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QUATERNIONIC MANIFOLDS FOR TYPE II SUPERSTRING VACUA OF CALABI–YAU SPACES

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In type II theories compactified on Calabi–Yau manifolds the moduli fields are accompanied by (Ramond–Ramond) scalar superpartners. $N=2$ space-time supersymmetry requires the low-energy effective mutual interaction of these excitations be described by a quaternionic σ-model in target space. In this paper we explicitly construct these manifolds from the restricted Kähler geometry of the moduli space of arbitrary (2,2) superconformal systems.

1. Introduction

Recently it has been shown [1–4] that the metric of the moduli space of superstring compactified on Calabi–Yau vacua★ [5] has a restricted Kähler structure as implied by $N=2$ space-time supersymmetry [6,7] when these manifolds are thought of as vacua of type II superstrings [8,9]. These restrictions have also been recently derived [2] (for superstrings compactified on arbitrary internal (2,2) superconformal field theories [10]) by exploiting the Ward identities of $N=2$ world-sheet supersymmetry on the four-point function of the moduli fields which is related to the curvature tensor of the moduli space.

$N=2$ space-time supersymmetry requires the moduli space, related to deformations of the Kähler class [(1,1) harmonic forms] and of the complex structure [(2,1) form] to be a product space [1,3,9]

$$\mathrm{M} = \mathrm{M}_{\mathrm{A}} \times \mathrm{M}_{\mathrm{B}},$$

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★ In this paper we will (somewhat improperly) use the word Calabi–Yau and their moduli space for generic (2,2) internal superconformal field theories.

with restricted Kähler potentials of the form [1]

$$K_{\mathrm{A(B)}} = -\log Y_{\mathrm{A(B)}} \qquad \begin{pmatrix} \mathrm{A} = (1,1) \text{ moduli} \\ \mathrm{B} = (2,1) \text{ moduli} \end{pmatrix}$$

with $Y_{\mathrm{A(B)}}$ given, in a certain choice of coordinates for the moduli, by [6,7]

$$Y_{\mathrm{A(B)}} = f_{\mathrm{A(B)}} + \bar{f}_{\mathrm{A(B)}} - \frac{1}{2}\left(\frac{\partial f_{\mathrm{A(B)}}}{\partial \phi_{\mathrm{A(B)}}} - \frac{\partial \bar{f}_{\mathrm{A(B)}}}{\partial \bar{\phi}_{\mathrm{A(B)}}} \right) \cdot \left(\phi_{\mathrm{A(B)}} - \bar{\phi}_{\mathrm{A(B)}} \right).$$

$f_{\mathrm{A(B)}}(\phi_{\mathrm{A(B)}})$ are two holomorphic functions of the complex moduli fields $\phi_{\mathrm{A(B)}}$ which determine all low-energy couplings of heterotic [11] strings as well as type II strings [12] compactified on an arbitrary (2,2) system.

In the field theory limit the functions $Y_{\mathrm{A(B)}}$ are given by [1,3,4]

$$Y_{\mathrm{A}} = V = \int J \wedge J \wedge J, \qquad Y_{\mathrm{B}} = i \int \Omega \wedge \bar{\Omega},$$

where V is the volume of the Calabi–Yau manifold, J is the Kähler form as a function of the (1,1) moduli and Ω is the holomorphic (3,0) form as a function of the (2,1) moduli.

In type II theories, due to the occurrence of extra massless scalar fields (Ramond–Ramond scalars), the Kähler spaces of the moduli are enlarged to quaternionic spaces [1,8], as required by $N=2$ space-time supersymmetry [6,7,13].

This phenomenon occurs when the moduli fields are members of $N=2$ hypermultiplets, which is the case for the (2,1) moduli in type IIA strings and for the (1,1) moduli in type IIB strings.

If one also includes the dilaton and the space-time axion (four-dimensional antisymmetric tensor) in the sector of massless excitations, the complete manifolds are [1,8]

$$\mathrm{M} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \mathrm{M_A} \times \mathrm{M_B}$$

in heterotic strings,

$$\mathrm{M}' = \mathrm{M_A} \times \mathrm{Q_B}$$

in type IIA strings, and

$$\mathrm{M}'' = \mathrm{Q_A} \times \mathrm{M_B}$$

in type IIB strings, where $\mathrm{M_A}$ and $\mathrm{M_B}$ are restricted Kähler manifolds of complex dimension $h_{(1,1)}$, $h_{(2,1)}$ respectively and $\mathrm{Q_B}$, $\mathrm{Q_A}$ are quaternionic manifolds of (real) dimension $4(h_{(2,1)}+1)$, $4(h_{(1,1)}+1)$ which contain, as submanifolds, the Kähler manifolds $(\mathrm{SU}(1,1)/\mathrm{U}(1)) \times \mathrm{M_B}$ and $(\mathrm{SU}(1,1)/\mathrm{U}(1)) \times \mathrm{M_A}$ respectively. The $(\mathrm{SU}(1,1)/\mathrm{U}(1))$ part refers to the dilaton, axion sector. The relation between these manifolds are called c and s-map in ref. [1]. These quaternionic manifolds were called dual quaternionic manifolds in ref. [1], where it was pointed out that their metric should be completely determined in terms of the same holomorphic functions $f_{\mathrm{A(B)}}$ which determine their Kähler submanifolds $\mathrm{M}_{\mathrm{A(B)}}$ (s-map). Particular cases of such quaternionic manifolds which correspond to symmetric or homogeneous Kähler manifolds have been discussed in the literature [14, 15].

Although the existence of such manifolds was established, no explicit construction of these spaces have been carried out yet. In ref. [1] the simpler problem of determining the s-map for rigid supersymmetry was solved. In that case the dual hyper-Kähler manifolds of dimension $4n$ were obtained from complex (rigid) Kähler manifolds of dimension n.

In the present paper we solve the problem of the s-map in local supersymmetry. The result is not only of physical but also of mathematical interest, in view of the fact that it allows the construction of continuous families of quaternionic manifolds, allowed by $N=2$ supergravity (Einstein spaces of negative curvature with a specific value), whose geometry in a certain coordinate system depends entirely on the very same holomorphic function of the restricted Kähler manifolds of the moduli space of (2, 2) superconformal theories [1–4].

We anticipate some of the properties shared by all these dual manifolds. Let us call the preferred coordinate systems for these manifolds (Z^a, C_i, S), where Z^a $(a=1,\dots,n)$ are the original coordinates of the restricted Kähler manifold (moduli fields), C_i $(i=1,\dots,n+1)$ are the Ramond–Ramond scalars and S is a complex field related to the dilaton and the (space-time) axion.

(i) At each point of the moduli space $Z^a = Z^{a^0}$ $(\partial_\mu Z^{a^0}=0)$, the Ramond–Ramond scalar parametrize a $\mathrm{SU}(1, n+2)/(\mathrm{U}(1)\times \mathrm{SU}(n+2))$ manifold.

(ii) A each point $(\mathrm{Re}\, C_i = 0,\ \mathrm{Im}\, C_i = \mathrm{Im}\, C_i^0)$ $(\partial_\mu C_i = 0)$ the (Z^a, S) fields parametrize a $(\mathrm{SU}(1,1)/\mathrm{U}(1))\times \mathrm{M}_n$ manifold when M_n is the original Kähler manifold. This is the Kähler manifold of the heterotic string when the charged matter fields are set to zero.

(iii) The quaternionic manifold $\mathrm{Q}_{4(n+1)}$ has at least $2n+4$ isometries acting on all coordinates but the Z^a coordinates.

(iv) The dual quaternionic manifolds are Einstein spaces with negative curvature $R = -8(n+3)(n+1)$.

(v) Those moduli which have vanishing Yukawa couplings [16] together with their (Ramond) $N=2$ partners from a Kähler quaternionic submanifold $\mathrm{SU}(2, n')/\mathrm{SU}(2)\times \mathrm{SU}(n')\times \mathrm{U}(1)$ of the original (non-Kähler) quaternionic manifold.

2. Properties of dual quaternionic spaces

Let us anticipate a particular result stated in ref. [1]. In the case of symmetric quaternionic spaces, all of them are s-maps of symmetric Kähler spaces with the exception of the $\mathrm{Sp}(n+1)/\mathrm{Sp}(1)\times(n)$ series. All symmetric [15] restricted (or (or homogeneous) [19–21] Kähler manifolds correspond to a holomorphic function f which is a polynomial of degree two or degree three in Z [15]. These particular polynomials lead to a vanishing (for quadratic functions) or constant (for cubic functions) Yukawa coupling [16] in heteroric strings [1,2,9]. In particular, vanishing Yukawa coupling means that the f function is quadratic and in that case the restricted Kähler manifold is always $SU(1,n)/U(1)\times SU(n)$ and the dual quaternionic manifold $SU(2,n+1)/SU(2)\times SU(n+1)\times U(1)$ which is also Kähler [1]. These results have been confirmed in special cases in refs. [1,7,17], where the moduli metric was computed for orbifolds and for Calabi–Yau spaces obtained by tensoring several copies of the $N=2$ minimal series [2].

Now let us consider the general case in which, for arbitrary choice of the holomorphic function $f(Z)$, the restricted Kähler manifold is not symmetric nor homogeneous. This is of course the case of interest for generic Calabi–Yau spaces.

The metric of the dual quaternionic manifold is derived [1,14] by performing a dimensional reduction from $D=4$ to $D=3$ dimensions of $N=2$ supergravity coupled to n-vector multiplets with holomorphic function $f(Z^a)$ $(a=1,\dots,n)$.

The bosonic part of the $N=2$ lagrangian for vector multiplets is [6,7]

$$e^{-1}\mathscr{L}=\tfrac{1}{2}R-K_{a\bar b}\partial_\mu Z^a\partial_\mu\bar z^b+\tfrac{1}{4}\mathrm{Re}\,\mathscr{N}_{ij}F_{\mu\nu}{}^iF_{\mu\nu}{}^j+\tfrac{1}{4}\mathrm{Im}\,\mathscr{N}_{ij}F_{\mu\nu}{}^i\tilde F_{\mu\nu}{}^j, \tag{2.1}$$

with

$$\mathscr{N}_{ij}=\tfrac{1}{4}\bar F_{ij}-(NZ)_i(NZ)_j/(ZNZ),\qquad N_{ij}=\tfrac{1}{4}\left(F_{ij}+\bar F_{ij}\right)\quad (2.2),(2.3)$$

$$K=-\ln 2\bar ZNZ=-\ln\left[f+f^*-\tfrac{1}{2}(Z^a-Z^{*a})(f_a-f_a^*)\right]. \tag{2.4}$$

F is a homogeneous holomorphic function of degree two in $n+1$ variables x^i $(Z^i=x^i/x^0)$.

$$F(\lambda x)=\lambda^2F(x),\qquad f(z)=(x^0)^{-2}F(x). \tag{2.5}$$

Dimensional reduction from $D=4$ $(N=2)$ to $D=3$ $(N=4)$ is obtained using a triangular gauge for the vierbein

$$e^{\hat a}{}_{\hat\mu}=\begin{bmatrix} e^a{}_\mu & 0\\ \phi^{1/2}B_\mu & \phi^{1/2}\end{bmatrix},\qquad \hat a,\hat\mu=1,\dots,4;\qquad a,\mu=1,\dots,3 \tag{2.6}$$

and for four-vectors we have

$$A_{\hat\mu}^{\ i}=\left(A_\mu^{\ i}+B_\mu\zeta^i,\zeta^i\right),\qquad \zeta^i=A_4^{\ i}. \tag{2.7}$$

The lagrangian (2.1) reduced to three dimensions, after a Weyl rescaling $e^{a}{}_{\mu} \to \phi^{-1/2} e^{a}{}_{\mu}$ becomes

$$e^{-1}\mathscr{L} = \tfrac{1}{2}R - \frac{1}{4\phi^2}(\partial_\mu \phi)^2 + \tfrac{1}{4}\phi^2 H_\mu{}^2 + - K_{a\bar{b}}\,\partial_\mu Z^a\,\partial_\mu \bar{Z}^b - \tfrac{1}{2}\phi R_{ij}\left(F_\mu^i + H_\mu \zeta^i\right)\left(F_\mu^j + H_\mu \zeta^j\right)$$

$$+ \frac{1}{2\phi} R_{ij}\,\partial_\mu \zeta^i\,\partial_\mu \zeta^j - \operatorname{Im}\mathscr{N}_{ij}\left(F_\mu^i + H_\mu \zeta^i\right)\partial_\mu \zeta^j, \tag{2.8}$$

where $H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $H_\mu = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho} H_{\nu\rho}$ and $R_{ij} = \operatorname{Re}\mathscr{N}_{ij}$. We now use three-dimensional duality in order to convert three-dimensional vector fields into scalar fields. For this purpose we add the Lagrange multipliers

$$-F_\mu^i\,\partial_\mu \tilde{\zeta}_i + \tfrac{1}{2}H_\mu\,\partial_\mu\left(\tilde{\phi} - \zeta^i \tilde{\zeta}_i\right) \tag{2.9}$$

and integrate over the F_μ^i, H_μ fields to get

$$e^{-1}\mathscr{L} = \tfrac{1}{2}R - K_{a\bar{b}}\,\partial_\mu Z^a\,\partial_\mu \bar{Z}^b + \frac{1}{4\phi^2}(\partial_\mu \phi)^2 + \frac{1}{4\phi^2}\left(\partial_\mu \tilde{\phi} + \zeta^i \overleftrightarrow{\partial}_\mu \tilde{\zeta}_i\right)^2$$

$$- \frac{1}{2\phi} R_{ij}\,\partial_\mu \zeta^i\,\partial_\mu \zeta^j - \frac{1}{2\phi}\left(\partial_\mu \tilde{\zeta}_i + \operatorname{Im}\mathscr{N}_{ik}\,\partial_\mu \zeta^k\right)(R^{-1})^{ij}\left(\partial_\mu \tilde{\zeta}_j + \operatorname{Im}\mathscr{N}_{jl}\,\partial_\mu \zeta^l\right). \tag{2.10}$$

Finally, let us define the fields

$$C_i = \mathscr{N}_{ij}\zeta^j + i\tilde{\zeta}_i, \qquad S = \phi + i\tilde{\phi} - \tfrac{1}{2}(C+\bar{C})_i (R^{-1})^{ij} (C+\bar{C})_j, \tag{2.11}$$

then the lagrangian describing the scalar manifold is

$$e^{-1}\mathscr{L} = -K_{a\bar{b}}\,\partial_\mu Z^a\,\partial_\mu \bar{Z}^b - \left[S + \bar{S} + \tfrac{1}{2}(C+\bar{C})R^{-1}(C+\bar{C})\right]^{-2}$$

$$\times \left|\partial_\mu S + (C+\bar{C})R^{-1}\,\partial_\mu C - \tfrac{1}{4}(C+\bar{C})R^{-1}\,\partial_\mu \mathscr{N} R^{-1}(C+\bar{C})\right|^2$$

$$+ \left[S + \bar{S} + \tfrac{1}{2}(C+\bar{C})R^{-1}(C+\bar{C})\right]^{-1}\left(\partial_\mu C - \tfrac{1}{2}\partial_\mu \mathscr{N} R^{-1}(C+\bar{C})\right)$$

$$\times R^{-1}\left(\partial_\mu \bar{C} - \tfrac{1}{2}\partial_\mu \bar{\mathscr{N}} R^{-1}(C+\bar{C})\right). \tag{2.12}$$

Positivity of the kinetic energy (in eq. (2.10)) required $K_{a\bar{b}}$ and $-R_{ij}$ to be positive-definite matrices.

Eq. (2.12) defines a manifold for the $2(n+1)$ complex scalar fields S, Z^a, C_i that according to ref. [1], should be the dual quaternionic manifold Q_{4n+4}, obtained by the s-map from the restricted Kähler manifold K_n (specified by the holomorphic function $f(z)$). More specifically, as mentioned in sect. 1, the lagrangian given by eq. (2.12) (in $D=4$ dimensions) describes the (low-energy) interactions of moduli fields (Z^a), dilaton and axion (S) and the Ramond scalars C_i present in type II superstrings. For a generic compactification on a $(2,2)$ system, the Z^a moduli refer (in the language of Calabi–Yau manifolds) to deformation of the Kähler class in type IIB strings and to deformation of the complex structure in type IIA strings. The dual quaternionic manifolds have therefore (quaternionic) dimensions $h_{(2,1)}+1$ in type IIA and $h_{(1,1)}+1$ in type IIB, where $h_{(1,1)}$ and $h_{(2,1)}$ are the Hodge numbers of the manifold. It is our aim to prove that the metric given by eq. (2.12) is quaternionic and to discuss its properties.

Let us first discuss properties (a), (b), (c), and (e). An alternative form for the metric given by eq. (2.12) is

$$e^{-1}\mathscr{L}= -K_{a\bar{b}}\partial_\mu Z^a\partial_\mu \bar{Z}^b - \tilde{K}_{S\bar{S}}D_\mu S D_\mu \bar{S} - \tilde{K}_{S\bar{C}_i}D_\mu S D_\mu \bar{C}_i - \tilde{K}_{C_i\bar{S}}D_\mu C_i S_\mu \bar{S}$$
$$- \tilde{K}_{C_i\bar{C}_j}D_\mu C_i D_\mu \bar{C}_j, \tag{2.13}$$

where

$$D_\mu C = \partial_\mu C - \tfrac{1}{2}\partial_\mu \mathscr{N} R^{-1}(C+\bar{C}),$$
$$D_\mu S = \partial_\mu S + \tfrac{1}{4}(C+\bar{C})R^{-1}\,\partial_\mu \mathscr{N} R^{-1}(C+\bar{C}),$$
$$\tilde{K} = -\ln\left(S+\bar{S}+\tfrac{1}{2}(C+\bar{C})R^{-1}(C+\bar{C})\right). \tag{2.14}$$

The above equations show that for fixed C ($\partial_\mu C=0$) and $C+\bar{C}=0$, the manifold Q contains the Kähler submanifold $(\mathrm{SU}(1,1)/\mathrm{U}(1))\times K_n$ with coordinates (S, Z^a), while for fixed Z^a it contains the Kähler submanifold $\mathrm{SU}(1,n+2)/(\mathrm{U}(1)\times \mathrm{SU}(n+2))$ with coordinates (S,C)*. The first manifold is the Kähler manifold for heterotic strings which contains the dilaton, the axion and he moduli fields, the second manifold is the manifold of (Ramond–Ramond) scalars for fixed values of the moduli.

We can also remark that if the matrix $\mathscr{N}(Z,\bar{Z})$ is holomorphic, i.e. does not depend on $\bar{Z}$ then the manifold Q is Kähler with Kähler potential $K+\tilde{K}$**. In view of eq. (2.2), this is the case if the function $f(Z)$ is a quadratic poly-

* The standard metric for the manifold is best seen by making the following (holomorphic for fixed Z) field redefinition: $S\to S-\tfrac{1}{2}CR^{-1}C$.

** Note that the covariant derivative in eq. (2.13) just reproduces the terms in the Kähler metric proportional to $\tilde{K}_{S\bar{Z}}$, $\tilde{K}_{C\bar{Z}}$ and their hermitian conjugates.

nomial [7,15] and the Kähler quaternionic manifold turns out to be [1,13] $SU(2, n+1)/SU(2) \times SU(n+1) \times U(1)$. This corresponds to statement (e) in sect. 1.

Let us now consider another important property of the Q manifold: its isometries. We may note from eq. (2.10) that among the $4(n+1)$ scalar fields only $2n+1$ may appear with non-polynomial interactions. These are the true moduli $(Z^a, \operatorname{Re} S)$ which correspond to the deformation of the manifolds and the dilaton. These are actually at least $2n+3$ isometries associated with the axion and the Ramond fields

$$S \to S + i\alpha - 2C\gamma - \gamma\mathcal{N}\gamma, \qquad C \to C + i\beta + \mathcal{N}\gamma, \tag{2.15}$$

where α, β_i, γ^i are $2n+3$ real parameters. An additional isometry is the scale transformation $S \to \lambda S$, $C \to \lambda^{1/2}C$. Therefore, the Q manifold has at least $2n+4$ isometries.

We may compare these results with the simpler case of rigid supersymmetry described in appendix B of ref. [1]. In rigid supersymmetry, the $N=2$ gravity sector (graviton and graviphoton) was missing (this sector contains four degrees of freedom), so we had a correspondence between a (rigid) restricted Kähler manifold of complex dimension n with a hyper-Kähler (rather than quaternionic) manifold [18] of (quaternionic) dimension n. Since a hyper-Kähler manifold can be regarded as a limiting case (for vanishing Sp(1) connection) of a quaternionic manifold [13] (as in local supersymmetry) the Einstein space structure of the latter is replaced by the Ricci flatness of the former [1,15].

3. The geometry of the Q manifolds

In this section we will prove that the manifold defined by eq. (2.10) is indeed a quaternionic manifold.

Let us recall that for quaternionic (4d-dimension) manifolds [13,19–21] there are three locally defined (1,1) tensors $(J^u)^\mu{}_\nu$ which satisfy the quaternionic algebra

$$J^u \times J^v = -\delta^{uv} + \varepsilon^{uvw} J^w. \tag{3.1}$$

Moreover, the three two-forms*

$$J^u = \tfrac{1}{2} J^u_{\mu\nu}\, dx^\mu dx^\nu, \qquad (J^u)_{\mu\nu} = g_{\mu\rho}(J^u)^\rho{}_\nu \tag{3.2}$$

are covariantly constant with respect to a Sp(1) connection ω

$$dJ + \omega J - J\omega = 0, \qquad J = J^u \sigma^u.$$

* Wedge product of forms $dx \wedge dy$ will be denoted by $dx\, dy$.

The Sp(1) curvature is proportional to the J two-forms

$$\mathrm{d}\omega + \omega\omega = i\lambda J \tag{3.3}$$

for some constant λ.

The holomony group of a quaternionic manifold is a subgroup of $\mathrm{Sp}(1) \times \mathrm{Sp}(d)$. In addition, quaternionic manifolds are Einstein spaces with $R_{\mu\nu} = 2\lambda(d+2)g_{\mu\nu}$. Consistent coupling to supergravity requires [13] λ to be negative and fixed to -1. So only quaternionic manifolds with negative curvature can be coupled to supergravity. We will see later that this property is automatically satisfied for the Q manifolds irrespective of the holomorphic function $f(Z)$*.

Let us consider the original Kähler manifold K_n with Kähler (closed) two-form given by

$$J = ie^A e^{\bar{A}}, \qquad e^A = e_a{}^A \mathrm{d}Z^a. \tag{3.4}$$

The Kähler metric is

$$K_{a\bar{b}} = e_a{}^A \left(e_b{}^A\right)^* . \tag{3.5}$$

It is very convenient to define a $n \times (n+1)$ matrix P_i^A as follows:

$$P_a{}^A = e_a{}^A, \qquad P_0{}^A = -e_a{}^A Z^a. \tag{3.6}$$

Note that P satisfies

$$P \cdot Z = 0 \qquad (Z^0 = 1), \qquad P^\dagger \cdot P = -\frac{1}{\bar{Z}NZ}\left(N - \frac{(NZ)(\bar{Z}N)}{\bar{Z}NZ}\right) \quad (3.7),(3.8)$$

$$PN^{-1}P^\dagger = -\frac{1}{\bar{Z}NZ}. \tag{3.9}$$

The vierbein one-forms are then given by

$$\begin{aligned} e &= P\mathrm{d}Z, \qquad E = \mathrm{e}^{(\tilde{K}-K)/2} PN^{-1}\left(\mathrm{d}C - \tfrac{1}{2}\mathrm{d}\mathcal{N} R^{-1}(C+\bar{C})\right), \\ u &= 2\,\mathrm{e}^{(\tilde{K}+K)/2} Z\left(\mathrm{d}C - \tfrac{1}{2}\mathrm{d}\mathcal{N} R^{-1}(C+\bar{C})\right), \\ v &= \mathrm{e}^{\tilde{K}}\left(\mathrm{d}S + (C+\bar{C})R^{-1}\mathrm{d}C - \tfrac{1}{4}(C+\bar{C})R^{-1}\mathrm{d}\mathcal{N} R^{-1}(C+\bar{C})\right). \end{aligned} \tag{3.10}$$

* We will use capital letters for flat indices, small letters for curved indices, initial letters of the alphabet a, A run from 1 up to n while middle letters i, I run from 1 up to $n+1$.

The lagrangian for the quaternionic manifold takes the form*

$$-e^{-1}\mathscr{L} = e\otimes\bar{e} + E\otimes\bar{E} + u\otimes\bar{u} + v\otimes\bar{v}$$
$$= \sum_{\alpha=1,2;\ I=1,\dots,n+1} e^{\alpha I}(e^{\alpha I})^* \tag{3.11}$$

in terms of the $2(n+1)$ component vierbeins

$$e^{\alpha I} = (e^{+I}, e^{-I}), \qquad e^{+I} = \begin{pmatrix} u \\ e^A \end{pmatrix}, \qquad e^{-I} = \begin{pmatrix} v \\ E^A \end{pmatrix}. \tag{3.12}$$

To find the connections we compute the exterior derivatives of the vierbein one-forms

$$\mathrm{d}e = -\omega e, \qquad \mathrm{d}v = v\bar{v} + u\bar{u} + E\bar{E}$$
$$\mathrm{d}u = \left(-\tfrac{1}{2}(v+\bar{v}) + \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{2\bar{Z}NZ}\right)u - \bar{E}e$$
$$\mathrm{d}E = \left(-\omega - \tfrac{1}{2}(v+\bar{v}) - \frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{2\bar{Z}NZ}\right)E - \bar{u}e - \tfrac{1}{2}\bar{Z}NZPN^{-1}\mathrm{d}(N-iY)N^{-1}P^{\mathrm{T}}\bar{E} \tag{3.13}$$

Here ω is the connection for the original Kähler manifold K_n

$$\omega = -\frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{2\bar{Z}NZ} + \frac{\bar{Z}NZ}{2}\{\mathrm{d}PN^{-1}P^\dagger - PN^{-1}\mathrm{d}P^\dagger - iPN^{-1}\mathrm{d}YN^{-1}P^\dagger\} \tag{3.14}$$

and

$$N_{ij} + iY_{ij} = \tfrac{1}{2}F_{ij}. \tag{3.15}$$

The curvature two-form for K_n is

$$(\mathrm{d}\omega + \omega\omega)^A{}_B = -\delta^A{}_B e^C\bar{e}^C - e^A\bar{e}_B - \frac{\bar{f}_{ACE}f_{EDB}}{16(\bar{Z}NZ)^2}\bar{e}^C e^D, \tag{3.16}$$

with $f_{ABC} = (f_{abc}e^a)({}_Ae^b)({}_Be^c)_C$, $f_{abc} = (\partial/\partial Z^a)(\partial/\partial Z^b)(\partial/\partial Z^c)f(Z)$ and $e^a{}_A = (e^A{}_a)^{-1}$. Eq. (3.16) is in agreement with ref. [7].

* The $\otimes$ symbol denotes the sum of the product of components of two one-forms.

The connections for the Q manifold are given by

$$\mathrm{d}e^{\alpha I} + P^{\alpha}{}_{\beta}e^{\beta I} + q^{I}{}_{J}e^{\alpha J} + t^{I}{}_{J}\varepsilon^{\alpha\alpha'}\overline{e^{\alpha' J}} = 0\,, \tag{3.17}$$

where

$$p = \begin{bmatrix} \frac{1}{4}(v-\bar{v}) - \frac{1}{4}\frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{\bar{Z}NZ} & -u \\ \bar{u} & -\frac{1}{4}(v-\bar{v}) + \frac{1}{4}\frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{\bar{Z}NZ} \end{bmatrix} \tag{3.18}$$

(p is the Sp(1) connection)

$$q = \begin{bmatrix} -\frac{3}{4}(v-\bar{v}) - \frac{1}{4}\frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{\bar{Z}NZ} & \bar{E} \\ -E & \omega - \frac{1}{4}(v-\bar{v}) + \frac{1}{4}\frac{\bar{Z}N\mathrm{d}Z - ZN\mathrm{d}\bar{Z}}{\bar{Z}NZ} \end{bmatrix} \tag{3.19}$$

$$t = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\bar{f}_{ABC}\bar{E}^{C}}{4\bar{Z}NZ} \end{bmatrix}. \tag{3.20}$$

The Sp(1) $\times$ Sp($n+1$) connection is most easily seen by defining a $4(n+1)$ component vierbein*

$$V^{\alpha\Gamma} = \begin{bmatrix} e^{\alpha I} \\ \varepsilon^{\alpha\beta}(e^{\beta I})^{*} \end{bmatrix}. \tag{3.21}$$

The flat space metric is $\frac{1}{2}\varepsilon_{\alpha\alpha'}\rho_{\Gamma\Gamma'}$ with $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $V^{\alpha\Gamma}$ is covariantly constant

$$(\mathrm{d} + \Omega)V = 0 \tag{3.22}$$

with connection

$$\Omega = p \times \mathbf{1}_{2(n+1)} + \mathbf{1}_{2} \times \begin{pmatrix} q & t \\ -t^{\dagger} & -q^{\mathrm{T}} \end{pmatrix},$$

$$p^{\dagger} = -p\,, \qquad q^{\dagger} = -q\,, \qquad t^{\mathrm{T}} = +t\,, \qquad \Omega^{\dagger} = -\Omega\,, \tag{3.23}$$

* $V^{\alpha\Gamma}$ correspond to $\gamma_{i}^{AZ}\,\mathrm{d}\phi^{i}$ in the notation of ref. [13]. They satisfy the reality condition $V_{\alpha\Gamma} = \frac{1}{2}(V^{\alpha\Gamma})^{*}$.

where the two terms correspond to the Sp(1) and Sp($n+1$) connections respectively. The curvature two-form for the Q manifold is

$$R = \mathrm{d}\Omega + \Omega\Omega = -iJ \times \mathbf{1}_{2(n+1)} + \mathbf{1}_2 \times \tilde{R} \tag{3.24}$$

where $\tilde{R}$ is the Sp($n+1$) curvature.

We are mainly interested in the Sp(1) curvature

$$-iJ = \mathrm{d}p + pp = \tfrac{1}{2}\begin{pmatrix} \bar{e}^+ \\ \bar{e}^- \end{pmatrix} \sigma^u \begin{pmatrix} e^+ \\ e^- \end{pmatrix} \cdot \sigma^u$$

or

$$(-iJ)^{\alpha}{}_{\beta} = -V^{\alpha\Gamma} V_{\beta\Gamma}. \tag{3.25}$$

J^u define the three "complex structures" satisfying the quaternionic algebra given by eq. (3.1). The construction of the three two-forms given by eq. (3.25) is the final proof that Q is a quaternionic manifold of quaternionic dimension $n+1$.

It is of interest to give the expression for the Sp($n+1$) curvature as well. This is a $(2(n+1) \times 2(n+1))$ matrix-valued two-form

$$\tilde{R} = \begin{pmatrix} r & r' \\ -r'^{\dagger} & -r^{\mathrm{T}} \end{pmatrix}$$

in which r, r' are two $[(n+1) \times (n+1)]$ matrix-valued two-forms. Their expression is

$$r^0{}_0 = -\tfrac{3}{2}(u\bar{u} + v\bar{v}) - \tfrac{1}{2}(e\bar{e} + E\bar{E}), \qquad r^A{}_0 = -(r^0{}_A)^* = \bar{u}e^A + \bar{v}E^A,$$

$$r^A{}_B = -\tfrac{1}{2}\delta^A{}_B(e\bar{e} + E\bar{E} + u\bar{u} + v\bar{v}) - e^A\overline{e^B} - E^A\overline{E^B} - \frac{\bar{f}_{ACE} f_{EDB}}{16(\bar{Z}NZ)^2}\left(\overline{e^C}e^D + \overline{E^C}E^D\right),$$

$$r'^A{}_{\bar{B}} = \frac{1}{4\bar{Z}NZ}\bar{f}_{ABC}\left(u\overline{e^C} + v\overline{E^C}\right) + \frac{1}{16(\bar{Z}NZ)^2}\bar{f}_{ABC} f_{CDE} e^D E^C - \frac{1}{4\bar{Z}NZ}\frac{1}{4}X^A{}_{\bar{B}\bar{C}\bar{D}}\overline{e^C E^D}$$

$$X_{\bar{a}\bar{b}\bar{c}\bar{d}} = \bar{f}_{abcd} + \frac{(NZ)_a\bar{f}_{bcd} + (NZ)_b\bar{f}_{acd} + (NZ)_c\bar{f}_{abd} + (NZ)_d\bar{f}_{abc}}{\bar{Z}NZ}$$

$$-\tfrac{1}{4}\left(\bar{f}_{abe}\bar{f}_{fcd} + \bar{f}_{ace}\bar{f}_{fdb} + \bar{f}_{ade}\bar{f}_{fbc}\right)(N^{-1})^{\bar{e}\bar{f}}. \tag{3.26}$$

The Sp($n+1$) curvature can alternately be written as [13]

$$(\tilde{R})^{\Gamma}{}_{\Gamma'} = -V^{\alpha\Gamma}V_{\alpha\Gamma'} + \rho^{\Gamma\Gamma^0}\Omega_{\Gamma^0\Gamma'\Gamma''\Gamma'''}V^{\alpha\Gamma''}V_{\alpha}{}^{\Gamma'''}, \tag{3.27}$$

where $\Omega_{\Gamma\Gamma'\Gamma''\Gamma'''}$ is completely symmetric due to the Bianchi identity $RV=0$. The quaternionic manifold, which is always an Einstein space, has scalar curvature given by

$$R = -8(n+1)(n+3), \tag{3.28}$$

in agreement with ref. [13]. We remark that the Q curvature, unlike the K_n curvature (given by eq. (3.15)) depends, through r', up to the fourth derivative of the holomorphic function $f(Z)$.

4. Some examples

In sect. 2 we have seen that for a quadratic holomorphic function $F(X)$ the Q manifolds become Kähler. This result can be obtained in a more general way by observing that the two-form

$$e\bar{e} + E\bar{E} + u\bar{u} + v\bar{v} \tag{4.1}$$

is always closed. However, this does not mean in general that the manifold is Kähler unless the vierbeins are holomorphic. The vierbeins are holomorphic if and only if $\mathrm{d}\mathcal{N}$ is holomorphic which is actually the case only for quadratic F functions. On the other hand, the non-holomorphic part of the Q connection given by t in eq. (3.24) is proportional to the third derivative of f which consistently shows that for quadratic f, the connection becomes a $\mathrm{SU}(2) \times \mathrm{SU}(n+1) \times \mathrm{U}(1)$ connection, as appropriate for the Kähler quaternionic manifold $\mathrm{SU}(2, n+1)/\mathrm{SU}(2) \times \mathrm{SU}(n+1) \times \mathrm{U}(1)$. In view of the relation between the third derivative of f and the Yukawa couplings in heterotic superstrings, the previous result implies that those moduli fields which correspond to vanishing Yukawa coupling correspond to a $\mathrm{SU}(1, n_0)/\mathrm{U}(1) \times \mathrm{SU}(n_0)$ restricted Kähler submanifold whose s-map is the quaternionic Kähler manifold

$$\frac{\mathrm{SU}(2, n_0+1)}{\mathrm{SU}(2) \times \mathrm{SU}(n_0+1) \times \mathrm{U}(1)}. \tag{4.2}$$

The general form of the $\mathrm{Sp}(n+1)$ connection in eq. (3.24) actually tells us that the above manifolds are the only Q manifolds which are also Kähler.

The Kähler potential of these manifolds is (see sect. 2)

$$K_{\mathrm{Q}} = K + \tilde{K} = -\ln 2(1 - Z^a \bar{Z}^a) - \ln\big(S + \bar{S} + \tfrac{1}{2}(C + \bar{C}) R^{-1} (C + \bar{C})\big), \tag{4.3}$$

with

$$R^{-1} = -\frac{2}{1 - Z^a \bar{Z}^a} \begin{bmatrix} 1 + Z^c \bar{Z}^c & Z^b + \bar{Z}^b \\ Z^a + \bar{Z}^a & \delta_{ab}(1 - Z^c \bar{Z}^c) + Z^a \bar{Z}^b + \bar{Z}^a Z^b \end{bmatrix}. \tag{4.4}$$

The other example we would like to mention is the s-map for the one-dimensional Kähler manifolds SU(1, 1)/U(1) corresponding to the holomorphic functions

$$f_1(Z) = 1 - Z^2, \qquad f_2(Z) = iZ^3, \qquad (4.5), (4.6)$$

respectively. These Kähler manifolds only differ from the value of the scalar curvature (-2 and $-\frac{2}{3}$, respectively); however, this s-map gives rise to completely different two-dimensional Q spaces.

According to ref. [1], the dual quaternionic manifolds should be SU(2, 2)/SU(2) × SU(2) × U(1) and G_2/SU(2) × SU(2), respectively.

In order to understand this point, it is sufficient to see the holomony group in these different cases. For the quadratic case we already know that the holomony groups is SU(2) × SU(2) × U(1). For the cubic case, we have a non-vanishing t matrix. The Sp(2) connection is given in terms of 2×2 matrices q, t (see eq. (3.24)) given by

$$q = \begin{bmatrix} 3X & E^* \\ -E & X \end{bmatrix}, \qquad X = -\tfrac{1}{4}(v - \bar{v}) - \frac{\mathrm{d}Z + \mathrm{d}\bar{Z}}{Z - \bar{Z}}, \tag{4.7}$$

$$t = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{2}{\sqrt{3}} E^* \end{bmatrix}. \tag{4.8}$$

The Sp(2) connection is given by a 4×4 matrix-valued one-form which only depends on the three differential forms

$$X, E, E^*. \tag{4.9}$$

This is the four-dimensional (spin-$\frac{3}{2}$) representation of the Sp(1) algebra with defining (spin-$\frac{1}{2}$) representation given by

$$\begin{bmatrix} +X & -\dfrac{E}{\sqrt{3}} \\ \dfrac{E^*}{\sqrt{3}} & -X \end{bmatrix}. \tag{4.10}$$

To realize the above statement it is sufficient to set $E=0$ (Cartan subalgebra L_3) and to see that the Sp(2) matrix has diagonal entries given by $(3X, X, -3X, -X)$ which are the eigenvalues of L_3 in the spin-$\frac{3}{2}$ representation.

In agreement with ref. [13] we see that the holomony group is $Sp(1)\times Sp(1)$ in this case, corresponding to the exceptional quaternionic space $G_2/SO(4)$.

5. Conclusions

In this paper we have worked out the geometry of the dual quaternionic manifolds which describe the effective interactions of the moduli fields of (2,2) superstring vacua and of their (Ramond–Ramond) scalar superpartners in type II strings. These results have been obtained by space-time supersymmetry arguments, but it is likely that they can be reproduced with a S-matrix approach as discussed recently by Dixon et al. [2].

Quaternionic spaces which correspond to non-vanishing Yukawa couplings have a complicated structure unless the Yukawa couplings are independent of the moduli. In the latter case, for untwisted sectors of orbifold compactifications, one may even get symmetric spaces of the type discussed in refs. [1,8]. However, when twisted fields are involved or general Calabi–Yau compactifications are considered, these spaces are not symmetric nor seem to be homogeneous. For those moduli fields which correspond to vanishing Yukawa couplings, the quaternionic subspaces are both Kähler and symmetric. More generally, at each point of the moduli space, the pure Ramond–Ramond sector is a $SU(1,n+2)/U(1)\times SU(n+2)$ submanifold. The results we described can alternatively be obtained by a Kaluza–Klein compactification of the ten-dimensional $N=2$ supergravity on Calabi–Yau spaces. In that case the extra Ramond–Ramond scalars come, in the case of type IIA supergravity, from the (2,1) and (3,0) internal components of the three-form A_{ijk}. The third and fourth derivative of the holomorphic function F, which appear in the Q curvature (eqs. (3.24) and (3.26)), are related to the following overlapping integrals on Calabi–Yau spaces [3,4]:

$$\int_K \Omega \wedge \frac{\partial\Omega}{\partial Z_i\,\partial Z_j\,\partial Z_k}, \qquad \int_K \Omega \wedge \frac{\partial\Omega}{\partial Z_i\,\partial Z_j\,\partial Z_k\,\partial Z_l},$$

where Ω is a three-form as a function of the deformation of the complex structure. It is of interest to investigate whether the field theory formulae, the formulae obtained by three-dimensional duality and the ones obtained by an S-matrix approach give the same answer. Besides the physical motivation related to the point-field limit of superstring theories, the interest on the construction of the dual quaternionic manifolds is that they offer examples of continuous families of manifolds whose geometry depends on a holomorphic function. These manifolds are the generalization of the class of manifolds studied in refs. [19–21], which were related to symmetric or homogeneous (restricted) Kähler manifolds. We have given here a component description of these manifolds. From the paper of ref. [13], this is sufficient to write the entire $N=2$ supergravity action. However, if one wants to describe arbitrary invariants and coupling to other multiplets, a general tensor calculus is required. The tensor calculus developed in ref. [6] is not at present suitable to describe general quaternionic manifolds. The superfield formulation based on a harmonic superspace [22] seems to be more adequate. A formulation of the c-map in superspace would be a very useful simplifying tool.

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