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Special geometry, cubic polynomials and homogeneous quaternionic spaces

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ABSTRACT

The existing classification of homogeneous quaternionic spaces is not complete. We study these spaces in the context of certain $N = 2$ supergravity theories, where dimensional reduction induces a mapping between *special* real, Kähler and quaternionic spaces. The geometry of the real spaces is encoded in cubic polynomials, those of the Kähler and quaternionic manifolds in homogeneous holomorphic functions of second degree. We classify all cubic polynomials that have an invariance group that acts transitively on the real manifold. The corresponding Kähler and quaternionic manifolds are then homogeneous. We find that they lead to a well-defined subset of the normal quaternionic spaces classified by Alekseevskiĭ (and the corresponding special Kähler spaces given by Cecotti), but there is a new class of rank-3 spaces of quaternionic dimension larger than 3. We also point out that some of the rank-4 Alekseevskiĭ spaces were not fully specified and correspond to a finite variety of inequivalent spaces. A simpler version of the equation that underlies the classification of this paper also emerges in the context of W_3 algebras.

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1 Introduction.

Supersymmetric field theories in a variety of space-time dimensions give rise to non-linear sigma models with a restricted target-space geometry. There are many examples in the literature where this phenomenon led to surprising results, sometimes with interesting connections to mathematics. Furthermore, the fact that some of these supersymmetric theories in different space-time dimensions are related by (supersymmetric) dimensional reduction offers a way of connecting seemingly unrelated geometries.

In the context of this paper $N = 2$ supergravity is relevant. In five space-time dimensions, one may consider the coupling of a certain number (say $n-1$) of supersymmetric abelian vector multiplets. As was shown some time ago [1], these theories are characterized by a cubic polynomial in n (real) variables, which gives rise to a non-linear sigma model corresponding to a real $(n-1)$-dimensional space. Some of these polynomials correspond to symmetric spaces and are related to Jordan algebras. After dimensional reduction of this theory, one finds $N = 2$ supergravity in four space-time dimensions, coupled to n abelian vector multiplets. It is known that the non-linear sigma models in the four-dimensional theory correspond to Kähler spaces of complex dimension n, characterized by a homogeneous holomorphic function of second degree, depending on $n + 1$ complex variables [2]. Such Kähler manifolds are called *special* [3]. Special Kähler geometry is relevant to string theory, where compactifications of type-II superstrings on $(2, 2)$ superconformal field theories with central charge $c = 9$ lead to $N = 2$ supergravity coupled to vector multiplets. The massless scalars of these vector multiplets play the role of coordinates of the moduli space of the conformal theories, so that the study of supergravity may thus lead to interesting results for the moduli geometry of certain superconformal theories [4]. Because certain (tree-level) results for compactifications of the heterotic string on a $(2, 2)$ superconformal system depend only on the choice of the conformal theory, special geometry plays a role for all string compactifications of this type, which include those on Calabi-Yau spaces. Indeed, this fact has been verified in several studies where various aspects of this intriguing connection have been explored [4-8].

After dimensional reduction of four-dimensional $N = 2$ supergravity coupled to n vector supermultiplets to three space-time dimensions, one finds a non-linear sigma model corresponding to a quaternionic manifold of quaternionic dimension $n + 1$. In this way one thus obtains a class of quaternionic manifolds whose structure is encoded in the homogenous holomorphic function of the special Kähler manifold. Hence there exists a map between special Kähler manifolds of complex dimension n and certain quaternionic manifolds of quaternionic dimension $n+1$, which in [5] was called the **c** map. It was also shown that the **c** map plays an interesting role in string theory. When compactifying IIA and IIB strings on the same conformal theory, the resulting non-linear sigma models consist of a product space of a Kähler manifold and a quaternionic manifold. The latter is also special, in the sense that it is characterized in terms of a homogeneous holomorphic function. When comparing the result of the compactification of the IIA to that of the IIB string, it turns out that the two manifolds are interchanged according to the action of the **c** map [4, 5].

Likewise one can introduce the **r** map, which yields for every real space that couples to $d = 5$ supergravity the corresponding Kähler space that one finds upon reduction to four space-time dimensions. The **r** map thus assigns a Kähler space of complex dimension n to a real space of dimension $n-1$. Supersymmetry combined with dimensional reduction, which preserves supersymmetry, are the essential ingredients in these two maps.

We shall use the term "special geometry" for both the real spaces originating from five-dimensional supergravity, the Kähler spaces originating from four-dimensional supergravity and the quaternionic spaces that are in the image of the **c** map[1]. It should be clear that the inverse **r** and **c** maps are not always defined as there are spaces that couple to supergravity, but the corresponding supergravity theory does not necessarily originate from a higher-dimensional theory. When the coupling of a certain space to supergravity is not unique, the result of the maps will depend on the type of coupling as, for instance, characterized by the way in which the subgroup of the sigma model isometries that can be extended to a symmetry of the full supergravity action, is realized. In four space-time dimensions these invariances usually act on the field strengths of the abelian vector fields, and not on the fields themselves, so that they only leave the equations of motion and not the action invariant. These transformations, called duality transformations, constitute a subgroup of $Sp(2n+2, \mathbb{R})$. The complex nature of the Kähler manifolds is thus related to the complex nature of the (anti-)selfdual Minkowskian field strengths [9, 2]. In five space-time dimensions, we are dealing with a real manifold, so that the transformations are realized directly on the vector fields, whereas in three dimensions the vector fields are converted into scalar fields (the relation of all these symmetries upon dimensional reduction will be discussed in [10]; see also [11]). Under these maps the dimensionality of the manifold increases.

For homogeneous spaces the isometries act transitively on the manifold so that every two points are related by an element of the isometry group. The orbit swept out by the action of the isometry group G from any given point is (locally) isomorphic to the coset space G/H, where H is the isotropy group of that point. For non-compact homogeneous spaces where H is the maximal compact subgroup of G, there exists a solvable subgroup that acts transitively, whose dimension is equal to the dimension of the space. Such spaces are called *normal*. It implies that there exists a solvable algebra s such that $\frac{G}{H} = e^s$. The construction of this algebra follows from the Iwasawa

[1] In the literature, special Kähler spaces were sometimes called Kähler spaces of restricted type; the special quaternionic spaces were also called dual-quaternionic spaces.

decomposition of the algebra $g = h + s$ (see e.g. [12]). The dimension of the Cartan subalgebra of s equals the *rank* of the homogeneous space. It will turn out that the rank of the symmetry algebra and of its solvable subalgebra increase by one unit under the **c** and **r** maps. In the context of this paper the following considerations are important. If the result of the **c** map is a homogeneous quaternionic space, then the duality invariance (the symmetry of the scalar-vector sector of the theory) of the original theory must act transitively on the corresponding manifold parametrized by the scalar fields. The proof of this result, which applies also to the **r** map, is given in [10]. Also the converse is true: if the vector-scalar symmetries act transitively on the manifold parametrized by the scalars, then one can show that the symmetry group after dimensional reduction gives rise to additional symmetries, which leave the original scalar fields invariant but act transitively on the new scalar fields. In this respect it is important that the process of dimensional reduction always entails new symmetries whose number is larger than or equal to the number of new coordinates.

The above results show that homogenous quaternionic spaces that are in the image of the **c** map correspond to special homogeneous Kähler spaces. On the other hand, special homogeneous Kähler spaces give rise to special homogeneous quaternionic spaces, provided that the scalar-vector symmetry transformations act transitively on the Kähler manifold. Likewise, such Kähler spaces that are themselves in the image of the **r** map correspond to special homogeneous real spaces. Again, special homogeneous real spaces give rise to homogeneous Kähler spaces, provided that the vector-scalar symmetries act transitively on the real manifold.

In this connection Alekseevskiĭ's classification of homogeneous quaternionic spaces [13] is relevant, as was first pointed out in [5]. In [13] it was conjectured that the homogeneous quaternionic spaces consist of compact symmetric quaternionic spaces and (non-compact) normal quaternionic spaces. Normal quaternionic spaces are quaternionic spaces that admit a transitive completely solvable group of motions. According to Alekseevskiĭ there are two different types of normal quaternionic spaces characterized by their so-called canonical quaternionic subalgebra. The first type has as canonical subalgebra C_1^1, the solvable algebra corresponding to $Sp(1,1)/(Sp(1)\otimes Sp(1))$, and corresponds to the quaternionic projective spaces $Sp(m,1)/(Sp(m)\otimes Sp(1))$. These spaces are *not* in the image of the **c** map. The second type has a canonical subalgebra A_1^1, the solvable subalgebra of $SU(2,1)/(SU(2)\otimes U(1))$. Denoting the dimension of the normal quaternionic algebra as $4(n{+}1)$, the structure of the solvable algebra is such that it always contains a normal Kähler algebra $\mathcal{W}^{\rm s}$ of dimension $2n$, whose action on the remaining part of the algebra naturally defines a $(2n{+}2)$-dimensional representation corresponding to a solvable subgroup of $Sp(2n+2,\mathbb{R})$.[2] Therefore each normal quater-

[2]This representation thus acts on $2n{+}2$ of the generators. The two remaining generators of in the quaternionic algebra, e_0 and e_+, are inert under $\mathcal{W}^{\rm s}$. The sum of the Cartan

nionic space of this type defines the basic ingredients of a special normal Kähler space, encoded in its solvable transitive group of duality transformations. Alekseevskiĭ's analysis thus strongly indicates that the corresponding $N=2$ supergravity theory should exist, so that under the **c** map one will recover the original normal quaternionic space. To establish the existence of the supergravity theory, one must prove that a corresponding holomorphic function $F(X)$ exists that allows for these duality transformations. This program was carried out by Cecotti [14], who explicitly constructed the function $F(X)$ corresponding to each of the normal quaternionic spaces with canonical subalgebra A_1^1 that appears in the classification of Alekseevskiĭ. With the exception of the so-called minimal coupling, where $F(X)$ is a quadratic polynomial, all the Kähler spaces are in the image of the **r** map. The corresponding special Kähler manifolds were denoted by $H(p,q)$ and $K(p,q)$. Under the **c** map, they lead to the normal quaternionic manifolds $V(p,q)$ and $W(p,q)$ defined in [13]. If Alekseevskiĭ's classification is complete, there can be no other special Kähler spaces with solvable transitive duality transformations.

In this paper we start at the other end and derive a classification of all homogeneous quaternionic spaces that are in the image of the $\mathbf{c}{\circ}\mathbf{r}$ map. The analysis can be performed completely at the level of the special real spaces, and amounts to classifying all the cubic polynomials whose invariance group acts transitively on the corresponding special real spaces. This invariance group leaves the full $d=5$ supergravity Lagrangian invariant. The corresponding real spaces are obviously homogeneous, but because of the results quoted above, so are the corresponding Kähler and quaternionic spaces that emerge under the action of the **r** map and the $\mathbf{c}{\circ}\mathbf{r}$ map. When comparing the result to the classification of Alekseevskiĭ (and the corresponding one of Cecotti) we find that their classification is incomplete!

The cubic functions that are classified in this paper, are parametrized by

$$C(h) = d_{ABC}\, h^A\, h^B\, h^C\ . \qquad (A,B,C=1,\ldots,n) \tag{1.1}$$

The corresponding sigma model, which is contained in the five-dimensional supergravity Lagrangian [1], is defined by the Lagrangian

$$\mathcal{L} = -\tfrac{3}{2} d_{ABC}\, h^A\, \partial_\mu h^B\, \partial^\mu h^C\ , \tag{1.2}$$

where the scalar fields h^A are restricted by $C(h)=1$, so that the sigma model corresponds to a $(n-1)$-dimensional real space. Linear redefinitions of the fields h^A that leave $C(h)$ invariant constitute invariances of the full $N=2$ supergravity Lagrangian. However, it is not excluded that the sigma model Lagrangian (1.2) has additional symmetries, which cannot be extended to

subalgebra of $\mathcal{W}^{\rm s}$ and e_0 constitutes the Cartan subalgebra of the quaternionic algebra, whose rank is thus 1 higher than that of the Kähler algebra. The weight of the Kähler algebra under e_0 is thus zero, while the weight of the generators that constitute the $(2n{+}2)$-dimensional representation is $1/2$ times the weight of e_+.

symmetries of the full supersymmetric Lagrangian. The polynomial $C(h)$ is left invariant by linear transformations of the fields h^A, whose infinitesimal form is parametrized by matrices B^A_B,

$$\delta h^A = B^A_B\, h^B \ , \tag{1.3}$$

restricted by the condition

$$B^D_{(A}\, d_{BC)D} = 0. \tag{1.4}$$

As explained above, our aim is to determine all tensors d_{ABC} whose invariance group acts transitively on the manifold defined by (1.2). To analyze this question we first redefine the scalar fields in some reference point where the metric associated with the sigma model has positive signature[3]. One may choose this reference point equal to $h^A = (1, 0, \ldots, 0)$. In that case the coefficients d_{ABC} can be redefined according to the so-called canonical parametrization

$$d_{11a} = 0, \quad d_{1ab} = -\tfrac{1}{2}\, d_{111}\, \delta_{ab} \ . \qquad (a, b = 2, \ldots, n) \tag{1.5}$$

with $d_{111} > 0$. To preserve this parametrization only orthogonal redefinitions of the fields h^a are allowed.

The condition (1.4) that the $C(h)$ be invariant is then analyzed in the canonical parametrization. Putting $d_{111} = 1$ for convenience, (1.4) implies that $B^A{}_B$ takes the following form (see [15] where the corresponding Kähler spaces were analyzed)

$$B^1{}_1 = 0 \ , \quad B^a{}_1 = B^1{}_a \ , \quad B^a{}_b = B^c{}_1\, d_{abc} + A_{ab} \tag{1.6}$$

where A_{ab} is an antisymmetric matrix with $a, b, \ldots = 2, \ldots, n$. This matrix is subject to the condition

$$\Gamma_{abcd}\, B^d{}_1 = d_{d(ab}\, A_{c)d}, \tag{1.7}$$

where

$$\Gamma_{abcd} \equiv d_{e(ab}\, d_{cd)e} - \tfrac{1}{2}\, \delta_{(ab}\, \delta_{cd)} \ . \tag{1.8}$$

Now we observe that transformations associated with the matrices A_{ab} that are independent of the parameters $B^a{}_1$, leave the canonical reference point invariant and thus correspond to the isotropy group. Hence we are left with the requirement that the symmetry group should contain $n-1$ independent parameters $B^a{}_1$. Writing $A_{ab} = B^c{}_1\, A_{ab;c}$, where $A_{ab;c}$ is antisymmetric in its first two indices, this leads to the equation

$$\Gamma_{abcd} = D_{abc;d} \ , \tag{1.9}$$

[3]Positive signature is required to ensure a positive-definite Hilbert space of physical states. The necessary and sufficient condition for this is that the variables h^A are restricted to a domain where

$$(3d_{ACD}\, d_{BEF} - 2d_{ABC}\, d_{DEF})\, h^C h^D h^E h^F$$

is a positive definite matrix.

where Γ_{abcd} was defined above and

$$D_{abc;d} = d_{e(ab}\, A_{c)e;d} \ . \tag{1.10}$$

From the above results, it is clear that the homogeneous real manifold corresponding to (1.2) is locally isomorphic to G/H, where G is the invariance group of the tensor d_{ABC} and H is the orthogonal invariance group of the tensor d_{abc}.

From the arguments given earlier it follows that there is a corresponding analysis for the special Kähler and quaternionic spaces that follow from the real spaces that we introduced above. One may thus consider the Kähler spaces and require that the symmetry group of the $d = 4$ supergravity Lagrangian acts transitively on the space. The cubic polynomial $C(h)$ is directly related to the holomorphic function $F(X)$, which encodes the information of the special Kähler manifolds that follow from the real manifolds by the **r** map. It reads

$$F(X) = i d_{ABC} \frac{X^A\, X^B\, X^C}{X^0} \ , \tag{1.11}$$

where X^0 and X^A are complex variables. The Kähler manifold is only n-dimensional because two points (X^0, X^A) that are related by multiplication with an arbitrary complex number are identified. The **r** map thus introduces n+1 new coordinates, but at the same time it leads to at least n+1 additional symmetries [10, 11] so that the analysis proceeds along the same steps. Similarly, for quaternionic manifolds, the requirement of transitivity rests upon the same analysis as presented for the real manifolds[4] Therefore there is no need for going into further details.

A special case of (1.9) (namely $\Gamma_{abcd} = 0$) was analyzed in [1] in the context of Jordan algebras and in [15] for the special Kähler spaces. The connection with Jordan algebras arose because the (1.9) is equivalent to the condition that the torsion tensor associated with the special real space is covariantly constant. In that case the real space is symmetric (but this does not exhaust the special symmetric spaces). Likewise the corresponding Kähler and quaternionic spaces that one obtains by means of the **r** map and the **c**$\circ$**r** map are symmetric (and in this case there are no other (special) symmetric spaces). Surprisingly, the equation $\Gamma_{abcd} = 0$ emerges also in a different context, namely that of W_3 algebras [17], where it corresponds to the condition that ensures that the higher-spin invariances of a two-dimensional conformal field theory can be consistently truncated to the energy-momentum tensor and a spin-3 charge [18].

In view of these and possible other applications of (1.9), we shall keep the analysis of (1.9) self-contained without using the connection with the special geometries. The central result of this paper, namely the classification of the tensors d_{abc} that satisfy (1.9), is presented in section 2. The reader who is only interested in the results can skip this section as well as section 3,

[4]The isometries for the special quaternionic spaces were analyzed in [16].

where we rewrite the results for d in a simpler form and present the solutions for the tensors A in (1.9). The final result for the cubic polynomial $C(h)$ is given in section 4. It can be expressed as follows (not in the canonical parametrization). First we decompose the coordinates h^A into h^1, h^2, h^μ and h^m, where the range of the indices μ and m is equal to $q+1$ and r, respectively. Hence we have

$$n = 3 + q + r, \tag{1.12}$$

so that $n \geq 2$. Then $C(h)$ can be written as

$$C(h) = 3\Big\{h^1\,(h^2)^2 - h^1\,(h^\mu)^2 - h^2\,(h^m)^2 + \gamma_{\mu mn}\,h^\mu\,h^m\,h^n\Big\}\ , \tag{1.13}$$

where the coefficients $\gamma_{\mu mn}$ are the generators of a $(q+1)$-dimensional real Clifford algebra with positive signature.

Further results and and implications are given in section 4.

2 Classification.

In this section we study

$$\Gamma_{abcd} = D_{abc;d}\ , \tag{2.1}$$

where the indices $a, b, \ldots$ take $n-1$ values $2, \ldots, n$,

$$\begin{aligned} \Gamma_{abcd} &= d_{e(ab}\,d_{cd)e} - \tfrac{1}{2}\delta_{(ab}\,\delta_{cd)}\ , &(2.2)\\ D_{abc;d} &= d_{e(ab}\,A_{c)e;d}\ , &(2.3) \end{aligned}$$

and $A_{ab;c}$ is a tensor that is antisymmetric in its first two indices. Observe that $D_{abc;d}$ is only manifestly symmetric in three indices; full symmetry is only obtained after imposing (2.1). The tensors d_{abc} are symmetric and are concisely summarized by the cubic polynomial,

$$\mathcal{Y}(x) = d_{abc}\,x_a\,x_b\,x_c\ . \tag{2.4}$$

We will now give a complete classification of the tensors d_{abc} that satisfy (2.1) up to orthogonal redefinitions. Obviously, the tensors $A_{ab;c}$ can only be determined up to the generators of orthogonal transformations that leave d_{abc}, and thus the function $\mathcal{Y}(x)$ invariant. The analysis is done in two steps. First we show that after a suitable $O(n-1)$ rotation, it is always possible to bring the tensors d_{abc} into a form such that

$$\begin{aligned} d_{22a} &= \frac{1}{\sqrt{2}}\,\delta_{a2}\ , \\ \Gamma_{222a} &= A_{a2;2} = 0\ . \end{aligned} \tag{2.5}$$

The second step is then to bring the d_{abc} coefficients in a form where d_{2ab} is diagonal for general a and b and examine the consequences of (2.1).

Let us start by using $O(n-1)$ transformations to define a "2" direction (which will not necessarily coincide with the "2" direction chosen in (2.5)) such that

$$d_{abb} = \lambda\, \delta_{a2}\ . \tag{2.6}$$

A contraction of (2.1) over two indices then implies that the following three tensors must be equal,

$$\begin{aligned} 3\Gamma_{abcc} &= 2d_{acd}\, d_{bcd} + \lambda\, d_{2ab} - \tfrac{1}{2}(n+1)\, \delta_{ab}\ , \\ 3D_{cca;b} &= \lambda\, A_{a2;b}\ , \\ 3D_{abc;c} &= 2d_{ec(a}\, A_{b)e;c} + d_{abc}\, A_{dc;d}\ . \end{aligned} \tag{2.7}$$

Now we distinguish between three different cases, denoted by I, II and III, which will play a role throughout this analysis.

In case I we have

$$\lambda = 0 \Longleftrightarrow d_{abb} = 0\ . \tag{2.8}$$

According to (2.7) we then have

$$\Gamma_{ccab} = D_{cca;b} = D_{abc;c} = 0\ . \tag{2.9}$$

Using a notation where d_a and A_a are $(n-1)\times(n-1)$ matrices defined by $(d_a)_{bc} \equiv d_{abc}$ and $(A_a)_{bc} \equiv A_{bc;a}$, the first equation (2.9) reads

$$\langle d_a\, d_b\rangle = \tfrac{1}{4}(n+1)\, \delta_{ab}\ , \tag{2.10}$$

where $\langle A\rangle$ denotes the trace of a matrix A. Making use of this result we contract the tensors appearing in (2.1) with d_{cdf}, leading to

$$\begin{aligned} 3\Gamma_{abcd}\, d_{cdf} &= \tfrac{1}{4}(n-3)\, d_{abf} + 2\langle d_a\, d_f\, d_b\rangle\ , \\ 3D_{acd;b}\, d_{cdf} &= \tfrac{1}{4}(n+1)\, A_{af;b} + 2\langle d_a\, d_f\, A_b\rangle\ . \end{aligned} \tag{2.11}$$

According to (2.1) these two tensors should be equal. However, the first one is symmetric and the second one antisymmetric in a and f. Therefore they should vanish separately. Combining the above results, we derive

$$\Gamma_{abcd}\, \Gamma_{abce} = \Gamma_{abcd}\, D_{abc;e} = -\tfrac{1}{2} D_{aad;e} = 0. \tag{2.12}$$

For case I we therefore obtain

$$\Gamma_{abcd} = D_{abc;d} = d_{abb} = 0. \tag{2.13}$$

These are the equations that were analyzed in the appendix of [15]. The first part of this analysis coincides with the one that we are about to present for case II and III in the limit where the $A_{ab;c}$ tensors are put to zero or coincide with generators of the $O(n-1)$ subgroup that is left invariant by d_{abc}. A minor complication is that the "2" direction is not yet defined for case I, in view of the fact that $d_{abb} = 0$. However, the analysis only requires that $d_{222} \neq 0$.

Hence we proceed to case II and III where $\lambda \neq 0$. Therefore we know from (2.7) that $A_{2a;b}$ is symmetric in a and b. From this it follows that $A_{ab;c} = 0$ whenever two of its indices are equal to 2, which leads to $D_{222;2} = 0$.

In case II we assume that $d_{22i} = 0$, where $i = 3, \ldots, n$. Using $\Gamma_{2222} = D_{222;2} = 0$ one finds that

$$(d_{222})^2 = \tfrac{1}{2}. \tag{2.14}$$

As we can choose the sign of d_{222} at will, we thus find that case II also leads to (2.5).

In case III we assume that not all d_{22i} vanish. Diagonalizing the symmetric matrix $d_{2ij} - A_{2i;j}$ gives

$$A_{2i;j} = d_{2ij} - \lambda_i \, \delta_{ij}. \tag{2.15}$$

Then $\Gamma_{222i} = D_{222;i}$ yields

$$d_{22i} \, (\alpha + \lambda_i) = 0, \tag{2.16}$$

where we used the notation $\alpha \equiv d_{222}$. For those values of i for which $d_{22i} \neq 0$, we have the same eigenvalue $\lambda_i = -\alpha$. Hence, by means of a rotation, we can define a "3" direction such that

$$d_{22i} = \beta \, \delta_{i3} \ , \qquad \lambda_3 = -\alpha, \tag{2.17}$$

where $\beta \neq 0$ (otherwise we would be dealing with case II). Using $\Gamma_{222i} = D_{22i;2}$ gives

$$d_{233} = -\alpha \ , \qquad A_{\alpha 3;2} = 3 d_{23\alpha} \ , \tag{2.18}$$

with $\alpha = 4, \ldots, n$. Hence we also have

$$A_{23;3} = d_{233} - \lambda_3 = 0 \ . \tag{2.19}$$

Then from $\Gamma_{2222} = D_{222;2} = 0$, one derives

$$\alpha^2 + \beta^2 = \tfrac{1}{2} \ . \tag{2.20}$$

Now we analyze (2.1) with indices (2233). First $D_{332;2} = D_{223;3}$ takes the form

$$-\,2 \, (d_{23\alpha})^2 = \tfrac{2}{3} \, (d_{23\alpha})^2 \ . \tag{2.21}$$

Combining the above equations gives

$$d_{23\alpha} = A_{\alpha 3;2} = A_{23;\alpha} = A_{2\alpha;3} = 0 \ . \tag{2.22}$$

This implies that also $D_{223;\alpha} = 0$ and thus

$$3 D_{22\alpha;3} = \beta \, A_{\alpha 3;3} = 0. \tag{2.23}$$

Hence the tensor $A_{ab;c}$ vanishes whenever two of its indices are equal to 2 or 3. Moreover we have $A_{2\alpha;\beta} = A_{2\beta;\alpha}$.

Subsequently we deduce from $D_{332;2} = D_{223;3} = 0$ that $\Gamma_{2233} = 0$. Combining this with (2.20) shows that

$$d_{333} = -\beta \ . \tag{2.24}$$

Then $\Gamma_{223\alpha} = 0$ gives

$$d_{33\alpha} = 0 \ . \tag{2.25}$$

Hence our results for the d coefficients take the form

$$\begin{aligned}
&d_{222} = \alpha \ , \ d_{223} = \beta \ , \ d_{233} = -\alpha \ , \ d_{333} = -\beta \\
&\alpha^2 + \beta^2 = \tfrac{1}{2} \ , \\
&d_{22\alpha} = d_{23\alpha} = d_{33\alpha} = 0 \\
&d_{2bb} = d_{2\beta\beta} = \lambda \neq 0 \ , \\
&d_{3bb} = d_{3\beta\beta} = 0 \ , \ d_{\alpha bb} = d_{\alpha\beta\beta} = 0 \ .
\end{aligned} \tag{2.26}$$

Now we may perform an $O(2)$ transformation in the $(2,3)$ space such that the new coefficient d_{223} vanishes. In terms of the cubic function $\mathcal{Y}(x)$ this transformation corresponds to an orthogonal redefinition of x_2 and x_3,

$$\begin{aligned}
x_2' &= x_2 \cos\phi \pm x_3 \sin\phi \ , \\
x_3' &= -x_2 \sin\phi \pm x_3 \cos\phi \ .
\end{aligned} \tag{2.27}$$

Using (2.20) and defining $\alpha = \frac{1}{\sqrt{2}}\cos\theta$ and $\beta = \frac{1}{\sqrt{2}}\sin\theta$, we obtain a one-parameter family of coefficients

$$\alpha' = \frac{1}{\sqrt{2}}\cos(\pm\theta - 3\phi) \ ; \qquad \beta' = \frac{1}{\sqrt{2}}\sin(\pm\theta - 3\phi). \tag{2.28}$$

We can thus choose a parametrization such that

$$\begin{aligned}
&d_{222} = \frac{1}{\sqrt{2}} \ , \qquad d_{233} = -\frac{1}{\sqrt{2}} \ , \\
&d_{223} = d_{22\alpha} = d_{333} = d_{23\alpha} = d_{33\alpha} = d_{\alpha\beta\beta} = 0 \ ,
\end{aligned} \tag{2.29}$$

so that case III also allows the parametrization (2.5). Observe that after this redefinition d_{abb} may only differ from zero for $a = 2$ or 3. Hence

$$d_{abb} = \lambda_2\,\delta_{a2} + \lambda_3\,\delta_{a3} \ . \tag{2.30}$$

Case I is now characterized by $\lambda_2 = \lambda_3 = 0$, case II by $\lambda_2 \neq 0$, $\lambda_3 = 0$, and case III by $\lambda_3 \neq 0$.

Note, however, that the angle ϕ in (2.28) is not uniquely determined. There are 6 solutions. This means that there is still the possibility of redefining x_2 and x_3, such that we remain within the parametrization (2.5). Those redefinitions consist of products of reflections,

$$x_3 \to -x_3 \ , \tag{2.31}$$

and $2\pi/3$ rotations,

$$
\begin{aligned}
x_2 &\;\to\; -\tfrac{1}{2}\, x_2 + \tfrac{1}{2}\sqrt{3}\, x_3 \ , \\
x_3 &\;\to\; -\tfrac{1}{2}\sqrt{3}\, x_2 - \tfrac{1}{2}\, x_3 \ .
\end{aligned} \tag{2.32}
$$

These replacements do not change the part of $\mathcal{Y}$ that is quadratic or cubic in x_2 and x_3,

$$
\mathcal{Y}(x) = \frac{1}{\sqrt{2}} \left(x_2^3 - 3x_2\, x_3^2 \right) + \ldots . \tag{2.33}
$$

Later we shall see that the above redefinitions allow one to rewrite some of the solutions belonging to case II into those belonging to case III. ■

This concludes the proof of (2.5). The second step in the classification starts by diagonalizing d_{2ab} for all a and b (this is consistent with (2.5)). Hence we adjust the frame of reference, such that

$$
d_{2ij} = \mu_i\, \delta_{ij} \ , \tag{2.34}
$$

where we recall that $i, j = 3, \ldots, n$.

Now we consider (2.1) with indices $(22ij)$, according to which the following three tensors should be equal,

$$
\begin{aligned}
3\Gamma_{22ij} &= \Big(\frac{1}{\sqrt{2}}\, \mu_i + 2\mu_i^2 - \tfrac{1}{2} \Big) \delta_{ij} \ , \\
3D_{22i;j} &= \Big(-\frac{1}{\sqrt{2}} + 2\mu_i \Big) A_{2i;j} \ , \\
3D_{ij2;2} &= (\mu_j - \mu_i)\, A_{ij;2} \ .
\end{aligned} \tag{2.35}
$$

As the last tensor vanishes for $i = j$, while the first one takes its non-zero values in that case, the three tensors should vanish separately. The vanishing of the first one implies that μ_i can only take two possible values, $-\frac{1}{\sqrt{2}}$ or $\frac{1}{2\sqrt{2}}$. Therefore it is convenient to split the indices i according to these values into indices $\mu, \nu, \ldots$ and $m, n, \ldots$ such that

$$
\mu_\mu = -\frac{1}{\sqrt{2}} \ , \qquad \mu_m = \frac{1}{2\sqrt{2}} \ . \tag{2.36}
$$

Furthermore we obtain

$$
A_{\mu m;2} = A_{2\mu;\nu} = A_{2\mu;m} = 0. \tag{2.37}
$$

It is clear that the special index value $i = 3$ that occurred in the analysis of case III, is contained in the index set labeled by $\mu, \nu, \ldots$.

The next step is the analysis of (2.1) with indices $(2ijk)$. The corresponding tensors are

$$
\begin{aligned}
3\Gamma_{2ijk} &= d_{ijk}\, (\mu_i + \mu_j + \mu_k) \ , \\
3D_{ijk;2} &= 3 d_{l(ij} A_{k)l;2} \ , \\
3D_{2ij;k} &= d_{lij} A_{2l;k} + (\mu_i - \mu_j)\, A_{ji;k} \ .
\end{aligned} \tag{2.38}
$$

Using (2.37) it follows that $d_{\mu\nu\rho}\, D_{\mu\nu\rho;2} = d_{\mu\nu m}\, D_{\mu\nu m;2} = d_{mnp}\, D_{mnp;2} = 0$ by virtue of the antisymmetry of the coefficients $A_{ij;2}$. Therefore the tensor Γ_{2ijk} should vanish when contracted with these d coefficients. As Γ_{2ijk} is itself proportional to the d coefficients, it follows that certain components should vanish, i.e.,

$$d_{\mu\nu\rho} = d_{\mu\nu m} = d_{mnp} = 0\ . \tag{2.39}$$

As $\Gamma_{2\mu mn}$ already vanishes by virtue of the fact that $\mu_\mu + \mu_m + \mu_n = 0$, we have thus established that all components of the Γ tensor with one or more indices equal to 2 now vanish. Most of the components of $D_{ijk;2}$ and $D_{2ij;k}$ now vanish identically. The equation $D_{\mu mn;2} = 0$ implies that the d tensors should be left invariant by orthogonal transformations characterized by the $A_{ij;2}$. The latter can be put to zero and do not restrict the d-coefficients. Furthermore, there is

$$D_{2\mu m;i} = 0 \iff A_{m\mu;i} = \tfrac{2}{3}\sqrt{2}\, d_{\mu mn} A_{2n;i}\ . \tag{2.40}$$

The only components of Γ_{ijkl} that do not vanish identically at this point, are

$$\begin{aligned} \Gamma_{\mu\nu mn} &= \tfrac{2}{3} d_{mp(\mu}\, d_{\nu)np} - \tfrac{1}{4}\delta_{\mu\nu}\, \delta_{mn}\ , && (2.41)\\ \Gamma_{mnpq} &= d_{\mu(mn}\, d_{pq)\mu} - \tfrac{3}{8}\delta_{(mn}\, \delta_{pq)}\ . && (2.42) \end{aligned}$$

According to (2.1), they should satisfy

$$\begin{aligned} \Gamma_{\mu\nu mn} &= D_{\mu\nu m;n} = -\Gamma_{\mu\nu mp}\, H_{pn}\ , && (2.43)\\ \Gamma_{\mu\nu mn} &= D_{mn\mu;\nu} = \tfrac{2}{3}\, d_{\mu q(m}\, A_{n)q;\nu} + \tfrac{1}{3}\, d_{\rho mn}\, A_{\mu\rho;\nu}\ , && (2.44)\\ \Gamma_{mnpq} &= D_{mnp;q} = \Gamma_{mnpr}\, H_{rq}\ , && (2.45) \end{aligned}$$

where we made use of (2.40) and defined $H_{mn} \equiv \frac{2}{3}\sqrt{2} A_{2m;n}$.

Contractions of the above equations will give useful information. Denoting the range of the indices μ by $q+1$, and the range of the indices m by r, so that

$$n = 3 + q + r\ , \tag{2.46}$$

we have

$$\begin{aligned} \Gamma_{\mu\nu mm} &= \tfrac{2}{3}\,\mathrm{tr}\,(d_\mu\, d_\nu) - \tfrac{1}{4} r\, \delta_{\mu\nu} = \tfrac{1}{3}\, d_{\rho mm}\, A_{\mu\rho;\nu}\ , && (2.47)\\ \Gamma_{\mu\mu mn} &= \tfrac{2}{3}(d\, d)_{mn} - \tfrac{1}{4}(q+1)\, \delta_{mn}\ , && (2.48)\\ \Gamma_{ppmn} &= \tfrac{2}{3}(d\, d)_{mn} - \tfrac{1}{8}(r+2)\, \delta_{mn} + \tfrac{1}{3} d_{\mu pp}\, d_{\mu mn}\ , && (2.49) \end{aligned}$$

where $(d\, d)_{mn} \equiv d_{\mu mp}\, d_{\mu np}$.

The remaining equations for which the corresponding Γ tensors vanish, $D_{m\mu\nu;\rho} = D_{mnp;\mu} = D_{\mu mn;p} = 0$, are solved by $A_{2m;\mu} = A_{mn;p} = A_{\mu\nu;p} = 0$. Other solutions that satisfy these equation, correspond to non-trivial invariances of the d_{abc} tensor.

Let us now turn again to the three cases discussed previously. The only non-vanishing components of d_{abc} are $d_{\mu mn}$ and

$$d_{222} = \frac{1}{\sqrt{2}} \;, \quad d_{2\mu\nu} = -\frac{1}{\sqrt{2}}\,\delta_{\mu\nu} \;, \quad d_{2mn} = \frac{1}{2\sqrt{2}}\,\delta_{mn} \;, \tag{2.50}$$

corresponding to

$$\mathcal{Y}(x) = \frac{1}{\sqrt{2}}\Big(x_2^3 - 3x_2\,(x_\mu^2 - \tfrac{1}{2}\,x_m^2)\Big) + 3d_{\mu mn}\,x_\mu\,x_m\,x_n \;. \tag{2.51}$$

The three cases are characterized by the possible non-vanishing values of d_{abb}, which are

$$d_{2bb} = \frac{1}{2\sqrt{2}}(r-2q), \quad \text{and} \quad d_{\mu bb} = d_{\mu mm}. \tag{2.52}$$

In case I we have $r = 2q$, so that $n = 3(q+1)$, and $d_{\mu mm} = 0$. As we established already, one must have

$$\Gamma_{\mu\nu mn} = \Gamma_{mnpq} = 0 \;. \tag{2.53}$$

Hence the $d_{\mu mn}$ may be regarded as $r\times r$ matrices, which generate a $(q+1)$-dimensional Clifford algebra. In view of the second condition, the dimension of this algebra is severely constrained. According to [15], only $q = 1$, 2, 4 and 8 are possible, corresponding to $n = 6$, 9, 15 and 27, respectively. This conclusion follows from the possible dimension of the reducible representations of the Clifford algebra. This case is related to Jordan algebras and the magic square [1]. In addition we have the trivial case with $q = 0$ and $n = 3$.

For case II we have $d_{\mu mm} = 0$ and $r - 2q \neq 0$. It turns out that it is sufficient to restrict our analysis to the case $q = -1$. Then there are no indices μ, so that the non-vanishing coefficients d_{abc} are

$$d_{222} = \frac{1}{\sqrt{2}} \;, \quad d_{2mn} = \frac{1}{2\sqrt{2}}\,\delta_{mn} \;, \tag{2.54}$$

with $r = n - 2$ arbitrary. Obviously, we have

$$\Gamma_{mnpq} = -\tfrac{3}{8}\,\delta_{(mn}\,\delta_{pq)} \;, \quad H_{mn} = \delta_{mn} \;, \tag{2.55}$$

while all other components of Γ vanish.

The reason why we do not have to consider $q \geq 0$, is that, after identifying one of the indices μ with 3, we can always perform a redefinition (2.32). After this redefinition we no longer have $d_{\mu mm} = 0$, so that we can perform the same steps as before, but now for case III. Nevertheless for clarity of the presentation we briefly derive the consequences for case II with arbitrary q. We first use (2.45) to obtain

$$\Gamma^{(2)}_{mn} = \Gamma^{(2)}_{mp}\,H_{pn}, \quad \text{with} \quad \Gamma^{(2)}_{mn} \equiv \Gamma_{mpqr}\,\Gamma_{pqrn}. \tag{2.56}$$

Let us now decompose the space associated with the indices $m, n, \ldots$ into the null space of $\Gamma^{(2)}$ and its orthogonal complement. The indices $m, n, \ldots$ are split accordingly into indices $A, B, \ldots$ and $M, N, \ldots$, so that $\Gamma^{(2)}_{Am} = \Gamma^{(2)}_{mA} = 0$ and $\det(\Gamma^{(2)}_{MN}) \neq 0$. This implies that

$$\Gamma_{mnpA} = 0, \tag{2.57}$$

while (2.56) restricts the matrix H according to $H_{MA} = 0$ and $H_{MN} = \delta_{MN}$. Combining $d_{\mu mm} = 0$ and (2.48), (2.49) and (2.57), we find

$$\Gamma_{\mu\mu AB} = \tfrac{1}{8}(r - 2q)\,\delta_{AB}, \qquad \Gamma_{\mu\mu AM} = \Gamma_{\mu\mu MA} = 0. \tag{2.58}$$

From (2.43) it then follows that the non-vanishing matrix elements of H are given by $H_{AB} = -\delta_{AB}$ and $H_{MN} = \delta_{MN}$. while

$$\Gamma_{\mu\nu MN} = 0\ . \tag{2.59}$$

Therefore $\Gamma_{\mu\mu mn}$ is now fully known and non-vanishing. On the other hand, $\Gamma_{\mu\mu mm}$ is restricted to vanish by (2.47). This implies that the null space of $\Gamma^{(2)}$ is in fact empty, so that there are no indices $A, B, \ldots$. Hence we find that $H_{mn} = \delta_{mn}$. From (2.43) it then follows that $\Gamma_{\mu\nu mn}$ vanishes,

$$\Gamma_{\mu\nu mn} = 0\ , \tag{2.60}$$

while Γ_{mnpq} remains arbitrary. Hence the coefficients $d_{\mu mn}$ may again be regarded as $r \times r$ matrices generating a $(q+1)$-dimensional Clifford algebra. This puts restrictions on r and q, but those are considerably weaker than in the previous case.

Now let us turn to case III, where $d_{\mu mm} \neq 0$. By a suitable rotation of the components labeled by $\mu, \nu, \ldots$, we choose the direction in which $d_{\mu mm}$ does not vanish to be equal to $\mu = 3$. The remaining indices μ will be denoted by $\hat\mu$. Subsequently we diagonalize d_{3mn},

$$d_{3mn} = \sqrt{\tfrac{3}{8}}\lambda_m\,\delta_{mn}\ . \tag{2.61}$$

We then obtain from (2.47) that $A_{3\mu;\nu}$ is symmetric in μ and ν (this conclusion requires $d_{\mu mm} \neq 0$), which implies that $A_{3\mu;3} = 0$. Substituting this result into (2.44) for $\mu = \nu = 3$, we obtain

$$\tfrac{1}{4}(\lambda_m^2 - 1)\delta_{mn} = (\lambda_m - \lambda_n) A_{nm;3}\ . \tag{2.62}$$

As $A_{nm;3}$ is antisymmetric in n and m, it follows that both sides of the equation should vanish separately, so that

$$\lambda_m^2 = 1\ , \qquad \Gamma_{\mu\nu mn} = 0 \quad \text{for} \quad \mu = \nu = 3\ . \tag{2.63}$$

Splitting the range of indices $m, n, \ldots$ into indices $x, y, \ldots$ and $\dot x, \dot y, \ldots$ such that

$$\lambda_x = 1\ ; \qquad \lambda_{\dot x} = -1. \tag{2.64}$$

it follows from (2.62) that $A_{x\dot{y};3}=0$. Subsequently, consider again (2.44) but now with $\mu=\hat{\mu}\neq 3$, $\nu=3$, $m=x$, $n=y$,

$$2d_{\hat{\mu}xy} = 2d_{\hat{\mu}z(x}A_{y)z;3} + d_{\hat{\rho}xy}A_{\hat{\mu}\hat{\rho};3}. \tag{2.65}$$

Multiplying the right-hand side with $d_{\hat{\mu}xy}$ gives zero by virtue of the anti-symmetry of the A coefficients. This implies $d_{\hat{\mu}xy}=0$. The same derivation can be repeated for two dotted indices, so we are left with the coefficients $d_{\hat{\mu}x\dot{y}}$ with mixed indices. This then yields $\Gamma_{3\hat{\mu}mn}=0$.

In cases I and II we proved that $\Gamma_{\mu\nu mn}=0$ in general. Therefore, in all cases with $q\geq 0$, one can identify a suitable index $\mu=3$ and bring d_{3mn} in diagonal form like in (2.61), so that one can employ the parametrization in terms of dotted and undotted indices and derive the restrictions for $d_{\hat{\mu}mn}$ as found above. The present formulation is thus fully applicable to all three cases with $q\geq 0$ (for the moment we ignore the results obtained above for the A tensors, which apply only to case III). Let us therefore proceed and present the relevant equations in this formulation for the general case.

From $D_{3x\dot{y};\hat{\mu}}=0$ it follows that

$$A_{x\dot{y};\hat{\mu}} = d_{\hat{\nu}x\dot{y}}\,H_{\hat{\nu}\hat{\mu}}\ , \tag{2.66}$$

where $H_{\hat{\mu}\hat{\nu}}\equiv\sqrt{2/3}\,A_{3\hat{\mu};\hat{\nu}}$. In addition we have $D_{\hat{\mu}x\dot{y};3}=0$, which can be solved by $A_{\dot{x}\dot{y};3}=A_{xy;3}=A_{\hat{\mu}\hat{\nu};3}=0$ and has no consequences for the d tensor. When non-zero values are possible for the A tensors, they define non-trivial invariances of the d coefficients.

Let us give the non-vanishing components of the Γ tensor in this notation (cf. (2.41, 2.42)),

$$\begin{aligned}
\Gamma_{\hat{\mu}\hat{\nu}xy} &= \tfrac{2}{3}\,d_{\dot{z}x(\hat{\mu}}\,d_{\hat{\nu})y\dot{z}} - \tfrac{1}{4}\,\delta_{\hat{\mu}\hat{\nu}}\,\delta_{xy}\ ,\\
\Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{y}} &= \tfrac{2}{3}\,d_{z\dot{x}(\hat{\mu}}\,d_{\hat{\nu})\dot{y}z} - \tfrac{1}{4}\,\delta_{\hat{\mu}\hat{\nu}}\,\delta_{\dot{x}\dot{y}}\ ,\\
\Gamma_{xy\dot{z}\dot{w}} &= \tfrac{2}{3}\,d_{\hat{\mu}x(\dot{z}}\,d_{\dot{w})y\hat{\mu}} - \tfrac{1}{4}\,\delta_{xy}\,\delta_{\dot{z}\dot{w}}\ .
\end{aligned} \tag{2.67}$$

We denote the range of indices $\hat{\mu}$, x and $\dot{x}$ by q, p and $\dot{p}$, respectively, so that $r=p+\dot{p}$ and $n=3+q+p+\dot{p}$.

These equations have a remarkable symmetry under interchange of the indices $\hat{\mu}$, x and $\dot{x}$. This is not a coincidence and is related to the redefinitions that were explained previously. To see this, consider the cubic polynomial $\mathcal{Y}$, which has acquired the following form (for $q\geq 0$),

$$\begin{aligned}
\mathcal{Y}(x) \;=\; & \frac{1}{\sqrt{2}}\,x_2\,(x_2+\sqrt{3}\,x_3)\,(x_2-\sqrt{3}\,x_3)\\
& +\frac{3}{\sqrt{2}}\Big(-x_2\,x_{\hat{\mu}}^2 + \tfrac{1}{2}(x_2+\sqrt{3}\,x_3)\,x_x^2 + \tfrac{1}{2}(x_2-\sqrt{3}\,x_3)\,x_{\dot{x}}^2\Big)\\
& +6\,d_{\hat{\mu}m\dot{m}}\,x_{\hat{\mu}}\,x_x\,x_{\dot{x}}.
\end{aligned} \tag{2.68}$$

The replacement (2.32) induces an interchange of the quantities x_2 and $-\frac{1}{2}(x_2\pm\sqrt{3}x_3)$, which leaves the form of the function $\mathcal{Y}(x)$ unchanged, except that the labels $\hat{\mu}$, x and $\dot{x}$ are interchanged. Similarly, the replacement

(2.31) corresponds to an interchange of labels x with $\dot{x}$ (of course, the range of the various indices changes accordingly).

Case I is now characterized by $q = p = \dot{p}$, case II by $p = \dot{p} \neq q$, and case III by $p \neq \dot{p}$. Contraction of the above tensors leads to the following equations,

$$\begin{aligned} \Gamma_{\hat{\mu}\hat{\mu}xy} &= \Gamma_{\dot{z}\dot{z}xy} + \tfrac{1}{4}(\dot{p} - q)\, \delta_{xy} \ , \\ \Gamma_{\hat{\mu}\hat{\mu}\dot{x}\dot{y}} &= \Gamma_{zz\dot{x}\dot{y}} + \tfrac{1}{4}(p - q)\, \delta_{\dot{x}\dot{y}} \ , \\ \Gamma_{\hat{\mu}\hat{\nu}xx} &= \Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{x}} + \tfrac{1}{4}(\dot{p} - p)\, \delta_{\hat{\mu}\hat{\nu}} \ . \end{aligned} \tag{2.69}$$

In this notation, the equations (2.43-2.45) take the form

$$\begin{aligned} \Gamma_{\hat{\mu}\hat{\nu}xy} &= \Gamma_{\hat{\rho}\hat{\nu}xy}\, H_{\hat{\rho}\hat{\mu}} = -\Gamma_{\hat{\mu}\hat{\nu}zy}\, H_{zx} \ , \\ \Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{y}} &= -\Gamma_{\hat{\rho}\hat{\nu}\dot{x}\dot{y}}\, H_{\hat{\rho}\hat{\mu}} = -\Gamma_{\hat{\mu}\hat{\nu}\dot{z}\dot{y}}\, H_{\dot{z}\dot{x}} \ , \\ \Gamma_{xy\dot{x}\dot{y}} &= \Gamma_{zy\dot{x}\dot{y}}\, H_{zx} = \Gamma_{xy\dot{z}\dot{y}}\, H_{\dot{z}\dot{x}} \ , \end{aligned} \tag{2.70}$$

where we suppressed the equations involving $H_{x\dot{y}}$ and $H_{\dot{x}y}$, which have no consequences for the d-coefficients.

The symmetry noted above should be taken into account when identifying inequivalent d tensors. However, its presence also facilitates our work, as it allows us to apply the following lemma in three possible situations:

Lemma: *Consider one of the three matrices H, say, $H_{\hat{\mu}\hat{\nu}}$. Then, either the two other matrices H are of equal dimension ($p = \dot{p}$), in which case $\Gamma_{\hat{\mu}\hat{\nu}xx} = \Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{x}} = 0$, or they are not of equal dimension ($p \neq \dot{p}$), in which case $H_{\hat{\mu}\hat{\nu}}$ is equal to plus or minus the identity matrix, with $\Gamma_{\hat{\mu}\hat{\nu}xy}$ or $\Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{y}}$ vanishing, respectively.*

To prove this lemma, multiply the third equation (2.69) with $H_{\hat{\mu}\hat{\rho}}$ and apply (2.70). When $p = \dot{p}$ the corresponding equations lead to $\Gamma_{\mu\nu xx} = \Gamma_{\mu\nu\dot{x}\dot{x}} = 0$, as claimed above. On the other hand, when $p \neq \dot{p}$, it follows that $H_{\hat{\mu}\hat{\nu}}$ is a symmetric matrix which can be diagonalized. Consider first the case where $\hat{\mu}$ and $\hat{\nu}$ belong to an eigenspace of H with eigenvalue different from ± 1. Then it follows from (2.70) that $\Gamma_{\hat{\mu}\hat{\nu}xx} = \Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{x}} = 0$, which leads to $p = \dot{p}$ and thus to a contradiction. Hence $H_{\hat{\mu}\hat{\nu}}$ has only eigenvalues equal to ± 1. Assume now that both eigenvalues occur. Consider then indices $\hat{\mu}$ and $\hat{\nu}$ corresponding to the subspace with eigenvalue $+1$, and an index $\hat{\rho}$ belonging to the subspace with eigenvalue -1. Then (2.70) implies that (no sum over repeated $\hat{\rho}$ index)

$$\Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{y}} = \Gamma_{\hat{\rho}\hat{\rho}xy} = \Gamma_{\hat{\mu}\hat{\rho}\dot{x}\dot{y}} = \Gamma_{\hat{\mu}\hat{\rho}xy} = \Gamma_{\hat{\nu}\hat{\rho}\dot{x}\dot{y}} = \Gamma_{\hat{\nu}\hat{\rho}xy} = 0 \ . \tag{2.71}$$

According to the last four equations $d_{\hat{\rho}}$ anticommutes as a matrix with $d_{\hat{\mu}}$ and $d_{\hat{\nu}}$. Thus we perform the following calculation (no sum over repeated $\hat{\rho}$ index),

$$\begin{aligned} 0 &= d_{\hat{\rho}x\dot{x}}\, \Gamma_{\hat{\mu}\hat{\nu}\dot{x}\dot{y}}\, d_{\hat{\rho}\dot{y}y} \\ &= \tfrac{2}{3}(d_{\hat{\rho}}\, d_{(\hat{\mu}}\, d_{\hat{\nu})}\, d_{\hat{\rho}})_{xy} - \tfrac{1}{4}\, (d_{\hat{\rho}}\, d_{\hat{\rho}})_{xy}\ \delta_{\hat{\mu}\hat{\nu}} \\ &= \Gamma_{\hat{\mu}\hat{\nu}xz}\, (d_{\hat{\rho}}\, d_{\hat{\rho}})_{zy} = \tfrac{3}{8}\, \Gamma_{\hat{\mu}\hat{\nu}xy} \ . \end{aligned} \tag{2.72}$$

Hence both $\Gamma_{\hat\mu\hat\nu xy}$ and $\Gamma_{\hat\mu\hat\nu\dot x\dot y}$ vanish, which requires that $p=\dot p$, thus leading to a contradiction. Hence the eigenvalues of H must all be equal, which completes the proof of the lemma. ■

With the help of this lemma it is straightforward to analyze the various solutions of (2.67,2.70). First we assume that q, p and $\dot p$ are non-vanishing. Application of the lemma then reveals that there are no solutions with different values for q, p and $\dot p$, simply because two Γ tensors must then vanish, which, by (2.69) implies that at least two of the parameters q, p or $\dot p$ should be equal. Because of the symmetry we can choose either two of the parameters equal. Let us assume, for instance, $\underline{q\neq p=\dot p\neq 0}$. Then, from the lemma applied to the three matrices H one finds four possibilities, two of which implying that two uncontracted Γ tensors vanish, which is inconsistent with $q\neq p$ or $q\neq\dot p$. Then one has the third possibility corresponding to

$$p=\dot p\ :\quad \Gamma_{xy\dot x\dot y}=\Gamma_{\hat\mu\hat\nu xx}=\Gamma_{\hat\mu\hat\nu\dot x\dot x}=0\ ,\quad H_{xy}=-\delta_{xy}\ ,\quad H_{\dot x\dot y}=-\delta_{\dot x\dot y}\ . \tag{2.73}$$

On the other hand the first line of (2.69) implies that $\Gamma_{\hat\mu\hat\mu xx}=\frac14(p-q)p$, which must vanish according to the above equation. Hence $p=0$; this is one of the cases to be discussed below. The remaining possibility, which leaves q arbitrary corresponds to

$$p=\dot p\ :\quad \Gamma_{\hat\mu\hat\nu xy}=\Gamma_{\hat\mu\hat\nu\dot x\dot y}=0\ ,\quad H_{xy}=\delta_{xy}\ ,\quad H_{\dot x\dot y}=\delta_{\dot x\dot y}\ . \tag{2.74}$$

Clearly, this solution belongs to case II, while the equivalent solution with $q=p\neq\dot p$ or $q=\dot p\neq p$ belongs to case III.

The case $\underline{q=p=\dot p}$ is case I, for which we showed already before that all Γ symbols are zero with traceless d coefficients.

What remains is to investigate the situation where at least one of the parameters q, p or $\dot p$ vanishes (this may occur in case I, II or III depending on the values of the other two parameters). In that case only one of the tensors Γ remains (unless one of the other parameters vanishes as well). Let us choose $\underline{q=0}$. There is only

$$q=0\ :\quad \Gamma_{xy\dot x\dot y}=-\tfrac14\,\delta_{xy}\,\delta_{\dot x\dot y}\ ,\quad H_{xy}=\delta_{xy}\ ,\quad H_{\dot x\dot y}=\delta_{\dot x\dot y}\ , \tag{2.75}$$

This completes the classification of the coefficients d_{abc} satisfying the equation (2.1).

3 Results of the classification.

3.1 d-coefficients and Clifford algebras.

In the previous section we obtained the possible tensors d_{abc} that are solutions to (2.1), up to arbitrary $O(n-1)$ rotations. The indices $a, b, \ldots$, are decomposed into indices 2, μ and m, where μ and m take $q+1$ and r values, respectively. We thus have $n = 3 + q + r$.

The general results for the d tensors are summarized in (2.50) and (2.51), where, as we shall see shortly, the coefficients $d_{\mu mn}$ satisfy the defining relation (up to a proportionality factor) of the generators of a Clifford algebra and can thus be expressed as (symmetric, real) gamma matrices according to

$$d_{\mu mn} = \sqrt{\tfrac{3}{8}}\,(\gamma_\mu)_{mn}\,. \tag{3.1}$$

Therefore the function $\mathcal{Y}(x)$ acquires the generic form

$$\mathcal{Y}(x) = \frac{1}{\sqrt{2}}\left\{x_2^3 - 3x_2\,(x_\mu^2 - \tfrac{1}{2}\,x_m^2) + \tfrac{3}{2}\sqrt{3}\,(\gamma_\mu)_{mn}\,x_\mu\,x_m\,x_n\right\}\ , \tag{3.2}$$

where the gamma matrices generate a real representation of the Clifford algebra $\mathcal{C}(q+1,0)$ with positive metric. Let us now analyze the various solutions found in the previous section.

The first case is $\underline{q=-1}$ (i.e., indices μ are absent). As $d_{\mu mn}$ does not exist, the d coefficients are completely given by (2.50), and the corresponding function $\mathcal{Y}$ reads

$$\mathcal{Y}(x) = \frac{1}{\sqrt{2}}\left(x_2^3 + \tfrac{3}{2}x_2\,x_m^2\right)\ . \tag{3.3}$$

As shown previously, $\Gamma_{mnpq} = -\frac{3}{8}\delta_{(mn}\,\delta_{pq)}$, which vanishes only for $r = 0$. We denote these solutions by $L(-1, r)$ with $r \geq 0$ and $n = 2 + r$.

For $q \geq 0$ there is one value of μ which we denote by "3" and we split indices m into x or $\dot{x}$, taking p and $\dot{p}$ values, respectively. These indices are distinguished by

$$d_{3xy} = \sqrt{\tfrac{3}{8}}\,\delta_{xy}\ , \qquad d_{3\dot{x}\dot{y}} = -\sqrt{\tfrac{3}{8}}\,\delta_{\dot{x}\dot{y}}\ , \qquad d_{3x\dot{y}} = 0. \tag{3.4}$$

For $\underline{q=0}$ there is no further restriction; p and $\dot{p}$ are arbitrary and we denote the solution as $L(0, P, \dot{P}) \equiv L(0, \dot{P}, P)$, where we replace p and $\dot{p}$ by P and $\dot{P}$ in order to have a uniform notation for the reducible representations of the Clifford algebra (to be discussed below). Whenever P or $\dot{P}$ are zero we write $L(0, P) \equiv L(0, P, 0)$. The diagonal matrix

$$(\gamma_3)_{mn} \equiv \sqrt{\tfrac{8}{3}}\,d_{3mn}\ , \tag{3.5}$$

can be viewed as a gamma matrix that generates a one-dimensional Clifford algebra $\mathcal{C}(1,0)$. This algebra has two inequivalent irreducible representations corresponding to $+1$ and -1. The numbers P and $\dot{P}$ specify the multiplicities of these representations in γ_3. The corresponding function $\mathcal{Y}$ follows directly from (3.2),

$$\begin{aligned}\mathcal{Y}(x) &= \frac{1}{\sqrt{2}}\Big\{x_2\,(x_2+\sqrt{3}\,x_3)\,(x_2-\sqrt{3}\,x_3)\\ &\qquad +\tfrac{3}{2}(x_2+\sqrt{3}\,x_3)\,x_x^2+\tfrac{3}{2}(x_2-\sqrt{3}\,x_3)\,x_{\dot{x}}^2\Big\}\ . \end{aligned}\tag{3.6}$$

The non-vanishing Γ tensor is

$$\Gamma_{xy\dot{x}\dot{y}} = -\tfrac{1}{4}\delta_{xy}\,\delta_{\dot{x}\dot{y}}\ , \tag{3.7}$$

which vanishes whenever p or $\dot{p}$ vanishes. Note that we have $n = 3+p+\dot{p}$.

For $\underline{q>0}$ we may restrict ourselves to $r>0$, as the case $q>0,\ r=0$ is equivalent to $L(0,q,0)$ by a rotation (2.32). Denoting the values of $\mu\neq 3$ by $\hat{\mu}$, the tensors $d_{\hat{\mu}mn}$ satisfy (2.74), which implies that we can define the following $r\times r$ gamma matrices,

$$\gamma_{\hat{\mu}} = \sqrt{\tfrac{8}{3}}\begin{pmatrix} 0 & d_{\hat{\mu}x\dot{v}} \\ d_{\hat{\mu}\dot{y}w} & 0\end{pmatrix}\ ,\qquad \gamma_3 = \sqrt{\tfrac{8}{3}}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\ . \tag{3.8}$$

We have thus established (3.2). To classify all cases with $q>0$ one must consider all possible gamma matrices that generate a real Clifford algebra $\mathcal{C}(q+1,0)$. The irreducible representations (with positive definite metric) are listed in table 1 [19]. They are unique except when the Clifford module consists of a direct sum of two factors. As shown in table 1 this is the case for $q=0$ mod 4, where there exist two inequivalent irreducible representations[5]. This implies that for $q\neq 0$ mod 4, the gamma matrices are unique once we specify the number of irreducible representations. The solution for the d coefficients is then denoted by $L(q,P)$, where P denotes the number of irreducible representations. We have thus $r = P\,\mathcal{D}_{q+1}$, or, equivalently, $n = 3+q+P\,\mathcal{D}_{q+1}$. However, when q is a multiple of 4 (i.e., $q=4m$ with m integer), then there exist two inequivalent irreducible representations and the solutions are characterized by specifying the multiplicities P and $\dot{P}$ of each of the two representations. The solutions are therefore denoted by $L(4m,P,\dot{P})\equiv L(4m,\dot{P},P)$ and we have $n = 3+4m+(P+\dot{P})\mathcal{D}_{4m+1}$. Whenever P or $\dot{P}$ vanishes, we denote the solutions by $L(4m,P)\equiv L(4m,P,0)$.

This concludes the classification of the various solutions. The only components of Γ_{abcd} that possibly differ from zero are

$$\Gamma_{mnpq} = \tfrac{3}{8}\left[(\gamma_\mu)_{(mn}\,(\gamma_\mu)_{pq)} - \delta_{(mn}\delta_{pq)}\right]. \tag{3.9}$$

As one easily verifies, this tensor vanishes only for $L(-1,0)$, $L(0,r)$, $L(1,1)$, $L(2,1)$, $L(4,1)$ and $L(8,1)$, corresponding to $n=2$, $3+r$, 6, 9, 15 and 27,

[5] These are the only dimensions for which the product of all gamma matrices, $Q\equiv \gamma_1\cdots\gamma_{q+1}$, commutes with every individual matrix and has square $\mathbb{1}$; the two inequivalent representations are related by an overall sign change in the gamma matrices: $\gamma_\mu\to-\gamma_\mu$, so that Q changes from $+\mathbb{1}$ to $-\mathbb{1}$, or vice versa.

q	$q+1$	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathbb{R}(\mathcal{D}_{q+1})$	$\mathbf{C}$
-1	0	$\mathbb{R}$	1	$\mathbb{R}$	$\mathbb{R}$
0	1	$\mathbb{R}\oplus\mathbb{R}$	1	$\mathbb{R}$	$\mathbb{R}$
1	2	$\mathbb{R}(2)$	2	$\mathbb{R}(2)$	$\mathbb{R}$
2	3	$\mathbb{C}(2)$	4	$\mathcal{C}(3,1)$	$\mathbb{C}$
3	4	$\mathbb{H}(2)$	8	$\mathbb{H}\otimes\mathbb{H}(2)$	$\mathbb{H}$
4	5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	8	$\mathbb{H}\otimes\mathbb{H}(2)$	$\mathbb{H}$
5	6	$\mathbb{H}(4)$	16	$\mathbb{H}\otimes\mathbb{H}(4)$	$\mathbb{H}$
6	7	$\mathbb{C}(8)$	16	$\mathcal{C}(8,0)$	$\mathbb{C}$
7	8	$\mathbb{R}(16)$	16	$\mathbb{R}(16)$	$\mathbb{R}$
$n+7$	$n+8$	$\mathbb{R}(16)\otimes\mathcal{C}(n,0)$	$16\ \mathcal{D}_n$	$\mathbb{R}(16)\otimes\mathbb{R}(\mathcal{D}_n)$	as for n

Table 1: Real Clifford algebras $\mathcal{C}(q+1,0)$. Here $\mathbf{F}(n)$ stands for $n\times n$ matrices with entries over the field $\mathbf{F}$, while $\mathcal{D}_{q+1}$ denotes the real dimension of an irreducible representation of the Clifford algebra. We decompose the matrices $\mathbb{R}(\mathcal{D}_{q+1})$ acting on the real irreducible representation space, either as a direct product with the Clifford algebra *representation* as a factor, or in the form of a higher-dimensional Clifford algebra. This decomposition is used to determine the centralizer $\mathbf{C}$ of the Clifford algebra in this representation.

respectively. Note that the contracted tensor

$$\begin{aligned}\Gamma_{mnpp} &= \tfrac{1}{8}(2q-r)\,\delta_{mn} \qquad \text{for } q\neq 0 \\ &= -\tfrac{1}{8}P\,(\delta_{mn}-(\gamma_3)_{mn}) - \tfrac{1}{8}\dot{P}\,(\delta_{mn}+(\gamma_3)_{mn}) \qquad \text{for } q=0\end{aligned} \tag{3.10}$$

($q=0$ is special, because it represents the only case where a gamma matrix can have a non-zero trace) has only zero eigenvalues in those cases where we already know that $\Gamma_{mnpq}=0$. This implies that the equation $\Gamma_{mnpq}Z^q=0$ has only non-trivial solutions Z^q when Γ_{mnpq} vanishes.

3.2 A-coefficients and symmetry groups

Now that we have found the non-vanishing components of Γ_{abcd} we consider the solutions for the corresponding tensors $A_{ab;c}$ as defined by (2.1). They are determined modulo solutions of the homogeneous equation,

$$d_{d(ab}\,A_{c)d}=0, \tag{3.11}$$

which define the invariances of the coefficients d_{abc}. We recall that these symmetries must preserve the metric δ_{ab}, so that the matrices A_{ab} are anti-symmetric. There are only two types of invariances. First there is

$$A_{2m}=-A_{m2}=\sqrt{3}\,\zeta_m\ , \qquad A_{m\mu}=-A_{\mu m}=(\gamma_\mu)_{mn}\,\zeta_n\ , \tag{3.12}$$

where ζ_m must satisfy

$$\Gamma_{mnpq}\,\zeta_q = 0\ . \tag{3.13}$$

However, from the discussion at the end of the previous subsection it follows that this equation has only non-trivial solutions for ζ_m when Γ_{mnpq} vanishes.

The second type of solutions of (3.11) corresponds to the invariance group of the tensor $d_{\mu mn} \propto \gamma_{\mu mn}$ associated with the matrices $A_{\mu\nu}$ and A_{mn},

$$A_{\mu\nu}\ (\gamma_\nu)_{mn} + \gamma_{\mu p(m}\,A_{n)p} = 0. \tag{3.14}$$

For any $A_{\mu\nu}$ there is the solution

$$A_{mn} = \tfrac{1}{4} A_{\mu\nu}\,(\gamma_\mu\gamma_\nu)_{mn}. \tag{3.15}$$

Obviously, the group associated with $A_{\mu\nu}$ is the rotation group $SO(q+1)$, which acts on the spinor coordinates labeled by m according to its cover group. Besides there can be additional invariances that act exclusively in spinor space and commute with the gamma matrices and thus with the corresponding representation of the Clifford algebra. Hence we are interested in the metric-preserving elements of the centralizer of the Clifford algebra in the r-dimensional real representation (i.e., the antisymmetric matrices A_{mn} belonging to $\mathbb{R}(r)$ that commute with γ_μ). Let us first determine the centralizers for the irreducible representations.

According to Schur's lemma, matrices that commute with an *irreducible* representation of the Clifford algebra must form a division algebra. Table 1 lists the centralizers of the real irreducible representations, which are thus equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We briefly present the arguments leading to this result[6]. First consider $q+1$ even. The only commuting element in the Clifford algebra representation is $\mathbb{1}$, while the centralizer is just the factor in $\mathbb{R}(\mathcal{D}_{q+1})$ that multiplies the Clifford algebra representation (cf. table 1). In this way we find that the centralizer is $\mathbb{R}$ for $q+1 = 0$ or 2 mod 8, and $\mathbb{H}$ for $q+1 = 4$ or 6 mod 8. Now take $q+1$ odd. For $q = 4m$ the irreducible representation of the Clifford algebra corresponds to only one of the terms of the direct sum in $\mathcal{C}(4m+1)$. Just as above, the only commuting element in this representation is $\mathbb{1}$, and the centralizer is obtained as the factor that multiplies the Clifford algebra representation in $\mathbb{R}(\mathcal{D}_{q+1})$, i.e., $\mathbb{R}$ for $q+1 = 1$ mod 8 and $\mathbb{H}$ for $q+1 = 5$ mod 8. What remains is the case $q = 2+4m$. Then, as indicated in table 1, the representation space is isomorphic to a higher-dimensional Clifford algebra, which makes it easy to verify that only $\mathbb{1}$ and $Q \equiv \gamma_1 \cdots \gamma_{q+1}$ span the centralizer. (Note that for $q = 2+4m$, $Q^2 = -\mathbb{1}$, while for $q = 4m$, Q itself is represented by $\pm\mathbb{1}$). We thus conclude that the centralizer is equal to $\mathbb{C}$ for $q+1 = 3$ or 7 mod 8.

To analyze the reducible representations, we first rewrite $\mathbb{R}(r)$ as $\mathbb{R}(p) \otimes \mathbb{R}(\mathcal{D}_{q+1})$, where p is the number of irreducible representations, thus $p = P$

[6] Many results on real irreducible representations of the Clifford algebras and their centralizers have been explicitly worked out in [20]. Another useful reference is [21].

for $q \neq 4m$ and $p = P + \dot{P}$ for $q = 4m$. Consider first $q \neq 4m$ such that $\gamma_\mu = \mathbb{1}_P \otimes \gamma_\mu^{irr}$. This shows that the centralizer is the direct product of $\mathbb{R}(P)$ with the centralizer of γ_μ^{irr}. leading to $\mathbb{R}(P)$ for $q = 1, 7$ mod 8, to $\mathbb{C}(P)$ for $q = 2, 6$ mod 8, and to $\mathbb{H}(P)$ for $q = 3, 5$ mod 8. What remains are the cases $q = 0$ mod 4, when we have $\gamma_\mu = \eta \otimes \gamma_\mu^{irr}$, where $\eta = \mathrm{diag}(1, \ldots 1, -1, \ldots -1)$. Writing A_{mn} as $A = H \otimes S$, where $H \subset \mathbb{R}(p)$ and $S \subset \mathbb{R}(\mathcal{D}_{q+1})$, we have the condition

$$[A, \gamma_\mu] = H\eta \otimes S\gamma_\mu - \eta H \otimes \gamma_\mu S = 0. \tag{3.16}$$

In the sector proportional to $(\mathbb{1} \pm \eta)H(\mathbb{1} \mp \eta)$, it follows that S anticommutes with γ_μ^{irr}; $S\,S^T$ is then a symmetric matrix that commutes with γ_μ^{irr}, so that it must be proportional to $\mathbb{1}$. Therefore S is orthogonal and $S\,\gamma_\mu^{irr} S^{-1} = -\gamma_\mu^{irr}$. This leads to a contradiction, as it implies that γ_μ^{irr} and $-\gamma_\mu^{irr}$ are equivalent representations. Consequently the matrices H are restricted to the $\mathbb{R}(P) \oplus \mathbb{R}(\dot{P})$ matrices commuting with η. For these matrices the same considerations apply as for $q \neq 4m$. The result is then that the centralizer is the direct product of $\mathbb{R}(P) \oplus \mathbb{R}(\dot{P})$ with the centralizer of γ_μ^{irr}, which corresponds to $\mathbb{R}(P) \oplus \mathbb{R}(\dot{P})$ for $q = 0$ mod 8, and $\mathbb{H}(P) \oplus \mathbb{H}(\dot{P})$ for $q = 4$ mod 8.

Now we determine the antisymmetric matrices in these centralizers corresponding to the generators of the metric-preserving subgroups. In each case these centralizers can be written as the direct product of real matrices with a division algebra (in the real representation, so that the imaginary units become antisymmetric matrices). Therefore in the complex or the quaternionic representation the antisymmetry requirement takes the form of an antihermiticity requirement. The metric-preserving groups are therefore

$$\begin{array}{rcl} \text{for } q = 1, 7 \text{ mod } 8 & : & SO(P) \\ \text{for } q = 0 \text{ mod } 8 & : & SO(P) \otimes SO(\dot{P}) \\ \text{for } q = 2, 6 \text{ mod } 8 & : & U(P) \\ \text{for } q = 3, 5 \text{ mod } 8 & : & U(P, \mathbb{H}) \equiv USp(2P) \\ \text{for } q = 4 \text{ mod } 8 & : & USp(2P) \otimes USp(2\dot{P}) \end{array} \tag{3.17}$$

In conclusion, we summarize the symmetries of the tensors d_{abc}. First there are the symmetries (3.12) for the cases $L(-1, 0)$, $L(0, r)$, $L(1, 1)$, $L(2, 1)$, $L(4, 1)$ and $L(8, 1)$. Secondly there is the group $SO(q + 1)$ and the group mentioned in (3.17) represented by matrices S_{mn}. This gives

$$\begin{array}{rcl} A_{\mu\nu} & = & \text{arbitrary} \\ A_{mn} & = & \frac{1}{4}\left(\gamma_\mu\gamma_\nu\right)_{mn} A_{\mu\nu} + S_{mn} \\ A_{2m} & = & \sqrt{3}\,\zeta_m \\ A_{m\mu} & = & \left(\gamma_\mu\right)_{mn} \zeta_n . \end{array} \tag{3.18}$$

Now that we have determined the solutions of the homogeneous equation (3.11), we turn to the inhomogeneous equation (2.1). A particular solution is

$$A_{2m;n} = -A_{m2;n} = \tfrac{3}{4}\sqrt{2}\,\delta_{mn}\ , \qquad A_{m\mu;n} = -A_{\mu m;n} = \tfrac{1}{4}\sqrt{6}\,(\gamma_\mu)_{mn}\ . \quad (3.19)$$

When $\Gamma_{mnpq} = 0$ these solutions correspond to an invariance of the d_{abc} coefficients and are already contained in the previous transformations.

4 Implications for homogeneous special spaces

Now we return to special geometry and the cubic polynomial $C(h)$, defined in (1.1). Using the canonical parametrization, we first introduce an extra coordinate x_1, and add the corresponding terms $x_1^3 - \frac{1}{2}x_1\,x_a^2$ to the polynomial (2.4). Giving up the canonical parametrization, we no longer have to restrict ourselves to $O(n-1)$ redefinitions, and we can make arbitrary linear redefinitions of the $x_1, \ldots, x_n$. Using

$$\begin{aligned} &h^1 = 3^{-1/3}\,(x_1 + \sqrt{2}\,x_2)\ ,\\ &h^2 = 3^{-1/3}\,(x_1 - \tfrac{1}{2}\sqrt{2}\,x_2)\ ,\\ &h^\mu = 2^{-1/2}\cdot 3^{1/6}\,x_\mu\ , \qquad h^m = 2^{-1/2}\cdot 3^{1/6}\,x_m\ , \end{aligned} \quad (4.1)$$

the polynomial $C(h)$ acquires the generic form given in section 1 (cf. 1.13),

$$C(h) = 3\Big\{h^1\,(h^2)^2 - h^1\,(h^\mu)^2 - h^2\,(h^m)^2 + \gamma_{\mu mn}\,h^\mu\,h^m\,h^n\Big\}\ . \quad (4.2)$$

We stress that this parametrization no longer coincides with the canonical one. The possible realizations for the gamma matrices were discussed in the previous section. Note that we have

$$n = 3 + q + r\ , \qquad \text{with} \quad r = (P + \dot{P})\,\mathcal{D}_{q+1}\ , \quad (4.3)$$

where the integers P and $\dot{P}$ characterize the representations for the gamma matrices, as discussed in the previous section.

We now summarize the linear transformations of h^A that leave (4.2) invariant. They can either be determined directly from (4.2), or can be evaluated from $\delta x^A = B^A{}_B x^B$, using (1.6) with $A_{ab} = B^c{}_1\,A_{ab;c}$, where $A_{ab;c}$ is taken from (3.19), plus a homogeneous solution as in (3.18),

$$\begin{aligned} \delta h^1 &= 2\xi_2\,h^1 + 2\xi_m h^m\ ,\\ \delta h^2 &= -\xi_2\,h^2 - \zeta_m\,h^m + 2\xi_\mu\,h^\mu\ ,\\ \delta h^\mu &= -\xi_2\,h^\mu + 2\xi_\mu\,h^2 - \zeta_n\,\gamma_{\mu mn}\,h^m + A_{\mu\nu}\,h^\nu\ ,\\ \delta h^m &= \tfrac{1}{2}\xi_2\,h^m + \xi_m\,h^2 - \zeta_m\,h^1 + \xi_n\,\gamma_{\mu mn}\,h^\mu + \xi_\mu\,\gamma_{\mu mn}\,h^n + A_{mn}\,h^n\ . \end{aligned} \quad (4.4)$$

The symmetries corresponding to the parameters ζ_m only exist when the tensor Γ_{mnpq} vanishes. As before, A_{mn} and $A_{\mu\nu}$ are antisymmetric matrices that leave the gamma matrices invariant (cf. (3.14)). As explained in the previous section, $A_{\mu\nu}$ and A_{mn} generate the product of $SO(q+1)$ and the metric-preserving group in the centralizer of the corresponding Clifford algebra representation given in (3.17). The parameters are defined as follows

$$\begin{aligned} &B^1_2 = \sqrt{2}\,\xi_2 \ , \qquad B^1_m = \sqrt{\tfrac{2}{3}}\,(\xi_m - \zeta_m) \ , \qquad B^1_\mu = 2\sqrt{\tfrac{2}{3}}\,\xi_\mu \ , \\ &A_{2m} = -A_{m2} = \tfrac{1}{2}\sqrt{3}\,(\xi_m + \zeta_m) \ , \qquad A_{m\mu} = -A_{\mu m} = \tfrac{1}{2}\gamma_{\mu mn}\,(\xi_m + \zeta_m) \ . \end{aligned} \tag{4.5}$$

It is illuminating to decompose the generators with respect to the abelian generator e_2 associated with the parameter ξ_2. The algebra then decomposes according to

$$\mathcal{X} = \mathcal{X}_{-3/2} + \mathcal{X}_0 + \mathcal{X}_{3/2} \ , \tag{4.6}$$

where $\mathcal{X}_{-3/2}$ contains the generators associated with the parameters ζ_m (which is thus only present when $\Gamma_{mnpq} = 0$), $\mathcal{X}_0$ consists of the generators associated with ξ_2, ξ_μ, $A_{\mu\nu}$ and A_{mn}, and $\mathcal{X}_{3/2}$ contains the generators corresponding to the parameters ξ_m. Obviously $\mathcal{X}_{3/2}$ constitutes a solvable algebra of dimension r. Also $\mathcal{X}_0$ contains a solvable algebra (of dimension q+2). This follows directly from the observation that the subalgebra consisting of the generators associated with the parameters ξ^μ and $A_{\mu\nu}$ constitute $so(q+1,1)$, which, by its Iwasawa decomposition, contains a solvable subalgebra of dimension q+1 and rank 1 (for $q \geq 0$; for $q = -1$ the algebra is empty, so that the rank is 0).[7] Indeed, the subspace of the special real manifold corresponding to $h^m = 0$ and h^1 fixed and non-zero, corresponds precisely to the coset space $SO(q+1,1)/SO(q+1)$.

The complete solvable transitive group of motions thus consists of the transformations (4.4) corresponding to the parameters ξ_a, combined with (for $q \geq 0$)

$$A_{\mu\nu} = 4\delta_{3[\mu}\xi_{\nu]} \ ; \qquad A_{mn} = \Big(\gamma_{[3}\gamma_{\mu]}\Big)_{mn}\,\xi_\mu \ , \tag{4.7}$$

where 3 denotes some arbitrary direction in the space of vectors labeled by indices μ.

Let us now discuss the implications of our results for the homogeneous special real spaces with a transitive isometry group that constitutes an invariance of the polynomial $C(h)$, and thus of the corresponding $N=2$ supergravity theory in five space-time dimensions. These spaces are classified in terms of the polynomials $C(h)$, as given in (4.2). The rank of these spaces is equal to 1 or 2, because the Cartan subalgebra of the solvable algebra consists of the transformations ξ_2, and the Cartan subalgebra of the solvable algebra corresponding to $SO(q+1,1)/SO(q+1)$. The rank-1 spaces have $q = -1$ and the corresponding expression for $C(h)$ is

$$L(-1,r): \qquad C(h) = 3\,h^2\,(h^1\,h^2 - (h^m)^2) \ . \tag{4.8}$$

[7] The action of $SO(q+1,1)$ on the spinor coordinates follows from the explicit terms in (4.4) proportional to ξ_μ and the generators (3.15) contained in A_{mn} corresponding to the cover of $SO(q+1)$. The additional generators in A_{mn} corresponding to (3.17) are compact; they commute with $SO(q+1,1)$ and have no bearing on the solvable subalgebra of $\mathcal{X}_0$.

Their solvable algebra is that of

$$L(-1,r): \qquad \frac{SO(r+1,1)}{SO(r+1)}\,, \qquad (n=r+2) \tag{4.9}$$

and we therefore identify them with these spaces. They are thus symmetric and were exhibited in the context of $d=5$ supergravity in [22]. A simple counting argument shows, however, that not all the $\frac{1}{2}(r+1)(r+2)$ symmetries of this space correspond to invariances of the cubic polynomial $C(h)$, as there are only r invariances associated with $\mathcal{X}_{3/2}$ and $\frac{1}{2}r(r-1)+1$ with $\mathcal{X}_0$ (corresponding to $A_{mn} \sim SO(r)$ and ξ_2, respectively). Indeed, explicit calculations [10] show that the missing r isometries do *not* correspond to linear transformations of the coordinates h^A. The case $r=0$ is an exception in this respect, as all isometries of the real manifold coincide with the invariances of $C(h)$. The non-linear transformations of h are not full invariances of the full $d=5$ supergravity action (only of the scalar part (1.2)), and the lower dimensional actions do therefore not exhibit these invariances. This is the reason why the Kähler and quaternionic spaces resulting from the **c** map and the **c**∘**r** map applied to $L(-1,r)$ are in general *not* symmetric, with the exception of the spaces corresponding to $L(-1,0)$ [8]. Their quaternionic counterparts are missing in the classification of homogeneous spaces in [13] and the corresponding Kähler spaces are therefore also missing in [14].

The rank-2 spaces with $q=0$ are special, because $C(h)$ factorizes in certain cases (corresponding to the symmetric spaces where either P or $\dot{P}$ vanishes),

$$\begin{aligned} L(0,P,\dot{P}): \quad C(h) \;&=\; -3\Big\{h^1\,(h^2+h^3)\,(h^2-h^3) \\ &\qquad +(h^2-h^3)\,(h^x)^2+(h^2+h^3)\,(h^{\dot{x}})^2\Big\}\,, \end{aligned} \tag{4.10}$$

where we have decomposed the indices m into P indices x and $\dot{P}$ indices $\dot{x}$, as explained in the preceding sections, with $n=3+P+\dot{P}$. The quaternionic and Kähler spaces corresponding to $L(0,P,\dot{P})$ were called $W(P,\dot{P})$ and $K(P,\dot{P})$ in [13] and [14], respectively. We shall denote the real spaces by $Y(P,\dot{P})$.

The rank-2 spaces corresponding to $L(q,P)$ with $q>0$ have a rank-4 quaternionic extension and a rank-3 Kähler extension, which were denoted by $V(P,q)$ and $H(P,q)$ in [13] and [14], respectively. We shall denote the corresponding real spaces by $X(P,q)$.

According to the classification of [13] and [14], for $q=4m\geq 4$ one has precisely one quaternionic and one Kähler space of given (allowed) dimension. However, the existence of inequivalent real representations of the Clifford algebra for $q=4m$ implies the existence of inequivalent real, Kähler and quaternionic spaces corresponding to $L(4m,P,\dot{P})$. We already encountered an example of the same phenomenon for $q=0$.

[8] In [22] it was assumed that the space remains symmetric after reduction; therefore the corresponding Kähler spaces were incorrectly identified with the minimal couplings of $d=4,\ N=2$ supergravity.

real	Kähler	quaternionic	$n+1$	R
		SG	0	0
		$\frac{USp(2n+2,2)}{USp(2n+2)\otimes SU(2)}$	$n+1 \geq 0$	1
	SG	$\frac{U(1,2)}{U(1)\otimes U(2)}$	1	1
	$\frac{U(n,1)}{U(n)\otimes U(1)}$	$\frac{U(n+1,2)}{U(n+1)\otimes U(2)}$	$n+1 \geq 2$	2
SG	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(+2)}}{SU(2)\otimes SU(2)}$	2	2

Table 2: Normal quaternionic spaces with rank $R \leq 2$ and quaternionic dimension $n{+}1$ and the corresponding special real and Kähler spaces (whenever they exist).

$C(h)$	real	Kähler	quaternionic	R	
$L(-1,m-1)$	$\frac{SO(m,1)}{SO(m)}$	$\star$	$\star$	3	$m \geq 2$
$L(-1,0)$	$SO(1,1)$	$\left[\frac{SU(1,1)}{U(1)}\right]^2$	$\frac{SO(3,4)}{(SU(2))^3}$	3	
$L(0,P,\dot{P})$	$Y(P,\dot{P})$	$K(P,\dot{P})$	$W(P,\dot{P})$	4	$P,\dot{P} \geq 0$
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$	4	$P,q \geq 1$
$L(4m,P,\dot{P})$	$\star$	$\star$	$\star$	4	$m,P,\dot{P} \geq 1$

Table 3: Homogeneous special real spaces with corresponding Kähler and quaternionic spaces. Those that were discussed for the first time in this paper are indicated by a $\star$. R is the rank of the quaternionic space.

As follows from the above arguments quaternionic spaces originating from special real spaces via special Kähler spaces have rank 3 or 4. But as mentioned before, these do not constitute all possible homogeneous quaternionic spaces. In fact, we know rank-2 symmetric quaternionic spaces, which originate from special Kähler spaces, but not from real spaces, and rank-1 symmetric quaternionic spaces that also have no Kähler origin. We summarize these in table 2. The corresponding Kähler and real spaces have real and complex dimension $n-1$ and n, and their rank is equal to $R-2$ and $R-1$, respectively. Because of the low rank, only a real space with zero rank can occur (which necessarily has zero dimension). This corresponds precisely to the pure $N=2$ supergravity theory in five space-time dimensions. In the table, this case is represented by "SG". A similar situation occurs for $R=1$ and $R=0$, where the only possibility for a special Kähler and quaternionic space corresponds to pure supergravity in four and three dimensions, respectively. Hence none of the spaces discussed in the table are related to the spaces classified in this paper. Observe that all spaces in table 2 are symmetric. Together with the homogeneous spaces resulting from the analysis of this paper, which are summarized in table 3, they constitute all the homogeneous quaternionic and special Kähler spaces that are known. A proof that this list contains all the symmetric special Kähler spaces is given in [23]. The symmetric rank-4 quaternionic spaces and their related special real and Kähler spaces correspond to $L(0,P),\ L(1,1),\ L(2,1),\ L(4,1)$, and $L(8,1)$.

These tables show a remarkable pattern. We have the pure $N=2$ supergravity theory in 3 dimensions ("the empty quaternionic space") and the minimal couplings : the quaternionic projective spaces. Then the remaining rank-1 quaternionic symmetric space is the one originating from pure $d=4$ supergravity ("the empty special Kähler space"). The minimal couplings of vector multiplets in $d=4,\ N=2$ supergravity, the complex projective spaces, are the origin of an infinite series of rank-2 quaternionic spaces. The remaining rank-2 quaternionic space originates from pure $d=5$ supergravity ("empty real space"), while the real projective spaces are the origin of an infinite series of rank-3 homogeneous quaternionic spaces (as discussed before, the reduction does not preserve the property that the space be symmetric). Seeing the ensuing pattern, it looks as if the remaining rank-3 quaternionic space should arise from the reduction of pure $d=6$ supergravity. The rank-4 quaternionic spaces would then find their origin in matter coupled $d=6$ supergravity. This is then also the last step, because $d=6$ is the largest space-time dimension in which a supergravity theory can exist with 8 independent supersymmetries (corresponding to a $d=6$ spinor). These $d=6$ couplings would then be characterized by the possible real realizations of positive-definite Clifford algebras (while $L(-1,0)$ corresponds to the "empty Clifford algebra"). This is in accord with a conjecture in [24] (cf. eq.(5.6)) where $d=6,\ N=2$ tensor and vector multiplets are incorporated in the field strength

$$F^{\mu}_{abc} = 3\partial_{[a}A^{\mu}_{bc]} + (A^m)_{[a}\,(\gamma^{\mu})_{mn}\,(F^n)_{bc]}\,, \tag{4.11}$$

which leads to a coupling of $q+1$ tensor multiplets (with tensor field A^{μ}_{ab} and field strength F^{μ}_{abc}) to r vector multiplets (with vectors A^m_a and field strengths A^m_{ab}).

In this paper we presented a complete classification of the special real homogeneous spaces with a transitive group of motions that leaves the polynomial $C(h)$, and thus the corresponding $d=5$ supergravity theory, invariant. Therefore we also obtained the corresponding classification for the homogeneous special Kähler and quaternionic spaces that are in the image of the **c** map and the **c**∘**r** map. However, we expect that the tables 2 and 3 in fact comprise all possible homogeneous quaternionic spaces. This result should still follow from the analysis of [13], and we believe that the absence in [13] of the spaces indicated by a $\star$ in table 3 is merely due to a calculational error. The nice pattern described above lends support to our conjecture that the classification of homogeneous quaternionic spaces is now complete, as the new spaces exhibited above are precisely needed for completing the overall picture.

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