}}\right)\\ &= \left(\frac{XX^tN}{X^tNX} - \frac{X(X^tN\bar{X})X^tN}{(X^tNX)(X^tN\bar{X})}\right)\partial_{\bar{a}}\bar{X} = 0, \end{aligned}$$

where we have used the following expression for $\partial_{\bar{a}}K^{sugra}$,

$$\frac{\partial}{\partial\bar{t^a}}K^{sugra} = -\frac{\partial\bar{X}^J}{\partial\bar{t^a}}\frac{\partial}{\partial\bar{X}^J}\log\left[-X^tN\bar{X}\right] = -\frac{X^tN\partial_{\bar{a}}\bar{X}}{X^tN\bar{X}}.$$

We see that our system of equations (5.31) is solved by

$$\mathcal{N}^{sugra} = \overline{\mathrm{Hess}} + iNP = \overline{\mathrm{Hess}} + i\frac{NXX^tN}{X^tNX}. \qquad \square$$

Remark: In the physics literature the symmetric matrix $\mathcal{N}$ is often called a period matrix. In chapters 6 and 7 we will give a justification of this terminology by considering the symplectic vector $v = \begin{pmatrix} X^I \\ F_I \end{pmatrix}$ consisting of the periods (5.7) of the volume form *vol* of the Calabi-Yau$_3$ [Fré96]. Note that by the remark on page 66, v is exactly the symplectic vector occurring in the type IIB vector multiplet theory. $\Diamond$

In this section we have found that special Kähler geometry for the scalar manifold in $N = 2$ (rigid and local) supersymmetry vector multiplet theory is intimately related with *both* supersymmetry conditions *as well as* with duality transformations of the vectors in the vector multiplets. Whereas the extrinsic description of special Kähler geometry can be completely given in terms of a holomorphic prepotential $F(X)$, the definition in terms of a symplectic vector bundle succeeds in describing a special Kähler manifold using only symplectically invariant objects. Note that the existence of $F(X)$, which is not symplectically invariant, *does* follow from the symplectic vector bundle's definition. At the heart of the equivalence between both descriptions stands the period matrix $\mathcal{N}$ which couples the scalars of the vector multiplets to the vectors. It translates the duality transformations on the vectors to the special geometry of the scalar manifold. Figure 5.1 is a diagrammatic representation of the two viewpoints which lead to special Kähler geometry.

$$\begin{array}{ccc} \text{Supersymmetry} & \longleftrightarrow & \text{Duality transformations} \\ F \searrow & \mathcal{N} & \swarrow v \\ & \text{Special Kähler geometry of } \mathscr{M}_{vm} & \end{array}$$

Figure 5.1: Special Kähler geometry of the vector multiplet's scalar manifold $\mathscr{M}_{vm}$ may either be escribed using supersymmetry of the theory or using electric-magnetic duality of the vectors in the vector multiplets. By supersymmetry arguments a holomorphic prepotential F is used to describe a nonlinear sigma model whose scalar manifold is a special Kähler manifold. Alternatively from the electric-magnetic duality of the vectors the scalar manifold is restricted to admit a symplectic vector bundle. Both interpretations are connected by the discussion on page 66, which lead to alternative definitions of special Kähler geometry: 5.2 and 5.19 for affine special Kähler geometry and 5.7 and 5.21 for projective special Kähler geometry. At the heart of both constructions lies the symmetric matrix $\mathcal{N}$ which forms the connection between the vectors and scalars in the vector multiplets. In the supersymmetric formalism $\mathcal{N}$ is constructed from the prepotential, while it provides the link between the electric and magnetic degrees of freedom of the vectors from the electric-magnetic point of view.

Chapter 6

A geometric interpretation of the rigid c-map

The rigid c-map in $N = 2$ supersymmetry provides a convenient method to investigate the structure of the hypermultiplet moduli space of type IIA string theory decoupled from gravity. Starting from the vector multiplet moduli space of (decoupled) type IIB string theory, physicists have used it to construct the metric on the type IIA hypermultiplet moduli space [CFG89, DJdWKV98]. In this chapter we will see how this result is obtained and then we will focus on a more mathematical approach. By the work of Cortés [Cor98], the construction of the rigid c-map has led to a general statement in mathematics about the cotangent bundle of affine special Kähler manifolds. We will review his methods, which will provide us with the necessary background to interpret the rigid c-map (applied to type IIB decoupled from gravity) mathematically as a bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$'s.

6.1 The rigid c-map in physics

Decoupling gravity

In section 4.4 we have seen how T-duality between both type II string theories leads to a correspondence between the type IIA's hypermultiplet moduli space and the type IIB's gravity and vector multiplet sectors, theorem 4.21. The type II theories are examples of four-dimensional $N = 2$ supergravity theories which may be decoupled from gravity to give four-dimensional $N = 2$ rigid supersymmetry theories. The local c-map between the supergravity theories reduces to the rigid c-map in $N = 2$ rigid supersymmetry, which provides a construction for the $N = 2$ hypermultiplet scalar manifold of *arbitrary* $N = 2$ rigid supersymmetry theories. In this chapter we are interested in the rigid c-map in general, but more specifically in the rigid c-map acting on the rigid supersymmetry theories obtained from type IIA and type IIB superstring theory. Since superstring theory without gravity seems to be a contradictio in terminis, we will briefly discuss what we mean with rigid supersymmetric type IIA and type IIB theories.

Recall from section 5.3 that supergravity vector multiplets of compactified type II superstring theory are most conveniently described by introducing a compensating vector multiplet and an additional complex dilatational symmetry on top of the super Poincaré symmetries which act on the total system of $n_V + 1$ vector multiplets. Before gauge fixing the additional dilatational symmetry, the scalars of the $n_V + 1$ vector multiplets describe an *affine* special Kähler manifold, whose prepotential is homogeneous of degree 2. The additional complex dilatational gauge field is grouped together with the super Poincaré parameters into a supermultiplet of the superconformal algebra, which is called the *Weyl multiplet*. Upon gauge fixing, the Weyl multiplet would reduce to the gravity multiplet in which also the physical graviphoton vector field from the compensating vector multiplet is incorporated. The complex scalar of the compensating vector multiplet would disappear as a physical field [CRTVP97].

It is possible to decouple gravity *without* first gauge fixing the additional complex dilatational symmetry of the superconformal description of the gravity and vector multiplet sectors. By taking *global* instead of *local* supersymmetry the Weyl multiplet is removed from the theory, since gauge fields are associated with local symmetries. In first instance the theory reduces to *conformal* $N=2$ *rigid supersymmetry* in which a *global* dilatational symmetry is still present, but it is possible to break this symmetry as well and consider pure $N=2$ rigid supersymmetry. Only in the latter case the homogeneity property of the prepotential associated to the vector multiplets is no longer necessary.

We will investigate the rigid c-map applied to this general $N=2$ supersymmetric setting, but also in the more specific context of the type IIB's vector multiplet sector. In the latter case it is useful to consider the conformal effective type IIB theory, whose complex structure's prepotential $F^{(2,1)}$ is still assumed to be homogeneous of degree 2. The rationale behind this is that it emphasizes the rigid c-map's workings on the (homogeneous coordinates of the) complex structure moduli space of the family of Calabi-Yau$_3$'s, thus preparing us for an analysis of the local c-map. Although it may seem a little bit strange from a physics point of view, mathematically it is simply a restriction of the actual rigid c-map to one that works on an affine special Kähler manifold with *homogeneous* prepotential. The difference with the local c-map is that the latter acts on the Weyl multiplet as well, while the rigid c-map is incapable of incorporating gravity.

In the remainder of this section we will consider the rigid c-map in a string theoretical context. The homogeneity of the vector multiplet's prepotential will be of no importance (yet), inducing an obvious extension of the rigid c-map to arbitrary (nonconformal) rigid supersymmetric theories. Section 6.2 will present the mathematical approach of [Cor98] to the general rigid c-map, whereas in section 6.3 we will consider the rigid c-map acting on rigid supersymmetric type IIB vector multiplets. In the latter case the prepotential will explicitly be homogeneous of degree 2 and the (complex structure of the) Calabi-Yau$_3$ will be an essential ingredient of the rigid c-map's geometric interpretation.

Constructing a hypermultiplet Lagrangian

In this section we will see explicitly how T-duality may be used to construct the type IIA's hypermultiplet moduli space from the type IIB's vector multiplet sector in $N=2$ rigid supersymmetry. The type IIB rigid supersymmetric vector multiplet's (bosonic) Lagrangian is given by

$$\mathscr{L}^{susy}_{vm} = -N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} + \frac{1}{2}\mathrm{Im}\,(\bar{F}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}). \tag{5.9}$$

Here F is the prepotential of the complex structure moduli space, which serves as a prototype of an arbitrary prepotential. For this reason the suffix $(2,1)$ will not be used throughout this chapter and since no explicit use of the homogeneity of F is made in this section, results are easily generalized to arbitrary $N=2$ rigid supersymmetry theories. In that formalism the matrix N is given by $N_{IJ} = 2\mathrm{Im}\,F_{IJ}$, $\mathcal{F}^I$ are the field strengths of the vectors in the theory $\mathcal{F}^I = \mathrm{d}A^I$, $I \in \{0,\dots,h^{2,1}\}$, and the fields X^I are the vector multiplets's scalar fields. In the decoupling limit from supergravity one of these scalar fields is the extra compensator field which has *not* been gauged fixed.

The type IIA's hypermultiplet Lagrangian is obtained by compactifying one space-coordinate on a circle S^1_R of radius R. Thus we assume the type IIB theory to be defined on a four-dimensional spacetime of the form $M_4 = M_3 \times S^1_R$. Coordinates on M_4 are denoted with $\hat{x}^\mu = (x,y)$, where $x = (x^m)^2_{m=0}$ are the coordinates on M_3 and $y \in [0, 2\pi R]$ is the coordinate on the circle of radius R. Similarly as in the Calabi-Yau$_3$-compactification we perform a separation of variables on the fields in the theory. Since each field must be periodic in the circular direction we can use a Fourier expansion. The real gauge fields A^I_μ can be written as

$$A^I_\mu(\hat{x})\mathrm{d}\hat{x}^\mu = A^I_m(x,y)\mathrm{d}x^m + A^I_3(x,y)\mathrm{d}y = \sum_{k=-\infty}^{\infty} A^I_{k,m}(x)e^{iky/R}\mathrm{d}x^m + \sum_{k=-\infty}^{\infty} A^I_{k,3}(x)e^{iky/R}\mathrm{d}y.$$

All but the zero Fourier modes are suppressed, because the k-th Fourier mode comes with a mass term $\frac{4\pi^2 k^2}{R^2} A^I_{k,\mu} A^{J\mu}_k$, which means it represents a heavy particle for very small R. The remaining *massless modes* will be denoted with $A^I = 2A^I_{0,3}$ and $A^I_m = A^I_{0,m}$. The latter appear in the compactified action via their field strengths $\mathcal{F}^I_{lm} = \mathrm{d}A^I_{lm}$. The effective three-dimensional theory is obtained by performing the integral over the circle, which has now become trivial. The vector sector of (5.9) gives the following contribution.

$$S_v = \int_{M_3} \mathrm{d}^3x \int_{S^1_R} \mathrm{d}y\, \frac{1}{2} \mathrm{Im}\,(\bar{F}_{IJ} \mathcal{F}^{+I}_{\mu\nu} \mathcal{F}^{+J\mu\nu}) = \frac{2\pi R}{2} \int_{M_3} \mathrm{d}^3x\, \mathrm{Im}\, \left[\bar{F}_{IJ}(\mathcal{F}^{+I}_{lm}\mathcal{F}^{+Jlm} + 2\mathcal{F}^{+I}_{m3}\mathcal{F}^{+Jm3})\right],$$

where

$$\mathcal{F}^{\pm I}_{lm} = \mathcal{F}^I_{lm} \mp \frac{i}{2}\varepsilon_{lmk}\partial^k A^I, \qquad \mathcal{F}^{\pm I}_{m3} = \frac{1}{2}\partial_m A^I \mp \frac{i}{2}\varepsilon_{mlk}\mathcal{F}^{Ilk}.$$

Using the antisymmetry of $\mathcal{F}^I_{lm}$ and the symmetry of F_{IJ}, this leads to the three-dimensional effective Lagrangian for the vectors,

$$\mathscr{L}_{v,3d} = -\frac{1}{2}\left(N_{IJ}\mathcal{F}^I_{lm}\mathcal{F}^{Jlm} + \varepsilon^{lmk}(F+\bar{F})_{IJ}\partial_l A^I \mathcal{F}^J_{mk} + \frac{1}{2}N_{IJ}\partial^m A^I \partial_m A^J + 2B_I \varepsilon^{lmk}\partial_l \mathcal{F}^I_{mk}\right).$$

We have neglected an overall factor $2\pi R$ and we have explicitly included a Lagrange multiplier $B_I\varepsilon^{lmk}\partial_l\mathcal{F}^I_{mk}$ to ensure that the gauge fields A^I_m satisfy the Bianchi identity in three dimensions. The equations of motion for $\mathcal{F}^I_{lm}$ enable us to express $\mathcal{F}^I_{lm}$ in terms of A^I and B_I,

$$\mathcal{F}^I_{lm} = \varepsilon_{klm}N^{IJ}(\partial^k B_I - \frac{1}{2}(F+\bar{F})_{IJ}\partial^k A^J).$$

Substituting this back into the effective Lagrangian by integrating the last term by parts, decompactifying all (scalar) fields and including the homogeneous complex structure moduli X^I, we have obtained the following result [CFG89, DJdWKV98].

Proposition 6.1 (Image of the rigid c-map). *By compactifying the type IIB vector multiplet Lagrangian (5.9) on a circle and suppressing all massive modes, the Lagrangian for the type IIA hypermultiplet sector in $N = 2$ supersymmetry is*

$$\mathscr{L}_{rc} = -N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} - N^{IJ}(\partial_\mu B_I - F_{IK}\partial_\mu A^K)(\partial^\mu B_J - \bar{F}_{JL}\partial^\mu A^L). \tag{6.1}$$

From the construction of (6.1) an interesting duality between Dp-branes and D$(p\pm1)$-branes becomes evident. The scalars A^I and B_I are obtained from the type IIB (R-R) gauge fields A^I_μ of table 4.4 and are mapped to the type IIA scalar fields A^I and B_I of table 4.3. Performing explicitly the compactification of the ten-dimensional (R-R)-fields in type IIA and type IIB reveals that the type IIB 1-forms A^I_μ stem from the self dual antisymmetric tensor field $C^{(4)}$ of table 4.2, while the type IIA scalar fields A^I and B_I are massless scalar fields obtained from the (R-R) 3-form $A^{(3)}$ of table 4.1. Explicitly they are the periods of the real Calabi-Yau$_3$ 3-form[1] $A^{(3)}_{CY} \in H^3(CY, \mathbb{R})$ over the A- and B-cycles $\{\gamma_{A^I}, \gamma_{B_I}\}^{h^{2,1}}_{I=0}$ of $H_3(CY,\mathbb{Z}) \subset H_3(CY,\mathbb{Z})\otimes\mathbb{R}$, cf. (2.8),

$$A^I = \int_{\gamma_{A^I}} A^{(3)}_{CY}, \qquad B_I = \int_{\gamma_{B_I}} A^{(3)}_{CY}. \tag{6.2}$$

The rigid c-map shows that part of the degrees of freedom in $C^{(4)}$ are equivalent with the scalar contribution of $A^{(3)}$, which means there is a duality between $A^{(3)}$ and $C^{(4)}$. This is a specific example of the general phenomenon that T-duality relates Dp-branes with D$(p\pm1)$-branes [Pol98].

The rigid c-map has an interesting interpretation in terms of electric-magnetic duality of the gauge fields. These transformations are responsible for coupling of the gauge fields's field strengths

[1] Remember from the proof of theorem 4.12 that a ten-dimensional 3-form may be decomposed into a 0-, 1-, 2- and 3-form on the Calabi-Yau$_3$, which yield a 3-, 2, 1- and 0-form on Minkowski space.

$\mathcal{F}^I_{\mu\nu} = (\mathrm{d}A^I)_{\mu\nu}$ to the period matrix $\mathcal{N}_{IJ} = \bar{F}_{IJ}$. Under the c-map the electric-magnetic information is encoded in the scalar fields A^I and B_I, which are periodic due to the four-dimensional gauge transformations with nontrivial winding around the compactified direction. For this reason the second term of (6.1), being the image of $\mathrm{Im}\,(\mathcal{N}_{IJ}\mathcal{F}^{+I}\mathcal{F}^{+J})$ under the c-map, is interpreted as a torus whose periodicity lattice is interpreted as the lattice of dyonic charges [DJdWKV98, dWKV99]. It is to be expected that the periodicity lattice and the duality transformations upon it appear in the geometric interpretation of the hypermultiplet scalar manifold.

Calabi's ansatz

Written in complex coordinates, the Lagrangian of the hypermultiplet sector reveals the hyperkähler structure of the hypermultiplet scalar manifold.

Proposition 6.2 (Hyperkähler structure on the hypermultiplet scalar manifold). *The Lagrangian (6.1) can be written in complex coordinates as*

$$\begin{aligned}\mathscr{L}_{rc} \;=\; & -N_{IJ}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} \\ & -N^{IJ}(\partial_\mu Y_I + iN^{KL}(Y-\bar{Y})_L \partial_\mu F_{IK})(\partial^\mu \bar{Y}_J + iN^{MN}(Y-\bar{Y})_M \partial^\mu \bar{F}_{JN}),\end{aligned} \tag{6.3}$$

where the complex coordinates Y are defined in terms of the fields A and B by

$$Y_I = B_I - F_{IJ}A^J. \tag{6.4}$$

The metric associated to (6.3) is a hyperkähler metric with Kähler potential

$$K_{hyperkahler}(X,\bar{X},Y,\bar{Y}) = iX^I\bar{F}_I(\bar{X}) - i\bar{X}^I F_I(X) - \frac{1}{2}(Y-\bar{Y})_I N^{IJ}(X,\bar{X})(Y-\bar{Y})_J. \tag{6.5}$$

Proof. This result, which is valid for arbitrary $N = 2$ rigid supersymmetry theories, has been proven in [CFG89]. □

Historically the preceding proposition has given the c-map its name. [CFG89] were aware of the construction of Calabi that constructs a hyperkähler metric on the cotangent bundle T^*M of certain Kähler manifolds (M, g) [Cal79]. In a simple form *Calabi's ansatz* for the metric on the cotangent bundle of a Kähler manifold is

$$\mathrm{d}s^2_{Calabi} = g_{i\bar{j}}(z,\bar{z})\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}} + P^{i\bar{j}}(z,\bar{z})\mathrm{D}w_i\mathrm{D}\bar{w}_{\bar{j}}, \tag{6.6}$$

where the coordinates on the cotangent bundle are given by the coordinates z of the Kähler manifold and the coordinates w on the fibres. The covariant derivative of the fibre coordinates is given by

$$\mathrm{D}w_i = \mathrm{d}w_i - \Gamma^k_{ji}w_k\mathrm{d}z^j = \mathrm{d}w_i - g^{k\bar{l}}(\partial_j g_{i\bar{l}})w_k\mathrm{d}z^j.$$

Calabi showed that (6.6) defines a hyperkähler manifold if and only if $P = g^{-1}$, which is why the Kähler potential may be written as

$$K_{Calabi}(z,\bar{z},w,\bar{w}) = K^M(z,\bar{z}) + \bar{w}^t g^{-1}(z,\bar{z})w, \tag{6.7}$$

where K^M is the Kähler potential for (M, g).

A hyperkähler structure cannot be built on any Kähler manifold; it should satisfy some properties. Apparently these properties are satisfied by affine special Kähler manifolds. Proposition 6.2 shows that a slight modification of (6.7) describes the Kähler potential of a hyperkähler manifold obtained from applying the rigid c-map to the vector multiplet scalar manifold in $N = 2$ rigid supersymmetry. This *generalized Calabi ansatz* reads

$$K_{CFG}(z,\bar{z},w,\bar{w}) = K^M(z,\bar{z}) + (w+\bar{w})^t g^{-1}(z,\bar{z})(w+\bar{w}). \tag{6.8}$$

Since this Kähler potential is so similar to Calabi's work, [CFG89] coined the name *c-map* for the relation between the vector multiplet and hypermultiplet sectors of $N = 2$ supersymmetry theories.

6.2 Hyperkähler structure on a torus bundle

Lagrangian Kähler submanifolds and special Kähler geometry

In [Cor98] the generalized Calabi ansatz has been analyzed from a more mathematical point of view. Cortés was able to generalize the result to the cotangent bundle of an arbitrary affine special Kähler manifold that is embedded into a fixed vector space. In order for us to understand his approach it is necessary to give (yet) another description of affine special Kähler geometry. We start by considering a complex symplectic vector space (V, σ) with a *compatible real structure* $\tau : V \to V$.

Definition 6.3 (Complex symplectic vector space with compatible real structure). Let (V, σ) be a $2n$-dimensional complex symplectic vector space. A *real structure* τ is a $\mathbb{C}$-*antilinear involution*, i.e. $\tau(\lambda v) = \bar{\lambda}\tau(v)$ for all $\lambda \in \mathbb{C}, v \in V$ and $\tau^2 = \mathbb{I}_V$. If the restriction of σ to the fixed point set V^τ of τ is a *real* symplectic structure, we say that the real structure is compatible with the symplectic structure and (V, σ, τ) is called a *complex symplectic vector space with compatible real structure*. ⊘

For all practical purposes V may be identified with $\mathbb{C}^{2n}$ and since τ determines what we mean by complex conjugation the fixed point set V^τ is $\mathbb{R}^{2n}$.

The observation made by [Cor98] is that an affine special Kähler manifold may be described as a Lagrangian Kähler submanifold of the complex symplectic vector space with compatible real structure (V, σ, τ) in general position with respect to a Lagrangian splitting $V = L \oplus L'$ (see also [Hit99, Cor01, ACD02]). Before we can understand this statement we need to define a few concepts. First of all we note that a complex symplectic vector space with compatible real structure (V, σ, τ) defines a canonical Kähler structure on the manifold.

Definition 6.4 (Kähler metric for a complex symplectic vector space with compatible real structure). Suppose (V, σ, τ) is a $2n$-dimensional complex symplectic vector space with compatible real structure, then

$$h(u, v) = i\sigma(u, \tau v), \qquad u, v \in V,$$

defines a hermitian form of signature (n, n) on V, cf. lemma 3.19. The hermitian form h induces a Kähler metric g via (2.3). ⊘

Since $(V, \sigma, \tau) = (V, \sigma, g)$ can now be seen as a Kähler manifold, we may speak of a *Kähler submanifold* $M \subset V$. By definition this is a complex submanifold for which $h|_M$ is nondegenerate. Its Kähler potential is induced by the embedding of M into V.

Proposition 6.5 (Kähler potential of a Kähler submanifold). *Let M be a Kähler submanifold of a complex symplectic vector space with compatible real structure $(V, \sigma, \tau) = (V, \sigma, h)$. Then the function $K^M(m) = h(m, m)$, $m \in M$ is a Kähler potential for M.*

Proof. Let us assume that M is determined by the embedding $\phi : M \to V$, which is given as the graph of the function $f : \mathbb{C}^{\dim M} \to \mathbb{C}^{2n - \dim M}$,

$$M = \mathrm{graph}(f) = \{\phi(m) = (z, f(z)) \in V\} \subset V,$$

where $2n$ is the (complex) dimension of V. The hermitian *metric* of V is given by $g(u, v) = i\sigma(u, v)$ and it determines an induced metric [Nak03],

$$\begin{aligned} g^M\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{j}}}\right) &= (\phi^* g)\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{j}}}\right) = g\left(\left(\frac{\partial z}{\partial z^i}, \frac{\partial f}{\partial z^i}\right), \left(\frac{\partial \bar{z}}{\partial \bar{z}^{\bar{j}}}, \frac{\partial \bar{f}}{\partial \bar{z}^{\bar{j}}}\right)\right) \\ &= \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} h\Big((z, f(z)), (z, f(z))\Big). \end{aligned}$$

□

Next we need to explain what is meant by a *Lagrangian splitting* and what it means for a Lagrangian submanifold to be in general position with respect to it. The alternative formulation of affine special Kähler geometry then follows.

Definition 6.6 (Lagrangian splitting). Let (V, σ, τ) be a complex symplectic vector space with a compatible real structure $\tau : V \to V$. A *Lagrangian splitting* for V^τ is a decomposition

$$V^\tau = L_0 \oplus L_0'$$

into Lagrange planes of V^τ. By defining two Lagrange planes $L, L' \subset V$ via

$$\begin{array}{rclcrcl} L & = & \tau L, & \quad & L_0 & = & L^\tau, \\ L' & = & \tau L', & \quad & L_0' & = & L'^\tau, \end{array}$$

we can define the induced Lagrangian splitting for V

$$V = L \oplus L'.$$

$\oslash$

Definition 6.7 (General position). Let $M \subset V$ be a Lagrangian submanifold of a complex symplectic vector space (V, σ, τ) with a compatible real structure $\tau : V \to V$. Then it is said to be in *general position* with respect to a Lagrangian splitting $V = L \oplus L'$ if the projection $\pi_1 : V \to L$ induces an isomorphism from M onto its image. $\oslash$

Proposition 6.8 (Lagrangian submanifolds in general position). *Suppose M is a Kähler submanifold of a $2n$-dimensional complex symplectic vector space (V, σ, τ) with a compatible real structure $\tau : V \to V$. Then M is an affine special Kähler manifold if and only if it is a Lagrangian submanifold in general position to a Lagrangian splitting for V.*

Proof. For a complete proof we refer to [ACD02]. The equivalence of the general position statement and the existence of a holomorphic function is nevertheless easily understood.

Suppose the manifold M is a Lagrangian submanifold in general position. Since it is in general position, we can locally describe M by symplectic coordinates q^i and $p_i = f_i(q^1, \ldots, q^n)$, $i \in \{1, \ldots, n\}$ for some function $f : \mathbb{C}^n \to \mathbb{C}^n$.[2] Plugging in the q-dependance of the coordinates $p_i = f_i(q)$ in the standard symplectic form $\sigma = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$ and realizing that σ should be the zero form on the *Lagrangian* submanifold M, we find

$$0 = \sum_{i,j=1}^n \frac{\partial f_i}{\partial q^j} \mathrm{d}q^i \wedge \mathrm{d}q^j = \sum_{1 \le i < j \le n} \left(\frac{\partial f_i}{\partial q^j} - \frac{\partial f_j}{\partial q^i} \right) \mathrm{d}q^i \wedge \mathrm{d}q^j,$$

which means that locally

$$f_i(q) = \frac{\partial F}{\partial q^i},$$

for some (holomorphic) function $F : \mathbb{C}^n \to \mathbb{C}$.

Conversely suppose M is an affine special Kähler manifold and on every chart a holomorphic function $F(z)$ is given. The image of the exact section $\mathrm{d}F$ of $T^*\mathbb{C}^n \cong V$ can be coordinatized by

$$\mathrm{d}F(z) = \left(z^1, \ldots, z^n, \frac{\partial F}{\partial z^1}, \ldots, \frac{\partial F}{\partial z^n} \right),$$

which is an embedding in the space $V \cong T^*\mathbb{C}^n$. The image of $\mathrm{d}F$ determines a Lagrangian submanifold in general position with respect to the canonical symplectic form. $\square$

[2] Here f_i does *not* denote the first derivative of f with respect to q^i but merely the i-th component of f.

Comparison with definition 5.19 reveals that the same ingredients may be found in both definitions of affine special Kähler geometry. Both start with a Kähler manifold M having some symplectic structure placed upon it. In the present definition the construction is made explicit by assuming the Kähler manifold to be embedded into a symplectic vector space, while in definition 5.19 the symplectic structure is put upon a vector bundle over the Kähler manifold. Then both definitions assume that one half of the components of the symplectic structure depends on the other half. In proposition 6.8 this is done by assuming M to be a *Lagrangian* submanifold in general position to some Lagrangian splitting. The Lagrangian splitting is similar to the division into A- and B-periods in the context of a variation of Hodge structure of a Calabi-Yau$_3$, cf. section 5.2. The general position argument ensures that independent coordinates may be chosen entirely from one of these two pieces, while the submanifold being Lagrangian ensures that the other piece is determined from the first piece. In definition 5.19 the Lagrangian condition is made explicit by assuming condition (5.26).

The cotangent bundle of affine special Kähler manifolds

Using the characterization of affine special Kähler geometry of the previous section, we can understand how a canonical hyperkähler structure exists on the cotangent bundle T^*M of an (arbitrary) affine special Kähler manifold M.

Definition 6.9 (Canonical real structure on T^*M). Let $M \subset V$ be a Lagrangian Kähler submanifold of a complex symplectic vector space with compatible real structure (V, σ, τ) in general position with respect to a given Lagrangian splitting $V = L \oplus L'$. The projection $\pi_1 : V \to L$ then induces an isomorphism $T_m M \tilde{\to} L$, which makes it possible to define a real subset $(T_m M)^{\rho'} \subset T_m M$ by $\mathrm{d}\pi_1 (T_m M)^{\rho'} \cong L^\tau$. The canonical real structure ρ' on TM is defined as the $\mathbb{C}$-antilinear involution of $T_m M$ with fixed point set $(T_m M)^{\rho'}$. The canonical real structure ρ on T^*M is the dual of ρ'. ⊘

Theorem 6.10 (Generalized Calabi ansatz). *Let (V, σ, τ) be a $2n$-dimensional complex symplectic vector space with compatible real structure and let g be the associated Kähler metric from definition 6.4. Furthermore suppose a Lagrangian splitting $V = L \oplus L'$ is given and $M \subset (V, \sigma, g)$ is a Lagrangian Kähler submanifold in general position with respect to it. Then the function*

$$K(\xi) = K^M(\pi(\xi)) + g^{-1}_{\pi(\xi)}(\xi + \rho(\xi), \xi + \rho(\xi)), \qquad \xi \in T^*M, \tag{6.9}$$

on T^*M is the Kähler potential of a hyperkähler metric G on T^*M. In this definition K^M is the Kähler potential of M defined in proposition 6.5, $\pi : T^*M \to M$ is the canonical projection map, ρ is the canonical real structure on T^*M as defined in definition 6.9 and g_m^{-1} is the hermitian metric on T_m^*M that is the inverse of $g|_{T_m M}$. The standard complex symplectic structure Ω on the cotangent bundle T^*M is parallel with respect to the canonical connection of the Kähler manifold (T^*M, G). Moreover if the complex signature of the metric on M is (k, l), then the complex signature of G on T^*M is $(2k, 2l)$.

Proof. Let us choose symplectic coordinates of V, $\{q^i, p_i\}_{i=1}^n$ for which the symplectic form σ is in its standard form $\sigma = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$. To relate this theorem explicitly with the more physically oriented proposition 6.2, we will rewrite (6.9) in terms of the prepotential F that determines the Lagrangian submanifold M in general position. By proposition 6.8 M is the graph of the function $f : \mathbb{C}^n \to \mathbb{C}^n : q \mapsto \mathrm{grad}F(q)$ and therefore its tangent space at the point m is given by

$$T_m M = \{(t, \mathrm{D}.\mathrm{grad}F(q)t) \in T_m V | t \in L\} = \{(t^1, \dots, t^n, F_{1i}t^i, \dots, F_{ni}t^i) \in V\}.$$

Thus a tangent vector $\frac{\partial}{\partial z^i}$ may be expressed in terms of the symplectic basis $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial p_i}$ of $T_m V$ by

$$\frac{\partial}{\partial z^i} = \frac{\partial}{\partial q^i} + \sum_{j=1}^n F_{ij} \frac{\partial}{\partial p_j}. \tag{6.10}$$

From

$$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = i\sigma\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{j}}}\right) = i(\bar{F}_{ij} - F_{ij}),$$

and

$$K^M(z) = h\Big(\mathrm{d}F(z), \mathrm{d}F(z)\Big) = i\sum_{i=1}^{n} z^i \bar{F}_i - \bar{z}^i F_i,$$

we retrieve (6.8) and the proof is completed by using [CFG89]'s proof of proposition 6.2. Alternatively [Cor98] repeats their argumentation using his more mathematical approach. □

From a mathematical point of view Cortés's formulation of theorem 6.10 still obscures the generality of the statement a little bit. Using Freed's intrinsic definition of affine special Kähler geometry, the statement can be formulated more concisely [Fre99]. We will continue to use Cortés's formulation however, because it is better suited to understand the application of the generalized Calabi ansatz applied to a torus bundle.

Theorem 6.11 (Hyperkähler structure on the cotangent bundle of an affine special Kähler manifold). *The cotangent bundle T^*M of an affine special Kähler manifold (M, σ_J, ∇) carries a canonical hyperkähler structure.*

Peccei-Quinn isometries

We want to understand the rigid c-map in terms of the mathematical approach of [Cor98]. One important ingredient of the c-map which we have ignored until now, are the *continuous Peccei-Quinn isometries* associated to the underlying lattice of dyonic charges [DJdWKV98, dWKV99]. Mathematically these may be included by considering automorphisms of the hyperkähler structure on the cotangent bundle (T^*M, G). They are defined via the real subvector space $V^\tau \subset V$ and can therefore be seen to incorporate the real structure within the hyperkähler structure of the cotangent bundle, cf. (6.4). By proving that the action of V^τ on (T^*M, G) preserves the hyperkähler structure, we automatically induce a hyperkähler structure on the torus bundle T^*M/Λ, where $\Lambda \subset V^\tau$ is a lattice in V^τ.

Lemma 6.12 (Isomorphism between V^τ and $T^*_m M$). *Let (V, σ, τ) be a $2n$-dimensional complex symplectic vector space with compatible real structure and let M be a Lagrangian Kähler submanifold of V in general position with respect to some Lagrangian splitting $V = L \oplus L'$. Then for each $m \in M$, V^τ and $T^*_m M$ are isomorphic as real vector spaces. In local coordinates $v = (v^i, v_i)_{i=1}^n \in V^\tau$, we denote this isomorphism by*

$$\psi_m : V^\tau \to T^*_m M : v \mapsto \psi_m(v) = F_{ij} v^j - v_i.$$

Proof. Consider the quotient map $V \to V/T_m M$ at an arbitrary point $m \in M$ (note that we have identified V and $T_m V$ at this stage). We define the linear map $\psi_m : V/T_m M \to T^*_m M$ given by the symplectic form,

$$\psi_m([v]) = \sigma(v, \cdot).$$

First note that this is indeed a linear map and that it is well-defined: suppose $v + t$ is another representative of $[v] \in V/T_m M$, where $t \in T_m M$. Then $\sigma(v + t, s) = \sigma(v, s)$ for all $s \in T_m M$ because $T_m M$ is a Lagrange plane. Furthermore ψ_m is an injective map. Suppose $\psi_m([v])(t) = \sigma(v, t) = 0$ for all $t \in T_m M$. Since M is a Lagrange manifold this implies that $v \in T_m M$ and therefore $[v]$ is the zero element $0 = [v] \in V/T_m M$.

Next we claim that $V^\tau \cap T_m M = \{0\}$ and therefore $V^\tau \cong V^\tau/T_m M$ canonically. The inclusion $\{0\} \subset V^\tau \cap T_m M$ is trivial, so let us consider an arbitrary element $t \in V^\tau \cap T_m M$. Since M is a Lagrange submanifold of V and $t \in T_m M$,

$$\sigma(s, t) = 0, \ \forall s \in T_m M.$$

Moreover since $t \in V^\tau$, $t = \tau t$ and therefore

$$h(s,t) = i\sigma(s,\tau t) = i\sigma(s,t) = 0,\ \forall s \in T_m M.$$

By definition of a Kähler submanifold, the hermitian form h is nondegenerate on $T_m M$. This implies $t = 0$ and thus $V^\tau \cap T_m M \subset \{0\}$.

Combining these results we find that

$$V^\tau \cong V^\tau / T_m M \subset V/T_m M \xrightarrow{\sim} T_m^* M.$$

By noting that the real dimension of $\dim_{\mathbb{R}} V^\tau = 2n = \dim_{\mathbb{R}} T_m^* M$, we have established a linear isomorphism between two real vector spaces V^τ and $T_m^* M$.

Finally let us derive an explicit expression for $\psi_m(v)$. In local coordinates we write $v \in V^\tau$ as $v = (v^i, v_i)_{i=1}^n$, where the splitting is made with respect to the (real) Lagrangian splitting $V^\tau = L_0 \oplus L_0'$. We know what element $\psi_m(v) \in T_m^* M$ is when we know how it acts on an arbitrary element $t = t^i \frac{\partial}{\partial z^i} \in T_m M$. We follow the definition of $\psi_m(v)$ and use (6.10),

$$\psi_m(v)(t) = \sigma(v,t) = -\sum_{i=1}^n v_i t^i + \sum_{j=1}^n v^i F_{ij} t^j = -\sum_{i=1}^n \left(v_i - \sum_{j=1}^n F_{ij} v^j \right) t^i. \qquad \square$$

Proposition 6.13 (Automorphic action of the hyperkähler structure). *In the situation of lemma 6.12, the action of V^τ on T^*M defined by*

$$v : T_m^* M \to T_m^* M : \xi \mapsto \xi + i\psi_m(v), \tag{6.11}$$

is a fibre preserving action on T^*M and simply transitive on each fibre. It is an automorphism of the hyperkähler structure of (T^*M, G), which means it is a holomorphic isometry, preserving the complex symplectic structure Ω.

Proof. We denote the local coordinates of T^*M by $(z^i, w_i)_{i=1}^n$. In terms of these coordinates the action of v on T^*M is

$$\tilde{z}^i = z^i, \qquad \tilde{w}_i = w_i - i(v_i - \sum_{j=1}^n F_{ij} v^j).$$

From this we see that the action is (by definition) fibre preserving. The preservation of $\Omega = \sum_i \mathrm{d}z^i \wedge \mathrm{d}w_i$ and G may be checked by straightforward calculation (note that G is determined by (6.9)). The simple transitivity of the action on the fibres follows from the fact that ψ_m provides a real isomorphism between the vector spaces V^τ and $T_m^* M$. $\square$

Since the hyperkähler structure is unaffected by the action (6.11) of V^τ on the cotangent bundle, it induces a hyperkähler structure on the equivalence classes of the cotangent bundle modulo a lattice $\Lambda \subset V^\tau$. Physically this quotient space is of interest because of the periodicity of the fibre coordinates due to the unbroken integer Peccei-Quinn symmetry at the nonperturbative level.

Theorem 6.14 (Hyperkähler structure on the torus bundle). *Let (V, σ, τ) be a complex symplectic vector space with compatible real structure and let M be a Lagrangian Kähler submanifold of V in general position to some Lagrangian splitting $V = L \oplus L'$. Then the hyperkähler structure on T^*M constructed in theorem 6.10 induces a hyperkähler structure on the torus bundle T^*M/Λ for any lattice $\Lambda \subset V^\tau$.*

Proof. First we need to explain what we mean by T^*M/Λ. Let $\Lambda \subset V^\tau$ be a lattice. Since $\Lambda \cap T_m M \subset V^\tau \cap T_m M = \{0\}$, the lattice Λ projects to a lattice $[\Lambda]$ in $V/T_m M$. Under the isomorphism induced by σ, cf. lemma 6.12, $V/T_m M$ and $T_m^* M$ are identified and therefore $[\Lambda]$ corresponds to a lattice $\psi_m(\Lambda) \subset T_m^* M$. We denote with T^*M/Λ the orbit space of the action (6.11) of $\Lambda \subset V^\tau$ on T^*M with fibres $T_m^* M/\psi_m(\Lambda)$.

Proposition 6.13 guarantees that V^τ defines an action that preserves the hyperkähler structure on (T^*M, G), which means that division by the action of a lattice $\Lambda \subset V^\tau$ preserves the hyperkähler structure as well. Therefore the hyperkähler structure on (T^*M, G) is translated to the same hyperkähler structure on T^*M/Λ [Cor98]. $\square$

6.3 The bundle of Griffiths intermediate Jacobians

Gauged Calabi-Yau$_3$'s

Having developed the necessary formalism we will now apply theorem 6.10 and 6.14 to the moduli space of complex structure deformations of a family of Calabi-Yau$_3$ manifolds. This will put us back into the original physical context of the rigid c-map from IIB to IIA. For remember: the moduli space of complex structure deformations forms the type IIB vector multiplet moduli space. Investigating the moduli space of complex structure deformations and the canonical hyperkähler structure on its cotangent space, means that we are analyzing a fibre bundle on the vector multiplet moduli space of type IIB. Mathematically the complex structure deformations will be studied by focusing on the Calabi-Yau$_3$'s third cohomology group, due to the article [BG83] on the variation of Hodge structure, cf. section 5.2. Note that we will consider here the affine special Kähler geometry associated to a homogenization of the complex structure moduli space and then taking all coordinates independent. That is the particular *value* of the holomorphic 3-form *is* important and will not be gauged away. The prepotential F of the affine special Kähler manifold will still be homogeneous of degree 2.

Definition 6.15 (Moduli space of gauged Calabi-Yau$_3$'s). Let CY be an arbitrary Calabi-Yau$_3$ manifold and consider its moduli space $\mathscr{M}^{2,1}$ of complex structure deformations, which can locally be identified with $\mathscr{M}^{2,1} \cong H^{2,1}(CY)$. We denote with $\mathscr{CY} = (CY_t)_{t\in\mathscr{M}^{2,1}} \to \mathscr{M}^{2,1}$, $CY_0 = CY$ the corresponding deformation of complex structure. With $H^{3,0}(\mathscr{CY}) - \mathscr{M}^{2,1}$ we denote the $\mathbb{C}^*$-bundle over $\mathscr{M}^{2,1}$ which is obtained from the complex line bundle $H^{3,0}(\mathscr{CY})$ by removing the zero section $\mathscr{M}^{2,1} \ni t \mapsto 0 \in H^{3,0}(CY_t)$. It is the *moduli space of gauged Calabi-Yau$_3$ manifolds* (CY_t, vol_t), where vol_t denotes the holomorphic volume form $vol_t \in H^{3,0}(CY_t) - 0$, $t \in \mathscr{M}^{2,1}$. ⊘

Lemma 6.16 (Alternative description of the moduli space of gauged Calabi-Yau$_3$'s). *Define the* period map

$$Per : \mathscr{M}^{2,1} \to \mathbb{P}(H^3(CY,\mathbb{C})) : t \mapsto H^{3,0}(CY_t).$$

The cone $\mathscr{M}^{2,1}_{gauged} = \bigcup_{t\in\mathscr{M}^{2,1}} Per(t) - \{0\} \subset H^3(CY,\mathbb{C})$ *over* $Per(\mathscr{M}^{2,1}) = \mathbb{P}(\mathscr{M}^{2,1}_{gauged})$ *is canonically identified with the moduli space* $H^{3,0}(\mathscr{CY}) - \mathscr{M}^{2,1}$ *of gauged Calabi-Yau$_3$ manifolds.*

Proof. The period map is well-defined since the moduli space $\mathscr{M}^{2,1}$ is local, i.e. we may assume that the bundle $H^3(\mathscr{CY},\mathbb{C}) \to \mathscr{M}^{2,1}$ is trivial. In particular the different fibres can be canonically identified with the typical fibre, $H^3(CY_t,\mathbb{Z}) \cong H^3(CY,\mathbb{Z})$ and $H^3(CY_t,\mathbb{C}) \cong H^3(CY,\mathbb{C})$. Moreover by Kodaira, Spencer and Griffiths's deformation theory [BG83, Cor98], the deformation of complex structure shifts $H^{3,0}(CY_t)$ at most into $H^{2,1}(CY_t)$, cf. section 5.2. In particular

$$\mathrm{d}Per(T_t\mathscr{M}^{2,1}) = \mathrm{d}\pi(H^{3,0}(CY_t) \oplus H^{2,1}(CY_t)),$$

where $\pi : H^3(CY,\mathbb{C}) \to \mathbb{P}(H^3(CY,\mathbb{C}))$ is the canonical projection. Looking at the dimensions at both sides, $\mathrm{d}Per$ is seen to be injective and therefore the period map is an immersion. Since we are only considering local deformations we may promote Per to an isomorphism $\mathscr{M}^{2,1} \cong Per(\mathscr{M}^{2,1}) \subset \mathbb{P}(H^3(CY,\mathbb{C}))$ [Cor98]. □

Thus the isomorphism Per makes precise the idea that a deformation of complex structure of a Calabi-Yau$_3$ manifold changes the direction of $H^{3,0}(CY_t)$ in a fixed $H^3(CY,\mathbb{C})$. We have pinpointed one version of the third cohomology group $H^3(CY,\mathbb{C})$ and compare the direction of $H^{3,0}(CY_t)$ in this vector space. A change of complex structure changes the direction. Conversely a change of the direction indicates that we have deformed the complex structure, since the direction $H^{3,0}(CY_t)$ explains what we mean with “holomorphic”. If we *do* care about the particular *value* of $vol_t \in H^{3,0}(CY_t)$ instead of just the direction of the line spanned by it, we obtain the moduli space of gauged Calabi-Yau$_3$ manifolds. For each point $t \in \mathscr{M}^{2,1}$ we have a Calabi-Yau$_3$ CY_t and possible values for $vol_t \in H^{3,0}(CY_t)$. The moduli space of gauged Calabi-Yau$_3$'s is therefore a cone over the moduli space of ordinary Calabi-Yau$_3$ manifolds.

We will now apply proposition 6.8 to the Lagrangian Kähler submanifold $\mathscr{M}^{2,1}_{gauged}$ of the complex symplectic vector space with compatible real structure $H^3(CY,\mathbb{C})$. The real structure is such that $(H^3(CY,\mathbb{C}))^\tau = H^3(CY,\mathbb{R})$ and the symplectic form is the *intersection form* Q defined in definition 3.15. Not surprisingly we may conclude that $\mathscr{M}^{2,1}_{gauged}$ is an affine special Kähler manifold when it is in general position with respect to a certain Lagrangian splitting of $H^3(CY,\mathbb{R})$.

Lemma 6.17 ($\mathscr{M}^{2,1}_{gauged}$ is a Lagrangian Kähler submanifold of $H^3(CY,\mathbb{C})$). *Let CY be a Calabi-Yau$_3$ manifold and consider $H^3(CY,\mathbb{C})$. Let Q be the intersection form and let τ be the standard real structure on $H^3(CY,\mathbb{C})$, i.e. $(H^3(CY,\mathbb{C}))^\tau = H^3(CY,\mathbb{R})$. $\mathscr{M}^{2,1}_{gauged} \subset H^3(CY,\mathbb{C})$ is a Lagrangian Kähler submanifold of the complex symplectic vector space with compatible real structure $(H^3(CY,\mathbb{C}),Q,\tau)$. The complex signature of the Kähler metric of $\mathscr{M}^{2,1}_{gauged}$ is $(h^{2,1},1)$.*

Proof. Let $m \in \mathscr{M}^{2,1}_{gauged}$ be an element of the moduli space of gauged Calabi-Yau$_3$ manifolds, such that $m \in Per(t) - \{0\}$. By Kodaira, Spencer and Griffiths's deformation theory the tangent space of $\mathscr{M}^{2,1}_{gauged}$ at $m \in \mathscr{M}^{2,1}_{gauged}$ is

$$T_m\mathscr{M}^{2,1}_{gauged} = H^{3,0}(CY_t) \oplus H^{2,1}(CY_t). \tag{6.12}$$

Restricted to this subspace of $H^3(CY,\mathbb{C})$, the Hodge-Riemann bilinear relations 3.17 ensure that $T_m\mathscr{M}^{2,1}_{gauged}$ is a Lagrange plane for any $m \in \mathscr{M}^{2,1}_{gauged}$ and that $\mathscr{M}^{2,1}_{gauged}$ is a Kähler submanifold whose metric has signature $(h^{2,1},1)$. □

We have now completely translated the variation of Hodge structure of a Calabi-Yau$_3$ manifold, i.e. the deformation of complex structure of gauged Calabi-Yau$_3$ manifolds, in terms of the Lagrangian Kähler submanifolds of theorems 6.10 and 6.14. A straightforward application yields the following geometric interpretation of the image of the rigid c-map from IIB to IIA.

Theorem 6.18 (Hyperkähler structure on the bundle of Griffiths intermediate Jacobians). *Let CY be a Calabi-Yau$_3$ manifold and suppose $H^3(CY,\mathbb{R}) = L_0 \oplus L_0'$ is a Lagrangian splitting with respect to the intersection form Q such that the moduli space of gauged Calabi-Yau$_3$ manifolds $\mathscr{M}^{2,1}_{gauged}$ is in general position with respect to it. Then there is a hyperkähler structure on the bundle of Griffiths intermediate Jacobians $\mathscr{J}_G \to \mathscr{M}^{2,1}_{gauged}$ with complex signature $(2h^{2,1},2)$, cf. definition 3.23.*

Proof. By lemma 6.17 the moduli space of gauged Calabi-Yau$_3$'s $\mathscr{M}^{2,1}_{gauged}$ is a Lagrangian Kähler submanifold of the complex symplectic vector space $(H^3(CY,\mathbb{C}),Q)$ with compatible real structure. We have assumed $\mathscr{M}^{2,1}_{gauged}$ to be in general position with respect to the Lagrangian splitting induced by $H^3(CY,\mathbb{R}) = L_0 \oplus L_0'$. By theorem 6.10 a hyperkähler structure with complex signature $(2h^{2,1},2)$ is then induced on the cotangent bundle $T^*\mathscr{M}^{2,1}_{gauged}$.

Using the isomorphism induced by Q, cf. lemma 6.12, the fibre at $m \in Per(t) - \{0\} \subset \mathscr{M}^{2,1}_{gauged}$ of $T^*\mathscr{M}^{2,1}_{gauged}$ is identified with

$$T^*_m\mathscr{M}^{2,1}_{gauged} \cong \frac{H^3(CY,\mathbb{C})}{T_m\mathscr{M}^{2,1}_{gauged}} = \frac{H^3(CY,\mathbb{C})}{H^{3,0}(CY_t)\oplus H^{2,1}(CY_t)}.$$

Furthermore the isomorphism of lemma 6.12 induces an embedding of the lattice $H^3(CY,\mathbb{Z}) \subset H^3(CY,\mathbb{R})$ into $T^*_m\mathscr{M}^{2,1}_{gauged}$. By theorem 6.14 the quotient of $T^*\mathscr{M}^{2,1}_{gauged}$ by the group action of $H^3(CY,\mathbb{Z}) \subset H^3(CY,\mathbb{R})$ is endowed with a hyperkähler structure induced from $T^*\mathscr{M}^{2,1}_{gauged}$. In our case under the isomorphism of lemma 6.12 that embeds $H^3(CY,\mathbb{Z})$ into $\frac{H^3(CY,\mathbb{C})}{H^{3,0}(CY_t)\oplus H^{2,1}(CY_t)}$, we have constructed a hyperkähler structure on the bundle of Griffiths intermediate Jacobians, cf. corollary 3.22 and definition 3.23. The fibre at a point $m \in Per(t) - \{0\} \subset \mathscr{M}^{2,1}_{gauged}$ is given by the Griffiths intermediate Jacobian of the Calabi-Yau$_3$ CY_t,

$$\mathcal{J}_G(CY_t) = \frac{H^3(CY,\mathbb{C})}{H^{3,0}(CY_t)\oplus H^{2,1}(CY_t)\oplus H^3(CY,\mathbb{Z})}.$$

□

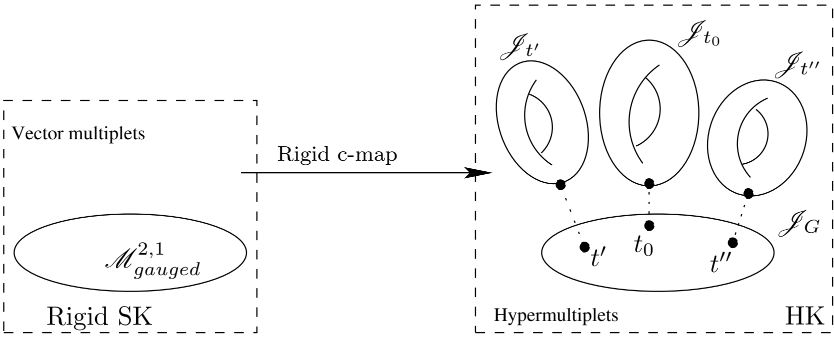

Figure 6.1: Geometrically we can understand the rigid c-map from type IIB to type IIA as an extension of the moduli space of gauged Calabi-Yau$_3$ manifolds $\mathscr{M}^{2,1}_{gauged}$ with the Griffiths intermediate Jacobians $\mathcal{J}_G$. At each point $m \in Per(t) - \{0\}$ of the moduli space, a number of extra degrees of freedom is incorporated by the Griffiths intermediate Jacobian $(\mathscr{J}_G)_t \cong \mathcal{J}_G(CY_t)$ associated to that point m in the moduli space. The total bundle $\mathscr{J}_G$ (having typical fibre $\mathcal{J}_G(CY)$) is the bundle of Griffiths intermediate Jacobians and is considered to be a more complete moduli space of the string theoretical moduli space. It is the hyperkähler manifold parameterized by the hypermultiplet moduli in the $N = 2$ rigid supersymmetric effective type IIA theory.

Metric on the bundle of Griffiths intermediate Jacobians

Following the strain of reasoning of the previous section, the rigid c-map in four-dimensional type II superstring theoretical context may be interpreted as a construction of the bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$'s, cf. figure 6.1. For this reason we suggest to rename the rigid c-map to *cG-map*, where the G stands for Griffiths. To make the relation with the hyperkähler Lagrangian (6.1) complete, we will construct a canonical metric on the fibres of this bundle, i.e. on the Griffiths intermediate Jacobians. Since these are *tori*, the formalism developed in section 3.1 can be put to the test. In lemma 3.13 we have given a method of defining a canonical metric on a nondegenerate complex torus in terms of its period matrix. Thus finding the period matrix of the Griffiths intermediate Jacobian of a particular member of the family of Calabi-Yau$_3$'s, automatically yields the metric on that fibre in the bundle of intermediate Jacobians.

Proposition 6.19 (Period matrices for the Griffiths intermediate Jacobians). *Let $\mathscr{CY} = (CY_t)_{t\in\mathscr{M}^{2,1}} \to \mathscr{M}^{2,1}$, $CY_0 = CY$ be a family of complex structure deformations of a Calabi-Yau$_3$ manifold CY and let $\mathscr{M}^{2,1}_{gauged}$ be the moduli space of gauged Calabi-Yau$_3$'s, in general position to a Lagrangian splitting of $H^3(CY,\mathbb{R})$. Then the period matrix Π^G_t of the Griffiths intermediate Jacobian of the gauged Calabi-Yau$_3$ manifold $(CY_t, vol_t) \in \mathscr{M}^{2,1}_{gauged}$ is given by*

$$\Pi^G_t = \begin{pmatrix} \mathbb{I}_{h^{2,1}+1} \\ \mathrm{T}_t \end{pmatrix}, \tag{6.13a}$$

where $\bar{\mathrm{T}}_t \in \mathrm{Mat}(h^{2,1}+1,\mathbb{C})$ is given by the Hessian of a holomorphic function $F : \mathbb{C}^{h^{2,1}+1} \to \mathbb{C} : x \mapsto F(x)$ which is homogeneous of degree 2,

$$(\bar{\mathrm{T}}_t)_{ij} = \frac{\partial^2 F(x(t))}{\partial x^i \partial x^j} = F_{ij}|_t, \qquad i,j \in \{0,\ldots,h^{2,1}\}. \tag{6.13b}$$

In the basis $\{\alpha_i,\beta^i\}_{i=0}^{h^{2,1}} \subset H^3(CY,\mathbb{R})$ corresponding to the Lagrangian splitting of $H^3(CY,\mathbb{R})$, the holomorphic function is the prepotential of the affine special Kähler manifold $\mathscr{M}^{2,1}_{gauged}$, which is homogeneous of degree 2.

Proof. Let $m \in Per(t) - \{0\}$ be an element in $\mathscr{M}^{2,1}_{gauged}$ and consider a Lagrangian splitting of $H^3(CY,\mathbb{R})$, such that $\mathscr{M}^{2,1}_{gauged}$ is in general position to it. This corresponds to a choice of a symplectic basis $\{\alpha_i, \beta^i\}_{i=0}^{h^{2,1}} \subset H^3(CY, \mathbb{R})$ with respect to the intersection form, $Q(\alpha_i, \beta^j) = \delta_i^j$, such that the coordinates of any element $T_m\mathscr{M}^{2,1}_{gauged} \ni \omega = \sum_i q^i\alpha_i - p_i\beta^i$ are dependent in the following sense,

$$p_i = p_i(q^0, \ldots, q^{h^{2,1}}) = \frac{\partial F(q)}{\partial q^i}, \qquad 0 \le i \le h^{2,1},$$

where F is the prepotential of the affine special Kähler manifold $\mathscr{M}^{2,1}_{gauged}$ for the symplectic basis $\{\alpha_i, \beta^i\}_{i=0}^{h^{2,1}}$. The period matrix Π_t^G of the complex torus $\mathscr{J}_G(CY_t) \cong T_m^*\mathscr{M}^{2,1}_{gauged}/H^3(CY, \mathbb{Z})$ is found by expressing a basis of $T_m^*\mathscr{M}^{2,1}_{gauged}$ in terms of this basis $\{\alpha_i, \beta^i\}$ of $H^3(CY, \mathbb{Z})$. To this end consider the element $vol_t \in H^{3,0}(CY_t)$ written with respect to the $\{\alpha_i, \beta^i\}$-basis,

$$vol_t = \sum_{i=0}^{h^{2,1}} x^i(t)\alpha_i - w_i(t)\beta^i$$

and define

$$\omega_i(t) = \frac{\partial vol_t}{\partial x^i(t)}, \qquad i \in \{0, \ldots, h^{2,1}\}.$$

Since $\mathscr{M}^{2,1}_{gauged}$ is in general position with respect to the symplectic basis $\{\alpha_i, \beta^i\}_{i=0}^{h^{2,1}}$, we find

$$\omega_i(t) = \alpha_i - \sum_{j=0}^{h^{2,1}} \frac{\partial}{\partial x^i(t)} w_j(t)\beta^j = \alpha_i - \sum_{j=0}^{h^{2,1}} \frac{\partial^2 F(x(t))}{\partial x^i \partial x^j}\beta^j.$$

We see that the $h^{2,1} + 1$ ω_i's are all independent and are all an element of $T_m\mathscr{M}^{2,1}_{gauged}$ by Griffiths transversality. We take these to be our basis for $T_m\mathscr{M}^{2,1}_{gauged}$. By (6.12) and the isomorphism induced by the intersection form Q, cf. lemma 6.12, the basis of $T_m^*\mathscr{M}^{2,1}_{gauged} \cong H^3(CY, \mathbb{C})/T_m\mathscr{M}^{2,1}_{gauged}$ is the complex conjugate of this. The period matrix thus reads

$$\Pi_t^G = \begin{pmatrix} \mathbb{I}_{h^{2,1}+1} \\ \mathrm{T}_t \end{pmatrix}, \qquad \text{where} \quad (\bar{\mathrm{T}}_t)_{ij} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} F(x(t)), \quad i, j \in \{0, \ldots, h^{2,1}\}.$$

We can explicitly verify that the function $F : \mathbb{C}^{h^{2,1}+1} \to \mathbb{C}$ is homogeneous of degree 2, because

$$0 = Q(vol_t, \omega_i(t)) = \sum_{j=0}^{h^{2,1}} x^j(t)\frac{\partial^2}{\partial x^j \partial x^i} F(x(t)) - \frac{\partial}{\partial x^i} F(x(t)), \qquad i \in \{0, \ldots, h^{2,1}\},$$

which can be integrated to yield $2F = \sum_{j=0}^{h^{2,1}} x^j \frac{\partial}{\partial x^j} F$. □

We have found the period matrix of each fibre $\mathscr{J}_G(CY_t)$ over the base point $m \in Per(t) - \{0\}$. The second part of each period matrix is the complex conjugate of the Hessian of the prepotential evaluated at the corresponding base point t in the moduli space of complex structure deformations $\mathscr{M}^{2,1} \cong \mathbb{P}(\mathscr{M}^{2,1}_{gauged})$. Application of lemma 3.13 to the Griffiths intermediate Jacobian of CY_t implies that the metric on the fibres of the bundle of Griffiths intermediate Jacobians may be given by

$$\mathrm{d}s^2_{fibre} = -(\mathrm{d}B_I - F_{IK}\mathrm{d}A^K)(-2)N^{IJ}(\mathrm{d}B_J - \bar{F}_{JL}\mathrm{d}A^L),$$

where A and B are coordinates of the real extension of the lattice $H^3(CY, \mathbb{Z})$ with respect to the basis $\{\alpha_I, \beta^I\}_{I=0}^{h^{2,1}}$. The factor 2 may be incorporated into a rescaling of these coordinates. If we identify the A-periods of the holomorphic form vol_t as the moduli $X^I = \int_{\gamma_{A^I}} vol_t \in \mathscr{M}^{2,1}_{gauged}$ of gauged Calabi-Yau$_3$'s, the metric on the base space $\mathscr{M}^{2,1}_{gauged}$ retrieves its standard form (5.1) in special coordinates,

$$\mathrm{d}s^2_{base} = N_{IJ}\mathrm{d}X^I\mathrm{d}\bar{X}^{\bar{J}}.$$

Combining the metric on the base space and the fibres in a direct sum, we have found a canonical metric on the bundle of Griffiths intermediate Jacobians,

$$\mathrm{d}s^2_{\mathscr{J}_G} = N_{IJ}\mathrm{d}X^I\mathrm{d}\bar{X}^{\bar{J}} + N^{IJ}(\mathrm{d}B_I - F_{IK}\mathrm{d}A^K)(\mathrm{d}B_J - \bar{F}_{JL}\mathrm{d}A^L). \tag{6.14}$$

Remark: In most of the physics literature [Fré96, CRTVP97, DJdWKV98, dWKV99, Asp00], the rigid c-map is identified with a fibration over an affine special Kähler manifold, whose fibres are Jacobians of (deformations of) an auxiliary *Seiberg-Witten curve*. The Seiberg-Witten curve is a Riemann surface which has been introduced in the context of *Seiberg-Witten theory* for any $N = 2$ supersymmetric gauge theory [SW94a, SW94b, Pes97, Ler97]. Although our result comes about more naturally in the context of string theory and compactifications on Calabi-Yau$_3$'s, the precise relation between the Jacobian of the Seiberg-Witten curve and the Griffiths intermediate Jacobian of the Calabi-Yau$_3$ is at the moment not clear to us. One important feature is that the affine special Kähler manifold in general rigid supersymmetry need not have a homogeneous prepotential, while we consider a conformal four-dimensional type IIB vector multiplet theory with homogeneous prepotential, cf. section 6.1. ♢

The elliptic modular surface

The approach of [Cor98] provides insight in the underlying structures of the c-map's construction. The variation of complex structure of the family of Calabi-Yau$_3$'s is combined with the underlying symplectic structure of electric-magnetic duality of the vectors in the vector multiplets. The cG-map takes both ingredients into account by constructing the bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$ manifolds. The Griffiths intermediate Jacobian measures the change in complex structure compared to a fixed symplectic structure of $H^3(CY, \mathbb{C})$.

Mathematically a parallel can be made with the *elliptic modular surface* of the Jacobian of an *elliptic curve*, a smooth projective algebraic curve of genus 1. Consider an elliptic curve $\mathcal{E}$ and its first cohomology group $H^1(\mathcal{E}, \mathbb{C}) = H^{1,0}(\mathcal{E}) \oplus H^{0,1}(\mathcal{E})$. The Jacobian of the elliptic curve measures the position of $H^{1,0}(\mathcal{E})$ in $H^1(\mathcal{E}, \mathbb{C})$ relative to the fixed position of the lattice $H^1(\mathcal{E}, \mathbb{Z})$. In slightly different terms, the line $H^{1,0}(\mathcal{E})$ in $H^1(\mathcal{E}, \mathbb{C})$ is a Lagrange plane with respect to the intersection form on $H^1(\mathcal{E}, \mathbb{C})$ and the Jacobian may be seen to be the quotient with respect to this Lagrange plane, $H^1(\mathcal{E}, \mathbb{C})/H^{1,0}(\mathcal{E})$, modulo the lattice $H^1(\mathcal{E}, \mathbb{Z})$. The direction of the line $H^{1,0}(\mathcal{E})$ is given by a complex number $\tau \in \mathscr{H} = \{\tau \in \mathbb{C} | \mathrm{Im}\, \tau > 0\}$, which is the (second half of the) period "matrix" of the torus. The symplectic structure becomes evident by the $\mathrm{SL}(2, \mathbb{R})$-action on $\mathscr{H}$ [Lan75], which allows us to write $\mathscr{H} \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$. The actual moduli space of possible Jacobians is $\mathbb{P}^1 \cong \mathscr{H}/\mathrm{SL}(2, \mathbb{Z})$, since basis transformations of the lattice that keep the intersection form invariant do not change the relative position of the line and the lattice.[3] The *elliptic modular surface* is defined by taking a tautological bundle of Jacobians over the moduli space of Jacobians $\mathcal{EMS} \to \mathbb{P}^1$. The fibres of this bundle are exactly the Jacobians corresponding to the value of $\tau \in \mathbb{P}^1$. $\mathcal{EMS}$ is locally isomorphic to $(\mathrm{SL}(2, \mathbb{Z}) \oplus \mathbb{Z}^2)\backslash(\mathrm{SL}(2, \mathbb{R}) \times \mathbb{C})/\mathrm{U}(1)$, where $\mathbb{Z}^2$ divides by the periods of the lattice $H^1(\mathcal{E}, \mathbb{Z})$. Its group action on $\mathbb{C}$ depends on $\mathrm{SL}(2, \mathbb{R})$ via τ, since τ determines the lattice direction,

$$(k, l) \cdot z = z + k + l\tau. \tag{6.15}$$

As a result the fibre bundle is nontrivial.

The bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$'s is similar to the elliptic modular surface. This time we consider a three-dimensional Calabi-Yau manifold CY and its *third* cohomology group $H^3(CY, \mathbb{C})$. The moduli space of gauged Calabi-Yau$_3$'s $\mathscr{M}^{2,1}_{gauged}$ defines a Lagrange plane $L_t = H^{3,0}(CY_t) \oplus H^{2,1}(CY_t) \subset H^3(CY, \mathbb{C})$ (with respect to the intersection form Q) for each point in the moduli space. We thus obtain a tautological bundle over $\mathscr{M}^{2,1}_{gauged}$ which has $H^3(CY, \mathbb{C})/L_t$ as fibres. Taking an exponential map we consider

[3] There are some subtleties here since $\mathscr{H}/\mathrm{SL}(2, \mathbb{Z})$ is actually isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, but we will not go into these.

the embedding of L_t into $H^3(CY,\mathbb{C}^*) \cong H^3(CY,\mathbb{C})/H^3(CY,\mathbb{Z})$, i.e. we consider the compact analogue of the tautological bundle. The bundle of Griffiths intermediate Jacobians is then the bundle of $H^3(CY,\mathbb{C}^*)/L_t$ over the moduli space of gauged Calabi-Yau$_3$'s; locally

$$\mathscr{J}_G \cong \mathscr{M}^{2,1}_{gauged} \times \frac{H^3(CY,\mathbb{C})}{L_t \oplus H^3(CY,\mathbb{Z})}.$$

The analogy with the elliptic modular surface is given by the interpretation of $\mathscr{M}^{2,1}_{gauged}$. By the work of [BG83] the complex structure deformations of a family of Calabi-Yau$_3$'s can be studied in terms of the third cohomology group $H^3(CY,\mathbb{C})$. The position of $H^{3,0}(CY_t)$ in $H^3(CY,\mathbb{Z})\otimes\mathbb{C}$ determines the complex structure and vice versa. In our case the moduli in $\mathscr{M}^{2,1}_{gauged}$ describe the slightly coarser information of the position of Lagrange planes L_t in $H^3(CY,\mathbb{C})$. Since the positions of Lagrange planes are given by the period matrices of the Griffiths intermediate Jacobians, $\mathscr{M}^{2,1}_{gauged}$ gets the interpretation of the moduli space of period matrices for the Griffiths intermediate Jacobians, cf. section 3.1,

$$\mathscr{M}^{2,1}_{gauged} \cong \mathrm{Sp}(2h^{2,1}+2,\mathbb{Z})\backslash\mathrm{Sp}(2h^{2,1}+2,\mathbb{R})/\mathrm{U}(h^{2,1},1). \tag{6.16}$$

As a result the bundle of Griffiths intermediate Jacobians is locally of the form

$$\mathscr{J}_G \cong (\mathrm{Sp}(2h^{2,1}+2,\mathbb{Z}) \oplus \mathbb{Z}^{2h^{2,1}+2})\backslash(\mathrm{Sp}(2h^{2,1}+2,\mathbb{R}) \times \mathbb{C}^{h^{2,1}+1})/\mathrm{U}(h^{2,1},1). \tag{6.17}$$

The cG-map describes T-duality in $N=2$ rigid supersymmetry. In rigid supersymmetry the vector multiplet's scalar manifold is the moduli space of *gauged* Calabi-Yau$_3$'s and not just its complex structure deformations. In order to describe the actual moduli space of complex structure deformations, the cG-map is too rigid. It is not refined enough to actually recognize which part of the Lagrange plane L_t is $H^{3,0}(CY)$ and which is $H^{2,1}(CY)$, the *true* moduli space of complex structure deformations $H^{2,1}(CY) \cong \mathscr{M}^{2,1}$. Including such a refinement changes the bundle construction to one that describes a tautological bundle on the actual complex structure moduli space. This construction is known as the local c-map, which will be the subject of the next chapter.

This concludes our analysis of the rigid c-map, or cG-map, in $N=2$ rigid supersymmetric type II theory. Geometrically the moduli space of gauged Calabi-Yau$_3$'s is extended with the Griffiths intermediate Jacobians $\mathcal{J}_G(CY_t)$ of the Calabi-Yau$_3$'s. In this way the electric-magnetic structure is translated into a symplectic structure of the third cohomology group of the Calabi-Yau$_3$. The total information of the vector multiplet sector at the IIB side consists of the complex structure deformations of the Calabi-Yau$_3$ *and* the symplectic duality of the vectors in the vector multiplets. At the type IIA side the symplectic structure is accounted for by the symplectic structure of $(H^3(CY,\mathbb{C}),Q)$, coincidentally the cohomology group that describes the Calabi-Yau$_3$'s complex structure by [BG83]. The rigid c-map can be understood as the incorporation of the Griffiths intermediate Jacobians into the total moduli space to ensure that the interplay between complex and symplectic structure of $H^3(CY,\mathbb{C})$ is not lost. The bundle of Griffiths intermediate Jacobians that arises in this way is understood to be the hypermultiplet moduli space of the total type IIA string theoretical moduli space in $N=2$ rigid supersymmetry.

Chapter 7

A geometric interpretation of the local c-map

The local c-map expresses the full effect of T-duality on the gravity, vector and hypermultiplet moduli spaces of four-dimensional effective type II string theories. Whereas the rigid c-map is a gravitationless simplification of T-duality that only relates the moduli spaces of the super*symmetry* theories, the local c-map includes gravity, thereby relating the supergravity theories. In this chapter we will investigate the local c-map and we will try to find a similar geometric interpretation as for the rigid c-map. In the literature a bundle of intermediate Jacobians is again anticipated for the local c-map [GMV96, Asp98], although precise statements lack when they are suggested. We will analyze the local c-map with the previous chapter in our mind. Led by the construction there, we will find that intermediate Jacobians play again an important role and we will make precise the manner in which they do. Moreover since the local c-map incorporates gravity, the total moduli space is expected to depend on gravitational input, rather than just geometric input of the Calabi-Yau$_3$'s upon which the ten-dimensional type II theories have been compactified. We will consider the relation between the gravitational and Calabi-Yau$_3$ contributions and will then try to sew all contributions together in a seamless fibration over the complex structure moduli space.

This chapter is organized as follows: first we will consider the local c-map in physics, which gives us an idea about the different protagonists in our story. Then we will consider the local c-map's equivalent of the bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$'s. After we have investigated the structure of the gravitational contribution in combination with the (R-R) fields, we conclude with a total description of the type IIA hypermultiplet moduli space in terms of a fibration over the moduli space of complex structure deformations.

7.1 The local c-map in physics

Constructing a hypermultiplet Lagrangian

We consider the type IIB gravity and vector multiplet theory whose bosonic part is given by the Lagrangian, cf. (4.16),

$$\mathscr{L}^{sugra}_{vm} = \sqrt{g}R_4 + \frac{\partial^2 K^{(2,1)}(X,\bar{X})}{\partial X^I \partial \bar{X}^{\bar{J}}}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} + \frac{1}{2}\mathrm{Im}\,(\mathcal{N}^{(2,1)}_{IJ}\mathcal{F}^{+I}_{\mu\nu}\mathcal{F}^{+J\mu\nu}), \tag{5.16}$$

where the coordinates $X = (X^I)^{h^{2,1}}_{I=0}$ are projective coordinates for the complex structure moduli space $\mathscr{M}^{2,1}$ and the $\mathcal{F}^I$ are the field strengths of the gauge vectors, $\mathcal{F}^I = \mathrm{d}A^I$. Note that the zeroth vector A^0_μ is the graviphoton belong to the gravity multiplet. The projective special Kähler potential $K^{(2,1)}$ and symmetric matrix $\mathcal{N}^{(2,1)}$ are given in (5.2) and (5.17) in terms of the prepotential $F^{(2,1)}$ of $\mathscr{M}^{2,1}$.

Notation: In this chapter we will be concerned with the complex structure moduli, which is why we will suppress the suffix $(2,1)$ in $\mathcal{N}$ and K from now on. Furthermore the number of *vectors* is denoted with $\bar{n} = h^{2,1} + 1$. ◊

According to the c-map's formalism we need to compactify one spatial direction of the four-dimensional manifold on a circle of radius R. Just as in section 6.1, we write $\hat{x}^\mu = (x, y) \in M_3 \times S^1_R$. The compactification of the last term of (5.16) is done rather similar as in the rigid situation. The zero modes of the Fourier transform have no y-dependence so these can be easily integrated out. A Lagrange multiplier ensures that we obtain three-dimensional field strengths, from which we obtain scalar fields A^I and B_I representing the electric-magnetic content of the original field strengths $\mathcal{F}^I$ from the type IIB point of view.

New in the local c-map's construction is the presence of an Einstein-Hilbert term $\sqrt{g}R_4$. As we have explained in the "proof" of theorem 4.21, the metric $g_{\mu\nu}$ may be decomposed into an M_3- and S^1-part,

$$g_{\mu\nu} = \begin{pmatrix} g_{lm} & \sigma_l \\ \sigma_m & \phi \end{pmatrix}.$$

A quite tedious calculation then shows that the compactification of the Ricci scalar R_4 leads to three terms [FS90, BCF91, BBS07],

$$R_4 \to R_3 + \alpha \partial_m \phi \partial^m \phi + \beta e^{-2\phi} \sigma_m \sigma^m.$$

The 1-form σ_m should again be a field strength in three dimensions. For this reason an extra Lagrange multiplier is added that substitutes a *scalar* σ for the 1-form σ_m. The precise details may be found in [FS90] in which the local c-map has been carried out for the first time. The contribution of R_3 is put in the gravity multiplet of the type IIA theory. The other terms stemming from R_4, are added to the contributions of $\frac{1}{2}\mathrm{Im}\,(\mathcal{N}_{IJ}\mathcal{F}^{+I}\mathcal{F}^{+J})$ and $\frac{\partial^2 K}{\partial X^I \partial \bar{X}^{\bar{J}}} \partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}}$. Decompactifying these (scalar) contributions to four dimensions yields the hypermultiplet Lagrangian of the type IIA theory, cf. (4.15d).

Proposition 7.1 (Image of the local c-map). *By compactifying the type IIB gravity and vector multiplet Lagrangian (5.16) on a circle and suppressing all massive modes, the Lagrangian for the type IIA hypermultiplet sector in $N = 2$ supergravity is*

$$\begin{aligned} \mathscr{L}_{lc} \;=\; & \frac{\partial^2 K(X,\bar{X})}{\partial X^I \partial \bar{X}^{\bar{J}}} \partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} && (7.1a) \\ & + \frac{1}{4} i e^{-\phi} (\mathcal{N} - \bar{\mathcal{N}})^{IJ} \left(\partial_\mu B_I - 2\mathcal{N}_{IK} \partial_\mu A^K\right)\left(\partial^\mu B_J - 2\bar{\mathcal{N}}_{JL} \partial^\mu A^L\right) && (7.1b) \\ & - \frac{1}{4} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2\phi} \left(\partial_\mu \sigma - \frac{1}{2} A^I \overleftrightarrow{\partial}_\mu B_I\right)\left(\partial^\mu \sigma - \frac{1}{2} A^I \overleftrightarrow{\partial}^\mu B_I\right), && (7.1c) \end{aligned}$$

where $\overleftrightarrow{\partial}$ denotes a graded Leibniz rule $A \overleftrightarrow{\partial} B = A\partial B - B\partial A$.

This Lagrangian describes a nonlinear sigma model over four-dimensional Minkowski space. The A- and B-coordinates have been chosen by physicists in an unbalanced fashion as can be seen from the factor 2 in front of A in (7.1b). We restore balance by replacing B with $2B$. By corollary 4.7 the scalar fields $\{X^I, A^I, B_I, \sigma, \phi\}_{I=0}^{h^{2,1}}$ parameterize a (real) $(4h^{2,1}+4)$-dimensional[1] manifold $\mathscr{M}_{hm}$ with metric

$$\begin{aligned} \mathrm{d}s^2_{hm} \;=\; & -4 \frac{\partial^2 K(X,\bar{X})}{\partial X^I \partial \bar{X}^{\bar{J}}} \mathrm{d}X^I \mathrm{d}\bar{X}^{\bar{J}} && (7.2a) \\ & -4 i e^{-\phi} (\mathcal{N} - \bar{\mathcal{N}})^{IJ} \left(\mathrm{d}B_I - \mathcal{N}_{IK} \mathrm{d}A^K\right)\left(\mathrm{d}B_J - \bar{\mathcal{N}}_{JL} \mathrm{d}A^L\right) && (7.2b) \\ & + \mathrm{d}\phi^2 + e^{-2\phi} \left(\mathrm{d}\sigma - A^I \overleftrightarrow{\mathrm{d}} B_I\right)^2. && (7.2c) \end{aligned}$$

[1]The coordinates X^I are a *projective* set of *complex* coordinates.

Results from physics

The manifold $\mathscr{M}_{hm}$ described by the metric (7.2) is the scalar manifold corresponding to the hypermultiplets of type IIA superstring theory. As we have already remarked in theorem 4.18 and as was explicitly shown in [FS90], we know it to be a quaternion-Kähler manifold. However its precise structure and in what manner the quaternionic structure comes about is still subject to investigation. In the rest of this chapter we hope to obtain a better understanding of $\mathscr{M}_{hm}$ in terms of a geometric interpretation of the local c-map. Before we turn to this interpretation though, we will give physical intuition about the coordinates of the manifold and we will quote some results known from physics that will guide us to our goal.

From the construction of the local c-map, it is obvious that the projective special Kähler manifold $\mathscr{M}^{2,1}$ is a submanifold of $\mathscr{M}_{hm}$. The string theoretical construction of toroidally compactifying the gravity and vector multiplet sectors thus seems to construct a fibre bundle over the complex structure moduli space $\mathscr{M}^{2,1}$, whose typical fibre is parameterized by coordinates $(A^I, B_I, \sigma, \phi)_{I=0}^{h^{2,1}}$. In order to profit from our understanding of the rigid c-map, we will consider different parts of the fibre at a time. First we consider the AB-fibre subbundle of $\mathscr{M}_{hm}$, consisting of fibres of A- and B-coordinates over the complex structure moduli space. Then we consider the total (typical) fibre by adding the contributions of ϕ and σ. The last step of bundling all total fibres together over the complex structure moduli space will provide the sought-for geometric interpretation.

The A- and B-coordinates have the same interpretation as in the rigid c-map, cf. section 6.1. From the type IIB point of view the scalars A^I and B_I are the electric-magnetic information of the gauge vector fields A^I_μ. The c-map provides an identification of these fields with the type IIA's scalars A^I and B_I that result from periods of the (R-R) 3-form of table 4.1 after compactifying on a Calabi-Yau$_3$. The second term of (7.2) shows a clear resemblance with the second term of (6.1) for which we have proven that it corresponds to the metric on the Griffiths intermediate Jacobians of the Calabi-Yau$_3$. In the next section we will discuss whether a similar interpretation holds for this AB-fibre subbundle.

As in the rigid case, the A- and B-fields are periodic, which means that the fibre subbundle is a bundle of tori, the *AB-tori*. Moreover the axion σ present in the local case is periodic as well. This periodicity follows from invariance of the nonperturbative theory under the integer *Peccei-Quinn symmetry* [GSW87, Pol98]. We have encountered this symmetry before in chapter 6 for the Ramond fields A^I and B_I, where we have associated it to the lattice of dyonic charges. In the local c-map the Peccei-Quinn symmetry of the axion σ is important as well [FS90]. The integer symmetry becomes a continuous symmetry at the perturbative level, which are called the *continuous Peccei-Quinn isometries* of the metric (7.2). Together with a dilatational isometry they form a set of $2h^{2,1}+4$ isometries of (7.2) for the classical theory (without quantum corrections) [FS90].

Lemma 7.2 (Isometries of $\mathscr{M}_{hm}$). *Consider the typical fibre in $\mathscr{M}_{hm}$ coordinatized by the coordinates $(A^I, B_I, \sigma, \phi)_{I=0}^{h^{2,1}}$. There are at least $2h^{2,1}+4 = 2\bar{n}+2$ isometries of (7.2) given by*

$$h_{abs}(A, B, \sigma, \phi) = (A+a, B+b, \sigma+s+a^tB-b^tA, \phi) \quad \text{and} \tag{7.3a}$$
$$d_\lambda(A, B, \sigma, \phi) = (\lambda A, \lambda B, \lambda^2\sigma, \phi+2\log\lambda), \tag{7.3b}$$

with $a, b \in \mathbb{R}^{\bar{n}}$, $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}_{>0}$. The isometries given by h are the Peccei-Quinn isometries of the axion and the Ramond fields [GSW87], while the last isometry is a dilatational one.

Proof. We verify explicitly the effect of h_{abs} on $\mathrm{d}\sigma - A^t \overleftrightarrow{\mathrm{d}} B$,

$$h_{abs}\left(\mathrm{d}\sigma - A^t \overleftrightarrow{\mathrm{d}} B\right) = \mathrm{d}\sigma + a^t\mathrm{d}B - b^t\mathrm{d}A - (A^t \overleftrightarrow{\mathrm{d}} B + a^t\mathrm{d}B - b^t\mathrm{d}A).$$

Invariance of the other terms and the effect of d_λ follow in a similar manner. $\square$

These isometries put a restriction on the possible structure of the typical fibre over the complex structure moduli space. We will use them to find that structure, since the fields cannot be explained geometrically in terms of the Calabi-Yau$_3$'s and we will need additional input. In the type IIA spectrum, ϕ and σ are identified with the dilaton and axion, which are not determined by the Calabi-Yau$_3$, cf. theorem 4.12. From the type IIB point of view, they result from the gravitational content of the four-dimensional metric (or graviton) by the c-map's toroidal compactification. Their presence in the fibre bundle over $\mathscr{M}^{2,1}$ is therefore independent of the Calabi-Yau$_3$ manifold and its complex structure. As a result they may be understood at first by purely considering the typical fibre of the fibre bundle without any reference to the base point $X(t) \in \mathscr{M}^{2,1}$ of the fibre.

7.2 AB-tori

Period matrices for the AB-tori

In this section we focus on the fibre subbundle of $\mathscr{M}_{hm}$ that arises by neglecting the contribution (7.2c) to the metric (7.2) and setting the dilaton ϕ to zero. Thus we consider the fibre bundle of AB-tori, whose fibres admit a metric,

$$\mathrm{d}s^2_{AB} = -4i(\mathcal{N}-\bar{\mathcal{N}})^{IJ}\left(\mathrm{d}B_I - \mathcal{N}_{IK}\mathrm{d}A^K\right)\left(\mathrm{d}B_J - \bar{\mathcal{N}}_{JL}\mathrm{d}A^L\right). \tag{7.4}$$

The next lemma justifies our interpretation of the fibre bundle as a torus bundle.

Lemma 7.3 (Period matrices for the AB-tori). *The line element (7.4) is the canonical metric on a complex torus. The second half of the period matrix for this torus is given by $\mathcal{N}$.*

Proof. By writing

$$\mathrm{d}s^2_{AB} = -4i\bar{Y}_I(\mathcal{N}-\bar{\mathcal{N}})^{IJ}Y_J = -2\bar{Y}_I(2i)(\mathcal{N}-\bar{\mathcal{N}})^{IJ}Y_J = -2\bar{Y}^t(\operatorname{Im}\mathcal{N})^{-1}Y, \tag{7.5}$$

where $Y = \mathrm{d}B - \bar{\mathcal{N}}\mathrm{d}A$, we see directly the correspondence with (3.10). The extra factor of 2 can be further incorporated into a basis transformation of A and B. $\square$

Each of the AB-fibres, fibred over an element $X(t) \in \mathscr{M}^{2,1}$ of the projective special Kähler manifold $\mathscr{M}^{2,1}$, is a complex torus with period matrix $\mathcal{N}$. In the rigid case we were able to interpret the period matrix found for the line element of the rigid c-map's metric (6.1) as the period matrix of the Griffiths intermediate Jacobian of the Calabi-Yau$_3$, i.e. there was a direct geometric interpretation for the form of the rigid c-map's metric. In this case, the local case, we want to have a similar interpretation for the period matrix of the local c-map. In particular, *can we still interpret the period matrices of the AB-tori as period matrices for the intermediate Jacobians of the Calabi-Yau$_3$'s?* The answer to this question turns out to be "yes, but with a twist".

Proposition 7.4 (Period matrices for the Weil intermediate Jacobians). *Let $\mathscr{CY} = (CY_t)_{t\in\mathscr{M}^{2,1}} \to \mathscr{M}^{2,1}$, $CY_0 = CY$ be a family of complex structure deformations of a Calabi-Yau$_3$ manifold CY. The period matrix Π^W_t of the Weil intermediate Jacobian of the Calabi-Yau$_3$ manifold CY_t is given by*

$$\Pi^W_t = \begin{pmatrix} \mathbb{I}_{\bar{n}} \\ \mathcal{N}_t \end{pmatrix}, \tag{7.6}$$

where $\mathcal{N}_t$ is given by (5.17) in terms of the holomorphic prepotential $F(X(t))$ of the projective special Kähler manifold $\mathscr{M}^{2,1}$.

Proof. Let $t = (t^a)_{a=1}^{h^{2,1}}$ be local coordinates for $\mathscr{M}^{2,1}$ and consider a point in the moduli space $t \in \mathscr{M}^{2,1}$. The period matrix Π^W_t of the complex torus $\mathcal{J}_W(CY_t)$ is found by expressing a basis of $H^{3,0}(CY_t) \oplus H^{1,2}(CY_t)$ in terms of a basis $\{\alpha_I, \beta^I\}_{I=0}^{h^{2,1}}$ of $H^3(CY,\mathbb{Z})$, cf. proposition 3.21. To this end we consider the holomorphic 3-form $vol_t \in H^{3,0}(CY_t)$ written with respect to the $\{\alpha_I, \beta^I\}$-basis,

$$vol_t = \sum_{I=0}^{h^{2,1}} X^I(t)\alpha_I - W_I(t)\beta^I.$$

The X- and W-coordinates are the A- and B-periods of the holomorphic 3-form, cf. (5.7). By lemma 5.10, the *Kähler covariant derivatives* of *vol* form a basis $\{\omega_a\}_{a=1}^{h^{2,1}}$ of $(2,1)$-forms,

$$\omega_a(t) = \mathrm{D}_a^K vol_t = \frac{\partial vol_t}{\partial t^a} + \left(\frac{\partial K(t,\bar{t})}{\partial t^a}\right) vol_t \in H^{2,1}(CY_t).$$

Therefore the set $\{vol, \bar{\omega}_{\bar{a}}\}_{\underline{a}=1}^{h^{2,1}}$ forms a basis for $H^{3,0}(CY_t) \oplus H^{1,2}(CY_t)$. The period matrix Π_t^W consists of periods X, W, $\bar{f}$ and $\bar{h}$,

$$f^I{}_a = \int_{\gamma_{A^I}} \mathrm{D}_a^K vol_t = \mathrm{D}_a^K X, \qquad h_{Ia} = \int_{\gamma_{B^I}} \mathrm{D}_a^K vol_t = \mathrm{D}_a^K W,$$

of the forms $\{vol, \bar{\omega}_{\bar{a}}\}_{a=1}^{h^{2,1}}$ with respect to the basis $\{\alpha_I, \beta^I\}_{I=0}^{h^{2,1}}$ of $H^3(CY, \mathbb{Z})$,

$$\Pi_t^W = \begin{pmatrix} X & \bar{f} \\ W & \bar{h} \end{pmatrix}.$$

Note that we interpret $f^I{}_a$ and h_{Ia} as $(h^{2,1}+1) \times h^{2,1}$-matrices and X and W as column vectors. The period matrix may be put in its standard form (3.3) by right-multiplication of the inverse of $(\, X \;\bar{f}\,)$ [CRTVP97]. From the work of [BG83] we know that with respect to a suitable basis $\{\alpha_I, \beta^I\}$ of $H^3(CY, \mathbb{Z})$ there exists a prepotential $F(X)$ such that W_I is the derivative of F with respect to X. The existence of this prepotential implies that we may use proposition 5.22 to find the explicit expression (5.17) for the second half of the period matrix in terms of $X(t)$ and $F(X(t))$. □

Thus the AB-tori are actually Weil intermediate Jacobians of the members of the family of Calabi-Yau$_3$'s. This implies that part of the hypermultiplet moduli space is a *fibre bundle of Weil intermediate Jacobians* over the complex structure moduli space $\mathscr{M}^{2,1}$,

$$\pi_{\mathscr{J}_W} : \mathscr{J}_W \to \mathscr{M}^{2,1}.$$

The typical fibre of this fibre bundle is the Weil intermediate Jacobian $\mathcal{J}_W(CY)$ of the initial Calabi-Yau$_3$ manifold CY.

cW-map vs. cG-map

The appearance of the Weil intermediate Jacobian in the local c-map construction is an interesting phenomenon, which we have not been able to find in the literature. Those authors who do mention intermediate Jacobians as a geometric description of the local c-map [GMV96, Asp98], often simply define the intermediate Jacobian as a *real* torus $H^3(CY, \mathbb{R})/H^3(CY, \mathbb{Z})$. From section 3.2 it is clear however that the Griffiths and Weil intermediate Jacobians differ by the complex structures that are put upon $H^3(CY, \mathbb{R})$. Without reference to a complex structure, it is unclear which *complex* torus is meant. Since the popular Griffiths intermediate Jacobian has completely overshadowed the Weil intermediate Jacobian, it is unlikely that the average reader will recognize this ambiguity. Hopefully by proposition 7.4 we have cleared up possible confusion and have given a clear geometric interpretation for the need of two distinct objects F_{IJ} and $\mathcal{N}$ for $N=2$ rigid and local supersymmetry respectively [ST83, dWVP84]. The rigid c-map can geometrically be described in terms of Griffiths intermediate Jacobians, while (part of) the local c-map has an interpretation in terms of Weil intermediate Jacobians. To emphasize their distinction, we propose to rename the local c-map into *cW-map*.

Although we have been able to *identify* both intermediate Jacobians in the construction of the c-maps, the different roles they have to play in both constructions is not completely clear to us. One remarkable difference between the Griffiths and Weil intermediate Jacobians is the signature of their canonical metric, cf. section 3.2. Whereas the Weil intermediate Jacobian has a positive definite metric, the Griffiths intermediate Jacobian has (real) index 2. This is consistent with the signatures of the hyperkähler manifold $\mathscr{M}_{hm}^{susy}$ and the quaternion-Kähler manifold $\mathscr{M}_{hm}^{sugra}$ in $N = 2$ supersymmetry and supergravity. The hypermultiplet scalar manifold in rigid supersymmetry has an index 4 metric. Two of the negative contributions stem from the metric on the fibres of the cotangent bundle, the other two are due to the affine special Kähler manifold itself. Contrarily the quaternion-Kähler manifold has a positive definite metric, which means that the Griffiths intermediate Jacobian cannot possible occur in a description of $\mathscr{M}_{hm}^{sugra}$.

When turning on gravity, i.e. when going from a supersymmetry to a supergravity theory, the rigid c-map's main constituent becomes an incompatible geometric object purely because of the signature of its metric. It is a relief that the geometric description of the local c-map stays fairly close to our rigid description, but exactly why the other intermediate Jacobian is chosen as the supergravity equivalent of the Griffiths intermediate Jacobian is not completely clear. Intuitively we can understand why the Weil intermediate Jacobian is at least an improvement of the Griffiths intermediate Jacobian. The polarization of the Griffiths intermediate Jacobian is negative definite on its $H^{0,3}$-part. By *twisting* the complex structure on that part (or equivalently, by replacing that part with its complex conjugate) the negative definiteness may be resolved, while the intersection form Q stays the same. The result is the Weil intermediate Jacobian [BL99]. Another way of seeing this is related to the discussion at the end of chapter 6. It was argued there that the complex structure of the Calabi-Yau$_3$ could be described in first approximation by Lagrange planes in $H^3(CY, \mathbb{C})$, while actually the position of $H^{3,0}(CY)$ *within* these Lagrange planes is of vital importance. Comparing the period matrices of the Griffiths and Weil intermediate Jacobians reveals how the Weil intermediate Jacobian succeeds in distinguishing the $H^{3,0}$-part from the $H^{2,1}$-part: by complex conjugating one of these parts. At the same time complex conjugation succeeds in the change of signature of the polarization.

Due to its positive definite polarization the Weil intermediate Jacobian is an abelian variety, just like the Jacobian of a smooth projective curve. Therefore the Weil intermediate Jacobians are more likely to allow a generalization of Seiberg-Witten theory than the Griffiths intermediate Jacobians, see also the remark on page 84. As such study of Weil intermediate Jacobians, for instance by finding the algebraic equation describing them, could be interesting for both the moduli spaces of supergravity theories as well as in the theory of integrable systems.

Finally the behavior of the Weil intermediate Jacobian under holomorphic variations could provide a means to study the quaternion-Kähler manifold $\mathscr{M}_{hm}^{sugra}$. As already mentioned in chapter 3 the Weil intermediate Jacobians do not vary properly under holomorphic variations. Our hypothesis is that to understand what "improper variation" means, resolves the question why the integrable quaternionic structure of the hyperkähler manifold $\mathscr{M}_{hm}^{susy}$ changes into a nonintegrable quaternionic structure of the quaternion-Kähler manifold $\mathscr{M}_{hm}^{sugra}$. A better understanding of the necessity of replacing the Griffiths intermediate Jacobian by the Weil intermediate Jacobian will probably illuminate the relation between $\mathscr{M}_{hm}^{sugra}$ and $\mathscr{M}_{hm}^{susy}$, cf. figure 4.4.

7.3 Dilatated Heisenberg group structure

We have succeeded in interpreting the A- and B-coordinates as periodic coordinates of the Weil intermediate Jacobians fibred over the complex structure moduli space. Nevertheless the total fibre in the fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$ consists of another two scalar fields ϕ and σ. Their geometric contribution is given in (7.2c) and (7.2b), where they are seen to mix with the contribution of the A- and B-coordinates. For simplicity we neglect the dependence on the complex structure moduli X via the period matrix $\mathscr{N}$ for the moment, since we already know how that comes about: it produces the line element on the Weil intermediate Jacobian of the particular Calabi-Yau$_3$ CY_t over which it is fibred. For the moment we simply investigate the structure of the typical fibre $\mathcal{T}$, coordinatized by the fields $(A^I, B_I, \sigma, \phi)_{I=0}^{h^{2,1}}$. We are particularly interested in how the second term of (7.2c) and the scaling $e^{-\phi}$ in (7.2b) come about. That is, how the Ramond A-, B-coordinates mix with the gravitational coordinates ϕ and σ.

The structure on the typical fibre is induced by the isometries of lemma 7.2 on the typical fibre. The isometries given by h are the continuous Peccei-Quinn symmetries of the (R-R) fields

A, B and the (NS-NS) axion σ. It is known that the generators of the Peccei-Quinn isometries form an algebra, which is the Lie algebra of the *Heisenberg group* [BB99, Ket01a]. The dilatations d can be incorporated by forming the semi-direct product of the Heisenberg group and the positive real axis [RVV06b]. Let us define the *dilatated Heisenberg group* and show how its structure is translated to the typical fibre.

The dilatated Heisenberg group

Definition 7.5 (Heisenberg group). Let (V, ω) be a $2n$-dimensional real symplectic vector space with symplectic form ω. We define the *Heisenberg group* $\mathrm{Heis}(2n, V, \omega) = (V \times \mathbb{R}, \cdot)$, where the multiplication $\cdot$ is defined by

$$(v_1, s_1) \cdot (v_2, s_2) = (v_1 + v_2, s_1 + s_2 + \omega(v_1, v_2)).$$

Elements of $\mathrm{Heis}(2n, V, \omega)$ are denoted with h_{vs} where $(v, s) \in V \times \mathbb{R}$. When $\omega|_{V_\mathbb{Z} \times V_\mathbb{Z}} \subset \mathbb{Z}$ for a lattice $V_\mathbb{Z}$ in V, the *integer Heisenberg group* $\mathrm{Heis}(2n, V_\mathbb{Z}, \omega) = (V_\mathbb{Z} \times \mathbb{Z}, \cdot)$ is defined as well. ⊘

Proof. $\mathrm{Heis}(2n, V, \omega)$ does indeed constitute a group: closure is obvious, associativity is easily checked, $(0, 0)$ is the identity element and $(-v, -s)$ is the inverse element of (v, s). □

Remark: The group we have defined here is commonly referred to as a *symplectic Heisenberg group*. It is a generalization of the perhaps more familiar, ordinary three-dimensional *Heisenberg group*. ♢

We may write the Heisenberg group as a matrix subgroup of $\mathrm{SL}(2n+2, \mathbb{R})$. For convenience we use the notation v^* for the *symplectic conjugate* of a vector in a symplectic vector space (V, ω). It is uniquely determined by the (nondegenerate) symplectic inner product ω via $v^* w = \omega(v, w)$ for arbitrary $w \in V$. Note that we view v^* as a row vector if v is a column vector.

Lemma 7.6 (Matrix representation of the Heisenberg group). *The group* $\mathrm{Heis}(2n, V, \omega)$ *is isomorphic to a subgroup of* $\mathrm{SL}(2n+2, \mathbb{R})$. *An element* $h_{vs} \in \mathrm{Heis}(2n, V, \omega)$ *may be represented by the matrix*

$$h_{vs} \doteq \begin{pmatrix} 1 & v^* & s \\ 0 & \mathbb{I}_{2n} & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. The representation already defines the necessary group isomorphism $\psi : \mathrm{H}(2n, V) \to \mathrm{SL}(2n+2, \mathbb{R})$. It is obviously injective and it is surjective by definition. We only need to verify the preservation of the multiplication law,

$$\begin{aligned}(v_1, s_1) \cdot (v_2, s_2) &\doteq \begin{pmatrix} 1 & v_1^* & s_1 \\ 0 & \mathbb{I}_{2n} & v_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v_2^* & s_2 \\ 0 & \mathbb{I}_{2n} & v_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (v_1+v_2)^* & s_1 + s_2 + \omega(v_1, v_2) \\ 0 & \mathbb{I}_{2n} & (v_1 + v_2) \\ 0 & 0 & 1 \end{pmatrix} \\ &\doteq (v_1 + v_2, s_1 + s_2 + \omega(v_1, v_2)).\end{aligned}$$ □

We want to include dilatations by forming a semi-direct product, which is why we quickly review the definition of a semi-direct product.

Definition 7.7 (Semi-direct product). Let H, J be groups and $\Phi : J \to \mathrm{Aut}(H)$ be a group homomorphism, i.e. $\Phi(j_1 j_2)(h) = \Phi(j_1)(\Phi(j_2)(h))$ for all $j_1, j_2 \in J$ and $h \in H$. Then we define the *semi-direct product* $H \rtimes_\Phi J$ as the set of ordered pairs $(h, j) \in H \times J$ with group multiplication $(h_1, j_1) \cdot (h_2, j_2) = (h_1 \Phi(j_1)(h_2), j_1 j_2)$. ⊘

Proof. This indeed constitutes a group with identity (e_H, e_J) and inverse

$$(h, j)^{-1} = (\Phi(j)^{-1}(h^{-1}), j^{-1}).$$

Closure is obvious and associativity follows from the fact that Φ is a group homomorphism. □

Projecting $H \rtimes_\Phi J \to J : (h,j) \mapsto j$ defines a group homomorphism onto J whose kernel $\{(h, e_J)|h \in H\}$ is isomorphic to H. By the first isomorphism theorem [Arm88] we see that $\{(h, e_J)|h \in H\} \cong H$ must be a normal subgroup of $H \rtimes_\Phi J$. J may also be identified with a subgroup $\{(e_H, j)|j \in J\}$ in $H \rtimes_\Phi J$, though it need not be a normal subgroup. The semi-direct product is an extension of the ordinary product group of two groups in the sense that if $\Phi(j)$ is the identity automorphism of H for all $j \in J$, then $H \rtimes_\Phi J = H \times J$.

We are interested in the case that H, J are both subgroups of another group G, in which case we take Φ to be the conjugation map, $\Phi(j)(h) = jhj^{-1}$. Under the following conditions this situation produces a semi-direct product.

Proposition 7.8 (Construction of a semi-direct product). *Let J be a subgroup of a group G, H be a normal subgroup of the same group G and suppose $G = HJ = \{hj|h \in H, j \in J\}$ and $H \cap J = \{e\}$. Then G is isomorphic to the semi-direct product of H and J with homomorphism $\Phi(j)(h) = jhj^{-1}$.*

Proof. The isomorphism is given by $\psi : H \rtimes_\Phi J \to G : (h,j) \mapsto hj$. It is a group homomorphism because of the description of Φ and because H is a normal subgroup. It is surjective because $G = HJ$ and it is injective because its kernel is

$$\ker(\psi) = \{(h,j) \in H \rtimes_\Phi J | hj = e\} = \{(h,j) \in H \times J | h = j^{-1}\} \subset H \cap J = \{e\}. \qquad \square$$

Notation: We reserve the notation $H \rtimes J$ for the semi-direct product of two subgroups $H \lhd G$, $J < G$ of a group G for which the conditions of proposition 7.8 hold. We will not bother to distinguish between (h,j) and hj, since both are to be identified under the isomorphism between G and $H \rtimes J$. ◊

Lemma 7.9 ($\mathbb{R}_{>0}$ as a subgroup of $\mathrm{SL}(m,\mathbb{R})$). *For each $m \geq 2$ the group $\mathbb{R}_{>0}$ is isomorphic to a subgroup of $\mathrm{SL}(m,\mathbb{R})$.*

Proof. The isomorphism is given by the map,

$$\psi : \mathbb{R}_{>0} \to \mathrm{SL}(m,\mathbb{R}) : \lambda \mapsto \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathbb{I}_{m-2} & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}. \qquad \square$$

Definition 7.10 (Dilatated Heisenberg group). Consider the Heisenberg group $\mathrm{Heis}(2n, V, \omega)$ and the group of positive real numbers $\mathbb{R}_{>0}$, both seen as a subgroup of $\mathrm{SL}(2n+2,\mathbb{R})$ via lemmas 7.6 and 7.9. The *dilatated Heisenberg group* $\mathrm{Hd}(2n, V, \omega)$ is defined as the semi-direct product of these two groups,

$$\mathrm{Hd}(2n, V, \omega) = \mathrm{Heis}(2n, V, \omega) \rtimes \mathbb{R}_{>0}.$$

Any element $g_{vs\lambda} \in \mathrm{Hd}(2n, V, \omega)$ is written as $g_{vs\lambda} = h_{vs} d_\lambda$ where $h_{vs} \in \mathrm{Heis}(2n, V, \omega)$ and $d_\lambda \in \mathbb{R}_{>0}$. ⊘

Lemma 7.11 (The dilatated Heisenberg group is a Lie group). *The dilatated Heisenberg group $\mathrm{Hd}(2n, V, \omega)$ is a $2n+2$-dimensional real Lie group.*

Proof. Elements $g_{vs\lambda}$ of $\mathrm{Hd}(2n, V, \omega)$ are of the form

$$g_{vs\lambda} = h_{vs} d_\lambda = \begin{pmatrix} 1 & v^* & s \\ 0 & \mathbb{I}_n & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathbb{I}_n & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}.$$

This space is locally of the form $V \times \mathbb{R} \times \mathbb{R}_{>0}$, which means it is a $2n+2$-dimensional real smooth manifold. Group multiplication is simply group multiplication of matrices, $g_{vs\lambda} g_{v's'\lambda'} = (h_{vs} d_\lambda h_{v's'} d_\lambda^{-1}) d_\lambda d_{\lambda'}$, so this is a smooth map $\mu : \mathrm{Hd} \times \mathrm{Hd} \to \mathrm{Hd}$. The inverse of $h_{vs} \in \mathrm{Hd}(2n, V, \omega)$ is given by $h_{vs}^{-1} = h_{-v,-s}$, which is a smooth operation. The same holds for the inverse maps of an element $d_\lambda \in \mathbb{R}_{>0}$, $d_\lambda^{-1} = d_{1/\lambda}$. Therefore $\iota : \mathrm{Hd} \to \mathrm{Hd} : g_{vs\lambda} \mapsto (d_\lambda^{-1} h_{vs}^{-1} d_\lambda) d_\lambda^{-1}$ is a smooth map as well. $\square$

Group action on the typical fibre

The reason we are considering the dilatated Heisenberg group is that it has a natural *group action* on the fibre $\mathcal{T}$, which corresponds to the isometries of lemma 7.2. The underlying symplectic vector space of the Heisenberg group is $(H^3(CY,\mathbb{R}),\Sigma)$, where $\Sigma = \left(\begin{smallmatrix} 0 & \mathbb{I}_{\bar{n}} \\ -\mathbb{I}_{\bar{n}} & 0 \end{smallmatrix}\right)$ is the standard matrix form of the intersection form Q, cf. corollary 2.28 and (6.2). For notational convenience we denote a vector $v \in H^3(CY,\mathbb{R})$ by $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ where a, b are the coordinates with respect to a symplectic basis $\{\alpha_I, \beta^I\}_{I=0}^{h^{2,1}}$ of $H^3(CY,\mathbb{R})$. Note that in that case $v^* = (-b^t, a^t)$.

Definition 7.12 (Dilatated Heisenberg group action on the typical fibre). We define a *group action* of $\mathrm{Hd}(2\bar{n}, H^3(CY,\mathbb{R}),\Sigma)$ on the fibre $\mathcal{T}$ by

$$h_{abs}d_\lambda(A,B,\sigma,\phi) = (\lambda A + a, \lambda B + b, \lambda^2\sigma + s + \lambda(a^tB - b^tA), \phi + 2\log\lambda,). \tag{7.7}$$

⊘

Proof. We need to verify the axioms of a group action. First we note that $h_{000}d_1(A,B,\sigma,\phi) = (A,B,\sigma,\phi)$ for all $(A,B,\sigma,\phi) \in \mathcal{T}$. Next we verify that

$$\begin{aligned}
& h_{abs}d_\lambda\Big(h_{a'b's'}d_{\lambda'}(\phi,A,B,\sigma)\Big) \\
= \;& h_{abs}d_\lambda\Big(\phi + 2\log\lambda', \lambda'A + a', \lambda'B + b', \lambda'^2\sigma + s' + \lambda'(a'^tB - b'^tA)\Big) \\
= \;& \Big(\phi + 2\log\lambda' + 2\log\lambda, \lambda\lambda'A + \lambda a' + a, \lambda\lambda'B + \lambda b' + b, \\
& \quad \lambda^2\lambda'^2\sigma + \lambda^2 s' + \lambda^2\lambda'(a'^tB - b'^tA) + s + \lambda\left(a^t(\lambda'B + b') - b^t(\lambda'A + a')\right)\Big) \\
= \;& \Big(\phi + 2\log(\lambda\lambda'), \lambda\lambda'A + \lambda a' + a, \lambda\lambda'B + \lambda b' + b, \\
& \quad (\lambda\lambda')^2\sigma + \lambda^2 s' + s + \lambda(a^tb' - b^ta') + \lambda\lambda'\left((\lambda a' + a)^tB - (\lambda b' + b)^tA\right)\Big) \\
= \;& h_{\lambda a'+a, \lambda b'+b, \lambda^2 s' + s + \lambda(a^tb' - b^ta')}d_{\lambda\lambda'}(\phi,A,B,\sigma) = \Big(h_{abs}d_\lambda h_{a'b's'}d_{\lambda'}\Big)(\phi,A,B,\sigma).
\end{aligned}$$

□

Because the (Lie) group action is transitive, its structure is translated to the coordinates of the typical fibre.

Lemma 7.13 (Transitivity of the dilatated Heisenberg group action). *The dilatated Heisenberg group has a* transitive *group action on the typical fibre $\mathcal{T}$ of the local c-map's fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$.*

Proof. Let (A,B,σ,ϕ) and (A',B',σ',ϕ') be two arbitrary elements of $\mathcal{T}$. We need to show that there is an element $h_{abs}d_\lambda \in \mathrm{Hd}(2\bar{n}, H^3(CY,\mathbb{R}),\Sigma)$ such that

$$(A',B',\sigma',\phi') = h_{abs}d_\lambda(A,B,\sigma,\phi) = (\lambda A + a, \lambda B + b, \lambda^2\sigma + s + \lambda(a^tB - b^tA), \phi + 2\log\lambda).$$

We immediately find $\lambda = e^{\frac{1}{2}(\phi'-\phi)}$, which determines a and b to be $a = A' - e^{1/2(\phi'-\phi)}A$, $b = B' - e^{1/2(\phi'-\phi)}B$. From this we can express s totally in terms of $\phi, A, B, \sigma, \phi', A', B', \sigma'$. □

Such an identification of the parameters of the Lie group with the coordinates of the typical fibre $\mathcal{T}$ makes $\mathcal{T}$ into a *homogeneous space*.

Definition 7.14 (Homogeneous space). A *homogeneous space* is a manifold M upon which a Lie group G acts transitively. ⊘

Proposition 7.15 (Homogeneous spaces and quotient spaces). *Let G be a Lie group acting transitively on a homogeneous space M and let $p_0 \in M$ be a fixed point in M. Then M is isomorphic (as a manifold) to the quotient space G/G_{p_0}, where G_{p_0} is the* isotropy subgroup *or* little group *of G of p_0. The isotropy subgroup is defined as*

$$G_{p_0} = \{g \in G | gp_0 = p_0\}.$$

Proof. Note that G_{p_0} is indeed a subgroup of G and for this reason the quotient G/G_{p_0} is a well-defined *set* of equivalence classes, where two elements $g, h \in G$ are said to be related if $g^{-1}h \in G_{p_0}$. The map $\phi : G/G_{p_0} \to M : [g] \mapsto gp_0$ is well-defined on the set of equivalence classes, because picking another representative $h \in [g]$ of the equivalence class $[g] \in G/G_{p_0}$ gives $hp_0 = hh^{-1}gp_0 = gp_0$. Due to transitivity of the group action, ϕ defines a surjective map and it is injective because $p_1 = gp_0 = hp_0 = p_2 \in M$ implies $g^{-1}h \in G_{p_0}$ and hence $[g] = [h]$. $\square$

Proposition 7.16 (Dilatated Heisenberg structure on the typical fibre). *Let $\mathcal{T}$ be the typical fibre of the local c-map's fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$ over the complex structure moduli space. It is isomorphic (as a manifold) to the quotient space*

$$\mathcal{T} \cong \frac{\mathrm{Heis}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma) \rtimes \mathbb{R}_{>0}}{\mathrm{Heis}(2\bar{n}, H^3(CY,\mathbb{Z}), \Sigma)}. \tag{7.8}$$

Proof. By lemma 7.11 and 7.13 the dilatated Heisenberg group action is a transitive Lie group action on the typical fibre $\mathcal{T}$. By proposition 7.15 this makes $\mathcal{T}$ into a homogeneous space $\mathrm{Heis}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma) \rtimes \mathbb{R}_{>0}/\mathrm{Iso}$, where Iso is the isotropy group of the element $(0,0,0,0) \in \mathcal{T}$. By the periodicity of the Ramond fields A, B and the axion σ the isotropy group is isomorphic to the integer Heisenberg group $\mathrm{Heis}(2\bar{n}, H^3(CY,\mathbb{Z}), \Sigma)$. $\square$

As a manifold the dilatated Heisenberg group is locally isomorphic to $H^3(CY,\mathbb{R}) \times \mathbb{R} \times \mathbb{R}_{>0}$ and therefore dividing by the integer Heisenberg group $H^3(CY,\mathbb{Z}) \times \mathbb{Z} \times \{1\}$ yields

$$\mathcal{T} \cong \frac{H^3(CY,\mathbb{R})}{H^3(CY,\mathbb{Z})} \times \frac{\mathbb{R}}{\mathbb{Z}} \times \mathbb{R}_{>0}.$$

Projecting on the first factor gives rise to the Weil intermediate Jacobian, although we have left out its precise complex structure at this stage. Projecting on the last two factors gives an $\mathbb{R}/\mathbb{Z} \times \mathbb{R}_{>0} \cong \mathbb{C}^*$-bundle. It is the fibre of the dilaton-axion system already known from the literature [GMV96, Ket01b]. Defining $\tau = \sigma + ie^{\phi}$ provides (7.2c) with the interpretation of a standard *Fubini-Study metric* on the upper half-plane [Ket01a, Ket01b, Dui04b].

Due to the position of ϕ and σ within the dilatated Heisenberg group (with respect to the AB-coordinates), the structure of ϕ and σ does not seem to be exactly that of an additional $\mathbb{C}^*$-bundle *on top of* the Weil intermediate Jacobians. The Heisenberg group and the dilatated Heisenberg group are characterized by the *exact sequences of groups*,

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & \mathrm{Heis}(2n,V,\omega) & \to & V & \to & 0 \quad \text{and} \\ 0 & \to & \mathrm{Heis}(2n,V,\omega) & \to & \mathrm{Hd}(2n,V,\omega) & \to & \mathbb{R}_{>0} & \to & 0, \end{array}$$

where $\mathbb{R}$ and $\mathrm{Heis}(2n,V,\omega)$ are normal subgroups of $\mathrm{Heis}(2n,V,\omega)$ and $\mathrm{Hd}(2n,V,\omega)$ respectively. In general an exact sequence $0 \to H \to G \to G/H \to 0$ of groups gives rise to a fibre bundle $G \to G/H$ with typical fibre H. In our case the Heisenberg group is the typical fibre over the base space $\mathbb{R}_{>0}$ parameterized by the dilaton, while the axion σ occurs as the coordinate of the typical fibre of the fibre bundle $\mathrm{Heis}(2n,V,\omega) \to (V,\omega)$. Thus the dilaton-axion system is *split* by the AB-coordinates of the symplectic vector space $(V,\omega) = (H^3(CY,\mathbb{R}),\Sigma)$: ϕ is the coordinate of the base space over which the Weil intermediate Jacobians are fibred and on top of these there is an additional S^1-bundle parameterized by σ.

Proposition 7.16 interprets the hypermultiplet moduli space $\mathscr{M}_{hm}$ as a principal-like fibre bundle. On top of each point in the complex structure moduli space a quotient of the dilatated

Heisenberg group modulo its integer subgroup is fibred. Part of this quotient is interpreted as the Weil intermediate Jacobian of the Calabi-Yau$_3$, which splits an additional $\mathbb{R}_{>0} \times S^1$-bundle that is independent of the complex structure moduli. We have to call $\mathscr{M}_{hm}$ a principal-*like* fibre bundle because the subgroup $\mathrm{Heis}(2\bar{n}, H^3(CY,\mathbb{Z}), \Sigma)$ is *not* a normal subgroup of $\mathrm{Hd}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma)$. For this reason the quotient does not have a (Lie) group structure. However since the dilatated Heisenberg group is mod out by an *integer* subgroup, the tangent spaces of the Heisenberg group and its quotient are equal. The formalism available for Lie groups and their Lie algebras is therefore not lost completely.

Geometry of the typical fibre

If the dilatated Heisenberg group is really the structure of the local c-map's typical fibre, we should be able to construct a metric on the typical fibre $\mathcal{T}$ that resembles (7.2). The starting point of this construction is the invariance of (7.2) under the action (7.7) of the dilatated Heisenberg group. A useful object in order to define such an invariant metric on the tangent bundle of a Lie group G is the *Maurer-Cartan form* Ω, cf. definition 2.5. The Maurer-Cartan form is invariant under the pushforward of left multiplication by another (constant) element $h \in G$, $(l_h)^*\Omega_{hg} = \Omega_g$. We use the Maurer-Cartan form to obtain an invariant 1-form on the tangent space of our typical fibre $\mathcal{T}$, which is the quotient of the dilatated Heisenberg group modulo its *integer* subgroup. With this Lie algebra-valued 1-form we construct an invariant metric that is similar to the kinetic term in a *Wess-Zumino-Witten model*.

Definition 7.17 (Killing bilinear form). Let $\mathfrak{g}$ be a Lie algebra. The Killing bilinear form $\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is defined by

$$\mathcal{K}(X, Y) = \mathrm{Tr}\left[\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right].$$

⊘

Lemma 7.18 (Killing bilinear form of a matrix Lie algebra). *Suppose elements of a Lie algebra $\mathfrak{g}$ are represented by $n \times n$-matrices. The Killing bilinear form on two matrices $X, Y \in \mathfrak{g}$ is given by*

$$\mathcal{K}(X, Y) = 2n\mathrm{Tr}[XY] - 2\mathrm{Tr}[X]\mathrm{Tr}[Y].$$

Proof. For matrices X, Y the expression $(\mathrm{ad}(X)(Y))_{ij} = \sum_{k,l} \mathrm{ad}(X)_{ijkl} Y_{kl}$ is given by the commutator $(XY - YX)_{ij} = \sum_k X_{ik}Y_{kj} - \sum_l Y_{il}X_{lj}$. Hence in terms of the components of X, the adjoint of X is $\mathrm{ad}(X)_{ijkl} = X_{ik}\delta_{jl} - X_{lj}\delta_{ki}$. Therefore the Killing bilinear form is given by

$$\begin{aligned}
\mathcal{K}(X, Y) &= \sum_{i,j,k,l} \mathrm{ad}(X)_{ijkl}\mathrm{ad}(Y)_{klij} = \sum_{i,j,k,l} (X_{ik}\delta_{jl} - X_{lj}\delta_{ki})(Y_{ki}\delta_{lj} - Y_{jl}\delta_{ik}) \\
&= n\sum_{i,k} X_{ik}Y_{ki} + n\sum_{j,l} X_{lj}Y_{jl} - \sum_{i,j} X_{ii}Y_{jj} - \sum_{i,j} X_{jj}Y_{ii} = 2n\mathrm{Tr}[XY] - 2\mathrm{Tr}[X]\mathrm{Tr}[Y].
\end{aligned}$$

□

Definition 7.19 (Invariant metric on the typical fibre). We define an invariant metric on the typical fibre $\mathcal{T}$ of the local c-map's fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$ in terms of the Maurer-Cartan form Ω of the dilatated Heisenberg group $\mathrm{Hd}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma)$ and the Killing bilinear form $\mathcal{K}$ of its Lie algebra,

$$\mathrm{d}s^2_{\mathcal{T}} = \mathcal{K}(\Omega^t, \Omega). \tag{7.9}$$

⊘

To see what our definition (7.9) has to do with (7.2), we calculate the former explicitly. The

Maurer-Cartan form Ω of an element $g_{abs\lambda} = h_{abs}d_\lambda \in \mathrm{Hd}(2\bar{n}, H^3(CY,\mathbb{R}),\Sigma)$ is given by,

$$\Omega = d_\lambda^{-1}h_{abs}^{-1}\mathrm{d}(h_{abs}d_\lambda) = \begin{pmatrix} \frac{\mathrm{d}\lambda}{\lambda} & \frac{-\mathrm{d}b^t}{\lambda} & \frac{\mathrm{d}a^t}{\lambda} & \ell \\ 0 & 0 & 0 & \frac{\mathrm{d}a}{\lambda} \\ 0 & 0 & 0 & \frac{\mathrm{d}b}{\lambda} \\ 0 & 0 & 0 & \frac{-\mathrm{d}\lambda}{\lambda} \end{pmatrix}, \quad \text{where} \tag{7.10a}$$

$$\ell = \frac{1}{\lambda^2}\left(\mathrm{d}s - a^t \overset{\leftrightarrow}{\mathrm{d}}\, b\right). \tag{7.10b}$$

Since $\mathrm{Tr}[\Omega] = \mathrm{Tr}[\Omega^t] = 0$, lemma 7.18 tells us that (7.9) is proportional to

$$\begin{aligned} \mathrm{d}s^2_{\mathcal{T}} &= \mathrm{Tr}\left[\Omega^t\Omega\right] = \frac{1}{\lambda^2}\mathrm{d}\lambda^2 + \frac{1}{\lambda^2}\mathrm{Tr}\left[\begin{pmatrix} -\mathrm{d}b \\ \mathrm{d}a \end{pmatrix}\begin{pmatrix} -\mathrm{d}b^t & \mathrm{d}a^t \end{pmatrix}\right] + \ell^2 + \frac{1}{\lambda^2}\left(\mathrm{d}a^t\mathrm{d}a + \mathrm{d}b^t\mathrm{d}b\right) + \frac{1}{\lambda^2}\mathrm{d}\lambda^2 \\ &= \frac{2}{\lambda^2}\left(\mathrm{d}a^t\mathrm{d}a + \mathrm{d}b^t\mathrm{d}b\right) + \frac{2}{\lambda^2}\mathrm{d}\lambda^2 + \frac{1}{\lambda^4}\left(\mathrm{d}s - a^t\overset{\leftrightarrow}{\mathrm{d}}\, b\right)^2. \end{aligned} \tag{7.11}$$

This metric is indeed invariant under the dilatated Heisenberg group action. The easiest way to see this is perhaps to notice the resemblance between (7.11) and (7.2), which is particularly evident when one remembers to take $\lambda \propto e^{1/2\phi}$. The action of the dilatated Heisenberg group on the typical fibre succeeds in interpreting the strange intertwining-term between the A, B- and ϕ, σ-coordinates in the metric (7.2). The dilatations are incorporated as a non-isotropic homogeneous structure on the Heisenberg group coordinates [Fol89, CDPT99]. Moreover the term $\alpha = \mathrm{d}\sigma - A^t\overset{\leftrightarrow}{\mathrm{d}}B$ appears naturally in the Maurer-Cartan form (7.10), telling us that invariant objects are not $(\mathrm{d}A, \mathrm{d}B, \mathrm{d}\sigma)$ but $(\mathrm{d}A, \mathrm{d}B, \mathrm{d}\sigma - A^t\overset{\leftrightarrow}{\mathrm{d}}B)$. α has a natural interpretation as a *contact form* for the subspace parameterized by $(A^I, B_I, \sigma)_{I=0}^{h^{2,1}}$ (i.e. the Heisenberg group as a subspace of the dilatated Heisenberg group) [CDPT99]. It defines the "horizontal" direction for the direction of σ.

Of course (7.11) is not the complete story. Its first term is associated to the Weil intermediate Jacobian, although it is somewhat hard to recognize. Due to the fact that we have ignored all dependence on the complex structure moduli, the torus-term is not in its true form. Incorporating the complex structure moduli will be the last step in understanding the full construction of the local c-map.

7.4 Construction of the local c-map's fibre bundle

Symplectic invariance

In section 7.3 we have considered the typical fibre of the local c-map's fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$. The typical fibre admitted a dilatated Heisenberg structure endowed on it by a transitive group action. The resulting metric (7.11) is invariant under the action of the dilatated Heisenberg group. We now wish to bundle all fibres together in a consistent manner as a fibre bundle over the complex structure moduli. In order to understand the full bundle, another invariance is important: symplectic invariance. Under symplectic coordinate transformations the projective special Kähler geometry does not change and the fibres transform invariantly too, since the dilatated Heisenberg group is compatible with symplectic transformations. The torus-term as it stands in (7.11) is a non-symplectic object. The pairing between A- and A-coordinates and between B- and B-coordinates is a symplectically inconsistent one, since the coordinates A^I and B_I are given with respect to a symplectic basis and should therefore pair crosswise. Physicists have a mnemonic to see this, as placing of indices remembers them how objects may be contracted with each other. We could have known this in advance, since $\mathrm{d}s^2_{\mathcal{T}}$ is defined in terms of the non-symplectic Maurer-Cartan form.

A construction of the total bundle should satisfy two requirements. As we know from section 7.2, the torus-term is the metric for the Weil intermediate Jacobians. Above each complex structure modulus the corresponding Weil intermediate Jacobian is fibred. This Weil intermediate Jacobian and its metric *depend* on the complex structure modulus $X(t) \in \mathscr{M}^{2,1}$ of the Calabi-Yau$_3$

through the Jacobian's period matrix, which is why we should incorporate the period matrix in the definition of the metric. Secondly the inclusion of the period matrix should be made in a symplectically invariant manner. If we start with symplectically invariant objects from the start, we are bound to find a symplectically invariant metric at the end.

The main problem of the torus-term is in the behavior of dA and dB under symplectic transformations. Under symplectic transformations $S = \left(\begin{smallmatrix} U & Z \\ W & V \end{smallmatrix}\right) \in \mathrm{Sp}(2\bar{n}, \mathbb{R})$ of the basis of $H^3(CY, \mathbb{R})$, A and B transform according to

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

The period matrix $\mathcal{N}$ transforms as (5.22). Note that this follows not only from the physical argumentation in section 5.4, but also from the fact that we can use basis transformations of the lattice Λ and the complex vector space V in our complex torus V/Λ (i.e. the Weil intermediate Jacobian) to keep the period matrix Π in its simple form (3.3), cf. lemma 3.4,

$$\begin{pmatrix} \mathbb{I}_{\bar{n}} \\ \tilde{\mathcal{N}} \end{pmatrix} = S \begin{pmatrix} \mathbb{I}_{\bar{n}} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} U + Z\mathcal{N} \\ W + V\mathcal{N} \end{pmatrix} \sim \begin{pmatrix} \mathbb{I}_{\bar{n}} \\ (W + V\mathcal{N})(U + Z\mathcal{N})^{-1} \end{pmatrix}.$$

We want to define a symplectically invariant combination of dA and dB that depends on the period matrix $\mathcal{N}$ to incorporate the complex structure moduli in a symplectically compatible manner. A symplectically *co*variant object is [DJdWKV98],

$$Y = \mathrm{d}B - \bar{\mathcal{N}}\mathrm{d}A. \tag{7.12}$$

It transforms under symplectic transformations as

$$\tilde{Y} = \left[V - (W + V\bar{\mathcal{N}})(U + Z\bar{\mathcal{N}})^{-1}Z\right](\mathrm{d}B - \bar{\mathcal{N}}\mathrm{d}A) = (U + Z\bar{\mathcal{N}})^{-t}Y.$$

A useful intermediate result to derive this transformation rule is the fact that for symplectic matrices $\left(\begin{smallmatrix} U & Z \\ W & V \end{smallmatrix}\right)$ (and a symmetric matrix $\bar{\mathcal{N}}$),

$$(U + Z\bar{\mathcal{N}})^t(W + V\bar{\mathcal{N}}) = (W + V\bar{\mathcal{N}})^t(U + Z\bar{\mathcal{N}}).$$

Using this latter expression it is also possible to derive the transformation rule of the imaginary part H of $\mathcal{N}$,

$$\tilde{\mathrm{H}} = \frac{1}{2i}(\tilde{\mathcal{N}} - \bar{\tilde{\mathcal{N}}}) = (U + Z\bar{\mathcal{N}})^{-t}\mathrm{H}(U + Z\mathcal{N})^{-1}.$$

From these expressions it is clear why the canonical complex torus line element (3.10) is invariant under symplectic transformations. For our purposes it would however be convenient to absorb one of the factors $U + Z\mathcal{N}$ into our symplectically covariant 1-form Y. Since the *matrix* H is real, symmetric and negative definite by lemma 5.17, we can do this by taking its "square root".

Theorem 7.20 (Cholesky decomposition). *Let $M \in \mathrm{Mat}(n, \mathbb{R})$ be a symmetric $n \times n$-matrix with real coefficients and positive eigenvalues. Then there exists a unique lower triangular matrix $\Theta \in \mathrm{Mat}(n, \mathbb{R})$ with positive diagonal entries such that $M = \Theta\Theta^t$.*

Proof. Cf. [GVL89]. □

Thus we define a lower triangular matrix Θ by

$$\mathrm{H} = -\Theta\Theta^t.$$

Using the transformation rule of H, we may deduce that Θ transforms as $\Theta \to (U + Z\bar{\mathcal{N}})^{-t}\Theta$ under symplectic transformations. *Thus $\Theta^{-1}Y$ is a symplectically invariant combination of* dA *and* dB *which depends on the complex structure moduli through the period matrix $\mathcal{N}$.* It can be written with respect to a real basis by separating its real and imaginary parts,

$$i\Theta^{-1}Y = \Theta^t\mathrm{d}A + i(\Theta^{-1}\mathrm{d}B - \Theta^{-1}\Xi\mathrm{d}A) \doteq \begin{pmatrix} \Theta^t & 0 \\ -\Theta^{-1}\Xi & \Theta^{-1} \end{pmatrix} \begin{pmatrix} \mathrm{d}A \\ \mathrm{d}B \end{pmatrix} = L^{-1} \begin{pmatrix} \mathrm{d}A \\ \mathrm{d}B \end{pmatrix}, \tag{7.13}$$

where $L = \Pi\Upsilon$ is a matrix obtained from multiplying the period matrix $\Pi = \left(\begin{smallmatrix} \mathbb{I} & 0 \\ \Xi & \mathrm{H} \end{smallmatrix}\right)$ with a matrix Υ consisting of the Cholesky decomposition of H,

$$L = \begin{pmatrix} \Theta^{-t} & 0 \\ \Xi\Theta^{-t} & \Theta \end{pmatrix}, \qquad \Upsilon = \begin{pmatrix} \Theta^{-t} & 0 \\ 0 & -\Theta^{-t} \end{pmatrix}. \tag{7.14}$$

The elements $L = \Pi\Upsilon$ form a group, which is a subgroup of the symplectic group.

Definition 7.21 (Lower triangular symplectic group). For every $n \in \mathbb{N}$ the *lower triangular symplectic group* $\mathrm{SpL}(2n, \mathbb{R})$ is the subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ defined by

$$\mathrm{SpL}(2n, \mathbb{R}) = \left\{ \begin{pmatrix} V^{-t} & 0 \\ W & V \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) \middle| V \text{ is lower triangular with positive diagonal entries} \right\}.$$

Despite its name an element $L \in \mathrm{SpL}(2n, \mathbb{R})$ is not lower triangular, but it is as lower triangular as an arbitrary *symplectic* matrix can be. ⊘

Proof. Obviously $\mathbb{I}_{2n} \in \mathrm{SpL}(2n, \mathbb{R})$, so the real check whether $\mathrm{SpL}(2n, \mathbb{R})$ is a subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ is that LM and L^{-1} are elements of $\mathrm{SpL}(2n, \mathbb{R})$ for arbitrary $L, M \in \mathrm{SpL}(2n, \mathbb{R})$. This follows from the fact that the multiplication of two lower triangular matrices with positive diagonal elements and the inverse of a lower triangular matrix with positive diagonal elements are again lower triangular matrices with positive diagonal elements. □

Note that $\left(\begin{smallmatrix} V^{-t} & 0 \\ W & V \end{smallmatrix}\right) \in \mathrm{Sp}(2n, \mathbb{R})$ means that V and W satisfy $WV^t = VW^t$, which is obeyed by the elements $L = \Pi\Upsilon$ of (7.14) due to the symmetry of Ξ. These group elements incorporate the dependence on the period matrix in a symplectically invariant manner, just as we wanted. Conjugating the Maurer-Cartan form with the period matrix Π, transforms the basis of the complex vector space to that of the lattice with respect to which the symplectic form Σ of the Heisenberg group is defined. As such the conjugated Maurer-Cartan form depends on the complex structure moduli through the period matrix. The additional factors of Θ are necessary to ensure that we are working with symplectically invariant objects from the start, which was the second point on our wish list. Due to the presence of conjugation, we incorporate this extra dependence on the period matrix via a semi-direct product. We call it the *toroidal dilatated Heisenberg group* because it is the semi-direct product of the dilatated Heisenberg group and a group of matrices constructed from the period matrices of complex tori.

Definition 7.22 (Toroidal dilatated Heisenberg group). Let $L = \left(\begin{smallmatrix} V^{-t} & 0 \\ W & V \end{smallmatrix}\right)$ be an element of $\mathrm{SpL}(2n, \mathbb{R})$. We may include two extra dimensions into their matrix representations by adding identities,

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V^{-t} & 0 & 0 \\ 0 & W & V & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The *toroidal dilatated Heisenberg group* $\mathrm{HdT}(2n, V, \omega)$ is the semi-direct product of $\mathrm{Hd}(2n, V, \omega)$ and $\mathrm{SpL}(2n, \mathbb{R})$,

$$\mathrm{HdT}(2n, V, \omega) = \mathrm{Hd}(2n, V, \omega) \rtimes \mathrm{SpL}(2n, \mathbb{R}). \tag{7.15}$$

Elements of $\mathrm{HdT}(2n, V, \omega)$ are denoted with $g_{abs\lambda}L_4$. ⊘

Proof. It is important for the semi-direct product construction that for each element $g_{abs\lambda} \in \mathrm{Hd}(2n, V, \omega)$ the conjugation $L_4^{-1} g_{abs\lambda} L_4$ is again an element of $\mathrm{Hd}(2n, V, \omega)$. Explicit calculation reveals that this is the case: the last column of $g_{abs\lambda}$ is changed by L_4^{-1} as in (7.13), while the same happens for the first row by the multiplication of L_4. The combined result is an element $g_{V^t a, -W^t a + V^{-1} b, s\lambda} \in \mathrm{Hd}(2n, V, \omega)$. □

The toroidal dilatated Heisenberg group is the amalgamation of the dilatated Heisenberg structure on the fibres with the symplectic structure on the total fibre bundle. Dividing by its discrete subgroup gives our geometric interpretation of the local c-map's fibre bundle. To establish the correspondence we will again consider the local c-map's line element.

An invariant metric on the fibre bundle

Following a similar reasoning as in section 7.3, we construct a metric on $\mathrm{HdT}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma)$ that is symplectically invariant and invariant under the dilatated Heisenberg group action (7.7). First we calculate the Maurer-Cartan form $\mho$ on $\mathrm{HdT}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma)$,

$$\begin{aligned}
\mho &= (g_{abs\lambda}L_4)^{-1}\mathrm{d}(g_{abs\lambda}L_4) = L_4^{-1}\Omega L_4 + L_4^{-1}\mathrm{d}L_4, && (7.16a)\\
L_4^{-1}\mathrm{d}L_4 &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \Theta^t\mathrm{d}\Theta^{-t} & 0 & 0\\ 0 & \Theta^{-1}(\mathrm{d}\Xi)\Theta^{-t} & \Theta^{-1}\mathrm{d}\Theta & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, && (7.16b)\\
L_4^{-1}\Omega L_4 &= \begin{pmatrix} \frac{\mathrm{d}\lambda}{\lambda} & \frac{-1}{\lambda}(\Theta^{-1}\mathrm{d}b - \Theta^{-1}\Xi\mathrm{d}a)^t & \frac{1}{\lambda}(\Theta^t\mathrm{d}a)^t & \ell\\ 0 & 0 & 0 & \frac{1}{\lambda}\Theta^t\mathrm{d}a\\ 0 & 0 & 0 & \frac{1}{\lambda}(\Theta^{-1}\mathrm{d}b - \Theta^{-1}\Xi\mathrm{d}a)\\ 0 & 0 & 0 & \frac{-\mathrm{d}\lambda}{\lambda}\end{pmatrix}, && (7.16c)
\end{aligned}$$

where Ω is the Maurer-Cartan form (7.10) of the dilatated Heisenberg group.

Definition 7.23 (Invariant metric on the toroidal dilatated Heisenberg group). We define an invariant metric on $\mathrm{HdT}(2\bar{n}, H^3(CY,\mathbb{R}), \Sigma)$ in terms of the Maurer-Cartan form of the toroidal dilatated Heisenberg group and the Killing bilinear form $\mathcal{K}$ of its Lie algebra,

$$\mathrm{d}s^2_{\mathrm{HdT}} = \mathcal{K}(\mho^t, R\mho). \tag{7.17}$$

Here R is a diagonal $(2h^{2,1}+4)\times(2h^{2,1}+4)$-matrix $R = \mathrm{diag}(\alpha, \beta\mathbb{I}_{2\bar{n}}, \gamma)$ consisting of constants $\alpha, \beta, \gamma \in \mathbb{R}$. ⊘

Using lemma 7.18, $\mathrm{d}s^2_{\mathrm{HdT}}$ is proportional to

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{HdT}} = \; & \mathrm{Tr}\left[(L_4^{-1}\Omega L_4)^t R L_4^{-1}\Omega L_4\right] + \mathrm{Tr}\left[(L_4^{-1}\mathrm{d}L_4)^t R L_4^{-1}\mathrm{d}L_4\right] \\
& +2\underbrace{\mathrm{Tr}\left[(L_4^{-1}\Omega L_4)^t R L_4^{-1}\mathrm{d}L_4\right]}_{=0} - \frac{1}{2h^{2,1}+4}\underbrace{\mathrm{Tr}\left[\mho^t\right]}_{=0}\mathrm{Tr}\left[R\mho\right].
\end{aligned} \tag{7.18}$$

Since $\mathrm{Tr}\left[(\mathrm{d}\Theta^{-1}\Theta)^t\right] + \mathrm{Tr}\left[\Theta^{-1}\mathrm{d}\Theta\right] = \mathrm{Tr}\left[\mathrm{d}\mathbb{I}_{\bar{n}}\right] = 0$, the Maurer-Cartan form (7.16) is traceless and therefore the last term in (7.18) does not contribute to the line element. The third term vanishes as well, (partly) due to the huge number of zeroes in $L_4^{-1}\Omega L_4$. The only remaining contribution is therefore the first line of (7.18). The first term contains differentials only of the fibre coordinates (A, B, σ, ϕ), since Ω is the Maurer-Cartan form of the dilatated Heisenberg group. The second term contains only differentials of the coordinates X of the base through the period matrix $\mathcal{N}$. As a result there will be no contributions in the metric $\mathrm{d}s^2_{\mathrm{HdT}}$ that mix differentials of the base and fibre coordinates, yielding a Pythagorean structure on the fibre bundle. The base space is justifiably called base space, because its metric does not depend on the fibre coordinates. For the moment we will ignore the second term of (7.18) and we will consider only derivatives in the fibre direction.

The metric on the fibres depends on the complex structure moduli via the period matrix $\mathcal{N}$. By calculating the first term of (7.18) we may verify this explicitly,

$$\begin{aligned}
\mathrm{d}s^2_{fibre} &= (\alpha+\gamma)\frac{\mathrm{d}\lambda^2}{\lambda^2} + \alpha\ell^2 + \frac{\alpha}{\lambda^2}\mathrm{Tr}\left[\begin{pmatrix}\Theta^{-1}\Xi\mathrm{d}a - \Theta^{-1}\mathrm{d}b\\ \Theta^t\mathrm{d}a\end{pmatrix}\left((\Theta^{-1}\Xi\mathrm{d}a - \Theta^{-1}\mathrm{d}b)^t, (\Theta^t\mathrm{d}a)^t\right)\right] \\
&\quad + \frac{\beta}{\lambda^2}\left((\Theta^t\mathrm{d}a)^t, (\Theta^{-1}\mathrm{d}b - \Theta^{-1}\Xi\mathrm{d}a)^t\right)\begin{pmatrix}\Theta^t\mathrm{d}a\\ \Theta^{-1}\mathrm{d}b - \Theta^{-1}\Xi\mathrm{d}a\end{pmatrix} \\
&= -\frac{\alpha+\beta}{\lambda^2}\bar{y}^t(\mathrm{Im}\,\mathcal{N})^{-1}y + (\alpha+\gamma)\frac{\mathrm{d}\lambda^2}{\lambda^2} + \frac{\alpha}{\lambda^4}\left(\mathrm{d}s - a^t\overleftrightarrow{\mathrm{d}}b\right)^2,
\end{aligned} \tag{7.19}$$

where $y = \mathrm{d}b - \bar{\mathcal{N}}\mathrm{d}a$ is the 1-form occurring in the canonical line element of a complex torus (3.10).

This line element shows close resemblance to the metric on the fibres of the local c-map, (7.2b) and (7.2c). The first term of (7.19) is the canonical line element on the Weil intermediate Jacobians, cf. (7.5), including a dilatational effect and including the dependence on the complex structure moduli through the period matrix $\Pi = \binom{\mathbb{I}_{\bar{n}}}{\mathcal{N}}$. The other two terms are the intertwining between the (R-R)-fields and the dilaton-axion system through a dilatational contribution and a typical contact form $\mathrm{d}s - a^t \overleftrightarrow{\mathrm{d}}\, b$. The relation between (7.2) and (7.19) can be made explicit by setting,

$$\alpha = \beta = \frac{1}{3}\gamma = 1, \qquad a = A, \quad b = B, \quad s = \sigma, \quad \lambda = e^{1/2\phi}.$$

Based upon these observations we have arrived at the following geometric interpretation of the local c-map.

Conjecture 7.24 (Geometric interpretation of the local c-map). *The local c-map constructs a principal-like fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$ on the complex structure moduli space $\mathscr{M}^{2,1}$. Its typical fibre is given by a quotient of the dilatated Heisenberg group modulo its integer subgroup, whose underlying symplectic vector space is $(H^3(CY,\mathbb{R}), Q)$. The bundle of Weil intermediate Jacobians appears as a torus-subbundle of this fibre bundle and the additional dilaton-axion system is intertwined with these tori via a dilatated Heisenberg group structure. The total fibre bundle is isomorphic to the toroidal dilatated Heisenberg group (modulo its integer subgroup) and the metric on the fibres is given by a (part of a) Wess-Zumino-Witten model,*

$$\boxed{\mathrm{d}s^2_{fibres} = \mathcal{K}(\mho^t, R\mho),} \tag{7.20}$$

where $\mho$ is the Maurer-Cartan form of the toroidal dilatated Heisenberg group, $\mathcal{K}$ the Killing form of its Lie algebra and R a constant diagonal square $(2h^{2,1}+4)$-matrix, $R = \mathrm{diag}(1, \mathbb{I}_{2\bar{n}}, 3)$.

Conjecture 7.24 expresses the metric on the fibres of the local c-map's fibre bundle in terms of the Killing form and Maurer-Cartan form of its structure group. In the preceding discussion we have seen a large part of the proof of this conjecture, but we have also seen that a full understanding is still lacking. We have incorporated the dilatated Heisenberg structure of the typical fibre, the dependence of the fibres on the complex structure moduli and the overall symplectic structure of the bundle of fibres. However the effect of the second term of (7.18), the appearance of the constant matrix R in our model and the precise necessity for the *lower triangular* symplectic group are still to be discussed.

The second term of (7.18) is a metric on the base space of the local c-map's fibre bundle $\mathscr{M}_{hm} \to \mathscr{M}^{2,1}$. We have not been able to calculate this term explicitly, but ideally one would hope that it provides the projective special Kähler structure on $\mathscr{M}^{2,1}$. It seems unlikely however that the total metric on the type IIA hypermultiplet's moduli space is given by a Wess-Zumino-Witten model. If that were the case the moduli space would be a homogeneous space, which is not true for arbitrary prepotential F. Therefore next to the second term of (7.18), which does yield a symplectically invariant metric, another contribution to the metric on the base space is to be expected that stems from a yet unravelled contribution on top of our Wess-Zumino-Witten model.[2] A closer examination of the second term in (7.18) could possibly reveal a more complete interpretation of the local c-map.

Because the second term of (7.18) depends solely on the Maurer-Cartan form of the lower triangular symplectic group, an issue related to this question is the appearance of the lower triangular matrices rather than the ordinary symplectic group itself. We will present our current understanding of their appearance in a moment, together with the analogue of the elliptic modular surface for the Weil intermediate Jacobians as in section 6.3. The integer subgroup of the toroidal dilatated Heisenberg group will evolve naturally in that discussion. Before we consider the symplectic structure, we will give a few comments on the appearance of the constant matrix R in our model.

[2] We thank V. Cortés for pointing this out to us.

Quaternionic structure and quantum corrections

So far we have analyzed the local c-map using three ingredients: symplectic structure, complex structure and dilatated Heisenberg group structure. The compatibility between symplectic and complex structure has led to the introduction of Weil intermediate Jacobians, while the combination of symplectic and dilatated Heisenberg structure has amalgamated the (R-R)-fields and dilaton-axion system into a total fibre bundle. The appearance of the diagonal matrix R of constants in the definition of the metric (7.20) is an indication that our interpretation still misses some ingredients in order to really understand the *quaternionic* structure of $\mathscr{M}_{hm}$.

Only by choosing the correct matrix R, we make sure that the metric (7.19) contributes to a quaternion-Kähler metric. For example if we would have chosen (more trivially) $R = \mathbb{I}_{\bar{n}+2}$, it would have been impossible to find a basis transformation which transforms (7.19) exactly into the metric (7.2) on the hypermultiplet moduli space, including the correct relative prefactors. This implies that the quaternion-Kähler structure is not captured by merely considering the toroidal dilatated Heisenberg group. *The quaternion-Kähler structure depends sensitively on the relative prefactors of the three different terms in (7.19), whereas each of the terms is already invariant on its own under the symplectic and dilatated Heisenberg group action.* Invariance under the toroidal dilatated Heisenberg group does not fix the relative prefactors, which is exactly what we need in order to find a quaternionic metric. For arbitrary matrix R we will find a metric that is invariant under the symplectic and dilatated Heisenberg group transformations, while only one particular R leads to a quaternion-Kähler metric $\mathcal{K}(\mho^t, R\mho)$. Apparently the quaternion-Kähler structure puts a restriction on the form of R, but at the moment the mechanism that explains this restriction is unclear to us. In our construction the investigation of the quaternionic structure of $\mathscr{M}_{hm}$ could be approached by analyzing the appearance of the matrix R.

From a physical point of view the remaining freedom in our construction is very much welcome. If our construction would have completely fixed the metric, it would have been impossible to include quantum corrections later on. Quantum corrections have already been observed at various places in the literature, e.g. [RLSV06, RLSTV07]. They change the moduli space and break some of the symmetries of the hypermultiplet metric (7.2), although the moduli space remains quaternion-Kähler due to general supersymmetry arguments [BW83]. At the perturbative level the dilatational invariance is broken, but the continuous Peccei-Quinn symmetry, i.e. the Heisenberg group structure, survives [Str98, AMTV03, ARV04, DTV04]. Thus at the perturbative level the matrix R could for example contain terms depending on ϕ instead consisting purely of constant numbers. A further investigation of the perturbative [Str98, AMTV03, RLSV06] and even nonperturbative effects [RLSTV07] on the moduli space should reveal whether these corrections are indeed to be incorporated into the construction of conjecture 7.24 by means of adapting R. Although we have not focused on quantum corrections in this research, it is good to know that our construction allows room for these effects.

Symplectic structure and the analogue of the elliptic modular surface

To conclude our discussion of the local c-map, we will again try to establish a parallel with the elliptic modular surface. In section 6.3 we have argued that the rigid c-map could be written as a bundle of Griffiths intermediate Jacobians over the complex structure moduli (6.17). In this expression the moduli space of period matrices of Griffiths intermediate Jacobians, $\mathrm{Sp}(2\bar{n},\mathbb{Z})\backslash\mathrm{Sp}(2\bar{n},\mathbb{R})/\mathrm{U}(h^{2,1},1)$, is interpreted as the moduli space of Lagrange planes in the third cohomology group $H^3(CY,\mathbb{R})$. The inclusion of the complex vector space $\mathbb{C}^{\bar{n}}$ in (6.17) means that a tautological bundle of Lagrange planes over its moduli space is constructed. Due to a $\mathbb{Z}^{2\bar{n}}$-action which depends on the period matrix $\mathrm{T} = \overline{\mathrm{Hess}}$ of the Griffiths intermediate Jacobians, the Lagrange planes are turned into tori by identifying points in the lattice directions, cf. (6.15).

In the local situation this construction should be refined. To identify the real complex structure deformations, the $H^{3,0}$-direction within the Lagrange plane L_t should be determined by including a projector

$$P^{3,0} = \frac{XX^tN}{X^tNX}, \tag{7.21}$$

on the $H^{3,0}(CY)$-line. We see that in order to characterize the complex structure of the Calabi-Yau$_3$ we need

1. a symplectic splitting of $(H^3(CY,\mathbb{R}),Q)$,
2. a Lagrange plane L_t of the symplectic form, which moves with respect to
3. a fixed lattice $H^3(CY,\mathbb{Z}) \subset H^3(CY,\mathbb{R})$ *and finally*
4. a projector $P^{3,0}$ which determines the $H^{3,0}(CY)$-direction within the Lagrange plane.

Since the projector is defined in terms of the A-periods $X = \int_{\gamma_A} vol$ of the holomorphic volume form vol (which are identified themselves as projective coordinates for the complex structure moduli), we need to include the complex structure moduli explicitly in the fibre bundle. Therefore the local c-map, or cW-map, constructs a bundle $\mathscr{M}_{hm}$ over the complex structure moduli space $\mathscr{M}^{2,1}$ which may locally be written as,

$$\mathscr{M}_{hm} \cong (\mathrm{Hd}(H^3(CY,\mathbb{Z})) \times \mathrm{Sp}(\mathbb{Z}))\backslash(\mathbb{C}^{\bar{n}} \times \mathrm{Hd}(H^3(CY,\mathbb{R})) \rtimes \mathrm{Sp}(\mathbb{R}))/(\mathbb{C}^* \times \mathrm{U}(\bar{n})). \tag{7.22}$$

A number of different structures may be recognized in this expression. The $\mathbb{C}^{\bar{n}}$ has a $\mathbb{C}^*$-action defined upon it to project out the compensator component in $(X^I)_{I=0}^{h^{2,1}}$. The Ramond fields A and B are included into the dilatated Heisenberg group, which intertwines them with the dilaton-axion system. They form a complex torus by the orbit-identifications of the $H^3(CY,\mathbb{Z})$-action that is captured in $\mathrm{Hd}(H^3(CY,\mathbb{Z}))$. It is at this point that the extra dependence on $\mathscr{M}^{2,1}$ enters, as the action of $H^3(CY,\mathbb{Z})$ on A and B can be defined by,

$$(k,l)\cdot(B-(\overline{\mathrm{Hess}}+iNP^{3,0})A) = B+k-(\overline{\mathrm{Hess}}+iNP^{3,0})(A+l). \tag{7.23}$$

The symplectic group is responsible for connecting different fibres, since the moduli space of period matrices of the different Weil intermediate Jacobians is $\mathrm{Sp}(2\bar{n},\mathbb{Z})\backslash\mathrm{Sp}(2\bar{n},\mathbb{R})/\mathrm{U}(\bar{n})$, cf. section 3.1 and note that $\mathrm{U}(0,\bar{n}) = \mathrm{U}(\bar{n},0) = \mathrm{U}(\bar{n})$. We suspect that $\mathrm{Sp}(2\bar{n},\mathbb{R})/\mathrm{U}(\bar{n})$ corresponds with our lower triangular symplectic group. This correspondence comes about by considering the following *free* and transitive group action of $\mathrm{SpL}(2\bar{n},\mathbb{R})$ on the space of period matrices with negative definite imaginary part.

Proposition 7.25 (Free transitive group action). $\mathrm{SpL}(2n,\mathbb{R})$ *has a free and transitive group action on the moduli space of period matrices with negative definite imaginary part,*

$$L = \begin{pmatrix} V^{-t} & 0 \\ W & V \end{pmatrix} \cdot \mathrm{T} = WV^t + V\mathrm{T}V^t. \tag{7.24}$$

Proof. Let $\mathrm{T} = \Xi + i\mathrm{H}$ be an arbitrary element in the space of period matrices with negative definite imaginary part. By the Cholesky decomposition there exists a unique lower triangular matrix Θ with positive diagonal elements such that $\mathrm{H} = -\Theta\Theta^t$. Therefore the element T may be obtained from a fixed element $-i\mathbb{I}_n$ via a transformation by the element,

$$L = \begin{pmatrix} \Theta^{-t} & 0 \\ \Xi\Theta^{-t} & \Theta \end{pmatrix}.$$

Furthermore the group action is free, since $WV^t - iVV^t = -i$ implies that $V = \mathbb{I}_n$ (it must be lower triangular with positive diagonal elements) and $W = 0$. Therefore the isotropy group $\mathrm{SpL}(2n,\mathbb{R})_{-i}$ is trivial. □

By proposition 7.15, $\mathrm{SpL}(2\bar{n},\mathbb{R})$ parameterizes the period matrices of Weil intermediate Jacobians just as $\mathrm{Sp}(2\bar{n},\mathbb{R})/\mathrm{U}(\bar{n})$ does, cf. section 3.1. The correspondence between $\mathrm{Sp}(2\bar{n},\mathbb{R})/\mathrm{U}(\bar{n})$ and $\mathrm{SpL}(2\bar{n},\mathbb{R})$ becomes more evident by writing an element $S \in \mathrm{Sp}(2\bar{n},\mathbb{R})$ as a product $S = LA$ of $L \in \mathrm{SpL}(2\bar{n},\mathbb{R})$ and $A = \left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right) \in \mathrm{U}(\bar{n}) \subset \mathrm{Sp}(2\bar{n},\mathbb{R})$. It may be checked that the first term of our model (7.18) does not depend on A when using $S = LA \in \mathrm{Sp}(2\bar{n},\mathbb{R})$ instead of a lower triangular symplectic matrix $L \in \mathrm{SpL}(2\bar{n}, \mathbb{R})$. For this reason our model seems to be well-defined on the equivalence classes of $\mathrm{Sp}(2\bar{n}, \mathbb{R})/\mathrm{U}(\bar{n})$. Nevertheless we have not been able to explicitly check whether or not each $S \in \mathrm{Sp}(2\bar{n}, \mathbb{R})$ can be uniquely decomposed in a lower triangular and unitary part. Since $\mathrm{U}(\bar{n})$ and $\mathrm{SpL}(2\bar{n}, \mathbb{R})$ are not normal subgroups of $\mathrm{Sp}(2\bar{n}, \mathbb{R})$, it seems unlikely that this is the case. The question is therefore in what manner $\mathrm{SpL}(2\bar{n}, \mathbb{R})$ and $\mathrm{Sp}(2\bar{n}, \mathbb{R})/\mathrm{U}(\bar{n})$ are isomorphic (as manifolds) to each other and to the moduli space of period matrices with negative definite imaginary part. A resolution to this question would probably provide a better understanding of the second term of (7.18) and of our geometric interpretation of the local c-map in general.

Chapter 8

Conclusions and outlook

In this thesis we have analyzed the rigid and local c-maps between the moduli spaces of the four-dimensional effective $N = 2$ supersymmetry theories resulting from type II superstring theories after compactification on a family of Calabi-Yau$_3$'s. We have tried to give a geometric interpretation for the form of the metric on the fibre bundle that is constructed by mapping the type IIB (gravity and) vector multiplet sectors to the type IIA hypermultiplet moduli space. As was already known in the literature, the rigid c-map from IIB to IIA may be understood as a bundle of Griffiths intermediate Jacobians over the moduli space of gauged Calabi-Yau$_3$'s [Cor98]. In the local c-map a bundle of Weil intermediate Jacobians is constructed over the complex structure moduli space. The complex structure moduli space is the scalar manifold of the IIB vector multiplet sector and it appears as the base manifold in the fibre bundle construction. In addition to the vector multiplets, the gravity multiplet contributes one vector field, the graviphoton, which is combined with the Ramond vectors of the vector multiplets into scalar field coordinates of the Weil intermediate Jacobians. Moreover the gravity multiplet produces two extra scalar fields which are responsible for an additional $\mathbb{R}_{>0} \times S^1$-bundle that is split by the bundle of Weil intermediate Jacobians. It is found that the structure of the combined gravitational and Weil intermediate Jacobian's typical fibre is that of a dilatationally extended Heisenberg group, divided by its integer subgroup due to Peccei-Quinn and dilatational invariance. As we understand now, the fibres are combined symplectically in a semi-direct product of the dilatated Heisenberg group and (a subgroup of) the symplectic group, yielding a construction quite similar to the elliptic modular surface. We have tried to describe the fibre bundle's geometry via a Wess-Zumino-Witten model which consists of the Killing bilinear form acting on the Maurer-Cartan form of the semi-direct product. For the fibres this works out perfectly, while its geometric contribution to the base manifold is expected to be more involved.

In future research our geometric interpretation could be helpful for a better understanding of the hypermultiplet moduli space. The behavior of the Weil intermediate Jacobians under variations of the underlying Calabi-Yau$_3$'s complex structure could possibly illuminate the nonintegrable quaternionic nature of the type IIA supergravity hypermultiplet moduli space as opposed to the integrable quaternionic structure of the bundle of Griffiths intermediate Jacobians. Moreover in our discussion the quaternion-Kähler structure of the hypermultiplet moduli space boils down to choosing a specific constant matrix R in our model. Understanding this matrix will be interesting for an understanding of the quaternionic structure and possibly also for an inclusion of quantum corrections to the metric. Some results about quantum corrections are known already [RLSV06, RLSTV07] but it is not fully understood at the moment how to include all D-brane and spacetime instantons. Freedom in the choice of R and the formalism of intermediate Jacobians could provide a useful mathematical context in which quantum corrections can be studied, for example by combining them with the formalism of *gerbes* [Asp00].

Moreover our geometric translation of the local c-map could pave the way for a general construction of a canonical quaternion-Kähler structure on a fibre bundle over an arbitrary projective special Kähler manifold, just as [Cor98] was able to generalize the rigid c-map's result to the cotangent bundle of arbitrary affine special Kähler manifolds. Our result suggests that an additional $\mathbb{R}_{>0} \times S^1$-bundle is necessary for constructing a quaternion-Kähler structure to account for missing dimensions. The role this additional bundle has to play over the bundle of Weil intermediate Jacobians would therefore be an interesting starting point to investigate general constructions of quaternion-Kähler fibre bundles associated to projective special Kähler manifolds.

Another interesting phenomenon in our construction is the appearance of Weil intermediate Jacobians, which are projective varieties described by an algebraic equation. Since Seiberg-Witten theory relates affine special Kähler geometry with Jacobians of projective curves (which are projective varieties as well), the Weil intermediate Jacobians are the most natural candidate for a generalization of Seiberg-Witten theory to higher dimensions. This could provide new results in a great number of fields, among others integrable systems and the wrapping of branes around 3-cycles instead of 2-cycles.

Finally another ambitious project which could use our results as a starting point, would be the implementation of the intermediate Jacobians of the c-map into the formalism provided by mirror symmetry, cf. lemma 2.40 and below. In this thesis we have considered the c-map from the complex structure moduli space, i.e. the vector multiplet moduli space of type IIB, to the type IIA hypermultiplet moduli space. According to mirror symmetry there should be a similar geometric interpretation for the c-map from the type IIA vector multiplet moduli space consisting of Kähler class deformations to the type IIB hypermultiplet moduli space. This means that there is a "mirror intermediate Jacobian" on the *second* (or fourth) cohomology group of the Calabi-Yau$_3$. Finding such an object would be interesting for physicists, because it enables them to analyze the type IIB hypermultiplet moduli space in an analogous manner as the type IIA hypermultiplet moduli space. It would be interesting for mathematicians as well, since it would lead to a better understanding of mirror symmetry in general. Mirror symmetry between hodge *numbers* would then be extended to the cohomology groups themselves.

This thesis provides a hint in which direction we should look to define a "mirror intermediate Jacobian" on the second (and/or fourth) cohomology group of a Calabi-Yau$_3$. The intermediate Jacobian is defined by considering a polarization on the third cohomology group, while changing the complex structure of that cohomology group. The Jacobian on the second cohomology group could be obtained by defining a similar polarization using a *symmetric* intersection form,

$$Q = \int_{CY} \sigma_J \wedge \xi \wedge \omega,$$

where σ_J is the Kähler form of the Calabi-Yau$_3$. Changing the Kähler structure induces a change of the *polarization*, which provides a similar structure to the Griffiths and Weil intermediate Jacobians. It measures the relative change in symplectic and complex structure, not by fixing the symplectic structure and changing the complex structure but vice versa. Future research should decide whether these ideas are indeed relevant in the construction of the local c-map from type IIA to type IIB and its mirror symmetric relation with the local c-map from type IIB to type IIA.

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