Symplectic Geometry

Prof. Thomas Vogel*

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Ludwigs-Maximilians-Universität München

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*Inofficially written by L.Stimpfle

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1.1 Linear Symplectic Geometry

1.1.1 Symplectic Vector Spaces

Definition 1.1. A vector space V is called *symplectic* if it admits a *symplectic form*, i.e. a nondegenerate, skew-symmetric, bilinear form ω. A linear isomorphism $\Phi : (V, \omega) \to (V, \omega)$ is called a *(linear) symplectomorphism* if $\Phi^*\omega = \omega$. The symplectomorphisms of a vector space (V, ω) form a group which is denoted by $\mathrm{Sp}(V, \omega)$.

Example 1.2. The prototypical example of a symplectic vector space is $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{1.1}$$

for a basis $\{x_1, y_1 \dots, x_n, y_n\}$ of $\mathbb{R}^{2n}$. We call ω_0 the *standard symplectic form*. For $n = 1$, we have $\omega_0 = dx \wedge dy$. For a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ it holds that $f^*\omega_0 = \det(f)\omega_0$, hence $f \in \mathrm{Sp}(2)$ if and only if $\det f = 1$, i.e. $\mathrm{Sp}(2) = SL(2, \mathbb{R})$. In particular, $\mathrm{Sp}(V, \omega)$ is not compact in general.

Definition 1.3. Let (V, ω) be a symplectic vector space, $W \subseteq V$ a linear subspace. The *symplectic complement* W^ω of W in V is defined as

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}. \tag{1.2}$$

A subspace W is called

(i) *isotropic* if $W \subseteq W^\omega$.

(ii) *coisotropic* if $W^\omega \subseteq W$.

(iii) *Lagrangian* if $W^\omega = W$.

(iv) *symplectic* if $W^\omega \cap W = \{0\}$.

Lemma 1.4. *For any subspace $W \subseteq V$ it holds that*

$$\dim W + \dim W^\omega = \dim V \qquad \text{and} \qquad W^{\omega\omega} = W. \tag{1.3}$$

Proof. Let $i_\omega : V \to V^*$ be defined by $i_\omega(v)(w) = \omega(v, w)$. Since ω is nondegenerate, i_ω is an isomorphism by which we identify the symplectic complement

$$W^\omega \cong \{\varphi \in V^* \mid W \subseteq \ker \varphi\} = W^\perp \subseteq V^* \tag{1.4}$$

(i.e. $W^\perp$ is just the *annihilator* of W in V^*). It is well-known that for any subspace

$$\dim V = \dim W + \dim W^\perp = \dim W + \dim W^\omega. \tag{1.5}$$

As for the second statement, we identify $V^{**} \cong V$, and note that $W^{\omega\omega} \cong W^{\perp\perp} \cong W$. $\square$

Lemma 1.5. *Let (V, ω) be a symplectic vector space, $W \subseteq V$ a linear subspace. Then the quotient space $W/W\cap W^\omega$ carries a natural symplectic structure.*

Proof. Let $W' = {}^{W}/_{W\cap W^\omega}$. Then $\omega_{|W'}$ is well-defined, since for $[w_i] = w_i + W \cap W^\omega \in W'$, $w_i \in W$, $i = 1, 2$, we have $\omega([w_1], [w_2]) = \omega(w_1, w_2)$. To check that $\omega_{|W'}$ is nondegenerate let $[w] \in W'$ such that $\omega([w], [v]) = 0$ for all $[v] \in W'$. Then $\omega(w, v) = 0$ for all $v \in W$, hence $w \in W^\omega$, i.e. $[w] = 0$. □

Theorem 1.6. *Let (V, ω) be a $2n$-dimensional symplectic vector space. Then there exists a basis $\{e_1, f_1, \ldots, e_n, f_n\}$ such that $\omega(e_j, e_k) = \omega(f_l, f_m) = 0$, $\omega(e_j, f_k) = \delta_{jk}$, and a linear isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that $\Phi^*\omega = \omega_0$.*

Definition 1.7. A basis $\{e_1, f_1, \ldots, e_n, f_n\}$ as in Theorem 1.6 is called *symplectic basis* (sometimes also *ω-standard basis*).

Proof. We use induction on n. For $n = 1$, there exists a basis $\{e_1, f_1\}$ of V such that $\omega(e_1, f_1) = 1$ because ω is nondegenerate. Now let $W = \{e_1, f_1\}^\omega \subseteq V$ denote the symplectic complement of $\{e_1, f_1\}$ in a $2n$-dimensional symplectic vector space V. Since $\omega_{|\{e_1,f_1\}}$ is nondegenerate, $\{e_1, f_1\}$ is a symplectic subspace, and Lemma 1.4 implies that W is symplectic, too. Therefore, $(W, \omega_{|W})$ is a $(2n-2)$-dimensional symplectic vector space and admits a symplectic basis $\{e_2, f_2, \ldots, e_n, f_n\}$, i.e. $\{e_1, f_1, \ldots, e_n, f_n\}$ is a symplectic basis of V. Finally, the linear map $\Phi : \mathbb{R}^{2n} \to V$, $\Phi(x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^n (x_i e_i + y_i f_i)$ is an isomorphism satisfying $\Phi^*\omega = \omega_0$. □

Corollary 1.8. *A skew-symmetric bilinear form ω on a $2n$-dimensional vector space V is nondegenerate if and only if its n-th exterior power $\omega^n = \omega \wedge \cdots \wedge \omega$ is nonvanshing.*

Proof. If ω is degenerate, i.e. there exists $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$, then, for a basis $\{v, v_2, \ldots, v_{2n}\}$ of V, we have $\omega^n(v, v_2, \ldots, v_{2n}) = 0$. Conversely, if ω is nondegenerate, then, since ω_0^n is nonvanishing, also ω^n by Theorem 1.6. □

1.1.2 Complex Structures

Definition 1.9. An automorphism $J : V \to V$ of a vector space V is called *complex structure* if $J^2 = -\mathrm{id}_V$. We denote the space of complex structures on V by $\mathcal{J}(V)$.

We have a natural isomorphism $(V, J) \cong (\mathbb{C}^n, \imath)$ for $2n = \dim_{\mathbb{R}} V$ with multiplication $\mathbb{C} \times V \to V$, $(x + \imath y, v) \mapsto xv + tJv$, i.e. (V, J) is a complex n-dimensional vector space. In particular on $\mathbb{R}^{2n}$ we have the standard complex structure

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{1.6}$$

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via $(x, y) \in \mathbb{R}^{2n} \mapsto x + \imath y \in \mathbb{C}^n$, J_0 corresponds to multiplication with $\imath$, i.e. $(x + \imath y, v) \in \mathbb{C} \times \mathbb{C}^n \mapsto (x + \imath y)v \in \mathbb{C}^n \mapsto xv + yJv \in \mathbb{R}^{2n}$.

Definition 1.10. Let (V, ω) be a symplectic vector space. A complex structure $J \in \mathcal{J}(V)$ is called *compatible* with ω if $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$ and $\omega(v, Jv) > 0$ for all $0 \neq v \in V$. We denote the space of compatible complex structures by $\mathcal{J}(V, \omega)$.

Note that if $J \in \mathcal{J}(V, \omega)$, then $g_J : V \times V \to \mathbb{R}$, $g_J(v, w) = \omega(v, Jw)$ defines an inner product on V. In fact, compatibility of $J \in J(V)$ with ω is equivalent to g_J being an inner product [MS05, p. 63].

Remark 1.11. Given a symplectic vector space (V, ω), $J \in \mathcal{J}(V, \omega)$, then the pairing

$$h = g_J - \imath\omega \tag{1.7}$$

is a Hermitian metric. A unitary basis $\{e_1, \dots, e_{2n}\}$ of V then provides an isomorphism $(V, \omega, J) \to (\mathbb{C}^n, \omega_0, J_0)$.

1.1.3 The Symplectic Linear Group

By Theorem 1.6 all symplectic vector spaces of the same dimension are isomorphic, and to examine the group of linear symplectomorphisms $\mathrm{Sp}(V, \omega)$, it suffices to consider the case $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$. We have seen in section **??** that a linear isomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if and only if

$$\Psi^\top J_0 \Psi = J_0. \tag{1.8}$$

With the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as in section 1.1.2 the complex linear group $GL(n, \mathbb{C})$ is a subgroup of $GL(2n, \mathbb{R})$ and the group of unitary matrices $U(n)$ is a subgroup of $\mathrm{Sp}(2n)$.

Lemma 1.12. *It holds that*

$$\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap GL(n, \mathbb{C}) = U(n), \tag{1.9}$$

where $O(2n)$ is the group of orthogonal matrices w.r.t. the inner product g_{J_0}.

Proof. Recall that a real $(2n \times 2n)$-matrix Ψ satisfies

$$\Psi \in \begin{cases} GL(n, \mathbb{C}) & \Leftrightarrow \quad \Psi J_0 = J_0 \Psi, \\ \mathrm{Sp}(2n) & \Leftrightarrow \quad \Psi^\top J_0 \Psi = J_0, \\ O(2n) & \Leftrightarrow \quad \Psi^\top \Psi = \mathbb{1}. \end{cases} \tag{1.10}$$

Any two of the three conditions in (1.10) implies the third. One can show that the subgroup $\mathrm{Sp}(2n) \cap O(2n)$ consists of matrices of the form

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in GL(2n, \mathbb{R}) \tag{1.11}$$

with $X^\top Y = Y^\top X$, $X^\top X + Y^\top Y = \mathbb{1}$. This is equivalent to $\Psi = X\mathbb{1} + Y J_0 \cong X + \imath Y$ being unitary. □

Lemma 1.13. *Let $\Psi \in \mathrm{Sp}(2n)$ and $\lambda \in \mathbb{R}$ be an eigenvalue of Ψ. Then λ^{-1} is an eigenvalue of Ψ with the same multiplicity. Moreover, if ± 1 is an eigenvalue of Ψ, it has even multiplicity.*

Proof. □

Lemma 1.14. *If $P \in \mathrm{Sp}(2n)$ is a symmetric, positive definite matrix, then $P^a \in \mathrm{Sp}(2n)$ for all $0 < a \in \mathbb{R}$.*

Proof. □

Recall that, for any $\Psi \in GL(n, \mathbb{R})$, there is a unique decomposition, the *polar decomposition*, as $\Psi = PO$ with $O \in O(n)$, and P symmetric and positive definite. (As for existence one sets $P = (\Psi^\top \Psi)^{1/2}$, and $O = (\Psi^\top \Psi)^{-1/2}\Psi$.)

Lemma 1.15. *Let $\Psi = PO \in \mathrm{Sp}(2n, \mathbb{R})$ be the polar decomposition. Then $P, O \in \mathrm{Sp}(2n, \mathbb{R})$.*

Proof. □

Corollary 1.16. *There is a deformation retract of $\mathrm{Sp}(2n)$ onto $U(n)$.*

Proof. Let $\iota : U(n) \to \mathrm{Sp}(2n)$ denote the inclusion and let $h : \mathrm{Sp}(2n) \to U(n)$ be defined by $\Psi = PO \in \mathrm{Sp}(2n) \mapsto O \in \mathrm{Sp}(2n) \cap O(n) = U(n)$. Then $\iota \circ h$ is homotopic to $\mathrm{id}_{U(n)}$ via the homotopy $H : \mathrm{Sp}(2n) \times [0, 1] \to \mathrm{Sp}(2n)$, $H(\Psi, t) = P^t O$. □

Lemma 1.17. *$U(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$, i.e. every compact subgroup $K \subseteq \mathrm{Sp}(2n)$ is conjugate to a subgroup of $U(n)$.*

Proof. Let $K \subseteq \mathrm{Sp}(2n)$ be a compact subgroup. Using a Haar measure we may choose a symmetric, positive definite matrix $P \in \mathrm{Sp}(2n)$ such that $\Psi^\top P \Psi = P$ for all $\Psi \in K$. By Lemma 1.14 also $P^{1/2} \in \mathrm{Sp}(2n)$. Since for $\Psi \in K$

$$(P^{1/2}\Psi P^{-1/2})(P\Psi P^{-1/2})^\top = P^{1/2}P^{-1}P^{1/2} = \mathbb{1}, \tag{1.12}$$

we have $P^{1/2}\Psi P^{-1/2} \in \mathrm{Sp}(2n) \cap O(2n) = U(n)$. □

Theorem 1.18. *Let (V, ω) be a symplectic vector space. Then $J(V, \omega)$ is contractible.*

Proof. We show that $\mathcal{J}(V, \omega)$ is homeomorphic to the space of symmetric, positive definite, symplectic matrices, which is contractible by Lemma 1.14. By Theorem 1.6 we may assume w.l.o.g. $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. A $(2n \times 2n)$-matrix J is a compatible complex structure if and only if $J^2 = -\mathbb{1}$, $J^\top J_0 J = J_0$, and $\omega_0(v, Jv) = \langle v, -J_0 J v\rangle > 0$ for all $v \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^{2n}$. It follows $(J_0 J)^\top = -J^\top J_0 = J^\top J_0 J^2 = J_0 J$, i.e. $P = J_0 J$ is symmetric, positive definite and symplectic. Conversely, a symmetric, positive definite and symplectic matrix P defines a compatible complex structure by setting $J = -J_0^{-1} P \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$. □

1.2 Symplectic Structures on Manifolds

Definition 1.19. A *symplectic structure* on a smooth manifold M is a nondegenerate, closed two-form $\omega \in \Omega^2(M)$. The pair (M, ω) is then called a *symplectic manifold*.

Example 1.20. Given two symplectic manifolds (M_i, ω_i), $i = 1, 2$, the product $M_1 \times M_2$ is a symplectic manifold $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$ for the natural projections $\pi_i : M_1 \times M_2 \to M_i$.

Nondegeneracy of a symplectic structure $\omega \in \Omega^2(M)$ implies that each tangent space $(T_p M, \omega_p)$ is a symplectic vector space for all $p \in M$. By Corollary 1.8 the n-th exterior power $\omega^n \in \Omega^{2n}(M)$ of symplectic structure $\omega \in \Omega^2(M)$ is a volume form. Moreover, the map $\Gamma(TM) \to \Omega^1(M)$, $X \mapsto i_X\omega$ is an isomorphism.

Closedness of a symplectic structure implies that ω represents a cohomology class $[\omega] \in H^2(M; \mathbb{R})$. If M is closed, $\int_M \omega^n > 0$ implies that ω cannot be exact, hence there is, in each even degree $2k$, a nonvanishing cohomology class $[\omega^k] = [\omega]^k \neq 0$. Thus, orientable $2n$-dimensional manifolds with $H^{2n}(M; \mathbb{R}) = 0$ do not admit a symplectic structure, e.g. S^{2n} for $n \geqslant 2$ does not admit a symplectic structure, since $H^k(S^{2n}; \mathbb{R}) = 0$ for all $k \neq 0, 2n$.

Definition 1.21. A diffeomorphism $\Phi : M \to M$ which preserves the symplectic structure, i.e. $\Phi^*\omega = \omega$, is called a *symplectomorphism*. The group of symplectomorphisms is denoted by $\mathrm{Symp}(M, \omega)$.

Example 1.22 (Cotangent Bundle)**.** Let M be a smooth n-dimensional manifold. Then its cotangent bundle $L = T^*M$ carries a canonical symplectic structure. In order to understand this we introduce the *tautological one-form* $\lambda \in \Omega^1(L)$ as follows. Consider the natural foot point projection $\pi : L = T^*M \to M$ and the induced map $d\pi : TL = TT^*M \to TM$. We define $\lambda : L \to T^*L$ by setting

$$\lambda_\eta(v) = \eta(d\pi(v)) \tag{1.13}$$

for $\eta \in L$, $v \in T_\eta L$. Then $\lambda : L \to T^*L$ is a section of $T^*L \to L$, $(\eta, v^*) \mapsto \eta$, i.e. $\lambda \in \Omega^1(L)$. Smoothness of λ can be made explicitly in a local chart of the tangent bundle L, $\eta = \sum_{i=1}^n x_i dy_i \in L \mapsto (x_i, y_i) \in \mathbb{R}^{2n}$. The tautological one-form λ is then

$$\lambda_\eta = \sum_{i=1}^n x_i dy_i. \tag{1.14}$$

The canonical symplectic structure $\omega \in \Omega^2(L)$ is then defined as

$$\omega = -d\lambda = \sum_{i=1}^n dx_i \wedge dy_i. \tag{1.15}$$

Clearly (1.15) defines a closed, nondegenerate two-form $\omega \in \Omega^2(L)$ turning (L, ω) into a symplectic manifold of dimension $2n$.

Lemma 1.23. *The tautological one-form $\lambda \in \Omega^1(L)$ on the cotangent bundle $L = T^*M$ is uniquely determined by*

$$\eta^*\lambda = \eta \tag{1.16}$$

for $\eta \in \Omega^1(M)$.

Proof. As in Example 1.22 a one-form $\eta \in \Omega^1(M)$ may be written in local coordinates as $\eta = \sum_{i=1}^n x_i dy_i \mapsto (x_1(y), \dots, x_n(y), y_1, \dots, y_n)$, i.e. $\eta : M \to T^*M = L$ is given by $x \mapsto (x(y), y)$. Thus $\eta^*\lambda = \sum_{i=1}^n x_i(y_i) dy_i = \eta$. $\square$

Lemma 1.24. *Let M be a smooth manifold and let $L = T^*M$ be its cotangent bundle. A diffeomorphism $\psi : M \to M$ lifts to a symplectomorphism $\Psi : (L, -d\lambda) \to (L, -d\lambda)$.*

Proof. Because pullback commutes with exterior derivative, it suffices to show $\Psi^*\lambda = \lambda$. Let $\eta \in T_p^*M$. We define $\Psi : L \to L$ by $\Psi(\eta) = \eta \circ d\psi^{-1}_{\psi(p)}$. Then $\pi \circ \Psi = \psi \circ \pi$, hence

$$(\Psi^*\lambda)_\eta(v) = \lambda_{\Psi(\eta)}(d\Psi_\eta(v)) = \Psi(\eta)\big(d\pi_{\Psi(\eta)} \circ d\Psi_\eta(v)\big) \tag{1.17}$$

$$= \eta\Big(d(\psi^{-1} \circ \pi \circ \Psi)_\eta(v)\Big) = \eta(d\pi_\eta(v)) \tag{1.18}$$

$$= \lambda_\eta(v), \tag{1.19}$$

i.e. $\Psi^*\omega = -d\Psi^*\lambda = -d\lambda = \omega$. $\square$

Lemma 1.25. *Let $M \subseteq T^*M = L$ denote the zero section, $\eta \in M \subseteq L$, $\pi(\eta) = p \in M$. Then there is an isomorphism*

$$T_\eta L = T_\eta T^*M \cong T_pM \oplus T_p^*M. \tag{1.20}$$

Proof. Let $\pi : L \to M$, $\eta \in L$, $\pi(\eta) = p \in M$. Then $d\pi_\eta : T_\eta L \to T_pM$ with $\ker d\pi_\eta$ consists of vectors $v \in T_\eta \pi^{-1}(p)$, where $\pi^{-1}(p) = T_p^*M$. Since $T_\eta T_p^*M \cong T_p^*M$, we obtain $\ker d\pi_\eta \cong T_p^*M$. Because $\dim T_\eta L = \dim T_pM + \dim T_p^*M$, we have $T_\eta L \cong T_pM \oplus T_p^*M$ for all $\eta \in M \subseteq L$ in the zero section M of L. □

Lemma 1.26. *Along the zero section M in $L = T^*M$, i.e. for $\eta \in M \subseteq T_qM$, and $v = (v_0, v_1^*), w = (w_0, w_1^*) \in T_qM \oplus T_q^*M$, the canonical symplectic form $-d\lambda \in \Omega^2(L)$ is given by*

$$-d\lambda_\eta(v, w) = w_1^*(v_0) - v_1^*(w_0). \tag{1.21}$$

Proof. Using the identification (1.20) of Lemma 1.25 with local coordinates $(U, (x, y))$ of $T_qM \oplus T_q^*M$, we have $-d\lambda = \sum_{i=1}^n dx_i \wedge dy_i$, $v = (v_0, v_1^*) = (v_{01}, v_{11}^*, \ldots, v_{0n}, v_{1n}^*)$, hence

$$-d\lambda(v, w) = -\sum_{i=1}^n (v_{0i}w_{1i}^* - w_{0i}v_{1i}^*) = w_1^*(v_0) - v_1^*(w_0).$$ □

Remark 1.27. Lemma 1.25 can be generalized to the whole bundle $TL = TT^*M$, i.e. there exists a bundle isomorphism $\Phi : TL \to TM \oplus T^*M$, where T^*M is identified with the subbundle of tangent spaces to the fiber.

1.2.1 Hamiltonian Vector Fields and Poisson Structures

Let (M, ω) be a symplectic manifold, and $H : M \to \mathbb{R}$ a smooth function. Due to non-degeneracy of ω there exists a unqiue vector field $X = X^H \in \Gamma(TM)$ corresponding to $H \in C^\infty(M)$ such that

$$dH = i_X\omega. \tag{1.22}$$

Definition 1.28. A smooth function $H : M \to \mathbb{R}$ is called *Hamiltonian* and the vector field $X^H \in \Gamma(TM)$ uniquely determined by (1.22) is called *Hamiltonian vector field.* Converserly, a vector field $X \in \Gamma(TM)$ is called

(i) *symplectic* if $L_X\omega = 0$.

(ii) *(locally) Hamiltonian* if $i_X\omega \in \Omega^1(M)$ is (locally) exact.

Remark 1.29. Symplectic vector fields define a Lie algebra and one can show that, for a closed symplectic manifold, the space of symplectic vector fields is the Lie algebra of the group of symplectomorphisms $\mathrm{Symp}(M, \omega)$ [MS05, Proposition 3.2].

Lemma 1.30. *A vector field $X \in \Gamma(TM)$ is locally Hamiltonian if and only if it is symplectic.*

Proof. By Cartan's formula and closedness of ω, $X \in \Gamma(TM)$ is symplectic if and only if $L_X\omega = di_X\omega = 0$, i.e. if and ony if $i_X\omega$ is closed which is equivalent to local exactness. □

Lemma 1.30 shows that the flow Φ_t generated by a Hamiltonian vector X^H field is a symplectomorphism.

Lemma 1.31. *The flow $\Phi_t : M \to M$ of $X = X^H \in \Gamma(TM)$ preserves $H : M \to \mathbb{R}$.*

Proof. By Cartan's formula $L_X H = i_X dH = \omega(X, X) = 0$. □

Lemma 1.31 shows that a Hamiltonian vector field is tangent to level sets of the corresponding Hamiltonian function H.

Example 1.32. Consider the two-sphere $S^2 \subseteq \mathbb{R}^3$ with height function $H : S^2 \to \mathbb{R}$, $H(x_1, x_2, x_3) = x_3$. Because

$$\omega = i_{x_1\partial_1+x_2\partial_2+x_3\partial_3}(dx_1 \wedge dx_2 \wedge dx_3) = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_3, \quad (1.23)$$

the corresponding Hamiltonian vector field $X = X^H$ is determined by

$$i_X \omega_{|S^2} = dx_{3|S^2} \qquad \Leftrightarrow \qquad i_X \omega = dx_3 + dr^2. \quad (1.24)$$

Equation (1.24) is solved by $X = -x_2\partial_1 + x_1\partial_2$, i.e. the flow of X is a rotation of S^2 around the x_3-axis, thus preserving level sets of H. In particular the flow is 2π-periodic.

Definition 1.33. Let F, G be smooth functions on a symplectic manifold (M, ω). We define the *Poisson bracket* $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$\{F, G\} = \omega(X^F, X^G) = i_{X^G} dF = L_{X^G} F. \quad (1.25)$$

Lemma 1.34. *The Poisson bracket is antisymmetric and bilinear. For all $F \in C^\infty(M)$, $\{F, \cdot\}$ is a derivation, i.e. $\{F, GH\} = \{F, G\}H + \{F, H\}G$. Moreover, for all F, G, H*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \quad (1.26)$$

Thus $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.

Lemma 1.35. *Let F, G be smooth functions on a symplectic manifold. Then $[X^F, X^G]$ is the Hamiltonian vector field of $-\{F, G\}$. Thus the map $H \in C^\infty(M) \mapsto X^H \in \Gamma(TM)$ is a Lie algebra antihomomorphism.*

Proof. The proof relies on the formula $i_{[X,Y]}\alpha = L_X i_Y \alpha - i_Y L_X \alpha$ for all $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$. Apply this to X^F, X^G, i.e.

$$i_{[X^F, X^G]}\omega = L_{X^F} i_{X^G} \omega - i_{X^G} L_{X^F} \omega = L_{X^F} dG = d\{G, F\} = -d\{F, G\} \quad (1.27)$$

since $L_{X^F}\omega = d i_{X^F} \omega = d^2 F = 0$. □

Lemma 1.35 also shows that Hamiltonian vector fields define a Lie subalgebra of symplectic vector fields (cf. Remark 1.29).

Definition 1.36. A *Poisson structure* P on a manifold is a Lie algebra structure on $C^\infty(M)$ such that $\{F, \cdot\}$ is a derivation for all $F \in C^\infty(M)$.

Example 1.37. Any symplectic Manifold (M, ω) has a Poisson structure as above.

Example 1.38. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite dimensional Lie algebra. Then $\mathfrak{g}^*$ has a canonical Poisson structure. Let $F, G \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. Since $\mathfrak{g}^{**} \cong \mathfrak{g}$, we can view

$$dF_\xi : T_\xi \mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R} \tag{1.28}$$

as an element of $\mathfrak{g}$. Now define

$$\{F, G\}(\xi) = \xi([dF_\xi, dG_\xi]). \tag{1.29}$$

This is a Poisson structure.

Remark 1.39. Example 1.38 shows that Poisson manifolds do not need to come from symplectic structures on manifolds. Weinstein showed: Given $(P, \{\cdot,\cdot\})$ a Poisson manifold, P can be decomposed into a collection of symplectic submanifolds such that on each submanifold $\{\cdot,\cdot\}$ is induced by the symplectic structure.

1.2.2 Normal Forms and Moser's Argument

In this section we want to investigate possible local invariants of symplectic structures on manifolds. More precisely, we will show that symplectic manifolds (opposed to e.g. Riemannian manifolds) do not have any local invariants. Here local can mean in a neighborhood around a point (cf. Darboux's Theorem 1.49) as well as around a symplectic or Lagrangian submanifold (cf. Theorems 1.74 and 1.75).

First let us introduce some general (non-symplectic) terminology. Let M be a smooth manifold. For a map $\Phi : M \times \mathbb{R} \to M$ we set $\Phi_t(p) = \Phi(p, t)$.

Definition 1.40. We call $(\Phi_t)_{t\in\mathbb{R}}$ a *smooth isotopy* if $\Phi_t : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$, and the map $(p, t) \in M \times \mathbb{R} \mapsto \Phi_t(p) \in M$ is smooth.

An isotopy Φ_t gives rise to a time dependent vector field X_t, i.e. a smooth family of vector fields $(X_t)_{t\in\mathbb{R}}$, by means of

$$X_t(\Phi_t(p)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \Phi_s(p). \tag{1.30}$$

Conversely, a time dependent vector field X_t determines (at least if X_t has compact support or M is comapct) an isotopy by means of (1.30).

Lemma 1.41. *Let Φ_t be an isotopy with time dependent vector field X_t and let $\eta_t \in \Omega^k(M)$ a smooth family of differential forms. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^*\eta_t = \Phi_t^*\left(L_{X_t}\eta_t + \frac{\mathrm{d}}{\mathrm{d}t}\eta_t\right). \tag{1.31}$$

Proof. Note first that by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_t^*\eta_t\right) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{s+t}^*\eta_t\right|_{s=0} + \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_t^*\eta_{s+t}\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{s+t}^*\eta_t\right|_{s=0} + \Phi_t^*\frac{\mathrm{d}}{\mathrm{d}t}\eta_t. \tag{1.32}$$

To compute the first summand on the right hand side of (1.32), we write $\Psi_s = \Phi_{s+t} \circ \Phi_t^{-1}$. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Psi_s(p)\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{t+s} \circ \Phi^{-1}(p)\right|_{s=0} = X_t(p), \tag{1.33}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi^*_{t+s}\eta_t\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\Psi^*_s \circ \Phi_t^{-1}\right)^* \eta_t\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi^*_t\Psi^*_s\eta_t\Big|_{s=0} = \Phi^*_t\,\frac{\mathrm{d}}{\mathrm{d}s}\Psi^*_s\eta_t\Big|_{s=0} \tag{1.34}$$

$$= \Phi^*_t L_{X_t}\eta_t, \tag{1.35}$$

where we use (1.33). This yields the claim. □

Now we apply this notion to symplectic structures. Let M^{2n} be a smooth manifold with two symplectic forms $\omega_0, \omega_1 \in \Omega^2(M)$.

Definition 1.42. (M, ω_0) and (M, ω_1) are called

(i) *symplectomorphic* if there exists a symplectomorphism $\Psi : M \to M$ with $\Psi^*\omega_1 = \omega_0$.

(ii) *strongly isotopic* if there is an isotopy $\Phi_t : M \to M$ with $\Phi_1^*\omega_1 = \omega_0$.

(iii) *deformation equivalent* if there is a smooth family $\omega_t \in \Omega^2(M)$ of symplectic forms with joining ω_0 and ω_1.

(iv) *isotopic* if they are deformation equivalent with $[\omega_t] = [\omega_0] \in H^2(M;\mathbb{R})$ for all $t \in [0, 1]$.

From the Defintion 1.42 it is obvious that strongly isotopic implies symplectomorphic, and also that isotopic implies deformation equivalent.

Lemma 1.43. *Strongly isotopic symplectic structures are isotopic.*

Proof. If ω_0 and ω_1 are strongly isotopic with isotopy Φ_t, then $\omega_t = \Phi_t^*\omega_1$ is a smooth family of symplectic forms joining ω_0 and ω_1, i.e. they are deformation equivalent. Moreover, by (1.31) and Cartan's formula we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \Phi^*_t L_{X_t}\omega + \Phi^*_t\frac{\mathrm{d}}{\mathrm{d}t}\omega_1 = \Phi^*_t d i_{X_t}\omega_1 = d\Phi^*_t i_{X_t}\omega_1, \tag{1.36}$$

hence $[\omega_t] = [\omega_1]$, i.e. ω_0 and ω_1 are isotopic. □

As remarked at the beginning of section 1.2.2 we want to adress the question if there are local invariants of symplectic manifolds. In other words, given a $2m$-dimensional manifold M and a n-dimensional submanifold N of M with neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ that admit symplectic forms ω_0 and ω_1, respectively, is there a symplectomorphism $\Psi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\Psi(N) = N$? If we are able to answer this question, we can immediately derive Darboux's Theorem by setting N as a single point.

In terms of Definition 1.40 we might reformulate this question in terms of isotopies: given a family $\eta_t \in \Omega^k(M)$ of smooth forms (with some additional property), we seek an isotopy Φ_t of diffeomorphisms such that $\Phi_0 = \mathrm{id}_M$ and $\Phi^*_t\eta_t = \eta_0$, i.e. Φ_t is the solution of the differential equation

$$X_t\left(\Phi_t(p)\right) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\Phi_s(p). \tag{1.37}$$

This observation can be used to construct the isotopy as flows of time dependent vector fields. It thus suffices to solve the linearized version of $\Phi^*_t\eta_t = \eta_0$, i.e.

$$0 = \Phi^*_t\left(\dot\eta_t + L_{X_t}\eta_t\right), \tag{1.38}$$

which amounts to finding a suitable time dependent vector field X_t. This method is known as *Moser's trick* or *argument* [**Moser65**] and will be the basis of the local theory.

Theorem 1.44 (Moser). *Let M be a closed, oriented n-manifold, and let $\omega_0, \omega_1 \in \Omega^n(M)$ be volume forms such that*

$$\int_M \omega_0 = \int_M \omega_1. \tag{1.39}$$

Then there is an isotopy Φ_t of M such that $\Phi_t^* \omega_1 = \omega_0$.

Proof. We define $\omega_t = (1-t)\omega_0 + t\omega_1$. This is a family of volume forms. We want to show $\Phi_t^* \omega_t = \omega_0$ which by (1.31) is equivalent to

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \Phi_s^* \omega_s = 0 \qquad \Leftrightarrow \qquad \Phi_t^* \left(L_{X_t} \omega_t + \dot{\omega}_t \right) = 0. \tag{1.40}$$

Now since ω has top degree, $L_{X_t}\omega_t = d i_{X_t} \omega_t$, i.e. (1.40) is equivalent to

$$d i_{X_t} \omega_t = -\dot{\omega}_t = -(\omega_1 - \omega_0) = d\alpha. \tag{1.41}$$

Thus, it is sufficient to solve $i_{X_t}\omega_t = \alpha$. Since the map $TM \to \bigwedge^{n-1} T^*M$, $X \mapsto i_X \omega_t$ clearly is bijective, there exists a unique family of vector fields X_t such that $i_{X_t}\omega_t = \alpha$. The flow Φ_t of X_t is the sought isotopy. $\square$

Applying the same argument, we finally tackle the question whether (M, ω_0) and (M, ω_1) are symplectomorphic. If we additionally require the symplectomorphism to be homotopic to the identity, we observe a necessary condition: if $\Psi : M \to M$ is a symplectomorphism which is homotopic to id_M such that $\Psi^* \omega_1 = \omega_0$, then by the homotopy formula of de Rham cohomology, we obtain for a chain homotopy $h : \Omega^k(M) \to \Omega^{k-1}(M)$,

$$\mathrm{id}_M^* \omega_1 - \Psi^* \omega_1 = dh\omega_1 + hd\omega_1, \tag{1.42}$$

hence $\omega_1 = \Psi^*\omega_1 + dh\omega_1$ implying on cohomology $[\omega_1] = [\Psi^*\omega_1] = [\omega_0]$, i.e. the cohomology class of the family $\omega_t \in \Omega^2(M)$ of symplectic forms must be constant.

Theorem 1.45 (Moser Stability Theorem). *Let M be closed and $\omega_t \in \Omega^2(M)$ be a family of symplectic forms such that $[\omega_t] \in H^2_{\mathrm{dR}}(M)$ is constant. Then there is an isotopy Φ_t of M with $\Phi_t^* \omega_t = \omega_0$ for all $t \in [0,1]$.*

Proof. Moser's argument shows that the statement follows if we solve $L_{X_t}\omega_t + \dot{\omega}_t = 0$. Because ω_t is closed for all t, Cartan's formula implies $L_{X_t}\omega_t = d i_{X_t}\omega_t$. Since the cohomology class of ω_t is constant, we have $\dot{\omega}_t = d\eta_t$ for $\eta_t \in \Omega^1(M)$, $t \in [0,1]$. Thus, we need to solve $d i_{X_t}\omega_t + d\eta_t = 0$ which is equivalent to

$$i_{X_t}\omega_t + \eta_t = 0. \tag{1.43}$$

By nondegeneracy of ω_t, we can solve (1.43) for each $t \in [0,1]$ but the resulting family η_t is not necessarily smooth. To construct a smooth family one can either apply the Poincaré Lemma and the Mayer-Vietoris sequence or use the Hodge Decomposition Theorem. Then, by nondegeneracy of ω_t, there exists a unique vector field X_t that solves (1.43). Since η_t is smooth, also X_t is, and the flow of X_t is the sought isotopy. $\square$

Remark 1.46. To apply Moser's argument in the proof of Theorems 1.44 and 1.45, we need compactness for the flow of the time dependent vector field to define an isotopy.

Theorem 1.45 implies that one cannot perturb the symplectic structure within a given homology class, i.e. for $a \in H^2_{\mathrm{dR}}(M)$ and $\mathcal{S}_a(M) = \{\omega \in \Omega^2(M) \mid \omega \text{ symplectic}, [\omega] = a\}$ Theorem 1.45 implies that all symplectic forms in the same path-component of $\mathcal{S}_a(M)$ are symplectomorphic (on a closed manifold M).

Remark 1.47. Recall that for any compact submanifold $Q \subseteq M$ there exists a *tubular neighborhood* $\mathcal{U}$ *of* Q *in* M, i.e. a vector bundle $\pi : \mathcal{U} \to Q$ isomorphic to the normal bundle $TQ^\perp \subseteq TM$ of Q [Spi75, p. 354f.]. To construct a tubular neighborhood $\pi : \mathcal{U} \to Q$ choose a Riemannian metric g on M, and let

$$TQ^\perp_\epsilon = \{v \in TQ^\perp \mid g(v,v) < \epsilon\}, \qquad \mathcal{U} = \{q \in M \mid d(q,Q) < \epsilon\}, \tag{1.44}$$

where $d : M \times M \to \mathbb{R}$ denotes the metric on M (as a topological space) induced by the Riemannian metric g. Then one can show that $\exp : TQ^\perp_\epsilon \to \mathcal{U}$ is a diffeomorphism.

Applying Moser's argument to a family of closed two-forms which along a compact submanifold coincide and are nondegenerate we prove a central Lemma in the local theory of symplectic manifolds.

Lemma 1.48. *Let M^{2n} be a smooth manifold, and $Q \subset M^{2n}$ be a compact submanifold. Let $\omega_0, \omega_1 \in \Omega^2(M)$ be closed two-forms such that $(\omega_0)_q = (\omega_1)_q$ are nondegenerate for all $q \in Q$. Then there are neighborhoods $\mathcal{U}_0$, $\mathcal{U}_1$ of Q and a diffeomorphism $\Phi : \mathcal{U}_0 \to \mathcal{U}_1$ such $\Phi_{|Q} = \mathrm{id}_Q$ and $\Phi^*\omega_1 = \omega_0$.*

Proof. Let $\omega_t = (1-t)\omega_0 + t\omega_1$ for $t \in [0,1]$. By assumption this defines a smooth family of closed two-forms that is constant and nondegenerate on Q because $(\omega_t)_q = (\omega_0)_q = (\omega_1)_q$ for all $q \in Q$. Again we want to apply Moser's argument: we need to show that there exist a neighborhood $\mathcal{U}_0$ of Q and $\eta \in \Omega^1(\mathcal{U}_0)$ such that $\eta_{|Q} = 0$ and $d\eta = \omega_1 - \omega_0$. Then $\omega_t = \omega_0 + td\eta$ on $\mathcal{U}_0$ and we may assume (by shrinking $\mathcal{U}_0$ if necessary) that ω_t is nondegenerate on $\mathcal{U}_0$. By Moser's argument it then suffices to solve (cf. (1.43))

$$i_{X_t}\omega_t + \eta = 0 \tag{1.45}$$

on $\mathcal{U}_0$. Since ω_t is nondegenerate on $\mathcal{U}_0$ there exists a unique time dependent vector field X_t solving (1.45), and we may assume (again by shrinking $\mathcal{U}_0$ if necessary) that its flow Φ_t solves $\Phi_t^*\omega_t = \omega_0$ for $t \in [0,1]$.

Now let us show that such neighborhood $\mathcal{U}_0$ and $\eta \in \Omega^1(\mathcal{U}_0)$ exist. We define $\mathcal{U}_0$ to be a (sufficiently small) tubular neighborhood of Q in M (cf. Remark 1.47), and let $\Psi_t : \mathcal{U}_0 \to \mathcal{U}_0$ for $t \in [0,1]$ be defined by

$$\Psi_t(\exp_q(v)) = \exp_q(tv). \tag{1.46}$$

Then Ψ_t is a diffeomorphism for $t \in (0,1]$, $\Psi_0(\mathcal{U}_0) \subseteq Q$, $\Psi_1 = \mathrm{id}_{\mathcal{U}_0}$, and $\Psi_{t|Q} = \mathrm{id}_Q$. It follows that $\Psi_0^*(\omega_1 - \omega_0) = 0$ and $\Psi_1^*(\omega_1 - \omega_0) = \omega_1 - \omega_0$. Let us define a vector field X_t for $t \in (0,1]$ by $X_t \circ \Psi_t = \dot{\Psi}_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi_t^*(\omega_1 - \omega_0)\right) = \Psi_t^* L_{X_t}(\omega_1 - \omega_0) = d\Psi_t^* i_{X_t}(\omega_1 - \omega_0) = d\eta_t, \tag{1.47}$$

where we define $\eta_t \in \Omega^1(\mathcal{U}_0)$ by

$$(\eta_t)_q(v) = (\omega_1 - \omega_0)_{\Psi_t(q)}\big(X_t(\Psi_t(q)), d(\Psi_t)_q(v)\big). \tag{1.48}$$

for $q \in \mathcal{U}_0$. Note that η_t is smooth (even for $t = 0$) and vanishes on Q by definition. Since

$$\omega_1 - \omega_0 = \Psi_1^*(\omega_1 - \omega_0) - \Psi_0^*(\omega_1 - \omega_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \Psi_t^*(\omega_1 - \omega_0)\, dt = \int_0^1 \eta_t\, dt = d\eta, \tag{1.49}$$

we can solve (1.45) and Moser's argument yields the statement. □

Theorem 1.49 (Darboux). *Let (M^{2n}, ω) be a symplectic manifold. Then for all $q \in M$ there are local coordinates $(U, (x_i, y_i))$, $i = 1, \ldots, n$ such that*

$$\omega_{|U} = \sum_i dx_i \wedge dy_i. \tag{1.50}$$

Proof. We apply Lemma 1.48 to the case where Q is a single point $q \in M$. Choosing (by Theorem 1.6) a symplectic basis $\{dx_1, dy_1, \ldots, dx_n, dy_n\}$ of T_q^*M we set $\omega_1 = \omega_q = \sum_{i=1}^n dx_i \wedge dy_i$. Let (V, x) be a chart around q and set $\omega_0 = \omega_{|U}$. By Lemma 1.48 there exist a neighborhood $\mathcal{U}_0$ of q and a diffeomorphism $\Psi : \mathcal{U}_0 \to U$ such that $\Psi^*\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. □

1.2.3 Almost Complex Structures and Kähler Manifolds

Definition 1.50. Let M be a $2n$-dimensional smooth manifold. An *almost complex structure* J on M is an endomorphism $J \in \Gamma(\mathrm{End}(TM))$ of the tangent bundle, i.e. $J_p : T_pM \to T_pM$ is a complex structure for T_pM for all $p \in M$ varying smoothly in p. An almost complex structure $J \in \Gamma(\mathrm{End}(TM))$ is called

(i) *compatible* with a nondegenerate two-form $\omega \in \Omega^2(M)$ if $g_J \in \Gamma(T^*M \otimes T^*M)$, defined by $g_J(X, Y) = \omega(X, JY)$ for $X, Y \in \Gamma(TM)$, is a Riemannian metric.

(ii) *integrable* if there exists an atlas $\{(U_i, \varphi_i)\}$ on M such that in local coordinates J is represented by J_0, i.e. for $q \in M$

$$d(\varphi_i)_q \circ J_q = J_0 \circ d(\varphi_i)_q : T_qM \to \mathbb{R}^{2n}. \tag{1.51}$$

If $J \in \Gamma(\mathrm{End}(TM))$ is an integrable almost complex structure, the differential of coordinate changes $d(\varphi_i \circ \varphi_j^{-1})$ is in $GL(n, \mathbb{C})$ because

$$d(\varphi_i \circ \varphi_j^{-1}) \circ J_0 = d\varphi_i \circ J \circ d\varphi_j^{-1} = J_0 \circ d(\varphi_i \circ \varphi_j^{-1}), \tag{1.52}$$

hence the coordinate changes are holomorphic. Conversely, if the coordinate changes are holomorphic, $J \in \Gamma(\mathrm{End}(TM))$, defined by $d\varphi_i(J_q v) = J_0 d\varphi_i(v)$, is an integrable almost complex structure.

Definition 1.51. An integrable almost complex structure $J \in \Gamma(\mathrm{End}(TM))$ is called a *complex structure* and the tuple (M, J) a *complex manifold*.

Remark 1.52. A useful criterion whether a given almost complex structure J is integrable can be given in terms of the *Nijenhuis tensor* $N_J : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]. \tag{1.53}$$

One can show (cf. [MS05, p. 124]) that an almost complex structure J is integrable if and only if N_J vanishes identically.

Definition 1.53. A complex manifold (M, J) is called *Kähler* if there exists a symplectic structure $\omega \in \Omega^2(M)$ such that J is compatible with ω.

Example 1.54. The Kähler manifold $(\mathbb{R}^{2n}, \omega_0, J_0)$ can be identified with $\mathbb{C}^n$ in such a way that $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to multiplication with $\imath = \sqrt{-1}$ in $\mathbb{C}^n$. One considers $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ as independent variables and introduces forms $dz_i = dx_i + \imath dy_i$ and $d\bar{z}_i = dx_i - \imath dy_i$. The exterior derivative $d : \Omega^k(\mathbb{C}^n) \to \Omega^{k+1}(\mathbb{C}^n)$ of complex valued differential forms then can be expressed as $d = \partial + \bar{\partial}$, where

$$\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} + \imath \frac{\partial f}{\partial y_i} \right) dz_i, \quad \bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} - \imath \frac{\partial f}{\partial y_i} \right) d\bar{z}_i. \tag{1.54}$$

The standard symplectic form takes the form

$$\omega_0 = \frac{\imath}{2} \partial \bar{\partial} f = \sum_{i=1}^n dz_i \wedge d\bar{z}_i \tag{1.55}$$

for $f : \mathbb{C}^n \to \mathbb{C}$, $f(z) = \sum_{i=1}^n |z_i|^2$.

Example 1.55. Complex projective space $\mathbb{C}P^n$ is a Kähler manifold. First, it is a complex manifold with complex structure given by $J_{[z]} : T_{[z]}\mathbb{C}P^n \to T_{[z]}\mathbb{C}P^n$, $J_{[z]}v = \imath v$. The charts (U_i, φ_i), where $U_i = \{[z] \in \mathbb{C}P^n \mid z_i \neq 0\}$ and

$$\varphi_i : U_i \to \mathbb{C}^n, \quad [z] \mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i} \right), \tag{1.56}$$

provide a holomorphic atlas, hence the complex structure J is integrable. One can show that the standard symplectic structure on it is given by the *Fubini-Study 2-form*

$$\omega = \omega_{\mathrm{FS}} = \frac{\imath}{2 \sum_{i=0}^n |z_i|^2} \sum_{i=0}^n \sum_{j \neq i} \left(z_j \bar{z}_j dz_i \wedge d\bar{z}_i - z_i \bar{z}_j dz_j \wedge d\bar{z}_i \right), \tag{1.57}$$

where $dz_i = dx_i + \imath dy_i$, $d\bar{z}_i = dx_i - \imath dy_i$ for homogeneous coordinates $[z_0 : \cdots : z_n]$ of $\mathbb{C}P^n$, and morover that the complex structure J is compatible ω_{FS}. Therefore, $\mathbb{C}P^n$ is a Kähler manifold. (Later we provide different proves that $\mathbb{C}P^n$ is symplectic.)

Example 1.56. With the standard symplectic structure $\mathbb{C}^n$ becomes a Kähler manifold. Let $M = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 = 1\} \subseteq \mathbb{C}^n$. Then M is a submanifold of $\mathbb{C}^n$ with $J(TM) = TM$ because $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$ is holomorphic and hence $v \in TM$ iff. $df(v) = 0$ iff. $\imath df(v) = df(Jv) = 0$ iff. $Jv \in TM$.

Remark 1.57. If M is a submanifold of a Kähler manifold and satisfies $JTM = TM$, then M is again complex, and hence Kähler.

Remark 1.58. A symplectic manifold (M, ω) is in general not Kähler (e.g. odd Betti numbers of Kähler manifolds are even).

Example 1.59 (Thurston-Kodaira Manifold)**.** Let $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2$ be the group with multiplication given by

$$(j', k') \cdot (j, k) = (j + j', A_{j'}k + k') \qquad \text{with } A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix} \tag{1.58}$$

for $j = (j_1, j_2) \in \mathbb{Z}^2$. This group Γ acts on $\mathbb{R}^4$ via diffeomorphisms by $\Gamma \to \mathrm{Diff}(\mathbb{R}^4)$, $(j, k) \mapsto \rho_{jk}$, where $\rho_{jk}(x, y) = (x + j, A_j y + k)$ for $(x, y) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. This action $\Gamma \curvearrowright \mathbb{R}^4$ is free and properly discontinuous. Clearly $\rho_{jk}(x, y) = (x, y)$ for all (x, y) implies $(j, k) = (0, 0) \in \Gamma$. Furthermore,

$$\|\rho_{jk}(x, y) - (x, y)\|^2 = \|j\|^2 + \|k + (j_2 y_2, 0)\|^2 \geqslant 1 \tag{1.59}$$

i.e. the action is properly discontinuous, indeed. Therefore, $M_\Gamma = {}^{\mathbb{R}^4}\!/\!_\Gamma$ is a smooth manifold [Lee12, Theorem 21.13]. Let $\omega_0 = dx_1 \wedge dx_2 + dy_1 \wedge dy_2 \in \Omega^2(\mathbb{R}^4)$ be the symplectic form on $\mathbb{R}^4$. Then

$$\rho_{jk}^* \omega_0 = d(x_1 + j_1) \wedge d(x_2 + j_2) + d(y_1 + j_2 y_2) \wedge d(y_2 + k_2) = \omega_0, \tag{1.60}$$

i.e. the symplectic form ω_0 is preserved under the Γ-action. Thus, we have a natural symplectic structure $\omega \in \Omega^2(M_\Gamma)$ induced by ω_0 in terms of the covering $\pi : \mathbb{R}^4 \to M_\Gamma$, $\omega_{|\pi(U)} = (\pi_{|U}^{-1})^* \omega_{0|U}$ for a neighborhood $U \subseteq \mathbb{R}^4$ such that $\rho_{jk}(U) \cap U = \varnothing$ for all $0 \neq (j, k) \in \Gamma$. (The quotient map is a covering since the action is properly discontinous. Recall that one can construct evenly covered neighborhoods in M_Γ by images under the quotient map of neighborhoods U such that $\rho_{jk}(U) \cap U = \varnothing$ for all $0 \neq (j, k) \in \Gamma$.) The fundamental group $\pi_1(M_\Gamma)$ is isomorphic to the group of deck transformations of the covering π, and in fact, since $\mathbb{R}^4$ is simply connected, isomorphic to Γ. By the Hurewicz homomorphism the first homology $H_1(M; \mathbb{Z}) \cong {}^{\Gamma}\!/\!_{[\Gamma,\Gamma]} \cong \mathbb{Z}^3$. With Remark 1.58 we conclude that M_Γ is a symplectic manifold which is not Kähler.

Example 1.60. Let $M = S^1 \times S^3$. We want to show that M admits a complex structure which cannot be Kähler by writing M as the quotient of $\mathbb{C}^2 \setminus \{0\}$ by an appropriate group action. We claim $M = {}^{\mathbb{C}^2 \setminus \{0\}}\!/\!_{\mathbb{Z}}$ where $\mathbb{Z} \curvearrowright \mathbb{C}^2 \setminus \{0\}$ is given by $z \cdot (z_0, z_1) = (z^k z_0, z^k z_1)$. This action is $\mathbb{C}$-linear, hence the complex structure J_0 on $\mathbb{C}^2 \setminus \{0\}$ descends to a complex structure on the quotient. One can show that there exist $\mathbb{Z}$-invariant disjoint neighborhoods for disjoint orbits, hence ${}^{\mathbb{C}^2 \setminus \{0\}}\!/\!_{\mathbb{Z}} \cong {}^{S^3 \times [1,2]}\!/\!_{(z,1) \sim (z,2)} \cong S^3 \times S^1$. Künneth's formula implies $H^2_{\mathrm{dR}}(S^1 \times S^3) = 0$, hence M cannot be symplectic.

Example 1.61 (Hopf Surface)**.** Let

$$W_k = \{([a : b], [x : y : z]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid a^k y = b^k x\} \tag{1.61}$$

$$= \{p_k(a, b, x, y, z) = a^k y - b^k x = 0\}. \tag{1.62}$$

Because $0 \in \mathbb{C}$ is a regular value of $p_k : \mathbb{C}P^1 \times \mathbb{C}P^2 \to \mathbb{C}$, $W_k \subseteq \mathbb{C}P^1 \times \mathbb{C}P^2$ is a complex submanifold of complex dimension two. Moreover, since $\mathbb{C}P^1 \times \mathbb{C}P^2$ is Kähler and a complex submanifold of a Kähler manifold is Kähler itself, we conclude that W_k is a symplectic manifold. The natural projection $\pi : W_k \to \mathbb{C}P^1$ defines a $\mathbb{C}P^1$-bundle. Indeed,

$$\pi^{-1}([a_0 : b_0]) = \{a_0^k y = b_0^k x\} \cong \mathbb{C}P^1 \subseteq \{[a_0 : b_0]\} \times \mathbb{C}P^2 \tag{1.63}$$

and we have local trivializations given by

$$((a, b), [p : q]) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mapsto ([a : b], [p : (b/a)^k p : q]) \in W_k. \tag{1.64}$$

1.2.4 Symplectic Vector Bundles

Definition 1.62. A vector bundle $\pi : E \to M$ is called *symplectic* if there is a symplectic bilinear form ω_p on each fiber $E_p = \pi^{-1}(p)$ that varies smoothly in $p \in M$.

Lemma 1.63. *Let (E, ω) be a $2n$-dimensional symplectic vector bundle over a manifold M. Then there is a family of complex structures J_p, $p \in M$, that is compatible with ω_p and varies smoothly in $p \in M$.*

Proof. Cover M with open sets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^{2k}$. So far the transition functions $\pi^{-1}(U_i \cap U_j) \to \pi^{-1}(U_i \cap U_j)$ are of the form $(p, v) \mapsto (p, A_p(v))$ where $A_p \in GL(\mathbb{R}^{2k})$. Our first goal is to obtain transition functions so that $A_p \in \mathrm{Sp}(2k, \mathbb{R})$. For each $p \in U_i$ pick $(v_1, \ldots, v_{2k})$ a basis of E_p. Using bundle charts we get a basis on a small neighborhood on U_i. Because $\omega_{|E_p}$ is symplectic there exist v_l such that $\omega_p(v_1, v_l) \neq 0$. On a neighborhood of p, $\tilde{v}_l = f v_l$ such that $\omega_p(v_1, \tilde{v}_l) = 1$. Now consider, on this neighborhood the symplectic bundle $\langle v_1, v_l \rangle^{\perp_\omega}$ and project the remaining basis vectors to $\langle v_1, v_l \rangle^{\perp}$. After k steps we get a local trivialization $(e_1, f_1, \ldots, e_k, f_k)$ of E that defines a symplectic basis. We have achieved our first goal.

On each bundle chart there is a compatible complex structure (in our coordinates J_0). By paracompactness we can assume that our improved bundle charts form a locally finite, countable cover of M. Construct J inductively: start with U_1 and pick J_1 on $\pi^{-1}(U_1)$. Now take U_2 and take J_1 on $\pi^{-1}(U_1 \cap U_2)$. If $U_1 \cap U_2 \neq \varnothing$, pick a function ϱ to get a J over $U_1 \cap U_2$. Then modify J_2 over $U_1 \cap U_2$: in coordinates U_1, J_1 look constant. Let $\varphi : J(V_2, \omega_2) \to J(V, \omega)$ be the contraction onto J_0. Replace $J_2(p)$ by $\varphi(J_1(1 - \varrho(p)))$ and proceed inductively. □

Remark 1.64. Given a symplectic vector bundle (E, ω) one can pick a compatible complex structure J and find bundle charts (U_i, φ_i) such that ω and J are in standard form, namely

$$(\pi^{-1}(U_i), \omega, J) \cong \Big(U_i \times \mathbb{R}^{2k}, \sum_{j=1}^{k} e_j^* \wedge f_j^*, J_0\Big), \tag{1.65}$$

and the transition functions are of the form $(p, v) \mapsto (p, A_p(v))$ with $A_p(v) \in \mathrm{Sp}(2k, \mathbb{R}) \cap GL(k, \mathbb{C}) = U(k)$. We conclude that from a vector bundle point of view (TM, ω) as symplectic vector bundle looks like the tangent bundle of a Kähler manifold. However, $J \in \Gamma(\mathrm{End}(TM))$ is not induced from an integrable almost complex structure.

Corollary 1.65. *Let (M, ω) be symplectic. The space of compatible almost complex structures is contractible.*

Remark 1.66. We conclude that on any symplectic manifold (M, ω) there exists a compatible almost complex structure. Morover, on any submanifold $N \subseteq M$ which is invariant under an almost complex structure J, i.e. $J(TN) = TN$, there is a natural symplectic structure given by the pullback of ω under the inclusion $\iota : N \to M$. Indeed, $\iota^*\omega$ is nondegenerate since $\iota^*\omega_x(X, JX) \neq 0$ for all $0 \neq X \in T_xN$.

Definition 1.67. Two symplectic vector bundles (E_0, ω_0), (E_1, ω_1) over M are called *isomorphic* if there is an isomorphism $\Phi : E_0 \to E_1$ such that $\Phi^*\omega_1 = \omega_0$.

1.2.5 Submanifolds of Symplectic Manifolds

Definition 1.68. Let (M, ω) be a symplectic manifold. A submanifold $Q \subseteq M$ is called *symplectic* (*isotropoic*, *coisotropic*, *Lagrangian*) if $T_qQ \subseteq T_qM$ is a symplectic (isotropic, coisotropic, Lagrangian) subspace of (T_qM, ω_q) for all $q \in Q$.

Example 1.69. Let (M, ω) be a symplectic manifold. Then the product $M \times M$ is a symplectic manifold with symplectic form $-\omega \oplus \omega \in \Omega^2(M \times M)$. The submanifolds $M \times \{p\}, \{p\} \times M \subseteq M \times M$, $p \in M$, are symplectic, and the diagonal $\Delta = \{(p,p) \mid p \in M\}$ is Lagrangian.

Example 1.70. Let us consider the canonical symplectic structure (1.15) on the cotangent bundle $L = T^*M$ of a smooth manifold. The zero section $\sigma_0 : M \to T^*M$, $p \mapsto (p, 0)$ is a Lagrangian submanifold since $\omega_{|\sigma_0(M)} = -d\lambda_{|\sigma_0(M)} = -d\sigma_0^*\lambda = 0$ by Lemma 1.23. Moreover, the tautological one-form λ is constant along the fibers of T^*M (for the inclusion $\iota : T_p^*M \to T^*M$ we have $(\lambda_{|T_p^*M})_\eta(v) = \eta(d\pi_\eta \circ d\iota_\eta(v)) = 0$ because $\pi \circ \iota = p$), hence they are Lagrangian submanifolds.

Lemma 1.71. *Let $\Psi : M \to M$ be a diffeomorphism and consider the graph of Ψ in $M \times M$. Then Ψ is a symplectomorphism if and only if* graph Ψ *is Lagrangian.*

Proof. Let $\mathrm{pr}_1, \mathrm{pr}_2 : M \times M \to M$ denote the canonical projections. Note that graph Ψ is an embedded submanifold via the embedding $\iota : M \to M \times M$, $p \mapsto (p, \Psi(p))$. Then graph Ψ is Lagrangian if and only if $\iota^*(-\omega \oplus \omega) = 0$. Now

$$\iota^*(-\omega \oplus \omega) = -(\iota \circ \mathrm{pr}_1)^*\omega + (\iota \circ \mathrm{pr}_2)^*\omega = -\omega + \Psi^*\omega, \tag{1.66}$$

hence $\iota^*(-\omega \oplus \omega) = 0$ if and only if $\Psi^*\omega = \omega$. $\square$

Example 1.72. Let (M, ω) be a symplectic manifold. In Example 1.22 we saw that the cotangent bundle T^*M carries a canonical symplectic structure $-d\lambda \in \Omega^2(T^*M)$. A one-form $\eta \in \Omega^1(M)$ is a section $\eta : M \to T^*M$ of $\pi : T^*M \to M$. Let us consider the embedded submanifold given by $\sigma(M) = \{\sigma_p \in T_p^*M \mid p \in M\} \subseteq T^*M$. It is Lagrangian if and only if the symplectic form vanishes on $\sigma(M)$, i.e.

$$0 = -d\lambda_{\sigma(M)} = -\sigma^*d\lambda = -d\sigma^*\lambda = -d\sigma, \tag{1.67}$$

where we use Lemma 1.23. Thus $\sigma(M)$ is Lagrangian if and only if σ is closed.

Example 1.73. Let (M^{2n}, ω) be a symplectic manifold, $c : M \to M$ an *antisymplectic involution*, i.e. $c^*\omega = -\omega$, $c^2 = \mathrm{id}$. The fixed point set F of c is an isotropic submanifold. Indeed, if $v, w \in T_pF$, then $\omega_p(v,w) = -(c^*\omega)_p(v,w) = -\omega_p(dc_p(v), dc_p(w)) = -\omega_p(v,w)$, hence $T_pF \subseteq T_pF^\omega$. But F is n-dimensional and hence by Lemma 1.4 Lagrangian. E.g. consider $(\mathbb{C}^n, \omega = \sum_i dx_i \wedge dy_i)$, $z_i = x_i + \imath y_i$. Then complex conjugation on $\mathbb{C}^n$ is an antisymplectic involution. If $f(z_1, \dots, z_n)$ is a polynomial with real coefficients, then $c_{|f^{-1}(0)}$ is an antisymplectic involution on $f^{-1}(0)$.

Let (M, ω) be a symplectic manifold and $Q \subseteq M$ a symplectic submanifold. Then we may identify the normal bundle $TQ^\perp$ of Q in M with the complementary symplectic bundle TQ^ω of Q in M. Because the restriction of ω to Q is symplectic, we may consider $TQ^\perp$ as a symplectic vector bundle. In the following let $\mathcal{U}(Q) \subseteq M$ denote a sufficiently small neighborhood of Q in M.

Theorem 1.74 (Weinstein Symplectic Neighborhood Theorem). *For $i = 0, 1$ let (M_i, ω_i) be a symplectic manifold with a compact symplectic submanifold $Q_i \subseteq M_i$. Suppose that there exists an isomorphism of symplectic normal bundles $\Phi : TQ_0^\omega \to TQ_1^\omega$, and a symplectomorphism $\phi : (Q_0, \omega_0) \to (Q_1, \omega_1)$ such that $\phi \circ \pi_0 = \pi_1 \circ \Phi$ for $\pi_i : TQ_i^\omega \to Q_i$, i.e.*

$$\begin{tikzcd} TQ_0^\omega \arrow[r, "\Phi"] \arrow[d, "\pi_0"'] & TQ_1^\omega \arrow[d, "\pi_1"] \\ Q_0 \arrow[r, "\phi"'] & Q_1 \end{tikzcd} \tag{1.68}$$

Then ϕ extends to a symplectomorphism $\psi : (\mathcal{U}(Q_0), \omega_0) \to (\mathcal{U}(Q_1), \omega_1)$ with $d\psi_{|TQ_0^\omega} = \Phi$.

Proof. Let $\exp_i : TQ_i \to M$ be the exponential map of some Riemannian metric. Then a small neighborhood of the zero section $Q_i \subseteq TQ_i$ is mapped diffeomorphically onto some neighborhood $\mathcal{N}(Q_i)$ of Q_i in M. Thus ϕ extends to a diffeomorphism $\phi' : \mathcal{N}(Q_0) \to \mathcal{N}(Q_1)$ with $d\phi'_{|TQ_0^\omega} = \Phi$ by setting

$$\phi' = \exp_1 \circ \Phi \circ \exp_0^{-1}. \tag{1.69}$$

Since $\phi'^*\omega_1 = \omega_0$ on Q_0, we may consider ω_0 and $\phi'^*\omega_1$ as two-forms on $\mathcal{N}(Q_0)$ which coincide and are nondegenerate along Q_0. Lemma 1.48 implies that there is an isotopy $\Psi_t : \mathcal{U}(Q_0) \to \mathcal{U}'(Q_0)$ of open neighborhoods of Q_0 in $\mathcal{N}(Q_0)$ such that $\Psi_{t|Q} = \mathrm{id}_Q$ and $\Psi_1^*\phi'^*\omega_1 = \omega_0$. Then $\psi = \phi' \circ \Psi_1 : (\mathcal{U}(Q_0), \omega_0) \to (\mathcal{N}(Q_1), \omega_1)$ is a symplectomorphism such that $d\psi_{|TQ_0^\omega} = \Phi$. □

Theorem 1.75 (Weinstein Lagrangian Neighborhood Theorem). *Let (M, ω) be a symplectic manifold and $L \subseteq M$ a compact Lagrangian submanifold. Then there exists a diffeomorphism $\phi : \mathcal{N}(L) \to \mathcal{U}(L)$ from an open neighborhood $\mathcal{N}(L) \subseteq T^*L$ of the zero section to an open neighborhood $\mathcal{U}(L) \subseteq M$ of L such that $\phi_{|L} = \mathrm{id}_L$ and $\phi^*\omega = -d\lambda$.*

Proof. Note first that the normal bundle of L in M is isomorphic to TL. In order to see this pick, by Lemma 1.63, a compatible almost complex structure $J \in \mathcal{J}(M, \omega)$. Then $J_q(T_qL) = T_qL^{\perp_{g_J}}$ for the induced metric g_J on M. Let $\Phi_q : T_q^*L \to T_qL$ be the isomorphism induced by g_J, i.e. $g_J(\Phi_q(\eta), v) = \eta(v)$ for $\eta \in T_q^*L$, $v \in T_qL$. We define $\phi : T^*L \to M$ by

$$\phi(\eta) = \exp_q\left(J_q\Phi_q(\eta)\right), \tag{1.70}$$

where $\exp : TL \to M$ denotes the exponential map of g_J. Clearly, $\phi(\eta) = \exp_q(0) = q$ for all $\eta \in L \subseteq T^*L$, i.e. $\phi_{|L} = \mathrm{id}_L : L \subseteq TL \to L \subseteq M$. Using the identification of Lemma 1.25, $v = (v_0, v_1^*) \in T_qL \oplus T_q^*L \cong T_\eta T^*L$ along the zero section, $\eta \in T_q^*L$, we obtain for the differential $d\phi_\eta : T_qL \oplus T_q^*L \to T_qM$ of (1.70)

$$d\phi_\eta(v) = v_0 + J_q\Phi_q(v_1^*). \tag{1.71}$$

Because L is Lagrangian, we obtain, for $v = (v_0, v_1^*), w = (w_0, w_1^*) \in T_qL \oplus T_q^*L$,

$$\begin{aligned} (\phi^*\omega)_\eta(v, w) &= \omega_q\big(v_0 + J_q\Phi_q(v_1^*), w_0 + J_q\Phi_1(v_1^*)\big) && (1.72)\\ &= \omega_q\big(v_0, J_q\Phi_q(w_1^*)\big) - \omega_\eta\big(w_0, J_q\Phi_q(w_1^*)\big) && (1.73)\\ &= (g_J)_q\big(v_0, \Phi_q(w_1^*)\big) - (g_J)_q\big(w_0, \Phi_q(v_0^*)\big) && (1.74)\\ &= w_1^*(v_0) - v_1^*(w_0) && (1.75)\\ &= -d\lambda_\eta(v, w), && (1.76) \end{aligned}$$

where we use Lemma 1.26 in (1.75). □

Remark 1.76. By Lemma 1.24 diffeomorphic manifolds have symplectomorphic cotangent bundles. Thus, Theorem 1.75 implies that the only symplectic invariant of a neighborhood of a compact Lagrangian manifold is its diffeomorphism class (because if $L, L' \subseteq M$ are diffeomorphic Lagrangian submanifolds of a symplectic manifold M, there is a symplectomorphism between open neighborhoods of the zero section L and L' and Theorem 1.75 implies that there exists a symplectomorphism between L and L').

Corollary 1.77. *Let $L \subseteq (M, \omega)$ be a closed Lagrangian submanifold of a symplectic manifold. Then there exists a neighborhood $\mathcal{U} \subseteq M$ of L which does not contain any closed symplectic submanifold of M.*

Proof. Let $L \subseteq M$ be closed and Lagrangian. By Theorem 1.75 there exists a symplectomorphism from $\mathcal{U} \subseteq M$ of L and $\mathcal{N} \subseteq T^*L$ of the zero section, hence $\omega_{\mathcal{U}}$ is exact. Let $\Sigma \subseteq \mathcal{U}$ be a closed $2k$-dimensional submanifold. Then

$$\int_\Sigma \omega^k = \int_\Sigma (d\lambda)^k = \int_\Sigma d\left(\lambda \wedge (d\lambda)^{k-1}\right) = \int_{\partial\Sigma} \lambda \wedge (d\lambda)^{k-1} = 0, \tag{1.77}$$

hence $\omega_{|\Sigma}$ is degenerate, i.e. Σ cannot be symplectic. □

In Example 1.69 we saw that the diagonal $\Delta \subseteq M \times M$ is a Lagrangian submanifold. Since Δ is diffeomorphic to M, Theorem 1.75 implies that there exists a symplectomorphism $\Psi : \mathcal{U}(\Delta) \to \mathcal{N}(M)$ between a neighborhood $\mathcal{U}(\Delta) \subseteq M \times M$ of the diagonal and a neighborhood $\mathcal{N}(\Delta) \cong \mathcal{N}(M) \subseteq T^*M$ of the zero section in T^*M. This observation can be used to parametrize symplectomorphisms that are C^1-small to the identity.

Theorem 1.78. *Let (M, ω) be a compact symplectic manifold. Then a neighborhood of the identity in $\mathrm{Symp}(M, \omega)$ can be identified with a neighborhood of $0 \in \{\eta \in \Omega^1(M) \mid d\eta = 0\}$.*

Proof. By the observation above there is a diffeomorphism $\Psi : \mathcal{U}(\Delta) \to \mathcal{N}(M)$ with $-\Psi^* d\lambda = -\omega \oplus \omega$. Let $\Phi \in \mathrm{Symp}(M, \omega)$ be C^1-close to the identity such that $\operatorname{graph}\Phi \subseteq \mathcal{U}(\Delta)$. By Lemma 1.71 $\operatorname{graph}\Phi$ is a Lagrangian submanifold, hence $\Psi(\operatorname{graph}\Phi) \subseteq \mathcal{N}(M)$ is Lagrangian, too. Because $\Psi(\operatorname{graph}\Phi)$ is the image of the map $M \to T^*M$ given by $q \mapsto \Psi(q, \Phi(q))$, it is C^1-close to the zero section. Then one can show [MS05, p. 103] that it is the graph of a one-form $\eta \in \Omega^1(M)$ which, by Example 1.72, is closed. □

1.2.6 Generating Functions

Let M_1, M_2 be two smooth n-dimensional manifolds with cotangent bundles $L_i = T^*M_i$. With the natural identification $L_1 \times L_2 \cong T^*(M_1 \times M_2)$, the tautological one-form λ on $T^*(M_1 \times M_2)$ is given by

$$\lambda = \mathrm{pr}_1^* \lambda_1 + \mathrm{pr}_2^* \lambda_2 \tag{1.78}$$

for projection $\mathrm{pr}_i : L_1 \times L_2 \to L_i$, $i = 1, 2$ and tautological one-form $\lambda_i \in \Omega^1(L_i)$. Let $f \in \mathbb{C}^\infty(M_1 \times M_2)$. Then $df \in \Omega^1(M_1 \times M_2)$ is closed and with Example 1.72 we conclude that

$$Y_f = \{(x_1, x_2, d_{x_1} f, -d_{x_2} f) \mid (x_1, x_2) \in M_1 \times M_2\} \subseteq M_1 \times M_2 \times L_1 \times L_2 \tag{1.79}$$

for $d_{x_i} f = \mathrm{pr}_{i*} df_{(x_1, x_2)}$ is a Lagrangian submanifold. If Y_f is the graph of a diffeomorphism $\Psi : L_1 \to L_2$, Lemma 1.71 implies that Ψ is a symplectomorphism.

Definition 1.79. If $Y_f = \operatorname{graph}\Psi$ for $f \in C^\infty(M_1 \times M_2)$, $\Psi : L_1 \to L_2$, then f is called the *generating function* of the symplectomorphism Ψ.

Let (U_1, x), (U_2, y) be charts for M_1, M_2 with induced charts $(T^*U_1, (x, \xi))$, $(T^*U_2, (x, \eta))$ of the cotangent bundles L_1 and L_2. Then Y_f is the graph of $\Psi : L_1 \to L_2$ if and only if for $(x, \xi) \in L_1$, $(y, \eta) \in L_2$ we have

$$\Psi(x, \xi) = (y, \eta) \qquad \Leftrightarrow \qquad \xi = d_x f, \quad \eta = -d_y f. \tag{1.80}$$

In local coordinates the equation $\Psi(x, \xi) = (y, \eta)$ for given $(x, \xi) \in L_1$ amounts to solving

$$\xi_i = \frac{\partial f}{\partial x_i}(x, y) \qquad \text{and} \qquad \eta_i = -\frac{\partial f}{\partial y_i}(x, y). \tag{1.81}$$

If we have a solution $y = \Psi_1(x, \xi)$ of the first equation in (1.81), we can insert y into the second one and obtain $\eta = \Psi_2(x, \xi)$, hence a symplectomorphism $\Psi = (\Psi_1(x, \xi), \Psi_2(x, \xi))$. By the implicit function theorem such a solution exists only if

$$\det \frac{\partial^2 f}{\partial y_j \partial x_i} \neq 0. \tag{1.82}$$

Remark 1.80. A diffeomorphism of $2n$-dimensional manifolds depnds on $2n$ functions of $2n$ variables. The discussion of generating functions shows that locally a symplectomorphism is determined by one single function of $2n$ variables, the generating function.

Example 1.81. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y) = \sum_i x_i y_i$. Then $y_i = \xi_i$, $\eta_i = -x_i$, hence the symplectomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ generated by f is given by $\Psi(x, \xi) = (\xi, -x)$.

Example 1.82. Let $M_1 = M_2 = \mathbb{R}^n$, and consider $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x, y) = -\frac{|x - y|^2}{2}. \tag{1.83}$$

Equations (1.81) are given by

$$\xi_i = \frac{\partial f}{\partial x_i} = y_i - x_i \qquad \text{and} \qquad \eta_i = -\frac{\partial f}{\partial y_i} = y_i - x_i. \tag{1.84}$$

Therefore $y_i = \xi_i + x_i$, $\eta_i = \xi_i$ and the symplectomorphism $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ generated by f is given by $\Psi(x, \xi) = (x + \xi, \xi)$. Identifying $\mathbb{R}^{2n}$ with $T\mathbb{R}^n$, i.e. ξ is the velocity vector, then Ψ corresponds to a free translation in Euclidean space.

2 Group Actions on Symplectic Manifolds

2.1 Review of Smooth Group Actions

Let G be a Lie group, not necessarily connected.

Definition 2.1. We say G *acts on a smooth manifold* M (to the left), written $G \curvearrowright M$, if there is a smooth map $\varphi : G \times M \to M$, $(g, x) \mapsto g \cdot x$ such that $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$. Given a group action $G \curvearrowright M$, $x \in M$ the (Lie) subgroup $G_x = \operatorname{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ is the *stabilizer* of x. For $x \in M$ the subset $G \cdot x$ is the *orbit* through x.

Lemma 2.2. *Let $x \in M$ and $h \in G$. Then $G_{hx} = hG_xh^{-1}$.*

Proof. On the one hand, $g \in hG_xh^{-1}$ implies that there exists $g' \in G_x$ such that $g = hg'h^{-1}$, and thus $ghx = hg'x = hx$, i.e. $g \in G_x$. On the other hand, note that for $y = hx \in M$, we have $h^{-1}G_yh \subseteq G_{h^{-1}y} = G_x$, hence $G_{hx} \subseteq hG_xh^{-1}$. □

Let $x \in M$. Lemma 2.2 shows that to each orbit Gx there corresponds a conjugacy class of stabilizers. We say two subgroups $H, H' \subset G$ are equivalent, if H is G-conjugate to H'. This defines an equivalence relation. We denote by (H) the equivalence class of H, and call it the *conjugacy class* of $H \subset G$.

Definition 2.3. The *orbit type* of x is the conjugacy class of the subgroup G_x in G, denoted by (G_x). Given $H \subset G$, the union of all x with orbit type (H) is denoted by $M_{(H)}$.

Definition 2.4. We call an action $G \curvearrowright M$

- *transitive* if there is only one orbit.
- *free* if $G_x = \{e\}$ for all $x \in M$.
- *effective* if $\bigcap_{x\in M} G_x = \{e\}$.
- *locally free* if all stabilizers are discrete.

Definition 2.5. Let $X \in \mathfrak{g} = T_eG$ be a left invariant vector field of G. The *fundamental vector* field $\underline{X}$ of X is defined as

$$\underline{X}_x = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tX)x. \tag{2.1}$$

Remark 2.6. Fundamental vector fields are infinitesimal generators of the group action $\mathbb{R} \times M \to M$, $(t, x) \mapsto \exp(tX)x$, i.e. the flow of $\underline{X}$ is $\Phi_t^X(x) = \exp(tX)x$. Fundamental vector fields are compatible with the Lie bracket in the sense $[\underline{X}, \underline{Y}] = \underline{[X, Y]}$ for all $X, Y \in \mathfrak{g}$.

Lemma 2.7. *For $x \in M$ the map $f_x : G/G_x \to M$, $gG_x \mapsto gx$ is an injective immersion.*[1]

Proof. Consider $d(f_x)_e : T_eG \cong \mathfrak{g} \to T_xM$. Then $\ker d(f_x)_e = \{X \in \mathfrak{g} \mid \underline{X}_x = 0\}$ because

$$d(f_x)_e(X) = d(f_x)_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\bigg|_{t=0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)f_x(e)\bigg|_{t=0} = \underline{X}_x. \tag{2.2}$$

Now $\mathfrak{g}_x = T_eG_x = \{X \in \mathfrak{g} \mid \underline{X}_x = 0\} = \ker d(f_x)_e$, hence $d(f_x)_e : T_eG/G_x \to T_xM$ is injective. But since the orbit map f_x commutes with left translation $l_g : M \to M$, $x \mapsto gx$, $\ker d(f_x)_g = \ker d(f_x)_e$. □

Example 2.8. Consider $\mathbb{R} \times T^2 \to T^2$, $(t, (x, y)) \mapsto (x + t, y + \sqrt{2}t)$. The orbits of this action are not embedded submanifolds.

Corollary 2.9. *If G is compact, then Gx is a submanifold.*

Proof. If G is compact, f_x is a proper map, and a proper injective immersion is an embedding. □

[1] The stabilizer $G_x \subseteq G$ is closed, hence a Lie subgroup. One can show that then the quotient G/G_x is a smooth manifold. [Aud05, Theorem 1.4.8]

2.1.1 The Slice Theorem and Some Applications

In Remark 1.47 we discussed how to describe neighborhoods of submanifolds. In order to adapt this discussion to the case of orbits of a group action of a compact Lie group G, we need an equivariant formulation of tubular neighborhoods. Let $x \in M$ and set $V_x = {}^{T_xM}/_{T_x(Gx)}$. Then, for $g \in G_x$, the differential $d(l_g)_x : T_xM \to T_{gx}M = T_xM$ is an isomorphism that descends to an isomorphism on V_x. Therefore, we obtain a linear representation of the stabilizer $G_x \to GL(V_x)$, $g \mapsto d(l_g)_x$, and an induced *free* action $G_x \curvearrowright G \times V_x$, $h(g,v) = (gh^{-1}, d(l_h)_x(v))$. The quotient space $G \times_{G_x} V_x$ under this action of G_x defines a vector bundle $G \times_{G_x} V_x \to {}^{G}/_{G_x}$ with fiber V_x, i.e.

$$\begin{tikzcd} G \times V_x \arrow[r] \arrow[d] & G \times_{G_x} V_x \arrow[d] \\ G \arrow[r] & {}^{G}/_{G_x} \end{tikzcd} \tag{2.3}$$

and G acts on $G \times_{G_x} V_x$ by $h[g,v] = [hg,v]$. In this way we may identify ${}^{G}/_{G_x}$ with the zero section $\{[g,0] \mid g \in G\} \subseteq G \times_{G_x} V_x$. The Slice Theorem asserts that the orbit map $f_x : {}^{G}/_{G_x} \to M$ extends to a neighborhood of the zero section $s : {}^{G}/_{G_x} \to G \times_{G_x} V_x$, $[g] \mapsto [g,0]$, i.e.

$$\begin{tikzcd} G \times_{G_x} V_x \arrow[r] & M \\ {}^{G}/_{G_x} \arrow[u, "s"] \arrow[r, "f_x"'] & Gx \arrow[u, "\iota"'] \end{tikzcd} \tag{2.4}$$

Theorem 2.10 (Slice Theorem). *Let G be a compact Lie group acting on M. Then, there exists a G-equivariant diffeomorphism from an open neighborhood of the zero section ${}^{G}/_{G_x}$ in $G \times_{G_x} V_x$ onto an open neighborhood of the orbit Gx in M induced by the natural map f_x of Lemma 2.7.*

Remark 2.11. Note that for a compact Lie group there always exists a G-invariant metric on M. Let $\langle \cdot, \cdot \rangle$ be a Riemannian metric on M. Then

$$\langle\langle \cdot, \cdot \rangle\rangle = \int_G g^* \langle \cdot, \cdot \rangle \, dG \tag{2.5}$$

is a G-invariant metric, and the exponential map is then G-equivariant.

Proof. Let g be a G-invariant metric on M, and let $\exp_x : T_xM \to M$ be the exponential map. In view of the G-invariance of the metric, the exponential map is G-equivariant, i.e. $g \exp_x v = \exp_{gx}(d(l_g)_x(v))$. We identify V_x with the orthogonal complement $T_x(Gx)^\perp$, and define $f_x : G \times V_x \to M$, $(g,v) \mapsto g \exp_x v$. Note that f_x is G_x-invariant because $f_x(gh^{-1}, d(l_h)_x(v)) = gh^{-1} \exp_{hx}(d(l_h)_x(v)) = f_x(g,v)$, hence induces a well-defined map

$$f_x : G \times_{G_x} V_x \to M, \qquad [g,v] \mapsto g \exp_x v. \tag{2.6}$$

We want to apply the fact that a map which embeds the zero section is an embedding of an open neighborhood of the zero section (cf. [Cie10, p. 91f.]). To see that f_x embeds the zero section note that f_x is G-equivariant, $f_x(h[g,v]) = hg\exp_x v = hf_x([g,v])$, hence it suffices to verify bijectivity of $d(f_x)_e : T_e(G\times_{G_x} V_x) \to T_xM$. We have, for $[X,v] \in T_e(G\times_{G_x} V_x)$,

$$d(f_x)_e([X,v]) = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\exp_x(tv)\Big|_{t=0} = \underline{X}_x + v \in T_x(Gx)\oplus V_x \cong T_xM, \tag{2.7}$$

i.e. $d(f_x)_e$ is surjective, and because $\dim G\times_{G_x} V_x = \dim M$, also bijective. □

Remark 2.12. There is no analogue of Theorem 2.10 for topological group actions.

Theorem 2.13. *Let M be a compact manifold, and let G act smoothly on M. Then, there are only finitely many orbit types.*

Proof. We prove the statement by induction on $\dim M = n$. If $n = 0$, then M is a finite collection of points (because it is compact), hence there are only finitely many orbits. Now suppose the statement is proved for $n-1$, and let M be covered by finitely many equivariant tubular neighborhoods. It therefore suffices to show that each tubular neighborhood $N = G\times_{G_x} V_x$, $x\in M$, contains only finitely many orbit types. For this choose a G_x-invariant metric on V_x, let $SV_x \subseteq V_x$ denote the unit sphere for that metric, and let $SN = G\times_{G_x} SV_x$ be the sphere bundle of N. Clearly, SN is a $(n-1)$-dimensional manifold with smooth G-action, hence there are finitely many orbit types in SN. Because the G_x-action on V_x is linear, the orbit type of $[g,\lambda v]\in N$ is the same as that of $[g,v]\in SN$ for all $\lambda \neq 0$. Thus, the orbit types in N are those in SN plus the orbit type of ${}^{G}/_{G_x}$. □

Theorem 2.14. *Let G be a compact Lie group acting on M, and $H\subset G$ a Lie subgroup. Then $M_{(H)} = \{x\in M \mid G_x \in (H)\}$ is a submanifold.*

Proof. By the Slice Theorem 2.14 it suffices to prove that $M_{(H)}\subseteq M$ is a submanifold in a neighborhood of $Gx\subseteq M$ for all $x\in M_{(H)}$. Consider orbits Gx of type (H) in $G\times_H V$ (via $f_x : G\times_H V\to M$). Note that for $g'\in G_{[g,v]}$ we have $[gh^{-1}, hv] = [g,v] = g'[g,v] = [g'g, v]$, i.e. $g' = gh^{-1}g^{-1}$, and $v = hv$. Thus $G_{[g,v]} = gH_vg^{-1}$ for $H_v = \{h\in H \mid hv = v\}$, and we conclude that $G[g,v]$ is of type (H) if and only if $H = H_v$, i.e. $v\in V$ is a fixed point of the H-action on V. Let $F\subseteq V$ denote the set of fixed points of H. Then

$$(G\times_H V)\cap M_{(H)} = \{[g,v]\in G\times_H V \mid G_{[g,v]}\in (H)\} = G\times_H F. \tag{2.8}$$

Because F is a linear subspace, $G\times_H F$ is a subbundle and, in particular, a submanifold, i.e. $M_{(H)}$ is a submanifold in the equivariant tubular neighborhood of the orbit of $x\in M_{(H)}$. □

Remark 2.15. $M_{(H)}$ is neither compact nor connected in general.

Corollary 2.16. *The set of fixed points of an action of a compact Lie group G is a submanifold of M.*

Proof. The set of fixed points is $M_{(G)}$. □

Example 2.17. Let $G = S^1$ act on $M = \mathbb{R}P^2$ by

$$e^{\imath t}\cdot[x_0 : x_1 : x_2] = [x_0 : (x_1\cos t + x_2\sin t) : (-x_1\sin t + x_2\cos t)]. \tag{2.9}$$

The fixed points of this action $\mathbb{R}P^2_{S^1} = [1:0:0]$. The orbit type of the identity $e = e^{\imath 0}$ is

$$\mathbb{R}P^2_{(e)} = \{[a:b:c] \mid a \neq 0 \text{ and } (b \neq 0 \text{ or } c \neq 0)\}, \tag{2.10}$$

and the orbit type of $e^{\pm 2\pi\imath} = \pm 1$ is

$$\mathbb{R}P^2_{(\pm 1)} = \{[0:b:c]\} = \mathbb{R}P^1 \subset \mathbb{R}P^2. \tag{2.11}$$

A neighborhood of $\mathbb{R}P^2_{(\pm 1)} = S^1 \cdot [0:1:0]$ looks like a Möbius band.

Theorem 2.18. *Let $G \curvearrowright M$ be a smooth action of a compact Lie group G, and let ${}^{M}\!/_{G}$ be connected. Then there is an orbit type (H) such that $M_{(H)}$ is open and dense, and ${}^{M_{(H)}}\!/_{G}$ is connected.*

Definition 2.19. An open and dense orbit type as in Theorem 2.18 is called *principal orbit type*. Orbits of the same dimension as principal orbits are called *exceptional*, all other orbits *singular*.

Proof. Assume G is connected. The statement is proved by induction on $\dim M = n$. If $n = 0$, and ${}^{M}\!/_{G}$ is connected, there is only one orbit (which is in fact M, i.e. open and dense). Assume the statement is proved for $n-1$, and consider, as in the proof of Theorem 2.18, a equivariant tubular neighborhood $N = G \times_H V$. Again we want to apply the induction hypothesis to SN, and therefore need to check that ${}^{SN}\!/_{G}$ is connected. Consider the vector bundle $\pi : N \to {}^{G}\!/_{H}$, $[g, v] \mapsto [g]$. To connect $a, b \in SN$ by a path, choose a path form $\pi(a)$ to $\pi(b)$ in ${}^{G}\!/_{H}$, and lift it (by the path-lifting property of vector bundles) to a path starting at $a \in SN$. The endpoint $b' \in SN$ of that lift lies in the same fiber as b. Thus, to connect a and b, it suffices that each fiber SV is connected. Because SV is a sphere in V, it is disconnected if and only if $\dim V = 1$. Then $SN \to {}^{G}\!/_{H}$ is a twofold covering. If it is not trivial (i.e. equivalent to the projection ${}^{G}\!/_{H} \times \{0,1\} \to {}^{G}\!/_{H}$), it is connected. If it is trivial, then H acts trivially on the slice $V \cong \mathbb{R}$, and hence all orbits in N are the same. If $\dim V \neq 1$, the induction hypothesis applies to SN so that there exists an orbit type (H) such that $SN_{(H)}$ is open and dense in SN. Similarly to the proof of Theorem 2.14, we conclude that $N_{(H)}$ is open and dense in N. To complete the proof, we pick a locally finite (by paracompactness) cover of M by equivariant tubular neighborhoods. Since the intersection of open and dense sets is dense and open, and the tubular neighborhoods are G-equivariant, the dense and open orbit type (H) must be the same for neighborhoods with non-empty intersection. But because ${}^{M}\!/_{G}$ is connected (hence path-connected), the dense orbit type (H) must be the same for all tubular neighborhoods. □

Theorem 2.20. *Let $G \curvearrowright M$ be a smooth action of a compact Lie group G. Then the subset of singular orbits in M has codimension $\geqslant 2$.*

Proof. Exercise. □

Example 2.21. As in Example 2.17 consider $S^1 \curvearrowright \mathbb{R}P^2$ given by

$$e^{\imath t}[x_0 : x_1 : x_2] = [x_0 : x_1 \cos t + x_2 \sin t : -x_1 \sin t + x_2 \cos t]. \tag{2.12}$$

The principal orbit type is $\mathbb{R}P^2_{(e)}$, the exceptional orbit type is $\mathbb{R}P^2_{(\mathbb{Z}_2)}$, and singular orbit type $\mathbb{R}P^2_{(S^1)}$.

Example 2.22. Consider $S^1 \curvearrowright S^3 \subset \mathbb{C}^2$, $e^{\imath t}(z_1, z_2) = (e^{\imath t p} z_1, e^{\imath t q} z_2)$. This action is free if and only if $gg^\top(p, q) = 1$. The principal orbit type is $S^3_{(e)} = \{z_1 \neq 0, z_2 \neq 0\}$.

2.2 Hamiltonian and Symplectic Group Actions

Definition 2.23. A smooth action $G \curvearrowright (M, \omega)$ is called *symplectic* if

$$\psi : G \to \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M) \tag{2.13}$$

acts by symplectomorphisms, i.e. $\psi_g^* \omega = \omega$ for all $g \in G$.

The *infinitesimal action* defines a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$. Indeed, for $X \in \mathfrak{g} = T_e G$ we define the *fundamental vector field* $\underline{X}$ on M by

$$\underline{X}(x) = \frac{\mathrm{d}}{\mathrm{d}t} \psi_{\exp(tX)}(x)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \exp(tX) \cdot x\Big|_{t=0}. \tag{2.14}$$

Since ψ_g is a symplectomorphism for all $g \in G$, the fundamental vector field $\underline{X} \in \mathfrak{X}(M)$ is *symplectic*, i.e. $L_{\underline{X}} \omega = 0$. By Cartan's formula this is equivalent to $i_{\underline{X}} \omega$ being a closed form. One can directly verify $\underline{[X, Y]} = [\underline{X}, \underline{Y}]$ for all $X, Y \in \mathfrak{g}$. (Note that the flow of $\underline{X}$ is the action of the 1-parameter subgroup associated to $X \in \mathfrak{g}$.)

Definition 2.24. A symplectic action of G on M is called *weakly Hamiltonian* if each fundamental vector field $\underline{X}$ is Hamiltonian, i.e. $i_{\underline{X}} \omega = dH$ for a smooth function $H : M \to \mathbb{R}$. In other words, for every $X \in \mathfrak{g}$ there corresponds a smooth Hamiltonian function $H \in C^\infty(M)$ inducing a map

$$\mathfrak{g} \to C^\infty(M), \qquad X \mapsto H. \tag{2.15}$$

(Since $H_{\underline{X}}$ is determined only up to a constant, we can choose it in such a way that it becomes linear.) A weakly Hamiltonian group action is called *Hamiltonian*, if

$$\tilde{\mu} : (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\}), \qquad X \mapsto H_{\underline{X}} \tag{2.16}$$

is a Lie algebra antihomomorphism.[2]

Remark 2.25. A weakly Hamiltonian action need not to be Hamiltonian in general. If $G \curvearrowright M$ is weakly Hamiltonian, the obstruction for this action being Hamiltonian can be seen as follows. Let $X, Y \in \mathfrak{g}$, and $\tilde{\mu}(X), \tilde{\mu}(Y)$ be the corresponding Hamiltonian functions for $\underline{X}$ and $\underline{Y}$ respectively. Then $-\{\tilde{\mu}(X), \tilde{\mu}(Y)\}$ is a Hamiltonian function for $[X, Y]$, and we have

$$d\left(-\{\tilde{\mu}(X), \tilde{\mu}(Y)\} - \tilde{\mu}([X, Y])\right) = i_{[\underline{X}, \underline{Y}]} \omega - i_{[\underline{X}, \underline{Y}]} \omega = 0. \tag{2.17}$$

If M is connected, this difference is a constant function. The requirement that $\tilde{\mu}$ is a Lie algebra antihomomorphism implies that this constant is zero. In some important cases it is possible to get rid of this ambiguity (e.g. if M is compact and connected, and G is Abelian; if M is closed and connected; G is semisimple, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Example 2.26. If $H : M \to \mathbb{R}$ is a Hamiltonian with complete Hamiltonian vector field X_H, then the flow $\Phi^{X_H} : \mathbb{R} \times M \to M$ defines a Hamiltonian $\mathbb{R}$-action on M with moment map $H : M \to \mathbb{R}$.

[2] There are different sign conventions for the definition of a Hamiltonian group action. E.g. [Aud05] defines a Hamiltonian action as a Lie algebra homomorphism, but uses opposite signs in the definition of Poisson bracket, while [MS05] define the Poisson bracket in the same way as we do, but require $\tilde{\mu}$ to be a Lie algebra homomorphism.

Example 2.27. Let $G \curvearrowright M$ be smooth Lie group action. Then there is an induced action $G \curvearrowright T^*M$ given by $(g,\eta) \mapsto l_{g*}\eta$ for the left translation $l_g : M \to M$. This is action is symplectic since it preserves the tautological one-form $\lambda \in \Omega^1(T^*M)$. In order to see this consider, more generally, for a diffeomorphism $f : M \to M$ the commutative diagram

$$\begin{tikzcd} T^*M \arrow[r, "F=f^*"] \arrow[d, "\pi"'] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, "f^{-1}"'] & M \end{tikzcd} \tag{2.18}$$

Let $\eta \in T^*M$, $v \in T_\eta T^*M$. Then

$$\begin{aligned} (F^*\lambda)_\eta(v) &= \lambda_{f^*\eta}(dF_\eta(v)) = f^*\eta\,(d(\pi \circ F)_\eta(v)) = \eta\,(d(f \circ \pi \circ F)_\eta(v)) = \eta(d\pi_\eta(v)) && (2.19)\\ &= \lambda_\eta(v). && (2.20) \end{aligned}$$

In particular, for $F = l_{g*} = l^*_{g^{-1}}$, $F^*\lambda = \lambda$. Thus, a group action $G \curvearrowright M$ induces a symplectic action on the cotangent bundle.

2.2.1 Coadjoint Action

A particularly important example of a Hamiltonian group action is the coadjoint action. Let $G \curvearrowright G$ be given by conjugation $c_g(h) = ghg^{-1}$.

Definition 2.28. The *adjoint action* $\mathrm{Ad} : G \to GL(\mathfrak{g})$ of a Lie group G on its Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{Ad}_g X = dc_g(X) \tag{2.21}$$

for $g \in G$, $X \in \mathfrak{g}$. The *coadjoint action* $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ of G on the dual $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{Ad}^*_g \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle \tag{2.22}$$

for $g \in G$, $\xi \in \mathfrak{g}^*$, $X \in \mathfrak{g}$.

Remark 2.29. The fundamental vector field ${}^{\mathfrak{g}}\underline{X} \in \Gamma(T\mathfrak{g})$ of the adjoint action is given by

$${}^{\mathfrak{g}}\underline{X}_Y = [X, Y]. \tag{2.23}$$

The fundamental vector field ${}^{\mathfrak{g}^*}\underline{X} \in \Gamma(T\mathfrak{g}^*)$ of the coadjoint action is given by

$$\langle {}^{\mathfrak{g}^*}\underline{X}_\xi, Y\rangle = -\langle \xi, [X, Y]\rangle. \tag{2.24}$$

Example 2.30. Let $SO(3)$ act on its Lie algebra $\mathfrak{so}(3) = \{A \in \mathrm{Mat}(3,\mathbb{R}) \mid A^\top = -A\}$ by the adjoint action. To describe this action we identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ via

$$X = (x,y,z) \in \mathbb{R}^3 \mapsto \hat{X} \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \in \mathfrak{so}(3). \tag{2.25}$$

Then $\mathrm{Ad}_A \hat{X} = AX$, $[\hat{X}, \hat{Y}] = X \times Y$ and the standard inner product $-\operatorname{tr}(\hat{X}\hat{Y})$ on $\mathfrak{so}(3)$ corresponds to the Euclidean inner product on $\mathbb{R}^3$. Identifying $\mathfrak{so}(3)$ with its dual via this product the coadjoint action $SO(3) \curvearrowright \mathfrak{so}(3)^*$ is given by

$$\mathrm{Ad}^*_A \hat{\xi} = -A\xi. \tag{2.26}$$

Therefore orbits of the coadjoin action are spheres of radius $\|\xi\|$.

Definition 2.31. Let $\omega : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^* = (\bigwedge^2 \mathfrak{g})^*$ be the skew-symmetric bilinear form on $\mathfrak{g}^*$ given by $\xi \mapsto \omega_\xi$ with

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle. \tag{2.27}$$

Lemma 2.32. *Consider $\mathfrak{g} \to \mathfrak{g}^*$, $X \mapsto \omega_\xi(X, \cdot)$. Then $\ker \omega_\xi$ is the Lie algbera $\mathfrak{g}_\xi$ of the stabilizer G_ξ of $\xi \in \mathfrak{g}^*$ under the coadjoint action.*

Proof. Let $X, Y \in \mathfrak{g}$. Differentiating the definition of the coadjoint action (2.22) yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \mathrm{Ad}^*_{\exp(-tX)}\, \xi, Y \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \xi, \mathrm{Ad}_{\exp(tX)}\, Y \rangle = \langle \xi, [X, Y] \rangle = -\omega_\xi(X, Y). \tag{2.28}$$

The stabilizer of $\xi \in \mathfrak{g}^*$ under the coadjoint action consists of those $g = \exp(tX) \in G$ with $c_{\exp(-tX)*}\xi = \xi$, hence $\ker \omega_\xi$ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer. □

By Lemma 2.32 the form ω_ξ induces a nondegenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\xi$. Note that the quotient $\mathfrak{g}/\mathfrak{g}_\xi$ may be identified with the tangent space to the orbit $T_\xi(G \cdot \xi)$ because the sequence

$$0 \to \mathfrak{g}_\xi \to \mathfrak{g} \to T_\xi(G \cdot \xi) \to 0 \tag{2.29}$$

with $X \in \mathfrak{g} \mapsto \underline{X}_\xi \in T_\xi(G \cdot \xi)$ is exact. By Lemma 2.32 we obtain a nondegenrate two-form ω_ξ on the orbit $G \cdot \xi$ under the coadjoint action.

Corollary 2.33. *Orbits of the coadjoint action have even dimension.*

Lemma 2.34. *Orbits of the coadjoint action are symplectic manifolds.*

Proof. It remains to show that ω_ξ is closed on $G\xi$. Because $T_\xi G\xi$ is spanned by those fundamental vector fields associated to the coadjoint action on $\mathfrak{g}^*$, it suffices to show that $d\omega_\xi(\underline{X}, \underline{Y}, \underline{Z}) = 0$ for all $X, Y, Z \in \mathfrak{g}$. By the invariant formula for the exterior derivative (cf. [Lee12, p. 370]) we have

$$\begin{aligned} d\omega_\xi(\underline{X}, \underline{Y}, \underline{Z}) = {} & \underline{X}\left(\omega_\xi(\underline{Y}, \underline{Z})\right) - \underline{Y}\left(\omega_\xi(\underline{X}, \underline{Z})\right) + \underline{Z}\left(\omega_\xi(\underline{X}, \underline{Y})\right) \\ & - \omega_\xi([\underline{X}, \underline{Y}], \underline{Z}) - \omega_\xi([\underline{Y}, \underline{Z}], \underline{X}) + \omega_\xi([\underline{X}, \underline{Z}], \underline{Y}). \end{aligned} \tag{2.30}$$

By definition of ω_ξ it is invariant under the coadjoint action [Aud05, p. 52], i.e. constant along the fundamental vector fields so that the first three terms in (2.30) vanish. The last three terms vanish because of the Jacobi identity, hence $d\omega_\xi = 0$. □

2.2.2 Moment Maps

Definition 2.35. The map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ of a Hamiltonian action as in (2.16) is called *comomoment map*. The dual concept $\mu : M \to \mathfrak{g}^*$ such that $\mu(x)(X) = \tilde{\mu}(X)(x)$ is called *moment map*.

Example 2.36. Let $G_i \curvearrowright (M_i, \omega_i)$, $i = 1, 2$, be two Hamiltonian actions of Lie groups on symplectic manifolds with moment maps $\mu_i : M_i \to \mathfrak{g}_i^*$. In Example 1.20 we saw that the product $M_1 \times M_2$ carries a natural symplectic structure and the product action $G_1 \times G_2 \curvearrowright M_1 \times M_2$ is Hamiltonian with moment map $\mu_1 \times \mu_2 : M_1 \times M_2 \to \mathfrak{g}_1^* \times \mathfrak{g}_2^*$.

Remark 2.37. There is a simple way to construct further moment maps out of a given moment map. Let $G \curvearrowright (M,\omega)$ be a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$ and $H \subset G$ a closed subgroup. Then the action of H on (M,ω) is still Hamiltonian with moment map $\mu_H : M \to \mathfrak{g}^* \to \mathfrak{h}^*$ (with the natural projection $\mathfrak{g} \to \mathfrak{h}$).

Lemma 2.38. *Let G be connected. Then $\tilde{\mu}$ is a Lie algebra antihomomorphism if and only if μ is G-equivariant (w.r.t. the coadjoint action).*

Proof. Assume $\tilde{\mu}$ is a Lie algebra antihomomorphism. Let $X \in \mathfrak{g}$, and $l_g : M \to M$ left translation with $g \in G$. Then $\underline{\mathrm{Ad}_{g^{-1}} X} = l_g^* \underline{X}$. Indeed, because $c_g : G \to G$ is a Lie group homomorphism, its differential $Ad_g = d(c_g)_e$ at the identity is a Lie algebra homomorphism, hence $\mathrm{Ad}_g X = X \circ c_g$ (for the left-invariant vector corresponding to $X \in \mathfrak{g}$), i.e. the integral curve of $\mathrm{Ad}_g X$ is the integral curve of X conjugated by g. Thus

$$\begin{aligned}\underline{\mathrm{Ad}_{g^{-1}} X}_p &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(t \,\mathrm{Ad}_{g^{-1}} X)p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g^{-1}\exp(tX)gp = d(l_{g^{-1}})_{gp}(\underline{X})_{gp} &\text{(2.31a)}\\ &= (l_g^*\underline{X})_p. &\text{(2.31b)}\end{aligned}$$

It follows that the Hamilton functions $H_{\mathrm{Ad}_{g^{-1}} X}$ and $H_X \circ l_g$ generate the same Hamiltonian vector field, and hence

$$\begin{aligned}H_{\mathrm{Ad}_{g^{-1}}[X,Y]} &= \left\{H_{\mathrm{Ad}_{g^{-1}} X}, H_{\mathrm{Ad}_{g^{-1}} Y}\right\} = \omega\left(\underline{\mathrm{Ad}_{g^{-1}} Y}, \underline{\mathrm{Ad}_{g^{-1}} X}\right) &\text{(2.32a)}\\ &= \{H_X \circ l_g, H_Y \circ l_g\} = \{H_X, H_Y\} \circ l_g = H_{[X,Y]} \circ l_g. &\text{(2.32b)}\end{aligned}$$

For any $g \in G$, let $g : [0,1] \to G$ be a path joining the identity to $g = g(1)$, and let $Y = \dot{g}(t) \circ l_{g(t)}^{-1} \in T_eG \cong \mathfrak{g}$. Then

$$\underline{Y} \circ l_{g(t)}(p) = \frac{\mathrm{d}}{\mathrm{d}t} l_{g(t)}, \qquad \text{and} \qquad \mathrm{Ad}_{g(t)^{-1}}[X,Y] = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{g(t)^{-1}} X, \tag{2.33}$$

and thus using (2.32)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(H_X \circ l_g - H_{\mathrm{Ad}_{g^{-1}} X}\right)(p) &= d(H_X \circ l_g)_p(l_g^*\underline{Y})_p - H_{\mathrm{Ad}_{g^{-1}}[X,Y]}(p) &\text{(2.34)}\\ &= d(H_X)_{gp}(\underline{Y}_{gp}) - H_{[X,Y]} \circ l_g &\text{(2.35)}\\ &= \omega_{gp}\left(\underline{X}_{gp}, \underline{Y}_{gp}\right) + \{H_X, H_Y\}(gp) &\text{(2.36)}\\ &= -\{H_X(gp), H_Y(gp)\} + \{H_X, H_Y\}(gp) = 0, &\text{(2.37)}\end{aligned}$$

i.e. $\langle \mu \circ l_g, X\rangle = H_X \circ l_g = H_{\mathrm{Ad}_{g^{-1}} X} = \langle \mathrm{Ad}_g^* \mu, X\rangle$ for all $X \in \mathfrak{g}$, hence the moment map $\mu : M \to \mathfrak{g}^*$ is G-equivariant. Conversely, let μ be G-equivariant, and let $\mu^X : M \to \mathbb{R}$ be the Hamilton function defined by $\mu^X(p) = \langle \mu(p), X\rangle$ for $X \in \mathfrak{g}$. Then

$$\begin{aligned}d\mu^X(\underline{Y}_p) &= d\mu^X\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tY)p\right) = \frac{\mathrm{d}}{\mathrm{d}t}\mu^X(\exp(tY)p)\bigg|_{t=0} &\text{(2.38a)}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{Ad}^*_{\exp(tY)} \mu(p), X\rangle\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(p), \mathrm{Ad}_{\exp(-tY)} X\rangle\bigg|_{t=0} &\text{(2.38b)}\\ &= \left\langle \mu(p), \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\exp(-tY)} X\bigg|_{t=0}\right\rangle = \langle \mu(p), -[Y,X]\rangle &\text{(2.38c)}\\ &= \mu^{[X,Y]}(p), &\text{(2.38d)}\end{aligned}$$

and on the other hand $d\mu^X(\underline{Y}_p) = \omega_p(\underline{X}_p, \underline{Y}_p) = -\{\mu^X(p), \mu^Y(p)\}$, i.e. the comoment map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$, $\mu(X)(p) = \langle \mu(p), X\rangle = \mu^X(p)$, is a Lie algebra antihomomorphism $\tilde{\mu}([X, Y]) = -\{\tilde{\mu}(X), \tilde{\mu}(Y)\}$. □

Example 2.39. The symplectic action $G \curvearrowright T^*M$ induced from a smooth action $G \curvearrowright M$ discussed in Example 2.26 is Hamiltonian with moment map $\mu : T^*M \to \mathfrak{g}^*$ defined by $\mu(\eta)(X) = -\lambda_\eta(\underline{X}_\eta) = \tilde{\mu}(X)(\eta)$ for $\eta \in T^*M$, $X \in \mathfrak{g}$, $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$, $X \mapsto \lambda(\underline{X})$. Since the action preserves the tautological one-form $\lambda \in \Omega^1(T^*M)$, we have $0 = L_{\underline{X}}\lambda = i_{\underline{X}}d\lambda + d i_{\underline{X}}\lambda$, hence $i_{\underline{X}}\omega = -i_{\underline{X}}d\lambda = d i_{\underline{X}}\lambda = d(\tilde{\mu}(X))$. It remains to check that $\mu : T^*M \to \mathfrak{g}$ is equivariant w.r.t. the coadjoint action, i.e. $\mu(g \cdot \eta) = \mathrm{Ad}^*_{g^{-1}}\, \mu(\eta)$. We compute

$$\mathrm{Ad}^*_g\, \mu(\eta)(X) = \mu(\eta)(\mathrm{Ad}_{g^{-1}} X) = -\lambda_\eta(\underline{\mathrm{Ad}_{g^{-1}} X}_\eta) = -\lambda_\eta\left(d(l_{g^{-1}})_{g\cdot\eta}\underline{X}_{g\cdot\eta}\right) \tag{2.39}$$

$$= -\left(l^*_{g^{-1}}\lambda\right)_{g\cdot\eta}(\underline{X}_{g\cdot\eta}) = -\lambda_{g\cdot\eta}(\underline{X}_{g\cdot\eta}) = \mu(g\cdot\eta)(X), \tag{2.40}$$

where we use (2.31).

Remark 2.40. The computation in Example 2.39 can be generalized to arbitary exact symplectic manifolds $(M, d\lambda)$: an action $G \curvearrowright M$ is Hamiltonian if and only if $l^*_g\lambda = \lambda$ for all $g \in G$. Such an action is sometimes called *symplectic exact* [MS05, p. 162].

Lemma 2.41. *For the coadjoint action $G \curvearrowright \mathfrak{g}^*$, the action of G on an orbit $G\xi$ is Hamiltonian. The moment map μ is the inclusion $G\xi \hookrightarrow \mathfrak{g}^*$.*

Proof. Obviously, the inclusion $\iota : G\xi \hookrightarrow \mathfrak{g}^*$ is G-equivariant so that it remains to check that $\iota^X : G\xi \to \mathbb{R}$, $\iota^X(Y) = \langle \iota(Y), X\rangle$ is the Hamiltonian function for the fundamental vector field generated by $X \in \mathfrak{g}$. This is equivalent to the condition $\langle d\mu_\xi(Y), X\rangle = \omega_\xi(X, Y)$ for all $Y \in T_\xi G\xi$, $X \in \mathfrak{g}$ and the differential $d\mu_\xi : T_\xi G\xi \to \mathfrak{g}^*$. (Indeed, by definition of the comoment map $\omega_p(\underline{X}, Y) = d\tilde{\mu}(X)_p(Y) = Y(\tilde{\mu}(X)(p)) = Y\langle \mu(p), X\rangle = \langle d\mu_p(Y), X\rangle$.) The tangent space to the coadjoint orbit is spanned by the fundamental vector fields of the coadjoint action so that we need to show $\langle {}^{\mathfrak{g}^*}\underline{Y}_\xi, X\rangle = \omega_\xi(X, Y) = \langle \xi, [X, Y]\rangle$. But this is just the defining relation for fundamental vector fields of the coadjoint action (2.24). □

Lemma 2.42. *The moment map $\mu : M \to \mathfrak{g}^*$ is a submersion at x if and only if $G_x \subseteq G$ is discrete.*

Proof. Consider $d\mu^*_x : \mathfrak{g} \to T^*_x M$ defined by $d\mu^*_x(X) = i_{\underline{X}}\omega_x$. Then the differential of the moment map $d\mu_x : T_xM \to \mathfrak{g}^*$ is, by definition, the *transpose* of $d\mu^*_x$, i.e. $d\mu_x(v) = v \circ d\mu^*_x : \mathfrak{g} \to \mathbb{R}$, $X \mapsto \omega_x(\underline{X}, v)$. The image of $d\mu_x$ is the *annihilator* in $\mathfrak{g}^*$ of $\ker d\mu^*_x = \{X \in \mathfrak{g} \mid i_{\underline{X}}\omega_x = 0\} = \{X \in \mathfrak{g} \mid \underline{X}_x = 0\} = \mathfrak{g}_x$, i.e.

$$\operatorname{im} d\mu_x = \{\xi \in \mathfrak{g}^* \mid \mathfrak{g}_x \subseteq \ker \xi\}. \tag{2.41}$$

This implies that the rank of $d\mu_x$ equals $\dim \mathfrak{g}/\mathfrak{g}_x$, i.e. the dimension of the orbit through x. Thus $\operatorname{rank} d\mu_x = \dim \mathfrak{g}^*$ if and only if $\dim \mathfrak{g}_x = 0$ if and only if G_x is discrete. □

Corollary 2.43. *Let G be a commutative and compact group acting effectively on a symplectic manifold M, and assume the action is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Then μ is a submersion on $M_{(e)}$, and $M_{(e)}$ is the principal orbit type.*

Proof. Let (H) ba a principal orbit type. Because G is commutative, all points in $M_{(H)}$ have stabilizer H. Then $H \curvearrowright M_{(H)}$ acts as identity and because $M_{(H)}$ is dense, this implies $H = \{e\}$ since $G \curvearrowright M$ is effective, i.e. $M_{(e)}$ is the principal orbit type. Lemma 2.42 implies that the moment map $\mu : M \to \mathfrak{g}^*$ is a submersion on $M_{(e)}$. $\square$

Theorem 2.44. *Let G be a compact connected Lie group. The principal orbit of the coadjoint representation is $\mathfrak{g}^*_{(T)}$, where $T \subset G$ is a maximal torus (unique up to conjugation). In particular $\mu : G\xi \to \mathfrak{g}^*$ is not a submersion.*

Proof. $\square$

Theorem 2.45 (Noether)**.** *Let (M, ω) be a symplectic manifold, $G \curvearrowright M$ a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$, $H : M \to \mathbb{R}$ some Hamiltonian function which is G-invariant, i.e. $H(gx) = H(x)$ for all $g \in G$, $x \in M$. Then μ is constant along the flow lines of X_H.*

Proof. Let $\gamma : I \to M$ be an integral curve of X_H. Then, for all $X \in \mathfrak{g}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu \circ \gamma(t), X\rangle = \langle d\mu_{\gamma(t)}(X_H \circ \gamma(t)), X\rangle = \langle X_H \circ \gamma(t), d\tilde{\mu}_{\gamma(t)}(\underline{X})\rangle \tag{2.42a}$$

$$= \langle X_H \circ \gamma(t), i_{\underline{X}}\omega_{\gamma(t)}\rangle = \omega_{\gamma(t)}(\underline{X}, X_H) = -dH_{\gamma(t)}(\underline{X}), \tag{2.42b}$$

and since $H(\exp(sX)\gamma(t)) = H(\gamma(t))$, differentation of (2.42) at $s = 0$ yields

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} H(\exp(sX)\gamma(t)) = dH_{\gamma(t)}(\underline{X}) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu \circ \gamma(t), X\rangle \tag{2.43}$$

for all $X \in \mathfrak{g}$, i.e. μ is constant along γ. $\square$

Remark 2.46. Theorem 2.49 implies that the Hamiltonian vector field is tangent to the level sets of the moment map. The physical interpretation is that to every symmetry of the *energy* H there corresponds a conserved quantity.

Example 2.47. Consider the Hamiltonian $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$\phi(\xi, \eta) = \frac{1}{2}\left(\|\xi\|^2\|\eta\|^2 - \langle \xi, \eta\rangle^2\right). \tag{2.44}$$

Let $SL(2, \mathbb{R})$ act on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a\xi + b\eta \\ c\xi + d\eta \end{pmatrix}. \tag{2.45}$$

We want to show that this action $SL(2, \mathbb{R}) \curvearrowright \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is Hamiltonian and that ϕ is preserved by it. We can express the Hamiltonian as

$$\phi(\xi, \eta) = \frac{1}{2}\det\begin{pmatrix} \langle \xi, \xi\rangle & \langle \eta, \xi\rangle \\ \langle \xi, \eta\rangle & \langle \eta, \eta\rangle \end{pmatrix}. \tag{2.46}$$

Then

$$\phi(A(\xi, \eta)) = \frac{1}{2}\det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \langle \xi, \xi\rangle & \langle \eta, \xi\rangle \\ \langle \xi, \eta\rangle & \langle \eta, \eta\rangle \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\top\right] = \phi(\xi, \eta) \tag{2.47}$$

since $A \in SL(2,\mathbb{R})$. Let $\omega_0 = \sum_{i=1}^{n+1} d\xi_i \wedge d\eta_i$ be the standard symplectic structure on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \cong T^*\mathbb{R}^{n+1}$. Since the action is linear, the fundamental vector fields are given by $\underline{\xi} = \alpha\xi + \beta\eta$, $\underline{\eta} = \gamma\xi - \alpha\eta$ for

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}). \tag{2.48}$$

We claim that

$$G = \alpha\langle\xi,\eta\rangle - \frac{\gamma}{2}\|\xi\|^2 + \frac{\beta}{2}\|\eta\|^2 \tag{2.49}$$

generates the Hamiltonian vector field. We compute $dG = \alpha(\eta d\xi + \xi d\eta) - \gamma\xi d\xi + \beta\eta d\eta$ and solve $i_{\underline{X}}\omega = dG$. We get

$$\underline{X} = \sum_{i=1}^{n+1} (\alpha\xi_i + \beta\eta_i) \frac{\partial}{\partial\xi_i} + \sum_{i=1}^{n+1} (\gamma\xi_i - \alpha\eta_i) \frac{\partial}{\partial\eta_i}. \tag{2.50}$$

Finally $\mathfrak{sl}(2,\mathbb{R}) \to C^\infty(M)$, $X \mapsto G_{\underline{X}}$ is a Lie algebra homomorphism. In order to see this let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.51}$$

be a basis of $\mathfrak{sl}(2,\mathbb{R})$. Then $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$ and

$$\{G_{\underline{H}}, G_{\underline{X}}\} = \omega(\underline{H},\underline{X}) = \omega\left(\alpha\xi_i\frac{\partial}{\partial\xi_i} - \alpha\eta_i\frac{\partial}{\partial\eta_i}, \eta_i\frac{\partial}{\partial\xi_i}\right) = \alpha\sum_i \eta_i\eta_i = 2G_{\underline{X}} = G_{2\underline{X}} \tag{2.52}$$

$$= G_{[\underline{H},\underline{X}]} \tag{2.53}$$

and similarly for the other basis elements. Hence the action is Hamiltonian with moment map $\mathfrak{sl}(2,\mathbb{R}) \to C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$, $X \mapsto G_{\underline{X}}$. With Noether's Theorem 2.45 we conclude that $\|\xi\|^2$, $\|\eta\|^2$, $\langle\xi,\eta\rangle$ are conserved along the flow lines of ϕ.

Example 2.48 (Periodic Hamiltonian). Let $S^1 \curvearrowright M$ be a Hamiltonian action. Then the moment map $\mu : M \to \mathbb{R}$ is just the Hamiltonian function of the fundamental vector field of this action. Because S^1 is Abelian, the coadjoint action is trivial, and the Hamiltonian therefore S^1-invariant. Thus, the fundamental vector field of such a S^1-action is tangent to the level sets of the Hamiltonian. We call such a function *periodic Hamiltonian* [Aud05, p. 79]. It follows that level sets of periodic Hamiltonians are (oriented) submanifolds with a free S^1-action.

2.2.3 Symplectic Reduction

In this section we formulate a mathematical version of the classical physical observation that given a k-dimensional symmetry group acting on symplectic (phase) space, the number of degrees of freedom may be reduced by $2k$.

Theorem 2.49 (Marsden-Weinstein). *Let (M,ω) be a symplectic manifold, $G \curvearrowright M$ a Hamiltonian action of a compact Lie group with moment map $\mu : M \to \mathfrak{g}^*$, and assume G acts freely on $\mu^{-1}(0)$. Then the orbit space ${}^{\mu^{-1}(0)}/_{G}$ is a manifold, $\pi : \mu^{-1}(0) \to {}^{\mu^{-1}(0)}/_{G}$ is a principal G-bundle, and there is a unique symplectic form ω_{red} on ${}^{\mu^{-1}(0)}/_{G}$ such that*

$$\pi^*\omega_{\mathrm{red}} = \omega_{|\mu^{-1}(0)}. \tag{2.54}$$

Remark 2.50. Note that because μ is G-equivariant, the action $G \curvearrowright \mu^{-1}(0)$ is well-defined. If G is Abelian, it makes sense to consider $\mu^{-1}(\xi)$ for $0 \neq \xi \in \mathfrak{g}^*$, or varying ξ.

In order to prove Theorem 2.49 we state several Lemmata. Let us begin with

Lemma 2.51. *Let $x \in M$, and let $\mathfrak{g}_x$ be the Lie algebra of the stabilizer G_x. Then*

$$\ker d\mu_x = (T_x Gx)^{\perp_\omega} \qquad \textit{and} \qquad \operatorname{im} d\mu_x = \mathfrak{g}_x^0 = \{\alpha \in \mathfrak{g}^* \mid \mathfrak{g}_x \subseteq \ker \alpha\}. \tag{2.55}$$

Proof. Since $\langle d\mu_x(v), X\rangle = \langle v, i_{\underline{X}}\omega_x\rangle = \omega_x(\underline{X}, v)$, it follows that $v \in \ker d\mu_x$ if and only if $v \in (T_x Gx)^{\perp_\omega}$. As for the second statement, we have $\operatorname{im} d\mu_x \subseteq \mathfrak{g}_x^0$ since $\langle d\mu_x(v), X\rangle = \omega_x(\underline{X}, v) = 0$ for $X \in \mathfrak{g}_x$ (because then $\underline{X}_x = 0$). Now $\dim \operatorname{im} d\mu_x + \dim\ker d\mu_x = \dim M$ and $\dim\ker d\mu_x = \dim M - \dim Gx = \dim M - \dim G + \dim \mathfrak{g}_x$, hence

$$\dim \operatorname{im} d\mu_x = \dim \mathfrak{g} - \dim \mathfrak{g}_x = \dim \mathfrak{g}_x^0, \tag{2.56}$$

where the last equality holds because the sequence $0 \to \mathfrak{g}_x^0 \to \mathfrak{g}^* \to \mathfrak{g}_x^* \to 0$ is exact. □

As a consequence of Lemma 2.51, the action $G \curvearrowright M$ is locally free (i.e. $\mathfrak{g}_x = 0$) if and only if $d\mu_x$ is surjective if and only if x is a regular point of μ. Since $G \curvearrowright \mu^{-1}(0)$ is assumed to be free, it follows that 0 is a regular value, and hence $\mu^{-1}(0)$ is a submanifold with $\operatorname{codim} \mu^{-1}(0) = \dim G$. Moreover, $\mu^{-1}(0)$ is a coisotropic submanifold because $T_x\mu^{-1}(0) = \ker d\mu_x = (T_x Gx)^{\perp_\omega}$ implies $(T_x\mu^{-1}(0))^{\perp_\omega} = T_x Gx \subseteq T_x\mu^{-1}(0)$. This shows

Lemma 2.52. *Under the assumptions of Theorem 2.49, $\mu^{-1}(0)$ is a coisotropic manifold, and $\ker \omega_{|\mu^{-1}(0)} = T_x Gx$.*

Lemma 2.53. *Let (V, ω) be a symplectic vector space, and let $W \subset V$ be a coisotropic subspace. Then ω induces a canonical symplectic form on ${}^{W}\!/\!_{W^\perp}$.*

Proof. Let $w_1, w_2 \in W$, $[w_1], [w_2] \in {}^{W}\!/\!_{W^\perp}$, and define ω' on ${}^{W}\!/\!_{W^\perp}$ by setting $\omega'([w_1], [w_2]) = \omega(w_1, w_2)$. Then ω' is well-defined because for $v_1, v_2 \in W^\perp$, we have

$$\omega'([w_1 + v_1], [w_2 + v_2]) = \omega(w_1, w_2) = \omega([w_1], [w_2]). \tag{2.57}$$

Furthermore, ω' is nondegenerate because if $0 = \omega'([w_1], [w_2]) = \omega(w_1, w_2)$ for all $w_1, w_2 \in W$, then $w_1 \in W^\perp$, i.e. $[w_1] = 0$. □

Theorem 2.54. *Let $G \curvearrowright N$ be free, G compact. Then ${}^{N}\!/\!_{G}$ is a manifold and $\pi : N \to {}^{N}\!/\!_{G}$ is a principal G-bundle.*

Sketch of Proof. We want to apply the Slice Theorem 2.10. Let $x \in N$ and choose a G-equivariant tubular neighborhood around the orbit Gx. The Slice Theorem provides charts for ${}^{N}\!/\!_{G}$, and locally trivializations for the G-bundle ${}^{N}\!/\!_{G}$. □

Proof of Theorem 2.49. Since $G \curvearrowright \mu^{-1}(0)$ is free, 0 is a regular value and $\pi : \mu^{-1}(0) \to {}^{\mu^{-1}(0)}\!/\!_{G}$ is a prinicpal G-bundle. Since $\mu^{-1}(0)$ is coisotropic, $\ker \omega_{x|\mu^{-1}(0)} = (T_x\mu^{-1}(0))^{\perp_\omega} = T_x Gx$ is the tangent space of the G-orbit through x. Lemma 2.53 implies that $\omega_{|\mu^{-1}(0)}$ induces a symplectic form on ${}^{T_x\mu^{-1}(0)}\!/\!_{T_x Gx^\perp}$, and because the tangent space $T_{[x]}{}^{\mu^{-1}(0)}\!/\!_{G}$ in

$[x] \in {}^{\mu^{-1}(0)}/_G$ may be identified with ${}^{T_x\mu^{-1}(0)}/_{T_xGx^\perp}$, we obtain a nondegenerate 2-form ω_{red} on ${}^{\mu^{-1}(0)}/_G$ such that $\pi^*\omega_{\mathrm{red}} = \omega_{|\mu^{-1}(0)}$. Since

$$\pi^* d\omega_{\mathrm{red}} = d\pi^*\omega_{\mathrm{red}} = d\omega_{|\mu^{-1}(0)} = \iota^* d\omega = 0 \tag{2.58}$$

for the inclusion $\iota : \mu^{-1}(0) \hookrightarrow M$, closedness of ω_{red} follows from injectvity of π^* (because π is a submersion). □

Example 2.55 (Symplectic Reduction for S^1-Actions)**.** Let $S^1 \curvearrowright M$ be a Hamiltonian action on a $2n$-dimensional symplectic manifold (M, ω) with moment map $\mu : M \to \mathbb{R}$. Let $x \in \mathbb{R}$ be a regular value of μ and suppose that S^1 acts freely on $\mu^{-1}(x)$. Since in this case the stabilizer S^1_x is trivial, the Slice Theorem 2.10 implies that there exists a S^1-equivariant diffeomorphism $S^1 \times D^{2n-2} \to \mathcal{U}(S^1x) \subseteq M$ inducing coordinates $(\vartheta, \mu, x_1, \ldots, x_{2n-2})$ on M near the orbit S^1x. Thus the symplectic form is locally given by

$$\omega = a d\vartheta \wedge d\mu + \sum_{i=1}^{2n-2} (b_i d\vartheta \wedge dx_i + c_i d\mu \wedge dx_i) + \sum_{1\leqslant i<j\leqslant 2n-2} d_{ij} dx_i \wedge dx_j. \tag{2.59}$$

Because $S^1 \curvearrowright M$ is Hamiltonian and the fundamental vector field is ∂_ϑ in those coordinates, we have $i_{\partial_\vartheta}\omega = d\mu$ which implies $a = 1$ and $b_i = 0$ for all $i = 1, \ldots, 2n-2$. On $\mu^{-1}(x) \subseteq M$ the moment map μ is constant, hence $\omega_{|\mu^{-1}(x)} = \sum_{i<j} d_{ij} dx_i \wedge dx_j$. Because ω is S^1-invariant and closed, also $\omega_{|\mu^{-1}(x)}$ is S^1-invariant and closed, hence

$$0 = d\omega_{|\mu^{-1}(x)} = \sum_{i<j} \frac{\partial d_{ij}}{\partial\vartheta} d\vartheta \wedge dx_i \wedge dx_j, \tag{2.60}$$

i.e. $\partial_\vartheta d_{ij} = 0$ for all i, j, and we conclude $d_{ij} = d_{ij}(x_1, \ldots, x_{2n-2}) = \pi^* d_{ij}$ for the natural projection $\pi : \mu^{-1}(x) \to {}^{\mu^{-1}(x)}/_{S^1}$. Thus there exists a symplectic form ω_{red} on $\pi(\mathcal{U}(S^1x))$ such that $\omega_{|\mu^{-1}(x)} = \pi^*\omega_{\mathrm{red}}$ which is unique since π^* is injective.

Example 2.56. As a special case of Example 2.55 consider the natural action $S^1 \curvearrowright \mathbb{C}^n$, $(e^{\imath\vartheta}, z) \mapsto e^{\imath\vartheta} z$. This action is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathbb{R}$, $z \mapsto |z|^2$ and 1 is a regular value of μ, where $\mu^{-1}(1) \cong S^{2n-1}$, hence ${}^{S^{2n-1}}/_{S^1} \cong \mathbb{C}P^{n-1}$ is a symplectic manifold.

Example 2.57. Let $\mathrm{Mat}(k \times n, \mathbb{C}) \cong \mathbb{C}^{kn}$ denote the space of $(k \times n)$-matrices. Then the real part of $\omega(A, B) = -\operatorname{tr}(\imath \bar{A}^\top B)$ is the standard symplectic form on $\mathrm{Mat}(k \times n, \mathbb{C})$. To see this, note that for $A = (a_{ij})$, $B = (b_{ij})$ we have $\operatorname{tr}(\bar{A}^\top B) = \sum_{i,j=1}^{kn} \bar{a}_{ij} w_{ij} = \langle A, B\rangle_{\mathbb{C}^{kn}}$. Recall that the Hermitian inner product on $\mathbb{C}^{kn}$ can be expressed as

$$\langle z, w\rangle_{\mathbb{C}^{kn}} = \langle z, w\rangle_{\mathbb{R}^{2kn}} + \imath\omega_0(z, w) \tag{2.61}$$

for $z, w \in \mathbb{C}^{kn}$ (cf. Remark 1.11). Thus

$$\omega_0(A, B) = \Im \operatorname{tr}(\bar{A}^\top B) = \Re(-\imath \operatorname{tr}(\bar{A}^\top B)) = \Re\omega(A, B). \tag{2.62}$$

Let $U(k) \curvearrowright \mathrm{Mat}(k \times n, \mathbb{C}) \cong \mathbb{C}^{kn}$ be given by left multiplication. We want to show that this action is Hamiltonian with moment map

$$\mu : \mathrm{Mat}(k \times n, \mathbb{C}) \to \mathfrak{u}(k)^* \cong \mathfrak{u}(k), \qquad A \mapsto \frac{1}{2\imath} A\bar{A}^\top, \tag{2.63}$$

where we identify $\mathfrak{u}(k)^*$ with $\mathfrak{u}(k)$ via the Hermitian inner product $\langle A, B\rangle = \omega(A, \imath B)$ on $\mathfrak{u}(k) = \{A \in \mathrm{Mat}(k, \mathbb{C}) \mid \bar{A}^\top = -A\}$. Since the action is linear, the fundamental vector field $\underline{A} \in \Gamma(\mathrm{Mat}(k \times n, \mathbb{C}))$ corresponding to $A \in \mathfrak{u}_k$ is given by $\underline{A}(B) = A \cdot B$ for $B \in \mathrm{Mat}(k \times n, \mathbb{C})$. First note that μ is $U(k)$-equivariant because

$$\mu(B \cdot A) = \frac{1}{2\imath} BA\bar{A}^\top \bar{B}^\top = B\mu(A)B^{-1} = \mathrm{Ad}_B\, \mu(A). \tag{2.64}$$

Thus it remains to check that $i_{\underline{A}}\omega = d\mu^A$ for $\mu^A(B) = \mu(B)(A) = \imath\, \mathrm{tr}(B\bar{B}^\top A)/2$. We have

$$d\mu^A_B(C) = \frac{\mathrm{d}}{\mathrm{d}t}\mu^A(B + tC)\bigg|_{t=0} = \frac{\imath}{2}\,\mathrm{tr}\left(C\bar{B}^\top A - B\bar{C}^\top A\right) = \imath\, \mathrm{tr}(\bar{B}^\top AC), \tag{2.65}$$

and

$$i_{\underline{A}}\omega_B(C) = \omega_B(\underline{A}, C) = -\imath\, \mathrm{tr}\left(\bar{B}^\top \bar{A}^\top C\right) = \imath\, \mathrm{tr}\left(\bar{B}^\top AC\right). \tag{2.66}$$

Thus the action $U(k) \curvearrowright \mathrm{Mat}(k \times n, \mathbb{C})$ is Hamiltonian. One can show that $\mathbb{1}_k/2\imath$ is a regular value of μ, hence the *Grassmannian* $\mathrm{Gr}(k, n) = {}^{\mu^{-1}(\mathbb{1}_k/2\imath)}/_{U(k)}$ is a symplectic manifold. Note that $\mathrm{Gr}(1, n) = \mathbb{C}P^n$ (cf. Example 2.56). The submanifold $\mu^{-1}(\mathbb{1}_k/2\imath)$ is the space of unitary k-frames in $\mathbb{C}^n$, called *Stiefel manifold*.

2.3 Morse Theory for Hamiltonians

2.3.1 Almost Periodic Hamiltonians

Definition 2.58. Let $H : (M, \omega) \to \mathbb{R}$ be a Hamiltonian function, and let Φ_t be the flow of the Hamiltonian vector field generated by H. We call the Hamiltonian H *almost periodic* if $\{\Phi_t \mid t \in \mathbb{R}\} \subseteq \mathrm{Diff}(M)$ has compact closure (in a suitable topology on $\mathrm{Diff}(M)$).

Remark 2.59. Note that $\overline{\{\Phi_t \mid t \in \mathbb{R}\}} \subseteq \mathrm{Diff}(M)$ is a compact, connected and Abelian group. Endowed with a suitable smooth structure it becomes a Lie group, and is therefore isomorphic to a k-torus T^k.

The definition of an almost periodic Hamiltonian is motivated by the study of torus actions on symplectic manifolds. If $T \curvearrowright M$ is a Hamiltonian torus action with moment map $\mu : M \to \mathfrak{t}^*$, an almost periodic Hamiltonian $H : M \to \mathbb{R}$ with $\overline{\{\Phi_t \mid t \in \mathbb{R}\}} \cong T$ makes it possible to study the whole torus action by means of one single function.

Theorem 2.60. *If H is almost periodic, then the fixed points of the corresponding torus action are exactly the critical points of H.*

Proof. The critical points of an almost periodic Hamiltonian H are the zeroes of the Hamiltonian vector field X_H generated by H. These correspond to the fixed points of the induced action of Φ_t on M, and by density to the fixed points of the action $T \curvearrowright M$. □

Recall that if $T \curvearrowright M$ is a smooth torus action, the set of fixed points $M_{(T)}$ of T is a submanifold by Corollary 2.16 (basically an application of the Slice Theorem 2.10).

Theorem 2.61. *The set of critical points $M_{(T)}$ of an almost periodic Hamiltonian is a symplectic submanifold.*

Proof. Pick a T-invariant metric, and derive an invariant compatible almost complex structure (to which is associated a Hermitian metric). Let $x \in M_{(T)}$ be a fixed point. The torus T then acts on T_xM by means of the linear representation $T \to GL(T_xM)$, $g \mapsto d(l_g)_x$ (cf. Section 2.1.1). This action $T \curvearrowright T_xM$ (obviously) preserves the complex structure and the Hermitian metric, i.e. acts as a subgroup of $U(n)$. Since the exponential map of a compact, connected Lie group is surjective, and elements in $\mathfrak{t} \subseteq \mathfrak{u}(n)$ are diagonalizable, each matrix in T is diagonalizable. Because T is Abelian, there is a basis of T_xM in which all elements in T are diagonal, hence we obtain a decomposition

$$T_xM = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \tag{2.67}$$

where $V_0 = T_xM_{(T)}$ is the tangent space to the submanifold of fixed points of T, and each V_j is T-invariant. Since the action of T on $T_xM_{(T)}$ is the identity, $T_xM_{(T)}$ is the eigenspace of the T-action by unitary transformations to eigenvalue one. Since the almost complex structure is T-invariant, i.e. $J_{gx} \circ d(l_g)_x = d(l_g)_x \circ J_x$, it follows that $J_x(T_xM_{(T)}) = T_xM_{(T)}$, i.e. $T_xM_{(T)}$ is invariant under the almost complex structure, hence $M_{(T)}$ is a symplectic submanifold (cf. Remark 1.66). $\square$

The decomposition (2.67) of the tangent space to a fixed point has important implications. Let X_H be the Hamiltonian vector field of an almost periodic Hamiltonian H, which we may consider as an element of the Lie algebra $\mathfrak{t}$ of the torus generated by it, and consider the action of $\exp X_H$ on the decomposition (2.67): on each (complex) subspace V_j, $\exp X_H$ acts as multiplication by $\exp(\imath\lambda_j)$ for some $\lambda_j \in \mathbb{R}$, and $\lambda_j \neq 0$ if $j \neq 0$ (because X_H generates the whole torus T). Since, on each subspace V_j, the torus T acts by rotation, it follows that in local coordinates $v_0^1, \dots, v_0^r, v_1, \dots, v_k$ corresponding to the decomposition (2.67) with $v_j = q_j + \imath p_j$ the Hamiltonian vector field takes the form

$$X_H = \sum_{j=1}^{k} \lambda_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right), \tag{2.68}$$

i.e.

$$dH = \sum_{j=1}^{k} \lambda_j \left(p_j dp_j + q_j dq_j \right). \tag{2.69}$$

Therefore, in these local coordinates the Hamiltonian up to second order takes the form

$$H = \frac{1}{2} \sum_{j=1}^{k} \lambda_j \left(p_j^2 + q_j^2 \right) + \mathcal{O}(|v|^3), \tag{2.70}$$

and we conclude that the *second derivative* of H at a critical point x is the quadratic form

$$v \in T_xM \mapsto \frac{1}{2} \sum_{j=1}^{k} \lambda_j \left| v_j \right|^2 \in \mathbb{R}. \tag{2.71}$$

Clearly, in directions transverse to the submanifold $M_{(T)}$ of critical points of H the second derivative is nondegenerate.

Definition 2.62. A smooth function $f : M \to \mathbb{R}$ is called *Morse-Bott* if the set of critical points of f is a submanifold, and restricted to manifolds transverse to this critical manifold the second derivative is a nondegenerate quadratic form.

Corollary 2.63. *Almost periodic Hamiltonians are Morse-Bott functions.*

2.3.2 Digression on Morse Theory

We want to briefly summarize basic notions of Morse theory. Let $f : M \to \mathbb{R}$ be a smooth function, $x \in M$ a crticial point of f, i.e. $df_x = 0$. A critical point x is *nondegenerate* if

$$d^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j} \tag{2.72}$$

is nondegenerate in x for coordinates x_i near x. The symmetric matrix (2.72) determines a quadratic form which depends on the choice of coordinates. However, its nondegeneracy is independent of the choice of coordinates (because x is critical). Two other properties of $(\partial^2 f/\partial x_i \partial x_j)$ are also independent of coordinates, namely its *nullity*, the dimension of $\ker(\partial^2 f/\partial x_i \partial x_j)$, and its *index*, the dimension of a maximal subspace where $(\partial^2 f/\partial x_i \partial x_j)$ is negative definite. If f is a Morse function, then the nullity is trivial.

Example 2.64. The definition of a Morse-Bott function is a generalization of the notion of *Morse function*. A smooth function $f : M \to \mathbb{R}$ is said to be Morse if its second derivative is a nondegenerate quadratic form for all critical points. Thus, a Morse-Bott function with isolated critical points simply is a Morse function.

Theorem 2.65. *Let $f : M \to \mathbb{R}$ be a proper Morse function, and suppose $[a, b] \subset \mathbb{R}$ contains no critical values of f. Then $f^{-1}([a, b]) \cong f^{-1}(b) \times [a, b]$.*

Sketch of Proof. Pick a Riemannian metric g and let $X = -\nabla f/\|\nabla f\|^2$. Then X is a vector field without zeroes and $L_X f = i_X df = i_X g(\nabla f, \cdot) = -1$. We defne a diffeomorphism $f^{-1}(b) \times [a, b] \to f^{-1}([a, b])$, $(x, t) \mapsto \Phi_{b-t}(x)$ for flow Φ_t of X. □

Example 2.66. A closed n-dimensional manifold M that admits a Morse function with precisely 2 critical points is homeomorphic to S^n.

Lemma 2.67 (Morse Lemma). *Let $f : M \to \mathbb{R}$ be smooth, $p \in M$ a nondegenerate critical point of index k. Then there are coordinates $x_1, \ldots, x_k, x_{k+1}, \ldots, x_n$ on a neighborhood of p such that*

$$f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2, \tag{2.73}$$

and $x_i(p) = 0$.

Lemma 2.68. *Given a smooth function $f : U \subset \mathbb{R}^n \to \mathbb{R}$, $0 \in U$, $f(0) = 0$. Then there are smooth functions g_i, $i = 1, \ldots, n$ with*

$$f(x_1, \ldots, x_n) = \sum_i x_i g_i(x_1, \ldots, x_n) \qquad \text{and} \qquad \frac{\partial f}{\partial x_i}(0) = g_i(0). \tag{2.74}$$

Proof. Consider

$$\begin{aligned} f(x_1, \ldots, x_n) &= f(x_1, \ldots, x_n) - f(0, \ldots, 0) & (2.75) \\ &= \int_0^1 \frac{\mathrm{d}f}{\mathrm{d}t}(tx_1, \ldots, tx_n)\, dt = \int_0^1 \sum_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) x_i\, dt & (2.76) \\ &= \sum_i x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt = \sum_i x_i g_i(x_1, \ldots, x_n). \end{aligned}$$

□

Proof of Lemma 2.67. Let $f(p) = c$ be the critical value, and pick coordinates $(y_1, \dots, y_n)$ near p such that $y_i(p) = 0$. Then

$$f(p) - c = \sum_i y_i g_i(y_1, \dots, y_n) = \sum_{i,j} y_i y_j h_{ij}(y_1, \dots, y_n) \tag{2.77}$$

$$= \frac{1}{2} \sum_{i,j} y_i y_j \left(h_{ij} + h_{ji}\right). \tag{2.78}$$

The Morse Lemma follows now by induction. Assume that there exists coordinates $u_1, \dots, u_n$ such that

$$f(y) - c = \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \sum_{i,j=r}^{n} u_i u_j H_{ij}(u_1, \dots, u_n), \tag{2.79}$$

where H_{ij} is symmetric and nondegenerate at the origin. Because H_{ij} is nondegenerate at 0, w.l.o.g. we may assume $H_{rr}(0) \neq 0$. Now define new coordinates $v_1 = u_1, \dots, v_{r-1} = u_{r-1}$,

$$v_r = \sqrt{|H_{rr}(u_1, \dots, u_n)|} \left(u_r + \sum_{i>r} u_i \frac{H_{ir}(u_1, \dots, u_n)}{H_{rr}(u_1, \dots, u_n)} \right), \tag{2.80}$$

and $v_{r+1} = u_{r+1}, \dots, v_n = u_n$. Conversely,

$$u_r = \frac{v_r}{\sqrt{|H_{rr}|}} - \sum_{i>r} v_i \frac{H_{ir}}{H_{rr}}. \tag{2.81}$$

Now a simple computation shows

$$f(v) - c = \pm v_1^2 \pm \dots \pm v_{r-1}^2 \pm v_{rr}^2 \pm \sum_{i>r+1} v_i v_j \bar{H}_{ij}, \tag{2.82}$$

and induction on r yields the statement (2.73). □

Theorem 2.69. *Let $p \in M$ be the only critical point in $f^{-1}([a,b])$ and assume that p is nondegnerate with index k. Then we have a homotopy equivalence*

$$f^{-1}([a,b]) \simeq f^{-1}(a) \cup D^k_{\sqrt{b-a}}. \tag{2.83}$$

Theorem 2.70. *Let $f : M \to \mathbb{R}$ be proper and smooth. Then every (strong) C^∞-neighborhood of f contains a Morse function.*

Example 2.71. Consider the function $f : \mathbb{C}P^n \to \mathbb{R}$ given by

$$[z_0 : \dots : z_n] \mapsto \sum_{j=0}^{n} \frac{j\,|z_j|^2}{|z_0|^2 + \dots + |z_n|^2}. \tag{2.84}$$

Let $U_i = \{[z_0 : \dots : z_n] \mid z_i \neq 0\} \subseteq \mathbb{C}P^n$ be an affine chart. Then

$$0 = \frac{\partial f}{\partial z_0} = \left(\frac{1}{\sum_{i=0}^n |z_i|^2} - \frac{\sum_{j=0}^n j\,|z_j|^2}{(\sum_{i=0}^n |z_i|^2)^2} \right) \bar{z}_0 \tag{2.85}$$

if and only if $z_0 = \bar{z}_0 = 0$. Similarly, $\partial_{z_i} f = 0$ if and only if $z_i = 0$ for all $i = 0, \ldots, n$ and hence $p = (z_1, \ldots, z_n) \in U_i$ is critical if and only if $p = [e_i] = [0 : \cdots : 1 : \cdots : 0]$. Now

$$df_{e_i} = 2\sum_j j(x_j dx_j + y_j dy_j) - 2i\sum_j (x_j dx_j + y_j dy_j) \tag{2.86}$$

$$= -\sum_{j<i}(i-j)(x_j dx_j + y_j dy_j) + \sum_{j>i}(j-i)(x_j dx_j + y_j dy_j) \tag{2.87}$$

for $z_i = x_i + \imath y_i$. Hence $0 \leqslant \mathrm{ind}_{e_i} f = 2(i-1) \leqslant 2n$ and we have precisely one critical point of index $2k$ for each $0 \leqslant k \leqslant n$. This function is Morse, and all indices of critical points are even. (Recall that $H_k(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}$ for $k = 0, 2, \ldots, 2n$, and $H_k(\mathbb{C}P^n;\mathbb{Z}) = 0$ else.)

Definition 2.72. Let $f : M \to \mathbb{R}$ be a Morse-Bott function, $Z \subseteq M$ be the critical manifold of f, and assume Z is connected. Then for $z \in Z$ we define the index (of a Morse-Bott function) as the index of f restricted to the fiber of the normal bundle of Z in M above z.

The following Theorems are analogues of Theorems 2.69 and 2.70 in the case of Morse-Bott functions. Let $M_a = f^{-1}(-\infty, a] = \{p \in M \mid f(p) \leqslant a\}$ for some $a \in \mathbb{R}$.

Theorem 2.73. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function. If M is compact and $[a, b] \subseteq \mathbb{R}$ contains no critical points of f, then M_a is diffeomorphic to M_b.*

For the analogue of Theorem 2.70, we need to replace data for Morse coordinates as follows. The domain of Morse coordinates is replaced by a tubular neighborhood of Z that is isomorphic to the normal bundle $TZ^\perp$. The disk is replaced by the sub-vector bundle of negative definite eigenspaces in the tubular neighborhood which we call $TZ^\perp_-$.

Theorem 2.74. *Let $f : M \to \mathbb{R}$ be a proper Morse-Bott function, and suppose $c \in [a, b]$ is the unique critical value of f in $[a, b]$. Then the homotopy type of M_b is described by the addition to M_a of the negative normal bundle $TZ^\perp_-$ of the critical submanifold $f^{-1}(c)$.*

Example 2.75. Consider $f : \mathbb{C}P^n \to \mathbb{R}$, $[z_0 : \cdots : z_n] \mapsto |z_0|^2 / (|z_0|^2 + \cdots + |z_n|^2)$. The critical points of f are its maximum at $[1 : 0 : \cdots : 0]$ with index $2n$, and its minimum $[0 : z_1 : \cdots : z_n]$ with index 0. We learn that the boundary of a tubular neighborhood of $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$ is diffeomorphic to S^{2n-1}.

2.3.3 Applications to Almost Periodic Hamiltonians and Convexity Theorems

We saw in Section 2.3.1 that almost periodic Hamiltonians are Morse-Bott functions. Now we want to further analyze their porperties in terms of the notions introduced in Section 2.3.2.

Theorem 2.76. *The critical points of an almost periodic Hamiltonian have even index.*

Proof. The set of critical values $Z = \mathrm{Crit}(H)$ of H are the fixed points of Φ_t, i.e. the set of fixed points of $T \curvearrowright M$. By Theorem 2.14 $\mathrm{Crit}(H)$ is a submanifold. From the Slice Theorem 2.10, we get a T-invariant tubular neighborhood

$$\varphi : Z_0 \times (T_p Z_0)^{\perp_g} \to U(Z), \tag{2.88}$$

where $Z_0 \subset Z$ is a connected component and $p \in Z_0$ (g is a T-invariant metric), such that φ is T-equivariant and $(T_pZ_0)^\perp = \mathbb{C}_{(1)} \oplus \cdots \oplus \mathbb{C}_{(l)}$ with an action $t = (t_1, \ldots, t_k) \in (S^1)^k = T$ on $w_{(i)} \in \mathbb{C}_{(i)}$ given by $t \cdot w_{(i)} = t_1^{\alpha_1} \cdots t_k^{\alpha_k} \cdot w_{(i)}$ for $\alpha_i \in \mathbb{Z}$. Furthermore, $Z_0 \subset M$ is a symplectic submanifold by Theorem 2.61, and $\varphi^*\omega_{|TM_{|Z_0}}$ is a symplectic form on $(Z_0 \times T_pZ_0)^{\perp_g}$ which is translational invariant in the fiber direction. Then $H_{|U(Z_0)}$ is determined up to an additive constant by $\omega(\underline{X}, \cdot) = dH$. We choose

$$H(w_1, \ldots, w_l) = \frac{1}{2}\left(\lambda_1 |w_1|^2 + \cdots + \lambda_l |w_l|^2\right) \tag{2.89}$$

with $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. □

Theorem 2.77 (Frankel). *Let M be connected. The level sets of an almost periodic Hamiltonian $H : M \to \mathbb{R}$ are either empty or connected.*

Proof. Assume that $H^{-1}(\xi) \subseteq M$ is nonempty and disconnected for a regular value ξ. Since H is Morse-Bott, no critical values have index 1, hence all level sets are disconnected. By Theorem 2.76 all level sets are disconnected once one level set is disconnected, hence M is not connected. □

Theorem 2.78 (Hausdorff). *Let $A \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{R}^n$, and*

$$H_\lambda = \{X \in \mathbb{C}^{n\times n} \mid \bar{X}^\top = X,\ \lambda \text{ is set of eigenvalues}\}. \tag{2.90}$$

Then $f_A : H_\lambda \to \mathbb{C}$, $X \mapsto \operatorname{tr}(AX)$ has convex image.

Proof. H_λ is a symplectic manifold: $U(n) \curvearrowright \mathfrak{u}(n)^*$ where $\mathfrak{u}(n)^*$ is isomorphic (as a vector space) to the space of Hermitian matrices. $U(n)$ acts on orbits with moment map μ : Orbit $\hookrightarrow \mathfrak{u}(n)^*$. Note that $\mathfrak{h} \to \mathfrak{u}(n) \to \mathfrak{u}(n)^*$, $B \mapsto \imath B \mapsto (C \mapsto \operatorname{tr}(\imath(\imath B)C))$. Let now $f_A = f_1 + \imath f_2$ for f_1, f_2 real valued smooth functions, and

$$A = \left(\frac{\bar{A}^\top + A}{2}\right) + \left(\frac{A - \bar{A}^\top}{2\imath}\right)\imath. \tag{2.91}$$

Going through the identifications of $\mathfrak{h}$, $\mathfrak{u}(n)$, and $\mathfrak{u}(n)^*$ one sees that f_1 is a Hamiltonian function for $(\bar{A}^\top + A)/2 \in \mathfrak{h}$, and f_2 for $(A - \bar{A}^\top)/2\imath\mathfrak{h}$. We want to apply Frankel's Theorem 2.77. As a $U(n)$-orbit H_λ is connected. Now $f_1\alpha_1 + \imath f_2\alpha_2$ is the Hamiltonian of some vector field namely $X(\alpha_1, \alpha_2) = \alpha_1(\bar{A}^\top + A)/2 + \alpha_2(A - \bar{A}^\top)/2\imath \in \mathfrak{u}(n)$. Thus $\bar{X}(\alpha_1, \alpha_2)$ generates a subgroup of $U(n)$, hence $\alpha_1 f_1 + \alpha_2 f_2$ is almost periodic. We conclude that the preimage of each value is connected. □

Theorem 2.79 (Schur). *Let $\lambda_1, \ldots, \lambda_n$ be a collection of real numbers. Again let H_λ denote the space of Hermitian matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ (cf. (2.90)). Then the image of $\varphi : H_\lambda \to \mathbb{R}^n$, $A \mapsto \{$diagonal entries of $A\}$ is the convex hull of $((\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}))$, $\sigma \in S^n$.*

Theorem 2.80 (Atiyah, Guillemin-Sternberg). *Let (M, ω) be a closed connected symplectic manifold, $T^n \curvearrowright M$ be a Hamiltonian group action with moment map*

$$\mu = (\mu_1, \ldots, \mu_n) : M \to \mathbb{R}^n \cong \mathfrak{t}^n. \tag{2.92}$$

Let $Z_1, \ldots, Z_n$ be the connected components of the fixed point set $M_{(T)}$. Then $\mu(M)$ is the convex hull of $\{\mu(Z_i)\}$.

Proof. We prove, following [Aud05, p. 114ff.], the statement by induction on n showing that for the statements defined below (A_n) implies (B_{n+1}), (B_n) implies (C_n), and (A_n) holds.

(A_n) The level set $\mu^{-1}(x)$ is empty or connected for all $x \in \mathbb{R}^n$.

(B_n) The image $\mu(M)$ is convex.

(C_n) If $Z_1, \dots, Z_N$ are the connected components of the set of common critical points of μ_i, then $\mu(Z_i)$ is a point c_j and $\mu(M)$ is the convex hull of the points c_j.

Let us start proving that (A_n) implies (B_{n+1}). Let $\mu = (\mu_1, \dots, \mu_{n+1}) : M \to \mathbb{R}^{n+1}$ and consider $f = \pi \circ \mu : M \to \mathbb{R}^n$ for any linear projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Clearly, $\pi(\mathbb{R}^{n+1})$ corresponds to a closed connected subgroup of the torus T^{n+1}, hence is itself a torus so that (A_n) holds for f. The sets $f(M) \cap \pi^{-1}(x) = \mu(f^{-1}(x))$ describe, for varying $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, and $x \in \mathbb{R}^n$, the intersection of all straight lines $\pi^{-1}(x) \in \mathbb{R}^{n+1}$ with $\mu(M)$. Because of (A_n), $f^{-1}(x)$ is empty or connected, hence $\mu(f^{-1}(x))$ is empty or connected implying that $\mu(M)$ is convex.

Next we show that (B_n) implies (C_n). Let $X_1, \dots, X_n$ denote the Hamiltonian vector fields corresponding to $\mu_1, \dots, \mu_n$, and let T be the torus generated by them. Then the Z_i's are the connected components of the set of fixed points of T, i.e. $X_{i|Z_j} \equiv 0$. Hence μ_i is constant on Z_j, i.e. $\mu(Z_j) = c_j \in \mathbb{R}^n$, which shows the first part of (C_n). We want to show that $\mu(M)$ is the convex hull of $\{\mu(Z_i)\}$. To this end let $\varphi = \sum_i \lambda_i \mu_i$ be such that the corresponding vector field $X = \sum_i \lambda_i X_i$ generates a dense subgroup of T (e.g. by choosing the λ_i's $\mathbb{Q}$-linearly independent). Then the Z_i's are the components of of the set of critical points of φ, and, in particular, φ attains its maximum on some Z_j. Considering $\sum_i \lambda_i \xi_i$ as a form on $\mathbb{R}^n$, its restriction to $\mu(M)$ reaches its maximum at some $c_j = \mu(Z_j)$. Because this is true for almost every choice of the λ_i's, $\mu(M)$ is contained in the convex hull of all the points c_j.

Finally, we show that (A_n) holds by induction over n. Statement (A_1) precisely is Frankel's Theorem 2.77 since $\mu = H : M \to \mathbb{R}$ for $n = 1$. Assume (A_n) is true, and let $\mu_1, \dots, \mu_{n+1} : M \to \mathbb{R}$ be functions such that $\mu_i^{-1}(x_i)$ for $x_i \in \mathbb{R}$ is either empty or connected. Let $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$. We need to show that $\mu^{-1}(x) = \mu_1^{-1}(x_1) \cap \dots \cap \mu_{n+1}^{-1}(x_{n+1})$ is either empty or connected. If μ has no regular value, at least one $d\mu_i$ is a linear combination of the others, and we may apply (A_n). So we assume $x \in \mathbb{R}^{n+1}$ to be a regular value of μ. Then $N = \mu_1^{-1}(x_1) \cap \dots \cap \mu_n^{-1}(x_n)$ is a submanifold, and by (A_n) connected. (Not done!) □

Remark 2.81. Theorem 2.80 implies that a Hamiltonian torus action does have fixed points since the set of fixed points is not empty. In fact, Hamiltonian torus actions tend to have many fixed points as the next Corollary 2.82 shows.

Corollary 2.82. *Let $T^n \curvearrowright M$ be effective and Hamiltonian. Then there exist at least $n+1$ fixed points.*

Proof. Because the action is effective, the principal orbit type $M_{(e)}$ is nonempty, and μ is a submersion at points in $M_{(e)}$ by Corollary 2.43. Hence $\mu(M_{(e)})$ is open and nonempty in $\mathbb{R}^m$, i.e. $\mu(M)$ has an interior point. We conclude that the convex polytope containing it has at least $n+1$ corners, i.e. fixed points by Theorem 2.80. □

Theorem 2.83. *Let $T \curvearrowright M$ be a Hamiltonian action with moment map μ, and let $z \in M_{(T)}$. Then there is a neighborhood U of z in M, and V of $\mu(z)$ in $\mathfrak{t}^*$ such that*

$$\mu(U) = V \cap C_p(\alpha_1, \ldots, \alpha_m), \tag{2.93}$$

where $C_p(\alpha_1, \ldots, \alpha_m)$ is the convex cone containing integral vectors $\alpha_1, \ldots, \alpha_m \in \mathfrak{t}^*$.

Proof. Pick a T-invariant Riemannian metric. Then $\exp_z : T_zM \to M$ is a T-equivariant diffeomorphism of neighborhoods of 0 and z. Then $\exp_z^* \omega$ is locally equivalent to a translation invariant symplectic form ω_0 on T_zM, i.e. there exists a T-invariant symplectomorphism ψ of neighborhoods of $0 \in T_zM$ such that $\psi^* \exp^* \omega = \omega_0$. Then $T \curvearrowright (T_zM, \omega_0)$ with moment map

$$\mu : T_zM \overset{\psi}{\to} T_zM \overset{\exp}{\to} M \to \mathfrak{t}^*, \tag{2.94}$$

and one can check that the map

$$\tilde{\mu}(v_1, \ldots, v_n) = \frac{1}{2}\left(\sum_{i=1}^m \alpha_i^{(1)} |v_i|^2, \ldots, \sum_{i=1}^m \alpha_i^{(m)} |v_i|^2\right) \tag{2.95}$$

generates a linear action. But because they are moment maps for the same action, they coincide up to a constant. $\square$

Example 2.84. Let $T^2 = S^1 \times S^1 \subseteq \mathbb{C}^2$ act on $\mathbb{C}P^1 \times \mathbb{C}P^1$ by

$$(u, v) \cdot ([a : b], [x : y]) = ([a : ub], [x : vy]). \tag{2.96}$$

We want to show that this action is Hamiltonian. For this we first consider the action $S^1 \curvearrowright \mathbb{C}P^1$ given by $u \cdot [z_0 : z_1] \mapsto [z_0 : uz_1]$ and then want to apply Example 2.36 to the case $G_1 \times G_2 = S^1 \times S^1 = T^2$. On $U_0 = \{[1 : w] \mid w \in \mathbb{C}\} \cong \mathbb{C} \subseteq \mathbb{C}P^1$ the S^1-action is rotation in the complex plane, hence the Hamiltonian vector field locally is given by $X = y\partial_x - x\partial_y$. Moreover, the Fubini-Study form (1.57) becomes

$$\omega_{|U_0} = -\frac{\imath}{2(1+|w|^2)^2} dw \wedge d\bar{w} = -\frac{1}{(1+|w|^2)^2} dx \wedge dy, \tag{2.97}$$

and we obtain

$$i_X \omega_{|U_0} = \frac{xdx + ydy}{(1+|w|^2)^2}. \tag{2.98}$$

We claim that the moment map $\mu : \mathbb{C}P^1 \to \mathbb{R}$ is given by

$$\mu([z_0 : z_1]) = \frac{1}{2}\frac{|z_1|^2}{|z_0|^2 + |z_1|^2}. \tag{2.99}$$

We need to check that $d\mu = i_X\omega$. Because $U_0 \subseteq \mathbb{C}P^1$ is open, dense and μ clearly is continuous, it suffices to verify this locally on U_0. We compute

$$d(\mu_{|U_0}) = \frac{1}{2}\left(\frac{d\,|w|^2}{1+|w|^2} - \frac{|w|^2\, d\,|w|^2}{(1+|w|^2)^2}\right) = \frac{1}{2}\frac{d\,|w|^2}{(1+|w|^2)^2} = \frac{xdx + ydy}{(1+|w|^2)^2} = i_X\omega_{|U_0}. \tag{2.100}$$

Thus $T^2 \curvearrowright \mathbb{C}P^1 \times \mathbb{C}P^1$ is Hamiltonian with moment map $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{R} \times \mathbb{R}$,

$$([a,b],[x,y]) \mapsto \frac{1}{2}\left(\frac{|b|^2}{|a|+|b|^2} + \frac{|y|^2}{|x|^2+|y|^2}\right). \tag{2.101}$$

The image of the moment map (2.101) is a square of side length 1/2 (since the fixed points of the action are $[1:0]$ and $[0:1]$ in each factor).

Example 2.85. Consider the action $T^2 \curvearrowright \mathbb{C}P^2$ given by $(u,v)\cdot[z_0:z_1:z_2] = [z_0:uz_1:v^2z_2]$. Again we consider the two circle actions separately. On U_0 the action of the second factor then is given by $v\cdot(z_1,z_2) = (z_1, v^2z_2)$. Note that this action preserves slices $P_{z_1} = \{z_1\}\times\mathbb{C} \subseteq U_0 \cong \mathbb{C}^2$, i.e. the fundamental vector field generated by this action is a rotation around the axis perpendicular to the slices P_{z_1}. In local coordinates, $X = 2(y\partial_x - x\partial_y)$. On each slice P_{z_1} the Fubini-Study form (2.97) is given by

$$\omega_{|U_0} = -\frac{c}{(c+|z|^2)^2}dx \wedge dy \tag{2.102}$$

for $c = 1+|z_1|^2$ and $z = z_2 = x + \imath y$. A computation analogous to (2.100) shows that the moment map $\mu_2 : \mathbb{C}P^2 \to \mathbb{R}$ is given by

$$\mu_2([z_0:z_1:z_2]) = \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}. \tag{2.103}$$

Similarly, for the first factor $\mu_1 : \mathbb{C}P^2 \to \mathbb{R}$,

$$\mu_1([z_0:z_1:z_2]) = \frac{1}{2}\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}. \tag{2.104}$$

The fixed points of the action $T^2 \curvearrowright \mathbb{C}P^2$ are $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, hence the image of the moment map is a rectangle.

Example 2.86. Combining Examples 2.85 and 2.84 let $T^2 \curvearrowright \mathbb{C}P^1 \times \mathbb{C}P^2$ be the action given by $(u,v)\cdot([a:b],[z_0:z_1:z_2]) = ([ua:b],[u^kz_0:z_1:vz_2])$. It is Hamiltonian with moment map $\mu : \mathbb{C}P^1 \times \mathbb{C}P^2 \to \mathbb{R}\times\mathbb{R}$,

$$\mu([a:b],[z_0:z_1:z_2]) = \frac{1}{2}\left(\frac{|a|^2}{|a|^2+|b|^2} + \frac{k\,|z_0|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right). \tag{2.105}$$

Note that this action preserves the Hopf surface (1.61) discussed in Example 1.61. The fixed points of $T^2 \curvearrowright W_k$ are given by those fixed points of $T^2 \curvearrowright \mathbb{C}P^1 \times \mathbb{C}P^2$ lying in W_k, i.e. $\{([0:1],[e_1]),([0,1],[e_2]),([1:0],[e_0]),([1:0],[e_2])\}$.

3 Construction of Symplectic Manifolds

3.1 Blowing Up

3.1.1 Blow Up in the Complex Category

We start our discussion with the blow up of a point in $\mathbb{C}^n$. We define $\tilde{\mathbb{C}}^n \subseteq \mathbb{C}^n \times \mathbb{C}P^{n-1}$ by

$$\tilde{\mathbb{C}}^n = \left\{(z,[w]) = ((z_1,\ldots,z_n),[w_1:\cdots:w_n]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z_iw_j = z_jw_i \ \forall i,j\right\}, \tag{3.1}$$

i.e. $(z,[w]) \in \tilde{\mathbb{C}}^n$ if and only if $z \in [w] \subseteq \mathbb{C}^n$.

Lemma 3.1. *$\tilde{\mathbb{C}}^n$ is a complex submanifold in $\mathbb{C}^n \times \mathbb{C}P^{n-1}$.*

Proof. Let $F_1 : \{(z,[w]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid w_1 \neq 0\} \to \mathbb{C}^{n-1}$ be defined by

$$((z_1,\dots,z_n),[w_1:\dots:w_n]) \mapsto \begin{pmatrix} z_1w_2 - z_2w_1 \\ \vdots \\ z_1w_n - z_nw_1 \end{pmatrix} \tag{3.2}$$

and similarly F_i for $i = 2,\dots,n$. Then F_i is holomorphic and provides a chart for $\tilde{\mathbb{C}}^n$ because $F_i^{-1}(0) \subseteq \tilde{\mathbb{C}}^n$. □

The definition of $\tilde{\mathbb{C}}^n$ comes with natural holomorphic projections, $\pi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ and $\mathrm{pr} : \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$.

Remark 3.2. The map $\mathrm{pr} : \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1}$ is the *tautological line bundle*. To see that pr defines a complex line bundle consider local sections $\sigma_i : U_i \subseteq \mathbb{C}P^{n-1} \to \tilde{\mathbb{C}}^n$,

$$[w_1:\dots:w_n] \mapsto \left(\left(\frac{w_1}{w_i},\dots,1,\dots\frac{w_n}{w_i}\right),[w_1:\dots:w_n]\right). \tag{3.3}$$

These define local trivializations of the tautological line bundle by setting

$$([w_1:\dots:w_n],\lambda) \in U_i \times \mathbb{C} \mapsto ([w_1:\dots:w_n],\lambda\sigma_i([w_1:\dots:w_n])) \in \tilde{\mathbb{C}}^n. \tag{3.4}$$

Note that the transition maps $g_{ij} : U_i \cap U_j \to GL(1,\mathbb{C}) = \mathbb{C}^*$, $\sigma_i = g_{ij}\sigma_j$ are given by $g_{ij}([w_1:\dots:w_n]) = z_j/z_i$. The tautological line bundle is the complex line bundle over $\mathbb{C}P^{n-1}$, where the fiber above $[w] \in \mathbb{C}P^{n-1}$ is given by the complex line in $\mathbb{C}^n$ determined by $[w]$.

The zero section of the tautological line bundle $E = \pi^{-1}(0) \cong \mathbb{C}P^{n-1} \subseteq \tilde{\mathbb{C}}^n$ is called the *exceptional divisor*. Since π restricts to a diffeomorphism $\tilde{\mathbb{C}}^n \setminus E \to \mathbb{C}^n \setminus \{0\}$, hence one can think of $\tilde{\mathbb{C}}^n$ being obtained from $\mathbb{C}^n$ by replacing the origin with a copy of all lines $\mathbb{C}P^{n-1}$ through the origin.

Definition 3.3. $\tilde{\mathbb{C}}^n$ is called the *blow up* of $\mathbb{C}^n$ at the origin.

To translate the blow of $\mathbb{C}^n$ to the case of a complex manifold M, we adapt the construction to a chart (U,φ) around $p \in M$ with $\varphi(p) = 0$, i.e. we define the blow up of p in M to be the blow up of $\varphi(p) = 0 \in \mathbb{C}^n$. For this to be well-defined we need

Lemma 3.4. *Every biholomorphism $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ with $\Psi(0) = 0$ uniquely lifts to a biholomorphism $\tilde{\Psi} : \tilde{\mathbb{C}}^n \to \tilde{\mathbb{C}}^n$ with $\tilde{\Psi}(E) = E$.*

Proof. The idea [Cie10, p. 58] is to define the lift $\tilde{\Psi}$ by

$$(z,[w]) \mapsto \begin{cases} (\Psi(z),[\Psi(z)]) & \text{if } z \neq 0; \\ (0, d\Psi_0([w])) & \text{if } z = 0. \end{cases} \tag{3.5}$$

One needs to check that this map is holomorphic. □

Thus we can define the *blow up* $\pi : \tilde{M} \to M$ of M at $p \in M$ by taking a chart (U, φ) around p with $\varphi(p) = 0$ and take $\varphi^{-1} \circ \pi : \tilde{\mathbb{C}}^n \to U$ as a chart for the blown up manifold.

Remark 3.5 (Connected Sum)**.** Recall that the *connected sum* $M_1 \# M_2$ of connected manifolds M_1, M_2 is obtained by removing from each M_i a coordinate ball $U_i \subseteq M_i$ centered at $p_i \in M_i$ attaching $M_1 \setminus U_1$ and $M_2 \setminus U_2$ along their boundaries via a diffeomorphism. More care is needed in case of oriented manifolds M_1, M_2. As before remove (oriented) coordinate balls (U_i, φ_i) centered at p_i and choose small neighborhoods (diffeomorphic to annuli) around the boundary of each $M_i \setminus U_i$. We define the connected sum $M_1 \# M_2$ of oriented manifolds by identifying the annuli in such a way that the outer boundary of one is identified with the inner boundary of the other. One can show that the diffeomorphism class of $M_1 \# M_2$ is independent of all the choices made in this construction.

Because holomorphic maps are orientation preserving (because the orientation is induced by the complex structure, hence complex manifolds are canonically oriented) so that there is no immediate generalization of connected sum to complex manifolds.

Theorem 3.6. *Let X be a complex manifold, $z \in X$. Then $\tilde{X}$ is diffeomorphic to $X \# \overline{\mathbb{C}P}^n$, where $\overline{\mathbb{C}P}^n$ is $\mathbb{C}P^n$ with the orientation opposite to the canonical orientation given by the complex structure.*

Corollary 3.7.

(i) $\pi_1(X) = \pi_1(\tilde{X})$.

(ii) $H_i(\tilde{X}) = H_i(X) \oplus H_i(\mathbb{C}P^n)$ *for* $i \neq 0, 2n$.

3.1.2 Blow Up in the Symplectic Category

To translate the blow up of a point into the symplectic category we first note an obvious difficulty. If (M, J, ω) is a Kähler manifold and $\pi : \tilde{M} \to M$ its (complex) blow up, then $\pi^*\omega$ vanishes along the exceptional divisor $E = \mathbb{C}P^{n-1}$, hence is not symplectic. As discussed in Example 1.55, however, $\mathbb{C}P^{n-1}$ carries a natural symplectic structure given by the Fubini-Study form ω_{FS}. In order to find out how one can use ω_{FS} to obtain a symplectic form on the blow up $\tilde{M}$ we again consider first the local model.

By the discussion in Section 3.1.1 above we may identifiy $\tilde{\mathbb{C}}^n$ and $\mathbb{C}^n \setminus \{0\}$, consider pr as a map $\mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^{n-1}$. Let $B^{2n}(r) \subseteq \mathbb{C}^n$ be a ball of radius r and let

$$\tilde{\mathbb{C}}^n_\delta = \{(z, [w]) \in \tilde{\mathbb{C}}^n \mid \|z\| \leqslant \delta\}. \tag{3.6}$$

Lemma 3.8. *For each $\lambda > 0$ the form*

$$\omega_\lambda = \pi^*\omega_0 + \lambda^2 \operatorname{pr}^* \omega_{\mathrm{FS}} \tag{3.7}$$

is Kähler on $\tilde{\mathbb{C}}^n$. Moreover, for all $\delta > 0$ there exists a symplectomorphism $(\tilde{\mathbb{C}}^n_\delta \setminus E, \omega_\lambda)$ and $(B^{2n}(\sqrt{\lambda^2 + \delta^2}) \setminus B^{2n}(\lambda), \omega_0)$.

Proof. See [Cie10, Lemma 7.6]. □

Using Lemma 3.8 we can define the symplectic blow up. Let (M, ω) be a symplectic manifold $\iota : B^{2n} = B^{2n}(\sqrt{\lambda^2 + \delta^2}) \to M$ a symplectic embedding for some $\lambda, \delta > 0$. We set

$$\tilde{M} = M \setminus \iota(B^{2n}) \cup \tilde{\mathbb{C}}^n_\delta \tag{3.8}$$

(attached via symplectomorphisms from Lemma 3.8) and define a symplectic form $\tilde{\omega}$ on $\tilde{M}$ by

$$\tilde{\omega} = \begin{cases} \omega & \text{on } M \setminus \iota(B^{2n}); \\ \omega_\lambda & \text{on } \tilde{\mathbb{C}}^n_\delta. \end{cases} \tag{3.9}$$

Definition 3.9. $(\tilde{M}, \tilde{\omega})$ is called the *symplectic blow up* of (M, ω).

Remark 3.10. One should be aware that the terminology of blowing up is not suited to what happens in the symplectic category. The volume of the blow up of a symplectic manifold decreases (by the volume of the symplectic ball removed) and one collapses the boundary to the exceptional divisor $E = \mathbb{C}P^{n-1}$.

3.2 Connected Sum

In Remark 3.5 we briefly discussed the connected sum of oriented manifolds. In the symplectic category this construction is possible only in very special cases. To see this we remove balls and consider the boundary spheres. They have tubular neighborhoods symplectomorphic to an annulus $B^{2n}(R) \setminus B^{2n}(r)$, $r < R$, and we need to find a symplectomorphism between the annuli interchanging their boundaries. However, such a symplectomorphism exists if and only if $n = 1$ [Cie10, Lemma 7.8].

Theorem 3.11 (Taubes). *Let M_1, M_2 be compact 4-manifolds such that $M_1 \# M_2$ admits a symplectic structure, and suppose $b_2^+ > 1$. Then M_2 is homeomorphic to S^4.*

3.2.1 Fiber Connected Sum

Let us introduce a notion of connected sum which works better in the symplectic category. Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds, and (Q, τ) be a compact $(2n-2)$-dimensional symplectic manifold. Suppose there exist symplectic embeddings $\iota_i : Q \to M_i$, $i = 1, 2$, such that the symplectic normal bundle TQ_i^ω for $Q_i = \iota_i(Q)$ are trivial. By the Symplectic Neighborhood Theorem 1.75 there exists a neighborhood $\mathcal{N}_\epsilon(Q_i) \subseteq M_i$ and a symplectomorphism $\Psi_i : Q_i \times B^2(\epsilon) \to \mathcal{N}_\epsilon(Q_i)$ for sufficiently small $\epsilon > 0$ such that $\Psi_i^* \omega_i = \tau \times dx \wedge dy$. Let $\phi : B^2(\epsilon) \setminus B^2(\delta) \to B^2(\epsilon) \setminus B^2(\delta)$, $0 < \delta < \epsilon$ be an orientation preserving diffeomorphism that interchanges the boundary components of the annulus $B^2(\epsilon) \setminus B^2(\delta)$.

Definition 3.12. The *fiber connected sum* of two symplectic manifolds (M_i, ω_i), $i = 1, 2$, is defined as

$$M_1 \#_Q M_2 = M_1 \setminus \mathcal{N}_\delta(Q_1) \cup_\phi M_2 \setminus \mathcal{N}_\delta(Q_2). \tag{3.10}$$

Since on the overlap $M_1 \setminus \mathcal{N}_\delta(Q_1) \cap M_2 \setminus \mathcal{N}_\delta(Q_2) \cong Q \times B^2(\epsilon) \setminus B^2(\delta)$ the symplectic forms $\Psi_1^* \omega_1$ and $\Psi_2^* \omega_2$ agree, we obtain a well-defined symplectic structure on $M_1 \#_Q M_2$.

Remark 3.13. The fiber connected sum depends on choices of Ψ_i and δ. The parameter δ determines the symplectic volume of the fiber connected sum. Different choices of the symplectomorphisms Ψ_i may lead to non-diffeomorphic manifolds. E.g. the Thurston-Kodaira manifold discussed in Example 1.59 may be described as the fiber connected sum of $M_1 = T^2 \times S^2$ with $M_2 = T^2 \times T^2$ along $Q = T^2 \times \{z_0\}$. However, if one takes the standard symplectomorphisms Ψ_i one obtains just $T^2 \times T^2$.

Theorem 3.14 (Gompf). *Let G be a finitely presented group. Then there exists a closed symplectic 4-manifold (M^4, ω) with $\pi_1(M) \cong G$.*

Proof. Recall that we constructed $E(1) \cong \mathbb{C}P^2 \# q\overline{\mathbb{C}P}^2 \supseteq T^2 \times D^2$ with $T^2 \times \{*\}$ symplectic, and $E(1) \setminus (T^2 \times D^2)$ simply connected. Let us now construct M. We start with a closed surface Σ of genus k with area form ω_1. Let ω_2 be an area form on T^2. Then

$$(\Sigma \times T^2, \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2) \tag{3.11}$$

is a symplectic 4-manifold with

$$\pi_1(\Sigma \times T^2) = \langle g_1, \dots, g_k, h_1, \dots, h_k; a, b \mid [a,b], [a, g_i] = [G, g_i] = [a, h_i] = [b, h_i], [g_i, h_i] \rangle. \tag{3.12}$$

Eliminating a, b is easy because those elements of $\pi_1(\Sigma \times T^2)$ are represented by curves in a fiber of $\mathrm{pr}_1 : \Sigma \times T^2 \to \Sigma$ (where a fiber is a symplectic 2-torus T^2 with trivial normal bundle; surgery similar to (symplectic) connected sum in codimension 2). Pick immersed curves $\gamma_1, \dots, \gamma_l$ in Σ representing $h_1, \dots, h_k, r_1, \dots, r_m$ (these curves avoid the region in Σ affected by the surgery), and one (embedded nontrivial) curve α in T^2. Then $\gamma_i \times \alpha : T^2 \to \Sigma \times T^2$ is an immersed torus (but unfortunately it is Lagrangian). Assume there is a 1-form ϱ on Σ such that ϱ is closed and $\varrho_{|\gamma_i} \neq 0$. Pick a closed 1-form ϑ on T^2 which doesn't vanish on α (easy on torus, i.e. if α is along the y-direction, we choose dy). Then one can replace $\omega = \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2$ by

$$\omega_{\mathrm{myst}} = \omega + (\varrho \wedge \vartheta)\epsilon \tag{3.13}$$

for $\epsilon > 0$ sufficiently small. In $(\Sigma \times T^2, \omega_{\mathrm{myst}})$ the immersed 2-tori $\gamma_i \times \alpha$ are symplectic. The normal bundle of $\gamma_i \times \alpha$ is trivial because there is a section: $\gamma_i \times \alpha$ is contained in the (embedded) hypersurface $(\Sigma \times \alpha)$ and the normal bundle has rank 2 (and is oriented). (Note that for an immersion there is not really a problem with talking about the normal bundle; also the normal bundle of an oriented manifold is oriented.) Now $\gamma_i \times \alpha$ has self-intersections (corresponding to self-intersections of γ_i) which we may assume to be transverse and simple, i.e. exactly two branches intersect. These self-intersections of $\gamma_i \times \alpha$ can be removed by pushing one branch of $\gamma_i \times \alpha$ out of the hypersurface $\Sigma \times \alpha$. Thus (because symplecticity is a C^1-open condition on a compact manifold) we obtain embedded symplectic 2-tori containing curves (freely) homotopic to γ_i. After sufficiently many surgeries we obtain a smooth symplectic 4-manifold with $\pi_1(M) = {}^{\pi_1(\Sigma \times T^2)}/_{a,b,[\gamma_1],\dots,[\gamma_l]} \cong G$.

We have to ensure that the assumption (existence of ϱ) is verified. This will lead to modifications of Σ. There are two steps.

1. We want: Given a collection of immersed curves $\gamma_1, \dots, \gamma_l$ in Σ which intersect transversley and simply find a 1-form $\hat{\varrho}$ so that

$$\int_\sigma \hat{\varrho} > 0 \tag{3.14}$$

for all smooth segments σ of $\bigcup \gamma_i$ which are either closed (S^1) or connects two interior points of $\bigcup \gamma_i$. But this is not possible, e.g. consider two curves γ_1, γ_2 around two holes and a bigger one γ_3 encircling both holes with opposite orientation, and let ∂ denote the boundary determined by those curves. Then

$$\int_{\gamma_1} \varrho + \int_{\gamma_2} \varrho + \int_{\gamma_3} \varrho = \int_{\partial} \varrho = 0 \tag{3.15}$$

because ϱ is closed. To get around this, we modify Σ and the system of curves. The basic building block B is a punctured two-torus which we will attach to the surface such that $\hat{\varrho} = 0$ on Σ and the boundary of the punctured two-torus and $\int \hat{\varrho} > 0$ on all segments in B. To construct this form $\hat{\varrho}$ on the building block B look at $\hat{\varrho}_{T^2} = d(x+y)$ (for the 2-torus as the square with suitably identified edges). To modify $\hat{\varrho}_{T^2}$ so that the result vanishes on some disc D in the square. Attaching B to every segment of $\bigcup \gamma_i$ we get $\hat{\varrho}$ with the desired properties.

2. Find $f : \Sigma \cup B_1 \cup \cdots \cup B_n \to \mathbb{R}$ such that $\hat{\varrho} + df > 0$ on segments.

□

3.3 Gromov's h-Principle for Symplectic Forms on Open Manifolds

Theorem 3.15 (Gromov). *Let M^{2n} be open (i.e. not closed), connected, and pick $a \in H^2_{\mathrm{dR}}(M)$ and a nondegenerate 2-form τ. Then M admits a symplectic structure ω such that $[\omega] = a$. Moreover, for any two such structures ω_0, ω_1 there is a family of symplectic forms ω_t, $t \in [0,1]$ such that $[\omega_t] = a$ for all t.*

Remark 3.16. Theoerem 3.15 is an example of an h-principle. Those are methods to construct geometric structure (or more precisely solutions of partial differential relations for sections in bundles). Here h refers to homotopy (homotopy of "formal solutions" to "honest solutions").

Before saying something about the Theorem let us remark: If M is open, then there exists a smooth function $f : M \to \mathbb{R}$ with no critical points at all. To see this let $g : M \to \mathbb{R}$ be a proper Morse function which is bounded from below, and consider $N_i = g^{-1}([a, n_i]) \subseteq M$, $n_0 < \cdots < n_i < \ldots$. We want to construct a function f_0 without critical points on N_0 by pushing critical points into $N_1 \setminus N_0$.

J.Milnor: *Lectures on h-corbodism theorem* (for the cancellation Lemma).
McDuff, Salamon: *Introduction to Symplectic Topology*
Eliashberg, Mishachev: *Lectures on h-principle*

Theorem 3.17. *Let M^{2n} be an open connected manifold, τ a non-degenerate 2-form, $a \in \Omega^2(M)$ such that $da = 0$. Then there exist a 1-form σ on M so that $a + d\sigma$ is symplectic, and a family of nondegenerate 2-form τ_t such that $\tau_0 = \tau$, $\tau_1 = a + d\tau$. Any two 1-forms σ_0, σ_1 with this properties are connected through a 1-form with the same properties.*

Proof. In the first step we construct a proper Morse function on M without critical points of index $2n$. To this end let $K_0 \subseteq K_1 \subseteq \ldots$ be an exhaustion of M by compact sets. (We need to eliminate singular points of index $2n$. The Morse Lemma shows that level sets of f near

$x_{\max}$ are spheres because $f(x_1,\ldots,x_{2n}) = x_{\max} - \sum_i x_i^2$. After a small perturbation of the (implicitely chosen) Riemannian metric g one can achieve that there is exactly 1-flowline of $-\nabla f$ connecting $x_{\max}$ to a critical point of index $2n-1$. These critical points form a pair which can be eliminated by the cancellation lemma.)

In the second step we apply an inductive construction: We construct $\sigma, \ldots$ on $M^c = \{f(x) \leqslant c\}$ for a regular value c. If the Theorem is proved on M^c then one can use the flow of $\beta(x)\nabla f/\|\nabla f\|$ to find a solution on $M^{c'}$.

Let us consider a critical point $x_{\min}$ of index 0. Then a is exact on a neighborhood of $x_{\min}$, hence we assume $a = 0$ near $x_{\min}$ (by adding $d\sigma_{x_{\min}}$). Now τ is 2-form, i.e. $\tau_{x_{\min}}$ is a 2-form on $T_{x_{\min}}M \cong T_{x_{\min}}\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ (by choosing Morse coordinates around $x_{\min}$). Then $\tau_{x_{\min}}$ is a form with constant coeffiecients on a neighborhood of $x_{\min} \in \mathbb{R}^{2n}$, hence $\tau_{x_{\min}}$ is closed, and consequently exact near $x_{\min}$. By Poincaré Lemma there exists a formula for the primitive $\tau_{x_{\min}} = d\sigma_{x_{\min}}$. We claim that $d\sigma_{x_{\min}}$ solves our problem near $x_{\min}$. At $x_{\min}$ we have $d\sigma_{x_{\min}} = \tau_{x_{\min}} = \tau$ and $(1-s)\tau + sd\sigma_{x_{\min}}$ is nondegenerate on a neighborhood of $x_{\min}$ (because nondegeneracy is an opne condition, and invariant under diffeomorphisms).

Pick Morse coordinates around the remaining critical points and push the solution around in M using the flow of $\nabla f/\|\nabla f\|$ until we arrive at a Morse chart of a next critical point x. We need to extend the constructed solution over the *stable manifold* of x (w.r.t. ∇f). To solve the extension problem from now on we will assume $a = 0$ and need

Lemma 3.18. $1 \leqslant m < 2n$, *τ a smooth nondegenerate 2-form I^{2n} such that τ is exact on an open neighborhood $\mathcal{O}(\partial I^n \times I^{2n-m})$. Then there is a family of nondegenerate 2-forms τ_t, $t \in [0,1]$, such that $\tau_0 = \tau$, $\tau_1 = d\sigma$ is exact on I^{2n}, $\tau_t = \tau$ near $\partial(I^m \times I^{2n-m})$, and σ extends the primitve of τ. If τ is already exact on I^{2n}, then so are τ_t.*

(In order to apply Lemma 3.18 to the induction, we need to introduce an additional paremeter τ_λ for $\lambda \in \Lambda$ where Λ is a compact parameter space in $\mathbb{R}^N$.)

Proof. The proof is by induction on m. We begin with the non-parametric version with $m = 1$ (the parametric version will then be clear, and the inductive step to $m = 2$ also): Let $(0,\xi) \in I^{2n-1} \times I^{m=1}$ there is a 1-form σ_ξ so that $d\sigma_\xi = \tau_\xi$ (at $(0,\xi)$) depending continuously on ξ. Then there exists $\epsilon > 0$ such that $d\sigma_\xi$ is nondegenerate on $[\xi, \xi+3\epsilon] \times [-\epsilon,\epsilon]^{2n-1}$. Moreover, $sd\sigma_\xi + (1-s)\tau_\xi$ as well as $s\tau_\xi + (1-s)\tau$ are all nondegenerate on $[\xi,\xi+3\epsilon] \times [-\epsilon,\epsilon]^{2n-1}$. There exists δ such that $0 < \delta < \epsilon$, and 1-forms $\sigma_{\xi,\xi'}$ on $[\xi, \xi'+\epsilon] \times [-\epsilon,\epsilon]^{2n-1}$ for $\xi \leqslant \xi' \leqslant \xi + \delta$. Let $\xi' > \xi$ such that $\xi' - \xi < \delta < \epsilon$. Pick a cut-off function $\beta_{\xi'}$ such that

$$\beta_{\xi'}(t) = \begin{cases} 0 & 0 < t < \xi' \\ \beta_{\xi'}(t) & \xi' < t < \xi' + \epsilon \\ 1, & \xi' + \epsilon < t < 1 \end{cases} \tag{3.16}$$

Let $\sigma_{\xi,\xi'} = (1-\beta_{\xi'})\sigma_\xi + \beta_{\xi'}\sigma_{\xi'}$. Then

$$d\sigma_{\xi,\xi'} = (1-\beta_{\xi'})d\sigma_\xi + \beta_{\xi'}d\sigma_{\xi'} + d\beta_{\xi'} \wedge (\sigma_\xi - \sigma_{\xi'}) \tag{3.17}$$

is nondegenerate on $[\xi, \xi'+3\epsilon] \wedge [-\epsilon,\epsilon]$. (Cf. Eliashberg/Mishachev for a geometric idea.) Pick $x_0 = -1 < x_1 < x_2 < \cdots < x_N = 1+3\epsilon$ in $[-1,1]$ such that $x_{i+1} - x_i < \delta$. Let $A = [0,(2N+3)\epsilon] \times [-\epsilon,\epsilon]^{2n-1}$ and $B = [-1, 1+3\epsilon]$. Let $\psi : A \to B$ be an affine map identifying

A and B, and let $\phi : A \to B$ be an immersion, defined on pieces $A_j = [2j\epsilon, (2j+3)\epsilon]$ into $[x_j, x_{j+1} + \epsilon] \subseteq B$ (the domain of $\sigma_{x_j x_{j+1}}$). Let ϕ_t be a family of immersions defined by $\phi_0 = \psi$, $\phi_1 = \phi$. Recall that $\sigma_{x_j x_{j+1}} = \sigma_{x_j}(1 - \beta_{x_{j+1}}) + \beta_{x_{j+1}} \sigma_{x_{j+1}}$. We define σ on A by

$$\sigma = \begin{cases} \phi_1^* \sigma_{x_{j-1}, x_j}, & [2j\epsilon, (2j+1)\epsilon] \\ \phi_1^* \sigma_{x_j}, & [(2j+1)\epsilon, (2j+\epsilon)\epsilon] \\ \phi_1^* \sigma_{x_j, x_{j+1}}, & [(2j+2)\epsilon, (2j+3)\epsilon]. \end{cases} \tag{3.18}$$

This is smooth, nondegenerate by properties of $\sigma_{\xi,\xi'}$, and the convex combination between $d\sigma$ and $\phi_1 \tau$ are all nondegenerate. Moreover, $\phi_1^* \tau$ and $\psi^* \tau = \phi_0^* \tau$ are connected through nondegenerate forms. For the inductive step $m = 1$ to $m = 2$, use that the previous step can be carried out parametrically (with compact parameter space Λ). Perform the previous step for the lines l_λ to obtain a family of solutions $\sigma_\lambda \to \sigma_{\lambda'}$. □

□

Let $\varphi : (S^1 \times [0,1], dx \wedge dy) = A \to A$ be area preserving. Then φ is called a *twist map* if there is a lift $\tilde{\varphi} : \tilde{A} \to \tilde{A}$ to the universal cover $\tilde{A} = (\mathbb{R} \times [0,1], dx \wedge dy)$ which moves boundary components in different directions.

Theorem 3.19 (Poincaré's Last Geometric Theorem). *Let $\varphi : A \to A$ be an area preserving twist map. Then φ has at least two fixed points.*

Example 3.20. $(x, y) \mapsto (x + \epsilon, y)$ is a non-twist area-preserving map without fixed points. $f : (x, y) \mapsto (x + y - 1/2, y)$, $g : (x, y) \mapsto (x, y + \alpha(y))$ are non-area preserving twist maps without fixed points.

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