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Geometric Quantization

MSc lecture course, summer semester 2016

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Contents

We work in the smooth category. So the considered manifolds and maps between manifolds are either assumed (or can be verified) to be of class C^∞. Manifolds will be always assumed to be Hausdorff and to admit a countable basis of the topology. That implies paracompactness and finally the existence of smooth partitions of unity.

1 Motivation

Quantization is the passage from a classical physical system to a quantum system. The quantization of classical mechanical systems leads to *quantum mechanics*, whereas quantization of classical field theories leads to *quantum field theory*. Geometric quantization is an approach which tries to provide an overall geometric picture starting from a symplectic description of the classical system rather than case by case quantization recipes for particular systems. In this lecture course we will be mainly concerned with the case of mechanical systems, which has the advantage that the phase space of the classical system is a *finite-dimensional* manifold and the geometric ideas will not be obscured by the technical difficulties involved with infinite-dimensional systems.

There are two main formulations of classical mechanics: Lagrangian and Hamiltonian mechanics. Hamiltonian mechanics is formulated in the language of symplectic geometry, which is the framework in which we will work. It will be reviewed in Section 2. The classical mechanical system is described by a symplectic manifold (M,ω), which is called the *phase space* of the system, and a function $H \in C^\infty(M)$. (M,ω,H) is called a *Hamiltonian system*. Points of M are interpreted as physical *states* and real-valued smooth functions on M as physical *observables*. The value of an observable $f \in C^\infty(M)$ on a state $p \in M$ is simply the real number $f(p)$. The *time-evolution* of the system is determined by the function $H \in C^\infty(M)$ called the *Hamiltonian*. In fact, as explained in Section 2, H gives rise to an ordinary differential equation known as *Hamilton's equation* for a curve γ in M. The solutions are interpreted as motions from an initial state $\gamma(t_0) = p$ at time t_0 to a state $\gamma(t)$ at time t.

Quantum mechanics is formulated in the framework of the theory of Hilbert spaces. The system is described by a (complex) Hilbert space $\mathcal{H}$. A *quantum state* is essentially a vector in $\mathcal{H}$ (or, more precisely, the line spanned by that vector). The *quantum observables* are now self-adjoint operators on $\mathcal{H}$. In particular, the *quantum Hamiltonian* $\hat{H}$ is a self-adjoint operator on $\mathcal{H}$ and the time-evolution of the system is governed by *Schrödinger's equation*

$$i\hbar\frac{\partial}{\partial t}\psi_t = \hat{H}\psi_t,$$

where $t \mapsto \psi_t$ is a curve in $\mathcal{H}$ and $\hbar > 0$ is a constant (called Planck's constant).

Example 1.1. *(Harmonic oscillator) The phase space of a classical harmonic oscillator is $M = \mathbb{R}^2$ with standard coordinates denoted by (q,p). The coordinate q is interpreted as position and p as momentum of a particle of mass $m > 0$ on the line. The symplectic form is the canonical volume form $\omega = dp \wedge dq$ on $\mathbb{R}^2$. The Hamiltonian is the total energy of the harmonic oscillator given by*

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2, \tag{1.1}$$

where $k > 0$ is a constant known as Hooke's constant.

The corresponding quantum Hamiltonian is the differential operator

$$\psi \mapsto \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}kx^2\psi,$$

defined on the dense subset $C_c^\infty(\mathbb{R},\mathbb{C})$ of the complex Hilbert space $L^2(\mathbb{R}) := L^2(\mathbb{R},\mathbb{C})$. Since we are working with functions with compact support, it follows by partial integration that the operator is Hermitian *(sometimes called* symmetric*), that is*

$$\langle\hat{H}\varphi,\psi\rangle = \langle\varphi,\hat{H}\psi\rangle \tag{1.2}$$

for all $\varphi, \psi \in C_c^\infty(\mathbb{R}, \mathbb{C})$. *Furthermore, it is known that* $\hat{H}$ *is* essentially self-adjoint*, that is can be extended in a unique way to a self-adjoint operator defined on some dense subspace of* $L^2(\mathbb{R})$.

From a naive point of view the classical and quantum Hamiltonians (1.1) and (1.2) are related by simply replacing the classical observable q *by the operator*

$$\hat{q} : \psi \mapsto x\psi$$

and p *by*

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

This prescription is known as canonical quantization. *The name refers to the coordinates* (q, p), *which are* canonical coordinates *in the sense that they are adapted to the symplectic form. One of the problems of this recipe is caused by the fact that the algebra of functions* $C^\infty(M)$ *(with the usual multiplication) is commutative whereas the algebra of operators is not. So applying canonical quantization to a Hamiltonian* $H = pq = qp$, *for instance, it is not clear whether* $\hat{H} = \hat{p}\hat{q}$ *or* $\hat{H} = \hat{q}\hat{p} \neq \hat{p}\hat{q}$, *or yet another of the infinitely many affine combinations of these, such as* $3\hat{p}\hat{q} - 2\hat{q}\hat{p}$. *In fact,*

$$[\hat{p}, \hat{q}] = -i\hbar \mathbf{1},$$

where $\mathbf{1}$ *denotes the identity operator. Another problem is that on a general symplectic manifold* (M, ω) *we may not have a global canonical coordinate system, and even if such a system is available the coordinate dependence of the quantization procedure needs to be analyzed.*

2 Hamiltonian formulation of classical mechanical systems

Definition 2.1. *A* symplectic manifold (M, ω) *is a smooth manifold* M *endowed with a* symplectic form ω, *that is a non-degenerate closed* 2*-form* ω.

Example 2.2 (Symplectic vector space)**.** *Let ω be a non-degenerate skew-symmetric bilinear form on a finite dimensional real vector space V. Then (V,ω) is called a (real)* symplectic vector space*. Every symplectic vector space is of even dimension and there exists a linear isomorphism $V \to \mathbb{R}^{2n}$, $2n = \dim V$, which maps ω to the* canonical symplectic form

$$\omega_{can} = \sum_{i=1}^{n} dx^i \wedge dx^{n+i}, \tag{2.1}$$

where $(x^1,\ldots,x^{2n})$ are standard coordinates on $\mathbb{R}^{2n}$.

It is a basic result in symplectic geometry, known as Darboux's theorem, see e.g. [AM, Thm. 3.2.2], that for every point in a symplectic manifold (M,ω) there exists a local coordinate system $(x^1,\ldots,x^{2n})$ defined in a neighborhood U of that point such that

$$\omega|_U = \sum_{i=1}^{n} dx^i \wedge dx^{n+i}.$$

So $\omega|_U$ looks like the canonical symplectic form on $\mathbb{R}^{2n}$.

Example 2.3 (Cotangent bundle)**.** *Let $\pi : N = T^*M \to M$ be the cotangent bundle of a manifold M. We define a 1-form λ on N by*

$$\lambda_\xi(v) := \xi(d\pi v),$$

for all $\xi \in N$, $v \in T_\xi N$. The 1-form is called the Liouville form*. Its differential $\omega = d\lambda$ is a symplectic form on N, which is called the* canonical symplectic form*. To check that ω is indeed non-degenerate, let us compute the Liouville form in coordinates $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ on $\pi^{-1}(U) = T^*U \subset T^*M$ associated with coordinates $(x^1,\ldots,x^n)$ on some open set $U \subset M$:*

$$q^i = x^i \circ \pi, \quad p_i(\xi) = \xi(\frac{\partial}{\partial x^i}), \quad \xi \in T^*U.$$

Since under the projection $d\pi : TN \to TM$ the vector fields $\partial/\partial q^i$ and $\partial/\partial p_i$ are mapped to $\partial/\partial x^i$ and zero, respectively, at every point $\xi = \sum \xi_j dx^j|_{\pi\xi} \in$

T^*U we have

$$\lambda_\xi(\frac{\partial}{\partial q^i}) = \xi(\frac{\partial}{\partial x^i}) = \xi_i = p_i(\xi), \quad \lambda_\xi(\frac{\partial}{\partial p_i}) = 0.$$

This shows that

$$\lambda|_U = \sum p_i dq^i, \quad \omega|_U = \sum dp_i \wedge dq^i,$$

proving that ω is a symplectic form.

Given a smooth function f on a symplectic manifold (M, ω) there is a unique vector field X_f such that

$$df = -\omega(X_f, \cdot).$$

Definition 2.4. *The vector field X_f is called the* Hamiltonian vector field *associated with f.*

According to Darboux's theorem, we can locally write $\omega = \sum dp_i \wedge dq^i$ for some local coordinate system $(q^1, \dots, q^n, p_1, \dots, p_n)$ on M. In such coordinates, which will be called Darboux coordinates, we can easily compute

$$X_f = \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{2.2}$$

Definition 2.5. *A* Hamiltonian system *(M, ω, H) is a symplectic manifold (M, ω) endowed with a function $H \in C^\infty(M)$, called the* Hamiltonian *of the system. An integral curve $\gamma : I \to M$ of the Hamiltonian vector field X_H, defined on an open interval $I \subset \mathbb{R}$, is called a* motion *of the system and the corresponding system*

$$\gamma' = X_H(\gamma)$$

of ordinary differential equations is called Hamilton's equation*. An* integral of motion *of the Hamiltonian system is a function $f \in C^\infty(M)$ which is constant along every motion. The symplectic manifold (M, ω) is called the* phase space *of the Hamiltonian system.*

Proposition 2.6. *Let (M,ω,H) be a Hamiltonian system. Then H is an integral of motion.*

Proof. Let $\gamma : I \to M$ be a motion of the system. Then

$$\frac{d}{dt}H(\gamma(t)) = dH\gamma'(t) = -\omega(X_H(\gamma(t)),\gamma'(t)) = -\omega(X_H,X_H)|_{\gamma(t)} = 0.$$

□

Proposition 2.7. *In Darboux coordinates, Hamilton's equation for a curve $\gamma : I \to M$ takes the form*

$$\dot{q}^i(t) = \frac{\partial H}{\partial p_i}(\gamma(t)), \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q^i}(\gamma(t)),$$

where $q^i(t) = q^i(\gamma(t))$, $p_i(t) = p_i(\gamma(t))$.

Proof. This follows immediately from (2.2) by comparing $\gamma' = \sum(\dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i})$ with $X_H(\gamma)$. □

Proposition 2.8. *Hamiltonian vector fields are symplectic, that is*

$$\mathcal{L}_{X_f}\omega = 0,$$

for all $f \in C^\infty(M)$.

Proof. Using the Cartan formula we obtain:

$$\mathcal{L}_{X_f}\omega = d\iota_{X_f}\omega = -ddf = 0.$$

□

Definition 2.9. *Given two functions f,g on a symplectic manifold (M,ω), the function*

$$\{f,g\} := \omega(X_f,X_g)$$

is called the Poisson bracket *of f and g. We say that f and g* Poisson commute *if $\{f,g\} = 0$*

Proposition 2.10. *For every symplectic manifold* (M,ω)*, the Poisson bracket is a Lie bracket on* $C^\infty(M)$ *and*

$$(C^\infty(M),\{\cdot,\cdot\}) \to (\mathfrak{X}(M),[\cdot,\cdot]), \quad f \mapsto X_f,$$

is a homomorphism of Lie algebras.

Proof. The Poisson bracket is obviously bilinear and skew-symmetric. We first check that the linear map $f \mapsto X_f$ satisfies

$$[X_f, X_g] = X_{\{f,g\}}$$

for all $f, g \in C^\infty(M)$. In fact, for all symplectic vector fields X, Y on M we have

$$\omega([X,Y],\cdot) = \mathcal{L}_X(\omega(Y,\cdot)) = d\iota_X(\omega(Y,\cdot)) + \iota_X \underbrace{d\iota_Y\omega}_{\mathcal{L}_Y\omega=0} = -d(\omega(X,Y)).$$

Specializing to $X = X_f$, $Y = X_g$ proves the claim.

Using the formula

$$\{f,g\} = -\omega(X_g, X_f) = dgX_f = X_f(g)$$

we can now prove the Jacobi identity for the Poisson bracket:

$$\begin{aligned}&\{f,\{g,h\}\} - \{\{f,g\},h\} - \{g,\{f,h\}\} = (X_fX_g - X_{\{f,g\}} - X_gX_f)h \\ &= ([X_f,X_g] - X_{\{f,g\}})h = 0, \quad f,g,h \in C^\infty(M).\end{aligned}$$

□

3 Prequantization

The aim of geometric quantization is to associate in a natural way a Hilbert space $\mathcal{H}$ with every Hamiltonian system (M,ω,H). By this correspondence classical observables $f \in C^\infty(M)$ should be mapped to Hermitian (or even essentially selfadjoint) operators $\hat{f}$ defined on a common dense subspace of $\mathcal{H}$ such that, in particular, we obtain a quantum Hamiltonian $\hat{H}$ defining the evolution of the quantum system. It is natural to require the map $f \mapsto \hat{f}$ to be a homomorphism of Lie algebras mapping the constant function 1 to the identity operator, where the Lie bracket on functions is the Poisson bracket and the Lie bracket of two Hermitian operators A, B is defined by

$$[A, B]_\hbar := \frac{i}{\hbar}[A, B] = \frac{i}{\hbar}(AB - BA).$$

Up to the factor $\hbar^{-1}$, this is precisely the Lie bracket induced from the Lie algebra of skew-Hermitian operators via the isomorphism $A \mapsto iA$ from Hermitian to skew-Hermitian operators.

We denote by $T^1 = \mathbb{R}/\mathbb{Z}$ the circle group. Its Lie algebra is canonically identified with $\mathbb{R}$. The first step of geometric quantization, is the construction of a T^1-principal bundle $P \to M$ with a connection θ, the curvature of which is the symplectic form ω of a given symplectic manifold (M, ω). (Recall that the curvature $d\theta$ is invariant and horizontal and can therefore be considered as a form on the base M.) As shown below, such a bundle exists if and only if ω satisfies a certain integrality condition. With the help of this bundle we will be able to specify a preliminary quantization procedure (called *prequantization* [K]) with the properties required so far. Later we will see that there are further desirable properties of a quantization procedure which will make it necessary to go beyond prequantization.

Theorem 3.1. *Let α be a real-valued closed two-form on a manifold M. Then the following conditions are equivalent:*

(i) α is the curvature of a connection in a principal T^1-bundle $P \to M$.

(ii) The de Rham class $[\alpha] \in H^2_{dR}(M, \mathbb{R})$ is integral, that is lies in the image of the natural homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \cong H^2_{dR}(M, \mathbb{R})$.

Proof. (cf. [W58, Ch. 5, Lemme 2]) We describe $H^2(M, \mathbb{R})$ and the canonical isomorphism $H^2(M, \mathbb{R}) \cong H^2_{dR}(M, \mathbb{R})$ in terms of Cech cohomology, see [W52, Wa, Wel] for the relation with other cohomology theories.

For this we consider a good open cover $\mathcal{U} = (U_i)$ of M, that is a cover such that all intersections of a finite number of the U_i are either empty or contractible. Then we have

$$H^k(M,A) \cong \check{H}^k(\mathcal{U},A)$$

for every non-negative integer k and every Abelian group $(A,+)$, where $H^k(M,A)$ denotes the sheaf cohomology of M with respect to the constant sheaf A and $\check{H}^k(\mathcal{U},A)$ the Cech cohomology of the covering. The latter is defined as the quotient

$$\check{H}^k(\mathcal{U},A) = Z^k(\mathcal{U},A)/B^k(\mathcal{U},A),$$

where $B^k(\mathcal{U},A) \subset Z^k(\mathcal{U},A)$ are the groups of Cech co-boundaries and co-cycles, respectively. Let us recall that a *(Cech) co-chain of degree* k (short k*-co-chain*) with values in A is a map f which associates with every $(k+1)$-tuple $(U_{i_0},\dots,U_{i_k})$ of elements of the good covering $\mathcal{U}$ such that

$$U_{i_0\dots i_k} := U_{i_0} \cap \dots \cap U_{i_k} \neq \varnothing$$

a constant $f_{i_0\dots i_k} \in A$. It is called a *co-cycle* if it is δ-closed, that is if the co-chain δf of degree $k+1$ defined by

$$(\delta f)_{i_0\dots i_{k+1}} := \sum_{\ell=0}^{k+1}(-1)^\ell f_{i_0\dots\hat{i}_\ell\dots i_{k+1}}$$

vanishes. In particular, a 0-co-chain (f_i) is δ-closed if and only if $f_i = f_j$, whenever $U_i \cap U_j \neq \varnothing$, and a 1-co-chain (f_{ij}) is δ-closed if and only if $f_{jk} - f_{ik} + f_{ij} = 0$, whenever $U_i \cap U_j \cap U_k \neq \varnothing$. A $(k+1)$-co-chain f is called a *co-boundary* if it is δ-exact, that is $f = \delta g$ for some k-co-chain g. Finally, one defines $B^0(\mathcal{U},A) = 0$. As a consequence, $\check{H}^0(\mathcal{U},A) = Z^0(\mathcal{U},A) = A$ if M is connected.

Next we describe the canonical isomorphism $H^2_{dR}(M,\mathbb{R}) \cong \check{H}^2(\mathcal{U},\mathbb{R})$. Given a cohomology class $[\alpha] \in H^2_{dR}(M,\mathbb{R})$, for every $U_i \in \mathcal{U}$ there exists a one-form β_i on U_i such that

$$d\beta_i = \alpha|_{U_i}. \tag{3.1}$$

This implies the existence of a function f_{ij} such that

$$\beta_i - \beta_j = df_{ij} \tag{3.2}$$

on U_{ij} for all $U_i, U_j \in \mathcal{U}$ such that $U_{ij} \neq \varnothing$ and we can choose the functions such that $f_{ij} = -f_{ji}$. For all $U_i, U_j, U_k \in \mathcal{U}$ such that $U_{ijk} \neq \varnothing$, the functions

$$c_{ijk} := f_{jk} - f_{ik} + f_{ij}, \tag{3.3}$$

defined on U_{ijk} are constant, since by (3.2)

$$dc_{ijk} = (\beta_j - \beta_k) - (\beta_i - \beta_k) + (\beta_i - \beta_j) = 0$$

on U_{ijk}. This shows that $c = (c_{ijk})$ defines a Cech 2-co-chain with values in $\mathbb{R}$ and it is easily verified from (3.3) that $\delta c = 0$. One can check (see tutorials) that the class $[c] \in \check{H}^2(\mathcal{U}, \mathbb{R})$ does not depend on the particular choices but only on the cohomology class $[\alpha]$ and that the map $[\alpha] \mapsto [c]$ defines an isomorphism

$$\phi : H^2_{dR}(M, \mathbb{R}) \xrightarrow{\sim} \check{H}^2(\mathcal{U}, \mathbb{R}).$$

Now let ω be the curvature of a connection θ in a T^1-principal bundle $\pi : P \to M$. Recall that ω is a two-form on M characterized by

$$\pi^* \omega = d\theta. \tag{3.4}$$

We have to show that $\phi([\omega]) \in \iota_* \check{H}^2(\mathcal{U}, \mathbb{Z})$, where $\iota_* : \check{H}^2(\mathcal{U}, \mathbb{Z}) \to \check{H}^2(\mathcal{U}, \mathbb{R})$ is the natural homomorphism. As explained above in the case of a general cohomology class $[\alpha] \in H^2_{dR}(M, \mathbb{R})$, for all $U_i \in \mathcal{U}$ we can find a one-form θ_i on U_i such that $d\theta_i = \omega|_{U_i}$. Such a system of one-forms (θ_i) can be obtained by choosing a section s_i of P over U_i and putting

$$\theta_i := s_i^* \theta.$$

Since U_{ij} is contractible, the function $s_i - s_j : U_{ij} \to \mathbb{R}/\mathbb{Z}$ can be lifted to a smooth function $f_{ij} : U_{ij} \to \mathbb{R}$ such that

$$s_i - s_j \equiv f_{ij} \pmod{\mathbb{Z}}.$$

This implies that

$$c_{ijk} = f_{jk} - f_{ik} + f_{ij} \in \mathbb{Z}.$$

One can check (see tutorials) that $df_{ij} = \theta_i - \theta_j$, which implies that $[c] = [(c_{ijk})] = \phi([\omega])$. Since the constants c_{ijk} are integers, this shows that $\phi([\omega]) \in \iota_* \check{H}^2(\mathcal{U}, \mathbb{Z})$.

Conversely, let $[\alpha] \in \phi^{-1}(\iota_* \check{H}^2(\mathcal{U}, \mathbb{Z})) \subset H^2_{dR}(M, \mathbb{R})$. Then we can choose data (β_i), (f_{ij}) and (c_{ijk}) such that (3.1), (3.2) and (3.3), with the additional property that the c_{ijk} are integers. Then (f_{ij}) satisfies the co-cycle condition[1]

$$f_{jk} - f_{ik} + f_{ij} \equiv 0 \pmod{\mathbb{Z}}.$$

This means that (f_{ij}) are transition functions of a T^1-principal bundle $P \to M$. The principal bundle P is obtained by gluing the trivial bundles $U_i \times T^1$ by means of the transition functions f_{ij}. In fact, $(x, t) \in U_i \times T^1$ is identified with $(x, t + [f_{ij}(x)]) \in U_j \times T^1$ for all $(x, t) \in U_{ij} \times T^1$, where $[f_{ij}(x)]$ denotes the class of $f_{ij}(x) \in \mathbb{R}$ in $T^1 = \mathbb{R}/\mathbb{Z}$. The one-forms β_i define connections on each of the trivial principal bundles $U_i \times T^1$, which are related by the gauge transformations $\beta_i \mapsto \beta_i - df_{ij} = \beta_j$. This shows that they define a principal connection θ on P such that $\beta_i = s_i^* \theta$, where s_i is trivialization of $P|_{U_i}$ associated with the embedding $U_i \times T^1 \xrightarrow{\sim} P|_{U_i} \subset P$ (see tutorials for the details). □

Given a T^1-principal bundle $P \to M$ and a one-dimensional complex representation

$$\rho : T^1 \to \mathrm{GL}(1, \mathbb{C}),$$

we can associate a complex line bundle $L_\rho = P \times_\rho \mathbb{C}$, where $P \times_\rho \mathbb{C}$ is the quotient of $P \times \mathbb{C}$ by the T^1-action

$$(a, (u, z)) \mapsto (u - a, \rho(a) z), \quad a \in T^1, \quad (u, z) \in P \times \mathbb{C}.$$

[1]The T^1-valued functions $[f_{ij}] : U_{ij} \to \mathbb{R}/\mathbb{Z}$ define a Cech co-cycle with values in the sheaf $C^\infty_M(T^1)$ of smooth T^1-valued functions on M (not in the constant sheaf T^1).

The equivalence class of an element $(u,z) \in P \times \mathbb{C}$ is denoted by $[(u,z)] \in L_\rho$. All one-dimensional T^1-representations are of the form

$$\rho_k : \mathbb{R}/\mathbb{Z} \ni [x] \mapsto \rho_k([x]) = e^{2\pi i k x} \in S^1 \subset \mathbb{C}^\times = \mathrm{GL}(1,\mathbb{C}),$$

for some integer k. Notice that ρ_k is equivalent to the representation on $\mathbb{C}^{\otimes k} = \mathbb{C} \otimes \cdots \otimes \mathbb{C}$ induced by the fundamental representation ρ_1. This implies that $L_{\rho_k} = L^{\otimes k}$, where $L := L_{\rho_1}$.

Proposition 3.2. *A principal connection θ on P induces a unitary connection ∇ in any of the line bundles $L^{\otimes k} = L_{\rho_k}$.*

Proof. To describe ∇ we choose a local section u of P defined on some open subset $U \subset M$. Then every section s of L_{ρ_k} on U can be described by a complex-valued function $f : U \to \mathbb{C}$ as $s = [(u,f)]$ and

$$\nabla_X s = [(u, \underbrace{X(f) + 2\pi i k (u^*\theta)(X) f}_{=:\nabla^u_X f})], \quad X \in \mathfrak{X}(M).$$

The factor $2\pi i k$ is precisely the differential $d\rho : \mathbb{R} = Lie\, T^1 \to i\mathbb{R} = Lie\, S^1$ of the homomorphism $\rho : T^1 \to S^1$. Let us check that the above formula for the connection is independent of the local section u of P. Let $a : U \to T^1$ be a smooth map defined on some open subset $U \subset M$. Then $[(u,f)] = [(u-a, \tilde{f})]$, where $\tilde{f} = e^{2\pi i k a} f$, and we have to check that

$$\begin{aligned} [(u, \nabla^u_X f)] &= [(u-a, \nabla^{u-a}_X \tilde{f})] \\ &= [(u, e^{-2\pi i k a} \nabla^{u-a}_X \tilde{f})]. \end{aligned}$$

This follows from

$$\begin{aligned} e^{-2\pi i k a} X(\tilde{f}) - X(f) &= 2\pi i k X(a) f \\ e^{-2\pi i k a} 2\pi i k (u-a)^*\theta(X) \tilde{f} - 2\pi i k u^*\theta(X) f &= -2\pi i k X(a) f, \end{aligned}$$

where we used that $(u-a)^*\theta - u^*\theta = -da$, see Exercise sheet 2. In fact, the sum of the left-hand sides of these equations is precisely $e^{-2\pi i k a}\nabla^{u-a}_X \tilde{f} - \nabla^u_X f$.

The unitarity of the connection follows from the fact that the connection form $2\pi iku^*\theta$ with respect to any local section u of P is purely imaginary. In fact, we can define a Hermitian scalar product on L_{ρ_k} as follows. Let $s_j = [(u, f_j)]$, $j = 1, 2$, be local sections of L_{ρ_k}. Then we define

$$\langle s_1, s_2 \rangle := f_1 \bar{f}_2.$$

This is well defined since $\tilde{f}_1 \overline{\tilde{f}_2} = f_1 \bar{f}_2$, where $\tilde{f}_j = e^{2\pi ika} f_j$. It remains to check that ∇ is Hermitian with respect to $\langle \cdot, \cdot \rangle$:

$$\begin{aligned} X\langle s_1, s_2\rangle &= X(f_1)\bar{f}_2 + f_1\overline{X(f_2)} = (\nabla_X^u f_1)\bar{f}_2 + f_1\overline{\nabla_X^u f_2} \\ &= \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle, \end{aligned}$$

for all $X \in \mathfrak{X}(M)$. □

Definition 3.3. *Let $\pi : P \to M$, $\pi' : P' \to M$ be G-principal bundles for some Lie group G. A G-equivariant diffeomorphism $\varphi : P \to P'$ is called an* isomorphism *(or* equivalence*) of principal bundles if $\pi' \circ \varphi = \pi$, that is if φ covers the identity map. Given connections θ and θ' on P and P', respectively, we say that an isomorphism of G-principal bundles $\varphi : P \to P'$ is an* isomorphism of principal bundles with connection *if $\varphi^*\theta' = \theta$.*

For every function $f \in C^\infty(M)$ we can now define a first order differential operator $\hat{f}$ which acts on smooth sections of L_{ρ_k} by

$$\hat{f} := -i\hbar\left(\nabla_{X_f} - 2\pi ikf\right) = -i\hbar\nabla_{X_f} - 2\pi k\hbar f.$$

The following theorem summarizes the main results of prequantization, see [K] for further information.

Theorem 3.4. *Let (M, ω) be a (connected) symplectic manifold with integral class $[\omega] \in H^2(M, \mathbb{R})$. Then there exists a principal T^1-bundle $P \to M$ with connection θ and curvature ω and with every $k \in \mathbb{Z}$ we can associate a complex line bundle $L_{\rho_k} \to M$ with a unitary connection ∇ induced by θ.*

(i) The map $f \mapsto \hat{f}$ is a Lie algebra homomorphism from the Poisson algebra $C^\infty(M)$ to the algebra of Hermitian operators on the space of smooth sections of L_{ρ_k} with compact support, which is equipped with the L^2 scalar product with respect to the symplectic volume form

$$dvol = \frac{\omega^n}{n!}.$$

(ii) The group $H^1(M, T^1) \cong \mathrm{Hom}(\pi_1(M), T^1)$ (where T^1 denotes the constant sheaf) acts simply transitively on the set $\mathfrak{P}_\omega$ of isomorphism classes of T^1-principal bundles (P, θ) with connection of curvature ω.

(iii) If $H_1(M, \mathbb{Z}) = 0$, then (P, θ) is unique up to isomorphism.

Proof. (i) We first show that $\hat{f}$ is Hermitian. Multiplication by the real factor $-2\pi k\hbar f$ is obviously Hermitian. Therefore it suffices to show that $-i\nabla_X$ is Hermitian for every Hamiltonian vector field X. In order to prove this by partial integration using the unitarity of ∇ we need to show that for every pair of considered sections s_1, s_2 of L_{ρ_k}, the function $X\langle s_1, s_2\rangle$ has zero integral. This follows from the next lemma.

Lemma 3.5. *Let X be a symplectic vector field on a symplectic manifold (M, ω) of dimension $2n$. Then for all $f \in C_c^\infty(M)$ we have*

$$\int_M X(f)\omega^n = 0.$$

Proof. If X is symplectic, we can write the integrand as

$$\mathcal{L}_X(f\omega^n) = d\iota_X(f\omega^n),$$

which is exact. Therefore, using that f has compact support, we can conclude by Stokes theorem that

$$\int_M X(f)\omega^n = 0.$$

□

Next we compute for $f, g \in C^\infty(M)$ the $\hbar$-commutator

$$
\begin{aligned}
[\hat{f}, \hat{g}]_\hbar &= \frac{i}{\hbar}[\hat{f}, \hat{g}] = \frac{i}{\hbar}\left([-i\hbar\nabla_{X_f}, -i\hbar\nabla_{X_g}] + i2\pi k\hbar^2 X_f(g) - i2\pi k\hbar^2 X_g(f)\right) \\
&= -i\hbar \underbrace{[\nabla_{X_f}, \nabla_{X_g}]}_{=2\pi ik\omega(X_f, X_g) + \nabla_{[X_f, X_g]}} -4\pi k\hbar\{f, g\} \\
&= 2\pi k\hbar\omega(X_f, X_g) - i\hbar\nabla_{[X_f, X_g]} - 4\pi k\{f, g\} \\
&= -i\hbar\nabla_{X_{\{f,g\}}} - 2\pi k\hbar\{f, g\} = \widehat{\{f, g\}}.
\end{aligned}
$$

(ii) Recall that the transition functions $f = (f_{ij})$ of a T^1-principal bundle with respect to some good covering $\mathcal{U} = (U_i)$ of M constitute a Cech co-cycle with values in the sheaf $C^\infty_M(T^1)$. One can check that the class $[f] \in H^1(M, C^\infty_M(T^1)) \cong \check{H}^1(\mathcal{U}, C^\infty_M(T^1))$ depends only on the isomorphism class of P as a T^1-principal bundle over M and not on the particular system of transition functions. Moreover, two principal bundles P and P' are isomorphic if and only if the corresponding classes in $H^1(M, C^\infty_M(T^1))$, which will be denoted by $[P], [P']$, are equal. Therefore $H^1(M, C^\infty_M(T^1))$ can be identified with the set of isomorphism classes of T^1-principal bundles over M and this set inherits the structure of an additive group.

Similarly, the set $\mathfrak{P}$ of isomorphism classes of T^1-principal bundles (P, θ) with connection has a natural Abelian group structure as we explain now. Let (P, θ) and (P', θ') be two such bundles with curvature α and α', respectively. Let (f_{ij}) and (f'_{ij}) be transition functions for P and P' with respect to a good covering $\mathcal{U} = (U_i)$. The connections θ and θ' give rise to one-forms θ_i and θ'_i on U_i such that

$$\theta_i - \theta_j = df_{ij}, \quad \theta'_i - \theta'_j = df'_{ij}$$

on non-trivial intersections $U_i \cap U_j$. Then $(f_{ij} + f'_{ij})$ are transition functions of a T^1-principal bundle P'' and the local system of one-forms $\theta_i + \theta'_i$ defines a connection θ'' on P'' with curvature $\alpha + \alpha'$. One can check that the class $[(P'', \theta'')] \in \mathfrak{P}$ depends only on the classes $[(P, \theta)], [(P', \theta')] \in \mathfrak{P}$ of the summands. Then $[(P, \theta)] + [(P', \theta')] := [(P'', \theta'')]$ defines a group structure on $\mathfrak{P}$ such that $\mathfrak{P}_\alpha + \mathfrak{P}_\beta = \mathfrak{P}_{\alpha+\beta}$ for all closed two-forms α, β on M with integral de Rham class, where $\mathfrak{P}_\alpha \subset \mathfrak{P}$ denotes the set of classes $[(P,\theta)]$ with curvature α. As a consequence, the set $\mathfrak{P}_0$ of isomorphism classes of flat bundles is a subgroup of $\mathfrak{P}$ and acts simply transitively on $\mathfrak{P}_\alpha$ for any α.

To prove (ii), it remains to show that $\mathfrak{P}_0 \cong H^1(M,T^1) \cong \mathrm{Hom}(\pi_1(M),T^1)$. With respect to some good covering $\mathcal{U} = (U_i)$ we can represent any element $[(P,\theta)] \in \mathfrak{P}_0$ by transition functions (f_{ij}) and a system of closed and, hence, exact one-forms $\theta_i = df_i$, $f_i \in C^\infty(U_i)$, such that $df_{ij} = \theta_i - \theta_j = d(f_i - f_j)$. Therefore, there exists a system of constants $k = (k_{ij})$ such that

$$f_{ij} = f_i - f_j + k_{ij}.$$

The system k defines a Cech co-cycle with values in T^1 and, hence, an element $[k] \in H^1(M,T^1)$. One can check that the class $[k]$ does not depend on the particular choice of the system of functions (f_{ij}, f_i) and that the resulting map

$$\mathfrak{P}_0 \to H^1(M,T^1) \tag{3.5}$$

is a group homomorphism. It is obviously surjective since every Cech co-cycle $k = (k_{ij})$ with values in T^1 defines a T^1-principal bundle P and this bundle can be endowed with the flat connection θ defined consistently by $\theta_i = 0$ (as the transition functions k_{ij} are constant). One can check that $[(P,\theta)]$ depends only on the class $[k] \in H^1(M,T^1)$ and that the resulting map $H^1(M,T^1) \to \mathfrak{P}_0$ is inverse to the homomorphism (3.5). This establishes that the groups $\mathfrak{P}_0$ and $H^1(M,T^1)$ are canonically isomorphic.

Remark. Since every Cech co-cycle (k_{ij}) with values in the constant sheaf T^1 is also a Cech co-cycle with values in $C^\infty_M(T^1)$ and every co-boundary $(k_i - k_j)$ with values in T^1 is also one with values in $C^\infty_M(T^1)$, there is a canonical homomorphism

$$H^1(M,T^1) \to H^1(M,C^\infty_M(T^1)).$$

Composing this with the canonical homomorphism (3.5), we obtain a homomorphism $\mathfrak{P}_0 \to H^1(M, C^\infty_M(T^1))$, which is simply the forgetful homomorphism $[(P,\theta)] \to [P]$.

To see that $\mathfrak{P}_0$ is canonically isomorphic to $\mathrm{Hom}(\pi_1(M), T^1)$ we first recall that a flat connection θ on a T^1-principal bundle $P \to M$ defines a parallel transport which is trivial along null homotopic loops and, hence, descends to a homomorphism $hol^\theta : \pi_1(M) \to T^1$, which depends only on the class $[(P,\theta)]$. The resulting map $\mathfrak{P}_0 \to \mathrm{Hom}(\pi_1(M), T^1)$ is a group homomorphism and can be inverted as follows. Given a homomorphism $\rho : \pi_1(M) \to T^1$ and considering the universal covering $\tilde{M} \to M$ as a $\pi_1(M)$-principal bundle, we can associate a T^1-principal bundle

$$P_\rho := \tilde{M} \times_\rho T^1,$$

which by construction inherits a flat connection with holonomy ρ from the canonical flat connection in the trivial principal bundle $\tilde{M} \times T^1$.

(iii) Since T^1 is Abelian, we have that

$$\mathrm{Hom}(\pi_1(M), T^1) = \mathrm{Hom}(\pi_1(M)/[\pi_1(M), \pi_1(M)], T^1)$$

but $\pi_1(M)/[\pi_1(M), \pi_1(M)] \cong H_1(M, \mathbb{Z})$ (Hurewicz theorem). This proves that $\mathfrak{P}_0 = 0$ if $H_1(M, \mathbb{Z}) = 0$ and, hence, that $\mathfrak{P}_\omega$ is a point. □

Remark. 1) The isomorphisms constructed in the above proof also show that

$$H^1(M, T^1) \cong \mathrm{Hom}(H_1(M, \mathbb{Z}), T^1).$$

The latter isomorphism can be also obtained using the universal coefficient theorem in cohomology.

2) If we take $\hbar = -\frac{1}{2\pi k}$, then the formula for $\hat{f}$ takes the standard form (cf. e.g. [BW, p. 94])

$$\hat{f} = -i\hbar \nabla_{X_f} + f,$$

which implies $\hat{1} = \mathrm{Id}$.

Example 3.6. *Consider the cotangent bundle $N = T^*M$ of a manifold M endowed with the canonical symplectic form $\omega = d\lambda$. Then $\theta = dt + \lambda$ defines a connection in the trivial T^1-principal bundle $P = N \times T^1 \to N$, where dt denotes the canonical one-form on $T^1 = \mathbb{R}/\mathbb{Z}$ induced by the standard coordinate t of $\mathbb{R}$. For the canonical section $u = (\mathrm{Id}, [0]) : N \to P = N \times T^1$, of P we have $u^*\theta = \lambda$. So for every section $s = [(u, f)]$, $f \in C^\infty(N, \mathbb{C})$, of L_{ρ_k} we have*

$$\nabla_X s = [(u, \underbrace{X(f) + 2\pi i k \lambda(X) f}_{X(f) - i\hbar^{-1}\lambda(X)f =: \nabla^u_X f})]$$

for all $X \in \mathfrak{X}(N)$. For the coordinate vector fields of canonical coordinates (q^j, p_j) on N associated to local coordinates on M we have

$$X_{q^j} = -\frac{\partial}{\partial p_j}, \quad X_{p_j} = \frac{\partial}{\partial q^j},$$

and, hence,

$$\nabla^u_{X_{q^j}} = -\frac{\partial}{\partial p_j}, \quad \nabla^u_{X_{p_j}} = \frac{\partial}{\partial q^j} - i\hbar^{-1} p_j.$$

As a consequence, we have

$$\widehat{q^j}[(u, f)] = [(u, i\hbar \frac{\partial}{\partial p_j} f + q^j f)], \quad \widehat{p_j}[(u, f)] = [(u, -i\hbar \frac{\partial}{\partial q^j} f)].$$

So the expressions for the differential operators $\widehat{q^j}$ and $\widehat{p_j}$ with respect to the trivialization u are

$$\begin{aligned} \widehat{q^j}: &\quad i\hbar \frac{\partial}{\partial p_j} + q^j, \\ \widehat{p_j}: &\quad -i\hbar \frac{\partial}{\partial q^j}. \end{aligned}$$

This does not precisely coincide with the canonical quantization prescription, which was

$$\begin{aligned} \widehat{q^j}: &\quad q^j, \\ \widehat{p_j}: &\quad -i\hbar \frac{\partial}{\partial q^j}. \end{aligned}$$

However, the latter prescription can be recovered by restricting the space of functions on which these differential operators act. In fact, it is sufficient to restrict to functions f which depend only on the q^i-coordinates, that is to $f \in C^\infty(M) \subset C^\infty(N)$. Here the inclusion is given by identification of a function f on M with its pull back $\pi^* f$ to $N = T^*M$, where $\pi : T^*M \to M$ denotes the canonical projection. The corresponding space of sections $s = [(u, f)]$ consists precisely of the sections which are parallel along the vertical distribution $T^v N = \ker d\pi \subset TN$. This distribution has two important properties which will play a key role in more general situations, as we will see in the sequel. It is integrable *and* Lagrangian.

We recall the following standard definitions and facts.

Definition 3.7. *A* distribution *of rank k on a manifold M is a vector subbundle $\mathcal{D} \subset TM$ of rank k. It is called* involutive *if $[X, Y] \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$. An injectively immersed submanifold $\varphi : N \to M$ is called an* integral submanifold *of a distribution $\mathcal{D} \subset TM$ if it is everywhere tangent to $\mathcal{D}$, that is $d\varphi T_p N \subset \mathcal{D}_{\varphi(p)}$ for all $p \in N$. A distribution $\mathcal{D}$ of rank k is called* integrable *if for all $p \in M$ it admits an integral submanifold of dimension k through p.*

Theorem 3.8 (Frobenius). *A distribution is integrable if and only if it is involutive.*

Proof. See [Wa] □

Given an integrable distribution $\mathcal{D}$ on a manifold M, we can decompose M into its maximal (injectively immersed but not necessarily embedded) integral submanifolds. Such a decomposition is called a *foliation* of M and the maximal integral submanifolds are called the *leaves* of the foliation. Notice that the dimension of the leaves coincides with the rank of $\mathcal{D}$.

Definition 3.9. *Let (V, ω) be symplectic vector space. A subspace $U \subset V$ is called* isotropic *if*

$$\omega(v, w) = 0$$

for all $v, w \in U$. The maximal isotropic subspaces are called Lagrangian *subspaces.*

One can easily show that a Lagrangian subspace of a symplectic vector space of dimension $2n$ has dimension n and that any two Lagrangian subspaces are related by a linear symplectic transformation.

Definition 3.10. *Let (M, ω) be a symplectic manifold. A distribution $\mathcal{D}$ on M is called* Lagrangian *if $\mathcal{D}_p \subset (T_pM, \omega_p)$ is a Lagrangian subspace for all $p \in M$. An immersed submanifold $\varphi : N \to M$ is called* Lagrangian *if $d\varphi T_pN \subset T_{\varphi(p)}M$ is Lagrangian for all $p \in N$. A foliation of M is called* Lagrangian *if all of its leaves are Lagrangian.*

Integrable Lagrangian distributions of a (real) symplectic manifold (M, ω) are also called *real polarizations* and will be studied in the next section together with more general polarizations. Sections of a line bundle over M which are parallel along a given polarization will be called *polarized* sections. The canonical polarization of (T^*M, ω_{can}) discussed in Example 3.6 is called the *vertical* polarization. The corresponding polarized sections are called *vertically polarized.*

To close this section let us point out that the restriction to polarized sections makes it also necessary to restrict the Poisson algebra of functions which is represented by the prequantization procedure. In Example 3.6, for instance, the operators $\hat{f}$ associated with functions $f \in C^\infty(T^*M)$ do not in general preserve the vector space of polarized sections but they do if we restrict to functions in the (Abelian) Poisson subalgebra $C^\infty(M) \subset C^\infty(T^*M)$. (See the next section for the quantization of a larger (non-Abelian) Poisson subalgebra, including canonical momenta.) This is in accordance with the following no go theorem explained in the lecture by Alexander Haupt.

Theorem 3.11 (Groenewald)**.** *Let $P_{\le k}$ denote the vector space of polynomials of degree $\le k$ on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ and $D(\mathbb{R}^n)$ the vector space of differential* *operators on* $\mathbb{R}^n$ *with complex-valued polynomial coefficients. Then there is no linear map* $Q : P_{\le 4} \to D(\mathbb{R}^n)$ *such that*

1. $Q(1) = \mathrm{Id}$,
2. $Q(q^j) = x^j$, $Q(p_j) = -i\hbar \frac{\partial}{\partial x^j}$ *(canonical quantization),*
3. $[Q(f), Q(g)]_\hbar = Q(\{f, g\})$ *for all* $f, g \in P_{\le 3}$.

Proof. See [H, Th. 13.13] and references therein. □

4 Polarizations of symplectic manifolds

4.1 The Weinstein connection

Let $\mathcal{D} \subset TM$ be a (real) polarization of a symplectic manifold (M, ω), that is an integrable Lagrangian distribution. The isomorphism

$$TM \to T^*M, \quad X \mapsto \omega X := \omega(X, \cdot),$$

induces an isomorphism

$$\phi_\omega : \mathcal{D} \to \mathcal{D}^0 \subset T^*M,$$

where $\mathcal{D}^0$ denotes the annihilator of $\mathcal{D}$.

By the involutivity of the distribution, the Lie derivative $\mathcal{L}_X$ in direction of a vector field $X \in \Gamma(\mathcal{D}) \subset \mathfrak{X}(M)$ tangent to the distribution acts on $\Gamma(\mathcal{D})$ and, hence, on $\Gamma(\mathcal{D}^0) \subset \Omega^1(M)$. Therefore, we can define a differential operator by the formula

$$\nabla_X := \phi_\omega^{-1} \circ \mathcal{L}_X \circ \phi_\omega : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}).$$

Theorem 4.1. *The map*

$$\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \quad (X, Y) \to \nabla_X Y,$$

induces a flat and torsion-free connection ∇^L *(known as the* Weinstein connection*) on every leaf* L *of the polarization* $\mathcal{D}$.

Proof. From the formula defining ∇_X, we easily obtain

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \phi_\omega^{-1} \circ ([\mathcal{L}_X, \mathcal{L}_Y] - \mathcal{L}_{[X,Y]}) \circ \phi_\omega = 0, \tag{4.1}$$

for all $X, Y \in \Gamma(\mathcal{D})$, in virtue of the Jacobi identity. Also the Leibniz rule for ∇_X is an immediate consequence of that for $\mathcal{L}_X$ and the $C^\infty(M)$-linearity in X follows from the calculation

$$\omega \nabla_X Y = \mathcal{L}_X(\omega Y) = \iota_X d(\omega Y),$$

which used that $\omega(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{D})$. Comparing this equation with

$$\mathcal{L}_X(\omega Y) = (\mathcal{L}_X \omega) Y + \omega \mathcal{L}_X Y = d(\omega X) Y + \omega[X, Y],$$

we obtain that

$$\omega[X, Y] = \iota_X d(\omega Y) - \iota_Y d(\omega X) = \omega(\nabla_X Y - \nabla_Y X). \tag{4.2}$$

Next, given two vector fields $X, Y \in \mathfrak{X}(L)$ on a leaf L, we define

$$\nabla^L_X Y\Big|_p := \nabla_{\tilde{X}} \tilde{Y}\Big|_p \in T_p L \tag{4.3}$$

at $p \in L$, where $\tilde{X}, \tilde{Y} \in \Gamma(\mathcal{D}|_U) \subset \mathfrak{X}(U)$ are arbitrary local extensions of X and Y to an open neighborhood U of p, that is $\tilde{X}|_{L\cap U} = X|_{L\cap U}$ and $\tilde{Y}|_{L\cap U} = Y|_{L\cap U}$. From the corresponding properties of $\nabla_{\tilde{X}}$ it is clear that ∇^L_X satisfies the Leibniz rule and is tensorial in X, provided that the above definition (4.3) does not depend on the choice of the local extensions of X and Y. Then the equations (4.1) and (4.2) show that ∇^L is flat and torsion-free. To prove that $\nabla_{\tilde{X}} \tilde{Y}|_p$, $p \in L$, is independent of the local extensions $\tilde{X}, \tilde{Y}$ of X, Y amounts to showing that $\nabla_{\tilde{X}} \tilde{Y}|_p = 0$ if $X|_{L\cap U} = 0$ or $Y|_{L\cap U} = 0$. In the case $X|_{L\cap U} = 0$ the claim follows immediately from the tensoriality of $\tilde{X} \mapsto \nabla_{\tilde{X}}$ and implies the case $Y|_{L\cap U} = 0$, since, by (4.2),

$$\nabla_{\tilde{X}} \tilde{Y}|_p = \nabla_{\tilde{Y}} \tilde{X}|_p + [\tilde{X}, \tilde{Y}]_p = \nabla_{\tilde{Y}} \tilde{X}|_p + [X, Y]_p.$$

$\square$

Remark. The map $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), (X, Y) \mapsto \nabla_X Y$, is an example of what is called a *partial connection* on the vector bundle $\mathcal{D} \to M$. It satisfies the Leibniz rule in Y and is $C^\infty(M)$-linear in X, but covariant differentiation is only defined in the direction of vector fields X tangent to $\mathcal{D}$. We will call ∇ the *Weinstein partial connection*.

Example 4.2. *Consider the symplectic manifold $N = T^*M$ endowed with the vertical polarization $\mathcal{D} = T^vN \subset TN$. Let $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ be canonical local coordinates on $T^*U \subset N$ associated with some choice of local coordinates on some open subset $U \subset M$. Then the canonical symplectic form is given by $\omega = \sum dp_i \wedge dq^i$ and*

$$\mathcal{D} = \operatorname{span}\left\{ \frac{\partial}{\partial p_i} \middle| i = 1, \ldots, n \right\}.$$

To compute the Weinstein partial connection in the local frame $(\partial_{p_i}) = (\partial/\partial p_i)$ of $\mathcal{D}$, we observe that

$$\phi_\omega \partial_{p_j} = dq^j \quad \text{and} \quad \mathcal{L}_{\partial_{p_i}} dq^j = 0.$$

This proves that

$$\nabla_{\partial_{p_i}} \partial_{p_j} = 0$$

and that the functions $p_i|_L$ form a global affine coordinate system with respect to the Weinstein connection ∇^L on every fibre L of T^*U.

Definition 4.3. *Let $\mathcal{D} \subset TM$ be a polarization of a symplectic manifold (M, ω). A smooth function $f : U \to \mathbb{R}$ defined on some open subset $U \subset M$ is called* polynomial of degree $\leq k$ along (the leaves of) $\mathcal{D}$ *if $(\nabla^L)^k (df|_{TL}) = 0$ for all leaves L of $\mathcal{D}|_U$, or, equivalently, if*

$$\nabla^k (df|_{\mathcal{D}}) = 0.$$

We denote by $P^k_{\mathcal{D}}(U) \subset C^\infty(U)$ the vector space of functions which are polynomial of degree $\leq k$ along $\mathcal{D}$.

Remark. The map $U \mapsto P^k_{\mathcal{D}}(U)$ is a subsheaf of the sheaf of smooth functions C^∞_M (when considered as a sheaf of vector spaces).

Example 4.4. *In the Example 4.2 of the cotangent bundle* $\pi : N = T^*M \to M$ *with the vertical polarization* $\mathcal{D}$ *we have that* $P^k_{\mathcal{D}}(T^*U)$ *consists precisely of polynomials of degree* $\le k$ *in the variables* p_i, $i = 1, \dots, n$, *with coefficients in* $C^\infty(U)$, *that is the coefficients depend only on the variables* q^i, $i = 1, \dots, n$. *As a consequence,* $P^0_{\mathcal{D}}(N) = \pi^* C^\infty(M) \subset C^\infty(N)$. *Notice also that the distribution* $\mathcal{D}$ *is spanned on* $T^*U \subset N$ *by the Hamiltonian vector fields of the functions* $q^1, \dots, q^n \in P^0_{\mathcal{D}}(T^*U)$.

Proposition 4.5 (Hamilton-Jacobi)**.** *Let* $\mathcal{D} \subset TM$ *be a polarization of a symplectic manifold* (M, ω) *and* $f \in C^\infty(M)$. *Then the following are equivalent:*

(i) $f \in P^0_{\mathcal{D}}(M)$, *that is* $df|_{\mathcal{D}} = 0$.

(ii) $X_f \in \Gamma(\mathcal{D})$.

Proof. Since $\omega X_f = -df$, we see that (i) $\Leftrightarrow$ $X_f \in \mathcal{D}^\perp = \mathcal{D}$, where $\perp$ stands for perpendicularity with respect to ω. □

Corollary 4.6. $P^0_{\mathcal{D}}(M) \subset C^\infty(M)$ *is an Abelian Poisson subalgebra.*

Proof. Given $f, g \in P^0_{\mathcal{D}}(M)$, one easily obtains $fg \in P^0_{\mathcal{D}}(M)$ by differentiation. In virtue of Proposition 4.5 (ii), we also have $X_f, X_g \in \Gamma(\mathcal{D})$ and, hence,

$$\{f, g\} = \omega(X_f, X_g) = 0.$$

□

Proposition 4.7. *Let* $\mathcal{D} \subset TM$ *be a polarization of a symplectic manifold* (M, ω) *and* $f \in C^\infty(M)$. *Then the following are equivalent:*

(i) $f \in P^1_{\mathcal{D}}(M)$, *that is* $\nabla(df|_{\mathcal{D}}) = 0$.

(ii) $[X_f, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.

Proof. By the proof of the Frobenius theorem, we know that locally there exists functions $f_1, \ldots, f_n \in P^0_{\mathcal{D}}(U)$, $U \subset M$ open, such that $\mathcal{D} = \cap_{i=1}^n \ker df_i$. In view of Proposition 4.5, this implies that $\mathcal{D}$ is locally spanned by the Hamiltonian vector fields X_{f_i}. Let us first observe that, by Corollary 4.6,

$$\nabla_{X_{f_i}} X_{f_j} = -\phi_\omega^{-1} \mathcal{L}_{X_{f_i}} df_j = -\phi_\omega^{-1} dX_{f_i}(f_j) = -\phi_\omega^{-1} d\{f_i, f_j\} = 0.$$

Hence for $\alpha = df|_{\mathcal{D}} \in \Gamma(\mathcal{D}^*)$ we have

$$(\nabla_{X_{f_i}} \alpha) X_{f_j} \quad = \quad X_{f_i} \underbrace{\alpha(X_{f_j})}_{=\{f_j, f\}} - \alpha(\nabla_{X_{f_i}} X_{f_j}) = \{f_i, \{f_j, f\}\} = \omega(X_{f_i}, [X_{f_j}, X_f]).$$

Since the vector fields X_{f_i} locally span the distribution $\mathcal{D}$, this implies that $\nabla \alpha = 0$ if and only if $[X_f, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$. $\square$

Corollary 4.8. *$P^1_{\mathcal{D}}(M) \subset C^\infty(M)$ is a Lie subalgebra which normalizes $P^0_{\mathcal{D}}(M)$.*

Proof. Let $f, g \in P^1_{\mathcal{D}}(M)$. Then $[X_f, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$ and $[X_g, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$, by Proposition 4.7. Since $X_{\{f,g\}} = [X_f, X_g]$, we conclude, using the Jacobi identity, that $[X_{\{f,g\}}, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$, that is $\{f, g\} \in P^1_{\mathcal{D}}(M)$.

Next let $f \in P^1_{\mathcal{D}}(M)$ and $g \in P^0_{\mathcal{D}}(M)$. Then $X_g \in \Gamma(\mathcal{D})$ (by Proposition 4.5) and $X_{\{f,g\}} = [X_f, X_g] \in \Gamma(\mathcal{D})$ (by Proposition 4.7). Therefore,

$$0 = \omega(X_{\{f,g\}}, \cdot)|_{\mathcal{D}} = -d\{f, g\}|_{\mathcal{D}},$$

that is $\{f, g\} \in P^0_{\mathcal{D}}(M)$. $\square$

Remark. Similarly, one can also show (exercise) that $f \in P^1_{\mathcal{D}}(M)$ if and only if $\{f, g\} \in P^0_{\mathcal{D}}(U)$ for all $g \in P^0_{\mathcal{D}}(U)$ and all open subsets $U \subset M$. (It follows that $P^1_{\mathcal{D}}$ is precisely the normalizer of the sheaf $P^0_{\mathcal{D}} \subset C^\infty_M$ of Lie subalgebras.) Notice that this condition is in general stronger than requiring $\{f, P^0_{\mathcal{D}}(M)\} \subset P^0_{\mathcal{D}}(M)$. In fact, it might happen that $P^0_{\mathcal{D}}(M)$ contains only constant functions, in which case the latter condition is void.

The local description of $P^k_{\mathcal{D}}(U)$ in Example 4.4 in terms of polynomial functions in the momentum variables p_i can be extended to arbitrary symplectic manifolds by the following refinement of the Darboux theorem.

Theorem 4.9 (Darboux-Weinstein)**.** *Let $\mathcal{D} \subset TM$ be a polarization of a symplectic manifold (M, ω), where $\dim M = 2n$. Then for every $p \in M$ there exists an open neighborhood $U \subset M$ of p and a symplectic diffeomorphism $\phi : U \to V \subset T^*\mathbb{R}^n$ onto an open neighborhood V of the zero section in $T^*\mathbb{R}^n$, which maps $\mathcal{D}|_U$ to the vertical distribution of $V \subset T^*\mathbb{R}^n$.*

Proof. (cf. eg. [Wo, Prop. 4.7.1]) Using the ordinary Darboux theorem one can easily show that there exists a Lagrangian submanifold $Q \subset M$ trough p which intersects the leaves of $\mathcal{D}$ transversally and which is diffeomorphic to $\mathbb{R}^n$. Using the exponential map of the Weinstein connection we can map a connected open neighborhood of $0 \in \mathcal{D}_q$, $q \in Q$, into the leaf L_q of $\mathcal{D}$ through $q \in Q$. This map provides a diffeomorphism from an open neighborhood of the zero section of $\mathcal{D}|_Q \to Q$ to an open neighborhood U of Q in M. Since $\mathcal{D}_q \subset T_qM$ is Lagrangian and complementary to the Lagrangian subspace $T_qQ \subset T_qM$, we can canonically identify $\mathcal{D}_q$ with T^*_qQ by means of ω and, hence, the vector bundle $\mathcal{D}|_Q$ with T^*Q. As a consequence, we have constructed a diffeomorphism from a neighborhood V of the zero section in T^*Q to U, which maps the zero section to $Q \subset M$ and the vertical polarization to $\mathcal{D}|_U$.

We have to show that under this diffeomorphism the canonical symplectic form ω_{can} of T^*Q corresponds to the symplectic form ω of M. This is, in fact, clear along the zero section $s : Q \to T^*Q$, since the Lagrangian decomposition $T_{s(q)}(T^*Q) = T_{s(q)}s(Q) \oplus T^v_{s(q)}(T^*Q) \cong T_qQ \oplus T^*_qQ$ corresponds precisely to the Lagrangian decomposition $T_qM = T_qQ \oplus \mathcal{D}_q$ and the duality pairing between T_qQ and T^*_qQ to the pairing between T_qQ and $\mathcal{D}_q$ induced by ω. Let us denote by ω' the symplectic form on $U \subset M$ which corresponds to ω_{can} under the above diffeomorphism. Hamiltonian vector fields with respect to ω and ω' will be denoted by X_f and X'_f, respectively. We claim that $X_f = X'_f$ if $f \in P^0_{\mathcal{D}}(U)$. In fact, X_f and X'_f are both tangent to $\mathcal{D}$ and coincide along Q.

Moreover, they are both parallel for the Weinstein partial connection ∇ of $(M, \omega, \mathcal{D})$. For X_f this follows from the calculation

$$\omega \nabla_X X_f = -\mathcal{L}_X df = -d(X(f)) = 0,$$

for all $X \in \Gamma(D)$. For X'_f this follows from the fact that the Weinstein connection $\nabla^L = \nabla^{L,\omega}$ of any leaf $L = L_q$ with respect to ω coincides with the standard flat connection of T_q^*Q (i.e. the Weinstein connection of $T_q^*Q \subset T^*Q$) under the identification of the cotangent spaces T_q^*Q with the leaves L_q, $q \in Q$. (Here we use that the identification $T_q^*Q \cong \mathcal{D}_q$ is linear and the exponential map for the Weinstein connection is affine, since the connection is flat and torsion-free.) This implies that $\nabla^L = \nabla^{L,\omega'}$. Therefore, we can conclude that $X_f = X'_f$. Now it suffices to remark that for every point $p \in L$, the tensor ω_p is obtained from $\omega_q = \omega'_q$ by the flow (of translations) generated by some Hamiltonian vector field X_f, $f \in P^0_{\mathcal{D}}(U)$, and ω'_p is obtained by the flow of $X'_f = X_f$. This proves that $\omega_p = \omega'_p$ for all $p \in U$. □

Corollary 4.10. $\{P^k_{\mathcal{D}}(M), P^\ell_{\mathcal{D}}(M)\} \subset P^{k+\ell-1}_{\mathcal{D}}(M)$.

Proof. By the Darboux-Weinstein theorem we can assume without loss of generality that M is a neighborhood of the zero section in $T^*\mathbb{R}^n$. Then we can compute the Poisson bracket of $f \in P^k_{\mathcal{D}}(M)$ and $g \in P^\ell_{\mathcal{D}}(M)$ from (2.2)

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right),$$

where $\partial_{p_i} f \in P^{k-1}_{\mathcal{D}}(M)$, $\partial_{p_i} g \in P^{\ell-1}_{\mathcal{D}}(M)$, $\partial_{q^i} f \in P^k_{\mathcal{D}}(M)$, $\partial_{q^i} g \in P^\ell_{\mathcal{D}}(M)$. □

In particular, we recover Corollary 4.6 and Corollary 4.8 by considering the cases when $k + \ell \leq 2$.

Theorem 4.11. *Let $\mathcal{D}$ be a real polarization of a symplectic manifold (M, ω) with integral symplectic form. Then the prequantization map $f \mapsto \hat{f} = -i\hbar \nabla_{X_f} + f$, defines a Lie algebra homomorphism from $P^1_{\mathcal{D}}(M) \subset C^\infty(M)$ to the space of differential operators acting on polarized sections of L_{ρ_k}, such* *that* $\hat{1} = \mathrm{Id}$. *As before, the vector space of differential operators is endowed with the Lie bracket* $[A,B]_\hbar = \frac{i}{\hbar}[A,B]$ *and* $\hbar = -\frac{1}{2\pi k}$, $k \in \mathbb{Z}$.

Proof. Let s be a polarized section, that is $\nabla_X s = 0$ for all local sections X of $\mathcal{D}$. We have to show that for every $f \in P^1_{\mathcal{D}}(M)$ the section $\hat{f}s$ is again polarized. Using that, by Proposition 4.7, $[X_f, X]$ is a section of $\mathcal{D}$ we compute

$$\begin{aligned}
\nabla_X(\hat{f}s) &= -i\hbar\nabla_X\nabla_{X_f}s + X(f)s + f\underbrace{\nabla_X s}_{=0} \\
&= -i\hbar(2\pi ik\omega(X,X_f) + \underbrace{\nabla_{[X,X_f]})s}_{=0} + X(f)s \\
&= (\omega(X_f,X) + X(f))s = 0.
\end{aligned}$$

$\square$

This means that we have succeeded in quantizing functions which are polynomial along the polarization and of degree at most 1. Here we have ignored for the moment the problem of the global existence of nontrivial polarized sections (locally such sections do always exist) and the problem of L^2-integrabilty of these sections. The latter problem leads to the consideration of the line bundle of half-forms (or half-densities) over $\mathcal{D}$.

4.2 Half-forms and the Hilbert space of polarized sections

In the Example 3.6 of the cotangent bundle $N = T^*M$ with the vertical polarization $\mathcal{D}$, the polarized sections $\sigma = [(u,s)] \in \Gamma(L_{\rho_k})$ are represented by complex-valued functions $s \in C^\infty(N,\mathbb{C})$ of the form $\pi^* s_M$, where $s_M \in C^\infty(M)$. (We now use the letter s for the function to distinguish it from the classical observable denoted f below.) Such functions are constant along the fibers and therefore never integrable on N, unless $s = 0$. The obvious idea to circumvent this problem is to consider integration on the base manifold

M rather than on the total space N. However, the differential operators $\hat{f}$ associated with functions $f \in P^1_{\mathcal{D}}(N)$ are no longer Hermitian if we redefine the scalar product in this way just using an arbitrary choice of volume form on M. This problem can be solved by taking the tensor product of L_{ρ_k} with a certain complex line bundle, as we will explain next for general polarized symplectic manifolds.

Let (M,ω) be a symplectic manifold of dimension $2n$ and $\mathcal{D} \subset TM$ a polarization. We consider the *canonical bundle* of $\mathcal{D}$, which is the complex line bundle

$$K_{\mathcal{D}} := \bigwedge^n (\mathcal{D}^0)^{\mathbb{C}} \subset \bigwedge^n (T^*M)^{\mathbb{C}},$$

where $(\mathcal{D}^0)^{\mathbb{C}} := \mathcal{D}0 \otimes \mathbb{C}$ denotes the complexification of the annihilator subbundle $\mathcal{D}^0 \subset T^*M$.

Proposition 4.12. *$K_{\mathcal{D}}$ is trivial if M is simply connected.*

Proof. This follows from the fact that $K_{\mathcal{D}}$ is the complexification of a real line bundle and that every real line bundle over a simply connected manifold is trivial (exercise). □

From now on we assume that the space of leaves $\overline{M} := M/\mathcal{D}$ is a (Hausdorff) manifold and that the projection

$$\pi : M \to \overline{M}$$

is a submersion.

Proposition 4.13. *Under this assumption, the canonical bundle $K_{\mathcal{D}}$ can be canonically identified with the pull back*

$$\pi^* \bigwedge^n (T^*\overline{M})^{\mathbb{C}}.$$

In particular, $K_{\mathcal{D}}$ is trivial if $\overline{M}$ is orientable.

Proof. First we observe that the differential $d\pi$ provides an isomorphism between the normal bundle

$$N_{\mathcal{D}} := TM/\mathcal{D} = (TM)/\mathcal{D}$$

of the distribution $\mathcal{D}$ and the pull back

$$\pi^* T\overline{M}$$

of the tangent bundle of the space of leaves. Using the canonical identification $N_{\mathcal{D}}^* \cong \mathcal{D}^0$, its dual map $\pi^* T^* \overline{M} \xrightarrow{\sim} N_{\mathcal{D}}^*$ can be considered as an isomorphism

$$\pi^* T^* \overline{M} \xrightarrow{\sim} \mathcal{D}^0$$

and induces isomorphisms $\pi^* \bigwedge^\ell T^* \overline{M} \cong \bigwedge^\ell \mathcal{D}^0$ for all ℓ. Taking $\ell = n$ and complexifying gives the desired result. $\square$

In the following we will assume that the real line bundle $\bigwedge^n \mathcal{D}^0$ admits a square root $(\bigwedge^n \mathcal{D}^0)^{\frac{1}{2}}$. We denote the corresponding square root of $K_{\mathcal{D}}$ by

$$K_{\mathcal{D}}^{1/2} = \left(\bigwedge^n \mathcal{D}^0 \right)^{\frac{1}{2}} \otimes \mathbb{C}.$$

This is obviously the case if M is simply connected or if $M/\mathcal{D}$ is orientable, since these conditions imply the triviality of $\bigwedge^n \mathcal{D}^0$. The sections of $K_{\mathcal{D}}^{1/2}$ are called *half-forms* along $\mathcal{D}$. Recall the partial connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$, which induces the Weinstein connection on the leaves of $\mathcal{D}$. Using the isomorphism $\phi_\omega : \mathcal{D} \to \mathcal{D}^0$, ∇ induces a partial connection (given by the Lie derivative along $\mathcal{D}$) on $\mathcal{D}^0 \to M$ and, hence, on the complex line bundles $K_{\mathcal{D}}$ and $K_{\mathcal{D}}^{1/2}$. We denote these partial connections again by ∇.

Remark. Using the fact that the vector bundle $\mathcal{D}^0 \to M$ is canonically isomorphic to $(TM/\mathcal{D})^* \to M$, the partial connection ∇ on $\mathcal{D}^0$ induces, by duality, a partial connection ∇ on the normal bundle $N_{\mathcal{D}} = TM/\mathcal{D} \to M$. In particular, ∇ induces a flat connection $\nabla^{L,B}$ in the normal bundle

$$N_L := (TM|_L)/TL$$

of each leaf L of $\mathcal{D}$. This flat connection, which exists on the normal bundle of the leaves of any integrable distribution, is know as the *Bott connection.* In terms of the quotient map

$$TM|_L \to N_L, \quad Y \mapsto [Y],$$

it is given by

$$\nabla_X^{L,B}[Y]\big|_p := [\mathcal{L}_{\tilde{X}}\tilde{Y}|_p], \quad p \in L, \tag{4.4}$$

where $\tilde{X} \in \Gamma(\mathcal{D}|_U)$ and $\tilde{Y} \in \Gamma(TM|_U)$ are local extensions of X and Y to an open neighborhood U of p in M. Using the isomorphism

$$\phi_\omega^* : N_{\mathcal{D}} \to \mathcal{D}^*,$$

which is dual to

$$\phi_\omega : \mathcal{D} \to \mathcal{D}^0 = N_{\mathcal{D}}^*$$

we can identify the normal bundle N_L and T^*L by means of

$$\phi_\omega^*|_{N_L} : N_L \to T^*L.$$

Under this identification, the Bott connection $\nabla^{L,B}$ is precisely the connection on T^*L which is dual to the Weinstein connection ∇^L on TL.

Lemma 4.14. *Let L be a leaf of an integrable distribution $\mathcal{D} \subset TM$ such that the quotient map $\pi : M \to \overline{M} = M/\mathcal{D}$ is a submersion. Then a section s of N_L is parallel with respect to the Bott connection if and only if it projects to a tangent vector in $T_L\overline{M}$. (Here we are considering L as a point in the space of leaves $\overline{M}$.) As a consequence, a section of $N_{\mathcal{D}}$ is parallel along $\mathcal{D}$ if and only if it projects to a vector field on $\overline{M}$.*

Proof. From (4.4) we see that a section $[Y]$ of N_L, $Y \in \Gamma(TM|_L)$, is parallel if and only if it is invariant under the flow φ_t^X of every tangent vector field $X \in \Gamma(\mathcal{D})$. Since such a flow is fibre preserving, $\pi \circ \varphi_t^X = \pi$, it induces the identity on $\overline{M}$. In particular, two vectors in $TM|_L$ related by the flow project to the same vector in $T_L\overline{M}$. Now it suffices to observe that any two points in L can be connected by a finite sequence of flow lines. □

Since $\mathcal{D}^0 = N^*_{\mathcal{D}} \cong \pi^* T^* \overline{M}$ and, hence, $\bigwedge^\ell \mathcal{D}^0 \cong \pi^* \bigwedge^\ell T^* \overline{M}$ we obtain, in particular, the following result.

Corollary 4.15. *A section of $K_{\mathcal{D}}$ is polarized if and only if it is the pull back of a (complex-valued) n-form on $\overline{M} = M/\mathcal{D}$.*

Now we replace the line bundle L_{ρ_k} by the line bundle

$$L_{\frac{1}{2}} := L_{\rho_k} \otimes K_{\mathcal{D}}^{1/2}$$

endowed with the tensor product of the given (partial) connections on the factors. In particular, we can consider polarized sections of $L_{\frac{1}{2}}$.

Proposition 4.16. *The polarized sections of $L_{\frac{1}{2}}$ can be locally written as $\sigma \otimes \nu$, where σ is a local polarized section of L_{ρ_k} and ν is a local polarized section of $K_{\mathcal{D}}^{1/2}$.*

Proof. This follows from the fact that the (partial) connections on L_{ρ_k} and $K_{\mathcal{D}}$ are flat along $\mathcal{D}$. □

Now we replace the differential operator $\hat{f} = -i\hbar \nabla_{X_f} + f$, $f \in C^\infty(M)$, acting on sections of L_{ρ_k}, by a differential operator $\tilde{f}$ acting on sections of $L_{\frac{1}{2}}$:

$$\tilde{f}(\sigma \otimes \nu) := (\hat{f}\sigma) \otimes \nu - i\hbar \sigma \otimes \mathcal{L}_{X_f} \nu, \quad \sigma \in \Gamma(L_{\rho_k}), \quad \nu \in \Gamma(K_{\mathcal{D}}^{1/2}). \tag{4.5}$$

Proposition 4.17. *For every $f \in P^1_{\mathcal{D}}(M)$ the differential operator $\tilde{f}$ acts on the space of polarized sections of $L_{\frac{1}{2}}$.*

Proof. By Proposition 4.16, we know that every polarized section of $L_{\frac{1}{2}}$ can be locally written as $\sigma \otimes \nu$ with polarized factors. We also know that $\hat{f}$ preserves the space of polarized sections of L_{ρ_k} if $f \in P^1_{\mathcal{D}}(M)$. So $(\hat{f}\sigma) \otimes \nu$ is clearly polarized. It remains to show that $\mathcal{L}_{X_f}\nu$ and, hence, $\sigma \otimes \mathcal{L}_{X_f}\nu$ is polarized. Let X be a section of $\mathcal{D}$. Then

$$\nabla_X (\mathcal{L}_{X_f} \nu) = \mathcal{L}_X \mathcal{L}_{X_f} \nu = \mathcal{L}_{[X, X_f]} \nu = 0$$

since $\mathcal{L}_X \nu = \nabla_X \nu = 0$ and $[X_f, X]$ is tangent to $\mathcal{D}$. □

Next we define a scalar product on the space of polarized sections of $L_{\frac{1}{2}}$ such that the operators $\tilde{f}$ associated with the functions $f \in P^1_{\mathcal{D}}(M)$ are Hermitian when acting on polarized sections with support projecting into a compact set of $\overline{M}$. First we note that the Hermitian scalar product on L_{ρ_k} induces a Hermitian scalar product on $L_{\frac{1}{2}}$ with values in the canonical line bundle $K_{\mathcal{D}}$:

$$\langle \sigma \otimes \nu, \sigma' \otimes \nu' \rangle := \langle \sigma, \sigma' \rangle \nu \otimes \bar{\nu}' \in \Gamma(K_{\mathcal{D}}),$$

where $\sigma, \sigma' \in \Gamma(L_{\rho_k})$ and $\nu, \nu' \in \Gamma(K_{\mathcal{D}}^{1/2})$. Here the complex conjugation is with respect to the real form $(\bigwedge^n \mathcal{D}^0)^{\frac{1}{2}} \subset K_{\mathcal{D}}^{1/2}$.

If s, s' are polarized sections of $L_{\frac{1}{2}}$ then $\langle s, s' \rangle$ is a polarized section of $K_{\mathcal{D}}$ and, hence, there exists an n-form $\varphi = \varphi_{s,s'}$ on $\overline{M}$ such that

$$\langle s, s' \rangle = \pi^* \varphi.$$

If φ is Integrable, we define

$$\langle s, s' \rangle_{L^2} := \int_{\overline{M}} \varphi. \tag{4.6}$$

We can now define the Hilbert space of polarized sections as the completion with respect to the scalar product (4.6) of the vector space of polarized sections s of $L_{\frac{1}{2}}$ such that $\pi(\operatorname{supp} s) \subset \overline{M}$ is contained in a compact set.

Theorem 4.18. *The map $f \mapsto \tilde{f}$ defines a homomorphism of Lie algebras, such that $\tilde{1} = \mathrm{Id}$, from the Lie subalgebra $P^1_{\mathcal{D}}(M) \subset C^\infty(M)$ to the algebra of Hermitian differential operators acting on polarized sections of $L_{\frac{1}{2}}$ with support projecting into a compact subset of $\overline{M}$.*

Proof. The condition on the support of the considered sections ensures that the scalar products $\langle \tilde{f} s, s' \rangle_{L^2}$ and $\langle s, \tilde{f} s' \rangle_{L^2}$ are defined for every pair of such sections s, s' of $L_{\frac{1}{2}}$ and every $f \in P^1_{\mathcal{D}}(M)$. To show that the scalar products coincide it suffices to prove that

$$\langle (\tilde{f} - f)s, s' \rangle - \langle s, (\tilde{f} - f)s' \rangle = -i\hbar \mathcal{L}_{X_f} \langle s, s' \rangle$$

is the pull back of an exact n-form on $\overline{M}$, $n = \dim \overline{M}$. To see this we recall that $\langle s, s' \rangle = \pi^* \varphi$ for some n-form φ on $\overline{M}$. Therefore,

$$\mathcal{L}_{X_f} \langle s, s' \rangle = \pi^* \mathcal{L}_{\overline{X}_f} \varphi = \pi^* d\iota_{\overline{X}_f} \varphi,$$

where $\overline{X}_f$ is the projection of X_f to a vector field on $\overline{M}$, which exists by Lemma 4.14. In fact, X_f defines a section $[X_f]$ of the normal bundle $N_{\mathcal{D}}$ and for all $X \in \Gamma(\mathcal{D})$ we have

$$\nabla_X [X_f] = [\mathcal{L}_X X_f] = 0,$$

by Proposition 4.7 (ii). □

Remark. Since the vector field X_f, $f \in P^1_{\mathcal{D}}(M)$, normalizes $\Gamma(\mathcal{D})$, its flow φ_t maps the leaves of the foliation defined by $\mathcal{D}$ to leaves of the foliation. Therefore φ_t induces a flow $\overline{\varphi}_t$ on the space of leaves $\overline{M}$. The corresponding vector field is precisely $\overline{X}_f$.

4.3 Complex polarizations

Definition 4.19. *A* complex polarization *of a symplectic manifold* (M, ω) *is an involutive complex Lagrangian distribution* $\mathcal{D} \subset (TM)^{\mathbb{C}}$*, where the complexified tangent spaces* $(T_pM)^{\mathbb{C}}$*,* $p \in M$*, are considered as complex symplectic vector spaces by means of complex bilinear extension of* ω_p*. A complex distribution* $\mathcal{D} \subset (TM)^{\mathbb{C}}$ *is called* purely real *if* $\mathcal{D} = \overline{\mathcal{D}}$ *and* purely complex *if* $\mathcal{D} \cap \overline{\mathcal{D}} = \{0\}$.

Notice that a purely real polarization $\mathcal{D} \subset (TM)^{\mathbb{C}}$ is the same as the complexification of a real polarization $\mathcal{D}_0 \subset TM$. In fact, $\mathcal{D}_0$ is precisely the fixed point set $\mathcal{D}^\tau$ of $\tau|_{\mathcal{D}}$, where τ denotes the complex conjugation

$$\tau : (TM)^{\mathbb{C}} \to (TM)^{\mathbb{C}}, \quad v \mapsto \bar{v}.$$

In this section we will concentrate on purely complex polarizations.

Proposition 4.20. *Let (M,ω) be a symplectic manifold. The following data are equivalent.*

(i) A purely complex polarization on (M,ω).

(ii) A complex structure J on M which is skew-symmetric with respect to ω.

(iii) A pseudo-Kähler structure (J,g) on M with Kähler form ω.

Proof. (ii) $\iff$ (i). Let J be an almost complex structure on M. Then the eigendistribution $T^{1,0}M := \mathrm{Eig}(J,i) \subset (TM)^{\mathbb{C}}$ is a purely complex distribution. Conversely, given a purely complex distribution $\mathcal{D} \subset (TM)^{\mathbb{C}}$ of rank $n = \frac{1}{2}\dim M$ there is a unique almost complex structure $J \in \Gamma(\mathrm{End}\, TM)$ such that $\mathcal{D} = \mathrm{Eig}(J,i)$ and $\overline{\mathcal{D}} = \mathrm{Eig}(J,-i)$. So we see that a purely complex distribution of rank n is equivalent to a an almost complex structure J. By the Newlander-Nirenberg theorem [N], J is integrable if and only if $T^{1,0}M$ is involutive. Finally, $T^{1,0}M$ is Lagrangian if and only if J is skew-symmetric with respect to ω (exercise).

(ii) $\implies$ (iii). The skew-symmetry of J with respect to ω implies that $g := \omega(J\cdot,\cdot)$ is a pseudo-Riemannian metric for which J is skew-symmetric and $\omega = g(\cdot, J\cdot)$. Since J is also integrable and ω is closed this defines a pseudo-Kähler structure on M with Kähler form ω. The converse statement is trivial. □

A purely complex polarization and the corresponding complex structure J will be called *positive* if the corresponding pseudo-Kähler metric g is positive definite.

Example 4.21. *Let V_0 be a real vector space of dimension $2n$ and V its complexification. A complex subspace $U \subset V$ is called* purely complex *if $U \cap \overline{U} = \{0\}$. Given a constant real symplectic form Ω_0 on V_0 and the corresponding complex symplectic form Ω on V, we have one-to-one correspondences between the following sets of data:*

(i) Purely complex Lagrangian subspaces $L \subset V$,

(ii) Constant complex structures J on V_0 which are skew-symmetric with respect to Ω_0.

(iii) Constant pseudo-Kähler metrics g on V_0 with Kähler form Ω_0.

The Lagrangian subspace $L \subset V$ and the complex structure J in this correspondence will be called *positive* if the corresponding metric g is positive definite. We denote by

$$\mathrm{Gr}_\Lambda^+(V) = \mathrm{Gr}_\Lambda^+(V,\Omega) \subset \mathrm{Gr}_\Lambda(V) \subset \mathrm{Gr}_n(V)$$

the Grassmanian of positive complex Lagrangian subspaces of V. It is an open submanifold of the Grassmannian $\mathrm{Gr}_\Lambda(V)$ of all complex Lagrangian subspaces of V, which is in turn a submanifold of the Grassmannian $\mathrm{Gr}_n(V)$ of all n-dimensional subspaces. The latter are homogeneous projective algebraic varieties (in particular, compact complex manifolds) of complex dimension respectively $\frac{1}{2}n(n+1)$ and n^2. The symplectic group $\mathrm{Sp}(V_0) = \mathrm{Aut}\,(V_0, \Omega_0) \cong \mathrm{Sp}(\mathbb{R}^{2n})$ acts transitively on $\mathrm{Gr}_\Lambda^+(V)$ by holomorphic transformations (exercise) and the stabilizer of a positive complex structure J_0 is the unitary group $\mathrm{U}(V_0) = \mathrm{Aut}\,(V_0, \Omega_0, J_0) \cong \mathrm{U}(n)$. We can therefore identify

$$\mathrm{Gr}_\Lambda^+(V) = \mathrm{Sp}(V_0)/\mathrm{U}(V_0) \cong \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n).$$

The resulting homogeneous space is an example of a Hermitian symmetric space of noncompact type. We will now show that it can be realized as the Siegel upper half-space

$$\mathrm{Sym}^+(\mathbb{C}^n) = \{\mathrm{Im}(Z) > 0\} \subset \mathrm{Sym}(\mathbb{C}^n) := \{Z \in \mathrm{M}(n,\mathbb{C}) | Z^t = Z\}.$$

For this we first remark that every symmetric matrix $S \in \mathrm{M}(n,\mathbb{C})$ defines a Lagrangian subspace

$$L_S := \mathrm{graph}(S) = \left\{ \begin{pmatrix} z \\ Sz \end{pmatrix} \middle| \, z \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n}.$$

In order to check that L_S is Lagrangian, we recall that the standard symplectic form Ω on $\mathbb{C}^{2n}$ is given by

$$\Omega\left(\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z' \\ w' \end{pmatrix}\right) = z^t w' - w^t z', \quad z, w, z', w' \in \mathbb{C}^n.$$

Therefore

$$\Omega\left(\begin{pmatrix} z \\ Sz \end{pmatrix}, \begin{pmatrix} z' \\ Sz' \end{pmatrix}\right) = z^t S z' - (Sz)^t z' = 0,$$

since $S^t = S$.

Proposition 4.22. *The map*

$$\phi : \mathrm{Sym}(\mathbb{C}^n) \to \mathrm{Gr}_\Lambda(\mathbb{C}^{2n}), \quad S \mapsto L_S,$$

is an open holomorphic embedding and induces a biholomorphism

$$\phi^+ := \phi|_{\mathrm{Sym}^+(\mathbb{C}^n)} : \mathrm{Sym}^+(\mathbb{C}^n) \to \mathrm{Gr}_\Lambda^+(\mathbb{C}^{2n}).$$

Under this biholomorphism, the action of $\mathrm{Sp}(\mathbb{R}^{2n})$ *on* $\mathrm{Gr}_\Lambda^+(\mathbb{C}^{2n})$ *corresponds to the following action on* $\mathrm{Sym}^+(\mathbb{C}^n)$*:*

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, S\right) \mapsto (C + DS)(A + BS)^{-1}.$$

Proof. Since the map $S : \mathbb{C}^n \to \mathbb{C}^n$ can be recovered from its graph L_S, the map ϕ is a bijection onto its image. The image consists of those Lagrangian subspaces $L \subset \mathbb{C}^{2n}$ for which the projection

$$\pi : \mathbb{C}^{2n} \to \mathbb{C}^n, \quad \begin{pmatrix} z \\ w \end{pmatrix} \mapsto z,$$

restricts to an isomorphism $\pi|_L : L \to \mathbb{C}^n$. The inverse map $L_S \mapsto S$ is a holomorphic local chart of the standard holomorphic atlas of $\mathrm{Gr}_\Lambda(\mathbb{C}^{2n})$, obtained by restricting the standard holomorphic atlas of $\mathrm{Gr}_n(V)$ to the submanifold $\mathrm{Gr}_\Lambda(\mathbb{C}^{2n})$. This implies that ϕ is an open holomorphic embedding.

Next we show that L_S is positive if and only if $\mathrm{Im}(S) > 0$. Let us denote by J the complex structure on V_0 associated with L_S. To check its positivity if suffices to compute the corresponding metric $g = \Omega_0(J\cdot,\cdot)$ on a pair $(v,\bar{v})$, where $v = \begin{pmatrix} z \\ Sz \end{pmatrix} \in L_S$:

$$g(v,\bar{v}) = i\Omega(v,\bar{v}) = i(z^t\bar{S}\bar{z} - (Sz)^t\bar{z}) = 2z^t\mathrm{Im}(S)\bar{z}.$$

This proves that $g > 0$ if and only if $\mathrm{Im}(S) > 0$.

To prove that ϕ^+ is a biholomorphism it suffices to show that every positive Lagrangian subspace $L \subset \mathbb{C}^{2n}$ lies in the image of ϕ. The positivity of L then implies that it is in the image of ϕ^+. Suppose that L is not in the image of ϕ. Then there exists a vector

$$v = \begin{pmatrix} 0 \\ w \end{pmatrix} \in L \setminus \{0\}.$$

But such a vector is necessarily a null vector with respect to the metric g associated with L, since $g(v,\bar{v}) = i\Omega(v,\bar{v}) = 0$, which contradicts the positivity assumption $g > 0$.

Finally we observe that a symplectic transformation

$$\varphi := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(\mathbb{R}^{2n})$$

maps a positive Lagrangian subspace $L = L_S$ to a positive Lagrangian subspace $L' = L_{S'}$ which consists of all vectors of the form

$$\begin{pmatrix} Az + BSz \\ Cz + DSz \end{pmatrix} = \begin{pmatrix} z' \\ S'z' \end{pmatrix}, \quad z \in \mathbb{C}^n.$$

Since the projections $\pi|_L$ and $\pi|_{L'}$ are both isomorphism, as well as the map $\varphi|_L : L \to L'$, we know that the map $z \mapsto z' = (A + BS)z$ is an isomorphism. Solving for z we obtain $z = (A + BS)^{-1}z'$ and

$$S'z' = (C + DS)z = (C + DS)(A + BS)^{-1}z',$$

that is $S' = (C + DS)(A + BS)^{-1}$. □

4.4 An example of holomorphic quantization

Let us consider $M = \mathbb{R}^{2n}$ as a Kähler manifold with the canonical complex structure J, metric g and symplectic form $\omega = \sum dx^j \wedge dy^j$. As in Example 3.6 we can consider the trivial T^1-principal bundle $P = T^1 \times M$ with the connection form $\theta := dt + \lambda$, where $\lambda := \sum x^j dy^j$. (This is the Liouville form if we identify $\mathbb{R}^{2n}$ and $T^*\mathbb{R}^n$ by the map $(x, y) \mapsto (p, q)$.) With respect to the canonical section u_0 of P we have $u_0^*\theta = \lambda$. Enlarging the structure group of P from $T^1 = \mathbb{R}/\mathbb{Z}$ to $T^1_{\mathbb{C}} := \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ we obtain the trivial principal bundle $P_{\mathbb{C}} = T^1_{\mathbb{C}} \times M$. The connection $\theta = dt + \lambda$ on P is canonically extended to a connection $\theta_{\mathbb{C}} = dt_{\mathbb{C}} + \lambda$ on $P_{\mathbb{C}}$ simply by extending the real coordinate t on $\mathbb{R}$ to a complex coordinate $t_{\mathbb{C}}$ on $\mathbb{C}$. For simplicity we will write again θ and t instead of $\theta_{\mathbb{C}}$ and $t_{\mathbb{C}}$. Next we consider the translational invariant positive polarization $\mathcal{D} \subset (TM)^{\mathbb{C}}$ which corresponds to the complex structure J, that is

$$\mathcal{D}_0 = T^{1,0}M = \operatorname{span}\left\{ \frac{\partial}{\partial z^a} \middle| a = 1, \ldots, n \right\},$$

where $z^a = x^a + iy^a$. The polarized sections $\sigma = [(u, s)]$ of the complex line bundle $L_{\rho_k} = P \times_{\rho_k} \mathbb{C}$ are represented by holomorphic functions s if we use a section u of $P_{\mathbb{C}}$ adapted to the polarization, that is such that $u^*\theta$ vanishes on $\overline{\mathcal{D}}$. We claim that

$$u = u_0 + f, \quad f = -\frac{1}{2}(iK + \sum x^a y^a),$$

has this property, where $K = r^2/2$ is the standard Kähler potential. Indeed, a straightforward calculation shows that

$$u^*\theta = u_0^*\theta + df = \lambda - \frac{i}{2}\partial K - \frac{i}{2}\bar{\partial} K - \frac{1}{2}\sum d(x^a y^a) = -i\partial K.$$

Recall that the L^2-norm of a section $[(u_0, s_0)]$ of L_{ρ_k} is given by

$$\int_M s_0 \bar{s}_0 dvol,$$

where $dvol = \omega^n/n!$ is the canonical symplectic volume form. To compute the norm of a polarized section $\sigma = [(u, s)]$ in terms of the holomorphic function s it suffices to remark that the the corresponding function s_0 such that $\sigma = [(u_0, s_0)]$ is given by:

$$\rho_k(-f)s = \exp(-2\pi i k f)s = \exp\left(\frac{i}{\hbar}f\right)s = \exp\left(\frac{1}{2\hbar}(K - i\sum x^a y^a)\right)s.$$

Therefore

$$\langle\sigma,\sigma\rangle_{L^2} = \int_M s\bar{s}\exp\left(\frac{K}{2\hbar}\right)dvol = \int_M s\bar{s}\exp\left(\frac{r^2}{4\hbar}\right)dvol,$$

The symplectic vector fields preserving the complex polarization $\mathcal{D}$ are precisely the holomorphic Killing vector fields on M. These are the Hamiltonian vector fields generated by the functions z^a, $\bar{z}^b$ and $z^a\bar{z}^b$, $a, b \in \{1, \ldots, n\}$. This means that we can now quantization also functions quadratic in momenta $p_a = x^a$, such as the Hamiltonian $H = K = \frac{r^2}{2}$. This includes the harmonic oscillator as the special case $n = 1$. In that case we have $\omega = \frac{i}{2}dz \wedge d\bar{z}$ and $X_H = i(z\partial_z - \bar{z}\partial_{\bar{z}})$, which implies for all holomorphic functions s:

$$\begin{aligned}\hat{H}s &= -i\hbar\nabla^u_{X_H}s + Hs = -i\hbar(X_H + 2\pi i k u^*\theta(X_H))s + Hs \\ &= -i\hbar X_H(s) - (-i\partial K)(X_H)s + Hs = \hbar z\partial_z s - (z\partial_z K)s + Hs = \hbar z\partial_z s.\end{aligned}$$

Including half-forms in the picture one obtains instead

$$\tilde{H}s = \hbar\left(z\frac{\partial}{\partial z} + \frac{1}{2}\right)s,$$

which is the answer expected by physicists. The extra term is obtained as follows. Generalizing the case of real polarizations, one defines the canonical bundle of a complex polarization as complex the line bundle

$$K_{\mathcal{D}} := \bigwedge^n \overline{\mathcal{D}}^0.$$

where $\overline{\mathcal{D}}^0 \subset (T^*M)^{\mathbb{C}}$ denotes the annihilator of $\overline{\mathcal{D}}$. Then one defines

$$L_{\frac{1}{2}} := L_{\rho_k} \otimes K_{\mathcal{D}}^{\frac{1}{2}}$$

and extends the differential operators $\hat{f}$ on L_{ρ_k} to differential operators $\tilde{f}$ on $L_{\frac{1}{2}}$ by the equation (4.17), as before. Now it suffices to remark that, in our

n-dimensional example, $\Omega = dz^1 \wedge \cdots \wedge dz^n$ is a polarized (i.e. holomorphic) section of $K_{\mathcal{D}}$ and $\mathcal{L}_{X_H}\Omega = i \sum \mathcal{L}_{z^a \partial_{z^a}} \Omega = in\Omega$. In particular, $\mathcal{L}_{X_H}\sqrt{dz} = \frac{1}{2}\sqrt{dz}$ in the one-dimensional case.

See [Wo, 9.2] for further information on this example. A quantization procedure for functions which are polynomial of arbitrary degree along a given polarization (and the Hamiltonian flows of which, therefore, do not in general preserve the polarization) is described in [K2]. See also [Wo] for different but related approaches.

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