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GEOMETRY OF TYPE II SUPERSTRINGS AND THE MODULI OF SUPERCONFORMAL FIELD THEORIES*

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We study general properties of the low-energy effective theory for 4D type II superstrings obtained by the compactification on an *abstract* (2, 2) superconformal system. This is the basic step towards the construction of their moduli space. We give an explicit and general algorithm to convert the effective Lagrangian for the type IIA into that of type IIB superstring defined by the same (2, 2) superconformal system (and vice versa). This map converts Kahler manifolds into quaternionic ones (and quaternionic into Kahlerian ones) and has a deep geometrical meaning. The relationship with the theory of normal quaternionic manifolds (and algebras), as well as with Jordan algebras, is outlined. It turns out that only a restricted class of quarternionic geometries is allowed in the string case. We derive a general and explicit formula for the (fully nonlinear) couplings of the vector-multiplets (IIA case) in terms of the basic three-point functions of the underlying superconformal theory. A number of illustrative examples is also presented.

1. Introduction

A very interesting aspect of two-dimensional (super-)conformal field theories is the possibility of describing the abstract space of all such theories in standard geometrical terms. In particular, Zamolodchikov[1] has shown that the space of the conformal field theories is equipped with a *natural* Riemannian structure. For the applications, one is particularly interested in the connected components of this space, which are the moduli spaces of the various (super-)conformal theories.

A very powerful tool[2] to construct the moduli spaces and to compute their geometrical properties, is the study of the low-energy field theory for a (super-)string compactified on the given (super-)conformal theory.

Roughly speaking, the moduli space is just the manifold of classical vacua for the low-energy theory. The method is especially profitable when the resulting low-energy

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effective theory is supersymmetric, since space-time SUSY gives rather severe restrictions on the space of possible vacua.

For the "physical" applications, the most important case is that of the (2, 2) superconformal theories with $c = 9$. They can be used to "compactify" the heterotic string down to four dimensions, while preserving $N = 1$ space-time supersymmetry.[3] (Throughout this paper we assume that the U(1) superconformal charges q and $\bar{q}$ are properly quantized so as to ensure space-time SUSY, see Ref. 3).

Indeed, this class of superconformal systems can be viewed[4] as sigma models on some kind of "abstract" Calabi-Yau manifold.[5]

To study the moduli space of a (2, 2) superconformal system, it is more convenient[2] to compactify on them type II rather than heterotic superstrings. The reason being that in this case we get $N = 2$ space-time supersymmetry (one from the left-movers and one from the right) and thus we have stronger SUSY constraints. One important difference with respect to the heterotic case is the following. From the theorem of Ref. 6 we know that all massless vectors are Abelian, and that all massless fields are inert under the gauge-symmetries associated to R-R vectors. Assuming that operators with

$$\begin{gathered}(h = 0 \quad \bar{h} = \tfrac{1}{2}; \quad q = 0 \quad \bar{q} = \pm 1) \\ (h = \tfrac{1}{2} \quad \bar{h} = 0; \quad q = \pm 1 \quad \bar{q} = 0)\end{gathered} \tag{1.1}$$

do not exist (otherwise we get a larger space-time and world-sheet supersymmetry) all massless vectors are in fact R-R. Then there are no minimal gauge-couplings for a type II superstring compactified on a (2, 2) system. Under this condition, in standard supergravity the potential just vanishes identically. Then the moduli space ($\sim$classical vacua) just equals the space in which the massless scalars take their value (neglecting, of course, the R-R scalars). The moduli space is simply the target manifold of the σ model describing the effective interactions of the space-time scalars, and the corresponding metric can be read directly from the scalars' kinetic terms.[2]

Following this strategy, Seiberg[2] was able to solve completely the problem for the moduli space of (4, 4) superconformal theories with $c = 6$. In this case one finds a high degree of uniqueness that can be understood physically in terms of the uniqueness of matter interactions in $N = 4$ supergravity or mathematically as due to the uniqueness of K3.

In the (2, 2) $c = 9$ case the above uniqueness is lost. Since the $N = 2$ supergravity Lagrangians are not unique, the moduli problem for the (2, 2) system cannot be solved by SUSY considerations alone.

Therefore, the first step in the construction of the moduli space should be a detailed study of the properties of the low-energy effective Lagrangians for type II superstrings.

The main purpose of the present paper is to give general results for the effective Lagrangians of type II superstrings which are valid for any (2, 2) superconformal systems. Our results come very short of giving the explicit low-energy effective Lagrangian for all cases in terms of the superconformal properties of the underlying

$2d$ theory. As by-products, we also obtain some new results in supergravity and supersymmetry.

In particular, we find that:

(i) The low-energy theory is described by a standard $N = 2$ supergravity with no *unusual* couplings.

(ii) The *universal* sector (the one related to the operator 1 in the (2, 2) superconformal system) is described by a standard coupling of a hypermultiplet to $N = 2$ SUGRA. The scalars of the universal hypermultiplet parametrize the quaternionic coset

$$\mathrm{SU}(2,1)/[\mathrm{SU}(2)\otimes \mathrm{U}(1)]. \tag{1.2}$$

Since the hypermultiplets' manifold in 4D $N = 2$ supergravity should be *irreducible*, in the general case, this coset is only a (totally geodesic) submanifold of the hypermultiplets' quaternionic space.

(iii) Under some mild assumptions, in type IIA superstrings the self-couplings of the vector-multiplets are described by a function $F(X^I)$ (for details on the $N = 2$ tensor calculus see Refs. 7 and 8) of the special form

$$F(X^I) = i\frac{d_{ABC}X^A X^B X^C}{X^0} + \lambda(X^0)^2 \tag{1.3}$$

with d_{ABC} and λ real numbers. For $\lambda = 0$ (as in all known examples) this is the typical form of the so-called flat-potential models.[9] In particular, *minimal* coupling is ruled out.

(iv) The numerical coefficients d_{ABC} can be seen as a kind of "structure constants" of a *generalized* Jordan algebra,[10] (in simple examples it is a true Jordan algebra). Then the numbers d_{ABC} are given directly as a three-point amplitude

$$d_{ABC} \sim \langle V^A_{(-1,-1)} V^B_{(-1/2,-1/2)} V^C_{(-1/2,-1/2)} \rangle \tag{1.4}$$

(see Sec. 2.2). Equations (2.3) and (2.4) specify completely the couplings of the vector multiplets for any type IIA superstring.

(v) The points (iii) and (iv) above do not apply to the IIB case. For this situation even examples with minimal couplings are known.

(vi) In the hypermultiplet sector not all the quaternionic manifolds (consistent with $N = 2$ SUGRA) can appear. Only a very special class of them is allowed. We call them *dual-quaternionic* manifolds. In particular, minimal coupling is ruled out. The beautiful geometry of these manifolds is explained in Sec. 2.3, and in the Appendices.

(vii) As it is well-known,[2] type IIA and type IIB superstrings—compactified on the *same* (2, 2) superconformal system—are related by the interchange of the vector multiplets with the hypermultiplets (other than the universal one). We describe an explicit operation, called the c map, which transforms the low-energy Lagrangian for one type II superstring into the Lagrangian for the other one (compactified on the same superconformal theory).

(viii) The points (iii), (iv) and (vii) give the explicit self-couplings of the hyper-multiplets for any 4D type IIB superstring.

(ix) The relation between the effective Lagrangians for the type II and the heterotic strings (compactified on the same (2, 2) system) is the following: the $N = 1$ Kahler manifold K_H for the heterotic chiral multiplets (corresponding to the moduli of the internal (2, 2) system) is

$$K_H = K_A \times K_B$$

where K_A, and K_B, are the Kahler manifolds of the $N = 2$ vector-multiplets corresponding to type IIA and IIB superstrings, respectively. Moreover there is always an additional factor SU(1, 1)/U(1) (the universal sector of the heterotic string). The geometrical role of the heterotic universal sector, and its relations to the geometry of type II superstrings is explained in Sec. 2.3.4.

Discussion

The above statements deserve a detailed discussion since they are quite different from what it is generally belived. Indeed, one would expect a coupling of the form $\phi F_{\mu\nu}F^{\mu\nu}$ (here ϕ is the "dilaton", i.e. the scalar whose vertex is essentially the trace part of the gravitational one) as well as a coupling $bF_{\mu\nu}\tilde{F}^{\mu\nu}$ (b is the scalar dual to the 2-form $B_{\mu\nu}$). This would not be compatible[2] with the standard supergravity since ϕ and b belong to a hypermultiplet, and no coupling is known between $F_{\mu\nu}F^{\mu\nu}$ and a hypermultiplet.

In fact, whereas it is true that in a generic dimension we have a coupling $\phi F_{\mu\nu}^2$, this does not happen in four dimensons, because four is the dimension in which

(i) the massless vectors are conformally invariant and

(ii) we have duality transformations between the vectors.

The simplest way to see that no $\phi F_{\mu\nu}^2$ coupling is present is just to compute directly the relevant three-point amplitude for the type II superstring (on S^2). First of all, recall that the vectors are R-R fields. For instance, the vertex of the "graviphoton" reads[6]

$$2k_\mu\varepsilon_\nu S_{(-1/2)\alpha}(\sigma^{\mu\nu})^{\alpha\beta}\tilde{S}_{(-1/2)\beta}e^{ik\cdot X} \tag{1.5}$$

where $S_{(-1/2)\alpha}(z)$ and $\tilde{S}_{(-1/2)\beta}(\bar{z})$ are the two (left- and right-moving) supercharges.[11] The expression

$$f_{\mu\nu}(k) \equiv 2k_{[\mu}\varepsilon_{\nu]}\exp(ik\cdot X) \tag{1.6}$$

can be seen as the asymptotic field strength for the graviphoton emitted by the vertex in Eq. (1.5). This means that the vertex

$$V_{(-1/2,-1/2)\mu\nu}(k,z) \equiv S_{(-1/2)\alpha}(\sigma_{\mu\nu})^{\alpha\beta}\tilde{S}_{(-1/2)\beta}e^{ik\cdot X}(z,\bar{z}) \tag{1.7}$$

can be interpreted directly as being the "*vertex*" for the field strength $F_{\mu\nu}$ of the graviphoton A_μ.

Then the on-shell coupling $\phi F_{\mu\nu} F_{\rho\sigma}$ reads

$$\eta_{\lambda\tau}\langle c\tilde{c}V^{\lambda\tau}_{(-1,-1)}(k_1, z_1)c\tilde{c}V_{(-1/2,-1/2)\mu\nu}(k_2, z_2)c\tilde{c}V_{(-1/2,-1/2)\rho\sigma}(k_3, z_3)\rangle_{S^2} \tag{1.8}$$

where $k^2{}_i = 0$, $c(z)$ and $\tilde{c}(\bar{z})$ are diffeomorphism ghosts (just to soak up their zero-modes) and

$$V^{\lambda\tau}_{(-1,-1)}(k, z) = e^{-\phi}\psi^{\lambda}e^{-\tilde{\phi}}\tilde{\psi}^{\tau}e^{ik\cdot X}(z, \bar{z}). \tag{1.9}$$

The amplitude in Eq. (1.8) factorizes between left- and right-movers. Recalling the SUSY current-Algebra[11,12]

$$S_{(-1/2)\alpha}(z)S_{(-1/2)\dot{\beta}}(w) \sim \frac{1}{\sqrt{2}}\frac{1}{z-w}e^{-\phi}(\gamma_{\mu})_{\alpha\dot{\beta}}\psi^{\mu}(w) + \text{finite} \tag{1.10}$$

one finds

$$\text{Eq. (1.8)} = \tfrac{1}{2}\eta_{\lambda\tau}\operatorname{Tr}[\gamma^{\lambda}\sigma_{\mu\nu}\gamma^{\tau}\sigma_{\rho\sigma}] \tag{1.11}$$

which—in four dimensions—vanishes since

$$\gamma^{\lambda}\sigma_{\mu\nu}\gamma_{\lambda} = 0. \tag{1.12}$$

By a similar computation one can see that there are no couplings of the form $bF\tilde{F}$. However, the easiest way to realize that this coupling is not allowed, is to go to the $N = 8$ case, i.e. to the trivial toroidal compactification. This cannot change the couplings for fields related to the identity operator of the internal conformal theory. The effective Lagrangian for the $N = 8$ case is just the $N = 8$ supergravity constructed by Cremmer and Julia.[13] By comparison with their work we see that b (the dual of $B_{\mu\nu}$) is actually a *scalar* rather than a *pseudoscalar* as it is in the heterotic string. (This can also be shown by more stringy arguments.) Therefore $bF\tilde{F}$ is ruled out by parity invariance. A stronger and more definitive argument against this coupling is that its presence would spoil the gauge-invariance of the model at the world-sheet level.

Indeed, if the 4D effective Lagrangian would be of the form (we write only the terms containing b)

$$\tfrac{1}{2}f(\phi)(\partial_{\mu}b)^2 + bF\tilde{F}, \tag{1.13}$$

after a duality transformation, we would get

$$\mathscr{L} = \tfrac{1}{2}f(\phi)^{-1}\{\partial_{[\mu}B_{\nu\rho]} - \omega^{(3)}_{\mu\nu\rho}\}^2 \tag{1.14}$$

where $\omega^{(3)}{}_{\mu\nu\rho} = 2A_{[\mu}F_{\nu\rho]}$ is the Abelian Chern-Simons 3-form for the vector A_{μ}. The situation would be analogous to that of the heterotic string. And, as in that case, gauge-invariance of the effective space-time Lagrangian—Eq. (1.14)—requires $B_{\mu\nu}$ to transform nontrivially under the Abelian gauge transformation $\delta A_\mu = \partial_\mu \omega$,

$$\delta B_{\mu\nu} = 2\omega F_{\mu\nu}\,. \tag{1.15}$$

But, then at the world-sheet level the action of the σ model would also change by

$$\delta S = 2\int_{S^2} \omega(X) F_{\mu\nu}(X) \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu\,. \tag{1.16}$$

This variation—contrary to the heterotic case—is not compensated by any $2d$ anomaly, since the $2d$ fermions are inert under the Abelian gauge symmetry. Therefore, a coupling $bF\tilde{F}$ would lead to a breaking of gauge-invariance at the space-time or at the world-sheet level. The absence of this coupling can be easily checked by a direct computation: it follows from the structure of the vertex for a R-R vector. A coupling $B_{\mu\nu}$-A_ρ-A_σ as predicted by the Lagrangian (1.14) cannot be written in terms of a vertex of the form (1.5), since these vertices imply that only the field strengths enter explicitly in the effective Lagrangian, whereas in Eq. (1.14) we have explicitly A_μ through the Chern-Simons form. In other words, the absence of $bF\tilde{F}$ couplings follows from the same argument that forbids minimal gauge couplings, Ref. 6.

The above are first remarks about the impossibility of having couplings of the form (hypermultiplet) F^2. In Sec. 2 this will be shown by explicitly constructing the low-energy effective Lagrangian for 4D type II superstring.

In standard supergravity, instead, we have nonvanishing coupling of the form ZF^2 and $ZF\tilde{F}$ where Z is a scalar belonging to a vector multiplet. By computing the amplitudes it is very easy to see that these couplings should be present in the string case too.

A very important issue for the geometry of the moduli space is the relation between the effective Lagrangians for type IIA and IIB superstrings compactified on the same (2, 2) system. The superconformal operators relevant for the massless sector (other than the identity) are those with[2]

$$(h = \bar{h} = \tfrac{1}{2},\ q = -\bar{q} = 1) \tag{1.17}$$

$$(h = \bar{h} = \tfrac{1}{2},\ q = \bar{q} = 1) \tag{1.18}$$

along with their Hermitian conjugates and those connected with them by space-time supersymmetry. In the case of a σ-model on a Calabi-Yau space, the operators (1.17) correspond to harmonic (1, 1) forms and the operators (1.18) to harmonic (2, 1) forms.

Now it is obvious from the form of the GOS projection that in type IIA case the operators in Eq. (1.17) lead to vector multiplets and those in Eq. (1.18) to hypermultiplets. In the type IIB string hypermultiplets and vector multiplets are interchanged.

We want to understand how this interchange of multiplets is realized at the level of effective Lagrangians.

In Sec. 2.3 we shall give the explicit recipe to construct one effective Lagrangian once the other is known. We call the operation transforming the type IIA effective Lagrangian into the type IIB the *c map*.

Roughly speaking, the c map transforms a Kahler manifold (of the special type allowed in $N = 2$ SUGRA) of dimension n into a quaternionic manifold (also of a special restricted type) of dimension $4(n + 1)$. For a more precise definition, see Sec. 2.3, as well as Appendices A, B, and C. For instance, the c map sends the Kahler coset $SU(1, 1)/U(1)$ (with $R = -2/3$) into the quaternionic space $G_{2(+2)}/SO(4)$ and the space $U(3, 3)/[U(3) \times U(3)]$ into $E_{6(+2)}/[SU(2) \times SU(6)]$. Of course, in the generic case we have no such simple symmetric manifolds; our construction of the c map is totally general and explicit.

From a mathematical point of view, the c map is quite an interesting algorithm since it allows one to construct *continuous* families of quaternionic and hyperkahler metrics (the hyperkahler case is studied in Appendix B below). It is also deeply connected with the theory of Jordan Algebras[10] (in particular to the *magic-square*[14]) and with some deep work by Alekseevskii[15] on quaternionic normal Lie Algebras.

The paper is organized as follows: in Sec. 2 we motivate our general statements about the low-energy effective theories corresponding to type II superstrings compactified on (2, 2) systems. In particular, in Sec. 2.1, we study the coupling in the universal sector and show that they are exactly those predicted by the $N = 2$ tensor calculus. In Sec. 2.2 we study in general the couplings of the vector multiplets in the type IIA case. We show that $F(X^0, X^A) = id_{ABC}X^AX^BX^C(X^0)^{-1} + \lambda(X^0)^2$, where the coefficients d_{ABC} can be seen as related to the structure constants of a Jordan-like Algebra. The d_{ABC} are computed in the same way as the structure constants for the gauge-group algebra.

In Sec. 2.3 we address the problem of the interchange of type IIA and type IIB, and construct explicitly the transformation (c map) between the two effective Lagrangians by microscopic (= stringy) methods. We also digress on some geometrical aspects of the c map. In Sec. 3 we present a number of explicit examples of effective Lagrangians to illustrate our ideas. We also describe how the type II effective Lagrangians are related to that for the heterotic superstring compactified on the same conformal system. Finally, in Sec. 4 we present our conclusions and we discuss the status of the moduli problem.

In Appendix A, we discuss the geometry of the vector couplings to $N = 2$ SUGRA, and in particular of their duality transformations. Moreover, here we collect some technical results that we need in the main body of the paper, as well as some explicit computations. In Appendix B we work out the c map in the rigid SUSY case, writing down explicitly the Kahler potential for the hyperkahler manifold which is the c image of any Kahler space of the $N = 2$ theory. This gives us a new class of explicit hyperkahler metrics, depending on an (arbitrary) holomorphic function of $n/4$ complex variables, where n is its dimension. This result is very interesting in its own for the theory of hyperkahler and quaternionic manifolds. Finally, in Appendix C we give a physicist's discussion of the c map for *symmetric* spaces, as well as an alternative argument to show that the couplings are those described in point (ii) above.

2. General Properties of the Low-Energy Effective Lagrangians for Type II Superstrings Compactified on a (2, 2) System

2.1 *The universal sector and relations with standard $N = 2$ sugra*

In this section we show the general properties of the low-energy effective Lagrangians for type II superstring we claimed in the introduction. Specific examples will be provided in Sec. 3.

We start by studying the couplings of the *universal* sector of the theory, the gravitational multiplet and the hypermultiplet whose vertices correspond to the operator 1 of the "internal" superconformal theory (and those connected to it by the spectral flow[3]). As we have discussed in the introduction, a specific question is the existence of a nontrivial coupling of the universal hypermultiplet with the graviphoton kinetic terms F^2 and $F\tilde{F}$.

Since the universal sector is model independent, to compute its couplings we can follow two strategies: we can choose a *simplified* model, compute its effective Lagrangian and then extract the universal part from it; or we can construct it directly, using only abstract properties of any 4D type II superstring, as e.g. symmetry considerations. Of course, the results of the two strategies agree.

The simplest way to implement the first strategy would be to construct a qualitatively correct Lagrangian, following the procedures employed by Witten[16] for the heterotic string compactified on a Calabi-Yau space—the prototype of a (2, 2) superconformal theory.

This method amounts to a truncation of the toroidal compactification of the 10D $N = 2$ supergravity down to its SU(3) invariant sector. Of course, this is quite a *poor* approximation to a real (2, 2) superconformal system (for real *string* examples, see Sec. 3). However, the approximation obviously does not affect the operator 1, and so the universal sector should be exact. This is certainly true in the heterotic case where the universal sector contains a chiral multiplet S parametrizing the Kahler coset

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \tag{2.1}$$

with a Kahler potential

$$K = -\ln(S + \bar{S}). \tag{2.2}$$

The same will be true here for the type II case.

However, as it will be clear in the following, the only point that really matters in the computation is that the theory is invariant under a "hidden" noncompact symmetry which acts on the vector fields through (generalized) duality transformations.[17] For the $N = 8$ case this is the celebrated E_7 symmetry first discovered by Cremmer and

Julia.[13] There is an increasing evidence[18] that these hidden symmetries are valid at the full string level (at least in the classical case). They certainly are symmetries of the low-energy effective theory. Our computations below depend in fact only on the rather general property of type II strings of having such hidden symmetries. In this sense, our computation can also be reinterpreted from the "abstract" point of view.

All the other models of Sec. 3 give identical results for the universal sector. A more geometrical argument is given at the end of Appendix C.

In principle, we could start from the 10D $N = 2$ theory. But that would be a cumbersome procedure. Indeed, it would be very difficult to re-write the resulting 4D theory in such a way as to make it manifest (at the full nonlinear level) that it is a *canonical* $N = 2$ theory.[7] Already in the Witten case,[16] it is not totally trivial to organize the various fields into complex coordinates on the relevant Kahler manifold. In the $N = 2$ case it would be much more difficult, since, because of the Bianchi identities,[8] there exist preferred complex coordinates[8,9] on the vectors' Kahler manifold, and the form of the canonical Lagrangian is not covariant under all holomorphic transformations of the coordinates but only under a certain group, which contains $\mathrm{PGL}(n+1, R)$. Therefore, we shall start from the toroidal compactification as given by Cremmer and Julia,[13] and shall work on the scalars' manifold in a coordinate-free way.

In practice, all we have to do is to truncate the local symmetry group SU(8) to its SU(3)-invariant subgroup H_{local} and to find the SU(3)-invariant subgroup of the global symmetry E_7, G_{global}. Then the scalars will parametrize the coset

$$G_{\mathrm{global}}/H_{\mathrm{local}}\,. \tag{2.3}$$

The action—through duality—of G_{global} on the field strengths of the SU(3)-invariant vectors of the $N = 8$ theory completely fixes all the couplings, and in particular, the two functions $f^{(1)}{}_{AB}$ and $f^{(2)}{}_{AB}$ defined by the vector kinetic terms

$$-f^{(1)}_{AB}F^{A}_{\mu\nu}F^{B}_{\mu\nu} - f^{(2)}_{AB}F^{A}_{\mu\nu}\tilde{F}^{B}_{\mu\nu}\,. \tag{2.4}$$

In the IIA case the embedding of the relevant SU(3) group (the holonomy of the internal *Calabi-Yau space*) in $\mathrm{SU}(8)_{\mathrm{local}}$ is given by the chain

$$\mathrm{SU}(3) \subset \mathrm{SU}(4) \simeq \mathrm{Spin}(6) \subset \mathrm{Spin}(8) \subset \mathrm{SU}(8)\,. \tag{2.5}$$

The SU(3) invariant subgroup of SU(8) is then,

$$H_{\mathrm{local}} = \mathrm{SU}(2) \otimes \mathrm{U}_R(1) \otimes \mathrm{U}(1)\,. \tag{2.6}$$

The two surviving gravitini are a doublet under SU(2) and have charges $(+3/4, 0)$ under the two U(1)'s.

The SU(3)-invariant subgroup of E_7 is

$$G_{\text{global}} = \mathrm{SU}(1,1) \otimes \mathrm{SU}(2,1), \tag{2.7}$$

whose maximal compact subgroup is isomorphic to H_{local}, with $\mathrm{U}_R(1)$ a subgroup of $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2) \times \mathrm{U}(1)$ the maximal compact subgroup of $\mathrm{SU}(2,1)$.

Thus, in this simplified model the massless scalars parametrize the coset

$$\frac{G_{\text{global}}}{H_{\text{local}}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}_R(1)} \otimes \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \otimes \mathrm{U}(1)} \tag{2.8}$$

Notice that this result is *consistent* with standard $N = 2$ supergravity, although not enough to conclude (for the moment) that it *is* standard $N = 2$ supergravity. The model contains, besides the gravitational multiplet, a hypermultiplet—the universal sector—and a vector multiplet corresponding to the single (1, 1) harmonic form, the Kahler form

$$i\delta_{i\bar{j}}\, dz^i \wedge d\bar{z}^j. \tag{2.9}$$

In standard supergravity, the hypermultiplets live on Kahler-quaternionic spaces[19] and the vector ones on Kahler spaces of a certain special type[7,8,9] (whose geometry is reviewed in Appendix A). Moreover, only two classes of *symmetric* Kahler spaces are allowed in $N = 2$ supergravity (for a proof see Ref. 20 or Sec. 2.3.4 below):

(i) the open unit ball in C^n, i.e. the coset $\mathrm{U}(n,1)/[\mathrm{U}(n) \times \mathrm{U}(1)]$, with its Bergman metric normalized so that the scalar curvature is $R = -n(n+1)$.

(ii) the spaces connected with Jordan algebras (with cubic norm).[10,21]

Case (i) is known as minimal coupling.[8]

Now, the coset $\mathrm{SU}(2,1)/\mathrm{U}(2)$ in Eq. (2.8) is a quaternionic space[22,15] of (real) dimension 4, the correct value for one hypermultiplet.

The particular noncompact form of the groups in the cosets is also typical of the tensor calculus, and indeed the coupling of a hypermultiplet parametrizing the coset $\mathrm{U}(n,2)/[\mathrm{U}(n) \times \mathrm{U}(2)]$ was constructed some time ago using the tensor calculus.[23] Moreover, the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$ is a Kahler space permitted by the theorem of Ref. 20 (indeed, there are two inequivalent[20] models based on $\mathrm{SU}(1,1)/\mathrm{U}(1)$, the minimal coupling with $R = -2$, and that associated with the Jordan algebra R, with $R = -2/3$). Also the embedding of the $\mathrm{U}_R(1) \times \mathrm{SU}(2)$ group—the authomorphism group of the SUSY algebra—is just that dictated by the tensor calculus. However, this still does not prove that the couplings are the usual ones. Below we shall compute the couplings and show that they agree with the prediction of the tensor calculus.

The way to figure out the couplings (in particular, the two functions in Eq. (2.4)) is to exploit the duality invariance of the model. Before the truncation E_7 acted on the vectors through generalized duality rotations.[13,24] In our model, the group G_{global}, Eq. (2.7), acts in the same way. The action on the field strengths is determined by the truncation. We have two vectors in the theory—the graviphoton and the matter one—whose field strengths will be denoted by $F_{(a)\mu\nu}$ $a = 1, 2$. Their variational duals will be denoted by $G_{(a)\mu\nu}$. Following Ref. 24, we define the linear combinations

$$F^{+}_{\pm(a)\mu\nu} = \tfrac{1}{2}(G^{+}_{(a)\mu\nu} \pm F^{+}_{(a)\mu\nu}) \tag{2.10}$$

and introduce the vector

$$\mathscr{F}^{+}_{\mu\nu} = (F^{+}_{-(2)}, F^{+}_{+(1)}, F^{+}_{-(1)}, F^{+}_{+(2)})_{\mu\nu}\,. \tag{2.11}$$

The four field strengths in $\mathscr{F}^{+}_{\mu\nu}$ transform under the global symmetry $SU(1,1) \times SU(2,1)$ according to the representation $(4,1)$. Thus, the field strengths are inert under the group $SU(2,1)$ and transform nontrivially only under $SU(1,1)$. Again, it is what it is expected from standard SUGRA.

Now we are in a position to prove the following:
"Theorem"

(1) The functions $f^{(1)}{}_{AB}$ and $f^{(2)}{}_{AB}$ depend only on the scalars of the vector multiplet, and not on those of the hypermultiplet.

(2) These functions, as well as the other couplings, are those of the $N = 2$ tensor calculus.

(3) The vector multiplet couplings cannot be *minimal*. (Then, by Ref. 20, it should be the model associated with the Jordan algebra R.)

In fact, this is a special instance of a more general result: If a supergravity model possesses an invariance group, acting by duality on the vectors' field strengths, which is transitive on the scalars' manifold, then all couplings are uniquely fixed by the invariance alone.

This statement is obvious in view of the results of Ref. 17. Here we prove it just for completeness.

Proof.

(1) Before the truncation we have (Eq. (2.13) of Ref. 24),

$$(1 - \Omega)\mathscr{V}\mathscr{F}^{+}_{\mu\nu} = \text{FERMIONS} \tag{2.11}$$

where the E_7 element $\mathscr{V}$ is a representative for the scalars' coset. In our case, $\mathscr{V}$ is truncated to be an element of the global group $G_{\text{global}} = SU(1,1) \times SU(2,1)$, and it represents the coset space of Eq. (2.8), parametrized by the scalar fields of the vector and matter multiplets. Since $\mathscr{F}^{+}_{\mu\nu}$ is invariant under $SU(2,1)$, $\mathscr{V}$ in Eq. (2.11) is actually an element of the $SU(1,1)$ subgroup, which is a representative for the vector multiplet coset $SU(1,1)/U(1)$.

We can solve Eq. (2.11) for the dual field strengths $G^{+}{}_{(a)\mu\nu}$ in terms of $F^{+}{}_{(a)\mu\nu}$. We get

$$\begin{aligned} G^{+}_{(a)\mu\nu} &= \mathscr{N}_{ab} F^{+}_{(b)\mu\nu} + \text{FERMIONS} \\ &\equiv -\frac{\partial \mathscr{L}}{\partial F^{+\mu\nu}_{(a)}} \end{aligned} \tag{2.12}$$

with $\mathcal{N}_{ab}$ a function of $\mathcal{V}$ which—by Eq. (2.11)—is in fact a function on the coset space SU(1, 1)/U(1), i.e. a function of the vector-multiplet scalars. Moreover, Eqs. (2.11) and (2.12) show that the function $\mathcal{N}_{ab}$ is completely determined by the group action of SU(1, 1) over $\mathcal{F}^+_{\mu\nu}$.

From Eq. (2.12) we have

$$f^{(1)}_{ab} + if^{(2)}_{ab} = 2\mathcal{N}_{ab} \tag{2.13}$$

so point (1) is proven.

(2) From the proof of (1), we see that the functions $f^{(1)}{}_{AB}, f^{(2)}{}_{AB}$, as well as the other couplings of the vector-multiplet, are *uniquely* fixed (up to inessential field re-definitions, which in the general case require also a duality transformation) once we know: (i) the duality group: (ii) the representation to which $\mathcal{F}^+_{\mu\nu}$ belongs.

In our case, the representation is the 4 of SU(1, 1). This fact alone proves (3) because in the "minimal" model the representation is reducible, being 2 + 2. Indeed, the linear combinations

$$G^+_{(a)\mu\nu} \pm \eta_{ab} F^+_{(b)\mu\nu} \tag{2.14}$$

correspond to invariant subspaces (see Eq. (3.30) of Ref. 8 or Appendix A).

Now, the point is that the tensor calculus does describe a coupling of a vector multiplet to $N = 2$ SUGRA, which is invariant under an SU(1, 1) generalized duality group, with the four field strengths transforming in the 4. It is the so-called flat-potential model of Ref. 9. (Actually, in Ref. 9 two such models are presented, called I and II; by the above result they should be equivalent. In Appendix A we describe the change of variables that maps one into the other).

Since there is *only one* coupling with these properties, the theory obtained by SU(3)-invariant truncation of $N = 8$ supergravity and that obtained from the $N = 2$ tensor calculus should be the *same*. We have also checked this explicitly. This completes the proof of the Theorem.

The above results justify our claim in Sec. 1 that no unusual coupling (at least in the universal sector) is present in the low-energy effective theories coming from type II strings. This is further confirmed by the examples of Sec. 3.

The universal sector can be extracted from the above simplified model. As we argued above, the result for this sector is exact.

The universal hypermultiplet parametrize the coset SU(2, 1)/U(2) and hence the universal sector is described by the Lagrangian of Breitenlohner and Sohnius.[23] {In general, this coset is only a subspace of the hypermultiplets' σ model, see Sec. 2.3.4 for more details).

We conclude this very preliminary analysis by considering the effective Lagrangian for the type IIB superstring in the above simplified compactification *à la* Witten.

Now, the harmonic (1, 1) form—Eq. (2.9)—gives a hypermultiplet, so the massless spectrum contains two hypermultiplets and no vector ones.

In the above computation the only thing that changes is the embedding of SU(3). We have to change the action of the internal Spin(6) on the right-moving supercharge from $\bar{4}$ to 4. Then the surviving local symmetry is

$$H_{\text{local}} = \mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{U}_R(1). \tag{2.15}$$

The compact subgroup of G_{global} is isomorphic to H_{local}. The noncompact generators are in the (2,4) representation of SU(2) $\times$ SU(2) and neutral under $\mathrm{U}_R(1)$. Thus,

$$G_{\text{global}} = G_{2(+2)} \otimes \mathrm{U}_R(1) \tag{2.16}$$

and the two hypermultiplets live on the coset

$$G_{2(+2)}/\mathrm{SO}(4) \tag{2.17}$$

which is again a quaternionic manifold,[22] allowed in standard supergravity.

In the case of the type IIB superstring there is a simpler argument for the absence of exotic couplings between the graviphoton and the scalars. Indeed, from the 10D point of view, the equation of motion of the graviphoton is just the condition of self-duality for the field strength of the 4-form. This equation does not contain 4D scalars, as it is easy to check.

2.2. *The general formula for the self-couplings of vector-multiplets in 4D type IIA superstrings*

In this section we prove the claims (iii) and (iv) of Sec. 1.

Through the present section we shall assume that the resulting low-energy Lagrangians are described by the standard tensor calculus[7,8,25] (at *least* in the vector multiplet sector). The example of Sec. 2.A, as well as those of Sec. 3 (to which the theorem of Sec. 2.1 applies as well) show that this assumption is in fact well-motivated.

For concreteness, we shall work in the case in which the (2,2) superconformal theory is actually a sigma-model on a Calabi-Yau space. A well-motivated conjecture by Gepner[4] states that this is, indeed, the general case, assuming the (2,2) superconformal theory to have a parity invariance at the world-sheet level.

In the case of type IIA superstrings the vector multiplets are associated to harmonic (1,1) forms $\omega^A{}_{i\bar{j}}(X^k, \bar{X}^l)$, $A = 1, \ldots, h_{1,1}$. In the $(-1,-1)$ picture,[11] the vertices for the two scalar particles in the multiplet are

$$\omega^A_{i\bar{j}}(X^k, \bar{X}^l) e^{-\phi} \psi^i e^{-\tilde{\phi}} \tilde{\psi}^{\bar{j}} \exp(ik \cdot X) \tag{2.18a}$$

$$\omega^A_{i\bar{j}}(X^k, \bar{X}^l) e^{-\phi} \psi^{\bar{j}} e^{-\tilde{\phi}} \tilde{\psi}^{i} \exp(ik \cdot X). \tag{2.18b}$$

Passing to the (0,0) picture, we see that the space-time scalar field associated to the difference of these two vertex operators, $b^A(X^\mu)$, enters the Lagrangian of the $2d$ sigma-model through a contribution to the internal torsion 2-form $B_{i\bar{j}}$

$$i \int_{S^2} \sum_A b^A(X^\mu)\omega^A_{i\bar{j}}(X^k, \overline{X^l})\varepsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j. \tag{2.19}$$

Now, neglecting nonperturbative corrections from world-sheet instantons,[a] we have the Peccei-Quinn symmetry of Ref. 26. Since the forms $i\omega^A_{i\bar{j}} dX^i \wedge d\overline{X^l}$ are closed, a shift of the fields $b^A(X^\mu)$ by a constant will not change the torsion three-form $H = dB$, and so it should leave invariant the string *equations* of motion (that is, the β functions of the $2d$ σ model[27]) to all orders in the σ model perturbation theory.

Therefore, the low-energy effective theory should be invariant under the symmetry

$$b^A(X^\mu) \to b^A(X^\mu) + C^A. \tag{2.20}$$

In the ten-dimensional case, not only the equations of motion but also the low-energy effective Lagrangian is invariant under $B \to B + \alpha$ with α closed, since this is just the two-form gauge-invariance. Then we also expect that in the 4D case the Peccei-Quinn transformation (2.20) will be a symmetry of the effective Lagrangian, not only of the equations of motion. The condition of being a symmetry at the *Lagrangian level* is very strong in $N = 2$ supergravity.

In terms of the homogeneous coordinates on the vectors' Kahler space, (X^0, X^A), all transformations which leave the Lagrangian invariant (up to a surface term) should be of the form (see Refs. 8 and 9 and Appendix A)

$$\delta X^I = B^I{}_J X^J \tag{2.21}$$

$(I, J = 0, 1, \ldots, h_{1,1})$ with B a real matrix. In terms of the physical scalars $Z^A = X^A/X^0$, Eq.(2.21) becomes

$$\delta Z^A = B^A{}_0 + (B^A{}_B - B^0{}_0 \delta^A{}_B)Z^B - B^0{}_B Z^B Z^A. \tag{2.22}$$

Then, the natural identification for the Peccei-Quinn symmetry in Eq. (2.20) is given by

$$\begin{aligned} b^A(X) &= \operatorname{Re} Z^A \\ C^A &= B^A{}_0 . \end{aligned} \tag{2.23}$$

With this identification, Eq. (2.20) is equivalent to

$$X^A \to X^A + C^A X^0, \qquad (C^A \text{ real}). \tag{2.24}$$

Notice that since X^A is the first component of a vector-multiplet, by supersymmetry,

[a] In many concrete examples, the nonperturbative corrections to our results are big and change the physical picture.

Eq. (2.24) also implies a transformation on the field strengths of the respective vector fields

$$A^A_\mu \to A^A_\mu + C^A A_\mu \tag{2.25}$$

where A_μ is the graviphoton field.

Under the transformation (2.21) the function F which encodes the vector couplings changes by[8,9]

$$\delta F = -iC_{IJ}X^I X^J \tag{2.26}$$

where C_{IJ} is a constant real symmetric matrix, related to $B^I{}_J$ and the function $F(X)$ by a certain consistency condition.[8,9] Equation (2.26) expresses the fact that under the above transformation the effective Lagrangian changes only by the surface term[8,9]

$$C_{IJ}F^I_{\mu\nu}\tilde{F}^{J\mu\nu} \tag{2.27}$$

which is a typical property of Peccei-Quinn symmetries.

Obviously, if under the transformation in Eq. (2.24) (with C^A real) $F(X)$ is to transform as in Eq. (2.26), it should be of the form

$$F(X) = i\frac{d_{IJK}X^I X^J X^K}{X^0} + \lambda(X^0)^2 \tag{2.28}$$

with d_{IJK} real constants. In Eq. (2.28) we used that F is homogeneous of degree 2.[7,8,9] Since

$$d_{IJK}X^I X^J X^K = d_{ABC}X^A X^B X^C + X^0 P_{IJ}X^I X^J \tag{2.29}$$

(P_{IJ} real) we have

$$F(X) = i\frac{d_{ABC}X^A X^B X^C}{X^0} + iP_{IJ}X^I X^J + \lambda(X^0)^2. \tag{2.30}$$

It is well-known that the Lagrangian of $N = 2$ SUGRA remains invariant (up to a term of the form (2.27)) if we add to the function $F(X)$ a quadratic polynomial with purely imaginary coefficients. Therefore, we can assume $P_{IJ} = 0$ and λ real. In all the explicit models discussed in this paper we have $\lambda = 0$. We do not know if this is a general feature. If $\lambda \neq 0$ we can put it equal to ± 1 by a rescaling of X^0.

The example of Sec. 2.1 corresponds to

$$F(X) = i(X^1)^3/X^0. \tag{2.31}$$

All the models of Sec. 3 are of the present form,[9] indeed they are all associated to Jordan algebras.[10,21]

In terms of the function $F(X)$, the vectors' kinetic terms read[8,9]

$$\tfrac{1}{4}F^{+I}_{\mu\nu}\mathcal{N}_{IJ}F^{+J}_{\mu\nu} + \text{H.c.} \tag{2.32}$$

where

$$\mathcal{N}_{IJ} = \frac{1}{4}\bar{F}_{IJ} - \frac{(NZ)_I(NZ)_J}{(ZNZ)} \tag{2.33}$$

and

$$F_{IJ} = \partial_I\partial_J F(X) \tag{2.34a}$$

$$N_{IJ} = \tfrac{1}{4}(F_{IJ} + \bar{F}_{IJ}) \tag{2.34b}$$

$$(NZ)_I = N_{IJ}Z^J; \qquad (ZNZ) = Z^I(NZ)_I \tag{2.34c}$$

and, by convention, $Z^0 = 1$.

From the above arguments it follows that (ZNZ) is a function of $\operatorname{Im} Z^A$ only, indeed

$$\begin{aligned}(ZNZ) - \lambda = \lambda - (\bar{Z}NZ)\\ = \frac{i}{4}d_{ABC}(Z^A - \bar{Z}^A)(Z^B - \bar{Z}^B)(Z^C - \bar{Z}^C)\end{aligned} \tag{2.35}$$

moreover, we have[9]

$$(NZ)_A = -\tfrac{3}{4}i d_{ABC}(Z^B - \bar{Z}^B)(Z^C - \bar{Z}^C). \tag{2.36}$$

However, $(NZ)_0$ has a nontrivial dependence on $\operatorname{Re} Z^A$. This can be understood from the fact that

$$(ZN)_I F^{+I}_{\mu\nu} \tag{2.37}$$

is invariant under the combination of Eqs. (2.24) and (2.25), then using Eq. (2.36), we get

$$(ZN)_A C^A F^+_{\mu\nu} + \delta_{C^A}[(ZN)_0]F^+_{\mu\nu} = 0 \tag{2.38}$$

i.e.

$$(ZN)_0 = -(ZN)_A \operatorname{Re} Z^A + \text{independent of } (\operatorname{Re} Z^A). \tag{2.39}$$

Then the terms in Eq. (2.32) containing the field strength of the graviphoton $F^0{}_{\mu\nu}$ have a nontrivial dependence on $\operatorname{Re} Z$.

Let us concentrate on the terms in Eq. (2.32) which are proportional to $F^A_{\mu\nu}\tilde{F}^B_{\mu\nu}$,

$A, B = 1, \ldots, h_{1,1}$. They are

$$\begin{aligned} &\tfrac{1}{32}(\bar{F}_{AB} - F_{AB})F^{A}_{\mu\nu}\tilde{F}^{B}_{\mu\nu} + \text{terms which do not contain } \mathrm{Re}\, Z \\ &\quad = \tfrac{3}{8} d_{ABC}(\mathrm{Re}\, Z^{C})F^{A}_{\mu\nu}\tilde{F}^{B}_{\mu\nu} + \text{terms which do not contain } \mathrm{Re}\, Z \end{aligned} \tag{2.40}$$

Hence, apart for an irrelevant overall factor (which can be absorbed in the definition of the fields) the coefficient d_{ABC} is just the value of the coupling

$$(\mathrm{Re}\, Z^{C})F^{A}_{\mu\nu}\tilde{F}^{B\mu\nu}.$$

In other words, let $V^{A}{}_{(-1,-1)}(k, z)$ be the vertex for the scalar $(\mathrm{Re}\, Z^{A})$ and

$$2k^{\mu}\varepsilon^{\nu}V^{A}_{(-1/2,-1/2)}(k, z)$$

the vertex for the vector $A^{A}{}_{\mu}$; then $(k^{2}{}_{i} = 0)$

$$d_{ABC} = \varepsilon^{\mu\nu\rho\sigma}\langle c\tilde{c}V^{A}_{(-1,-1)}(k_1, z_1)c\tilde{c}V^{B}_{(-1/2,-1/2)}(k_2, z_2)c\tilde{c}V^{C}_{(-1/2,-1/2)}(k_3, z_3)\rangle_{S^2}. \tag{2.41}$$

This equation, together with the formula

$$F(X) = i\frac{d_{ABC}X^{A}X^{B}X^{C}}{X^{0}} + \lambda(X^{0})^{2} \tag{2.42}$$

give our general expression for the coupling of the vector multiplets for any type IIA superstring compactified on a (2, 2) superconformal theory (having a world-sheet parity invariance). There is still the problem with the integration constant λ; however, even this parameter may be computed by a simple three-point amplitude.

As a word of caution, we remember that in actual superstring models the "natural" basis of fields need not coincide with the above one. In particular, one has to identify correctly the "graviphoton" i.e. $F^{0}{}_{\mu\nu}$ before using Eqs. (2.41) and (2.42) since it is not quite the same field strength one finds in the SUSY transformation of the gravitino field (at the nonlinear level).

We finish this section with some general comments on the models of the form in Eq. (2.42). We focus on the $\lambda = 0$ case, but $\lambda \neq 0$ would not change most of the features. The supergravity models of this general form can be thought of as coming from five dimensions by dimensional reduction[10] (notice that, on the contrary, the string models are not toroidal compactifications of 5D superstrings).

Gunaydin, Sierra and Townsend[10,21] have shown that the peculiar geometry of these 5D $N = 2$ supergravities is very deeply connected with the theory of Jordan algebras with cubic invariant norm. The general model comes short of being associated to a Jordan algebra, in the sense that all, except one, of the axioms of a Jordan algebra are fulfilled.[10] The last axiom (called M5 in Ref. 10) is also satisfied in the special models whose scalars parametrize a *symmetric* Kahler-manifold (which is always a bounded domain equipped with the Bergman metric). In particular, all our examples in the present paper correspond to Jordan algebras. Their cubic norm reads

$$N(X) = d_{ABC} X^A X^B X^C \tag{2.43}$$

with d_{ABC} being the same coefficients as in Eq. (2.42) (up to an irrelevant overall factor rescaling).

Given this structure, it is natural to think of the vector couplings as being determined by some kind of "*generalized*" Jordan algebra (at least for $\lambda = 0$). The coefficients d_{ABC} are related with the "structure constants" of this algebra.

Then Eq. (2.41) is analogous to the equation determining the structure constants of the gauge group G in the heterotic string

$$\langle cJ^a(z_1)cJ^b(z_2)cJ^c(z_3)\rangle = \tfrac{1}{2}kf_{abc} \tag{2.44}$$

which follows from the Kač-Moody current algebra

$$J^a(z)J^b(w) \sim \frac{f_{abc}J^c(w)}{z-w} + \frac{k}{2}\frac{\delta^{ab}}{(z-w)^2}\,. \tag{2.45}$$

Thus, in some sense, to compute the vector couplings in the type II case is as easy as finding the gauge group in the heterotic case.

2.3 The Relation Between the Low-Energy Effective Lagrangians for Type IIA and IIB Superstrings Compactified on the Same (2, 2) Superconformal Theory

In this section, we construct explicitly the c map which transforms the low-energy effective Lagrangian for a type IIA superstring into the corresponding Lagrangian for a type IIB superstring compactified on the same (2, 2) superconformal system. Of course, the map can be inverted to map the Lagrangian for type IIB in that of type IIA.

The present section is divided into four parts: first of all we discuss some general features of the c map in abstract terms. In the second part we discuss some supergravity aspects of the c map. In the third part we shall show that the c map constructed by supergravity arguments is in fact the mapping between type IIA and type IIB superstring. This third part will consist of a purely stringy construction of the c map, using only general abstract properties of the (2, 2) superconformal system. Finally, in the fourth part we shall discuss some mathematical property of the c map, in particular its relations with some results already present in the mathematical literature.

2.3.1 *Generalities about the c-map*

First of all, we recall the relation between the two low-energy theories. There is a universal sector which is the same for type IIA and IIB which corresponds to the operator 1 in the internal superconformal theory. This sector contains the gravitational supermultiplet plus the universal hypermultiplet. In type IIA strings the operators connected to the ones in Eq. (1.17) by spectral flow lead to vector multiplets and those related to the representations in Eq. (1.18) to hypermultiplets (that is matter multiplets). In type IIB case the role of the two representations gets interchanged, as it is obvious from the GOS projection. We shall denote the number of representations ($h = \bar{h} = 1/2$, $q = -\bar{q} = 1$) by $h_{1,1}$ and that of ($h = \bar{h} = 1/2$, $q = \bar{q} = 1$) with $h_{2,1}$ by analogy with the Calabi-Yau case. Then, in type IIA we have $h_{1,1}$ vector multiplets (i.e. $h_{1,1} + 1$ vectors, keeping into account the "graviphoton") and $h_{2,1} + 1$ hypermultiplets. In type IIB we have $h_{2,1}$ vector multiplets ($h_{2,1} + 1$ vectors) and $h_{1,1} + 1$ hypermultiplets.

Since in $N = 2$ supergravity the scalars of the vector multiplets parametrize a Kahler manifold of a certain restricted type (the geometry of these spaces is described in Appendix A) and the scalars of the hypermultiplets a quaternionic manifold (with negative scalar curvature),[19] roughly the c map can be seen as sending a restricted-type Kahler manifold of complex dimension $h_{1,1}$ into a quaternionic manifold of real dimension $4(h_{1,1} + 1)$ and a quaternionic manifold of dimension $4(h_{2,1} + 1)$ into a restricted-type Kahler manifold of dimension $h_{2,1}$.

To make this idea more precise, we have to define properly the base and target spaces for the c map. First of all, the specification of the geometry of the vector multiplet couplings requires more than just a Kahler manifold and metric. We have also to specify a kind of "canonical" structure of the scalars' manifold, which corresponds to specify the geometry of the vectors' kinetic terms. This "canonical" structure as well as the Kahler metric are encoded in a holomorphic function of $n + 1$ complex variables $F(X^0, X^A)$ ($A = 1, \ldots, n$) which should be homogeneous of degree 2[7,8] (see Appendix A). However, two different functions F_1 and F_2 may correspond to physically equivalent couplings: as shown in Appendix A, two physically equivalent F's are obtained one from the other by an $\mathrm{Sp}(2n + 2, R)$ transformation. Denoting by F_{n+1} the space of all homogeneous holomorphic functions in $n + 1$ variables (degree 2), the space of the inequivalent couplings of n vector multiplets to $N = 2$ SUGRA is

$$K_n = \frac{\mathbb{F}_{n+1}}{\mathrm{Sp}(2n + 2, \mathbb{R})} . \tag{2.46}$$

Notice that $\mathrm{Sp}(2n + 2, R)$ does not act freely on F_{n+1}; the fixed points correspond precisely to those SUGRA models having a generalized duality invariance, (in particular, isometries of the scalars' Kahler space). So the space K_n is a kind of orbifold space. The Peccei-Quinn symmetry of Sec. 2.2 thus implies that the low-energy effective Lagrangians for a 4D IIA superstring always corresponds to a "conical singularity" of the space of all $N = 2$ SUGRA models.

Analogously, we shall denote by Q_n the space of physically inequivalent couplings of n hypermultiplets to $N = 2$ SUGRA. By a Theorem by Bagger and Witten,[19] Q_n is the space of $4n$ manifolds equipped with a quaternionic metric normalized as

$$R = -8(n^2 + 2n) \tag{2.47}$$

identified up to differmorphisms. By a corollary of Alekseevskii (Ref. 47), Eq. (2.47) implies that these quaternionic manifolds are all *irreducible* (as Riemannian spaces). Equation (2.47) is equivalent to the statement that the reduced curvature[15] $v = -2$, independent of the dimension of the space.

Then the c map has the general form

$$c: K_{h_{1,1}} \otimes Q_{h_{2,1}+1} \to K_{h_{2,1}} \otimes Q_{h_{1,1}+1}\,. \tag{2.48}$$

However, as we shall see below, the fundamental operation is not c but s_n

$$s_n: K_n \to Q_{n+1} \tag{2.49}$$

mapping a restricted Kahler n manifold into a quaternionic space with $\dim_R = 4(n+1)$.

The image of the map s_n is some proper subset of the space Q_{n+1}, defined above. Let

$$\tilde{Q}_{n+1} \equiv \mathrm{Im}(s_n) \subset Q_{n+1}\,. \tag{2.50}$$

The special quaternionic metrics belonging to $\tilde{Q}_{n+1}$ will be called *dual-quaternionic*. The reason for this name will be clear from the construction of the s_n map in Sec. 2.3.2.

In $\tilde{Q}_{n+1}$ we can introduce an inverse to s_n. In fact s_n is injective in K_n (but not in F_{n+1}),

$$s_n^{-1}: \tilde{Q}_{n+1} \to K_n\,. \tag{2.51}$$

Thus s_n is an isomorphism between the two spaces K_n and $\tilde{Q}_{n+1}$.

We shall prove below that in type II superstrings the hypermultiplets should parametrize dual-quaternionic spaces. Then the low-energy effective Lagrangian is specified by a point in the direct product spaces

$$K_{h_{1,1}} \otimes \tilde{Q}_{h_{2,1}+1} \qquad \text{for IIA} \tag{2.52a}$$

$$\tilde{Q}_{h_{1,1}+1} \otimes K_{h_{2,1}} \qquad \text{for IIB} \tag{2.52b}$$

Acting on these spaces, the c and c^* maps read

$$c = s_{h_{1,1}} \otimes s_{h_{2,1}}^{-1} \tag{2.53a}$$

$$c^* = s_{h_{1,1}}^{-1} \otimes s_{h_{2,1}}\,. \tag{2.53b}$$

Notice that we can use the isomorphism s_n to specify the metric of a quaternionic space in $\tilde{Q}_{n+1}$ in terms of a homogeneous holomorphic function of $n + 1$ complex variables. This fact was not known before.

Then the $N = 2$ supergravity emerging in the low-energy limit of a 4D type II superstring is completely specified (up to field re-definitions) by two homogeneous (degree 2) holomorphic functions:

$$F_A(X^0, X^1, \ldots, X^{h_{1,1}}) \qquad \text{and} \qquad F_B(Y^0, Y^1, \ldots, Y^{h_{2,1}}). \tag{2.54}$$

In type IIA, F_A describes the vector-multiplet couplings and F_B the hypermultiplet couplings, whereas in type IIB they get interchanged. F_A and F_B are the two functions of the standard $N = 2$ tensor calculus[7,8] describing the couplings of the vector multiplets in the IIA and IIB superstring, respectively. Notice that the complex variable Y^0 in Eq. (2.54) is (in IIA) associated to the universal hypermultiplet and X^0 to the "graviphoton".

2.3.2 *Macroscopic arguments*

Now we shall construct the map s_n transforming the (restricted) Kahler metrics into the quaternionic ones. In this subsection the arguments will be based mainly on space-time supersymmetry. In Subsec. 2.3.3 we shall *recover* the same results by microscopic superconformal considerations.

To appreciate the deep geometrical nature of the map s_n, as well as to make contact with previous work, it is convenient to start with the simplified case in which the vectors' Kahler manifold is a symmetric space, and hence a bounded domain equipped with the Bergman metric. Once we have understood the s_n map in the symmetric case, the generalization to an arbitrary Kahler space (of the type described in Appendix A) will be quite easy.

Let us start with the simplified model we studied in Sec. 2.1, which has $h_{1,1} = 1$ and $h_{2,1} = 0$. For the type IIA string we have the one-dimensional space SU(1, 1)/U(1) for the vector multiplet and the quaternionic coset SU(2, 1)/U(2) for the universal hypermultiplet.

In the IIB case, we have no vector multiplets and the two hypermultiplets parametrize the quaternionic space $G_{2(+2)}/SO(4)$.

At first, it may seem that no simple relation exists between the two spaces SU(1, 1)/U(1) and $G_{2(+2)}/SO(4)$. However a very simple and quite general relation is present. To make the connection even more evident, let us go to the next example of Sec. 3, i.e. the low-energy effective Lagrangian for the compactification on the Z_3 orbifold[28] (we neglect the massless scalars coming from the twisted sectors). In this case we have

$$h^{(0)}_{1,1} = 9 \qquad h^{(0)}_{2,1} = 0 \tag{2.55}$$

where the superscript (0) means that only the "harmonic forms" from the untwisted sector are considered.

In type IIA, the nine vectors live on the Kahler space

$$\frac{U(3,3)}{[U(3) \otimes U(3)]} \tag{2.56}$$

whereas in the type IIB case, the ten hypermultiplets parametrize

$$\frac{E_{6(+2)}}{[SU(2) \otimes SU(6)]} \tag{2.57}$$

(the details of the low-energy effective Lagrangians for the type II superstrings compactified on the Coxeter[29] orbifolds are presented in Sec. 3).

The relation between the Kahler space $U(3,3)/U(3) \times U(3)$ and the quaternionic space $E_{6(+2)}/SU(2) \times SU(6)$ is transparent and well-known.[21] They are the Kahler and the quaternionic spaces which in the Magic-Square[14] are associated with the same Jordan algebra, namely $J^C{}_3$, the Jordan algebra of the hermitian 3×3 complex matrices.

As we have mentioned in Sec. 2.2, all the symmetric Kahler manifolds described by a function of the form

$$F(X^0, X^A) = id_{ABC} X^A X^B X^C (X^0)^{-1} \tag{2.58}$$

are associated to a Jordan-like algebra with cubic norm[10] (for $J^C{}_3$ the norm is just the determinant). This association is, in fact, a one-to-one correspondence.

Thus, in the particular case of a symmetric Kahlerian sigma-model of the type (2.58) the c map just replaces the given Kahler space with the quaternionic space associated to the same Jordan algebra. In particular, for the "magical" cases this just amounts to go to the next element in each column of the magical-square of Freundenthal, Rozenfeld and Tits.[14] This fact was noticed some time ago by Gunaydin, Sierra and Townsend,[21] in a different context. Adding to the magic-square a column for the field itself (viewed as a Jordan algebra) we just get the map

$$SU(1,1)/U(1) \to G_{2(+2)}/SO(4) \tag{2.59}$$

corresponding to the simplified model discussed in Sec. 2.1. Indeed, both spaces in Eq. (2.59) are associated with the Jordan algebra R.

For future reference, we mention that the Kahler manifold $SO^*(12)/U(6)$ of complex dimension 15 is sent by the s_{15} map into the quaternionic space

$$E_{7(-5)}/[SO(12) \otimes SU(2)] \tag{2.60}$$

of real dimension $64 = 4(15 + 1)$. In fact, both spaces are associated to the Jordan algebra $J^H{}_3$, the algebra of hermitian quaternionic 3×3 matrices.

An even deeper geometrical understanding of the c map for *homogeneous* (but not necessarily symmetric) manifolds can be obtained by exploiting the relations between the homogeneous quaternionic spaces and Kahlerian normal algebras having Q representations, as described in Ref. 15. We shall discuss these geometrical developments in Sec. 2.3.4.

The c map for all the symmetric Kahler spaces allowed in $N = 2$ SUGRA[20] is discussed in Appendix C from an elementary point of view. There also we prove that the quaternionic spaces which are the c image of the "minimal coupling" models are in fact, Kahler spaces as well.

Now we are ready to describe the c map (more precisely the s_n map) for any Kahler metric allowed in $N = 2$ SUGRA, whether or not it satisfies the stringy constraints of Sec. 2.2. Here we describe the c map in the context of supergravity; in Sec. 2.3.3 we shall prove by string considerations that: (i) it is the map transforming the IIA effective Lagrangian into the IIB one (and vice versa); (ii) in both the IIA and IIB cases, the hypermultiplets parametrize dual-quaternionic manifolds.

The c map was, essentially, already constructed by the authors of Ref. 21, where the "magical" situations are discussed in detail. The idea is the following.[21] Consider the given 4D $N = 2$ SUGRA. If we reduce it dimensionally to 3 dimensions we get a 3D $N = 4$ supergravity. Under this dimensional reduction the hypermultiplet sector just produces a 3D hypermultiplet system which parametrizes the same quaternionic manifold as in the original 4D theory. Instead, reducing the vector multiplet sector we get one extra scalar for each multiplet from the fourth component of the vector ${A^A}_4$. Moreover, in three dimensions the Abelian vector A_μ^A is dually equivalent to a real scalar. Hence, after the duality transformation the (bosonic part of the) vector-multiplet sector also reduces to a purely scalar theory, with $4n + 4$ degrees of freedom, if we started with n vector-multiplets. The additional four scalars come from the compactification of the gravitational sector, and can be seen as the contribution from the "graviphoton multiplet". These $4n + 4$ scalars should parametrize a quaternionic manifold since in three dimensions local $N = 4$ supersymmetry requires the sigma-model to be quaternionic. However, the situation in three dimensions is quite different from that in 4D $N = 2$ SUGRA. In this last case, the hypermultiplets' scalars live on a quaternionic manifold of prescribed *negative* curvature.[19] By Ref. 47, this means that we have an irreducible σ model. Instead, in the 3D $N = 4$ case we have two distinct classes of "hypermultiplets": those obtained from the 4D hypermultiplets and those from the 4D vector-multiplets. Then, for the scalars' σ model we have the *weaker* condition that the hypermultiplets of each class parametrize an *independent* quaternionic manifold with $v = -2$.

The resulting 3D $N = 4$ SUGRA is based on the product of two irreducible quaternionic manifolds

$$m_1 \otimes m_2$$

(for a discussion of this product structure, and explicit examples, see also Sierra, Ref. 30.) where m_1 is the 4D hypermultiplets' manifold and m_2 comes from the 4D vector-multiplets.

Then the s_n map associates to the original 4D vector-multiplets' self-couplings (represented by a point $k \in K_n$) the quaternionic space m_2 we constructed above

$$s_n : k \longmapsto m_2(k).$$

Comparing with Eq. (2.53) we see that the c map (i.e. the interchange IIA with IIB) just corresponds to the interchange

$$m_1 \leftrightarrow m_2 .$$

We shall prove this by $2d$ superconformal methods in Sec. 2.3.3 below.

Notice that this transformation is mathematically well-defined since each space m_i is irreducible,[47] and thus the decomposition of the σ model space into two quaternionic factors is *unique*.

By definition, the class of quaternionic metrics that can be obtained by this procedure of dimensional reduction and duality transformations is the space $\tilde{Q}_{n+1}$ of *dual-quaternionic* metrics. It is obvious that not all the quaternionic metrics are dual, since a dual-quaternionic metric has at least $2(n+1)$ commuting noncompact isometries corresponding to the constant translations of $A^I{}_4$ and of Φ_I (the scalar field dual to $F^I{}_{\mu\nu}$). These noncompact isometries should correspond to a nilpotent subalgebra of the Lie Algebra generated by the Killing vectors of the given quaternionic manifold. Thus, the condition of being a dual quaternionic space is rather restrictive and highly nontrivial. Besides these symmetries the dual quaternionic manifolds have also other special properties (see Sec. 2.3.4). Indeed, the quaternionic hyperbolic spaces

$$HH^n = \mathrm{Sp}(n,1)/[\mathrm{Sp}(n) \times \mathrm{Sp}(1)] \tag{2.61}$$

(which correspond to the *minimal* coupling of n hypermultiplets to $N = 2$ SUGRA) are not dual-quaternionic spaces, although they are the most symmetric quaternionic spaces. On the other hand, all the other noncompact Wolf spaces[22] *are* dual-quaternionic, as it is shown in Sec. 2.3.4, and by different methods in Appendix C.

This result, together with claim (ii) above, gives as a corollary that in a type II superstring, compactified on a (2, 2) superconformal theory, *minimal* coupling of the hypermultiplets is forbidden.

In practice the c map has the following structure. We start in 4D by giving a function $F(X^0, X^A)$ $A = 1, \ldots, n$, which specifies the original 4D $N = 2$ SUGRA coupled to n vector-multiplets. By the above procedure of dimensional reduction followed by a duality transformation, we get a quaternionic manifold of dimension $4(n+1)$. This map associating a quaternionic metric to a function F is the explicit s_n map we are looking for.

To write down explicitly the quaternionic metric associated to each function F is a bit cumbersome and the explicit formula will be given elsewhere. However, in order to give an idea of the explicit form of the c map, we shall present it here in the limit $k^2 \to 0$, that is, in *rigid* $N = 2$ supersymmetry.

In the rigid case the hypermultiplets live on a hyperkahler space rather than in a quaternionic one. So the rigid s_n operation maps Kahler spaces (of the restricted type allowed in rigid $N = 2$ SUSY, see Appendix A) into hyperkahler ones. Moreover, since the four scalars coming from the gravitational sector decouple as $k^2 \to 0$, s_n maps a manifold of complex dimension n into a manifold of (real) dimension $4n$.

The Kahler metrics that are allowed in rigid $N = 2$ SUSY have the following structure. One starts from a holomorphic function $F(Z^A)$ of the vector-multiplets' scalars. The Kahler potential $G(Z^A, \overline{Z^B})$ is defined by

$$G(Z^A, Z^B) = F_A(Z)\overline{Z^A} + \bar{F}_A(\bar{Z})Z^A \tag{2.62}$$

so that the Kahler metric is

$$G_{A\bar{B}} = (F_{AB} + \bar{F}_{AB}) \tag{2.63}$$

i.e. real-symmetric and with harmonic components.

The hypermultiplets' σ model, being hyperkahler, is in particular Kahler. Then the most convenient way to represent s_n, in the rigid case, is as an operation that to each holomorphic function $F(Z^A)$ associates a Kahler potential $K(Z, W, \bar{Z}, \overline{W})$ (depending on the $2n$ complex coordinates (Z^A, W_A), $A = 1, \ldots, n$) such that $K_{\alpha\bar{\beta}}$ is a metric of $\mathrm{Sp}(n)$ holonomy. By the procedure of dimensional reduction followed by a duality transformation, we get

$$K(Z, W, \bar{Z}, \overline{W}) = G(Z^A, \overline{Z^B}) + G^{A\bar{B}}(Z, \bar{Z})(W_A + \overline{W}_A)(W_B + \overline{W}_B) \tag{2.64}$$

where $G^{A\bar{B}}(Z, \bar{Z})$ is the inverse of the metric $G_{A\bar{B}}$, Eq. (2.63). For any holomorphic function $F(Z)$, Eqs. (2.62)–(2.64) give the Kahler potential of a metric with $\mathrm{Sp}(n)$ holonomy. In particular, as it is easy to check, the complex Monge-Ampere equation is fulfilled

$$\det(K_{\alpha\bar{\beta}}) = 2^n. \tag{2.65}$$

The proof that $K_{\alpha\bar{\beta}}$ is indeed hyperkahler, as well as the details of the computations of Eq. (2.64) are presented in Appendix B.

Equation (2.64) is rather similar to the Calabi Ansatz[30] for the hyperkahler metrics on cotangent bundles

$$K = f[G + G^{A\bar{B}}W_A\overline{W}_B] \tag{2.66}$$

which explains why we call the transformation $G \to K$ the c map.

By the same argument as in the local case, Eq. (2.64) cannot correspond to the *general* hyperkahler metric. Indeed, the metric $K_{\alpha\bar{\beta}}$ should have $2n$ noncompact commuting isometries. Their explicit form is

$$\delta W_A = F_{AB}(Z)k^B + ih_A \tag{2.67}$$

where k^A, h_A are *real* parameters. Under the above transformation, the Kahler potential K changes as

$$\delta K = 2k^A(W_A + \overline{W}_A) \tag{2.68}$$

which is, manifestly, a Kahler transformation.

2.3.3 *The c map at the microscopic string level*

It remains to prove the claims (i) and (ii) above. The most interesting aspects of the c map are related to the R-R states. The vertices for the R-R vectors are of the form (to fix the ideas, we specialize in the IIA case)

$$V^{A;\mu}_{(-1/2,-1/2)}(k,z) = 2e^{-1/2(\phi+\tilde{\phi})}k_\nu[S^\alpha(\sigma^{\mu\nu})_\alpha{}^\beta\tilde{S}_\beta\Sigma^A(z,\bar{z}) + S_{\dot{\alpha}}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\tilde{S}^{\dot{\beta}}\bar{\Sigma}^A(z,\bar{z})]\exp(ik\cdot X) \tag{2.69}$$

where S_α, $S^{\dot{\beta}}$ (respectively $\tilde{S}_\alpha$, $\tilde{S}^{\dot{\beta}}$) are left-moving (respectively right-moving) spin-fields for the space-time SO(4) current-Algebra (with definite chirality). The operators $\Sigma^A(z,\bar{z})$ are the ($h = \bar{h} = 3/8$, $q = -\bar{q} = -1/2$) (2, 2) superconformal operators which are related by the spectral flow[3] isomorphism to the operators $\Lambda^A(z,\bar{z})$, with ($h = \bar{h} = 1/2$, $q = -\bar{q} = 1$) which we identify with the "abstract" (1, 1) harmonic forms. The corresponding NS-NS vertices for the vector-multiplets' scalars are

$$V^A_{(-1,-1)}(k,z) = e^{-(\phi+\tilde{\phi})}\Lambda^A(z,\bar{z})e^{ik\cdot X} \tag{2.70a}$$

$$V^{\bar{A}}_{(-1,-1)}(k,z) = e^{-(\phi+\tilde{\phi})}\bar{\Lambda}^A(z,\bar{z})e^{ik\cdot X}. \tag{2.70b}$$

For the type IIB theory, compactified on the same (2, 2) superconformal system, the operators $\Lambda^A(z,\bar{z})$ correspond to the hypermultiplets. The vertices for the corresponding $4n$ scalars are those in Eq. (2.70) together with the two R-R vertices

$$W^A_{(-1/2,-1/2)}(k,z) = e^{-1/2(\phi+\tilde{\phi})}k_\mu S^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\tilde{S}^{\dot{\beta}}\Sigma^A(z,\bar{z})e^{ik\cdot X} \tag{2.71a}$$

$$W^{\bar{A}}_{(-1/2,-1/2)}(k,z) = e^{-1/2(\phi+\tilde{\phi})}k_\mu S_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\tilde{S}_\beta\bar{\Sigma}^A(z,\bar{z})e^{ik\cdot X}. \tag{2.71b}$$

For our argument, we shall also need the vertices for the universal sector. The universal R-R states consist of a vector (the graviphoton) and two scalars. The vertex for the graviphoton has the same form as in Eq. (2.69) with $\Sigma^A(z,\bar{z})$ replaced by

$$\Sigma^0(z,\bar{z}) = \exp\left\{\frac{\sqrt{3}}{2}[H(z) - \tilde{H}(\bar{z})]\right\} \qquad \text{for type IIA} \tag{2.72a}$$

$$\Xi^0(z,\bar{z}) = \exp\left\{\frac{\sqrt{3}}{2}[H(z) + \tilde{H}(\bar{z})]\right\} \qquad \text{for type IIB} \tag{2.72b}$$

where the free scalars $H(z)$ and $\tilde{H}(\bar{z})$ are defined in terms of the two superconformal U(1) currents as

$$J(z) = i\sqrt{3}\partial H(z) \tag{2.73a}$$

$$\tilde{J}(\bar{z}) = i\sqrt{3}\bar{\partial}\tilde{H}(\bar{z}). \tag{2.73b}$$

The two R-R scalars of the universal hypermultiplet have vertices of the form (2.71) with $\Sigma^A(z,\bar{z})$ replaced by $\Xi^0(z,\bar{z})$ (Eq. (2.72b)) for the IIA superstring and by $\Sigma^0(z,\bar{z})$ for the IIB case.

Finally, the NS-NS states of the universal sector are the graviton and the remaining two scalars of the universal hypermultiplet. Their vertices correspond to the symmetric traceless, antisymmetric, and trace parts of

$$e^{-(\phi+\tilde{\phi})}\psi_\mu(z)\tilde{\psi}_\nu(\bar{z})e^{ik\cdot X}. \tag{2.74}$$

Now we compactify trivially the two superstrings, types IIA and IIB, to three dimensions. In the massless sector, the vertices of the resulting 3D superstrings are just those of the 4D case with $k_4 = 0$.

The vector vertex in Eq. (2.69) splits as 3 + 1 under the Euclidean group SO(3)

$$V^{A,4}_{(-1/2,-1/2)}(k,z) = 2e^{-1/2(\phi+\tilde{\phi})}k_i[S^\alpha(\sigma^i)_\alpha{}^\beta\tilde{S}_\beta\Sigma^A - S_{\dot{\alpha}}(\sigma^i)^{\dot{\alpha}}{}_{\dot{\beta}}\tilde{S}^{\dot{\beta}}\bar{\Sigma}^A]e^{ik\cdot X} \tag{2.75a}$$

$$V^{A,i}_{(-1/2,-1/2)}(k,z) = 2e^{-1/2(\phi+\tilde{\phi})}\varepsilon^{ijk}k_j[S^\alpha(\sigma^k)_\alpha{}^\beta\tilde{S}_\beta\Sigma^A + S_{\dot{\alpha}}(\sigma^k)^{\dot{\alpha}}{}_{\dot{\beta}}\tilde{S}^{\dot{\beta}}\bar{\Sigma}^A]e^{ik\cdot X} \tag{2.75b}$$

(here $i, j, k = 1, 2, 3$). The peculiar structure of this last R-R vertex allows us to extract directly the vertex for the dual scalar particle (by the same argument we have used in Eqs. (1.5)–(1.7)).

$$V^{A,\mathrm{DUAL}}_{(-1/2,-1/2)}(k,z) = 2e^{-1/2(\phi+\tilde{\phi})}k_i[S^\alpha(\sigma^i)_\alpha{}^\beta\tilde{S}_\beta\Sigma^A + S_{\dot{\alpha}}(\sigma^i)^{\dot{\alpha}}{}_{\dot{\beta}}\tilde{S}^{\dot{\beta}}\bar{\Sigma}^A]e^{ik\cdot X} \tag{2.76}$$

By comparing Eqs. (2.71a)–(2.71b) (with $k_4 = 0$) with Eqs. (2.75a) and (2.76), we see that the transformation

$$\tilde{S}_\beta(\bar{z}) \leftrightarrow \tilde{S}^{\dot{\beta}}(\bar{z}) \tag{2.77}$$

($\beta = \dot{\beta}$ in value) maps the two vertices obtained from a vector vertex of the IIA superstring, Eqs. (2.75a)–(2.76), into the real and imaginary parts of the vertices for the two R-R scalars of the corresponding hypermultiplet in the type IIB theory.

Now, Eq. (2.77) is in fact an isomorphism of the SO(3) current algebra, since the two sides of Eq. (2.77) transform the same way under the SO(3) subgroup of SO(4). Moreover, in the bosonized version of the theory the SO(4) spin-fields read[11,12]

$$\tilde{S}_\alpha(\bar{z}) \sim \exp\{\pm\tfrac{1}{2}[\tilde{\phi}_1(\bar{z}) + \tilde{\phi}_2(\bar{z})]\}\ \text{cocycle} \tag{2.78a}$$

$$\tilde{S}^{\dot{\alpha}}(\bar{z}) \sim \exp\{\pm\tfrac{1}{2}[\tilde{\phi}_1(\bar{z}) - \tilde{\phi}_2(\bar{z})]\}\ \text{cocycle}. \tag{2.78b}$$

Therefore, the interchange in Eq. (2.77) corresponds in the bosonic language to the change of sign in the free-field $\tilde{\phi}_2(\bar{z})$

$$\tilde{\phi}_2(\bar{z}) \leftrightarrow -\tilde{\phi}_2(\bar{z}) \tag{2.79}$$

which is obviously an automorphism of the $2d$ superconformal theory. Thus the map (2.77), being an invariance of the underlying $2d$ model and leaving invariant the representations of the 3D Poincaré group is, in particular, an invariance of the 3D low-energy effective supergravity theory.

On the other hand, Eq. (2.77) also corresponds to the replacement of IIA with IIB in three dimensions. We have already seen this for the R-R vectors of the IIA string (which go into the hypermultiplets' R-R scalars of the IIB string). For the scalars of these vector-multiplets (the $(1,1)$ "forms") this is also true since the operators in Eqs. (2.70) are the vertices for both the vectors' NS-NS scalars of the IIA string and the hypermultiplets' NS-NS scalars of the IIB case. And, obviously, $V^A{}_{(-1,-1)}(k,z)$ is invariant under (2.77).

For the hypermultiplets of the IIA string, Eqs. (2.70) and (2.71) still hold, with the replacements

$$\Lambda^A(z,\bar{z}) \leftrightarrow \Xi^a(z,\bar{z})$$

$$\Sigma^A(z,\bar{z}) \leftrightarrow \Pi^a(z,\bar{z})$$

($a = 1,\ldots,h_{2,1}$) where

$$\Xi^a(z,\bar{z}) \qquad \text{has} \qquad (h = \bar{h} = 1/2, q = \bar{q} = 1) \tag{2.80a}$$

$$\Pi^a(z,\bar{z}) \qquad \text{has} \qquad (h = \bar{h} = 3/8, q = \bar{q} = -1/2) \tag{2.80b}$$

are the "abstract" $(2,1)$ harmonic forms and their spectral flow partners. With the same replacements, Eqs. (2.70) and (2.69) become the vertices for the vector multiplet bosons for the type IIB superstring. Again, we see that Eq. (2.77) interchanges the IIA and IIB vertices also in the sector corresponding to the "(2,1) harmonic forms".

It remains to see how Eq. (2.77) works in the universal sector. In the R-R subsector it interchanges the two scalars from the reduction of the graviphoton with the two R-R scalars of the universal hypermultiplet. To see this, we have just to specialize the above analysis to the operators in Eq. (2.72).

For the NS-NS vertices of Eq. (2.74), we first notice that Eq. (2.79) implies

$$\tilde{\psi}_4(\bar{z}) \leftrightarrow -\tilde{\psi}_4(\bar{z}) \tag{2.81}$$

which for the background fields means

$$g_{i4} \leftrightarrow b_{i4} \tag{2.82a}$$

$$g_{44} \leftrightarrow \phi \tag{2.82b}$$

(Eq. (2.82b) is most easily understood in terms of light-cone vertices).

Equation (2.82) states that Eq. (2.77) maps the two *NS-NS* scalars of the universal hypermultiplet, ϕ and b_{i4}, into the two scalars coming from the reduction of the metric, g_{44} and g_{i4}, the 3D SUSY partners of the two *R-R* scalars produced by the reduction of the graviphoton.

In conclusion, we have shown that the transformation in Eq. (2.77) is an invariance of the low-energy effective Lagrangian, and that it maps the IIA superstring into the IIB, and vice versa.

Moreover, the explicit action of this transformation on the various vertices shows that, as an invariance of the low-energy effective Lagrangian, it corresponds just to the permutation

$$\mathscr{M}_1 \leftrightarrow \mathscr{M}_2 \tag{2.83}$$

of the two factor manifolds in the hypermultiplets' 3D *s* model, where $\mathscr{M}_1$ and $\mathscr{M}_2$ are defined as in Sec. 2.3.2. The transformation (2.83) is, by definition, the *c* map we introduced before.

Then, we see that the microscopic isomorphism of the 2*d* superconformal theory, $\tilde{S}_\alpha(\bar{z}) \leftrightarrow \tilde{S}^{\dot{\alpha}}(\bar{z})$ acts at the level of the low-energy effective Lagrangian by the *c* map we have constructed above by SUGRA methods.

But we have seen that the interchange $\tilde{S}_\alpha(\bar{z}) \leftrightarrow \tilde{S}^{\dot{\alpha}}(\bar{z})$ is just the same thing as IIA $\leftrightarrow$ IIB. This, together with the fact that $\mathscr{M}_1$ is the same quaternionic space (and metric) as in the 4D hypermultiplets' σ model, proves our claim that the *c* map we defined above permutes the effective Lagrangians for the two type II four-dimensional superstrings.

The fact that only dual quaternionic manifolds are allowed for the hypermultiplets of a type II superstring (compactified on a (2, 2) system) follows as an obvious corollary. However, it should be emphasized that this is a very strong constraint on the possible low-energy effective Lagrangians. We shall return to the problem of a complete characterization of the dual-quaternionic metrics in a future paper.

2.3.4 *Mathematical (and physical) properties of the c map*

Now we describe some further mathematical property of the *c* map. For lack of space, we shall be rather sketchy, giving only the main ideas. The main purpose of this section is to elucidate the connections between the *c* map and some work by D. V. Alekseevskii.[15] This will give us a better understanding of what is going on, as well as further characterization of the dual-quaternionic manifolds. As a by-product, we shall also get some *new* results in $N = 2$ SUGRA, and a simpler (and by far deeper) proof of some old ones, as the Cremmer-van Proeyen theorem[20] on the symmetric Kahler manifolds allowed in $N = 2$ SUGRA.

In his classical paper on the classification of the quaternionic spaces with a transitive solvable group of motions (normal quaternionic spaces) Alekseevskii[15] characterizes completely the *normal* dual-quaternionic spaces and constructs explicitly their image under the inverse *s* map. The corresponding Kahler spaces are also normal. The Alekseevskii approach is not only deeper but also more general than the one based on

Jordan algebras, since it furnishes the c map also for the homogeneous, but not symmetric, spaces.

Let us recall his results. The normal Riemannian spaces are in one-to-one correspondence with the normal metric Lie algebras (up to isomorphism). The space is quaternionic if and only if the algebra is (see Ref. 15 for the precise definition). The (normal) quaternionic algebras are divided into two classes according to their canonical quaternionic subalgebra is $C^1{}_1$ or $A^1{}_1$. In more down-to-earth terms, the two classes of normal quaternionic manifolds are specified by having as a canonical totally geodesic submanifold either the quaternionic manifold

$$HH^1 = \mathrm{Sp}(1,1)/[\mathrm{Sp}(1) \otimes \mathrm{Sp}(1)] \tag{2.84}$$

or

$$\mathrm{SU}(2,1)/[\mathrm{U}(1) \otimes \mathrm{SU}(2)]. \tag{2.85}$$

In physical terms, the normal quaternionic manifolds having the canonical subalgebra $A^1{}_1$ are just those quaternionic manifolds which are consistent with the existence of a distinguished hypermultiplet—the *universal* one—which parametrizes the coset (2.85), as we found in Sec. 2.1. In other words, the canonical subalgebra is related, in a way to be specified below, to the *physical* symmetries associated to the fields ϕ and $B_{\mu\nu}$ (in the heterotic case dilatation and R invariance).

Thus, the normal quaternionic manifolds of the second class have just the properties we expect, on physical grounds, for a dual-quaternionic space. Indeed, they *are* dual-quaternionic, as we shall show in a moment.

Instead, those of the first class cannot be dual quaternionic. In Ref. 15 it is shown that the only spaces of this class are

$$HH^n = \mathrm{Sp}(n,1)/[\mathrm{Sp}(n) \otimes \mathrm{Sp}(1)]. \tag{2.86}$$

Thus, HH^n cannot appear in the low-energy Lagrangian for a type II superstring.

The most important result of Ref. 15 (for our present purposes) is the classification of the quaternionic algebras with canonical subalgebra $A^1{}_1$. They are associated in a natural way to Kahlerian normal Lie algebras (more precisely, to the Q representations of such algebras). The association is one-to-one. Moreover, the Kahlerian normal algebras are in one-to-one correspondence with normal Kahler spaces. Therefore, a normal quaternionic manifold, other than HH^n, is associated to a normal Kahler manifold (of a special type).

Moreover, the Kahler algebra is in fact a subalgebra (called the principal Kahler subalgebra) of the original quaternionic algebra. This implies that the Kahler manifold is in fact a submanifold of the associated quaternionic space. In fact it is a *totally geodesic submanifold.*[15]

The Kahlerian algebra of a normal quaternionic space has a special structure.

In particular, it should be a *direct* sum $F_0 + W$ where the two dimensional subalgebra F_0 is the Kahler algebra associated with the universal quaternionic submanifold $SU(2,1)/[SU(2) \times U(1)]$. The Kahler space corresponding to F_0 is the coset $SU(1,1)/U(1)$, with metric normalized to $R = -2$, i.e. it is the *heterotic* universal sector, with the Kahler potential as in Eq. (2.2). It is just the space parametrized by ϕ and the "axion" b (dual to $B_{\mu\nu}$). This relation with the heterotic string is not casual; this point will be clearer in Sec. 3 where we explain the relations between the low-energy effective Lagrangians for the type II and heterotic strings compactified on the *same* (2, 2) system. [In the heterotic case, the universal sector just gives an independent factor space of the chiral multiplets' Kahler manifold. This is easily seen from the fact that the Kahler algebra with a Q representation is always a *direct sum*[15] of F_0 with some other Kahler algebra. This cannot happen in the type II case, since the hypermultiplets' space is irreducible. Then, the universal sector plays a much more interesting role for the geometry of type II superstring than in the usual case. Its geometrical meaning is clear from the structure of the c map.]

W is the Kahlerian algebra whose associated Kahler manifold is the inverse image under the s map of the given (normal) quaternionic manifold. In fact, for a quaternionic manifold of dimension $4(n + 1)$ W has (real) dimension $2n$.

Therefore, for a homogeneous Kahler space (admissible in $N = 2$ SUGRA) K_n, we have that the embedding

$$K_n \otimes SU(1,1)/U(1) \hookrightarrow s_n(K_n) \tag{2.87}$$

is *totally geodesic*.

The image of the normal quaternionic spaces (of the second class) under the map s^{-1} gives us all the normal Kahler spaces allowed in $N = 2$ SUGRA. We have that the normal (= having a transitive solvable group of motions) Kahler spaces allowed in $N = 2$ SUGRA are in one-to-one correspondence with the Kahler normal algebras W, such that $U = F_0 + W$ has a Q representation.

From this observation, and the classification of such Kahler algebras in Ref. 15, we learn that the symmetric allowed spaces are just those listed by Cremmer and van Proeyen.[20] However, there are two additional infinite families of homogeneous, but *nonsymmetric* spaces, corresponding to the Alekseevskii spaces $W(p,q)$ and $V(p,q)$ (p, q = integers; for some special values of p and q we get the symmetric spaces related to Jordan algebras).

It is easy (in the symmetric case) to get back the classification in terms of Jordan algebras (more precisely, in terms of the four division algebras). In fact, it turns out that the relevant algebras (for rank 4 normal quaternionic spaces) are classified in terms of Clifford Modules, which in turn are classified with the help of division algebras.[15]

The correspondence Kahler $\leftrightarrow$ Quaternionic spaces as given by the Alekseevskii theory is in all cases (except for the nonsymmetrical ones, which are not studied there) the same we find in Appendix C by simple physicists' arguments (mainly stringy considerations). We have seen that in the normal case the embeddings

$$\mathrm{SU}(1,1)/\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2,1)/\mathrm{U}(2) \hookrightarrow s_n(K_n) \tag{2.88}$$

$$K_n \otimes \mathrm{SU}(1,1)/\mathrm{U}(1) \hookrightarrow s_n(K_n) \tag{2.89}$$

are totally geodesic. We finish this section by showing that this is true also in the *general* case. This is an important piece of information for the full characterization of dual-quaternionic manifolds.

We give a physicist' proof. First of all, physical consistency requires K_n to be a complete (Kahlerian) manifold. Since the superstring is belived to be a consistent theory, we assume completeness. Then we have only to show that under the above (isometric) embeddings the geodesics are mapped into geodesics of the quaternionic space $s_n(K_n)$.

The idea of the argument is the following: if we have a σ model

$$\mathscr{L} = g_{ij}(X)\partial_\mu X^i \partial_\mu X^j, \tag{2.90}$$

it is obvious that the configurations depending on only one coordinate

$$X^i(\tau, \mathbf{X}) \equiv X^i(\tau) \tag{2.91}$$

are solutions of the equations of motion if and only if the curve $X^i(\tau)$ is a geodesic for the Kahler metric of $\mathrm{SU}(1,1)/\mathrm{U}(1) \times K_n$.

Since the equations of motion after the duality transformation are *equivalent* to the original ones, the above geodesics are also a solution of the equations of motion for the resulting quaternionic σ model. Hence, the embedding given by the equation $F^I{}_{\mu\nu} = 0$ (or the corresponding one in the dual formulation) of $\mathrm{SU}(1,1)/\mathrm{U}(1) \times K_n$ into $s_n(K_n)$ is totally geodesic. Similar arguments work for $\mathrm{SU}(2,1)/\mathrm{U}(2)$ (notice that its geodesics are obtained from those of $\mathrm{SU}(1,1)/\mathrm{U}(1)$ by space-time SUSY).

3. Explicit Examples

In this section, we construct the low-energy effective Lagrangians for some simple classes of four-dimensonal type II superstrings just to illustrate the main ideas of the previous sections.

The first class of models we consider are the symmetric Z_N orbifolds[28] obtained by modding out the maximal torus of a semi-simple rank 6 Lie algebra by its Coxeter elements. Such orbifolds were described in Ref. 29 and are listed in Table 1. The generator θ of the Z_N group acts on the internal $2d$ fields as the element

$$\theta = \exp\{2\pi i(aJ_{12} + bJ_{34} + cJ_{56})\} \tag{3.1}$$

of Spin(6), with the coefficients (a,b,c) as listed in Table 1. Obviously, θ leaves two left-moving SO(6) spin fields $S^\pm(z)$ invariant, as well as two right-moving $\tilde{S}^\pm(\bar{z})$. $S^\pm(z)$ (respectively $\tilde{S}^\pm(\bar{z})$) corresponds to the Killing-spinor operators introduced in Ref. 34.

Table 1. Symmetric $\mathbb{Z}_N$ Coxeter orbifolds

Case	Lattice	$\mathbb{Z}_N$	(a,b,c)	N	χ	$h_{1,1}$	$h_{2,1}$	$h^{(0)}_{1,1}$	$h^{(0)}_{2,1}$
1	$[SU(3)]^3$	$\mathbb{Z}_3$	$1/3(1,1,-2)$	27	72	36	0	9	0
2	$SU(3) \times (G_2)^2$	$\mathbb{Z}_6$	$1/6(1,1,-2)$	3	48	29	5	5	0
3	$[SU(4)]^2$	$\mathbb{Z}_4$	$1/4(1,1,-2)$	16	48	31	7	5	1
4	$SU(3) \times SO(8)$ $SU(2) \times SU(6)$	$\mathbb{Z}_6$	$1/6(1,2,-3)$	12	48	35	11	3	1
5	$SO(9) \times SO(4)$ $SO(8) \times SO(4)$	$\mathbb{Z}_8$	$1/8(1,2,-3)$	8	48	28	4	3	1
6	$SO(4) \times F_4$	$\mathbb{Z}_{12}$	$1/12(1,4,-5)$	4	48	31	7	3	1
7	$SO(5) \times SO(9)$ $SO(5) \times SO(8)$	$\mathbb{Z}_8$	$1/8(1,3,-4)$	4	48	27	3	3	0
8	$SU(3) \times SO(8)$ $SU(3) \times F_4$ E_6	$\mathbb{Z}_{12}$	$1/12(1,4,-5)$	3	48	27	3	3	0
9	$SU(7)$	$\mathbb{Z}_7$	$1/7(1,2,-3)$	7	48	24	0	3	0

In Table 1, N is the number of fixed points, χ the Euler characteristic, $h_{1,1}$ and $h_{2,1}$ are the numbers of conformal operators in the two relevant representations of the (2, 2) superconformal algebra, Eqs. (1.17) and (1.18), respectively. Instead, $h^{(0)}{}_{1,1}$ and $h^{(0)}{}_{2,1}$ count the number of harmonic forms from the *untwisted* sector of the theory.

Putting the twisted sector scalars equal to zero (this is consistent with space-time supersymmetry, see, for instance, the discussion in Ref. 35) the resulting low-energy effective Lagrangian for the untwisted sector can be computed just by modding out the $E_{7(+7)}$ duality group by the Coxeter element θ, with Spin(6) embedded in $SU(8) \subset E_{7(+7)}$ as in Eq. (2.5) for the IIA case, and in the flipped embedding for the IIB superstring.

In this way we get the Tables 2 and 3 in which the sigma models for the vector and hypermultiplets are given for all the Coxeter orbifolds. We also list the Jordan algebras associated to the vectors' manifolds of type IIA and the hypermultiplets' manifolds for type IIB. This association with the Jordan algebras is a further check of our general results above, and in particular, of the role of the Peccei-Quinn symmetry of Sec. 2.2 in fixing the scalars' sigma-model geometry.

According to the general philosophy of the moduli spaces, the sigma model metric for the space-time fields corresponding to moduli should not depend on the *particular* string theory we consider. Thus the above results should also contain the scalars' sigma model for the *heterotic* string compactified on the Coxeter orbifolds. In the heterotic case the space-time theory is an $N = 1$ supergravity. The Kahler spaces corresponding to the low-energy theory for the compactification of the heterotic string on these orbifolds was recently studied by Cvetic, Louis and Ovrut.[35] Comparing their table with ours, we get

$$K_H = K_A \times K_B$$

where K_H, K_A, and K_B are respectively the Kahler manifolds for the chiral multiplets

Table 2. Scalars' σ models for the untwisted sector of type IIA string compactified on the orbifolds of Table 1

Case	Vector multiplets' manifold	$\dim_{\mathbb{C}}$	Jordan algebra	Hypermultiplet manifold	dim.
1	$U(3,3)/U[U(3)\times U(3)]$	9	$J_3^{\mathbb{C}}$	$SU(2,1)/U(2)$	4
2	$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,4)}{SO(2)\times SO(4)}$	5	$\mathbb{R}+Q(4)$	$SU(2,1)/U(2)$	4
3	$\frac{SU(1,1)}{U(1)}\times\frac{SO(2,4)}{SO(2)\times SO(4)}$	5	$\mathbb{R}+Q(4)$	$U(2,2)/[U(2)\times U(2)]$	8
4, 5, 6	$[SU(1,1)/U(1)]^3$	3	$\mathbb{R}+\mathbb{R}+\mathbb{R}$	$U(2,2)/[U(2)\times U(2)]$	8
7, 8, 9	$[SU(1,1)/U(1)]^3$	3	$\mathbb{R}+\mathbb{R}+\mathbb{R}$	$SU(2,1)/U(2)$	4

Table 3. Same as Table 2, for type IIB

Case	Vectors' manifold	$\dim_{\mathbb{C}}$	Hypermultiplet manifold	dim.	Jordan algebra
1	—	0	$E_{6(+2)}/[SU(2)\times SU(6)]$	40	$J_3^{\mathbb{C}}$
2	—	0	$SO(4,6)/[SO(4)\times SO(6)]$	24	$\mathbb{R}+Q(4)$
3	$SU(1,1)/U(1)$	1	$SO(4,6)/[SO(4)\times SO(6)]$	24	$\mathbb{R}+Q(4)$
4, 5, 6	$SU(1,1)/U(1)$	1	$SO(4,4)/[SO(4)\times SO(4)]$	16	$\mathbb{R}+\mathbb{R}+\mathbb{R}$
7, 8, 9	—	0	$SO(4,4)/[SO(4)\times SO(4)]$	16	$\mathbb{R}+\mathbb{R}+\mathbb{R}$

in the heterotic case, and for the vector multiplets of type IIA and IIB. This result is, in fact, general: the effective Lagrangian for the heterotic string (in the sector corresponding to the moduli) and the effective Lagrangians for the two type II strings are always related by the simple product formula above. This can be shown directly at the string level. This fact is a nontrivial check of the universality of the metric on the moduli space for the given abstract superconformal theories, and also of the consistency of the *macroscopic* approach to the moduli space. In principle, one can also compute the full Kahler potential for the heterotic string (including the E_6 generations) but we shall not do it here.

Comparing Tables 3 and 2, we see an aspect of the c map which is worth discussing both because it illustrates well our ideas (and the geometry of $N = 2$ supergravity) and because it shows explicitly that the vectors' manifolds of a *type* IIB superstring need not be of the form associated to a *generalized* Jordan algebra. Indeed, consider the cases 2, 4, 7, and 9. In these cases, in the type IIB string we have just one vector multiplet living on the Kahler coset $SU(1,1)/U(1)$. Comparing with Table 2, we see that the c map sends this space into the quaternionic coset $U(2,2)/[U(2)\times U(2)]$. At first this may seem paradoxical, since it would contradict the statement of Sec. 2.3 that the c map sends $SU(1,1)/U(1)$ into the quaternionic space $G_{2(+2)}/SO(4)$.

However, there is no contradiction. Recall that the c map is defined on the spaces K_n which contain more data than just the Kahler metric. These data are encoded in the holomorphic function $F(X^0, X^A)$. Two functions F with the same underlying Kahler manifold, but which are not in the same orbit as the $Sp(2n+2, R)$ group, are mapped under c into two *different* quaternionic metrics. In particular, for $SU(1,1)/U(1)$ we have two (and only two) distinct orbits:

(A) The minimal coupling

$$F(X^0, X^1) = (X^0)^2 - (X^1)^2 \tag{3.2a}$$

(B) The coupling associated to the Jordan Algebra R

$$F(X^0, X^1) = i(X^1)^3/X^0 . \tag{3.2b}$$

Again, using the results of Sec. 2.1, the couplings are uniquely identified (up to field re-definitions) by the transformation properties of the vector field strengths under the duality group SU(1, 1). In particular,[8,20]

(1) For the case (A) the duality group is actually U(1, 1) rather than SU(1, 1) as it is in the case (B).

(2) In the case (A) the four field strengths are in the reducible representation 2 + 2 of SU(1, 1). Instead, in the case (B) they are in the *irreducible* representation 4.

(3) The *radius* of the SU(1, 1)/U(1) manifold is different in the two cases. In fact, the scalar curvature is

$$R = -\frac{1}{s} \tag{3.3}$$

where s is the *spin* of the field strengths under SU(1, 1), $s = 1/2, 3/2$, respectively. (This is related[15] to the two possible values of the root for the Kahlerian algebras which are direct sums of two key algebras, $F_0 + F$, namely $\mu = 1$ and $\mu = (3)^{-1/2}$.)

For the model discussed in Sec. 2.1, the structures on the manifold SU(1, 1)/U(1) are those dictated by the Jordan algebra R; this is implied by the general theorem of Sec. 2.2, and it is also what we found by the explicit computations.

On the other hand, the general theorem does not apply to the cases of Table 3, since they correspond to type IIB superstrings. Then for these cases, *a priori*, both possibilities (A) and (B) are open. An explicit computation (with the particular embedding in $E_{7(+7)}$) shows that the SU(1, 1)/U(1) manifolds of Table 3 do in fact correspond to minimal coupling.

Thus under the c map we have

$$[SU(1,1)/U(1)]_{\text{Minimal}} \to U(2,2)/[U(2) \otimes U(2)] \tag{3.4a}$$

$$[SU(1,1)/U(1)]_{\text{Jordan}} \to G_{2(+2)}/SO(4) . \tag{3.4b}$$

Equation (3.4a) also follows from the fact (proven in Appendix C) that the c images of the "minimal coupling" Kahler manifolds are themselves Kahlerian spaces, and the fact that the Wolf space $U(2,2)/[U(2) \times U(2)]$ is the only noncompact (simply connected) quaternionic manifold (of real dimension 8) which is also a *Kahler* manifold. [See, Appendix C for a discussion]. {Compare also proposition 9.1 of Ref. 15}.

These examples show that the restriction on the possible Kahler metrics we have shown for the type IIA is not true for the type IIB. In fact, even *minimal* coupling is allowed in the IIB case. Below we shall present a IIB model in which the vectors' Kahler manifolds *is* associated to a Jordan algebra.

Conversely, in the type IIB case it is the hypermultiplets' quaternionic metric which is severely restricted, and easily computable from the three-point amplitudes. Putting together the Peccei-Quinn symmetry of Sec. 2.2 and the (partial) characterization of the dual-quaternionic manifolds, we see that the type IIB hypermultiplets' sigma-model has at least

$$3n + 2 \tag{3.5}$$

commuting isometries (without fixed points).

In principle, one could compute the d_{abc} coefficients for all the models in Table 2 from the three-point amplitudes. However, it is convenient to compute them in the following way. Since all the examples are in fact truncations of the $N = 8$, we can compute the d_{abc} coefficients directly in the $N = 8$ case. In order to do this we consider the following consistent truncation of the $N = 8$ theory. Firstly we fix a U(1) subgroup of the superconformal algebra of the internal free theory. Then throw away all the $N = 8$ vectors which are not *R-R* fields, as well as all states whose vertices are related to the *NS-NS* vectors by the spectral flow generated by the chosen U(1) (and its right-moving counterpart). The resulting $N = 2$ theory has 15 vector multiplets which parametrize the "magic" coset SO*(12)/U(6), i.e. the Kahler manifold associated to $J^H{}_3$. Then, we need only to compute the d's for this SO*(12)/U(6) theory, since all the other cases are just truncations of it. Then the d's are computed just from the current algebra of the SO(10) spin-fields, and thus are built out of the six-dimensional γ-matrices[a]. This is just another way to state their relationship with the Clifford algebras we found in Sec. 2.3.4.

All the examples above are just sectors of the theory, and not the complete effective Lagrangians. It is important to work out the *full* effective Lagrangian for some model. One possible strategy, for an arbitrary type IIA superstring is to use the general formula of Sec. 2.2 to compute the complete exact coupling of all the multiplets, including those coming from the exotic sectors of the theory. This method is particularly promising for a Z_p orbifold (with p prime), since, in the IIA case, no hypermultiplet is present (except the universal one) and therefore the analysis of Sec. 2.2 suffices to construct the full nonlinear low-energy Lagrangian for all the (massless) sectors. However, such a detailed study of specific models is beyond our present purposes.

Here we limit ourselves to a 4D type II superstring obtained using the fermionic construction of Ref. 36. One interesting aspect of this model is that it is a fixed point of the c map, in the sense that in this case the effective Lagrangians for type IIA and IIB are just equal.

In the language and conventions of Ref. 36, the model is presented by specifying a

[a] Six-dimensional γ matrices are, in turn, connected to the division algebra H,[33].

basis ξ of the group of spin-structure assignments. Our choice is

$$\xi = \{F, S, \tilde{S}, b_1, b_2\} \tag{3.6}$$

where

$$S = \{\psi_\mu, \chi_1, \ldots, \chi_6\} \tag{3.7a}$$

$$\tilde{S} = \{\tilde{\psi}_\mu, \tilde{\chi}_1, \ldots, \tilde{\chi}_6\} \tag{3.7b}$$

$$\begin{aligned} b_1 = \{&\psi_\mu, \chi_1, \chi_2, y_1, \omega_1, y_3, y_4, y_5, y_6, \\ &\tilde{\psi}_\mu, \tilde{\chi}_1, \tilde{\chi}_2, \tilde{y}_1, \tilde{\omega}_1, \tilde{y}_3, \tilde{y}_4, \tilde{y}_5, \tilde{y}_6\} \end{aligned} \tag{3.7c}$$

$$\begin{aligned} b_2 = \{&\psi_\mu, \omega_1, \omega_2, \chi_3, \chi_4, y_4, \omega_4, y_5, \omega_6, \\ &\tilde{\psi}_\mu, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\chi}_3, \tilde{\chi}_4, \tilde{y}_4, \tilde{\omega}_4, \tilde{y}_5, \tilde{\omega}_6\}. \end{aligned} \tag{3.7d}$$

The resulting string model obviously corresponds to a (2, 2) superconformal theory, with a world-sheet parity invariance. It is also manifestly $N = 2$ supersymmetric in the space-time sense. Moreover, there are no massless states from the "twisted" sectors. Essentially, the above model is the more general (2, 2) superconformal system with this last property that can be constructed using the formalism of Ref. 36. All the other models with these properties either have the same effective Lagrangian as the present model or a truncation of it.

The model has $h_{1,1} = h_{2,1} = 3$ with, in the notations of Sec. 2.3,

$$\Lambda^A(z, \bar{z}) = \chi_A(z)\tilde{\chi}_{\bar{A}}(\bar{z}) \tag{3.8a}$$

$$\Lambda^{\bar{A}}(z, \bar{z}) = \chi_{\bar{A}}(z)\tilde{\chi}_A(\bar{z}) \tag{3.8b}$$

$$\Xi^A(z, \bar{z}) = \chi_A(z)\tilde{\chi}_A(\bar{z}) \tag{3.8c}$$

$$\Xi^{\bar{A}}(z, \bar{z}) = \chi_{\bar{A}}(z)\tilde{\chi}_{\bar{A}}(\bar{z}) \tag{3.8d}$$

where $A = 1, 2, 3$ is an SU(3) index.

For the purpose of the computation of the effective Lagrangian, this fermionic model is equivalent to a $Z_2 \times Z_2$ orbifold with a point group generated by

$$\theta_1 = \exp\{i\pi J_{12}\} \tag{3.9a}$$

$$\theta_2 = \exp\{i\pi J_{34}\}. \tag{3.9b}$$

For such a $Z_2 \times Z_2$ orbifold, flipping the embedding (i.e. interchanging IIA and IIB) does not change anything, and so we have the same effective Lagrangian for both IIA and IIB.

By the same methods as above, one shows that the vector-multiplets' Kahler manifold is

$$[\mathrm{SU}(1,1)/\mathrm{U}(1)]^3 \tag{3.10}$$

which is associated to the Jordan algebra $R + R + R$ (i.e. the Kahlerian algebra[15] $W_4 = F_1 + F_2 + F_3$ where the F_i are key algebras with roots equal to 1), whereas the hypermultiplets' sigma-model is given by the quaternionic coset

$$\mathrm{SO}(4,4)/[\mathrm{SO}(4) \otimes \mathrm{SO}(4)] \tag{3.11}$$

which is associated to the same Jordan algebra (to the Q representation of the algebra $F_0 + W_4$), as it is obvious from the fact that the two coset (3.10) and (3.11) are interchanged by the c map. The fact that these two spaces are related by the c map also follows from the comparison of the Tables 2 and 3, as well as from the general discussion of the c map for symmetric spaces we present in Appendix C, or by proposition 9.2 of Ref. 15.

4. Conclusions: The Moduli Problem

The geometry of the moduli space for a given $2d$ superconformal system is encoded in the low-energy effective Lagrangian of *any* string model that can be constructed using that system as the "internal" sector of the underlying $2d$ theory. In particular, the *exact* metric of the moduli space can be read from the scalars' kinetic terms. The resulting metric is exact although the low-energy effective Lagrangian itself is just an approximation, since we usually neglect the higher derivative and curvature terms (as well as the quantum corrections).

The computation of the geometry of the moduli space from macroscopic considerations is simpler when the given superconformal theory can be used as a background for many (super-)string theories, since then we can use the interplay between the various effective Lagrangians to further constrain the moduli geometry.

In Ref. 2 the above program was completed for the simplest (but nontrivial) case, namely the $(4,4)$ superconformal models with $c = 6$. The natural direction to further carry on the program is, obviously, the next situation in increasing order of difficulty: the $(2,2)$ superconformal systems with $c = 9$. These are also the interesting ones from a "phenomenological" point of view. *Heuristically*, the two situations above correspond to Calabi-Yau spaces[37,38] of (complex) dimension 2 and 3, respectively. However, a Calabi-Yau 2-fold is automatically a hyperkahler manifold, whereas a 3-fold (obviously) cannot be hyperkahler. Since hyperkahler σ models have quite peculiar properties, in the second case new phenomena appear, both at the microscopic and macroscopic levels.

From a macroscopic point of view, the main difference between the two situations is the following. In the (4, 4) case the low-energy theory is essentially *unique* due to the rigidity of the allowed scalars' manifolds in $N = 4$ supergravity. This rigidity may be seen as a consequence of Berger's theorem[39]: since the σ model manifold should have a holonomy which is more special than that of a hyperkahler or of a quaternionic manifold (they would correspond to only $N = 2$), it should be *locally* symmetric. Thus, for $N = 4$ "minimal coupling" is more or less the only open possibility. This rigidity does not hold any longer in the $N = 2$ case, as it is suggested by the Berger's Theorem viewpoint.

To this (local) rigidity of the space-time $N = 4$ supergravity, it corresponds at the world-sheet level the topological uniqueness of the K3 surface: any *compact* hyperkahler manifold should be a K3 space.[40] The precise connection between these two facts is given by the result of Seiberg that the moduli space of an *abstract* (4, 4) $c = 6$ system is equal to the moduli space for the K3 CY metrics as it is computed in the mathematical literature (see Ref. 41, Theorem 4).

But, again, this topological uniqueness is no longer valid for the CY 3-folds. Indeed, a large numbr of such spaces has been constructed since the seminal paper by Candelas *et al.*[5]

Given these new aspects of the problem, the extension of the above program to the (2, 2) case requires a preliminary discussion of the general properties of the low-energy effective Lagrangians for the 4D superstrings defined by such superconformal systems, as well as to construct an algorithm to compute these Lagrangians explicitly. Of course, for this program the most interesting strings are the type II ones.

The simplest ansatz[2] for such effective Lagrangians, i.e. *minimal* coupling, does not work. Naively, one could have expected minimal coupling by analogy with the $N = 4$ case. However, this cannot be for a very simple reason (besides the more technical ones we already discussed in the main body of the paper). Take any $N = 4$ SUGRA (which is minimally coupled, since there is no other possibility). Then truncate it to the maximal $N = 2$ sub-supergravity; the resulting $N = 2$ SUGRA is *not* minimally coupled, instead it is of the Jordan algebra type, both for the vector-multiplets and the hypermultiplets (*via* the c map). In fact, the scalars' manifold is of the form

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(n_1,2)}{SO(n_1) \otimes SO(n_2)} \otimes \frac{SO(n_2,4)}{SO(n_2) \otimes SO(4)} \tag{4.1}$$

i.e. a product of a Kahler manifold and a quaternionic manifold *both* associated with Jordan algebras $R + Q$ where Q is a Jordan algebra with quadratic norm. Thus, minimal coupling in the $N = 4$ sense is, in fact, Jordan-type coupling in the $N = 2$ sense. Unfortunately, such couplings are not unique in the $N = 2$ case, as the variety of examples in Sec. 3 shows.

[Notice that the manifolds in Eq. (4.1) are typical of the effective Lagrangians for $N = 2$ superstrings obtained by modding out an $N = 4$ superstring by a Z_2 symmetry; see Ref. 42 for examples in the heterotic case].

In the present paper we paved the way for the study of the moduli geometry for the (2, 2) system, by a detailed discussion of the relevant low-energy effective Lagrangians.

The results of this paper come very short of giving an explicit algorithm to compute the low-energy effective Lagrangian of any 4D type II superstring from the building blocks of the underlying superconformal theory (i.e. the three-point functions). In particular:

(i) we have shown that the hypermultiplet couplings are severely restricted, their σ model should be based on a dual-quaternionic manifold, which is quite a severe constraint from a supergravity point of view;

(ii) we gave an explicit algorithm (the c map) to construct the effective Lagrangian for type IIB from that of type IIA and vice versa;

(iii) under some mild assumptions (Sec. 2.2) we gave an explicit formula for the couplings of an IIA vector-multiplets (and IIB hypermultiplets) in terms of three-point functions.

What do all these results teach us for the moduli problem?

The answer has two sides. Whereas from a physical point of view we get almost all the relevant information on the nonlinear structure of the low-energy theory, from a purely mathematical point of view the most interesting problems remain open. To see this, let us discuss a bit more on the structure of the moduli problem.

Very crudely, the moduli space of the (2, 2) $c = 9$ system may be seen as the space of all possible "Calabi data"[37]

$$\{K, J, [\omega]\} \tag{4.2}$$

where K is a compact complex 3-fold with vanishing first Chern class, J is a complex structure over K and $[\omega]$ is a *complex* (1, 1) Dolbeault class, with $\mathrm{Re}[\omega]$ identified with the Kahler class and $\mathrm{Im}[\omega]$ with the "torsion" class. This identification is manifest to the leading order of the large radius expansion for the $2d$ σ model. In this limit the superconformal condition became simply the Calabi-Yau condition, and then the identification follows from the Calabi uniqueness[37] and the Yau existence[38] theorems. [Let us mention, in passing, that this identification works quite differently on the K3 case. The mapping between the "Calabi data" and the CY metrics is not injective: since a hyperkahler metric is *simultaneously* Kahlerian with respect a 2-sphere of complex structures, many different $\{J, [\omega]\}$ do correspond to the *same* hyperkahler metric, i.e. to the same $2d$ superconformal theory].

Of course, this identification is rather crude, since the large radius expansion has probably many problems, and since (contrary to the (4, 4) case[2]) at some exceptional points in moduli space some massive field can become massless, so changing also the dimension of the superconformal moduli space.[43] This phenomenon is well-understood in the case of toroidal compactification.[44]

Anyhow, the identification with the Calabi data is convenient from a qualitative point of view. Of course, from a mathematical point of view, the most interesting part of the Calabi data are the moduli of the complex structure. As is well-known, the infinitesimal deformation of the complex structure belong to the cohomology group $H^1(K, \Theta)$; on a CY 3-fold this group is easily identified with that of the harmonic (2, 1) forms.[5] Above we have identified these forms with the ($h = \bar{h} = 1/2$, $q = \bar{q} = 1$) superconformal operators; again, this identification is straight only in the large radius limit. In general the formal Hodge numbers are different from the purely geometrical ones.[26]

Thus, under these identifications the moduli space for the CY 3-folds is essentially the same manifold on which the scalars corresponding to the "(2, 1) forms" (except near the exceptional points) take value. This space is that of the vector-multiplets' σ model for the type IIB string. This is the manifold we cannot compute using the Peccei-Quinn symmetry. Thus, no simple and general answer is found for the most interesting aspect of the moduli problem in the case of CY 3-folds. On the contrary, in the 2-fold case, the $H^1(\Theta)$ group was related to the harmonic (1, 1) forms, so the deformations of the complex structure were controlled completely (from a space-time point of view) in terms of the Peccei-Quinn symmetry.

However, the situation is not that bad. When we fix our ideas on a particular model, usually we can compute its moduli using some trick *ad hoc*. Indeed, in this paper we have discussed only the general features and we have not used all the techniques at our disposal to compute the effective Lagrangians. Powerful methods do exist, such as the auxiliary-field vertices,[45] "hidden-symmetries",[18] etc.

The conclusion is that we can compute the moduli spaces (using macroscopic methods), but this should be done on a case by case basis and at the price of some hard work, since no simple and *general* answer seems to exist for the (2, 2) $c = 9$ case.

This is quite natural, given the fact that the space of the (2, 2) $c = 9$ superconformal systems is not smoothly connected, just as the moduli space for the CY 3-folds. It appears, however, that the moduli space is, in some sense, continuously connected through surfaces with nodes.[43] Probably, if we understand what this connection means from the point of view of the low-energy effective Lagrangian, the above program would be in much better shape. However, it seems likely that, in order to do this, we should keep track of all the higher derivative terms (or equivalently, of all massive modes).

Appendix A: Geometry of $N = 2$ Supergravity, Duality Transformations and All That

In this Appendix we explain the geometry of the special type of Kahler manifolds on which the $N = 2$ vector-multiplets' scalars take value. In particular, we shall prove some results needed in the main body of the paper, about the duality invariance in $N = 2$ supergravity. For instance, we want to show that the two "flat-potential" models called I and II in Ref. 9 are indeed one and the same model (this is a consistency check, since this is also implied by our claims of Sec. 2.1). More in general, we want to find when two functions $F_1(X)$ and $F_2(X)$ give *physically* equivalent $N = 2$ supergravities. (In the literature there is some confusion about this point.) The issue is crucial for the definition of the c map.

A.1 *Geometry*

Let M be a complex manifold with $\dim_C M = n + 1$, ($\dim_C M = n$ in *rigid* SUSY). In most examples M it is just C^{n+1}. Let $T^*[M]$ be its cotangent bundle, which is obviously a complex manifold of dimension $2(n + 1)$ (respectively $2n$). On $T^*[M]$ we have the natural symplectic form

$$\alpha_1 = dX^I \wedge dP_I + d\overline{X^I} \wedge d\overline{P_I}\,. \tag{A.1}$$

Here, we are interested in the holomorphic maps $T^*[M] \to T^*[M]$ which are also canonical transformations, i.e. they leave the (2,0) form $dX^I \wedge dP_I$ invariant. As it is well-known from elementary mechanics, they are constructed out of a generating function S. The map $(X^I, P_I) \to (Y^I, Q_I)$ reads

$$P_I = \frac{\partial S(X, Q)}{\partial X^I} \tag{A.2a}$$

$$Y^I = \frac{\partial S(X, Q)}{\partial Q_I} \tag{A.2b}$$

where the function $S(X, Q)$ is required to be *holomorphic* (in both its arguments). If we fix the new complex canonical momenta Q_I to have some constant value, which without loss of generality could be chosen to vanish, the equation

$$P_I = \partial_I S(X, Q = 0)$$

describes the *immersion* in $T^*[M]$ of a complex manifold $\mathscr{F}[S]$ with $\dim_C \mathscr{F}[S] = (n + 1)$ (respectively n). $\mathscr{F}[S]$ is a complex manifold with the property of being (at least locally) both holomorphically and canonically equivalent to the subspace of the complex phase space (Y^I, Q_J) given by "configuration space" $(Y^I, 0)$.

On $T^*[M]$ we also put a metric structure. The natural metric turns out to be Kahlerian and has the Lorentzian signature $(n + 1, n + 1)$ [respectively (n, n)]. In terms of the standard complex structure on $T^*[M]$ the metric is specified by the Kahler form

$$\omega = \tfrac{1}{2}(d\overline{X^I} \wedge dP_I - d\overline{P_I} \wedge dX^I)\,. \tag{A.3}$$

Then we have that $T^*[M]$ equipped with the above metric is a *Lorentzian* hyperkahler manifold. The three Kahler forms are ω, $i\alpha_1/2$, $i\alpha_2/2$.

Indeed, define $E = (dX^I, d\overline{X_J})$ and $e = (-idP_I, id\overline{P_J})$. Then

$$ds^2 = \tfrac{1}{2}\bar{E} \otimes e \tag{A.5a}$$

$$\omega = \frac{i}{2}\bar{E} \wedge \sigma_3 e \tag{A.5b}$$

$$\frac{i\alpha_1}{2} = \frac{i}{2}\bar{E} \wedge \sigma_2 e \tag{A.5c}$$

$$\frac{i\alpha_2}{2} = \frac{i}{2}\bar{E} \wedge \sigma_1 e. \tag{A.5d}$$

Equations (A.5) show our assertion (for more details, see the analogue proof in Appendix B).

We are interested in a very special class of maps from $T^*[M]$ into itself: the holomorphic canonical maps that are also isometries of Eq. (A.3). In the language of hyperkahler geometry they are the three-holomorphic isometries. The corresponding motions are generated by Hamiltonian Killing vectors.

It is easy to see that the group of the three-holomorphic isometries is $\mathrm{Sp}(2n + 2, R)$.

In fact, the Jacobian of a holomorphic canonical transformation is a holomorphic $\mathrm{Sp}(2n + 2, C)$ matrix. To be an isometry of Eq. (A.3), the Jacobian should belong to $\mathrm{Sp}(2n + 2, R)$. Since it is holomorphic, it should be a *constant.* Moreover, in the local case, constant shifts are ruled out by an homogeneity argument, to be discussed later.

Let $h \in \mathrm{Sp}(2n + 2, R)$ be a three-holomorphic isometry. Given the submanifold $\mathscr{F}[S] \subset T^*[M]$, consider its image under h, $h\{\mathscr{F}[s]\}$. Now, there exists a holomorphic function $S_h(X, Q)$ such that

$$\mathscr{F}[S_h] = h\{\mathscr{F}[S]\}. \tag{A.6}$$

Indeed, h is (in particular) a holomorphic canonical transformation. The composition of such maps is again holomorphic ànd canonical. Then the composition of the map in Eq. (A.2) with h should be of the form in Eq. (A.2) with S replaced by some function S_h. By definition, $h\{\mathscr{F}[S]\}$ is the image of the "configuration space" $(Y^I, 0)$ under the composition of the two maps.

On $\mathscr{F}[S]$ we have the natural Kahler metric

$$i_S^*\omega \tag{A.7}$$

where i_S is the immersion

$$i_S\colon \mathscr{F}[S] \hookrightarrow T^*[M]. \tag{A.8}$$

The metric $i^*{}_S\omega$ does not need to be positive definite since ω itself is Lorentzian. Notice that this metric is not trivial, even if $\mathscr{F}[S]$ is holomorphically and canonically equivalent to configuration space, since the map in Eq. (A.2) is *not* (in general) an isometry of Eq. (A.3).

Obviously, the two submanifolds $\mathscr{F}[S]$ and $\mathscr{F}[S_h]$ are equivalent with respect to all the above geometrical structures. In particular they are isometric. Thus S and S_h lead to physically indistinguishable theories. We shall say that two functions S which are connected by a three-holomorphic isometry are *physically equivalent*. In particular, we have that the two functions $S_1(X)$ and $S_2(X)$ with

$$S_1(X) - S_2(X) = \text{quadratic polynomial with real coefficients} \tag{A.9}$$

are physically equivalent.

Indeed, consider the $Sp(2n+2, R)$ element $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ where the $(n+1) \times (n+1)$ matrix C is symmetric. It maps the point $(X^I, \partial_I S)$ of $\mathscr{F}[S]$ into

$$(X^I)^1 = X^I \tag{A.10a}$$

$$(P_I)^1 = \partial_I S + C_{IJ} X^J = \partial_I (S + \tfrac{1}{2} C_{IJ} X^I X^J) \tag{A.10b}$$

which shows that S and $S + C_{IJ} X^I X^J/2$ are equivalent.

Given a holomorphic function $S(X) = S(X, Q = 0)$, we define its *duality group* as the subgroup of three-holomorphic isometries which map $\mathscr{F}[S]$ into itself. The duality group is the real physical invariance of the model specified by the function S.

How the above language is related to the more usual one[7,8,9]? For 4D $N = 2$ supergravity (respectively $N = 2$ super Yang-Mills) we have the simple identification

$$S(X^I) = \frac{i}{2} F(X^I) \tag{A.11}$$

for the Hamilton-Jacobi function $S(X)$. Then, the space $\mathscr{F}[S]$ is given by the equation

$$P_I = \frac{i}{2} F_I(X) \equiv \frac{i}{2} \partial_I F(X). \tag{A.12}$$

Therefore, on $\mathscr{F}[iF/2]$,

$$dP_I = \frac{i}{2} F_{IJ}(X)\, dX^J \tag{A.13}$$

so the Kahler form in Eq. (A.7) reads

$$i^*_{iF/2}\omega = i N_{\bar{I}} \, d\overline{X^I} \wedge dX^J \tag{A.14}$$

where

$$N_{\bar{I}J} = \tfrac{1}{4}(F_{IJ} + \bar{F}_{IJ}). \tag{A.15}$$

In rigid $N = 2$ SUSY (i.e. super Yang-Mills) this is the complete story. The metric in

Eq. (A.15) is the Kahlerian metric of the σ model for the scalars of the vector-multiplets. However, since ω is a Lorentzian metric, N_{IJ} need not be positive definite; therefore the open domain of $\mathscr{F}[iF/2]$ where it is positive-definite is identified with the true Kahler space underlying the σ model. Of course, in general there may be a problem with completeness.

In the local case (supergravity) we take $M = C^{n+1}\backslash\{0\}$. Then we have the natural (Hopf) projection

$$\pi\colon M \to P\mathbb{C}^n \tag{A.16}$$

and the physical σ model will be identified with an open domain of PC^n. Contrary to the rigid case, we have a constraint on the function $F(X)$: since N_{IJ} should be well-defined on PC^n, on M it should be homogeneous of degree zero in X^I. Then $F(X)$ should be homogeneous of degree 2 in X^I.

Let $U_i = \{X\varepsilon PC^n\colon X^i \neq 0\}$ be the standard cover of PC^n, and $Z^I{}_i = X^I/X^i$ the corresponding local coordinates. The closed form defined (in U_i) by

$$\frac{1}{6\pi}\Omega = \frac{i}{2\pi}\partial\bar{\partial}\ln[N_{\bar{I}J}\overline{Z_i^I}Z_i^J] \tag{A.17}$$

obviously agrees on the intersections $U_i \cap U_j$. Indeed, it would be a global closed form in PC^n were it not for the fact that the quantity $Y_i = N_{\bar{I}J}\overline{Z_i^I}Z_j^J$ need not be positive definite. The form Ω can be defined only in the domain $\mathscr{D} \subset PC^n$ where Y_i is positive. The physical Kahler space is identified with the subdomain of $\mathscr{D}$ where Ω is positive-definite. Then Ω is the physical Kahler form. A priori, it is not clear that such a construction leads to a complete Kahler manifold. Nevertheless, with some mild technical assumptions, one can show that the Kahler spaces associated to functions of the form $F(X) = id_{ABC}X^AX^BX^C(X^0)^{-1}$ (d_{ABC} real) are in fact complete.

Let us return to the geometry of duality rotations. If the given F has some continuous group of duality rotations, the corresponding infinitesimal transformations should belong to the Lie algebra of $\mathrm{Sp}(2n + 2, R)$. Then we must have

$$\begin{aligned} \delta X^I &= B^I{}_J X^J + D^{IJ}P_J \\ \delta P_I &= -B^J{}_I P_J + C_{IJ}X^J \end{aligned} \tag{A.18}$$

with B, D and C real, C and D symmetric. When Eq. (A.18) is evaluated on $\mathscr{F}[iF/2]$ we get

$$\delta X^I = B^I{}_J X^J + \frac{i}{2}D^{IJ}F_J \tag{A.19}$$

which is exactly Eq. (6.5) of Ref. 9 since, by homogeneity, $F_I = F_{IJ}X^J$. The second equation (A.18) becomes

$$\frac{i}{2}\delta(F_I) = -\frac{i}{2}B^J{}_I F_J + C_{IJ}X^J. \tag{A.20}$$

The condition of the invariance of the submanifold $\mathscr{F}[iF/2]$ under the duality transformation requires Eq. (A.20) to be compatible with Eq. (A.18). This, using homogeneity, implies

$$C_{IJ}X^J - \frac{i}{2}B^J{}_I F_J = \frac{i}{2}F_{IJ}(B^J{}_K X^K + \frac{i}{2}D^{JK}F_K) \tag{A.21}$$

which is the integrability condition[8,9] which identifies for each given function $F(X)$ what subgroup of $\mathrm{Sp}(2n+2, R)$ is actually a duality group.

A duality transformation is automatically an isometry of Ω. In fact, it is an isometry of the metric (A.3) in $T^*[M]$, and thus of the expression $\overline{X^I}N_{IJ}X^J$ on $C^n\backslash\{0\}$. Then, comparing with Eq. (A.17) we see that the Kahler potential $\ln\{N_{IJ}\overline{Z^I}Z^J\}$ changes just by a harmonic function, and so Ω remains invariant. Analogously, from the fact that the duality transformation leaves invariant the symplectic structure in Eq. (A.1) we deduce that the vectors' kinetic terms have the correct transformation.

The transformation of F is

$$\begin{aligned} 2\delta(F) &= \delta(X^I F_I) \\ &= -2iC_{IJ}X^I X^J + \frac{i}{2}D^{IJ}F_I F_J \end{aligned} \tag{A.22}$$

which is equivalent to the theorem by Gaillard and Zumino.[17]

In the present formulation, it is quite easy to see that in the minimal coupling case, i.e. $F(X) = \eta_{IJ}X^I X^J$ the action of the duality group is given by the reducible representation $(n+1) + (n+1)$. Indeed, the linear span of X^I and $P_J = i^{\eta}{}_{IJ}X^J$ has dimension $(n+1)$. The duality group[8] is given by the symplectic authomorphism of $P_I = i^{\eta}{}_{IJ}X^J$, i.e. by the matrices U such that $U^+\eta U = \eta$. With the physical signature for η this is $\mathrm{U}(n,1)$. Notice the factor $\mathrm{U}(1)$ which for $n = 1$ distinguishes minimal coupling from the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ model corresponding to the Jordan algebra R.

A.2 *When two F's lead to the same supergravity?*

From the above discussion, we see that two F's describe the same supergravity model if and only if there exists a transformation $h \varepsilon \mathrm{Sp}(2n+2, R)$ which connects them.

For instance, in the case of the models of Ref. 9, it is easy to see (by comparing the explicit representations of $\mathrm{SU}(1,1)$ that the $\mathrm{Sp}(2n+2, R)$ transformation from II into I is

$$S = (\tfrac{1}{2})^{1/4}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2}/3 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}. \tag{A.23}$$

S gives the map $(X^0, X^1)_{\rm II} \to (Y^0, Y^1)_{\rm I}$ of the form

$$\begin{aligned} Y^0 &= (2)^{1/4} X^0 \\ Y^1 &= -(2)^{1/4}\frac{i}{6}\frac{\partial F_{\rm II}}{\partial X^1} \end{aligned} \tag{A.24}$$

where $F_{\rm II} = 4(X^1)^{3/2}(X^0)^{1/2}$ is the "solution II" of Ref. 9. Then

$$Y^1 = -i(2)^{1/4}(X^1)^{1/2}(X^0)^{1/2}. \tag{A.25}$$

Since the function F is not invariant under duality, we cannot compute $F_{\rm I}$ just by replacing X with Y in value in $F_{\rm II}$. However, $F_{\rm I}$ is easily computed by requiring

$$\begin{aligned} X^0 &= (2)^{-1/4} Y^0 \\ X^1 &= (2)^{1/4}\frac{i}{6}\frac{\partial F_I}{\partial Y^1} \end{aligned} \tag{A.26}$$

to be the inverse of Eq. (A.24). Then

$$F_I = 2\sqrt{2}i(Y^1)^3(Y^0)^{-1}. \tag{A.27}$$

Appendix B: The *C*-Map in Rigid Supersymmetry

In this Appendix we shall construct *explicitly* the c map (defined in Sec. 2.3), in the case of *rigid* $N = 2$ supersymmetry.

In the rigid case the map transforms the vector-multiplets' Kahler manifold (which is specified by a holomorphic function $F(Z^I)$ $I = 1, \ldots, n$) into a hyperkahler manifold of twice as many complex dimensions.

In terms of the function $F(Z^I)$ the original Kahler metric and form reads

$$ds^2 \equiv N_{\bar{I}J}\, d\overline{Z^I} \otimes dZ^J \tag{B.1}$$

$$\omega = iN_{\bar{I}J}\, d\overline{Z^I} \wedge dZ^J \tag{B.2}$$

where

$$N_{\bar{I}J} = F_{IJ} + \bar{F}_{IJ} \tag{B.3}$$

and, as always, $F_I = \partial_I F$, etc.

Equations (B.1) and (B.2) correspond to a Kahler potential

$$N(Z,\bar{Z}) = F_I(Z)\overline{Z^I} + \bar{F}_I(\bar{Z})Z^I. \tag{B.4}$$

Let $N^{I\bar{J}}$ be the inverse of the metric $N_{\bar{I}J}$. Then the c map sends the Kahler potential (B.4) into the Kahler potential $K(Z,\bar{Z},W,\overline{W})$ of a hyperkahler metric, given by

$$K = N(Z,\bar{Z}) + N^{\bar{I}J}(Z,\bar{Z})(W_I + \overline{W}_I)(W_J + \overline{W}_J). \tag{B.5}$$

Here we want to show that:

(i) for *any* holomorphic function $F(Z^I)$ Eqs. (B.3)–(B.5) give a hyperkahler metric;

(ii) the *physical* relation between the $N = 2$ nonlinear gauge theory whose couplings are specified by the function $F(Z)$ and the $N = 2$ σ model based on the hyperkahler space (B.5) is the one defining the c map (Sec. 2.3).

In order to show that the Kahler metric $K_{\alpha\bar{\beta}}$ is, in fact, sufficient to find a *closed* (2, 0) form α and an action of the group Sp(1) on the cotangent bundle, such that:[46]

(i) it leaves the metric invariant;

(ii) the Kahler form ω and the closed forms α and $\bar{\alpha}$ belong to the adjoint representation of Sp(1).

Simple algebra gives the following

Lemma Let $e^i{}_I$ be an unitary coframe for the Kahler metric (B.1)

$$N_{I\bar{J}} = e^i{}_I\overline{e^i{}_J} \tag{B.6}$$

and let $e^I{}_i$, be its dual frame. Then the Hermitian metric corresponding to the Kahler potential (B.5) reads

$$ds^2 = e^i \otimes \overline{e^i} + \bar{E}_i \otimes E_i \tag{B.7}$$

where the vierbein forms are

$$e^i = e^i{}_I dZ^I \tag{B.8a}$$

$$E_i = \sqrt{2}e_i{}^I[dW_I - F_{IMJ}N^{\overline{M}L}(W_L + \overline{W}_L)dZ^J] \tag{B.8b}$$

From Eqs. (B.7)–(B.8) it is manifest that the function K in Eq. (B.5) satisfies the complex Monge-Ampere equation

$$\det[K_{\alpha\bar{\beta}}] = 2^n \tag{B.9}$$

i.e. the metric (B.7) is Ricci-flat.

The action of the Sp(1) group on the cotangent bundle is given by

$$\begin{pmatrix} e^i \\ \bar{E}_i \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^i \\ \bar{E}_i \end{pmatrix} \tag{B.10}$$

which is obviously an invariance of the metric, Eq. (B.7).

On the other hand, it is obvious that the three 2-forms

$$\omega = ie^i \wedge \bar{e}^i + iE^i \wedge \bar{E}^i \qquad \text{(Kahler form)} \tag{B.11a}$$

$$\alpha = 2ie^i \wedge E_i \qquad (2,0) \text{ form} \tag{B.11b}$$

$$\bar{\alpha} = -2i\bar{e}^i \wedge \bar{E}_i \qquad (0,2) \text{ form} \tag{B.11c}$$

transform among themselves according to the adjoint representation of Sp(1).

Therefore, we have only to show that $e^i \wedge E_i$ is *closed*. From Eq. (B.8),

$$\begin{aligned} e^i \wedge E_i &= \sqrt{2}\, dZ^I \wedge [dW_I - F_{IMJ} N^{\bar{M}L}(W_L + \bar{W}_L)\, dZ^J] \\ &= \sqrt{2}\, dZ^I \wedge dW_I \end{aligned} \tag{B.12}$$

since $F_{IMJ} = F_{JMI}$. Then α is closed and the metric is hyperkahler for any $F(Z)$.

Next we prove that the above c map is what one obtains reducing to 3D[b] and performing a vector-scalar duality. Start with the (bosonic sector) Lagrangian

$$\mathscr{L} = (F_{IJ} + \bar{F}_{IJ})\partial_\mu Z^I \partial_\mu \overline{Z^J} - \tfrac{1}{4}(F_{IJ} + \bar{F}_{IJ})F^I_{\mu\nu}F^J_{\mu\nu} + \tfrac{1}{4}(F_{IJ} - \bar{F}_{IJ})F^I_{\mu\nu}\tilde{F}^J_{\mu\nu} \tag{B.13}$$

reducing to 3D ($A^I = A^I{}_4$)

$$\begin{aligned} \mathscr{L}_{(3)} = {} & N_{\bar{I}J}\partial_\mu Z^J \partial_\mu \overline{Z^I} + \tfrac{1}{2}N_{I\bar{J}}\partial_\mu A^I \partial_\mu A^J + \frac{i}{2}(F_{IJ} - \bar{F}_{IJ})\varepsilon^{\mu\nu\rho}F^I_{\mu\nu}\partial_\rho A^J \\ & - \tfrac{1}{4}N_{I\bar{J}}F^I_{\mu\nu}F^J_{\mu\nu} \end{aligned} \tag{B.14}$$

obviously, $F^I{}_{\mu\nu}$ is constrained by the Bianchi identity. We add a Lagrange multiplier ϕ_I enforcing this constraint. Then the Lagrangian becomes,

$$\mathscr{L} = \mathscr{L}_{(3)} + \frac{i}{2}\varepsilon^{\mu\nu\rho}F^I_{\mu\nu}\partial_\rho \Phi_I\,. \tag{B.15}$$

[b] Dimensional reduction to 3D of $N = 2$ nonlinear super Yang-Mills systems were also studied (from a different point of view) in Ref. 49.

Eliminating ${F^I}_{\mu\nu}$ using its algebraic equations of motion

$$N_{I\bar{J}}F^I_{\mu\nu} = \varepsilon_{\mu\nu\rho}[\partial_\rho \Phi_I + i(F_{IJ} - \bar{F}_{IJ})\partial_\rho A^J] \tag{B.16}$$

$\mathscr{L}$ reduces to

$$\begin{aligned}\mathscr{L} = {} & N_{I\bar{J}}\partial_\mu Z^I \partial_\mu \overline{Z^J} + \tfrac{1}{2}N_{I\bar{J}}\partial_\mu A^I \partial_\mu A^J \\ & + \tfrac{1}{2}N^{\bar{I}J}[\partial_\mu \Phi_I + i(F_{IK} - \bar{F}_{IK})\partial_\mu A^K][\partial_\mu \Phi_J + i(F_{JL} - \bar{F}_{JL})\partial_\mu A^L] .\end{aligned} \tag{B.17}$$

Now we have to rewrite this metric in a meaningful way. In particular, we have to find one of the S^2 complex structures in order to write down a Kahler potential for it.

It is convenient to replace A^I with η_I defined by

$$A^I = N^{I\bar{J}}\eta_J . \tag{B.18}$$

Then, after some rearrangements, Eq. (B.17) becomes

$$\begin{aligned} & N_{I\bar{J}}\partial_\mu Z^I \partial_\mu \overline{Z^J} + \tfrac{1}{2}N^{I\bar{J}}\partial_\mu(\eta_I + i\Phi_I)\partial_\mu(\eta_I - i\Phi_I) \\ & - \eta_I N^{IJ}(F_{JLM}\partial_\mu Z^M)N^{LN}\partial_\mu(\eta_N - i\phi_N) - \eta_I N^{IJ}(\bar{F}_{JLM}\partial_\mu \overline{Z^M})N^{LN}\partial_\mu(\eta_N + i\phi_N) \\ & + 2N^{IJ}N^{KL}N^{MN}(F_{IKP}\partial_\mu Z^P)(\bar{F}_{JMQ}\partial_\mu \bar{Z}^Q)\eta_L \eta_N \end{aligned} \tag{B.19}$$

with the identification

$$W_I = \tfrac{1}{2}(\eta_I + i\phi_I) . \tag{B.20}$$

Equation (B.19) gives the metric obtained from the Kahler potential K in Eq. (B.5). This shows that K is the c map of the Kahler metric (B.1).

Appendix C: The C Map for Symmetric Kahler Spaces

In this Appendix, we give the quaternionic spaces associated, through the c map, to the *symmetric* Kahler spaces. The general case will be discussed elsewhere.

In the first column of Table 4, all the symmetric Kahler spaces allowed in $N = 2$ SUGRA are listed (see Cremmer and van Proeyen[20]). There are two general families, minimal coupling $\mathrm{U}(1,n)/[\mathrm{U}(n) \times \mathrm{U}(1)]$ and the "factorizable" models

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(n-1,2)}{\mathrm{SO}(n-1)\otimes \mathrm{SO}(2)} \qquad n \geq 2 . \tag{C.1}$$

For $n = 3$ this gives the special model

Table 4. The c map for *symmetric* Kähler spaces (of the restricted type allowed in $N = 2$ supergravity)

Kähler space	$\dim_{\mathbb{C}}$	Quaternionic space
$\frac{U(1,n)}{U(n) \times U(1)}$ (minimal coupling)	n	$\frac{U(2,n+1)}{U(2) \times U(n+1)}$
$\frac{SU(1,1)}{U(1)} \times \frac{SO(n-1,2)}{SO(n-1) \times SO(2)}$	$n \geq 2$	$\frac{SO(n+1,4)}{SO(n+1) \times SO(4)}$
$\frac{SU(1,1)}{U(1)}$	1	$\frac{G_{2(+2)}}{SO(4)}$
$\frac{Sp(6,\mathbb{R})}{U(3)}$	6	$\frac{F_{4(+4)}}{USp(6) \times SU(2)}$
$\frac{U(3,3)}{U(3) \times U(3)}$	9	$\frac{E_{6(+2)}}{SU(6) \times SU(2)}$
$\frac{SO^*(12)}{U(6)}$	15	$\frac{E_{7(-5)}}{SO(12) \times SU(2)}$
$\frac{E_{7(-26)}}{E_6 \times SO(2)}$	27	$\frac{E_{8(-24)}}{E_7 \times SU(2)}$

$$\left[\frac{SU(1,1)}{U(1)}\right]^3. \tag{C.2}$$

Moreover, we have five *exceptional* models, related to the Jordan algebras, (in the language of Ref. 15, rank 4 type 1 and rank 2 type 3 Kahlerian algebras).

In the last column we have written the corresponding quaternionic spaces. Notice that *all* the noncompact Wolf[22] spaces are in the list, except for the quaternionic hyperbolic spaces

$$HH^n = Sp(n,1)/[Sp(n) \otimes Sp(1)] \tag{C.3}$$

which are the most natural spaces in the tensor calculus approach. Indeed, in Sec. 2.3.4 we have argued that they cannot be obtained from the c map. However, from Table 4 we learn that *all* the other noncompact Wolf spaces are dual, (compare with the discussion in Sec. 2.3.4).

The table is constructed as follows. For the exceptional cases just compare with the magic-square. All these cases except $Sp(6,R)/U(3)$ and $E_{7(-26)}/[E_6 \times SO(2)]$ also appear in our explicit examples of Secs. 2.1 and 3, thus giving another way to computing the corresponding quaternionic spaces.

The factorizable models, Eq. (C.1), are also associated to Jordan algebras,[10] the corresponding quaternionic spaces are

$$SO(n+1,4)/[SO(n+1) \otimes SO(4)] \tag{C.4}$$

as follows from the arguments of Ref. 21 as well as from Ref. 15. The cases with $n = 3$ and $n = 5$ also appear in the examples of Sec. 3.

The minimal models are not associated to Jordan Algebras (in fact, they are related to the Jordan triple systems[10]). The obvious guess (confirmed by the results of Ref. 15 for their c map is

$$\mathrm{U}(2, n+1)/[\mathrm{U}(2) \otimes \mathrm{U}(n+1)]. \tag{C.5}$$

The cases $n = 0$ and $n = 1$ are present between our example above; $n = 0$ is just the universal sector.

Here we give still another proof of Eq. (C.5), based on the results of Ref. 47. We like this proof since it shows all the power of supersymmetry in mathematics. First we shall prove the following:

LEMMA The quaternionic manifolds which are the c image of the "minimal coupling" Kahler spaces, $\mathrm{U}(n, 1)[\mathrm{U}(n) \times \mathrm{U}(1)]$, are themselves Kahler manifolds.

We show this by SUGRA methods. The bosonic terms in the Lagrangian of an ungauged $N = 2$ SUGRA coupled to vector multiplets are

$$\mathscr{L}_B = -\tfrac{1}{2}R - \mathcal{N}_{A\bar{B}}\partial_\mu Z^A \partial_\mu \overline{Z^B} - (\tfrac{1}{4}\mathcal{N}_{IJ}F^{I-}_{\mu\nu}F^{J-}_{\mu\nu} + \mathrm{H.c.}) \tag{C.6}$$

where $\mathscr{M}_{AB}(Z, Z)$ is the scalars' Kahler metric and

$$F^{I\pm}_{\mu\nu} = \tfrac{1}{2}(F^I_{\mu\nu} \pm \tilde{F}^I_{\mu\nu}). \tag{C.7}$$

In $N = 1$ SUGRA the gauge-modulator function $\mathcal{N}_{IJ}(Z, \bar{Z})$ would be a *holomorphic* function. So it is in *rigid* $N = 2$ SUSY (since rigid $N = 2$ is a special case of rigid $N = 1$ SUSY). However, in local $N = 2$ it is not so

$$\mathcal{N}_{IJ} = \tfrac{1}{4}F_{IJ}(Z) - \frac{(N\bar{Z})_I(N\bar{Z})_J}{(\bar{Z}N\bar{Z})}. \tag{C.8}$$

The holomorphic function $F_{IJ} = \partial_I\partial_J F$ is the gauge-modulator of the global case and the nonholomorphic term on the right-hand side of Eq. (C.8) comes from the elimination of the auxiliary fields $T^{ij}{}_{\mu\nu}$ of the $N = 2$ gravitational multiplet.

Since $\mathcal{N}_{IJ}$ is not holomorphic, in general Eq. (C.6) cannot be seen as the bosonic sector of the Lagrangian of an $N = 1$ SUGRA. There is an *exception* however. In the minimal case $F(X) = \eta_{IJ}X^I X^J$, both F_{IJ} and N_{IJ} are *constant* matrices. Then the gauge-modulator in Eq. (C.8) is *anti*-holomorphic.

Obviously, inverting the orientation of space-time we can replace

$$F^{I-}_{\mu\nu} \leftrightarrow F^{I+}_{\mu\nu}.$$

Thus, after the inversion of the orientation, Eq. (C.6) is exactly the bosonic part of an $N = 1$ SUGRA (with vanishing superpotential and gauge-couplings) specified by the Kahler potential

$$K = -\ln\left(1 - \sum_{K=1}^{n} |Z^K|^2\right) \tag{C.9}$$

and gauge-modulator

$$f_{IJ}(Z) = \frac{1}{2}\eta_{IJ} - \frac{\eta_{IK}Z^K\eta_{JL}Z^L}{1 - \sum_{K=1}^{n} (Z^K)^2} \tag{C.10}$$

where $\eta_{IJ} = \mathrm{diag}(+1, -1, \ldots, -1)$, and $Z^0 = 1$.

Of course, when (in the construction of the c map) we reduce the theory to 3D and then make a duality transformation of the vectors, only the bosonic terms matter to compute the metric of the resulting quaternionic space. Since, in the minimal case, these bosonic terms can be embedded in both an $N = 1$ and an $N = 2$ SUGRA, we can get the relevant quaternionic space both by reducing the $N = 2$ *and* the $N = 1$ supergravities. But the σ model of an $N = 2$ 3D supergravity is Kahlerian. This proves the Lemma.

In particular, the Lemma explains why the universal sector (the c map of the graviphoton itself), contains a hypermultiplet parametrizing the coset

$$\mathrm{SU}(2,1)/\mathrm{U}(2). \tag{C.11}$$

Indeed the above Lemma (and the string considerations of the end of Sec. 2.3) implies that the universal hypermultiplet parametrizes a space which is both Kahler and quaternionic. The manifold in Eq. (C.11) is the *only* space of dimension 4 and negative scalar curvature which is both quaternionic and Kahler.

This can be seen as follows. Since $\mathrm{Sp}(1)\mathrm{Sp}(n)$ is not a subgroup of $\mathrm{U}(n)$, in order to be Kahlerian and Quaternionic, a space should have an holonomy group which is a *proper* subgroup of $\mathrm{Sp}(1)\mathrm{Sp}(n)$. By a Theorem by Alekseevskii[47] all the quaternionic spaces with such an holonomy group are symmetric spaces. Comparing the classifications of *symmetric* manifolds of Cartan[48] (for the Kahler case) with that of Wolf for the quaternionic case[22,15] we conclude that the manifold in Eq. (C.11) is the only Quaternionic-Kahler 4-fold.

Of course, the same argument can be repeated verbatim for all the manifolds in Eq. (C.5), then proving that they are the c map of the "minimal" models, as claimed above.

All these results can be recovered in the more general approach of Ref. 15.

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