Invariant Geometric Structures and Chern numbers of G_2 Flag Manifolds

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Outline

1. Generalized flag manifolds
2. G_2 and its flag manifolds
3. The twistor space
4. The quadric

Generalized flag manifolds

Definition

A *flag* is a strictly increasing sequence

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V \qquad\qquad k \leq n$$

of subspaces of an n-dimensional vector space V (e.g. $\mathbb{C}^n$). Set $d_i = \dim V_i$; then $(d_1, \ldots, d_{k-1})$ is the signature of the flag. If $d_i = i$, we call the flag *complete* and otherwise we have a *partial* flag.

Generalized flag manifolds

Example (Flag manifolds of flags in $\mathbb{C}^n$)

a) The Grassmannian of k-planes parametrizes flags of signature (k):

$$\mathrm{Gr}_k(\mathbb{C}^n) \cong \frac{U(n)}{U(k) \times U(n-k)} \cong \frac{SU(n)}{S(U(k) \times U(n-k))}$$

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b) The *complete* flag manifold is $U(n)/T^n \cong SU(n)/T^{n-1}$.

Generalized flag manifolds

In general, any *flag manifold* of flags in $\mathbb{C}^n$ is of the form

$$\frac{SU(n)}{S(U(r_1) \times \ldots U(r_k))}$$

where $\{r_1, \ldots, r_k\}$ is an ordered partition of n. The isotropy subgroup is the centralizer of a torus $T^{k-1} \subset SU(n)$.

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This naturally generalizes to:

Definition

A *generalized flag manifold* is a homogeneous space of the form $G/C(T)$, where G is a compact, connected and semisimple Lie group, and $C(T)$ is the centralizer of a torus $T \subset G$.

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Theorem (Borel [1], Koszul [2], Matsushima [3])

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Furthermore, certain examples are known to carry multiple invariant almost complex structures with distinct Chern numbers (cf. Borel & Hirzebruch [4]).

Generalized flag manifolds

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Study generalized flag manifolds *geometrically* and give a concrete interpretation of their invariant geometric structures.

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This program was carried out for $SU(n)/S(U(n-2) \times U(1) \times U(1))$ by Kotschick & Terzić [6].

G_2 and its flag manifolds

The exceptional Lie group G_2 is the automorphism group of the octonions $\mathbb{O}$. Its generalized flag manifolds fit into the following diagram:

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 & G_2/U(2)_- = Q \arrow[dl, "\mathbb{CP}^2"'] \arrow[dr, "\mathbb{CP}^1"'] & & G_2/U(2)_+ = Z \arrow[dl, "\mathbb{CP}^1"] \\
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The twistor space

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Definition

A quaternionic Kähler (QK) manifold is an (oriented) Riemannian manifold of dimension $4n$, $n \geq 2$, whose holonomy group is contained in the subgroup $Sp(n) \cdot Sp(1)$ of $SO(4n)$.

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Reduced holonomy implies curvature restrictions: QK manifolds are Einstein.

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The total space $S(E)$ of the sphere bundle of E is called the *twistor space* over M.

A point $z = \alpha I + \beta J + \gamma K \in S(E)$ corresponds to an orthogonal complex structure on $T_{\pi(z)}M$.

The twistor space

Theorem (Salamon [7], Bérard-Bergery [8])

(i) *$S(E)$ admits a natural (integrable) complex structure.*

(ii) *If M has positive scalar curvature, then $S(E)$ admits a Kähler-Einstein metric with positive scalar curvature.*

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Proof of (i).

$TS(E) \cong \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{H}_z \cong T_{\pi(z)}M$; we define a complex structure $J = J_v \oplus J_h$ as follows: $J_v = J^{\mathrm{std}}_{\mathbb{C}\mathrm{P}^1}$, while $z \in S(E)$ *is* the image of J_h under the identification $\mathcal{H}_z \cong T_{\pi(z)}M$. The Nijenhuis tensor is explicitly shown to vanish. $\square$

The twistor space: Rigidity of the Kähler structure

Using the Gysin sequence we determine the cohomology ring and Pontryagin classes of $Z = G_2/U(2)_+$. Additively, the cohomology is that of $\mathbb{C}\mathrm{P}^5$, but the ring structure is different.

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This is the exact analog of rigidity results of Hirzebruch & Kodaira [9] (and Yau [10]) for $\mathbb{CP}^n$, and Brieskorn [11] for Q_n ($n \geq 3$).

The twistor space: Rigidity of the Kähler structure

Sketch of Proof.

We have $c_1(X) = d \cdot g_2$, where g_2 is the positive generator of $H^2(X;\mathbb{Z})$. The Chern number $c_1 c_4[X]$ is fixed by $h^{p,q}(X) = h^{p,q}(\mathbb{CP}^5)$, hence $c_1 c_4[X] = 90$. Thus, d divides 90. Every possibility except $d = 3$ is ruled out case-by-case, using the Pontryagin classes as well as the Todd genus, which is also determined by the Hodge numbers.

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Once $c_1(X)$ is fixed, Mukai's classification of Fano manifolds of coindex 3 finishes the proof. $\square$

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Proposition

The Chern numbers of the two invariant almost complex structures on the twistor space are:

Chern Number	Z	N
c_5	6	6
c_1^5	4374	-18
$c_1^3 c_2$	2106	-6
$c_1^2 c_3$	594	18
$c_1 c_4$	90	18
$c_1 c_2^2$	1014	-2
$c_2 c_3$	286	6

Reminder: G_2 flag manifolds

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 & & G_2/T^2 \arrow[dl, "\mathbb{C}\mathrm{P}^1"'] \arrow[dr, "\mathbb{C}\mathrm{P}^1"] & \\
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The quadric

$G_2/U(2)_- = Q$ is the space of oriented 2-planes in $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$.

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Let $\{e_1, e_2\}$ be a positive, orthonormal basis for $P \in Q$ (unique up to $U(1)$-transformation). $\mathbb{C}$-linearly extending the standard inner product $(-,-)_{\mathbb{R}^7}$ to $\mathbb{C}^7$, set $f(z) = (z, z) = \sum z_j^2$. Then $f(e_1 + ie_2) = 0$.

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This yields a diffeomorphism

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The invariant complex structure and Kähler-Einstein metric are inherited from $\mathbb{CP}^6$: The restriction of the Fubini-Study metric is Kähler-Einstein, and even $SO(7)$-invariant.

The quadric: Invariant almost complex structures

Equip S^6 with its G_2-invariant almost complex structure, given by $J_x(v) = x \cdot v$. Then $\mathbb{P}(TS^6)$ inherits an invariant almost complex structure.

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Remark

This shows that a complex structure on S^6 gives rise to (at least two) non-standard complex structures on the quadric $Q \subset \mathbb{CP}^6$.

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We have now found three invariant almost complex structures on $G_2/U(2)_-$, out of a total of four.

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The fourth is obtained from the quadric by flipping the fiber over S^6. In fact, all four invariant almost complex structures are related by flips:

$$\begin{tikzcd}
Q \arrow[r, leftrightarrow, "\text{flip } S^6\text{-fibration}"] \arrow[d, leftrightarrow, "\text{flip } G_2/SO(4)\text{-fibration}"'] & X \arrow[d, leftrightarrow, "\text{flip } G_2/SO(4)\text{-fibration}"] \\
\mathbb{P}(T^*S^6) \arrow[r, leftrightarrow, "\text{flip } S^6\text{-fibration}"'] & \mathbb{P}(TS^6)
\end{tikzcd}$$

The quadric: Chern numbers

The invariant almost complex structures are distinguished by their Chern numbers (compare GNO [12]):

Proposition

The Chern numbers of the four invariant almost complex structures on $G_2/U(2)_-$ are:

Chern Number	Q	$\mathbb{P}(TS^6)$	$\mathbb{P}(T^*S^6)$	X
c_5	6	6	6	6
c_1^5	6250	-486	486	-2
$c_1^3 c_2$	2750	-162	162	2
$c_1^2 c_3$	650	18	18	2
$c_1 c_4$	90	18	18	-6
$c_1 c_2^2$	1210	-54	54	-2
$c_2 c_3$	286	6	6	-2

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