Isometries of the 1-loop deformed universal hypermultiplet

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September 2017

We are interested in determining all isometries of the metric g^c of the 1-loop corrected universal hypermultiplet manifold $M := \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$:

$$g^c = F(\rho)^2 \mathrm{d}\rho^2 + G(\rho)^2 (\mathrm{d}\tilde{\phi} + \zeta^0 \mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0 \mathrm{d}\zeta^0)^2 + H(\rho)^2 \left((\mathrm{d}\zeta^0)^2 + (\mathrm{d}\tilde{\zeta}_0)^2 \right),$$

where $F(\rho), G(\rho), H(\rho)$ are functions of ρ given by:

$$F(\rho) := \frac{1}{2\rho} \sqrt{\frac{\rho + 2c}{\rho + c}}, \quad G(\rho) := \frac{1}{2\rho} \sqrt{\frac{\rho + c}{\rho + 2c}}, \quad H(\rho) := \frac{\sqrt{2(\rho + 2c)}}{2\rho}.$$

Lemma 1. The norm of the curvature of $(M, g^{c>0})$ is an injective function of $\rho > 0$.

Proof. As computed in [1], the curvature may be regarded as an operator $\mathscr{R} : \Lambda^2 T^* M \to \Lambda^2 T^* M$ with eigenvalues:

$$\begin{aligned}
\lambda^+_{234} &= -2 \left[1 + 2 \left(\frac{\rho}{\rho + 2c} \right)^3 \right], \\
\lambda^-_{234} = \lambda^-_{342} = \lambda^-_{423} &= -2, \\
\lambda^+_{342} = \lambda^+_{423} &= -2 \left[1 - \left(\frac{\rho}{\rho + 2c} \right)^3 \right].
\end{aligned}$$

The curvature norm is therfore given by:

$$\|\mathscr{R}\|^2 = \sum_{(J,K,L),\epsilon} |\lambda^\epsilon_{JKL}|^2 = 24 \left[1 + \left(\frac{\rho}{\rho + 2c} \right)^6 \right],$$

where (J, K, L) runs over cyclic permutations of $(2, 3, 4)$ and ϵ runs over $\pm$. This function can be checked to be injective over $\rho > 0$ for all $c > 0$. $\square$

Lemma 2. There is a one-to-one correspondence between isometries $\varphi^c_g : M \to M$ that preserve g^c for a fixed value of $c > 0$ and isometries $\varphi_h : \mathbb{R} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}$ that preserve h^k for all $k \in (0, 1/8c)$, where h^k is given by:

$$h^k = k(\mathrm{d}\tilde{\phi} + \zeta^0 \mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0 \mathrm{d}\zeta^0)^2 + (\mathrm{d}\zeta^0)^2 + (\mathrm{d}\tilde{\zeta}_0)^2.$$

Proof. Assume that $\varphi_h : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ is an isometry that preserves h^k for all $k \in (0, 1/8c)$ where $c > 0$ is fixed beforehand. Then since we can write g^c as:

$$g^c = F(\rho)^2 \mathrm{d}\rho^2 + H(\rho)^2 h^{G(\rho)^2/H(\rho)^2},$$

and since $G(\rho)^2/H(\rho)^2 \in (0, 1/8c)$ for all $\rho > 0$, it follows that $\varphi_g^c : M \to M$ given by $(\rho, \tilde{\phi}, \zeta) \mapsto (\rho, \varphi_h(\tilde{\phi}, \zeta))$ is an isometry preserving g^c.

Conversely, now fix a $c > 0$ and assume that $\varphi_g^c : M \to M$ is an isometry that preserves g^c. Then since it leaves the curvature norm invariant and the curvature norm is an injective function of ρ, it must send constant ρ hypersurfaces to themselves. Moreover, it must preserve the unit normal bundle of these hypersurfaces, which is to say it must preserve $F(\rho)^{-1}\partial_\rho$. These two facts imply that φ_g^c must necessarily be of the form $(\rho, \tilde{\phi}, \zeta) \mapsto (\rho, \varphi_h(\tilde{\phi}, \zeta))$ where φ_h preserves $h^{G(\rho)^2/H(\rho)^2}$ for all $\rho > 0$. But this is equivalent to saying φ_h preserves h^k for all $k \in (0, 1/8c)$. $\square$

Remark 3. Since any $(M, g^{c>0})$ is isometric to (M, g^1) under the rescaling $(\rho, \zeta) \mapsto (c\rho, c\zeta)$, we conclude by taking c to be arbitrarily small that if φ_h preserves h^k for $k \in (0, 1/8c)$, then it preserves h^k for all $k > 0$.

Lemma 4. If $X \in \Gamma T(\mathbb{R}\times\mathbb{C})$ is a Killing vector preserving h^k for all $k > 0$, then it must be an $\mathbb{R}$-linear combination of the following vector fields:

$$X_{\tilde{\phi}} := \partial_{\tilde{\phi}}, \quad X_{\zeta^0} := \partial_{\zeta^0} - \tilde{\zeta}_0 \partial_{\tilde{\phi}}, \quad X_{\tilde{\zeta}_0} := \partial_{\tilde{\zeta}_0} + \zeta^0 \partial_{\tilde{\phi}}, \quad X_\zeta := \tilde{\zeta}_0 \partial_{\zeta^0} - \zeta^0 \partial_{\tilde{\zeta}_0}.$$

Proof. We first note that $X_{\tilde{\phi}}, X_{\zeta^0}, X_{\tilde{\zeta}_0}$ form a $C^\infty(\mathbb{R}\times\mathbb{C})$-basis for $\Gamma T(\mathbb{R}\times\mathbb{C})$. So any vector field X can be expressed as:

$$X = f_{\tilde{\phi}} X_{\tilde{\phi}} + f_{\zeta^0} X_{\zeta^0} + f_{\tilde{\zeta}_0} X_{\tilde{\zeta}_0},$$

where $f_{\tilde{\phi}}, f_{\zeta^0}, f_{\tilde{\zeta}_0} \in C^\infty(\mathbb{R}\times\mathbb{C})$. Substituting the above into $0 = \mathscr{L}_X h^k$ and using the fact that $\mathscr{L}_{X_{\tilde{\phi}}} h^k = \mathscr{L}_{X_{\zeta^0}} h^k = \mathscr{L}_{X_{\tilde{\zeta}_0}} h^k = 0$, we get:

$$\begin{aligned} 0 &= 2\,\mathrm{d}f_{\tilde{\phi}} X_{\tilde{\phi}}^\flat + 2\mathrm{d}f_{\zeta^0} X_{\zeta^0}^\flat + 2\,\mathrm{d}f_{\tilde{\zeta}_0} X_{\tilde{\zeta}_0}^\flat \\ &= 2k(\mathrm{d}f_{\tilde{\phi}} - 2\tilde{\zeta}_0 \mathrm{d}f_{\zeta^0} + 2\zeta^0 \mathrm{d}f_{\tilde{\zeta}_0})(\mathrm{d}\tilde{\phi} + \zeta^0 \mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0 \mathrm{d}\zeta^0) + 2\,\mathrm{d}f_{\zeta^0}\,\mathrm{d}\zeta^0 + 2\,\mathrm{d}f_{\tilde{\zeta}_0}\,\mathrm{d}\tilde{\zeta}_0, \end{aligned}$$

where juxtaposition denotes the normalised symmetric tensor product. Since the above holds for all $k > 0$, the following two conditions need to hold separately:

$$(\mathrm{d}f_{\tilde{\phi}} - 2\tilde{\zeta}_0 \mathrm{d}f_{\zeta^0} + 2\zeta^0 \mathrm{d}f_{\tilde{\zeta}_0})(\mathrm{d}\tilde{\phi} + \zeta^0 \mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0 \mathrm{d}\zeta^0) = 0, \tag{1}$$

$$\mathrm{d}f_{\zeta^0}\,\mathrm{d}\zeta^0 + \mathrm{d}f_{\tilde{\zeta}_0}\,\mathrm{d}\tilde{\zeta}_0 = 0. \tag{2}$$

Now (2) implies that f_{ζ^0} depends only on $\tilde{\zeta}_0$, $f_{\tilde{\zeta}_0}$ depends only on ζ^0, and that $\partial_{\tilde{\zeta}_0} f_{\zeta^0} = -\partial_{\zeta^0} f_{\tilde{\zeta}_0}$. Of course, $\partial_{\tilde{\zeta}_0} f_{\zeta^0} = -\partial_{\zeta^0} f_{\tilde{\zeta}_0}$ can only hold if they are both equal to constants. So, f_{ζ^0} is an affine function of $\tilde{\zeta}_0$, while $f_{\tilde{\zeta}_0}$ is an affine function of ζ^0. Since we are only interested in $f_{\zeta^0}, f_{\tilde{\zeta}_0}$ up to constants, we may assume without loss of generality that $f_{\zeta^0} = a\tilde{\zeta}_0$ and $f_{\tilde{\zeta}_0} = -a\zeta^0$ for some constant $a \in \mathbb{R}$.

Substituting this into (1), we get:

$$\mathrm{d}(f_{\tilde{\phi}} - a|\zeta|^2)(\mathrm{d}\tilde{\phi} + \zeta^0\mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0\mathrm{d}\zeta^0) = 0.$$

This implies that $f_{\tilde{\phi}} - 2a|\zeta|^2$ is a constant which since we are only interested in $f_{\tilde{\phi}}$ up to a constant, we may take to be zero. So, we have $f_{\tilde{\phi}} = a|\zeta|^2$, giving us:

$$X = a|\zeta|^2 X_{\tilde{\phi}} + a\tilde{\zeta}_0 X_{\zeta^0} - a\zeta^0 X_{\tilde{\zeta}_0} = aX_\zeta.$$

Thus, any X satisfying $\mathscr{L}_X h^k = 0$ for all $k > 0$ must necessarily be an $\mathbb{R}$-linear combination of $X_{\tilde{\phi}}, X_{\zeta^0}, X_{\tilde{\zeta}_0}, X_\zeta$. $\square$

Lemma 5. The isometries that preserve h^k for all $k > 0$ are either of the form $(\tilde{\phi}, \zeta) \mapsto (\tilde{\phi} + t + \Im(\overline{\zeta}\tau), e^{\imath\theta}(\zeta + \tau))$ or of the form $(\tilde{\phi}, \zeta) \mapsto (-\tilde{\phi} - t + \Im(\zeta\overline{\tau}), e^{-\imath\theta}(\overline{\zeta} + \overline{\tau}))$, where $t, \theta \in \mathbb{R}$ and $\tau \in \mathbb{C}$.

Proof. Any isometry φ_h that preserves h^k for all $k > 0$ must act on the Lie algebra spanned by $X_{\tilde{\phi}}, X_{\zeta^0}, X_{\tilde{\zeta}_0}, X_\zeta$ via a Lie algebra isomorphism. In particular, the centre spanned by just $\partial_{\tilde{\phi}}$ must be mapped to itself. This means that φ_h must be of the form $(\tilde{\phi}, \zeta) \mapsto (u(\tilde{\phi}, \zeta), v(\zeta))$ where $u : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $v : \mathbb{C} \to \mathbb{C}$ are smooth maps. Since φ_h preserves h^k for all $k > 0$, it must separately preserve the following tensors:

$$(\mathrm{d}\tilde{\phi} + \zeta^0\mathrm{d}\tilde{\zeta}_0 - \tilde{\zeta}_0\mathrm{d}\zeta^0)^2 = (\mathrm{d}\tilde{\phi} + \Im(\overline{\zeta}\,\mathrm{d}\zeta))^2, \quad (\mathrm{d}\zeta^0)^2 + (\mathrm{d}\tilde{\zeta}_0)^2 = |\mathrm{d}\zeta|^2.$$

Since $\zeta \mapsto v(\zeta)$ must preserve $|\mathrm{d}\zeta|^2$ in particular, v must be a Euclidean motion (inclusive of reflections). In other words, v must be either of the form $\zeta \mapsto e^{\imath\theta}(\zeta + \tau)$ or of the form $\zeta \mapsto e^{-\imath\theta}(\overline{\zeta} + \overline{\tau})$, where $\theta \in \mathbb{R}$ and $\tau \in \mathbb{C}$ are constants. Meanwhile, since $(\tilde{\phi}, \zeta) \mapsto (u(\tilde{\phi}, \zeta), v(\zeta))$ must also preserve $(\mathrm{d}\tilde{\phi} + \Im(\overline{\zeta}\,\mathrm{d}\zeta))^2$, we must have (at least) one of the following two possibilities:

$$\mathrm{d}(u(\tilde{\phi}, \zeta) - \tilde{\phi}) = -\Im(\overline{v(\zeta)}\,\mathrm{d}v(\zeta) - \overline{\zeta}\,\mathrm{d}\zeta), \tag{3}$$

$$\mathrm{d}(u(\tilde{\phi}, \zeta) + \tilde{\phi}) = -\Im(\overline{v(\zeta)}\,\mathrm{d}v(\zeta) + \overline{\zeta}\,\mathrm{d}\zeta). \tag{4}$$

Note that the left-hand side in either of the two equations is an exact form, so the right-hand side has to be an exact form as well if the above are to hold. If v is of the form $\zeta \mapsto e^{\imath\theta}(\zeta + \tau)$ then this can only hold in case of (3). While if v is of the form $\zeta \mapsto e^{-\imath\theta}(\overline{\zeta} + \overline{\tau})$, then this can hold only in case of (4). In the first case, we get the solution $u(\tilde{\phi}, \zeta) = \tilde{\phi} + t + \Im(\overline{\zeta}\tau)$ where $t \in \mathbb{R}$ is some constant, whereas in the second case, we get the solution $u(\tilde{\phi}, \zeta) = -\tilde{\phi} - t + \Im(\zeta\overline{\tau})$ where $t \in \mathbb{R}$ is some constant. $\square$

Proposition 6. The group of isometries of $(M, g^{c>0})$ is $\mathrm{H}_3(\mathbb{R}) \rtimes \mathrm{O}(2)$, where $\mathrm{H}_3(\mathbb{R})$ is the continuous Heisenberg group.

Proof. Lemmata 2 and 5 together imply that the most general form that any isometry φ_g^c can take is either of the form $(\rho, \tilde{\phi}, \zeta) \mapsto (\rho, \tilde{\phi} + t + \Im(\overline{\zeta}\tau), e^{\imath\theta}(\zeta + \tau))$ or of the form $(\rho, \tilde{\phi}, \zeta) \mapsto (\rho, -\tilde{\phi} - t + \Im(\zeta\overline{\tau}), e^{-\imath\theta}(\overline{\zeta} + \overline{\tau}))$, where $t, \theta \in \mathbb{R}$ and $\tau \in \mathbb{C}$. These constitute precisely the group $\mathrm{H}_3(\mathbb{R}) \rtimes \mathrm{O}(2)$. $\square$

References

[1] Vicente Cortés, Arpan Saha, “Quarter-pinched Einstein metrics interpolating between real and complex hyperbolic metrics,” arXiv:1705.04186 (2017)