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Completeness in Supergravity Constructions

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Abstract: We prove that the supergravity r- and c-maps preserve completeness. As a consequence, any component $\mathcal{H}$ of a hypersurface $\{h = 1\}$ defined by a homogeneous cubic polynomial h such that $-\partial^2 h$ is a complete Riemannian metric on $\mathcal{H}$ defines a complete projective special Kähler manifold and any complete projective special Kähler manifold defines a complete quaternionic Kähler manifold of negative scalar curvature. We classify all complete quaternionic Kähler manifolds of dimension less or equal to 12 which are obtained in this way and describe some complete examples in 16 dimensions.

1. Introduction

The supergravity r-map and the supergravity c-map are geometric constructions known to theoretical physicists working in supergravity and string theory. They can be obtained by dimensional reduction of the vector multiplet sector of supergravity theories with eight real supercharges from 5 to 4 and from 4 to 3 spacetime dimensions, respectively. The reduction from 4 to 3 dimensions was worked out by Ferrara and Sabharwal [22], who used the resulting explicit local description of the c-map to prove that it maps a projective special Kähler manifold (see Definition 4) of real dimension $2n$ defined by a holomorphic prepotential $F = F(z^0, \ldots, z^n)$ homogeneous of degree two to a quaternionic Kähler manifold of dimension $4n+4$ of negative scalar curvature. Similarly, it was shown by de Wit and Van Proeyen [21] that the r-map maps projective special real manifolds (see Definition 1) of dimension n defined by a cubic polynomial $h = h(x^1, \ldots, x^n)$ to projective special Kähler manifolds of dimension $2n + 2$.

Despite some recent advances in the geometric understanding of the r-map [3] and the c-map [29], as well as in finding new formulations of the c-map within the formalisms of supergravity [16,18,19,36], very little is known about global geometric properties of these constructions. In recent approaches to hypermultiplet moduli spaces, there have been efforts aimed at the computation of quantum corrections to the Ferrara-Sabharwal metric inspired by the work of Gaiotto, Moore and Neitzke on wall crossing [24]. While work along this line has been successful in the limit where gravity decouples, that is in the framework of hyper-Kähler geometry, part of the quantum corrections to the Ferrara-Sabharwal metric have been computed using string dualities [25–27], and the connection to wall crossing has been discussed in [28]. Despite these encouraging results our understanding of the quantum corrected geometry of hypermultiplets coupled to supergravity is still limited. In particular, the problem of completeness of quaternionic Kähler metrics obtained as deformations (e.g. by quantum corrections) of quaternionic Kähler metrics constructed by the c-map is an interesting subject for future investigation.

The main results of this paper are Theorem 4 and Theorem 5, which state that the supergravity r-map and the supergravity c-map preserve completeness. As a consequence, we obtain an effective method for the construction of complete projective special Kähler and quaternionic Kähler manifolds starting from certain real affine hypersurfaces defined by homogeneous cubic polynomials h. Any such hypersurface of dimension n defines a quaternionic Kähler manifold of dimension $4n + 8$, that is a point defines an 8-fold, a curve defines a 12-fold, a surface defines a 16-fold, etc. The study of the completeness of the quaternionic Kähler manifold is reduced to that of the completeness of the cubic hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ with respect to the Riemannian metric $-\partial^2 h|_{\mathcal{H}}$. We show how to obtain interesting complete examples and even classification results in low dimensions. In particular, we find two complete inhomogeneous examples, see Corollary 4 b) and Example 3. The homogeneous examples are automatically complete, as is any complete Riemannian manifold. Moreover, all known examples of homogeneous projective special Kähler manifolds with exception of the complex hyperbolic spaces are in the image of the r-map and all known examples of homogeneous quaternionic Kähler manifolds of negative scalar curvature with exception of the quaternionic hyperbolic spaces are in the image of the c-map, see [21] and references therein. The known examples include the homogeneous projective special Kähler manifolds of semisimple groups classified in [2] and the quaternionic Kähler manifolds admitting a simply transitive (completely) solvable group of isometries classified in [1,9]. The first class contains only Hermitian symmetric spaces of noncompact type, whereas the second class contains all quaternionic Kähler symmetric spaces of noncompact type as well as all known nonsymmetric homogeneous quaternionic Kähler manifolds. We plan to work towards general completeness results for hypersurfaces in higher dimensions in the future.

One of the inhomogeneous complete examples is the 'quantum STU model' [6,5,31, 30]. This model, or actually family of models, can be constructed by compactification of the heterotic string on manifolds with holonomy contained in $SU(2)$ and with instanton numbers (12, 12) or (13, 11) or (10, 4). The special real, special Kähler and special quaternionic manifold occur as moduli spaces of compactifications on $K3 \times S^1$, $K3 \times T^2$ and $K3 \times T^3$, respectively. The qualification 'quantum' refers to the quantum corrections to the cubic part of the underlying Hesse potential

$$h = STU \to h = STU + \frac{1}{3}U^3,$$

which makes the corresponding manifolds inhomogeneous. This deformation is captured by the triple intersection forms of dual models, which are compactifications on Calabi-Yau threefolds which are elliptic fibrations over the Hirzeburch surfaces F_0, F_1, F_2, respectively [31,34,40]. The choice of the base of the fibration corresponds to the choice of instanton numbers in the dual heterotic model. The heterotic compactifications on $K3 \times T^2$ are dual to type-IIA compactifications on the corresponding Calabi-Yau threefolds, and the vector multiplet moduli spaces of these models are the complexified Kähler cones of the Calabi-Yau threefolds. The type-IIA model has an 'M-theory lift' to a compactification of eleven-dimensional M-theory on the same Calabi-Yau threefold, which is then dual to the heterotic compactification on $K3 \times S^1$. The moduli spaces of five-dimensional vector multiplets correspond to a fixed volume slice of the (real) Kähler cones of the underlying Calabi-Yau threefolds. Some properties of the special Kähler and special real metrics occurring in the vector multiplet sectors of these models have been discussed in the physics literature [7,32,33]. The geodesically complete spaces considered in this paper are the natural choices of scalar manifolds if these models are considered as supergravity models. The moduli spaces relevant for string theory are sub-domains of these manifolds, as discussed in [7,32–34,40].

In the last section we give a geometric interpretation of the complex $(n+1) \times (n+1)$-matrix $\mathcal{N} = (\mathcal{N}_{IJ})$, which defines the nontrivial part of the Ferrara-Sabharwal metric. We show that $\mathcal{N}$ defines a Weil flag which is precisely the image of the Griffiths flag associated with the variation of Hodge structure of weight 3 defined by the underlying affine special Kähler manifold under a natural $\mathrm{Sp}(\mathbb{R}^{2n+2})$-equivariant map from Griffiths to Weil flags, see Corollary 5. Furthermore, $\mathcal{N}$ is canonically associated with a positive definite metric which differs from the (indefinite) affine special Kähler metric by a canonical sign switch, see Corollary 6. Using this geometric insight, we are able to extend the c-map and our completeness result to special Kähler manifolds which do not admit a global description by a single prepotential, see Theorem 10.

2. Completeness of Metrics on Product Manifolds and Bundles

Let us recall that a Riemannian manifold (M, g) is called complete if it is complete as a metric space, i.e. if every Cauchy sequence in M converges. The basic result in Riemannian geometry concerning completeness is the following theorem of Hopf and Rinow, cf. [35] Ch. 5, Thm. 21.

Theorem 1 (Hopf-Rinow). *For a Riemannian manifold (M, g) the following conditions are equivalent:*

(i) *M is complete.*
(ii) *M is geodesically complete, i.e. every inextendible geodesic ray has infinite length.*
(iii) *Any closed and bounded subset of M is compact.*

We give another equivalent formulation of completeness in terms of (smooth) curves $\gamma : I \to M$, where $I \subset \mathbb{R}$ is a (nondegenerate) interval.

Lemma 1. *A Riemannian manifold (M, g) is complete if and only if every curve $\gamma : I \to M$ which is not contained in any compact subset of M has infinite length.*

Proof. "$\Rightarrow$" Let us assume that M is complete and that $\gamma : I \to M$ is not contained in any compact set. Then $\gamma(I) \subset M$ is unbounded, since it is not contained in any ball. Here we use the fact that closed balls are compact in any complete Riemannian manifold, by Theorem 1 (iii). Clearly, an unbounded curve has infinite length.

"$\Leftarrow$" If M is not complete, then there exists an inextendible geodesic ray $\gamma : [0, L) \to M$ of finite length L, by Theorem 1 (ii). The ray γ is not contained in any compact set K, because otherwise $\gamma(t)$ would converge to a limit point in K for $t \to L$ and γ could then be extended to a geodesic ray $\tilde{\gamma} : [0, \tilde{L}) \to M$ for some $\tilde{L} > L$. $\square$

Let $M = M_1 \times M_2$ be a product manifold and denote by $\pi_i : M \to M_i$, $i = 1, 2$, the projections. We consider Riemannian metrics g of the form

$$g = g_1 + g_2, \tag{2.1}$$

where g_1 is (the pullback of) a Riemannian metric on M_1 and $g_2 \in \Gamma(\pi_2^* S^2 T^* M_2)$ is a family of Riemannian metrics on M_2 depending on a parameter $p \in M_1$. Notice that the tensors g_1 and g_2 take the form

$$g_1 = \sum g^{(1)}_{ab}(x) dx^a dx^b, \quad g_2 = \sum g^{(2)}_{\alpha\beta}(x, y) dy^\alpha dy^\beta,$$

with respect to local coordinates $x = (x^a)$ on M_1 and $y = (y^\alpha)$ on M_2.

We will assume that the tensor field g_2 satisfies the following condition:

(C) For all compact subsets $A \subset M_1$ there exists a complete Riemannian metric g_A on M_2 such that $g_2 \geq \pi_2^* g_A$ on $A \times M_2 \subset M$.

Lemma 2. *Assume that (M_1, g_1) is complete and that g_2 satisfies Condition (C). Then (M, g) is complete.*

Proof. Let $\gamma = (\gamma_1, \gamma_2) : I \to M = M_1 \times M_2$ be a curve which is not contained in any compact subset $K \subset M$. In view of Lemma 1, it suffices to show that any such curve γ has infinite length. From the assumption on γ it follows that

(i) $\gamma_1 : I \to M_1$ is not contained in any compact subset $K_1 \subset M_1$ or
(ii) $\gamma_2 : I \to M_2$ is not contained in any compact subset $K_2 \subset M_2$.

(Otherwise, γ would be contained in a compact set $K = K_1 \times K_2$.) In case (i), $L(\gamma_1) = \infty$, by the completeness of (M_1, g_1) and Lemma 1. Comparing lengths we obtain

$$L(\gamma) \geq L(\gamma_1) = \infty$$

and, hence, $L(\gamma) = \infty$. If (i) is not satisfied, then $\gamma_1(I)$ is contained in a compact set $A = K_1 \subset M_1$. By the completeness of (M_2, g_A) and property (ii), we now have that $L_{g_A}(\gamma_2) = \infty$. Now it suffices to compare the lengths: $L(\gamma) \geq L_{g_A}(\gamma_2) = \infty$. □

Theorem 2. *Let (M_1, g_1) be a complete Riemannian manifold and $(g_2(p))_p$ a smooth family of G-invariant Riemannian metrics on a homogeneous manifold $M_2 = G/K$, depending on a parameter $p \in M_1$. Then the Riemannian metric $g = g_1 + g_2$ on $M = M_1 \times M_2$ is complete. Moreover, the action of G on M_2 induces an isometric action of G on (M, g).*

Proof. The last assertion is obvious. For the completeness of (M, g), it suffices to check that g_2 satisfies Condition (C) of Lemma 2. We use the natural one-to-one correspondence between G-invariant Riemannian metrics on $M_2 = G/K$ and K-invariant scalar products on the vector space $T_o M_2 \cong \mathfrak{g}/\mathfrak{k}$, $o = eK$. Under this correspondence, the family $(g_2(p))_p$ corresponds to a family $(\beta(p))_p$ of scalar products. For every compact subset $A \subset M_1$ the family $(\beta(p))_{p \in A}$ is uniformly bounded from below by a scalar product β_A:

$$\beta(p) \geq \beta_A, \quad \text{for all} \quad p \in A.$$

This implies

$$g_2(p) \geq g_A, \quad \text{for all} \quad p \in A,$$

for the G-invariant Riemannian metric g_A associated with β_A. Now it suffices to remark that g_A is complete, as is any G-invariant Riemannian metric on G/K. □

Corollary 1. *Let $g_U = \sum g_{ab} dx^a dx^b$ be a complete Riemannian metric on a domain $U \subset \mathbb{R}^n$. Then the metric*

$$g_M = \frac{3}{4} \sum g_{ab}(x)(dx^a dx^b + dy^a dy^b) \tag{2.2}$$

on $M = U \times \mathbb{R}^n$ is complete. The action of $\mathbb{R}^n$ by translations in the y-coordinates is isometric and the projection $M \to U$ is a principal fiber bundle with structure group $\mathbb{R}^n$. The submanifold $U = U \times \{0\} \subset U \times \mathbb{R}^n = M$ is totally geodesic.

(The factor $\frac{3}{4}$ is introduced only in order to obtain the usual normalization of the projective special Kähler metric for the r-map defined in the next section.)

Proof. For completeness it suffices to apply Theorem 2 to the case $M_2 = G = \mathbb{R}^n$, $g_2 = \sum g_{ab}(x) dy^a dy^b$. $U \subset M$ is totally geodesic as the fixed point set of the isometric involution $(x, y) \mapsto (x, -y)$. □

2.1. Generalisation to the case of bundles. More generally, for later applications we consider now a bundle $\pi : M \to M_1$ with standard fiber M_2 over a Riemannian manifold (M_1, g_1). We suppose that the total space M is endowed with a Riemannian metric g such that for all $p \in M_1$ there exists a neighbourhood $U \subset M_1$ and a local trivialisation $\pi^{-1}(U) \cong U \times M_2$ with respect to which the metric takes the form

$$g|_{\pi^{-1}(U)} = g_1|_U + g_2^U, \tag{2.3}$$

where g_2^U is a smooth family of Riemannian metrics on M_2 depending on a parameter in U. Such metrics g will be called bundle metrics. We will assume that g_2^U satisfies Condition (C) for all compact subsets $A \subset U$. Lemma 2 and Theorem 2 have the following straightforward generalisations:

Lemma 3. *Assume that (M_1, g_1) is complete and that the local fiber metrics g_2^U in Eq. (2.3) satisfy Condition (C) for all compact subsets $A \subset U$. Then the bundle metric g is complete.*

Theorem 3. *Let g be a bundle metric on a bundle $\pi : M \to M_1$ over a complete Riemannian manifold (M_1, g_1) and assume that the standard fiber is a homogeneous space $M_2 = G/K$ and that the fiber metrics g_2^U are G-invariant. Then (M, g) is complete.*

3. The Generalized r-map Preserves Completeness

Let $U \subset \mathbb{R}^n$ be a domain which is invariant under multiplication by positive numbers and let $h : U \to \mathbb{R}^{>0}$ be a smooth function which is homogeneous of degree $d \in \mathbb{R} \setminus \{0, 1\}$. Then

$$\mathcal{H} := \{x \in U | h(x) = 1\} \subset U$$

is a smooth hypersurface and $U = \mathbb{R}^{>0} \cdot \mathcal{H} \cong \mathbb{R}^{>0} \times \mathcal{H}$. We will assume that $-\frac{1}{d}\partial^2 h$ is positive definite on $T\mathcal{H}$. This easily implies that $-\frac{1}{d}\partial^2 h$ is a Lorentzian (if $d > 1$) or Riemannian (if $d < 1$) metric on U which restricts to a Riemannian metric $g_{\mathcal{H}}$ on $\mathcal{H}$.

Definition 1. *The Riemannian manifold* $(\mathcal{H}, g_{\mathcal{H}})$ *is called a* projective special real manifold *if, in addition, h is a polynomial function of degree $d = 3$.*

Example. Let $V = H^{1,1}(X, \mathbb{R})$ be the $(1, 1)$-cohomology of a compact Kähler manifold X of complex dimension $d \geq 2$ and $U \subset V$ the Kähler cone. We define a homogeneous polynomial h of degree d on V by

$$h(a) = a^{\cup d} = a \cup \cdots \cup a, \quad a \in V.$$

The polynomial h defines a positive function on U and the metric $g_{\mathcal{H}}$ defined above is positive definite on the hypersurface

$$\mathcal{H} := \{x \in U | h(x) = 1\} \subset V \cong \mathbb{R}^n, \quad n = h^{1,1}(X).$$

This follows from the Hodge-Riemann bilinear relations, see [39] Chap. V, Sect. 6, which imply that $g_{\mathcal{H}}$ is positive definite on the primitive cohomology $H_0^{1,1}(X, \mathbb{R}) = T_\kappa \mathcal{H}$ defined as the kernel of the cup product with $\kappa^{d-1} : H^2(X, \mathbb{R}) \to H^{2d}(X, \mathbb{R})$, where $\kappa \in U$ is a Kähler class. If $d = 3$ then $(\mathcal{H}, g_{\mathcal{H}})$ is a projective special real manifold.

We endow U with the Riemannian metric

$$g_U = -\frac{1}{d}\partial^2 \ln h \tag{3.1}$$

and $M = TU \cong U \times \mathbb{R}^n$ with the metric (2.2).

Proposition 1. *(U, g_U) is isometric to the product $(\mathbb{R} \times \mathcal{H}, dr^2 + g_{\mathcal{H}})$.*

Proof. This is a straightforward calculation using the diffeomorphism

$$\mathbb{R} \times \mathcal{H} \to U, \quad (r, x) \mapsto e^r x,$$

and the formula (3.1). □

Definition 2. *The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (M, g_M)$ is called the* generalized r-map. *The restriction to polynomial functions h of degree $d = 3$ is called the* supergravity r-map.

Next we need to recall the notion of a projective special Kähler manifold [4,20,23]. It is best explained starting from the notion of a conical special Kähler manifold, cf. [4,12]. For the purpose of this paper, we shall restrict the signature of the metric by the condition (iv) in the following definition.

Definition 3. *A* conical special Kähler manifold (M, J, g, ∇, ξ) *is a pseudo-Kähler manifold (M, J, g) endowed with a flat torsionfree connection ∇ and a vector field ξ such that*

(i) $\nabla\omega = 0$*, where $\omega = g(\cdot, J\cdot)$ is the Kähler form,*
(ii) $d^\nabla J = 0$*, where J is considered as a one-form with values in the tangent bundle,*
(iii) $\nabla\xi = D\xi = \mathrm{Id}$*, where D is the Levi-Civita connection,*
(iv) *g is positive definite on the plane $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^\perp$.*

It is shown in [4,12] that the geometric data of a conical special Kähler manifold can be locally encoded in a holomorphic function F homogeneous of degree 2 defined in a domain $M_F \subset \mathbb{C}^n$, see Theorem 2 and Proposition 6 [12]. In fact, F is the generating function of a holomorphic Lagrangian immersion $M_F \to \mathbb{C}^{2n}$, which induces on M_F the structure of a conical special Kähler manifold. F is called the holomorphic prepotential in the supergravity literature [20]. Under the assumptions of Definition 3, the vector fields ξ and $J\xi$ define a holomorphic action of a two-dimensional Abelian Lie algebra. We will assume that this infinitesimal action lifts to a principal $\mathbb{C}^*$-action on M with the base manifold $\bar{M} = M/\mathbb{C}^*$. Then the hypersurface $S := \{p \in M | g(\xi(p), \xi(p)) = 1\} \subset M$ is an S^1-principal bundle over $\bar{M}$. The principal action is isometric, since it is generated by the Killing vector field $J\xi$. Therefore, the Lorentzian metric $g_S = -g|_S$ induces a Riemannian metric $\bar{g}$ on $\bar{M}$, which is easily seen to be Kählerian. In fact, the negative definite Kähler manifold $(\bar{M}, -\bar{g})$ is precisely the Kähler reduction of (M, g) with respect to the above isometric Hamiltonian S^1-action for a positive level of the moment map, which is $\mu = \frac{g(\xi,\xi)}{2}$.

Definition 4. *The Kähler manifold* $(\bar{M}, \bar{g})$ *is called a* projective special Kähler manifold. *The metric* $\bar{g}$ *is called a* projective special Kähler metric.

A standard example of a projective special Kähler metric is the Weil-Petersson metric on the space of complex structure deformations of a Calabi-Yau 3-fold, that is of a compact Kähler manifold with holonomy SU(3), see, for instance, [10,37].

Theorem 4. *The generalized r-map maps complete Riemannian manifolds* $(\mathcal{H}, g_{\mathcal{H}})$ *as above to complete Kähler manifolds* (M, g_M) *with a free isometric action of the vector group* $\mathbb{R}^n$. *The supergravity r-map maps complete projective special real manifolds to complete projective special Kähler manifolds.*

Proof. The isometric action of $\mathbb{R}^n$ exists for general metrics as in Corollary 1. The Kähler property follows from the fact that the metrics g_U considered here are of Hessian type, cf. [3] Prop. 3. In fact, $(\mathcal{H}, g_{\mathcal{H}})$ is mapped to $M = U \times \mathbb{R}^n \cong \mathbb{R}^n + iU \subset \mathbb{C}^n$ with the metric g_M defined by the Kähler potential $-\ln h(x)$, where $(x, y) \in U \times \mathbb{R}^n = M$ is identified with $\zeta = y + ix \in \mathbb{R}^n + iU \subset \mathbb{C}^n$. If h is a cubic polynomial then a simple calculation shows that the metric g_M can be also obtained from the Kähler potential $K(1, \zeta)$ defined by

$$K(z) = -\ln\left(i \sum_{I=0}^{n} \left(z^I \bar{F}_I - F_I \bar{z}^I\right)\right),$$

where

$$F(z^0, \dots, z^n) = h(z^1, \dots, z^n)/z^0 \tag{3.2}$$

is a holomorphic function homogeneous of degree 2 on the domain

$$\tilde{M} := \{z^0 \cdot (1, \zeta) | z^0 \in \mathbb{C}^*, \quad \zeta \in \mathbb{R}^n + iU\} \subset \mathbb{C}^{n+1}.$$

(It suffices to check that $\frac{i}{|z^0|^2} \sum (z^I \bar{F}_I - F_I \bar{z}^I) = 8h(x)$.) This shows that (M, g_M) is a projective special Kähler manifold with the holomorphic prepotential F. The corresponding conical special Kähler manifold is the $\mathbb{C}^*$-bundle $\tilde{M} \to M$ endowed with the affine special Kähler metric $g_{\tilde{M}} = 2 \sum_{I,J=0}^{n} (\mathrm{Im} F_{IJ}) dz^I d\bar{z}^J$, which has signature $(2, 2n)$. By Proposition 1 and Corollary 1, the completeness of $(\mathcal{H}, g_{\mathcal{H}})$ implies that of (U, g_U) and the completeness of (U, g_U) implies that of (M, g_M). □

Corollary 2. *Any complete projective special real manifold* $(\mathcal{H}, g_{\mathcal{H}})$ *of dimension* $n-1 \geq 0$ *admits a canonical realisation as a totally geodesic submanifold of a complete projective special Kähler manifold* (M, g_M) *with a free isometric action of the group* $\mathbb{R}^n$. *Each orbit of* $\mathbb{R}^n$ *is flat and and intersects the submanifold* $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset M$ *orthogonally in exactly one point.*

Proof. The orbits are flat since the metric is translation invariant. The hypersurface $\mathcal{H} \subset U \cong \mathbb{R}^{>0} \times \mathcal{H}$ is totally geodesic by virtue of Proposition 1 and $U \subset M$ is totally geodesic by virtue of Corollary 1. Therefore $\mathcal{H} \subset M$ is totally geodesic. □

Notice that the above proof shows that $\mathcal{H} \subset M = r(\mathcal{H})$ is totally geodesic also for the generalized r-map.

4. The Supergravity c-map Preserves Completeness

A projective special Kähler manifold (M, g_M) which is globally defined by a single holomorphic prepotential F is called a projective special Kähler domain. Notice that the manifolds in the image of the r-map are defined by the prepotential (3.2) and, hence, are examples of projective special Kähler domains. Recall [22] that the supergravity c-map maps projective special Kähler domains (M, g_M) of dimension $2n$ to quaternionic Kähler manifolds (N, g_N) of dimension $4n+4$ and of negative scalar curvature. More generally, we may consider the situation when the projective special Kähler manifold (M, g_M) is covered by a collection of charts U_α on which the special Kähler geometry is encoded by a prepotential F_α. The investigation of this case is postponed to Sect. 6. For a projective special Kähler domain (M, g_M), the quaternionic Kähler metric g_N on $N = M \times \mathbb{R}^{2n+3} \times \mathbb{R}^{>0} \cong M \times \mathbb{R}^{2n+4}$ is of the form

$$g_N = g_M + g_G, \tag{4.1}$$

$$\begin{aligned} g_G = {}& \frac{1}{4\phi^2} d\phi^2 + \frac{1}{4\phi^2}\Big(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)\Big)^2 + \frac{1}{2\phi} \sum \mathcal{I}_{IJ}(p) d\zeta^I d\zeta^J \\ & + \frac{1}{2\phi} \sum \mathcal{I}^{IJ}(p)\Big(d\tilde{\zeta}_I + \sum \mathcal{R}_{IK}(p) d\zeta^K\Big)\Big(d\tilde{\zeta}_J + \sum \mathcal{R}_{JL}(p) d\zeta^L\Big), \end{aligned} \tag{4.2}$$

where $(\tilde{\zeta}_I, \zeta^I, \tilde{\phi}, \phi)$, $I = 0, 1, \ldots, n$, are coordinates on $\mathbb{R}^{2n+4} \supset \mathbb{R}^{2n+3} \times \mathbb{R}^{>0}$ and the metric is defined for $\phi > 0$. The matrices $(\mathcal{I}_{IJ}(p))$ and $(\mathcal{R}_{IJ}(p))$ depend only on $p \in M$ and $(\mathcal{I}_{IJ}(p))$ is invertible with the inverse $(\mathcal{I}^{IJ}(p))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i \frac{\sum_K N_{IK} z^K \sum_L N_{JL} z^L}{\sum_{IJ} N_{IJ} z^I z^J}, \quad N_{IJ} := 2\mathrm{Im} F_{IJ}, \tag{4.3}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates z^I on the underlying conical special Kähler manifold $\tilde{M} \to M$. Notice that the expressions are homogeneous of degree zero and, hence, well defined local functions on M. Also note that our conventions are slightly different from those in [22]. The prepotentials are related by $F = \frac{i}{4} F^{[FS]}$, while $N = N^{[FS]}$. Also note that $\mathcal{I} = -\mathcal{R}^{[FS]}$, and therefore $\mathcal{I}$ is positive definite.

Let (M, g_M) be a special Kähler domain and $N = M \times G$ the corresponding quaternionic Kähler manifold with the Ferrara-Sabharwal metric $g_N = g_M + g_G$. We will show that, for fixed $p \in M$, $g_G(p)$ can be considered as a left-invariant Riemannian metric on a certain Lie group diffeomorphic to $\mathbb{R}^{2n+4}$. We define the Lie group G by putting the following group multiplication on $\mathbb{R}^{2n+4}$:

$$
\begin{aligned}
&(\tilde{\zeta}, \zeta, \tilde{\phi}, \phi) \cdot (\tilde{\zeta}', \zeta', \tilde{\phi}', \phi') \\
&\quad := (\tilde{\zeta} + e^{\phi/2}\tilde{\zeta}', \zeta + e^{\phi/2}\zeta', \tilde{\phi} + e^{\phi}\tilde{\phi}' + e^{\phi/2}(\zeta^t\tilde{\zeta}' - \zeta'^t\tilde{\zeta}), \phi + \phi'),
\end{aligned}
$$

where $\zeta^t = (\zeta^0, \dots, \zeta^n)$ is the transposed of the column vector ζ. We remark that G is isomorphic to the solvable Iwasawa subgroup of $\mathrm{SU}(1, n+2)$, which acts simply transitively on the complex hyperbolic space of complex dimension $n+2$. G is a rank one solvable extension of the $(2n+3)$-dimensional Heisenberg group. We can realise it as a group of affine transformations of $\mathbb{R}^{2n+4}$ by associating to an element $(\tilde{v}, v, \alpha, \lambda) \in G = \mathbb{R}^{2n+4}$ the affine transformation

$$
(\tilde{\zeta}, \zeta, \tilde{\phi}, \phi) \mapsto (e^{\lambda/2}\tilde{\zeta} + \tilde{v}, e^{\lambda/2}\zeta + v, e^{\lambda/2}(\tilde{v}^t\zeta - v^t\tilde{\zeta}) + e^{\lambda}\tilde{\phi} + \alpha, e^{\lambda}\phi). \quad (4.4)
$$

By virtue of this action, we can identify G with the orbit of the point $(0, 0, 1, 0)$, which is $L = \mathbb{R}^{2n+3} \times \mathbb{R}^{>0} \subset \mathbb{R}^{2n+4}$. This identification is simply

$$
G \ni (\tilde{v}, v, \alpha, \lambda) \mapsto (\tilde{v}, v, \alpha, e^{\lambda}) \in L.
$$

One can easily check that the following pointwise linearly independent one-forms are invariant under the transformations (4.4) and, hence, define a left-invariant coframe on $G \cong L$:

$$
\begin{aligned}
\eta^I &:= \sqrt{\frac{2}{\phi}} d\zeta^I, \quad \xi_I := \sqrt{\frac{2}{\phi}} \left(d\tilde{\zeta}_I + \sum \mathcal{R}_{IK}(p) d\zeta^K \right), \\
\xi_{n+1} &:= \frac{d\phi}{\phi}, \quad \eta^{n+1} := \frac{1}{\phi} \left(d\tilde{\phi} + \sum (\zeta^I d\tilde{\zeta}_I - \zeta_I d\tilde{\zeta^I}) \right).
\end{aligned}
\quad (4.5)
$$

This shows, in particular, that the metric $g_G(p)$ is left-invariant for all $p \in M$.

Theorem 5. *The supergravity c-map maps any complete projective special Kähler domain (M, g_M) to a complete quaternionic Kähler manifold (N, g_N) of negative scalar curvature, which admits the free isometric action (4.4) of the group G.*

Proof. The above description of the c-map, starting from (4.1), shows that the quaternionic Kähler manifold (N, g_N) is of the form $N = M \times G$, $g_N = g_M + g_G$, where g_G is a smooth family of left-invariant metrics on the group G depending on a parameter $p \in M$. Therefore, Theorem 2 shows that (N, g_N) is complete if (M, g_M) is complete. □

Corollary 3. *Any complete projective special Kähler domain (M, g_M) admits a canonical realisation as a totally geodesic Kähler submanifold of a complete quaternionic Kähler manifold (N, g_N) with a free isometric action of the group G. Each orbit of G is isometric to a complex hyperbolic space of holomorphic sectional curvature -4 and intersects the submanifold $M \subset N$ orthogonally in exactly one point.*

Proof. The submanifold $M = M \times \{e\} \subset M \times G = N$ of the quaternionic Kähler manifold N defined by the supergravity c-map is the fixed point set of the isometric involution

$$
(\tilde{\zeta}_I, \zeta^I, \tilde{\phi}, \phi) \mapsto (-\tilde{\zeta}_I, -\zeta^I, -\tilde{\phi}, \phi^{-1}).
$$

This implies that M is totally geodesic. Next we compare $g_G(p)$ with the standard left-invariant Kähler metric of constant holomorphic sectional curvature -1 on G, which originates from the simply transitive action of G on the complex hyperbolic space. By a linear change of the coordinates ζ^I we may assume that the positive definite symmetric matrix $\mathcal{I}_{IJ}(p) = \delta_{IJ}$. Then

$$g_G(p) = \frac{1}{4}\sum_{i=0}^{n+1}(\xi_i^2 + (\eta^i)^2),$$

where the left-invariant coframe (4.5) has the following differentials:

$$d\xi_{n+1} = 0, \quad d\eta^{n+1} = -\sum_{I=0}^{n+1}\xi_I \wedge \eta^I, \quad d\xi_I = -\frac{1}{2}\xi_{n+1} \wedge \xi_I, \quad d\eta^I = -\frac{1}{2}\xi_{n+1} \wedge \eta^I.$$

This shows that, up to multiplication with the factor $1/2$, the above coframe is dual to the standard orthonormal basis of the elementary Kählerian Lie algebra $\mathfrak{g}$ and, hence, that $g_G(p)$ is a Kähler metric of constant holomorphic sectional curvature -4. $\square$

5. Examples: Complete Quaternionic Kähler Manifolds Associated to Cubic Polynomials

As an immediate corollary of Theorem 4 and Theorem 5 we have:

Theorem 6. *To any complete projective special real manifold* $(\mathcal{H}, g_{\mathcal{H}})$ *the composition of the r-map with the c-map associates a complete quaternionic Kähler manifold* (N, g_N) *of negative scalar curvature.*

Example. We can consider the point $\mathcal{H} = \{1\} \subset \{x^3 = 1\} \subset \mathbb{R}$ as an example of a projective special real manifold. The corresponding complete quaternionic Kähler eightfold obtained by the above construction is the symmetric space $\mathrm{G}_2^*/\mathrm{SO}(4)$ of noncompact type.

A homogeneous cubic polynomial $h \in S^3(\mathbb{R}^n)^*$ will be called hyperbolic (respectively, elliptic) if there exists a point $p \in \{h = 1\} := \{x \in \mathbb{R}^n | h(x) = 1\}$ such that $\partial^2 h$ is negative (respectively, positive) definite on the tangent space of the hypersurface $\{h = 1\}$ at p. Such points will be called hyperbolic (respectively, elliptic). Let us denote by $\mathcal{H} = \mathcal{H}(h) \subset \{h = 1\}$ the open subset of hyperbolic points. It is a projective special real manifold with the metric $g_{\mathcal{H}}$ given by the restriction of $-\frac{1}{3}\partial^2 h$.

The classification of complete projective special real manifolds reduces to the following two problems:

Problem 1. *Classify all hyperbolic homogeneous cubic polynomials up to linear transformations. In other words, describe the orbit space* $S^3(\mathbb{R}^n)^*_{hyp}/\mathrm{GL}(n)$*, where* $S^3(\mathbb{R}^n)^*_{hyp} \subset S^3(\mathbb{R}^n)^*$ *stands for the open subset of hyperbolic polynomials.*

Problem 2. *For each hyperbolic homogeneous cubic polynomial h, determine the components of the hypersurface* $\mathcal{H}(h)$ *which are complete and classify them up to linear transformations.*

We will solve these problems in the simplest case, that is for $n = 2$. This gives the classification of complete projective special real curves. The classification of complete projective special real surfaces is open, but we will give some examples of such surfaces.

5.1. Classification of complete cubic curves and corresponding 12-dimensional quaternionic Kähler manifolds.

Theorem 7. *The orbit space* $S^3(\mathbb{R}^2)^*_{hyp}/\mathrm{GL}(2)$ *consists of three points, which are represented by the polynomials*

$$x^2y, \quad x(x^2-y^2) \quad \textit{and} \quad x(x^2+y^2).$$

Proof. Let $h \in S^3(\mathbb{R}^2)^*_{hyp}$. Interchanging the variables x and y, if necessary, we can assume that $\deg_x h$, the degree of $h = h(x, y)$ in the variable x, is 2 or 3. Since the Hessian of h is nondegenerate we also have $\deg_y h \geq 1$.

Case 1) If $\deg_x h = 3$, then the polynomial $f(x) := h(x, 1)$ has degree 3. Any polynomial $f(x)$ of degree 3 can be brought to one of the following forms by an affine transformation in the variable x:

$$x^3, \quad cx^2(x-1), \quad c(x+a)x(x-1), \quad cx(x^2+1), \quad c(x+a)(x^2+1),$$

where $c \neq 0$ and $a > 0$ are real constants. The first form is excluded, since $\deg_y h \geq 1$. This implies that h can be brought to one of the following forms by a linear transformation:

$$x^2(x-y), \quad (x+ay)x(x-y), \quad x(x^2+y^2), \quad (x+ay)(x^2+y^2), \quad a>0.$$

The first polynomial is linearly equivalent to x^2y. The zero set of the second polynomial in the real projective line consists of the points $-a, 0, 1 \in \mathbb{R} \subset \mathbb{R}P^1 = P(\mathbb{R}^2)$. Since any three pairwise distinct points in the projective line are related by an element of GL(2), we can assume that $a = 1$. Finally, the last polynomial can be brought to the form $x(x^2 + y^2)$ by a linear conformal transformation. Therefore, we are left with the following 3 normal forms:

$$x^2y, \quad x(x^2-y^2) \quad \text{and} \quad x(x^2+y^2). \tag{5.1}$$

Case 2) If $\deg_x h = 2$, then the quadratic polynomial $f(x) = h(x, 1)$ can be brought to one of the following forms by an affine transformation:

$$\pm x^2, \quad cx(x-1), \quad c(x^2+1), \quad c \neq 0.$$

Therefore h can be brought to one of the following forms by a linear transformation:

$$x^2y, \quad xy(x-y), \quad (x^2+y^2)y.$$

The last two polynomials are equivalent to the polynomials $x(x^2 \mp y^2)$ already included in our list (5.1).

It remains to check that all 3 polynomials are indeed hyperbolic. For dimensional reasons ($n = 2$), this is equivalent to the existence of a point $p = (x, y) \in \mathbb{R}^2$ such that $h(p) > 0$ and $D(p) := \det \partial^2 h(p) < 0$.

a) For $h = x^2y$ we have $D = -4x^2$. Therefore all points of the curve $\{h = 1\}$ are hyperbolic.
b) The same is true for $h = x(x^2 - y^2)$, since $D = -4(3x^2 + y^2)$.

c) For $h = x(x^2 + y^2)$ we find $D = 4(3x^2 - y^2)$. The point $\frac{1}{\sqrt[3]{5}}(1, 2) \in \{h = 1\}$ is hyperbolic, whereas $\frac{1}{\sqrt[3]{2}}(1, 1) \in \{h = 1\}$ is elliptic. □

Next we investigate the components of $\mathcal{H}(h)$.

Theorem 8. a) *The curve $\{x^2 y = 1\}$ consists of hyperbolic points and has two equivalent components. They are homogeneous and, hence, complete projective special real curves.*
b) *The curve $\{x(x^2 - y^2) = 1\}$ consists of hyperbolic points and has three equivalent components, which are inhomogeneous complete projective special real curves.*
c) *Let $h = x(x^2 + y^2)$. The curve $\mathcal{H}(h) = \{p \in \mathbb{R}^2 | h(p) = 1, D(p) < 0\}$, which consists of the hyperbolic points of $\{h = 1\}$ has two equivalent components. They are incomplete. The curve $\{p \in \mathbb{R}^2 | h(p) = 1, D(p) > 0\}$ which consists of the elliptic points of $\{h = 1\}$ is connected and incomplete.*

Proof. a) $h = x^2 y$. The reflection $x \mapsto -x$ interchanges the two components of the curve $\{h = 1\}$ and the subgroup $\{\mathrm{diag}(\lambda, \lambda^{-2}) | \lambda > 0\} \subset \mathrm{GL}(2)$ acts transitively on each component.
b) $h = x(x^2 - y^2)$. The transformation

$$\begin{pmatrix} -1/2 & 1/2 \\ -3/2 & -1/2 \end{pmatrix} \in \mathrm{SL}(2)$$

generates a cyclic group of order three, which interchanges the three components of the curve $\{h = 1\}$. Let us consider the component

$$C := \{p \in \mathbb{R}^2 | h(p) = 1, x > 0\}.$$

It is symmetric with respect to the x-axis, intersects the x-axis at $x = 1$ and approaches the asymptotic lines $y = \pm x$ when $x \to \infty$. To prove the completeness of C, it suffices to show that the length of the upper half $C_+ = C \cap \{y > 0\}$ of the curve is infinite. A straightforward calculation shows that the metric $ds^2 = g = -\frac{1}{3}\partial^2 h|_C$ is given by the formula

$$\frac{3}{2} g = -3x dx^2 + x dy^2 + 2y dx dy = \frac{3(4x^3 - 1)}{4x^2(x^3 - 1)} dx^2.$$

This yields the following asymptotics:

$$\frac{g}{dx^2} = \frac{2}{x^2} + O\left(\frac{1}{x^5}\right),$$
$$\frac{ds}{dx} = \frac{\sqrt{2}}{x} + O\left(\frac{1}{x^4}\right),$$

which implies that the arc length $\int_1^x ds = \sqrt{2} \ln x + O(\frac{1}{x^3})$ grows logarithmically with x. This shows that C_+ has infinite length with respect to g. A simple calculation shows that the automorphism group of C is trivial.

c) $h = x(x^2+y^2)$. The two components of $\mathcal{H}(h) = \{h = 1, x < \frac{1}{\sqrt[3]{4}}\}$ are interchanged by the reflection $y \mapsto -y$. They are incomplete due to the points of inflections at the boundary points $(\frac{1}{\sqrt[3]{4}}, \pm\frac{\sqrt{3}}{\sqrt[3]{4}})$. The same is true for the curve $\{p \in \mathbb{R}^2 | h(p) = 1, D(p) > 0\} = \{h = 1, x > \frac{1}{\sqrt[3]{4}}\}$. $\square$

Corollary 4. *There exist precisely two complete projective special real curves, up to linear equivalence:*

a) $\{(x, y) \in \mathbb{R}^2 | x^2 y = 1, x > 0\}$ *and*
b) $\{(x, y) \in \mathbb{R}^2 | x(x^2 - y^2) = 1, x > 0\}$.

Under the composition of the r- and c-map both give rise to complete quaternionic Kähler manifolds of dimension 12. In the first case we obtain the symmetric quaternionic Kähler manifold

$$\frac{\mathrm{SO}_0(4,3)}{\mathrm{SO}(4) \times \mathrm{SO}(3)}.$$

The second example gives rise to a new complete quaternionic Kähler manifold.

5.2. Examples of complete cubic surfaces and corresponding 16-dimensional quaternionic Kähler manifolds.

Example 1 (STU model). The surface $\mathcal{H} = \{xyz = 1, x > 0, y > 0\} \subset \mathbb{R}^3$ is a homogeneous projective special real manifold. In fact, the group $\mathbb{R}^{>0} \times \mathbb{R}^{>0}$ acts simply transitively on $\mathcal{H}$ by unimodular diagonal matrices. The corresponding quaternionic Kähler manifold is the symmetric space

$$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}.$$

Example 2. The surface $\mathcal{H} = \{x(xy - z^2) = 1, x > 0\} \subset \mathbb{R}^3$ is another homogeneous projective special real manifold. It admits the following simply transitive solvable group of linear automorphisms: $(x, y, z) \mapsto (\lambda^{-2}x, \lambda^4(\mu x + y + 2\mu z), \lambda(\mu x + z))$, $\lambda > 0$, $\mu \in \mathbb{R}$. The corresponding quaternionic Kähler manifold is the nonsymmetric homogeneous manifold $\mathcal{T}(1)$ described in [9].

Example 3 (quantum STU model). The surface $\mathcal{H} = \{x(yz + x^2) = 1, x < 0, y > 0\}$ is an inhomogeneous complete projective special real manifold. Its automorphism group is one-dimensional. The maximal connected subgroup is given by: $(x, y, z) \mapsto (x, \lambda y, \lambda^{-1}z)$, $\lambda > 0$. Under the r-map, the surface $\mathcal{H}$ gives rise to a new complete projective special Kähler manifold, which is mapped to a new complete quaternionic Kähler manifold of dimension 16 under the c-map. Let us check the completeness of $\mathcal{H}$. A straightforward calculation shows that:

$$\begin{aligned}
-\partial^2 h|_{\mathcal{H}} &= 2(1 - x^3)\left(\frac{dx^2}{x^2} + \frac{dy^2}{y^2}\right) + 2(1 + 2x^3)\frac{dx\,dy}{xy} \\
&\geq 2\left(1 - x^3 - \frac{|1 + 2x^3|}{2}\right)\left(\frac{dx^2}{x^2} + \frac{dy^2}{y^2}\right) \\
&\geq \frac{dx^2}{x^2} + \frac{dy^2}{y^2}.
\end{aligned}$$

So the projective special real metric $g = -\frac{1}{3}\partial^2 h|_{\mathcal{H}}$ is bounded from below by the product metric

$$\frac{dx^2}{3x^2} + \frac{dy^2}{3y^2}$$

on $\mathbb{R}^{<0} \times \mathbb{R}^{>0}$, which is complete. In fact, $\frac{dx^2}{3x^2} + \frac{dy^2}{3y^2} = d\tilde{x}^2 + d\tilde{y}^2$ under the change of variables $\tilde{x} = \frac{1}{\sqrt{3}} \ln(-x)$, $\tilde{y} = \frac{1}{\sqrt{3}} \ln y$, which maps $\mathbb{R}^{<0} \times \mathbb{R}^{>0}$ to $\mathbb{R}^2$.

6. Globalisation of the Ferrara-Sabharwal Metric

In this section we will investigate the problem of gluing Ferrara-Sabharwal manifolds $(N_\alpha = M_\alpha \times G, g_\alpha = g_{M_\alpha} + g_G^\alpha)$ obtained from projective special Kähler domains $M_\alpha \subset M$ in a projective special Kähler manifold (M, g_M) to a global quaternionic Kähler manifold (N, g_N). Here $g_{M_\alpha} = g_M|_{M_\alpha}$. Recall that $G = \mathbb{R}^{2n+4}$ with the group structure defined in Sect. 4. Denote by $\tilde{M}_\alpha = \pi_M^{-1}(M_\alpha) \subset \tilde{M}$ the corresponding special coordinate domain in the underlying conical special Kähler manifold $\pi_M : \tilde{M} \to M$. The affine special coordinates on $\tilde{M}_\alpha$ will be denoted by q^a, or, more precisely, by q^a_α. The holomorphic special coordinates will be z^I or z^I_α. Let $(M_\alpha)_\alpha$ be a covering of M by projective special Kähler domains. Then we define the quotient

$$N = \bigcup_\alpha N_\alpha / \sim$$

by the equivalence relation

$$N_\alpha \ni (m, v) \sim (m', v') \in N_\beta :\Longleftrightarrow m = m' \quad \text{and} \quad v = \tilde{A}_{\alpha\beta} v',$$

where $A_{\alpha\beta}$ is the linear symplectic transformation such that $q_\alpha = A_{\alpha\beta} q_\beta$ and $\tilde{A} = \mathrm{diag}((A^T)^{-1}, \mathbb{1}_2)$.

Theorem 9. *The natural projection $\pi : N \to M$ is a symplectic vector bundle and at the same time a bundle of Lie groups. Each fiber is isomorphic to the solvable Lie group G. There exists a unique quaternionic Kähler structure (Q, g_N) on N such that $g_N|_{N_\alpha} = g_\alpha$. Up to an isomorphism of quaternionic Kähler manifolds consistent with the bundle structures, the quaternionic Kähler manifold (N, Q, g_N) does neither depend on the covering $(M_\alpha)_\alpha$ of M nor on the choice of special coordinates on the domains M_α.*

Proof. The transition functions $(\tilde{A}_{\alpha\beta})$ can be considered as a Čech 1-cocycle with values in the group

$$\mathrm{Sp}(2n+2, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2n+4, \mathbb{R}),$$

which defines the structure of a symplectic vector bundle on N. The above linear action of $\mathrm{Sp}(2n+2, \mathbb{R})$ on $G = \mathbb{R}^{2n+4}$ is by automorphisms of the solvable Lie group G, which means that the gluing preserves the group structure of the fibers. In order to prove that the local metrics g_α can be glued to a global Riemannian metric g_N, it suffices to check that $g_G^\beta = \tilde{A}^* g_G^\alpha$, since we know already that $g_{M_\alpha} = g_{M_\beta}$ on $M_\alpha \cap M_\beta$. It is useful to rewrite $g_G = g_G^\alpha$ in the following way:

$$g_G = \frac{1}{4\phi^2} d\phi^2 + \frac{1}{4\phi^2}(d\tilde{\phi} + \sum p_a \Omega^{ab} dp_b)^2 + \frac{1}{2\phi} \sum \hat{H}^{ab} dp_a dp_b \,. \tag{6.1}$$

Here $(p_a) = (\tilde{\zeta}_I, \zeta^J)$,

$$\begin{aligned} \Omega^{-1} = (\Omega^{ab}) &:= \begin{pmatrix} 0 & -\mathbb{1}_{n+1} \\ \mathbb{1}_{n+1} & 0 \end{pmatrix} , \\ \hat{H}^{-1} = (\hat{H}^{ab}) &:= \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix} \end{aligned} \tag{6.2}$$

that is

$$\hat{H} = (\hat{H}_{ab}) = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} . \tag{6.3}$$

We observe that 2Ω is the matrix representing the Kähler form $\omega = 2\sum dx^I \wedge dy_I$ of the conical special Kähler domain $\tilde{M}_\alpha$ in the affine special coordinates $q^a = (x^I, y_J)$. The first two terms of (6.1) are manifestly invariant under symplectic transformations $A \in \mathrm{Sp}(2n+2, \mathbb{R})$ since the 1-form $\sum p_a \Omega^{ab} dp_b$ is invariant. The invariance of the last term is stated in the next lemma establishing the existence of the metric g_N. The proof of the lemma will be given in the next section together with a geometric interpretation.

Lemma 4. *The tensors $\hat{H}_\alpha^{-1}$ and $\hat{H}_\beta^{-1}$ defined on M_α and M_β are related by*

$$\hat{H}_\alpha^{-1} = A_{\alpha\beta} \hat{H}_\beta^{-1} A_{\alpha\beta}^T$$

on overlaps $M_\alpha \cap M_\beta$.

The metric g_N is locally a quaternionic Kähler metric. To see that the local quaternionic structures are consistent, we observe that the coordinate transformations $N_\beta \to N_\alpha$ are orientation preserving isometries and that an orientation preserving isometry between two quaternionic Kähler manifolds of nonzero scalar curvature automatically maps the quaternionic structures to each other. This follows from the fact that the restricted holonomy group of a quaternionic Kähler manifold of nonzero scalar curvature together with the orientation uniquely determines the quaternionic structure. (Notice that the orientation is needed, since the symmetric quaternionic Kähler manifold

$$\frac{\mathrm{SO}_0(4, n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)}$$

admits precisely two parallel skew-symmetric quaternionic structures, which, however, induce opposite orientations.) □

The correspondence $(M, g_M) \mapsto (N, g_N)$ established in Theorem 9 is a global version of the c-map of Ferrara and Sabharwal. We will still call it the supergravity c-map.

Theorem 10. *The supergravity c-map maps (isomorphism classes of) complete projective special Kähler manifolds (M, g_M) of dimension $2n$ to (isomorphism classes of) complete quaternionic Kähler manifolds (N, g_N) of dimension $4n+4$ of negative scalar curvature such that N is a vector bundle over M with totally geodesic zero section isometric to M.*

Proof. This follows from Theorem 9 and Theorem 3. □

6.1. From Griffiths to Weil flags in special Kähler geometry. Let us consider the complex vector space $V = \mathbb{C}^{2n+2} = \mathbb{R}^{2n+2} \otimes \mathbb{C}$ with its standard symplectic structure $\Omega = \sum dz^I \wedge dw_I$ and pseudo-Hermitian sesquilinear metric

$$\gamma(u, v) = \sqrt{-1}\Omega(u, \bar{v}), \quad u, v \in V,$$

of split signature. We denote by $Gr_0^{k,l}(V)$ the Grassmannian of complex Lagrangian subspaces of signature (k, l), where $k + l = n + 1$. For $k \geq 1$, let $F_0^{k,l}(V)$ denote the complex manifold of flags (ℓ, L), where $L \in Gr_0^{k,l}(V)$ and $\ell \subset L$ is a positive definite line. Notice that we have a canonical holomorphic projection

$$F_0^{k,l}(V) = \frac{\mathrm{Sp}(\mathbb{R}^{2n+2})}{\mathrm{U}(1) \times \mathrm{U}(k-1, l)} \longrightarrow Gr_0^{k,l}(V) = \frac{\mathrm{Sp}(\mathbb{R}^{2n+2})}{\mathrm{U}(k, l)}, \quad (\ell, L) \mapsto L,$$

which is $\mathrm{Sp}(\mathbb{R}^{2n+2})$-equivariant.

Proposition 2. *There exists a canonical* $\mathrm{Sp}(\mathbb{R}^{2n+2})$*-equivariant diffeomorphism*

$$\psi : F_0^{k,l}(V) \longrightarrow F_0^{l+1,k-1}(V).$$

Proof. For $(\ell, L) \in F_0^{k,l}(V)$ we put

$$E := \{v \in L | v \perp \ell\}$$

and define $\psi(\ell, L) := (\ell, L')$ where

$$L' := \ell + \bar{E}.$$

$\square$

In particular, we obtain an equivariant diffeomorphism from the manifold of Griffiths flags to the manifold of Weil flags:

$$\psi : F_0^{1,n}(V) \longrightarrow F_0^{n+1,0}(V).$$

Remark. In order to motivate the terminology we observe that given $L \in F_0^{1,n}(V)$ and a lattice $\Gamma \subset \mathbb{R}^{2n+2}$ the quotient of $W = V/L$ by (the image of) Γ is a complex torus which is analogous to the Griffiths intermediate Jacobian

$$\frac{H^3(X, \mathbb{C})}{H^{3,0}(X, \mathbb{C}) + H^{2,1}(X, \mathbb{C}) + H^3(X, \mathbb{Z})},$$

whereas the quotient of $W' = V/L'$ by Γ is analogous to the Weil intermediate Jacobian

$$\frac{H^3(X, \mathbb{C})}{H^{3,0}(X, \mathbb{C}) + H^{1,2}(X, \mathbb{C}) + H^3(X, \mathbb{Z})}$$

associated to the Hodge structure of a Calabi-Yau 3-fold X. It is known that the bundle of Griffiths intermediate Jacobians over the (conical special Kähler) deformation space $M_X = \{(J, \nu)\}$ of complex structures J of X gauged by a J-holomorphic volume form ν carries a hyper-Kähler metric obtained from the affine version of the c-map [10]. Similarly, a certain bundle of Weil intermediate Jacobians is quaternionic Kähler by the supergravity c-map [29,38].

Recall that there is a totally geodesic $\mathrm{Sp}(\mathbb{R}^{2n+2})$-equivariant embedding

$$\iota : Gr_0^{k,l}(V) = \frac{\mathrm{Sp}(\mathbb{R}^{2n+2})}{\mathrm{U}(k,l)} \longrightarrow \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n+2}) = \frac{\mathrm{SL}(2n+2,\mathbb{R})}{\mathrm{SO}(2k,2l)} \tag{6.4}$$

into the space $\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n+2})$ of symmetric unimodular matrices of signature $(2k, 2l)$. The embedding ι is described geometrically in [14]. In the next lemma we give an explicit description of ι in terms of coordinates. Local holomorphic coordinates near a point $L_0 \in Gr_0^{k,l}(V)$ can be described as follows. Since $Gr_0^{k,l}(V)$ is a homogeneous space we can assume that

$$L_0 = \mathrm{span}\left\{\frac{\partial}{\partial z^0} + i\frac{\partial}{\partial w_0}, \ldots, \frac{\partial}{\partial z^{k-1}} + i\frac{\partial}{\partial w_{k-1}}, \frac{\partial}{\partial z^k} - i\frac{\partial}{\partial w_k}, \ldots, \frac{\partial}{\partial z^n} - i\frac{\partial}{\partial w_n}\right\}.$$

An open neighbourhood U of L_0 is given by

$$U := \{L \in Gr_0^{k,l}(V) | L \cap (\mathbb{C}^{n+1})^* = 0\},$$

where

$$(\mathbb{C}^{n+1})^* = \{(z, w) \in V | z = 0\} \subset V = T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^*.$$

We will now explain that any point $L \in U$ is described by a complex symmetric matrix $S = (S_{IJ})$ such that the real matrix $\mathrm{Im} S_{IJ}$ has signature (k, l). Let us denote by $\mathrm{Sym}_{k,l}(\mathbb{C}^{n+1})$ the complex vector space of all such matrices. Any $S \in \mathrm{Sym}_{k,l}(\mathbb{C}^{n+1})$ defines a Lagrangian subspace

$$L = L(S) = \{(z, w) \in V | w_I = \sum S_{IJ} z^J\} \subset V,$$

and one can easily check that the map $S \mapsto L(S)$ is a biholomorphism $\mathrm{Sym}_{k,l}(\mathbb{C}^{n+1}) \to U \subset Gr_0^{k,l}(V)$. Notice that $L_0 = L(S_0)$, $S_0 = i I_{k,l} = i\,\mathrm{diag}(\mathbb{1}_k, -\mathbb{1}_l)$. It is well known that the matrix S transforms as

$$S \mapsto S' = (C + DS)(A + BS)^{-1} \tag{6.5}$$

under a symplectic transformation

$$\mathcal{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{6.6}$$

Lemma 5. *The restriction of the map (6.4) to the the open subset $U \subset Gr_0^{k,l}(V)$ is given by*

$$\begin{aligned} \iota|_U : \; & U \cong \mathrm{Sym}_{k,l}(\mathbb{C}^{n+1}) \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n+2}), \\ & S = \mathcal{R} + i\mathcal{I} \mapsto \iota(L(S)) = g^S = (g^S_{ab}) := \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}. \end{aligned} \tag{6.7}$$

Proof. The above formula shows that $g^{S_0} = \mathrm{diag}(I_{k,l}, I_{k,l})$. To prove that $g^S = \iota(L(S))$ for all $S \in U$ it suffices to check that

$$g^{S'} = \mathcal{O}^{T,-1} g^S \mathcal{O}^{-1}$$

for S' defined in (6.5). This follows from Lemma 6 which is stated and proved below. $\square$

The relation between the manifolds of Griffiths and Weil flags, the associated Grassmannians, and spaces of symmetric matrices is summarized in the following diagram.

$$\begin{tikzcd}
F_0^{1,n}(V) \arrow[r, "\psi"] \arrow[d] & F_0^{n+1,0}(V) \arrow[d] \\
Gr_0^{1,n}(V) \arrow[d, hook, "\iota"] & Gr_0^{n+1,0}(V) \arrow[d, hook, "\iota"] \\
Sym^1_{2,2n}(\mathbb{R}^{2n+2}) & Sym^1_{2n+2,0}(\mathbb{R}^{2n+2})
\end{tikzcd} \tag{6.8}$$

Lemma 6. *Let* $\mathcal{N} = \mathcal{R} + i\mathcal{I}$ *be a complex symmetric* $(n+1) \times (n+1)$ *matrix with invertible imaginary part. Using the decomposition into real and imaginary parts, define the real symmetric* $(2n+2) \times (2n+2)$ *matrix*

$$\hat{H} = (\hat{H}_{ab}) = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} .$$

Then $\hat{H}$ *is invertible with inverse matrix*

$$\hat{H}^{-1} = (\hat{H}^{ab}) = \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix} . \tag{6.9}$$

Moreover, $\mathcal{N}$ *transforms fractionally linearly under symplectic transformations (6.6),*

$$\mathcal{N} \to (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$$

if and only if $\hat{H}$*, and, hence* $\hat{H}^{-1}$ *transform as tensors:*

$$\hat{H} \to \mathcal{O}^{T,-1}\hat{H}\mathcal{O}^{-1} , \quad \hat{H}^{-1} \to \mathcal{O}\hat{H}^{-1}\mathcal{O}^{T} .$$

Remarks. This lemma relates the transformation properties of vector multiplet couplings in special holomorphic and special real coordinates. While we need the lemma to establish the well-definiteness of the local c-map, it applies to the rigid c-map as well. In the rigid case the role of $\mathcal{N}$ is played by two times the matrix of second derivatives of the holomorphic prepotential, $2(F_{IJ})$, while the role of $\hat{H}$ is played by the Hessian metric $\partial^2 H$, where H is the Legendre transform of two times the imaginary part of the holomorphic prepotential. When passing to supergravity, $2(F_{IJ})$ is replaced by $\bar{\mathcal{N}}$, while the Hessian metric is replaced by $\hat{H}$. When using the superconformal calculus to construct vector multiplet couplings these replacements are induced by eliminating certain auxiliary fields. From a geometrical perspective these replacements can be understood as follows. The kinetic terms of both scalar fields and vector fields must be positive definite in a physically acceptable theory. In a theory of rigid supermultiplets, the relevant coupling matrix for both types of fields is the metric $2\mathrm{Im}F_{IJ}$ of the affine special Kähler manifold, which therefore must be positive definite. In the locally supersymmetric theory scalar and vector fields have different couplings matrices. The coupling matrix for the scalars is the metric $\bar{g}$ of the projective special Kähler manifold, while the coupling matrix for the vector fields is $\mathcal{N}$, with the kinetic terms given by the imaginary part $\mathcal{I}$. Therefore $\bar{g}$ and $\mathcal{I}$ must be positive definite, which is equivalent to imposing that the corresponding conical affine special Kähler metric has complex Lorentz signature.

Proof. We now prove Lemma 6. It is trivial to verify that (6.9) is the inverse matrix of $\hat{H}$. Note that $\mathcal{I}$ is invertible by assumption. The relation between the transformation properties of $\mathcal{N}$ and $\hat{H}$ can be verified by direct calculation. Such calculations have occurred in the supergravity literature, see for example [16], so that we only need to indicate the main steps. Let $\mathcal{N}' = \mathcal{R}' + i\mathcal{I}'$ be the matrix obtained by fractionally linear action of the symplectic matrix $\mathcal{O}$ on $\mathcal{N}$. This is equivalent to $\hat{H}$ transforming as a tensor if and only if the following three relations hold

$$
\begin{aligned}
(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})' &= D(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})D^T + D\mathcal{R}\mathcal{I}^{-1}C^T + C\mathcal{I}^{-1}RD^T \\
&\quad + C\mathcal{I}^{-1}C^T \ , \qquad (6.10) \\
(\mathcal{R}\mathcal{I}^{-1})' &= D(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})B^T + D\mathcal{R}\mathcal{I}^{-1}A^T + C\mathcal{I}^{-1}\mathcal{R}B^T \\
&\quad + C\mathcal{I}^{-1}A^T \ , \qquad (6.11) \\
(\mathcal{I}^{-1})' &= B(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})B^T + B\mathcal{R}\mathcal{I}^{-1}A^T + A\mathcal{I}^{-1}\mathcal{R}B^T \\
&\quad + A\mathcal{I}^{-1}B^T \ . \qquad (6.12)
\end{aligned}
$$

The matrices A, B, C, D are block sub-matrices of the symplectic matrix $\mathcal{O}$, which satisfies

$$
\mathcal{O}^T \Omega \mathcal{O} = \Omega \ , \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} .
$$

Therefore they satisfy

$$
A^T C = C^T A \ , \quad B^T D = D^T B \ , \quad A^T D - C^T B = \mathbb{1} \ .
$$

This can be used to verify the following the useful identities

$$
U^T(C + D\mathcal{N}) = (C + D\mathcal{N})^T U \ , \quad U^T(C + D\bar{\mathcal{N}}) = (C + D\mathcal{N})^T \bar{U} - 2i\mathcal{I} \ , \quad (6.13)
$$

where $U = U(\mathcal{N}) := A + B\mathcal{N}$. Two further identities are obtained by complex conjugation. By repeated use of these identities, we can show that

$$
2i\mathcal{I} = \mathcal{N} - \bar{\mathcal{N}} = U^T(C + D\mathcal{N})U^{-1}\bar{U} - U^T(C + D\bar{\mathcal{N}}) \ . \qquad (6.14)
$$

The imaginary part $\mathcal{I}$ of $\mathcal{N}$ transforms under symplectic transformations into

$$
\mathcal{I}' = -\frac{i}{2}[(C + D\mathcal{N})U^{-1} - (C + D\bar{\mathcal{N}})\bar{U}^{-1}] \ .
$$

Using identity (6.14) this can be rewritten as

$$
\mathcal{I}' = U^{-1,T}\mathcal{I}\bar{U}^{-1} = \bar{U}^{-1,T}\mathcal{I}U^{-1} \ ,
$$

where the second equation holds because $\mathcal{I}$ is real. Since $\mathcal{I}$ is invertible by assumption, we conclude

$$
(\mathcal{I}^{-1})' = \bar{U}\mathcal{I}^{-1}U^T = U\mathcal{I}^{-1}\bar{U}^T \ . \qquad (6.15)
$$

Writing out $U = A + B\mathcal{N}$ and $\mathcal{N} = \mathcal{R} + i\mathcal{I}$ we obtain (6.12). Next, we note that the real part $\mathcal{R}$ of $\mathcal{N}$ transforms into

$$
\mathcal{R}' = \frac{1}{2}[(C + D\mathcal{N})U^{-1} + (C + D\bar{\mathcal{N}})\bar{U}^{-1}] \ .
$$

Combining this with (6.15) we obtain

$$(\mathcal{R}\mathcal{I}^{-1})' = \frac{1}{2}[(C + D\mathcal{N})\mathcal{I}^{-1}\bar{U}^T + (C + D\bar{\mathcal{N}})^{-1}U^T] .$$

Expressing $U, \mathcal{N}$ in terms of A, B, C, D and $R, \mathcal{I}$, we obtain (6.11). Finally, we multiply $(\mathcal{R}\mathcal{I}^{-1})'$ by $\mathcal{R}'$ from the right and add $\mathcal{I}'$. After repeated use of the identities (6.13) we finally obtain (6.12). □

Let $L \subset V = \mathbb{C}^{2n+2}$ be a Lagrangian subspace defined by $S \in \mathrm{Sym}_{1,n}(\mathbb{C}^{n+1})$ and $\ell = \mathbb{C}(z, w)^T \subset L$. Then $w = Sz$, $(\ell, L) \in F_0^{1,n}(V)$ and $(\ell, L') = \psi(\ell, L) \in F_0^{n+1,0}(V)$, cf. (6.8).

Lemma 7. *The positive definite Lagrangian subspace $L' \subset V$ corresponds to the following matrix $S' \in \mathrm{Sym}_{n+1,0}(\mathbb{C}^{n+1})$:*

$$S'_{IJ} := \bar{S}_{IJ} + i\frac{\sum_K N_{IK}z^K \sum_L N_{JL}z^L}{\sum_{IJ} N_{IJ}z^I z^J}, \quad N_{IJ} := 2\mathrm{Im}\, S_{IJ}. \tag{6.16}$$

Proof. The following calculation shows that $\ell = \mathbb{C}(z, w)^T$ is contained in the Lagrangian subspace $L(S')$ defined by S':

$$S'z = \bar{S}z + 2i(\mathrm{Im}\, S)z = Sz.$$

Next we consider the orthogonal complement E of the line ℓ in L. A vector $(u, Su)^T \in L$ belongs to E if and only if $\sum N_{IJ}z^I\bar{u}^J = 0$. In that case we obtain

$$S'\bar{u} = \bar{S}\bar{u} = \overline{Su},$$

which proves that $\bar{E}$ is contained in $L(S')$. Therefore, $L(S') = L'$. □

Corollary 5. (i) *The complex matrix $S' = \mathcal{N} \in \mathrm{Sym}_{n+1,0}(\mathbb{C}^{n+1})$ occurring in the definition of the Ferrara-Sabharwal metric, see (4.3), is related to the matrix $S = F_{IJ} \in \mathrm{Sym}_{1,n}(\mathbb{C}^{n+1})$ by the correspondence $S \mapsto S'$ of the previous lemma, which is induced by the map ψ from Griffiths flags to Weil flags, cf. (6.17).*

(ii) *The real matrix (6.3) occurring in the formula (6.1) for the Ferrara-Sabharwal metric is given by*

$$\hat{H} = g^{S'} = g^{\mathcal{N}},$$

where the map $S \mapsto g^S$ is defined in (6.7).

The following diagram gives an overview of the relations between the Lagrangian subspaces $L = L(S)$, $L' = L(S')$ and the corresponding real matrices $g^S = (g_{ab})$ and $g^{S'} = \hat{H}_{ab}$:

$$\begin{array}{ccc}
(\ell, L) \in F_0^{1,n}(V) & \xrightarrow{\psi} & F_0^{n+1,0}(V) \ni (\ell, L') \\
\downarrow & & \downarrow \\
L \in Gr_0^{1,n}(V) & & Gr_0^{n+1,0}(V) \ni L' \\
\big\downarrow \iota & & \big\downarrow \iota \\
g^S \in Sym^1_{2,2n}(\mathbb{R}^{2n+2}) & & Sym^1_{2n+2,0}(\mathbb{R}^{2n+2}) \ni g^{S'},
\end{array} \tag{6.17}$$

where the line ℓ is generated by the vector $(z, Sz) = (z, S'z)$.

Corollary 6. *The (indefinite) affine special Kähler metric* $g = 2\sum g_{ab}dq^a dq^b$ *is related to the positive definite metric* $g' = 2\sum \hat{H}_{ab}dq^a dq^b$ *by*

$$g'|_{\mathcal{D}} = g|_{\mathcal{D}}, \quad g'|_{\mathcal{D}^\perp} = -g|_{\mathcal{D}^\perp},$$

where $\mathcal{D}$ *is defined in Definition 3* (iv).

Proof. This follows from the geometric description of the map $\iota : L = L(S) \mapsto g^S$ [14]. Recall that in the affine special coordinates $q^a = (x^I = \mathrm{Re}\, z^I, y_J = \mathrm{Re}\, w_J)$, the Kähler form is given by $\omega = 2\sum dx^i \wedge dy_i$. The matrix $g = g^S$ represents the scalar product $\mathrm{Re}\, \gamma|_L$ in the Darboux coordinates $\sqrt{2}q^a$ restricted to L. We can compare the restrictions of γ to L and L' by the following isomorphism of real vector spaces $\Psi : L \to L'$:

$$\Psi|_\ell := \mathrm{Id}, \quad \Psi(v) := \bar{v} \quad \text{for all} \quad v \in E.$$

We can easily see that Ψ is an isometry on $\ell = L \cap L'$ and an antiisometry $E \to \bar{E}$ on E. In fact, $\gamma(\bar{v}, \bar{v}) = -\gamma(v, v)$ for all $v \in V$. This shows that the metric $\Psi^*\mathrm{Re}\, \gamma|_{L'}$ is related to $g = \mathrm{Re}\, \gamma|_L$ by changing the sign on the orthogonal complement of ℓ. Finally, the Gram matrix $g^{S'} = (\hat{H}_{ab})$ of $\mathrm{Re}\, \gamma|_{L'}$ in the coordinates $\sqrt{2}q^a|_{L'}$ is the same as that of $\Psi^*\mathrm{Re}\, \gamma|_{L'}$ in the coordinates $\sqrt{2}q^a|_L$, since $q^a \circ \Psi = q^a$. □

We now finally prove Lemma 4 and thus complete the proof of Theorem 9.

Proof. Let us denote by F^α the holomorphic prepotential of the special Kähler domain M_α and put $S^\alpha := F^\alpha_{IJ}$. We know that the complex matrices S_α and S_β are related by a fractional linear transformation associated with a symplectic transformation $\mathcal{O} = A_{\alpha\beta} \in \mathrm{Sp}(\mathbb{R}^{2n+2})$, which relates the Gram matrices g^{S_α} and g^{S_β} of the corresponding affine special Kähler metrics $g_{\tilde{M}_\alpha}$ and $g_{\tilde{M}_\beta}$ in affine Darboux coordinates. Since all the maps in the diagram (6.17) are $\mathrm{Sp}(\mathbb{R}^{2n+2})$-equivariant, this implies that the matrices $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ are related by the same fractional linear transformation and that $g^{S'_\alpha}$ and $g^{S'_\beta}$ are related by the symplectic transformation $\mathcal{O}$. This shows that $\hat{H}_\alpha$ transforms as claimed in Lemma 4. □

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