Geometric construction of the r-map: from affine special real to special Kähler manifolds

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October 28, 2008

Abstract

We give an intrinsic definition of (affine very) special real manifolds and realise any such manifold M as a domain in affine space equipped with a metric which is the Hessian of a cubic polynomial. We prove that the tangent bundle $N = TM$ carries a canonical structure of (affine) special Kähler manifold. This gives an intrinsic description of the r-map as the map $M \mapsto N = TM$. On the physics side, this map corresponds to the dimensional reduction of rigid vector multiplets from 5 to 4 space-time dimensions. We generalise this construction to the case when M is any Hessian manifold.

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Introduction

Projective (or local) very special real geometry is the scalar geometry of five-dimensional supergravity coupled to vector multiplets [GST, dWvP, ACDV]. We will usually omit the word "very". It is locally the geometry of a nondegenerate hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ defined by a homogeneous cubic polynomial h. In relation with string compactifications the polynomial h could be, for instance, the cubic form

$$h([\alpha]) = \int_X \alpha \wedge \alpha \wedge \alpha$$

on the second cohomology of a Calabi-Yau 3-fold X.

In this paper we are concerned with *affine (or rigid) very special real geometry*, which is the scalar geometry of five-dimensional rigid vector multiplets [CMMS1]. The Lagrangian of rigid vector multiplets is encoded in a (not necessarily homogeneous) cubic polynomial and the metric of the scalar manifold is the Hessian of this polynomial.

In the first part of the paper we shall provide an intrinsic definition of the notion of an affine special real manifold and study, in particular, the geometry of such manifolds:

Definition 1 *An affine special real manifold (M, g, ∇) is a pseudo-Riemannian manifold (M, g) endowed with a flat torsion-free connection ∇ such that the tensor field $S = \nabla g$ is totally symmetric and ∇-parallel.*

We relate the intrinsic definition to the description in the physical literature in terms of a cubic prepotential. In fact, we show that any simply connected affine special real manifold (M, g, ∇) of dimension n admits an affine immersion ψ onto a domain $V \subset \mathbb{R}^n$, such that g is the pull back of the Hessian of a cubic polynomial h, see Corollary 1. The pair (V, h) is unique up to affine tranformations of $\mathbb{R}^n$. We calculate the curvature tensor of a special real manifold (and, more generally, of a Hessian manifold, see Corollary 4) and find a simple expression in terms of the tensor S. As an application, we obtain that a special real manifold with a positive definite metric has nonnegative Ricci curvature, see Corollary 7.

Dimensional reduction of (local/rigid) supersymmetric theories from 5 to 4 space-time dimensions induces a correspondence between the respective scalar geometries, which is know as the (local/rigid) *r-map* [dWvP, CMMS1]. The relevant scalar geometry in 4 space-time dimensions is (projective/affine) *special Kähler geometry*, see [C] for a survey. The (local/rigid) r-map associates a (projective/affine) special Kähler manifold to any (projective/affine) special real manifold. The r-map is explicitely known in terms of the prepotentials, which locally define special real and special Kähler geometry. In the affine case, for instance, the r-map associates to the real cubic polynomial $h(x^1, \ldots, x^n)$ defining the special real manifold the holomorphic prepotential $F(z^1, \ldots, z^n) = \frac{1}{2i}h(z^1, \ldots, z^n)$ defining the corresponding special Kähler manifold in terms of special holomorphic coordinates $z^1, \ldots, z^n$ [CMMS1]. However, an intrinsic geometric construction of the affine and projective r-maps is missing. In the second part of the paper we shall give such a construction in the affine case[1].

We show that the tangent bundle $N = TM$ of any (affine) special real manifold (M, g, ∇) carries the structure of an (affine) special Kähler manifold (N, J, g^N, ∇^N).

More precisely, the special Kähler structure (J, g^N, ∇^N) on N is canonically associated to the geometric data (g, ∇) on M, see Theorem 2. Recall that a special Kähler manifold (N, J, g^N, ∇^N) is (pseudo-)Kähler manifold (N, J, g^N) endowed with a connection ∇^N which is *special* (i.e. $\nabla^N J$ is symmetric), torsion-free, symplectic (with respect to the Kähler form) and flat. The map

$$\begin{array}{rcl} \mathbf{r} : \{\text{special real manifolds}\} & \to & \{\text{special Kähler manifolds}\} \\ (M, g, \nabla) & \mapsto & (N, J, g^N, \nabla^N), \end{array} \tag{0.1}$$

which associates to the special real manifold (M, g, ∇) the special Kähler manifold $(N = TM, J, g^N, \nabla^N)$ is our geometric construction of the r-map. As an application, we prove that there is no compact simply connected special real manifold with a positive definite metric, see Theorem 5.

We show that our r-map extends naturally to a map

$$\begin{array}{rcl} \mathbf{r} : \{\text{Hessian manifolds}\} & \to & \left\{ \begin{array}{c} \text{Kähler manifolds with a torsion-free,} \\ \text{symplectic and special connection} \end{array} \right\} \\ (M, g, \nabla) & \mapsto & (N, J, g^N, \nabla^N), \end{array} \tag{0.2}$$

A *Hessian manifold* (M, g, ∇) is a pseudo-Riemannian manifold such that $S = \nabla g$ is totally symmetric (but not necessarily ∇-parallel). The flatness of the connection ∇^N is lost when the r-map is applied to Hessian manifolds which are not special real. In fact, ∇^N is flat if and only if (M, g, ∇) is special real. Moreover, the manifold (N, J, g^N, ∇^N) associated to a Hessian manifold (M, g, ∇) by the r-map is again Hessian if and only if (M, g, ∇) is special real, see Corollary 9.

Finally, we characterise the image of the maps (0.1) and (0.2) in Theorems 3 and 4. We calculate the curvature tensors of the Levi-Civita connection of the metric g_N and of ∇^N, which have a simple expression in terms of the symmetric tensor $S = \nabla g$ and ∇S, respectively, see Corollaries 8, 10 and 11.

In particular, it follows from those theorems that a special Kähler manifold (N, J, g^N, ∇^N) of real dimension $2n$ can be locally obtained from the r-map if and only if it admits locally n holomorphic[2] Killing vector fields which span a Lagrangian distribution and which are ∇^N-parallel along this distribution.

[1]The projective case is the subject of work in progress.

[2]Recall that a real vector field X on a complex manifold (N, J) is called *holomorphic* if $\mathcal{L}_X J = 0$.

We prove that under some assumptions a simply connected n-dimensional Hessian manifold admits a canonical realisation as an improper affine hypersphere in $\mathbb{R}^{n+1}$ equipped with the Blaschke metric and the induced connection.

1 Hessian geometry and affine special real geometry

Definition 2 *A* Hessian manifold *(cf. [S])* (M, g, ∇) *is a pseudo-Riemannian manifold* (M, g) *with a flat torsion-free connection* ∇ *such that* $S = \nabla g$ *is a symmetric three-form (cubic form). An* affine (very) special real manifold (M, g, ∇) *is a Hessian manifold* (M, g, ∇) *with* ∇*-parallel cubic form* S.

Example 1 *Let* h *be a smooth function in affine space* $V \cong \mathbb{R}^n$*. We will say that* h *is* nondegenerate *at a point* $x_0 \in V$ *if the Hessian* $\partial^2 h(x_0)$ *is nondegenerate, where* ∂ *is the flat connection in* V*. We denote by* $V(x_0) \subset V$ *the connected component of* x_0 *in* $\{\det \partial^2 h \neq 0\} \subset V$*. The domain* $V(x_0)$ *is equipped with the pseudo-Riemannian metric* $g = \partial^2 h$*. Then* $(V(x_0), g, \partial)$ *is a Hessian manifold. Indeed* $S = \partial g = \partial^3 h$ *is completely symmetric. It is an affine special real manifold if and only if the cubic form* S *is constant.*

Proposition 1 *Any Hessian manifold (respectively, affine special real manifold)* (M, g, ∇) *is locally isomorphic to a domain* $(V(x_0), g, \partial)$ *associated with a smooth function* h *(respectively, cubic polynomial* h*), as in Example 1. The polynomial is given by*

$$h = \frac{1}{6} \sum S_{ijk} x^i x^j x^k + \frac{1}{2} b_{ij} x^i x^j .$$

Here x^i *are* ∇*-affine coordinates* $g = \sum g_{ij} dx^i dx^j$, $g_{ij} = \sum S_{ijk} x^k + b_{ij}$ *and* $S = \sum S_{ijk} dx^i dx^j dx^k$*. For any Hessian manifold the* n *linearly independent gradient vector fields* $\mathrm{grad}(x^i)$ *commute and the coordinate vector fields* $\frac{\partial}{\partial x^i}$ *are also commuting gradient vector fields.*

Proof: Since $S = \nabla g$ is totally symmetric, there exists locally a smooth function h such that $g = \nabla^2 h$. Moreover, if $\nabla S = 0$, then the function h is a cubic polynomial in affine local coordinates x^i. Then $\partial^2 h = \nabla^2 h = g = \sum g_{ij} dx^i dx^j$ and $\partial^3 h = \nabla^3 h = S = \sum S_{ijk} dx^i dx^j dx^k$, where $g_{ij} = \sum a_{ijk} x^k + b_{ij}$ and $S_{ijk} = a_{ijk}$. This shows that h coincides with the above expression up to terms of degree less or equal to 1 in the coordinates x^i, which do not contribute to g and S. Now we check that the vector fields $\mathrm{grad}(x^i) = \sum g^{ij} \partial_j$ commute:

$$\begin{aligned} [g^{ij}\partial_j, g^{kl}\partial_l] &= g^{ij}\partial_j g^{kl}\partial_l - g^{kl}\partial_l g^{ij}\partial_j = -g^{ij}g^{kk'}S_{k'l'j}g^{l'l}\partial_l + g^{kl}g^{ii'}S_{i'j'l}g^{j'j}\partial_j \\ &= -S^{kli}\partial_l + S^{ilk}\partial_l = 0, \end{aligned} \tag{1.1}$$

by the complete symmetry of S. (Here and in later calculations we use the Einstein summation convention.) The coordinate vector field $\frac{\partial}{\partial x^i}$ is the gradient of the function $\frac{\partial h}{\partial x^i}$. □

Corollary 1 *Let (M, g, ∇) be a simply connected Hessian manifold of dimension n. Then there exists an affine immersion $\psi : (M, \nabla) \to (\mathbb{R}^n, \partial)$ onto some domain $\psi(M) \subset \mathbb{R}^n$, unique up to affine transformations of $\mathbb{R}^n$. The gradients* $\mathrm{grad}(x^i)$ *of the coordinate functions $x^i = \psi^i$ span a canonical n-dimensional commutative Lie algebra of vector fields. Conversely, a pseudo-Riemannian manifold (M, g) with n pointwise linearly independent gradient vector fields* $\mathrm{grad}(x^i)$ *is canonically extended to a Hessian manifold (M, g, ∇).*

If (M, g, ∇) is an affine special real manifold, then, moreover, there exists a unique cubic polynomial h on $\mathbb{R}^n$ without linear and constant terms such that $g = \psi^*\partial^2 h$.

Proof: Since M is simply connected, there exists a ∇-parallel coframe $(\xi^1, \cdots, \xi^n)$. The one-forms ξ^i are closed and, hence, exact, i.e. $\xi^i = dx^i$ for globally defined functions x^i. These functions define an affine immersion $\psi = (x^1, \cdots, x^n)$ since $\nabla dx^i = 0$. Given a pseudo-Riemannian manifold (M, g) with n pointwise linearly independent gradient vector fields $\mathrm{grad}(x^i)$, there is a unique flat torsion-free connection ∇ such that $\nabla dx^i = 0$. Then ∇g is completely symmetric by (1.1).

On any domain $U \subset M$ such that $\psi|_U : U \to \psi(U)$ is a diffeomorphism there exists a smooth function h_U such that $g_{ij}|_U = \partial_i \partial_j h_U$. The function h_U is unique up to the addition of an affine function in the local coordinates x^i. In the special real case h_U is a cubic polynomial, which can be canonically chosen by the requirement that the linear and constant terms vanish: $h_U = \frac{1}{6} a_{ijk} x^i x^j x^k + \frac{1}{2} b_{ij} x^i x^j$. The coefficients a_{ijk}, b_{ij} are independent of U, since M is connected and $h_U = h_V$ on overlaps $U \cap V \neq \emptyset$. Therefore $h = \frac{1}{6} a_{ijk} x^i x^j x^k + \frac{1}{2} b_{ij} x^i x^j$ is canonically associated to the immersion ψ and satisfies $g = \psi^* \partial^2 h$. □

Remark: Shima and Yagi proved that the domain $\psi(M)$ is convex if g is positive definite, see [SY].

Corollary 2 *A pseudo-Riemannian manifold (M, g) admits the structure of an affine special real manifold if it admits an atlas with affine transition functions such that the metric coefficients $g_{ij} = a_{ijk} x^k + b_{ij}$ are affine functions and the coefficients a_{ijk} are symmetric. Then $g_{ij} = \partial_i \partial_j h$ where $h = \frac{1}{6} a_{ijk} x^i x^j x^k + \frac{1}{2} b_{ij} x^i x^j$*

Corollary 3 *Let (M, g, ∇) be a simply connected Hessian manifold with transitive action of $G = \mathrm{Aut}(M, g, \nabla)$. Then the affine immersion $\psi : M \to \psi(M) \subset \mathbb{R}^n$ of Corollary 1 is a G-equivariant covering $M = G/H \to \psi(M) = Gx_0 = G/G_{x_0}$ over the (open) orbit Gx_0 of a point $x_0 \in \psi(M)$ with respect to an affine action α of G on $\mathbb{R}^n$. In particular, there is no nonconstant G-invariant function on the domain $\psi(M)$ and at most one (up to scaling) relative invariant with character $\chi(a) = (\det A)^{-2}$, where $\alpha(a)x = Ax + b$, $a \in G$. If such a relative invariant $:\psi(M) \to \mathbb{R}$ exists, it is given (up to a constant factor) by the formula $\delta \circ \psi = \det(g_{ij})$ (which in general defines only a local relative invariant). For an affine special real manifold, the (globally defined) relative invariant δ is a polynomial of degree n.*

Recall that given a pseudo-Riemannian metric g and a connection ∇ on manifold M, the conjugate connection $\bar{\nabla}$ is defined by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \bar{\nabla}_X Z),$$

where X, Y, Z are vector fields on M. Notice that $\bar{\nabla}_X = D_X + \hat{S}_X^*$, where D is the Levi-Civita connection, $\hat{S} = D - \nabla$ and $\hat{S}_X^*$ is the metric adjoint of the endomorphism $\hat{S}_X$.

Proposition 2 *Let (M, g, ∇) be a Hessian manifold with cubic form $S = \nabla g$. Then $\hat{S}_X = \frac{1}{2}g^{-1} \circ S_X = \hat{S}_X^*$. The conjugate connection is flat and torsion-free and we have the following formulas:*

$$\begin{aligned}\bar{\nabla}_X &= D_X + \hat{S}_X \\ \nabla_X &= D_X - \hat{S}_X.\end{aligned}$$

Proof: The connection $\nabla + \frac{1}{2}g^{-1} \circ S$ is torsion-free, by the symmetry of S. We check that it preserves the metric g:

$$\nabla_X g + \frac{1}{2}(g^{-1} \circ S_X) \cdot g = S_X - \frac{1}{2}g(g^{-1} \circ S_X \cdot, \cdot) - \frac{1}{2}g(\cdot, g^{-1} \circ S_X \cdot) = S_X - S_X = 0.$$

This shows that $\nabla + \frac{1}{2}g^{-1} \circ S$ is the Levi-Civita connection $D = \nabla + \hat{S}$. Hence $\frac{1}{2}g^{-1} \circ S_X = \hat{S}_X = \hat{S}_X^*$. It is clear that the conjugate connection $\bar{\nabla}_X = D_X + \hat{S}_X$ has zero torsion. It remains to calculate its curvature $\bar{R}$. For this we write $\bar{\nabla} = \nabla + 2\hat{S}$ and compute:

$$\begin{aligned}\bar{R}(X, Y) &= R^\nabla(X, Y) + 2d^\nabla \hat{S}(X, Y) + 4[\hat{S}_X, \hat{S}_Y] \\ &= 2(d^\nabla \hat{S}(X, Y) + 2[\hat{S}_X, \hat{S}_Y]) \\ \nabla_X \hat{S} &= \frac{1}{2}\nabla_X(g^{-1} \circ S) = -\frac{1}{2}g^{-1} \circ S_X g^{-1} \circ S + \frac{1}{2}g^{-1} \circ \nabla_X S \\ &= -2\hat{S}_X \hat{S} + \frac{1}{2}g^{-1} \circ \nabla_X S. \end{aligned} \tag{1.2}$$

Therefore

$$d^\nabla \hat{S}(X, Y) = (\nabla_X \hat{S})Y - (\nabla_Y \hat{S})X = -2[\hat{S}_X \hat{S}_Y],$$

since ∇S is symmetric. Thus $\bar{R} = 0$. □

Corollary 4 *Under the assumptions of the proposition, the following formulas are satisfied:*

$$R^D(X, Y) = -[\hat{S}_X, \hat{S}_Y], \quad d^D \hat{S} = 0, \quad d^\nabla \hat{S} + 2[\hat{S}, \hat{S}] = 0.$$

Proof: The first two equations are obtained by taking the sum and difference of the equations

$$\begin{aligned}0 &= \bar{R} = R^D + d^D \hat{S} + [\hat{S}, \hat{S}] \\ 0 &= R^\nabla = R^D - d^D \hat{S} + [\hat{S}, \hat{S}].\end{aligned}$$

The third equation follows from $\bar{\nabla} = \nabla + 2\hat{S}$:

$$0 = \bar{R} = R^\nabla + 2d^\nabla \hat{S} + 4[\hat{S}, \hat{S}] = 2(d^\nabla \hat{S} + 2[\hat{S}, \hat{S}]).$$

□

The following results are analogues of the corresponding results in special Kähler geometry, see [BC1].

Theorem 1 *Let (M, g, ∇) be a simply connected Hessian manifold such that ∇ preserves the metric volume form. Then there exists a realisation of (M, g, ∇) as an improper affine hypersphere $\varphi : M \to \mathbb{R}^{n+1}$, unique up to unimodular affine transformations. Moreover, any automorphism of (M, g, ∇) has a unique extension to a unimodular affine transformation of $\mathbb{R}^{n+1}$ preserving $\varphi(M)$.*

Proof: By the fundamental theorem of affine geometry a simply connected pseudo-Riemannian manifold (M, g, ∇) with a torsion-free connection ∇ admits a Blaschke immersion $\varphi : M \to \mathbb{R}^{n+1}$ as a hypersurface with Blaschke metric g and induced connection ∇ if and only if the conjugate connection $\bar{\nabla}$ is torsion-free and projectively flat, and if the metric volume form is ∇-parallel, see [DNV]. Moreover, the Blaschke immersion is unique up to unimodular affine transformations and is an improper affine hypersphere if and only if the connection ∇ is flat. The assumptions of the fundamental theorem are satisfied in virtue of Proposition 2 and ∇ is flat by Definition 2. If $\psi : M \to M$ is an automorphism, then, due to the unicity statement in the fundamental theorem, there exists a unimodular affine transformation $\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\alpha \circ \varphi = \varphi \circ \psi$. It is unique since any affine transformation which fixes an affine frame is the identity and an affine frame in $\mathbb{R}^{n+1}$ is determined by a frame in $T_{\varphi(p)}M$ and the affine normal which is invariant under unimodular affine tranformations. □

Corollary 5 *If $G = Aut(M, g, \nabla)$ acts transitively on a simply connected Hessian manifold (M, g, ∇) and ∇ preserves the metric volume form, then the Blaschke immersion $\varphi : (M, g, \nabla) \to \mathbb{R}^{n+1}$ is a covering $M = G/H \to \varphi(M) = Gx_0 = G/G_{x_0}$ over the orbit Gx_0 of a point $x_0 \in \varphi(M)$ with respect to an affine action of G on $\mathbb{R}^{n+1}$. Moreover, $\varphi(M) \subset \mathbb{R}^{n+1}$ is an improper affine hypersphere.*

Corollary 6 *Let (M, g, ∇) be a Hessian manifold with complete and positive definite metric g and such that ∇ preserves the metric volume form. Then g is flat and $D = \nabla$. In particular, any homogeneous Hessian manifold with positive definite metric and volume preserving connection ∇ is finitely covered by the product of a flat torus and a Euclidian space.*

Proof: By the previous theorem, the universal covering of (M, g, ∇) can be realised as an improper affine hypersphere with complete and positive definite Blaschke metric. By the Calabi-Pogerelov theorem, see [NS] and references therein, such a hypersurface is a paraboloid and the Blaschke metric is flat. Now the existence of the finite covering follows from Bieberbach's theorem. □

2 Geometric structures on the tangent bundle

Now we show that the tangent bundle $\pi : N = TM \to M$ of a Hessian (pseudo-) Riemannian manifold (M, g, ∇) has a natural (pseudo-) Kähler structure and the tangent bundle of an affine special real manifold has a natural special (pseudo-) Kähler structure. We recall that an (affine) special (pseudo-) Kähler structure (g, J, ∇) on a manifold N

is given by a (pseudo-) Kähler structure (g, J) and a flat torsion-free symplectic ($\nabla\omega = \nabla g \circ J = 0$) connection such that ∇J is a symmetric (1,2)-tensor, see [C] and references therein.

Let $TN = T^hN \oplus T^vN$ be the decomposition of the tangent bundle of $N = TM$ into vertical and horizontal subbundles with respect to the flat connection ∇ on the Hessian manifold (M, g, ∇). We have a canonical isomorphism

$$T_\xi N = T^h_\xi \oplus T^v_\xi \cong T_{\pi(\xi)}M \oplus T_{\pi(\xi)}M.$$

Local affine coordinates x^i on M induce canonical coordinates (x^i, u^i) on N such that any vector $\xi \in TM$ is written as $\xi = u^i \frac{\partial}{\partial x^i}$. Then $\partial_i := \frac{\partial}{\partial x^i}$, $\partial_{i'} := \frac{\partial}{\partial u^i}$ forms a local frame of T^h and T^v, respectively. For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M we denote by $X^h = X^i \frac{\partial}{\partial x^i}$, $X^v = X^i \frac{\partial}{\partial u^i}$ the horizontal and vertical lifts of X, respectively. Then we have the formulas:

$$[X^h, Y^h] = [X, Y]^h, \quad [X^v, Y^v] = 0, \quad [X^h, Y^v] = (\nabla_X Y)^v.$$

The canonical isomorphism $\mathbb{1} = \frac{\partial}{\partial u^i} \otimes dx^i : T^h_\xi \cong T_{\pi(\xi)}M \cong T^v_\xi$ defines the complex structure

$$J := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

which is integrable since it has constant coefficients in the coordinates (x^i, u^i). We define a natural extension of the metric by

$$g^N := \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Proposition 3 *(cf. [S0]) For any Hessian (pseudo-) Riemannian manifold (M, g, ∇) the pair (g^N, J) is a (pseudo-) Kähler structure on $N = TM$.*

Proof: It is sufficient to check that the 2-form

$$\omega = g^N \circ J = \begin{pmatrix} 0 & -g \\ g & 0 \end{pmatrix} = -g_{ik}(x) dx^i \wedge du^k$$

is closed. Indeed,

$$d\omega = \frac{\partial g_{ik}}{\partial x^j} dx^i \wedge dx^j \wedge du^k = 0$$

due to the total symmetry of $\nabla g = \partial g$. □

Proposition 4 *With respect to the coordinates $x^I = (x^i, u^{i'})$, the Christoffel symbols $\Gamma_I = (\Gamma^K_{IJ})$ of the metric g_N are given by*

$$\Gamma_i = \begin{pmatrix} \hat{S}_i & 0 \\ 0 & \hat{S}_i \end{pmatrix}, \quad \Gamma_{i'} = \begin{pmatrix} 0 & -\hat{S}_i \\ \hat{S}_i & 0 \end{pmatrix} = J\Gamma_i.$$

If (M, g, ∇) is special real, that is $\nabla S = 0$, then the curvature R^N of the Levi Civita connection D^N is given by

$$\begin{aligned} R^N(X^h, Y^h) &= R^N(X^v, Y^v) \\ &= \begin{pmatrix} R^D(X,Y) & 0 \\ 0 & R^D(X,Y) \end{pmatrix} = -\begin{pmatrix} [\hat{S}_X, \hat{S}_Y] & 0 \\ 0 & [\hat{S}_X, \hat{S}_Y] \end{pmatrix} \qquad (2.1) \\ R^N(X^h, Y^v) &= \begin{pmatrix} 0 & \{\hat{S}_X, \hat{S}_Y\} \\ -\{\hat{S}_X, \hat{S}_Y\} & 0 \end{pmatrix}. \qquad (2.2) \end{aligned}$$

Corollary 7 *Let (M, g, ∇) be an affine special real manifold. Then the Ricci curvature of the (pseudo-) Kähler manifold (N, g_N, J) is given by*

$$ric(X^h + Y^v, X^h + Y^v) = 2tr\hat{S}_X^2 + 2tr\hat{S}_Y^2.$$

If the metric g is Riemannian then the Riemannian metric g_N has nonnegative Ricci curvature. The Ricci curvature is strictly positive if and only if the map $X \mapsto S(X, \cdot, \cdot)$ has trivial kernel.

Proof: Since the Ricci form of any Kähler manifold is given by $\rho(X,Y) = ric(JX, Y) = \frac{1}{2}trJR(X,Y)$ the formulas for the Ricci curvature follow from the previous proposition. Since $\hat{S}_X$ is symmetric with respect to g, $tr\hat{S}_X^2 \geq 0$ if g is definite. □

Now we extend the (1,2)-tensor field $\hat{S}$ on M considered as tensor on $T_\xi^h N \cong T_{\pi(\xi)}M$ to a (1,2)-tensor $\hat{S}^N$ on N such that

$$\hat{S}_J^N = \hat{S}^N J = -J\hat{S}^N.$$

Then it is given by

$$\hat{S}_{X^h}^N = \begin{pmatrix} \hat{S}_X & 0 \\ 0 & -\hat{S}_X \end{pmatrix}, \quad \hat{S}_{X^v}^N = \begin{pmatrix} 0 & -\hat{S}_X \\ -\hat{S}_X & 0 \end{pmatrix}. \qquad (2.3)$$

We define a connection ∇^N on N by

$$\nabla^N = D^N - \hat{S}^N.$$

Lemma 1 *Let (N, g^N, J) be the Kähler manifold associated to a Hessian manifold (M, g, ∇). Then the above connection ∇^N has the following Christoffel symbols with respect to the coordinates $x^I = (x^i, u^{i'})$:*

$$\Gamma_i = \begin{pmatrix} 0 & 0 \\ 0 & 2\hat{S}_i \end{pmatrix}, \quad \Gamma_{i'} = \begin{pmatrix} 0 & 0 \\ 2\hat{S}_i & 0 \end{pmatrix}.$$

Theorem 2 *Let $(M, g, \nabla = D - \hat{S})$ be an affine special real manifold. Then $(N, g^N, J, \nabla^N = D^N - \hat{S}^N)$ is an affine special (pseudo-) Kähler manifold.*

Proof: Due to Proposition 3 it suffices to show that the connection ∇^N is a) torsion-free, b) symplectic, c) special (i.e. $\nabla^N J$ is symmetric) and d) flat. The properties a), b) and c) are valid for any Hessian manifold (M, g, ∇). Indeed the Levi Civita connection D^N is torsion-free and preserves ω and J. Therefore:
a) follows from the symmetry $\hat{S}^N_X Y = \hat{S}^N_Y X$ for $X, Y \in TN$.
b) follows from the symmetry $\omega(\hat{S}^N_X Y, Z) = \omega(\hat{S}^N_X Z, Y)$ for all $X, Y, Z \in TN$.
c) follows from the symmetry of $\hat{S}^N_X Y$, since $\nabla^N J = [\hat{S}^N_X, J] = -2J\hat{S}^N_X$.
Now we check that the curvature

$$R_{IJ} = \partial_I \Gamma_J - \partial_J \Gamma_I + [\Gamma_I, \Gamma_J]$$

of ∇^N vanishes. Using the formula (1.2) for $\nabla S = 0$ and the previous lemma we get:

$$\partial_i \Gamma_j = \begin{pmatrix} 0 & 0 \\ 0 & -4\hat{S}_i \hat{S}_j \end{pmatrix}, \quad \partial_i \Gamma_{j'} = \begin{pmatrix} 0 & 0 \\ -4\hat{S}_i \hat{S}_j & 0 \end{pmatrix}, \quad \partial_{i'} \Gamma_J = \frac{\partial}{\partial u^i} \Gamma_J = 0.$$

Now it is easy to check that $R_{IJ} = 0$. □

The same calculation shows:

Corollary 8 *The curvature of the connection ∇^N of the Kähler manifold (N, g^N, J) associated to a Hessian manifold (M, g, ∇) is given by:*

$$R(X^h, Y^h) = \begin{pmatrix} 0 & 0 \\ 0 & P_{X,Y} - P_{Y,X} \end{pmatrix}, \quad R(X^v, Y^v) = 0, \quad R(X^h, Y^v) = \begin{pmatrix} 0 & 0 \\ P_{X,Y} & 0 \end{pmatrix},$$

where $P_{X,Y} Z = g^{-1}(\nabla_X S)(Y, Z, \cdot)$. The Ricci curvature of ∇^N is given by:

$$ric(X^h, Y^h) = -tr P_{X,Y}, \quad ric(X^v, Y^v) = ric(X^h, Y^v) = 0.$$

Corollary 9 *Let (g^N, J, ∇^N) be the geometric structures on $N = TM$ associated to a Hessian manifold (M, g, ∇). Then the following are equivalent:*

(i) *∇^N is flat.*

(ii) *(N, g^N, ∇^N) is Hessian.*

(iii) *(M, g, ∇) is special real.*

Proof: The equivalence of (i) and (iii) follows from the previous corollary and it is clear that (ii) implies (i). It remains to check that (i) implies (ii). The complete symmetry of the tensor field $\nabla^N g^N$ follows from the symmetry $\hat{S}^N_X Y = \hat{S}^N_Y X$, since

$$\begin{aligned} (\nabla^N_X g^N)(Y, Z) &= -(\hat{S}^N_X \cdot g^N)(Y, Z) = g^N(\hat{S}^N_X Y, Z) + g^N(Y, \hat{S}^N_X Z) \\ &= g^N(\hat{S}^N_X Y, Z) - \o^N(J^N Y, \hat{S}^N_X Z) \\ &= g^N(\hat{S}^N_X Y, Z) - \o^N(J^N \hat{S}^N_X Y, \hat{S}^N_X Z) = 2g^N(\hat{S}^N_X Y, Z). \end{aligned}$$

□

Corollary 10 *Under the assumptions of Theorem 2 the exterior covariant derivatives of the endomorphism valued one-form $\hat{S}^N$ are given by:*

$$
\begin{aligned}
d^{\nabla^N}\hat{S}^N(X^h, Y^h) &= d^{\nabla^N}\hat{S}^N(X^v, Y^v) = -2\begin{pmatrix} [\hat{S}_X, \hat{S}_Y] & 0 \\ 0 & [\hat{S}_X, \hat{S}_Y] \end{pmatrix} \\
d^{\nabla^N}\hat{S}^N(X^h, Y^v) &= 2\begin{pmatrix} 0 & \{\hat{S}_X, \hat{S}_Y\} \\ -\{\hat{S}_X, \hat{S}_Y\} & 0 \end{pmatrix} \\
d^{D^N}\hat{S}^N = 0 &
\end{aligned} \tag{2.4}
$$

Proof: The proof follows from (2.1), $R^{\nabla^N} = 0$ and the formulas

$$
\begin{aligned}
d^{\nabla^N}\hat{S}^N &= R^N - R^{\nabla^N} - [\hat{S}^N, \hat{S}^N] \\
d^{D^N}\hat{S}^N &= R^N - R^{\nabla^N} + [\hat{S}^N, \hat{S}^N].
\end{aligned}
$$

□

Corollary 11 *Under the assumptions of Theorem 2 the curvature of the Levi-Civita connection of g_N is given by*

$$R^N = -[\hat{S}^N, \hat{S}^N].$$

Definition 3 *The map which to any affine special real manifold (M, g, ∇) associates the affine special Kähler manifold $\mathbf{r}(M) := (N, g^N, J, \nabla^N)$ is called the* (affine) r-map.

Corollary 12 *Let (M, g, ∇) be a special real manifold which locally admits a homogeneous Hesse potential $h = \frac{1}{6}S_{ijk}x^i x^j x^k$. Then the Kähler manifold $\mathbf{r}(M)$ is not flat.*

Proof: By (2.1) $R^N = 0$ is equivalent to $\hat{S}_X\hat{S}_Y = 0$ for all $X, Y \in TM$. This is impossible since, by Proposition 1, locally we can identify the Hessian manifold with a domain $(V(x_0), g, \partial)$ and $2g \circ \hat{S}_X = S(X, \cdot, \cdot) = \partial^2 h(X) = g_X$ is invertible for all $X \in V(x_0)$. □

Flat special Kähler manifolds were classified in [BC2]. By the corollary, they cannot be obtained from a homogeneous cubic polynomial by the r-map.

Theorem 3 *Let (M, g, ∇) be a Hessian manifold of dimension n and $(N = TM, g^N, J, \nabla^N)$ the corresponding (pseudo-) Kähler manifold with the connection defined above. Then*

(i) The decomposition $TN = T^v + T^h = T^v + JT^v$ is a decomposition into two orthogonal integrable Lagrangian distributions which are totally geodesic and flat with respect to ∇^N. The horizontal distribution is also totally geodesic with respect to D^N.

(ii) For any leaf $L = M(\xi)$, $\xi \in N$, of the horizontal distribution there exists an involution $\sigma = \sigma_L \in \mathrm{Aut}(N, g^N, J, \nabla^N)$ which preserves the vertical and horizontal foliations and such that $L = N^\sigma$.

(iii) *The group generated by products $\sigma_L \circ \sigma_{L'}$ preserves each fiber T_pM, $p \in M$, and acts as the translation group of the fiber T_pM.*

Proof: (i) follows from the formulas for the Christoffel symbols and curvature of D^N and ∇^N and from the formulas for $g^N = g_{ij}(x)(dx^i dx^j + du^i du^j)$, J and $\omega = g^N \circ J$. In each coordinate domain one can check directly that the reflection $\sigma_U : (x, u) \mapsto (x, -u + 2u_0)$ with respect to an open domain $U = \{u = u_0\} \subset L$ in a leaf L has the properties claimed in (ii) . The reflections σ_U coincide on overlaps and, hence, define the global reflection σ_L. (iii) follows from the fact that the product of two central symmetries in the affine space T_pM is a parallel translation. $\square$

The converse can be stated as follows.

Theorem 4 *Let (N, J, g^N) be a $2n$-dimensional pseudo-Kähler manifold which admits a free holomorphic and isometric action of the vector group $\mathbb{R}^n$ with Lagrangian orbits such that the projection $\pi : N \to N/\mathbb{R}^n = M$ is a trivial (principal) bundle. Then there exists an induced pseudo-Riemannian metric g and flat connection ∇ on M such that (M, g, ∇) is Hessian and N is identified with TM with the pseudo-Kähler structure (J, g^N) induced from (M, g, ∇) by Proposition 3. If moreover (N, J, g^N, ∇^N) is special Kähler and the Killing vector fields U_i of the above action are ∇^N-parallel along the Lagrangian orbits, then $(M = N/\mathbb{R}^n, g, \nabla)$ is special real and (N, J, g^N, ∇^N) is obtained from $(M = N/\mathbb{R}^n, g, \nabla)$ by the r-map.*

Proof: We denote by U_i the commuting vector fields on N which are the generators of the action of $\mathbb{R}^n$. The holomorphicity of the U_i implies that the vector fields $U_i, X_j = -JU_j$ commute. They are linearly independent since the distribution $T^vN := \mathrm{span}\{U_i | i = 1, \cdots, n\}$ is Lagrangian. There exist local coordinates (x^i, u^j) such that $U_i = \frac{\partial}{\partial u^i}$ and $X_j = \frac{\partial}{\partial x^j}$. In these coordinates

$$g^N = g_{ij}(dx^i dx^j + du^i du^j),$$

where the functions $g_{ij} = g_{ij}(x)$ depend only on the x^i, since the U_i are Killing vector fields. Since U_i is a holomorphic Killing vector field, it is also symplectic with respect to the Kähler form. This implies that $X_i = -JU_i$ is a gradient vector field, hence the one-form $g(X_i, \cdot)$ is closed and $\frac{\partial}{\partial x^i} g_{jk}$ is completely symmetric. This shows that (M, g, ∇) is a Hessian manifold, where $g = (g_{ij}(x))$ is the metric on M which makes $N \to M$ a pseudo-Riemannian submersion and ∇ is the flat connection induced by the flat connection on the leaves of the distribution $T^h := JT^v$ with parallel vector fields $\frac{\partial}{\partial x^i}$. Identifying M with a section of the trivial bundle $N \to M$ we can identify N with TM. It is clear that the pseudo-Kähler structure on $N = TM$ is obtained by Proposition 3 from the Hessian manifold (M, g, ∇).

In the special Kähler case the assumption $\nabla^N_{U_i} U_j = 0$ implies

$$\nabla^N_{JU_i}(JU_j) = (\nabla^N_{JU_i} J)U_j + J\nabla^N_{JU_i} U_j = -J(\nabla^N_{U_j} J)U_i + J(\nabla^N_{U_j} J)U_i = 0.$$

Using the fact that for a special Kähler manifold $\hat{S}^N := D^N - \nabla^N = -\frac{1}{2} J\nabla^N J$, one can easily check that ∇^N is the connection from Lemma 1. Now Corollary 8 shows that $\nabla S = 0$, i.e. (M, g, ∇) is special real. $\square$ As an application we prove the following non-existence result.

Theorem 5 *There is no compact simply connected special real manifold of positive dimension.*

Proof: Let (M, g, ∇) be a compact simply connected special real manifold. We first prove that the metric of the special Kähler manifold $(N = TM, g^N, J, \nabla^N)$ is complete. As in the proof of Corollary 1, we have a ∇ parallel coframe $\xi^i = dx^i$, $i = 1, \ldots, n$. We denote by ∂_i the dual global frame on M and by u^i the corresponding globally defined functions on $N = TM$, which are the coordinates of a vector with respect to the frame ∂_i. There exists a free isometric action of $\mathbb{R}^n$ on $N = TM$ given by $u^i \mapsto u^i + c^i$, whose orbits are the tangent spaces. The quotient of (N, g^N) by the lattice of integral translations is a compact, hence, complete Riemannian manifold. Thus (N, g^N) is complete as the universal covering of a complete Riemannian manifold.

By [BC1], (N, g^N, ∇^N) can be realised as an improper affine hypersphere with complete Riemannian metric. In fact, one can check that the integrability conditions[3] for the existence of an affine hypersphere immersion $\varphi : N \to \mathbb{R}^{2n+1}$ follow from the properties of a special Kähler manifold. By the Calabi-Pogorelov theorem, see [NS] and references therein, the metric g^N admits a quadratic Hesse potential and the Levi-Civita connection of g^N coincides with ∇^N. Using the formulas (2.3), this implies that the Levi-Civita connection of the special real manifold coincides with ∇ and, hence, is flat. Since there is no compact simply connected Riemannian manifold of positive dimension we obtain the theorem. □

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