TWISTING HERMITIAN AND HYPERCOMPLEX GEOMETRIES

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Abstract

A twist construction for manifolds with torus action is described generalizing certain T-duality examples and constructions in hypercomplex geometry. It is applied to complex, SKT, hypercomplex, and HKT manifolds to construct compact simply connected examples. In particular, we find hypercomplex manifolds that admit no compatible HKT metric, and HKT manifolds whose Obata connection has holonomy contained in $\mathrm{SL}(n, \mathbb{H})$.

1. Introduction

The study of special metrics compatible with one or more complex structures has a long history. The most widely studied case is that of Kähler metrics, where the Levi-Civita connection also preserves the complex structure; here there is a rich, plentiful source of examples, although there are complex manifolds that admit no Kähler metric. However, the analogue of this situation for two anticommuting complex structures, namely hyper-Kähler geometry, is very restrictive and only a limited number of compact examples are known. Thus from the mathematical point of view, weaker compatibility conditions are of interest. On the other hand, models from theoretical physics, particularly in the presence of supersymmetry, lead to complex structures and metrics with potentially less restrictive constraints (see [16], [37]). In general, it is not hard to weaken the formal definitions, but then the question remains whether these really lead to new structures and how one might construct examples. In this article, we present a general construction that from a manifold with torus symmetry produces a new manifold, the twist, and we show how geometric data may be moved through this construction. We then specialize the construction to complex and hypercomplex geometry, using it to construct examples that are compact and simply connected.

In outline, the twist construction we describe is as follows. Consider a manifold M with an action of an n-torus T_M. Suppose $P \to M$ is a principal T^n-bundle with connection. If the T_M-action lifts to P commuting with the principal action, then we

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may construct the quotient space

$$W = P/T_M.$$

Furthermore, if the lifted T_M-action preserves the principal connection, then tensors on M may be transferred to tensors on W by requiring their pullbacks to P to agree on horizontal vectors. In this way, an invariant geometric structure on M, such as a complex structure or a metric, determines a corresponding geometric structure on the twist W.

Essentially this construction was considered mathematically by Joyce [25] in the context of hypercomplex manifolds using instanton connections. For the special case of circle actions and instanton connections this was specialized to hyper-Kähler with torsion (HKT) metrics for hypercomplex structures by Grantcharov and Poon [23]. As demonstrated in [38], specific noncompact examples in the HKT context include the T-duality construction evoked by Gibbons, Papadopoulos, and Stelle [19] based on the σ-model duality of Buscher [8] (see [4], [35], [43]). The construction we present is more general, applying in principle to any geometric structure, but even in the hypercomplex or HKT case we see that the instanton condition can be relaxed with advantage. Furthermore, in many situations the T^n-twist construction is not equivalent to n invocations of the S^1-twist. Versions of the S^1-twist have been announced and discussed in [38], [39] and applied to almost quaternion-Hermitian manifolds in [29].

We start the article by discussing the general framework for the twist construction. In particular, we study in detail the problem of lifting the T_M action to the principal bundle P in Section 2 and invoke the topological results of [26]. The twist construction itself is described in Section 3. We show how tensors and almost complex structures may be moved through the twist construction, and we study the effects on the exterior derivatives of forms and integrability of complex structures. In the latter case, we see that the instanton condition is not the most general requirement, in line with the constructions of Goldstein and Prokushkin [20]. Since the twist W is constructed as a quotient it is potentially singular. For our applications, we are only interested in smooth manifolds, but many of the results extend without change to the orbifold case.

In Section 4, we apply the twist construction in the context of Hermitian geometry. We concentrate mostly on the case of SKT manifolds, strong Kähler manifolds with torsion. These are characterized by $\partial\bar{\partial}\omega_I = 0$, where ω_I is the Kähler form. Gauduchon [17] showed that any compact Hermitian surface admits such a metric, but in higher dimensions the condition is more complicated. The SKT structures on 6-dimensional nilmanifolds were classified by Fino, Parton, and Salamon [13]; however, these are not simply connected. Grantcharov, Grantcharov, and Poon [22] used torus bundles to provide 6-dimensional examples. Using the instanton case, we reproduce their examples via the twist construction, extending them to higher dimensions, and point out how a number of relatively explicit examples may be produced. We also show that the noninstanton case gives rise to further compact simply connected SKT manifolds. This section closes with a discussion of the behavior of complex volume forms. Note that other known compact examples of SKT manifolds include even-dimensional compact Lie groups (see [36]) and certain instanton moduli spaces (see [28], [10]) and that Fino and Tomassini [14] have recently shown that the SKT condition is preserved by blowup.

The final part of the paper, Section 5, concerns hypercomplex and HKT geometry. Hypercomplex manifolds carry two anticommuting complex structures; the HKT condition on a compatible metric may be expressed via a simple first order relation on the exterior derivatives of the corresponding Kähler forms, equation (5.1). The HKT condition implies that there is a unique connection preserving the metric and the complex structures, with torsion determined by a 3-form. This geometry was originally introduced by Howe and Papadopoulos [24] in the physics literature and the article by Grantcharov and Poon [23] gave a mathematical description and several constructions. We now know that HKT structures behave in many ways as a good quaternionic analogue of Kähler geometry. In particular, there is a potential theory (see [2]), a version of Hodge theory (see [41]), and some work toward Calabi conjecture types of results (see [1]). However, the strongest versions of these results require a reduction of the holonomy of the hypercomplex structure: there is a unique torsion-free connection, the Obata connection, that preserves the given complex structures, and the requirement is that its holonomy should be contained in the subgroup $SL(n, \mathbb{H})$, consisting of the invertible $n \times n$ quaternion matrices $GL(n, \mathbb{H}) \subset GL(4n, \mathbb{R})$ that preserve a real volume form.

Verbitsky [40] used instanton connections on vector bundles to construct compact HKT manifolds; however, these have infinite fundamental group. We show in Section 5.1 that the instanton version of the twist construction leads to many simply connected examples. We also demonstrate that, at least locally, HKT metrics may be produced from noninstanton twists.

Using instanton twists, we show in Section 5.2, that there are nontrivial HKT metrics on compact simply connected manifolds such that the Obata holonomy lies in $SL(n, \mathbb{H})$. Barberis, Dotti, and Verbitsky [3] recently provided similar examples on compact nilmanifolds, but these have infinite fundamental group.

Finally, we construct via noninstanton twists examples of compact simply connected hypercomplex manifolds in all allowable dimensions that do not admit a compatible HKT metric. Examples with infinite fundamental group were previously constructed on nilmanifolds in dimension 8 by Fino and Grantcharov [12].

2. Lifting Abelian actions

Let A^n be a connected Abelian Lie group of dimension n. Suppose that $\pi : P \to M$ is a principal A^n-bundle with structural group A_P. Write $\mathfrak{a}_P$ for the Lie algebra of A_P, and let $\rho \colon \mathfrak{a}_P \to \mathfrak{X}(P)$, $Y \mapsto \rho_Y$, be the vector fields generated by the principal action.

We now assume that there is an action of $A_M \cong A^n$ on M and write $\xi \colon \mathfrak{a}_M \to \mathfrak{X}(M)$ for the infinitesimal action. If $\theta \in \Omega^1(P, \mathfrak{a}_P)$ is a connection 1-form on P with curvature $F \in \Omega^2(M, \mathfrak{a}_P)$, $\pi^* F = d\theta$, we wish to determine conditions so that the A_M-action is covered by an Abelian Lie group action on P preserving θ and commuting with A_P. We use $\tilde{X}$ to denote the horizontal lift of $X \in TM$ to $\mathcal{H} = \ker\theta \subset TP$.

First consider the problem of lifting to an $\mathbb{R}^n = \widetilde{A^n}$-action. Such a lift is given by a map $\mathring{\xi} \colon \mathfrak{a}_M \to \mathfrak{X}(P)$.

PROPOSITION 2.1

The A_M-action induced by ξ lifts to an $\mathbb{R}^n$-action preserving the connection form θ if and only if

(i) $L_\xi F = 0$,

(ii) $[\xi \lrcorner F] = 0 \in H^1(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$, *and*

(iii) $\xi^* F = 0$.

Moreover, if A_M is compact, then condition (iii) is redundant.

Proof

Write

$$\mathring{\xi} = \tilde{\xi} + \mathring{a}\rho \tag{2.1}$$

for some $\mathring{a} \in \Omega^0(P, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$, where $\tilde{\xi}$ is the horizontal lift and ρ is regarded as an element of $\mathfrak{X}(P) \otimes \mathfrak{a}_P^*$. The condition that $\mathring{\xi}$ preserves θ gives

$$0 = L_{\mathring{\xi}}\theta = \mathring{\xi} \lrcorner d\theta + d(\mathring{\xi} \lrcorner \theta) = \mathring{\xi} \lrcorner \pi^* F + d(\mathring{a}\rho \lrcorner \theta) = \pi^*(\xi \lrcorner F) + d\mathring{a}.$$

Thus

$$d\mathring{a} = -\pi^*(\xi \lrcorner F).$$

The first consequence of this is that

$$\pi^*(L_\xi F) = \pi^*(d\xi \lrcorner F) = -d^2\mathring{a} = 0,$$

giving condition (i). We also find differentiating in the vertical directions that $\rho\mathring{a} = \rho \lrcorner d\mathring{a} = -\rho \lrcorner \pi^*(\xi \lrcorner F) = 0$, so $\mathring{a}$ is constant on fibers and is the pullback of a function $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$:

$$\mathring{\xi} \lrcorner \theta = \mathring{a} = \pi^* a.$$

We now have

$$da = -\xi \lrcorner F, \tag{2.2}$$

so the class $[\xi \lrcorner F]$ is zero in $H^1(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$, which is condition (ii).

It remains to show that $\mathring{\xi}\colon \mathfrak{a}_M \to \mathfrak{X}(P)$ is a Lie algebra homomorphism. As we are working with Abelian groups, this is the same as $[\mathring{\xi}_X, \mathring{\xi}_Y] = 0$ for all $X, Y \in \mathfrak{a}_M$. We have

$$[\mathring{\xi}_X, \mathring{\xi}_Y] = [\tilde{\xi}_X, \tilde{\xi}_Y] + [\mathring{a}_X \rho, \tilde{\xi}_Y] + [\tilde{\xi}_X, \mathring{a}_Y \rho] + [\mathring{a}_X \rho, \mathring{a}_Y \rho]. \tag{2.3}$$

The last term vanishes since $\mathfrak{a}_P$ is Abelian and $\rho \mathring{a} = 0$. For one of the middle terms, we have

$$\pi_*[\mathring{a}_X \rho, \tilde{\xi}_Y] = [0, \xi_Y] = 0,$$

so the horizontal part is zero, and the vertical part is determined by

$$\begin{aligned}\theta([\mathring{a}_X \rho, \tilde{\xi}_Y]) &= -d\theta(\mathring{a}_X \rho, \tilde{\xi}_Y) - \tilde{\xi}_Y(\mathring{a}_X)\\ &= -(\pi^* F)(\mathring{a}_X \rho, \tilde{\xi}_Y) - \pi^*(\xi_Y a_X)\\ &= \pi^*\big(F(\xi_X, \xi_Y)\big).\end{aligned}$$

As $\mathfrak{a}_M$ is Abelian, the first term in (2.3) has

$$\pi_*([\tilde{\xi}_X, \tilde{\xi}_Y]) = [\xi_X, \xi_Y] = 0 \qquad \text{and} \qquad \theta([\tilde{\xi}_X, \tilde{\xi}_Y]) = -\pi^*\big(F(\xi_X, \xi_Y)\big).$$

Putting these together we get

$$\theta([\mathring{\xi}_X, \mathring{\xi}_Y]) = \pi^*\big(F(\xi_X, \xi_Y)\big),$$

while the horizontal part is zero. This gives condition (iii). Now note that $F(\xi_X, \xi_Y) = -da_X(\xi_Y) = -L_{\xi_Y} a_X$, so $d(F(\xi_X, \xi_Y)) = -L_{\xi_Y} da_X = L_{\xi_Y}(\xi_X \lrcorner F) = 0$. This shows that $F(\xi_X, \xi_Y)$ is constant. When A_M is compact, each component of $\xi_Y a_X$ has a zero on each A_M-orbit, for example, at a maximum of the component of a_X, so the constant $F(\xi_X, \xi_Y)$ is zero. □

Note that the lift is not unique, depending instead on a choice of $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$ in equation (2.2). In particular, if M is compact, we can add a constant element of $\mathfrak{a}_P \otimes \mathfrak{a}_M^*$ to a to ensure that a is invertible.

Suppose F is a closed 2-form with values in $\mathbb{R}^n \cong \mathfrak{a}_P$.

Definition 2.2
We say that an A_M-action is *F-Hamiltonian* if it satisfies conditions (i) and (ii) of Proposition 2.1.

2.1. Topological considerations for torus actions
In constructions with $A^n = T^n$ an n-torus, we would like to have as starting data a manifold M with T_M-action and a 2-form $F \in \Omega^2(M, \mathfrak{t}_n)$ that is T_M-invariant. We then wish to construct a principal T^n-bundle P with a connection θ whose curvature is F in such a way that T_M lifts to a T^n-action on P preserving θ and commuting with the principal action.

First, ignoring the T_M-action we need F to be a closed form with integral periods; we write this as $F \in \Omega^2_{\mathbb{Z}}(M, \mathfrak{t}_n)$. We then have that $[F] \in H^2(M, \mathbb{Z}^n) \otimes \mathbb{R} \subset H^2(M, \mathfrak{t}_n)$, and by Chern-Weil theory we can find a principal T^n-bundle P with $c_1(P) \otimes \mathbb{R} = [F]$ and connection θ_0 on P with $d\theta_0 = \pi^* F$. Let us write $T^n = \mathfrak{t}_n / \Lambda_n$ and $T_M = \mathfrak{t}_M / \Lambda_M$, so $\Lambda_n \cong \mathbb{Z}^n \cong \Lambda_M$.

The question of equivariantly lifting the T_M-action to P is largely addressed in the clear paper [26]. The main tool used is the spectral sequence of the fibration

$$M \longrightarrow M_G \longrightarrow BG,$$

where $G = T_M$, $EG = (S^\infty)^n = (\operatorname{colim} S^k)^n \to BG = BT_M \cong \mathbb{C}\mathrm{P}(\infty)^n$ is the classifying space and $M_G = EG \times_G M$. It is shown that the T_M-action lifts to P if and only if $c_1(P) \in H^2(M, \Lambda_n)$ is T_M-invariant and lies in

$$E_\infty^{0,2} = \ker(d_3\colon E_3^{0,2} \to E_3^{3,0}) \subset \ker(d_2\colon E_2^{0,2} \to E_2^{2,1}).$$

Since T_M is connected it acts trivially on $H^q(M, \Lambda_n)$, so the invariance of $c_1(P)$ is automatic, and the locally constant presheaf $H^q(M, \Lambda_n)$ over the simply connected space BG is constant. We thus have

$$\begin{aligned} E_2^{p,q} &= H^p\bigl(BT_M, H^q(M, \Lambda_n)\bigr) \\ &= \begin{cases} H^q(M, \Lambda_n) \otimes_{\mathbb{Z}} S^k \Lambda_M^* & \text{for } p = 2k \geqslant 0 \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This gives immediately that $d_3 = 0$. Since $E_3^{p,q} = \ker(d_2\colon E_2^{p,q} \to E_2^{p+2,q-1}) / \operatorname{im}(d_2\colon E_2^{p-2,q+1} \to E_2^{p,q})$, we have

$$\begin{aligned} E_\infty^{0,2} = E_3^{0,2} &= \ker(d_2\colon E_2^{0,2} \to E_2^{2,1}) \\ &= \ker\bigl(d_2\colon H^2(M, \Lambda_n) \to H^1(M, \Lambda_n) \otimes_{\mathbb{Z}} \Lambda_M^*\bigr). \end{aligned}$$

Tensoring with $\mathbb{R}$, the map d_2 is given by

$$(d_2^{\mathbb{R}})[F] = [\xi \lrcorner F] \in H^1(M, \mathfrak{t}_n \otimes \mathfrak{t}_M^*)$$

when F is T_M-invariant.

If the T_M-action is F-Hamiltonian (Definition 2.2), then we have $[\xi \lrcorner F] = 0$ and the class $d_2(c_1(P))$ is torsion in $H^1(M, \Lambda_n) \otimes_{\mathbb{Z}} \Lambda_M^*$. However, $H^1(M, \mathbb{Z})$ is isomorphic to the torsion-free part of $H_1(M, \mathbb{Z})$, so $d_2(c_1(P)) = 0$. Thus, the T_M-action lifts to P covering the action on M. The lifted T_M-action will not necessarily preserve θ_0, but averaging θ_0 over the lifted T_M gives an invariant connection form θ, still with curvature F.

In summary, we have the following.

PROPOSITION 2.3

Suppose M admits an F-Hamiltonian T_M-action for some closed 2-form with integral periods $F \in \Omega^2_{\mathbb{Z}}(M, \mathfrak{t}_n)$. Then there is a principal T^n-bundle $P \to M$ which admits

(i) *a $\mathring{T}_M$-action of the n-torus T_M commuting with the principal action, covering the T_M-action on M, and*

(ii) *a $\mathring{T}_M$-invariant T^n-connection θ on P with curvature F.*

In fact, such a lift exists for any T^n-bundle P with $c_1(P) \otimes \mathbb{R} = [F]$. □

The lifts above are not unique: when lifting to an $\mathbb{R}^n$-action, there is an ambiguity in $a \in \Omega^0(M, \mathfrak{t}_P \otimes \mathfrak{t}_M^*)$ as one may add any constant element of $\lambda \in \mathfrak{t}_P \otimes \mathfrak{t}_M^*$ to a. Given a torus lift, the integrality condition $\lambda \in \Lambda_P \otimes \Lambda_M^* \otimes \mathbb{Z}$ leads to a new such torus action on P. If the T_M-action on M is free, then each of these lifts $\mathring{T}_M$ is free on P. Allowing $\lambda \in \Lambda_P \otimes \Lambda_M^* \otimes \mathbb{Q}$ leads to torus actions on P whose infinitesimal generators map to those on M under π_* and which cover the action of a finite cover of T_M.

3. Twist construction

Suppose that M is a manifold with an effective F-Hamiltonian A_M-action where $F \in \Omega^2_{\mathbb{Z}}(M, \mathfrak{a}_n)$. Let P be a principal A^n-bundle with a connection θ whose curvature is F and with an $\mathring{A}_M$-action preserving θ and covering the A_M-action infinitesimally. Here $\mathring{A}_M$ is some connected Abelian group covering A_M. Assume that $\mathring{A}_M$ acts properly on P and that $\mathring{A}_M$ is transverse to

$$\mathscr{H} = \ker \theta.$$

Then $\mathring{A}_M$ has discrete stabilizers and $P/\mathring{A}_M$ has the same dimension as M. This transversality is the same as requiring $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$ in (2.2) to be invertible. If M is compact and $A^n \cong T^n$ is a torus, then the discussion of the previous section shows that there is always a proper lift and that we may add a constant rational element of $\mathfrak{a}_P \otimes \mathfrak{a}_M^*$ to a to ensure that $a^{-1} \in \Omega^0(M, \mathfrak{a}_M \otimes \mathfrak{a}_P^*)$ exists.

Definition 3.1
A *twist* of M with respect to A_M, F and invertible a, is the quotient space

$$W = P/\mathring{A}_M.$$

We say that W is a *smooth twist* if W is a manifold.

For torus actions, a twist W will at worst be an orbifold under the assumptions above. We are interested in constructing smooth manifolds, and therefore, we only discuss geometric structures in the case of smooth twists; however, many of our results carry over to the orbifold case without modification. Note that the following example shows that smooth twists may not exist.

Example 3.2
Suppose $M = \mathbb{C}\mathrm{P}(n)$ with circle action

$$[z_0, \dots, z_n] \mapsto [e^{2\pi i \lambda_0 \theta} z_0, \dots, e^{2\pi i \lambda_n \theta} z_n].$$

The canonical circle bundle over $\mathbb{C}\mathrm{P}(n)$ is $S^{2n+1} \subset \mathbb{C}^{n+1}$ with principal action $\mathbf{z} \mapsto e^{2\pi i \phi}\mathbf{z}$. The lifted action

$$(z_0, \dots, z_n) \mapsto (e^{2\pi i (\lambda_0 + k)t} z_0, \dots, e^{2\pi i (\lambda_n + k)t} z_n)$$

is free only if $k = -(\lambda_i + \lambda_j)/2$ for all $\lambda_i \neq \lambda_j$, which in general is impossible.

In the case that the lifting function a is nonconstant, we cannot consistently identify $\mathfrak{t}_M$ and $\mathfrak{t}_P$, and so twisting with T_M cannot be reduced to repeated twists by circle subgroups via circle bundles.

3.1. Geometric structures

Returning to W a smooth twist of a general M, we have projection maps

$$M \xleftarrow{\ \pi\ } P \xrightarrow{\ \pi_W\ } W.$$

Our assumptions imply that both maps are transverse to the distribution $\mathscr{H}$. We use this to relate objects on M and W. Invariant vector fields may be transferred simply by lifting horizontally and projecting.

Definition 3.3
Two $(p, 0)$-tensors α on M and α_W on W are said to be *$\mathscr{H}$-related*, written

$$\alpha \sim_{\mathscr{H}} \alpha_W,$$

if their pullbacks to P agree on $\mathscr{H}$; that is, $\pi^*\alpha = \pi_W^*\alpha_W$ on $\mathscr{H}$.

LEMMA 3.4
Each invariant p-form $\alpha \in \Omega^p(M)^{A_M}$ is $\mathscr{H}$-related to a unique p-form $\alpha_W \in \Omega^p(W)$ given by

$$\pi_W^*\alpha_W = \pi^*\alpha - \theta \wedge \pi^*(a^{-1}\xi \lrcorner \alpha).$$

Note that A_M-invariance of α is a necessary condition.

Proof
The form $\pi_W^*\alpha_W$ may be decomposed with respect to θ and $\mathscr{H}$ as an element of $\Omega^p(\mathscr{H}) + \theta \wedge \Omega^{p-1}(\mathscr{H}, \mathfrak{a}_P^*)$. By definition the first component is $\pi^*\alpha$; we write the second as $\theta \wedge \beta$. Using (2.1), we now compute

$$\begin{aligned} 0 = \mathring{\xi} \lrcorner \pi_W^*\alpha_W &= \mathring{\xi} \lrcorner \pi^*\alpha + \mathring{\xi} \lrcorner (\theta \wedge \beta) \\ &= \tilde{\xi} \lrcorner \pi^*\alpha + \mathring{a}\theta(\rho)\beta - \theta \wedge \tilde{\xi} \lrcorner \beta \\ &= \pi^*(\xi \lrcorner \alpha) + (\pi^*a)\beta - \theta \wedge (\tilde{\xi} \lrcorner \beta), \end{aligned}$$

since $\theta \circ \rho$ is the identity on $\mathfrak{a}_P$. Considering horizontal vectors, we have $\beta = -\pi^*(a^{-1}\xi \lrcorner \alpha)$ and the result follows. □

COROLLARY 3.5
If g is an invariant metric on M, then the unique metric g_W on W $\mathscr{H}$-related to g is given by

$$\pi_W^* g_W = \pi^* g - 2\theta \vee \pi^*(a^{-1}\xi^\flat) + \theta^2 \pi^*\big((a^{-1})^2 g(\xi \otimes \xi)\big).$$

Proof
Writing $\pi_W^* g_W = \pi^* g + \theta \vee \gamma + \theta^2 f$ with $\gamma \in \Omega^1(\mathscr{H}, \mathfrak{a}_P^*)$ and $f \in \Omega^0(P, S^2\,\mathfrak{a}_P^*)$, we have

$$0 = (\pi_W^* g_W)(\mathring{\xi}, \cdot) = (\pi^* g)(\tilde{\xi}, \cdot) + \frac{1}{2}\big(\mathring{a}\gamma + \theta\gamma(\tilde{\xi})\big) + \mathring{a}\theta f.$$

Taking the horizontal part gives $\gamma = -2\pi^*(a^{-1}g(\xi, \cdot)) = -2\pi^*(a^{-1}\xi^\flat)$. From the vertical part we then find $f = -\mathring{a}^{-1}\gamma(\tilde{\xi})/2 = \pi^*((a^{-1})^2 g(\xi \otimes \xi))$ and the result follows. □

COROLLARY 3.6
Suppose that $\alpha \sim_{\mathcal{H}} \alpha_W$. Then

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - a^{-1} F \wedge \xi \lrcorner \alpha.$$

Proof
Computing directly, we have

$$\begin{aligned}\pi_W^* \, d\alpha_W = d\pi_W^* \alpha_W &= d\bigl(\pi^*\alpha - \theta \wedge \pi^*(a^{-1}\xi \lrcorner \alpha)\bigr)\\ &= \pi^*(d\alpha - F \wedge a^{-1}\xi \lrcorner \alpha) + \theta \wedge \pi^*\bigl(d(a^{-1}\xi \lrcorner \alpha)\bigr),\end{aligned}$$

which agrees horizontally with the claimed result. Vertically we have θ wedge the pullback under π of

$$\begin{aligned}-a^{-1} da\, a^{-1} \wedge \xi \lrcorner \alpha + a^{-1} d(\xi \lrcorner \alpha) &= a^{-1}\xi \lrcorner F\, a^{-1} \wedge \xi \lrcorner \alpha - a^{-1}\xi \lrcorner d\alpha)\\ &= -\bigl(a^{-1}\xi \lrcorner (d\alpha - a^{-1} F \wedge \xi \lrcorner \alpha)\bigr),\end{aligned}$$

since $L_\xi \alpha = 0$. The result now follows. □

Thus the invariant part of the exterior algebra of W may be regarded as the invariant exterior algebra of M with the twisted differential $d - a^{-1} F \wedge \xi \lrcorner$.

3.2. Duality
Let us now show that W is dual to M in the sense that M may also be obtained from W via a twist. The distribution $\mathcal{H}$ on P is transverse to the action of $\mathring{A}_M$. If $\mathring{A}_M$ acts freely we have a principal bundle $\mathring{A}_M \to P \to W$. Its connection form corresponding to $\mathcal{H}$ is

$$\theta_W = \pi^*(a^{-1})\theta,$$

as may be seen by writing $\theta_W = f\theta$, for some $f \in \Omega^0(P, \mathfrak{a}_M \otimes \mathfrak{a}_P^*)$, and enforcing the condition $\theta_W(\mathring{\xi}) = \mathrm{id}$ on $\mathfrak{a}_M$. This has curvature

$$\pi_W^* F_W = \pi^*(a^{-1} F) - \pi^*(a^{-1} da\, a^{-1}) \wedge \theta,$$

which is simply the 2-form F_W that is $\mathcal{H}$-related to $a^{-1}F$. Since A_P commutes with $\mathring{A}_M$, it descends to an action of an Abelian group A_W on W preserving F_W. Write $\zeta \colon \mathfrak{a}_W \to \mathfrak{X}(W)$ for the infinitesimal action of A_W, so $\zeta = \pi_W \circ \rho$. This action is F_W-Hamiltonian with $\zeta \lrcorner F_W = -d(a^{-1})$. The original manifold M is obtained by twisting W with respect to A_W and F_W using a^{-1}.

3.3. Lie brackets and complex structures

Tangent vectors X on M and X_W on W are said to be $\mathcal{H}$*-related* if their horizontal lifts to $\mathcal{H}$ agree. Writing $\widehat{\cdot}$ for the horizontal lift from W, this says

$$\widehat{X_W} = \widetilde{X}.$$

LEMMA 3.7

Lie brackets between $\mathcal{H}$-related vector fields are related by

$$[X_W, Y_W] \sim_{\mathcal{H}} [X, Y] + \xi a^{-1} F(X, Y).$$

Proof

Lifting $[X, Y]$ horizontally gives the horizontal part of $[\widetilde{X}, \widetilde{Y}]$. The vertical part of this last Lie bracket is $-\rho\pi^* F(X, Y)$, so

$$\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}] + \rho\pi^* F(X, Y).$$

Similarly,

$$\begin{aligned}
\widehat{[X_W, Y_W]} &= [\widehat{X}_W, \widehat{Y}_W] + \mathring{\xi}\pi_W^* F_W(X_W, Y_W) \\
&= [\widehat{X}_W, \widehat{Y}_W] + (\tilde{\xi} + a\rho)\pi^*(a^{-1}F + a^{-1}da\, a^{-1} \wedge \theta)(\widehat{X}_W, \widehat{Y}_W) \\
&= [\widehat{X}_W, \widehat{Y}_W] + \big(\pi^*(\xi a^{-1} F) + \rho\pi^* F\big)(\widehat{X}_W, \widehat{Y}_W),
\end{aligned}$$

from which the result follows. □

Almost complex structures may now be $\mathcal{H}$*-related* in a similar way giving

$$\widehat{I_W A} = \widetilde{I\pi_* \hat{A}}. \tag{3.1}$$

For a p-form α and an index k we write

$$I_{(k)}\alpha(X_1, \dots, X_p) = -\alpha(X_1, \dots, IX_k, \dots, X_p)$$

and $I_{(ab\cdots c)} = I_{(a)} I_{(b)} \cdots I_{(c)}$. This convention ensures that $I_{(k)} J_{(k)}\alpha = (IJ)_{(k)}\alpha$, but is the opposite of the usual convention in complex geometry. In particular, $\Lambda^{1,0}$ becomes the $(-i)$-eigenspace for $I_{(1)}$.

PROPOSITION 3.8

The Nijenhuis tensors of $\mathcal{H}$-related almost complex structures I and I_W are related by

$$N_{I_W} \sim_{\mathcal{H}} N_I - (1 - \mathcal{L}_I)\mathcal{F}, \tag{3.2}$$

where $\mathcal{F} = \xi a^{-1} F \in \Gamma(TM \otimes \Lambda^2 T^* M)$ *and* $\mathcal{L}_I = I_{(12)} + I_{(13)} + I_{(23)}$.

Proof

The definition $N_I(X, Y) = [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y]$ and Lemma 3.7 show that N_{I_W} is $\mathcal{H}$*-related* to N_I plus the correction term

$$\begin{aligned} &\xi a^{-1} F(IX, IY) - I\xi a^{-1} F(IX, Y) - I\xi a^{-1} F(X, IY) - \xi a^{-1} F(X, Y) \\ &= \big((I_{(23)} + I_{(12)} + I_{(13)} - 1)\mathcal{F}\big)(X, Y), \end{aligned}$$

giving (3.2). □

Note that $\mathcal{L}_I$ acts on $TM \otimes \Lambda^2 T^* M$ with eigenvalues -3 and $+1$. The (-3)-eigenspace is $[\![T^{1,0} \otimes \Lambda^{0,2}]\!]$, where $[\![V]\!] \otimes \mathbb{C} = V + \overline{V}$. We thus see that if F is of type $(1, 1)$, then twisting preserves integrability. However, it is important for us that other choices of F can also give integrable complex structures.

To understand the integrability better, fix a point $x \in M$, let $\mathcal{A} = \operatorname{im} \xi \subset T_x M$, and put $\mathcal{A}_I = \mathcal{A} \cap I\mathcal{A}$. Define $s = \dim_{\mathbb{C}} \mathcal{A}_I$ and $r = \dim_{\mathbb{R}} \mathcal{A}$. For a basis $e_1, \dots, e_n$ of $\mathfrak{a}_M$, write $X_i = \xi_x(e_i)$. Using $\mathcal{A}_I \leqslant \mathcal{A} \leqslant T_x M$, we may choose this basis so that

(i) $X_1, \dots, X_{2s}$ is a complex basis of $\mathcal{A}_I$, with $IX_{2j-1} = X_{2j}$, for $j = 1, \dots, s$,

(ii) $X_1, \dots, X_r$ is a basis of $\mathcal{A}$, and

(iii) $X_{r+1}, \dots, X_n$ are zero.

We then write $a^{-1} F \in \Omega^2(M, \mathfrak{a}_M)$ as

$$a^{-1} F = \sum_{k=1}^{n} F_k \otimes e_k$$

with $F_k \in \Omega^2(M)$, so $\mathcal{F} = \sum_{k=1}^{r} X_k \otimes F_k$.

LEMMA 3.9

If (M, I) is complex, then the induced almost complex structure I_W on a twist W is integrable if and only if

(i) $(F_{2j-1} + i F_{2j})^{0,2} = 0$, *for* $j = 1, \dots, s$, *and*

(ii) $F_k \in \Lambda^{1,1}$, *for* $k = 2s + 1, \dots, r$.

Proof
By (3.2), the integrability condition for I_W is $(1-\mathcal{L}_I)\mathcal{F} = 0$, since I is complex. Note that

$$\begin{aligned}(1-\mathcal{L}_I)(X_k \otimes F_k) &= X_k \otimes (1-I)F_k - IX_k \otimes I_{(1)}(1-I)F_k \\ &= 2X_k \otimes (F_k^{2,0} + F_k^{0,2}) + 2IX_k \otimes (iF_k^{2,0} - iF_k^{0,2}).\end{aligned}$$

For $2s < k \leqslant r$, X_k and IX_k are linearly independent of X_j for all $j \neq k$, so $F_k^{2,0} = 0 = F_k^{0,2}$. For $j \leqslant s$, we should consider the components $2j-1$ and $2j$ together. Let $X = X_{2j-1}$, $F_{(1)} = F_{2j-1}$, $F_{(2)} = F_{2j}$, and $F_{(c)} = F_{(1)} + iF_{(2)}$. Now, since $F_{(1)}$ and $F_{(2)}$ are real, we get

$$\begin{aligned}&(1-\mathcal{L}_I)(X_{2j-1} \otimes F_{2j-1} + X_{2j} \otimes F_{2j}) \\ &\quad= 2X \otimes (F_{(1)}^{2,0} + F_{(1)}^{0,2} - iF_{(2)}^{2,0} + iF_{(2)}^{0,2}) \\ &\qquad+ 2IX \otimes (F_{(2)}^{2,0} + F_{(2)}^{0,2} + iF_{(1)}^{2,0} - iF_{(1)}^{0,2}) \\ &\quad= 4\,\mathrm{Re}\left((X - iIX) \otimes F_{(c)}^{0,2}\right)\end{aligned}$$

and the result follows. □

4. Hermitian and SKT structures

Suppose (g, I) is a Hermitian structure on M, meaning that I is integrable and $g(IX, IY) = g(X, Y)$ for all $X, Y \in TM$. Then there is a unique connection ∇^{B} with skew-symmetric torsion preserving I and g (see [18]). This is known as the *Bismut connection*, due to its appearance in [5], and is given by

$$\nabla^{\mathrm{B}} = \nabla^{\mathrm{LC}} + \frac{1}{2}T^{\mathrm{B}}, \qquad c = (T^{\mathrm{B}})^\flat = -Id\omega_I, \tag{4.1}$$

where $\omega_I(X, Y) = g(IX, Y)$.

Definition 4.1
The triple $(g, I, \nabla^{\mathrm{B}})$, or equivalently (g, I, c), is known as a *Kähler with torsion (KT) structure*. It is *strong* or SKT if $dc = 0$.

Let us consider how the torsion form c of a KT manifold changes under a twist. We use the notation of Section 3 so that $A_M \cong A^n$ is a connected Abelian group acting on M in such a way that there is a smooth twist W given via curvature $F \in \Omega^2_{\mathbb{Z}}(M, \mathfrak{a}_P)$, $A_P \cong A^n$, and lifting function $a \in \Omega^0(M, \mathfrak{a}_P \otimes \mathfrak{a}_M^*)$.

PROPOSITION 4.2
Suppose the action of A_M preserves a Hermitian structure (g, I) on M with torsion form c. If (F, a) are such that the twist has I_W integrable, then W is a KT manifold with torsion given by

$$c_W \sim_{\mathcal{H}} c - a^{-1} I F \wedge \xi^\flat. \tag{4.2}$$

Proof
By construction I_W is compatible with g_W. We thus have $\omega_I^W \sim_{\mathcal{H}} \omega_I$ and that c is invariant under A_M. Now the result follows from

$$c_W \sim_{\mathcal{H}} -I(d - a^{-1} F \wedge \xi \lrcorner)\omega_I,$$

by Corollary 3.6. □

Many examples of this construction may be given by considering any free T^n-action on a Hermitian manifold and choosing a Hamiltonian 2-form $F \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^n)$. As we see below, this often produces Hermitian structures that are not of Kähler type.

It follows from (4.2) that the exterior derivative of the torsion satisfies

$$\begin{aligned} dc_W \sim_{\mathcal{H}} dc - a^{-1}\big(F \wedge (\xi \lrcorner c) + d(IF) \wedge \xi^\flat + IF \wedge d\xi^\flat\big) \\ + (a^{-1})^2\big(g(\xi, \xi) F \wedge IF + F \wedge (\xi \lrcorner IF) \wedge \xi^\flat \\ - (\xi \lrcorner F) \wedge IF \wedge \xi^\flat\big). \end{aligned} \tag{4.3}$$

Note that if F is of type (1, 1), the *instanton case*, then this simplifies to

$$dc_W \sim_{\mathcal{H}} dc - a^{-1} F \wedge \big((\xi \lrcorner c) + d\xi^\flat - g(\xi, \xi) a^{-1} F\big). \tag{4.4}$$

Using these last two expressions it is now possible to give a number of examples of compact simply connected SKT manifolds.

4.1. Instanton twists from tori

First note that any Kähler manifold is SKT since it has $c = 0$. Let N be an SKT manifold, and consider the product $M = N \times T^2$, where T^2 is a 2-torus with an invariant, and thus flat, Kähler metric. Then M is SKT with torsion supported on N. Let ξ be the torus action on the T^2-factor. We have $\xi \lrcorner c = 0$ and $d\xi^\flat = 0$. If $F \in \Omega^2_{\mathbb{Z}}(N, \mathfrak{a}_P)$ with $\mathfrak{a}_P \cong \mathbb{R}^2$, then $\xi \lrcorner F = 0$ and a is a constant isomorphism $\mathfrak{a}_M \to \mathfrak{a}_P$. Choosing bases we may write $\xi = (X_1, X_2)$, $a^{-1}F = (F_1, F_2)$.

Suppose now that F is of type $(1, 1)$. Then the twist W is Hermitian and the SKT condition becomes

$$\sum_{i,j=1}^{2} g(X_i, X_j) F_i \wedge F_j = 0.$$

Taking $T^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ with the standard metric, we may choose X_i orthonormal and a the identity matrix, the strong condition reduces to $F_1^2 + F_2^2 = 0$ with $F_i \in \Omega^{1,1}_{\mathbb{Z}}(N)$.

Example 4.3

Let $N = \Sigma_1 \times \Sigma_2$ be a product of Riemann surfaces, and take $[F_i]$ to be the fundamental class on the ith factor. The twist W is then a product of nontrivial circle bundles over the Riemann surfaces. When each Σ_i has genus 0, the twist W is topologically the product $S^3 \times S^3$, and we obtain the Calabi-Eckmann complex structures (see [9]). The torsion form c_W is a sum of volume forms on the two factors. In general, the constructed SKT structure only has T^2-symmetry, and even that may be destroyed by adding $i\partial\bar{\partial} f$ to the Kähler form ω_I^W for $f \in C^\infty(W)$ with suitably small C^2-norm.

To construct further examples of this type, note that Eells and Lemaire [11] showed that for an almost Kähler manifold X, any nonconstant holomorphic map $f\colon X \to \mathbb{C}\mathrm{P}(1)$ has $[f^*\omega_{\mathbb{C}\mathrm{P}(1)}] \neq 0$ in $H^2(X, \mathbb{Z})$. Thus if N is Kähler and admits a nonconstant holomorphic map to $\mathbb{C}\mathrm{P}(1)$, we get a 2-form

$$\Phi(f) = 2\pi f^*\omega_{\mathbb{C}\mathrm{P}(1)} \in \Omega^{1,1}_{\mathbb{Z}}(N)$$

with $\Phi(f)^2 = 0$. Furthermore such a map f exists whenever N admits a nonconstant holomorphic map to some compact Riemann surface.

Example 4.4

Consider the Kummer construction of a K3 surface N as the resolution of $X/\{\pm 1\}$, where $X = T^4 = \mathbb{C}^2/(\mathbb{Z} + i\mathbb{Z})^2$, obtained by blowing up the 16 singular points. Note that the Weierstrass $\wp$-function $\wp\colon \mathbb{C} \to \mathbb{C}\mathrm{P}(1)$ descends to T^2 and satisfies $\wp(z) = \wp(-z)$, so each factor $\mathbb{C}$ of $\mathbb{C}^2$ defines a nonconstant holomorphic map $\wp_i\colon X/\{\pm 1\} \to \mathbb{C}\mathrm{P}(1)$, which we may then pull back to the desingularization N. The classes $[\Phi(\wp_i)]$ are inequivalent in $H^2_{\mathrm{orb}}(X/\{\pm 1\}, \mathbb{Z}) = (H^2(X, \mathbb{Z}))^{\pm 1}$, indeed $[\Phi(\wp_1) \wedge \Phi(\wp_2)]$ is nonzero in $H^4(N)$.

Taking $F_i = \Phi(\wp_i)$, $i = 1, 2$, we see that $M = N \times T^2$ may be twisted to an SKT 6-manifold W with finite fundamental group. Taking the universal cover, we thus obtain a compact simply connected SKT manifold that is a T^2-bundle over the K3 surface N. We see below that W is not Kähler.

These examples generalize as follows (cf. [22] and [20]).

PROPOSITION 4.5

Let N be a compact simply connected SKT manifold. Suppose that for some even integer $n = 2k$, there are n closed integral $(1,1)$-forms $F_i \in \Omega^{1,1}_{\mathbb{Z}}(N)$ with $[F_i] \in H^2(N,\mathbb{R})$ linearly independent and such that $\sum_{i,j=1}^n \gamma_{ij} F_i \wedge F_j = 0$ for some positive definite matrix $(\gamma_{ij}) \in M_n(\mathbb{R})$. Then there is a compact simply connected T^n-bundle $\widetilde{W}$ over N whose total space is SKT.

Moreover,

(i) *no complex structure on $\widetilde{W}$ compatible with the fibration to N is of Kähler type;*

(ii) *if $b_2(N) = n$, then the topological manifold $\widetilde{W}$ admits no Kähler metric.*

Proof

We take W to be the twist of $N \times T^n$, where the flat Kähler metric on $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is given by (γ_{ij}) with respect to the standard generators with a compatible complex structure: from the classification of quadratic forms $(\gamma_{ij}) = Q^T Q$, and we may then take $I = Q^{-1} I_0 Q$, where I_0 is the standard complex structure on $\mathbb{R}^n = \mathbb{C}^k$. Then the discussion above shows that W is SKT. Topologically W is a principal torus bundle over N with Chern classes $[F_i]$. By Lemma 4.7 below, its universal cover $\widetilde{W}$ is itself a T^n-bundle over N and thus a compact simply connected SKT manifold.

For part (i), the projection $\widetilde{W} \to N$ is holomorphic, with complex fibers that are homologous to zero. But any complex submanifold of a Kähler manifold is nonzero in homology, so the complex structure on $\widetilde{W}$ cannot be of Kähler type.

In part (ii) $b_2(N) = n$, Lemma 4.7 below gives $b_2(\widetilde{W}) = b_2(N) - n = 0$ and so $\widetilde{W}$ admits no Kähler metric. □

Example 4.6

Let N_0 be a simply connected projective Kähler manifold of real dimension 4. Fix an embedding $N_0 \subset \mathbb{C}\mathrm{P}(r)$. Then a generic linear subspace $\mathbb{P}(V) \subset \mathbb{C}\mathrm{P}(r)$ of complex dimension $r-2$ meets N_0 transversely at a finite number of points $p_1, \dots, p_d$. Choose homogeneous coordinates $[z_0, \dots, z_r]$ on $\mathbb{C}\mathrm{P}(r)$ so that $\mathbb{P}(V) = (z_0 = 0 = z_1)$. Then the blowup $\widehat{\mathbb{C}\mathrm{P}(r)}$ of $\mathbb{C}\mathrm{P}(r)$ along $\mathbb{P}(V)$ is

$$\widehat{\mathbb{C}\mathrm{P}(r)} = \{([z_0, \dots, z_r], [w_0, w_1]) : z_0 w_1 = z_1 w_0\} \subset \mathbb{C}\mathrm{P}(r) \times \mathbb{C}\mathrm{P}(1).$$

Let π_1 and π_2 denote the projections to the first and second factors of $\mathbb{C}\mathrm{P}(r) \times \mathbb{C}\mathrm{P}(1)$. Then $N_1 = \pi_1^{-1}(N_0)$ is the blowup of N_0 at $p_1, \dots, p_d$, and $f = \pi_2|_{N_1}$ is a nonconstant holomorphic map from N_1 onto $\mathbb{C}\mathrm{P}(1)$. As above, f defines $F_1 = \Phi(f) \in \Omega^{1,1}_{\mathbb{Z}}(N_1)$, nonzero in cohomology with $F_1^2 = 0$.

Iterating this construction, we find that for any $n = 2k$ there is a multiple blowup N of N_0 that satisfies the hypotheses of Proposition 4.5 in the form

$\sum_{i=1}^n F_i^2 = 0$; hence, N is the base of a simply connected SKT manifold of dimension $n+4$.

Here is the topological information needed in the proof of Proposition 4.5.

LEMMA 4.7

Let N be a compact simply connected manifold; then $H^2(N,\mathbb{Z})$ has no torsion. Suppose that $Y \to N$ is a T^n-bundle with Chern classes $\alpha_1, \ldots, \alpha_n$ linearly independent over $\mathbb{R}$. Then the universal cover $\widetilde{Y}$ of Y is a T^n-bundle over N with Chern classes $\beta_1, \ldots, \beta_n$ that form part of an integral basis for $H^2(N,\mathbb{Z})$. The second Betti numbers satisfy $b_2(\widetilde{Y}) = b_2(N) - r$.

Proof

By the universal coefficient theorem the torsion part of $H^2(N,\mathbb{Z})$ is isomorphic to the torsion part of $H_1(N,\mathbb{Z})$. However, $H_1(N)$ is zero for N simply connected, so $H^2(N,\mathbb{Z})$ is torsion free.

Regard $H^2(N,\mathbb{Z})$ as the integral lattice Λ in $H^2(N,\mathbb{R})$, and put $\Lambda_\alpha = \mathrm{Span}_{\mathbb{R}}\{\alpha_1,\ldots,\alpha_n\} \cap \Lambda$. This is a subgroup of the finitely generated free Abelian group Λ, so Λ_α is free. Furthermore, from [30, Theorem 4.2], there is a basis $\beta_1,\ldots,\beta_m$ for Λ and positive integers $k_1,\ldots,k_n$ such that $k_1\beta_1,\ldots,k_n\beta_n$ is a basis for Λ_α. In our case, for each $i \leqslant n$, the generator β_i of Λ is also a real linear combination of $\alpha_1,\ldots,\alpha_n$, so β_i in fact lies in Λ_α and $k_i = 1$.

Let Y_β be a T^n-bundle over N with Chern classes $\beta_1,\ldots,\beta_n$. As the Chern classes of Y are integer combinations of those of Y_β, we may choose Y_β as a covering space of Y. We claim that Y_β is the universal cover $\widetilde{Y}$ of Y. Since T^n is Abelian, the exact homotopy sequence for $Y_\beta \to N$ implies that $\pi_1(Y)$ is Abelian and so equal to $H_1(Y_\beta,\mathbb{Z})$. The free part of this is then $H^1(Y_\beta,\mathbb{Z})$, and the torsion part is the torsion part of $H^2(Y_\beta,\mathbb{Z})$. These cohomology groups may be computed via the Leray spectral sequence which has $E_2^{p,q} = H^p(N,\mathbb{Z}) \otimes H^q(T^n,\mathbb{Z})$. Choosing appropriate generators γ_i for $H^1(T^n)$, we have that $d_2\colon E_2^{0,q} \to E_2^{2,q-1}$ is given by $d_2\gamma_{i_1,\ldots,i_q} = \sum_{j=1}^q (-1)^{j+1}\beta_{i_j} \otimes \gamma_{i_1,\ldots,\hat{i_j},\ldots,i_q}$. Independence of the β_i implies that these maps are injective for $q \geqslant 1$. In particular, $E_3^{0,2} = \{0\}$ and d_3 is zero on $E_3^{p,q}$ for $p+q \leqslant 2$, and thus the spectral sequence stabilizes at level E_3 for these terms. This gives $H^1(Y_\beta,\mathbb{Z}) \cong E_3^{1,0} + E_3^{0,1} = 0$, $H^2(Y_\beta,\mathbb{Z}) \cong \sum_{j=0}^2 E_3^{j,2-j} = E_3^{0,2} \cong \mathrm{Span}_{\mathbb{Z}}\{\beta_{n+1},\ldots,\beta_m\}$. We conclude that Y_β is simply connected, so $Y_\beta = \widetilde{Y}$, and that $b_2(\widetilde{Y}) = b_2(Y_\beta) = m - r = b_2(N) - r$. □

4.2. Noninstanton twists

If M is a torus, then M can be repeatedly twisted to produce nilmanifolds. Indeed every nilmanifold may be produced this way (cf. [39]). From the results of Fino, Parton, and Salamon [13] for SKT structures on 6-dimensional nilmanifolds, we can see that noninstanton twists are necessary to produce all such examples. We now demonstrate that there are simply connected SKT manifolds obtainable from noninstanton twists over K3 surfaces.

Let N be a K3 surface. Yau's proof of the Calabi conjecture shows that N admits hyper-Kähler metrics $(g, \omega_I, \omega_J, \omega_K)$. From Looijenga's Torelli theorem in [27] for period maps of K3 surfaces there is a hyper-Kähler metric $\tilde{g}$ whose Kähler forms $\tilde{\omega}_I, \tilde{\omega}_J, \tilde{\omega}_K$ have integral periods. The corresponding Kähler classes are necessarily linearly independent in cohomology.

Consider $M = N \times T^2$ as above, with the product Kähler metric for $T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ standard and any Kähler metric on N which is Hermitian with respect to I from the hyper-Kähler triple. We twist M using $(F_1, F_2) = (F_1^0 + \tilde{\omega}_J, F_2^0 + \tilde{\omega}_K)$ with F_i^0 of type $(1, 1)$ with respect to I. Note that $\tilde{\omega}_J + i\tilde{\omega}_K$ is of type $(2, 0)$ with respect to I (with the conventions given before Proposition 3.8), so the almost complex structure I_W on the resulting twist W is integrable by Lemma 3.9. The condition for W to be SKT is now

$$d(IF_1) \wedge X_1^\flat + d(IF_2) \wedge X_2^\flat + F_1 \wedge IF_1 + F_2 \wedge IF_2 = 0.$$

We have $(IF_1, IF_2) = (F_1^0 - \tilde{\omega}_J, F_2^0 - \tilde{\omega}_K)$, so find $d(IF_i) = 0$ and the SKT condition reduces to $(F_1^0)^2 + (F_2^0)^2 = \tilde{\omega}_J^2 + \tilde{\omega}_K^2$.

As $\tilde{\omega}_I^2 = \tilde{\omega}_J^2 = \tilde{\omega}_K^2$ for any hyper-Kähler metric in dimension 4, we can solve the SKT equations by taking $F_i^0 = \varepsilon_i \tilde{\omega}_I$, $\varepsilon_i \in \{\pm 1\}$. The resulting T^2-twists are then SKT with finite fundamental group, and their universal covers $\widetilde{W}$ are the promised manifolds. As above, the complex structure admits no compatible Kähler metric.

Using $b_2(N) = 22$, Lemma 4.7 gives $b_2(\widetilde{W}) = 20$ (a similar computation gives $b_3(\widetilde{W}) = 42$). However, we can be more precise. Recall from Lemma 4.7 that the Chern classes β_1, β_2 of $\widetilde{W}$ extend to a basis of $H^2(N, \mathbb{Z})$. Now as N is a simply connected oriented 4-manifold its intersection form is unimodular. This implies that every homomorphism $H^2(N, \mathbb{Z}) \to H^4(N, \mathbb{Z}) = \mathbb{Z}\eta$ may be realized by cup products with elements of $H^2(N, \mathbb{Z})$, and in particular that there exist elements $a_1, a_2 \in H^2(N, \mathbb{Z})$ with $a_i \cup \beta_j = \delta_{ij}\eta$. By [22, Proposition 12] the cohomology of $\widetilde{W}$ is now torsion free, and the classification [21] of such simply connected 6-manifolds with free circle action shows that $\widetilde{W}$ is diffeomorphic to $20(S^2 \times S^4) \# 21(S^3 \times S^3)$.

4.3. Complex volume forms

If (M, I) is a complex manifold with trivial canonical bundle, it is natural to ask which twists W also enjoy this property.

PROPOSITION 4.8

Suppose the twist (W, I_W) of (M, I) via ξ, F and a is complex. If (M, I) carries an invariant complex volume form Θ, then the $\mathcal{H}$-related form Θ_W is a complex volume *form on* (W, I_W) *if and only if*

$$a^{-1}\xi \lrcorner F^{1,1} = 0,$$

where $F^{1,1} = (F + IF)/2$ *is the* $(1, 1)$*-part of* F.

Proof
Let $\dim_{\mathbb{C}} M = m$. A section of $\Lambda^{m,0}$ is holomorphic only if it is closed. We have

$$d\Theta_W \sim_{\mathcal{H}} d\Theta - a^{-1}F \wedge \xi \lrcorner \Theta = -a^{-1}F \wedge \xi \lrcorner \Theta,$$

so we need to determine when the right-hand side vanishes. This is a pointwise computation and we may thus use Lemma 3.9.

In the notation of Lemma 3.9, we wish to compute $\sum_{i=1}^{n} F_i \wedge X_i \lrcorner \Theta$. Note that $X \lrcorner \Theta$ is of type $(m-1, 0)$ and that $IX \lrcorner \Theta = iX \lrcorner \Theta$. Now, for $k > 2s$, we have $F_k = F_k^{(1,1)}$, whereas for $j \leqslant s$, $F_{2j-1} \wedge X_{2j-1} \lrcorner \Theta + F_{2j} \wedge X_{2j} \lrcorner \Theta = F_{(1)} \wedge X \lrcorner \Theta + F_{(2)} \wedge IX \lrcorner \Theta = F_{(c)} \wedge X \lrcorner \Theta = F_{(c)}^{1,1} \wedge X \lrcorner \Theta$, as $F_{(c)}^{2,0} \wedge X \lrcorner \Theta \in \Lambda^{m+1,0} = \{0\}$. However, for any $(1, 1)$-form F we have $F \wedge \Theta = 0$, so $(X \lrcorner F) \wedge \Theta + F \wedge (X \lrcorner \Theta) = 0$. This gives

$$\sum_{i=1}^{n} F_i \wedge X_i \lrcorner \Theta = \sum_{i=1}^{n} F_i^{1,1} \wedge X_i \lrcorner \Theta = -\Big(\sum_{i=1}^{n} X_i \lrcorner F_i^{1,1}\Big) \wedge \Theta.$$

The result follows from the fact that $\cdot \wedge \Theta \colon \Lambda^1 \to \Lambda^{m,1}$ is an $\mathbb{R}$-linear isomorphism. $\square$

5. Hypercomplex and HKT geometry

We now turn geometries with multiple complex structures. An *almost hypercomplex structure* on a manifold M is a triple of tangent bundle endomorphisms $I, J, K \in \operatorname{End}(TM)$ satisfying the identities

$$I^2 = -1 = J^2 = K^2, \qquad IJ = K = -JI.$$

The first identities say that I, J, and K are almost complex structures. If these three are integrable complex structures, we then have a *hypercomplex structure* on M. A metric g satisfying $g(IX, IY) = g(X, Y) = g(JX, JY) = g(KX, KY)$ for all $X, Y \in TM$ is said to be *almost hyper-Hermitian*.

Definition 5.1
An almost hyper-Hermitian structure (g, I, J, K) is *hyper-Kähler with torsion* or HKT if

$$I d\omega_I = J d\omega_J = K d\omega_K, \tag{5.1}$$

where $\omega_I(X, Y) = g(IX, Y)$, and so forth.

In [29] it was shown that HKT structures are always hypercomplex, a condition that was previously included in the definition of HKT (cf. [23]). Equation (5.1) is now the condition that the Bismut connections for (g, I), (g, J), and (g, K) agree.

The results of Sections 3 and 4 may now be applied to these structures. First, a direct consequence of Proposition 4.2 and equation (5.1) is the following statement.

PROPOSITION 5.2
Suppose M is an HKT manifold with twist data (ξ, F, a) and that the action ξ preserves the HKT structure. Then the twist W of M by (ξ, F, a) is HKT if and only if

$$a^{-1} I F \wedge \xi^\flat = a^{-1} J F \wedge \xi^\flat = a^{-1} K F \wedge \xi^\flat. \tag{5.2}$$

□

5.1. Instanton HKT twists
The condition (5.2) is satisfied by any *instanton*, meaning $F \in S^2 E \otimes \mathfrak{a}_P$, where $S^2 E = \Lambda_I^{1,1} \cap \Lambda_J^{1,1} \cap \Lambda_K^{1,1}$. This immediately gives many examples.

Consider the following building blocks: we need both integral Hamiltonian instantons and torus symmetries.

(i) *Tori* $T^{4k} = \mathbb{H}^k/\Lambda$, $\mathbb{H} = \mathbb{R}^4$, $\Lambda \cong \mathbb{Z}^{4k}$ a lattice. These are hyper-Kähler, so HKT. They carry no Hamiltonian instantons, but supply symmetries.

(ii) *Compact irreducible hyper-Kähler manifolds*. Passing to the universal cover, we may take these to be simply connected. These give a rich supply of integral instantons. One family of examples are provided by K3 surfaces M; here the instanton condition is just that F be self-dual; by the Torelli theorem there are examples where the integral instantons form a lattice of rank 19. Notice that these have no Killing vectors, since they are compact Ricci-flat and irreducible.

(iii) *Compact groups* and related homogeneous spaces. If G is a compact, simple Lie group, then there is a torus T^r such that $G \times T^r$ equipped with a biinvariant metric is HKT (see [23]). The hypercomplex structure was determined by Joyce [25] and the minimal value of r may be found in [36, Table 1]. In particular, $\mathrm{SU}(2n + 1)$ carries an HKT structure for all n. These structures admit HKT deformations with torus symmetry (cf. [34]). Also, some of these homogeneous

HKT structures descend to homogeneous spaces $(G/H)\times T^s$ (see [32]). Again, the spaces $G \times T^r$ carry no Hamiltonian instantons, but do supply symmetries.

(iv) *Squashed* 3-*Sasaki structures*. A Riemannian manifold $(\mathcal{S}, h_0)$ of dimension $4n-1$ is 3-Sasaki if the cone $(\mathcal{S} \times \mathbb{R}, g_0 = dr^2 + r^2h_0)$ is hyper-Kähler with complex structures invariant under $X = r\partial/\partial r$. The metric h_0 is then Einstein with positive scalar curvature, so if $\mathcal{S}$ is compact then $\pi_1(\mathcal{S})$ is finite. Passing to the universal cover we may assume $\mathcal{S}$ is simply connected. Rescaling g_0 with different weights along and transverse to the quaternionic span of X, one may produce an HKT metric $g = dt^2 + h$ that descends to $\mathcal{S} \times S^1$ such that X acts as a triholomorphic isometry (see [33]). Galicki and Salamon [15] showed that harmonic 2-forms for h_0 are instantons and orthogonal to the quaternionic span of X, so these are also instantons for the hypercomplex structure on $\mathcal{S} \times S^1$. Thus any $\mathcal{S}$ with $b_2(\mathcal{S}) > 0$ provides integral Hamiltonian instantons on $\mathcal{S} \times S^1$. Many such examples of 3-Sasaki manifolds have been constructed by Boyer, Galicki, and colleagues [7]; in particular, there are inhomogeneous examples in dimension 7 with arbitrarily large $b_2(\mathcal{S})$. Moreover, such $\mathcal{S}$ often admit nontrivial isometries preserving both the HKT structure of $\mathcal{S} \times S^1$ and h_0 and, hence, the above-mentioned instantons. Thus, the manifolds $\mathcal{S} \times S^1$ provide a rich source of symmetries too, including free actions that have $\xi \lrcorner F$ nonzero.

The hypotheses of the following theorem are satisfied by an HKT space M that is a product of manifolds of the four types above and the additional torus factors needed in types (iii) and (iv). The existence of an appropriate Hamiltonian instanton F is guaranteed if there are sufficiently many factors of type (ii) or type (iv) with large $b_2(\mathcal{S})$.

We say that a group action on a product $A \times B$ *projects transitively* to B if the action preserves the product structure, so $g \cdot (a, b) = (g \cdot a, g \cdot b)$, and the induced action on the second factor B is transitive.

THEOREM 5.3
Let $M = M_0 \times T^p$ be a compact HKT manifold of dimension $4m$, $p = b_1(M)$. Suppose T^n acts freely on M preserving the HKT structure, projecting transitively to T^p and preserving a Hamiltonian instanton $F \in \Omega^2_{\mathbb{Z}}(M)$ of rank n in $H^2(M, \mathfrak{t}_n)$. Then there is a finite cover $\widetilde{W}$ of a twist W of M via F that is a compact simply connected HKT manifold.

Here the rank of F in $H^2(M, \mathfrak{t}_n)$ is the rank of $[F]$ as a linear map $H^2(M,\mathbb{R}) \to \mathfrak{t}_n^* \cong \mathbb{R}^n$.

Proof

From the discussion of the building blocks we see that the Hamiltonian instanton F is the pullback of a form F_0 on M_0. Then the twisting bundle $P \to M = M_0 \times T^p$ is $P_0 \times T^p$, where $P_0 \to M_0$ is a principal T^n-bundle with curvature F_0. As $[F]$ has rank n, the fundamental group of P_0 is finite. The lifted T^n-action $\mathring{T}_M$ is free and projects transitively to T^p. In particular, $W = P/\mathring{T}_M$ is diffeomorphic to a quotient of P_0 by the free action of a compact group and so $\pi_1(W)$ is finite. However, by Proposition 5.2, the instanton condition also ensures that W is HKT. □

For example, this theorem gives HKT metrics not only on torus bundles of rank 4, 8, 12, and 16 over a single K3 surface, but also on certain bundles with fiber $\mathrm{SU}(n+1)$ or $\mathcal{S} \times S^1$ for a 3-Sasaki manifold $\mathcal{S}$.

Note that Verbitsky [40] constructs HKT metrics on vector bundles out of instanton connections, but the resulting compact quotients are never simply connected.

5.2. *Special Obata holonomy*

If (M^{4m}, I, J, K) is any hypercomplex manifold, then there is a unique torsion-free connection ∇^{Ob} preserving the complex structures (see [31]). This implies that the holonomy of ∇^{Ob} lies in the group $\mathrm{GL}(m, \mathbb{H})$. Verbitsky [42] showed, using a version of Hodge theory for HKT manifolds, that if M is HKT and the canonical bundle is trivial, then the holonomy reduces to the subgroup $\mathrm{SL}(m, \mathbb{H})$. The following is a simpler result that is sufficient for our purposes.

PROPOSITION 5.4

Let (M^{4m}, I, J, K) be a hypercomplex manifold with trivial canonical bundle. If M admits a complex volume form Θ with respect to I that satisfies $J\Theta = \overline{\Theta}$, then the Obata connection has holonomy in $\mathrm{SL}(m, \mathbb{H})$.

Proof

As ∇^{Ob} preserves the complex structure I, we have $\nabla^{\mathrm{Ob}}\Theta = \theta \otimes \Theta$ for some 1-form $\theta \in \Omega^1(M, \mathbb{C})$. Now ∇^{Ob} is torsion free, so $d\Theta$ is the alternation of $\nabla^{\mathrm{Ob}}\Theta$ and $0 = d\Theta = \theta \wedge \Theta = \theta^{0,1} \wedge \Theta \in \Omega^{2m,1}(M)$, where type decompositions are with respect to I. We conclude that $\theta^{0,1} = 0$.

Now ∇^{Ob} preserves J too, so $\theta(X)J\Theta = J\nabla^{\mathrm{Ob}}_X\Theta = \nabla^{\mathrm{Ob}}_X(J\Theta) = \nabla^{\mathrm{Ob}}_X\overline{\Theta} = \overline{\theta}(X)J\Theta$ for any tangent vector X. This shows that θ is a real 1-form, and from $\theta^{0,1} = 0$ we find that $\theta = 0$. Thus, Θ is Obata parallel and $\Theta \wedge \overline{\Theta}$ is a parallel real volume form for ∇^{Ob}. □

THEOREM 5.5
The spaces of Theorem 5.3 built using factors of types (i) and (ii) are simply connected hypercomplex $4m$*-manifolds with holonomy contained in* $\mathrm{SL}(m, \mathbb{H})$.

Proof
The given factors are hyper-Kähler, so $\Theta = (\omega_J + i\omega_K)^m$ is a complex volume form on $M = M_0 \times T^p$. In our case M_0 is a product of compact irreducible hyper-Kähler manifolds; the twisting form F is the pullback of a form on M_0, whereas the group action of T_M is simply that of the factor T^p. So $a^{-1}\xi \lrcorner F = 0$, and Proposition 4.8 implies that $\widetilde{W}$ has holomorphically trivial canonical bundle with complex volume form $\Theta_W = (\omega_J^W + i\omega_K^W)^m$. But $J\Theta_W = (J\omega_J^W + iJ\omega_K^W)^m = (\omega_J^W - i\omega_K^W)^m = \overline{\Theta_W}$. The result now follows from Proposition 5.4. □

In particular, this constructs examples on instanton torus bundles over products of K3 surfaces.

The other building blocks of types (iii) and (iv) described above typically do not have a holomorphically trivial canonical bundle and so cannot be used in the construction of Theorem 5.5.

5.3. Noninstanton HKT twists

Let us note that HKT twists of the type in Proposition 5.2 need not come from instantons, at least in the noncompact case.

Example 5.6
Let $M = \mathbb{R}_{>0} \times T^3 \subset \mathbb{H}/(\mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k)$ with the flat hyper-Kähler structure. Let $X_0 = \partial/\partial x^0$ be the generator of the first factor, so that $X_1 = IX_0$, $X_2 = JX_0$, and $X_3 = KX_0$ generated the three circle factors of T^3. Let b_0, b_1, b_2, b_4 be the (unit length) dual 1-forms. Put $F_I = b_0 \wedge b_1 = b_{01}$, $F_J = b_{02}$, and $F_K = b_{03}$. Then $X_1 \lrcorner F_I = -b_0 = -dx^0 = X_2 \lrcorner F_J = X_3 \lrcorner F_K$, with all other $X_i \lrcorner F_A$ zero. We may thus take $a = -\operatorname{diag}(x^0, x^0, x^0)$. Then

$$\begin{aligned} a^{-1}IF \wedge \xi^\flat &= -\tfrac{1}{x^0}(IF_I \wedge b_1 + IF_J \wedge b_2 + IF_K \wedge b_3) \\ &= -\tfrac{1}{x^0}(0 + b_{13} \wedge b_2 - b_{12} \wedge b_3) \\ &= \tfrac{2}{x^0} b_{123} = a^{-1}JF \wedge \xi^\flat = a^{-1}KF \wedge \xi^\flat. \end{aligned}$$

Thus we may twist to obtain a new HKT metric, even though $F \notin S^2E \otimes \mathfrak{a}_P$ since $JF_I \neq F_I$.

5.4. Hypercomplex manifolds that are not HKT

Proposition 3.8 directly tells when hypercomplex structures are preserved by the twist construction.

PROPOSITION 5.7
Suppose that M is hypercomplex with twist data (ξ, F, a) and that the action ξ preserves the hypercomplex structure. Then the twist W is hypercomplex if and only if

$$\mathcal{L}_I \mathcal{F} = \mathcal{F} = \mathcal{L}_J \mathcal{F} = \mathcal{L}_K \mathcal{F}.$$

□

Let us use this to give examples of compact simply connected hypercomplex manifolds that are not HKT.

Let N be a hyper-Kähler K3 surface for which the three Kähler forms ω_I, ω_J, and ω_K have integral periods (cf. Section 4.2). Take M to be the product $N \times T^4$ with the hyper-Kähler torus $T^4 = (S^1)^4 = \mathbb{H}/\mathbb{Z}^4$, and let X_0, $X_1 = IX_0$, $X_2 = JX_0$, $X_3 = KX_0$ be vector fields generating the four circles. Let ω_0 be any nonzero self-dual element of $\Omega^2_{\mathbb{Z}}(N)$. Then ω_0 is of type $(1, 1)$ with respect to each complex structure.

Now twist M by $\mathcal{F} = X_0 \otimes F_0 + X_1 \otimes F_I + X_2 \otimes F_J + X_3 \otimes F_K$, where $F_A = \pi_N^* \omega_A$ is the pullback of ω_A. Since $X_i \lrcorner F_A = 0$ we may take the twisting function a to be the identity matrix. The resulting twist W has a finite fundamental group and so its universal cover $\widetilde{W}$ is simply connected. We have

$$\begin{aligned}\mathcal{F} &= X_0 \otimes F_0 + X_1 \otimes F_I + \operatorname{Re}\big((X_2 - iX_3) \otimes (F_J + iF_K)\big) \\ &\in T \otimes \Lambda_I^{1,1} + [\![T_I^{1,0} \otimes \Lambda_I^{2,0}]\!],\end{aligned}$$

so $\mathcal{L}_I \mathcal{F} = \mathcal{F}$ and similarly for I, J, and K. Thus, W is hypercomplex by Proposition 5.7. However, Proposition 5.2 shows that the geometry on W is not HKT, since M is HKT and

$$\begin{aligned}a^{-1} IF \wedge \xi^\flat &= F_0 \wedge b_0 + F_I \wedge b_1 - F_J \wedge b_2 - F_K \wedge b_3 \\ &\neq a^{-1} JF \wedge \xi^\flat = F_0 \wedge b_0 - F_I \wedge b_1 + F_J \wedge b_2 - F_K \wedge b_3,\end{aligned} \tag{5.3}$$

where $b_i = X_i^\flat$.

We claim that the hypercomplex structure on $\widetilde{W}$ admits no compatible HKT metric. Suppose for a contradiction that g_0 is an HKT metric on $(\widetilde{W}, I, J, K)$. As the hypercomplex structure on W is constructed via the twist construction, it is T^4-invariant. Averaging g_0 over $\pi_1(W)$ and then the principal T^4-action gives a metric g_1 on W that is still HKT since the HKT condition is linear and invariant under triholomorphic pullbacks. Untwisting W gives the original product hypercomplex structure on $N \times T^4$ and a hyper-Hermitian metric g_2 on $N \times T^4$ which is T^4-invariant. As all hyper-Hermitian metrics on a 4-dimensional vector space are proportional, we write

$$g_2 = f\, g_N + h \sum_{i=0}^{3} b_i^2 + 2 \sum_{i=0}^{3} b_i \vee \alpha_i,$$

where g_N is the hyper-Kähler metric on N, $\alpha_i \in \Omega^1(N)$, $\alpha_1 = I\alpha_0$, and so forth, and $f, h \in C^\infty(N)$ are positive (pullback signs have been omitted).

Let $\omega_A^{(i)}$ denote the Kähler forms of g_i. Then

$$\begin{aligned} I_W\, d\omega_I^{(1)} \sim_{\mathcal{H}} I df \wedge \omega_I + I dh \wedge (b_{01} + b_{23}) \\ -2 \sum_{i=0}^{3} b_i \wedge I d\alpha_i + \sum_{i=0}^{3} I F_i \wedge (h\, b_i + \alpha_i). \end{aligned} \tag{5.4}$$

For W HKT, we have $I_W\, d\omega_I^{(1)} = J_W\, d\omega_J^{(1)} = K_W\, d\omega_K^{(1)}$. Using (5.4) and considering the coefficient of b_{01} gives $I dh = 0$, so h is constant. Looking at the coefficient of b_i, we find

$$\begin{aligned} 2(I d\alpha_i) - h\, I F_i &= 2(J d\alpha_i) - h\, J F_i \\ &= 2(K d\alpha_i) - h\, K F_i. \end{aligned} \tag{5.5}$$

For $i = 0$, this gives $I d\alpha_0 = J d\alpha_0 = K d\alpha_0$, so $d\alpha_0$ is self-dual on the compact space N and therefore zero. As $b_1(N) = 0$, we may write $\alpha_0 = d\phi$, for some $\phi \in C^\infty(N)$.

Equation (5.5) for $i = 1$, is

$$2(I d\alpha_1) - h\omega_I = 2(J d\alpha_1) + h\omega_I = 2(K d\alpha_1) + h\omega_I. \tag{5.6}$$

This first gives that $J d\alpha_1 = K d\alpha_1$, so $d\alpha_1 \in \Lambda_I^{1,1}$. Writing $d\alpha_1 = \beta + \lambda\omega_I$, with $\beta \in \Omega_+^2(N)$, gives $J d\alpha_1 = \beta - \lambda\omega_I$, and (5.6) implies $\lambda = h/2$, which is constant.

However, $\alpha_1 = I\alpha_0 = I d\phi$ and so $\lambda = \Delta_N \phi$, where Δ_N is the Laplacian of the hyper-Kähler metric g_N. Since Δ_N has image orthogonal to the constant functions, we conclude that $\lambda = 0$ and hence $h = 0$, contradicting the positive definiteness of g_2. Thus g_1 cannot be HKT and the hypercomplex structure on $\widetilde{W}$ admits no compatible HKT metric.

We summarize as follows.

THEOREM 5.8
There are compact simply connected hypercomplex 8*-manifolds that admit no compatible HKT metric. Moreover, these exist with Obata holonomy contained in* $\mathrm{SL}(2, \mathbb{H})$.

Proof

It remains to prove the final assertion. However, the twist W constructed above starts from a hyper-Kähler manifold and has $a^{-1}\xi \lrcorner F^{1,1} = 0$, so as in the proof of Theorem 5.5 the Obata holonomy reduces. □

THEOREM 5.9

There are nontrivial compact simply connected hypercomplex manifolds in all dimensions $4m \geqslant 8$ *that admit no compatible HKT metric. Furthermore, examples exist with holonomy in* $\mathrm{SL}(m, \mathbb{H})$.

The examples constructed are torus bundles over a product base and by "nontrivial" we mean that the structure does not split as a product of torus bundles over a product of factors of the base.

Proof

Let W_0 be the 8-dimensional example constructed above. Untwist the symmetry X_0 by F_0 to get the hypercomplex manifold $W_1 = S^1 \times B_7$. Let M_2 be a product of $(m-2)$-factors that are each K3 surfaces with a fixed choice of nonzero integral self-dual 2-form $F_{(i)}$. Twisting the S^1 factor of W_1 by $F = F_0 + F_{(1)} + \cdots + F_{(m-2)}$ we obtain a $4m$-dimensional hypercomplex manifold W_2 with finite fundamental group and Obata holonomy in $\mathrm{SL}(m, \mathbb{H})$. The universal cover $\widetilde{W}_2$ of W_2 is the required example.

The hypercomplex manifold $\widetilde{W}_2$ admits no compatible HKT metric, since any potential HKT metric may be averaged so that it descends to a torus invariant HKT metric on W_2 and then twisted to an HKT metric on $M \times W_0$. As W_0 is a hypercomplex submanifold of $M \times W_0$ such an HKT metric would restrict to an HKT metric on W_0 itself, but that is a contradiction. Thus $\widetilde{W}_2$ carries no compatible HKT metric. □

Using the squashed 3-Sasaki building blocks of Section 5.1 one may obtain examples with holomorphically nontrivial canonical bundle and so Obata holonomy not in $\mathrm{SL}(m, \mathbb{H})$.

In dimension 8 we can prove a more general nontriviality result.

THEOREM 5.10

Any compact hypercomplex 8*-manifold that does not admit an HKT metric is not a nontrivial product of smaller dimensional hypercomplex manifolds.*

Proof

Suppose that M^8 is such a product $N_1 \times N_2$ with N_i hypercomplex. Then each N_i has dimension 4. However, Boyer [6] showed that any compact hypercomplex 4-manifold is T^4, a K3 surface, or $S^3 \times S^1$. But each of these examples admits a compatible HKT metric, and the product structure on $M = N_1 \times N_2$ is then HKT, contradicting the hypotheses. □

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