Mathematical Finance and Black-Scholes Model

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Abstract

This article mainly explains two different methods of deriving the Black-Scholes pricing formula, non-arbitrage pricing and risk-neutral pricing. Non-arbitrage pricing construct an asset portfolio and use the no-arbitrage principle to obtain the Black-Scholes partial differential equation. Risk-neutral pricing use the equivalent martingale measure and martingale properties to obtain the risk-neutral pricing formula. We will start from definitions of martingale, Brownian motion and Itô's lemma.

1 Basic Concepts

Definition 1.1 (Martingale)**.** Let $\{X_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ be two sequences of random variables. Let $Z_n = \sum_{t=0}^{n} X_t$. We say $\{Z_n\}_{n\geq 0}$ is a martingale w.r.t. $\{X_n\}_{n\geq 0}$ if

$$\mathrm{E}\left[Z_{n+1} \mid X_0, X_1 \ldots, X_n\right] = Z_n.$$

Sometimes we say a single sequence $\{X_n\}_{n\geq 0}$ is a martingale if it is a martingale w.r.t. itself. Formally, if for every $n \geq 0$, it holds that

$$\mathbf{E}\left[X_{n+1} \mid X_0, \ldots, X_n\right] = X_n.$$

For convenience, from now on we use $\overline{X}_{i,j} = (X_i, X_{i+1} \ldots, X_j)$ to simplify the notations. The conditional expectation $\mathbf{E}\left[Z_{n+1} \mid \overline{X}_{0,n}\right]$ is equivalent to $\mathbf{E}\left[Z_{n+1} \mid \sigma\left(\overline{X}_{0,n}\right)\right]$ where $\sigma\left(\overline{X}_{0,n}\right)$ is the σ-algebra generated by $X_0, \ldots, X_n$. This motivates us to define martingale in a more general way.

Definition 1.2 (Martingale (defined by filtration))**.** Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a sequence of σ-algebras. We call such σ-algebra sequence a filtration if it satisfies

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \cdots.$$

Given a filtration $\{\mathcal{F}_n\}_{n\geq 0}$, let $\{Z_n\}_{n\geq 0}$ be a stochastic process that Z_n is $\mathcal{F}_n$-measurable for every $n \geq 0$. Then we say $\{Z_n\}_{n\geq 0}$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\geq 0}$ if for every $n \geq 0$

$$\mathrm{E}\left[Z_{n+1} \mid \mathcal{F}_n\right] = Z_n.$$

Definition 1.3 (Standard Brownian Motion/Wiener Process)**.** We say a stochastic process $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion or Wiener process if it satisfies[4]

- $W(0) = 0$;
- **Independent increments**: $\forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$,
  $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent;

- **Stationary increments**: $\forall s, t > 0, W(s+t) - W(s) \sim \mathcal{N}(0, t)$;
- $W(t)$ is continuous almost surely.

There are some properties of Brownian motion:

- $\mathbb{E}(W(t)) = 0$;
- $\mathrm{Var}(W(t)) = t = \mathbb{E}(W^2(t))$;
- $\mathrm{Cov}(W(s), W(t)) = \mathbb{E}(W(t)W(s)) = s \wedge t$.

Thus standard Brownian motion is a special family of Gaussian processes where the covariance of $W(s)$ and $W(t)$ is $s \wedge t$.

Theorem 1.1 (Brownian motion is a martingale)**.** $\mathrm{E}\left[W(t) \mid \mathcal{F}(s)\right] = W(s),\ \forall 0 \leq s \leq t.$

Proof.

$$\begin{aligned}
\mathbb{E}(W(t)|\mathcal{F}(s)) &= \mathbb{E}(W(t) - W(s) + W(s)|\mathcal{F}(s)) \\
&= \mathbb{E}(W(t) - W(s)|\mathcal{F}(s)) + \mathbb{E}(W(s)|\mathcal{F}(s)) \\
&= \mathbb{E}(W(t) - W(s)) + W(s) \\
&= W(s)
\end{aligned}$$

□

2 Stochastic Calculus

2.1 Quadratic Variation

As we have learned in Riemann integral, the first variation of differentiable functions equals to zero. Thus the quadratic variation of differentiable functions is also zero. Yet things are different for Brownian motion.

Definition 2.1 (Quadratic Variation)**.** The quadratic variation of a function f on an interval $[0, T]$ is

$$\langle f\rangle(T) = \lim_{\|\Pi\| \to 0} \sum_{k=0}^{n-1} \left|f\left(t_{k+1}\right) - f\left(t_k\right)\right|^2.$$

Remark 2.1. If f is differentiable, then $\langle f\rangle(T) = 0$, because

$$\begin{aligned}
\sum_{k=0}^{n-1} \left|f\left(t_{k+1}\right) - f\left(t_k\right)\right|^2 &= \sum_{k=0}^{n-1} \left|f'\left(t_k^*\right)\right|^2 \left(t_{k+1} - t_k\right)^2 \\
&\leq \|\Pi\| \cdot \sum_{k=0}^{n-1} \left|f'\left(t_k^*\right)\right|^2 \left(t_{k+1} - t_k\right)
\end{aligned}$$

and

$$\begin{aligned}
\langle f\rangle(T) &\leq \lim_{\|\Pi\| \to 0} \|\Pi\| \cdot \lim_{\|\Pi \to 0} \sum_{k=0}^{n-1} \left|f'\left(t_k^*\right)\right|^2 \left(t_{k+1} - t_k\right) \\
&= \lim_{\|\Pi\| \to 0} \|\Pi\| \int_0^T \left|f'(t)\right|^2 dt \\
&= 0
\end{aligned}$$

However, the paths of Brownian motion are not differentiable, so the quadratic variation of Brownian motion is different from differentiable functions.

Theorem 2.1 (Quadratic Variation of Brownian motion)**.** $\langle W\rangle(T) = T$

Remark 2.2 (Differential Representation). By the properties of Brownian motion, we know that

$$\mathbb{E}\left[\left(W\left(t_{k+1}\right)-W\left(t_k\right)\right)^2\right]=t_{k+1}-t_k.$$

$$\operatorname{Var}\left[\left(W\left(t_{k+1}\right)-W\left(t_k\right)\right)^2\right]=2\left(t_{k+1}-t_k\right)^2.$$

When $(t_{k+1}-t_k)$ is small, $(t_{k+1}-t_k)^2$ is very small, and we have the approximate equation

$$\left(W\left(t_{k+1}\right)-W\left(t_k\right)\right)^2 \simeq t_{k+1}-t_k$$

which we can write informally as

$$dW(t)dW(t) = dt$$

By definition of $\{W(t)\}$, we know that $dW(t) = W(t + dt)W(t) \sim N(0, dt)$. Let X and Y be two random variables that $X \sim N(0, dt)$ and $Y = X^2$. Then the formula $(dW(t))^2 \simeq dt$ tells us that Y is well concentrated on dt.

2.2 Construction of Itô Integral

The integrator is Brownian motion $W(t), t \geq 0$, with associated filtration $\mathcal{F}(t), t \geq 0$, and the following properties: [3]

1. $s \leq t \implies$ every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$;
2. $\forall t, W(t)$ is $\mathcal{F}(t)$-measurable;
3. For $t \leq t_1 \leq \ldots \leq t_n$, the increments $W(t_1) - W(t), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent of $\mathcal{F}(t)$.

The integrand is $\delta(t), t \geq 0$, where

1. $\forall t, \delta(t)$ is $\mathcal{F}(t)$-measurable(i.e., δ is adapted);
2. δ is square-integrable:
$$\mathbb{E}\int_0^T \delta^2(t)dt < \infty, \quad \forall T.$$

We define the Itô Integral:

$$\begin{aligned} I(t) &= \int_0^t \delta(u)dW(u), \quad t \geq 0 \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\delta(t_{k-1})(W(t_k) - W(t_{k-1})). \end{aligned}$$

Remark 2.3. The Itô Integral has a natural intuition in finance. The left endpoints of the intervals in the limiting process are chosen. This could be interpreted as the fact that in finance you don't know any future stock prices.

2.3 The Chain Rule for Itô Integral

Using the Taylor expansion:

$$
\begin{aligned}
\mathrm{d}f(W(t)) &= f(W(t)+\mathrm{d}W(t)) - f(W(t)) \\
&= f'(W(t))\mathrm{d}W(t) + \frac{1}{2}f''(W(t))(\mathrm{d}W(t))^2 + \frac{1}{6}f'''(W(t))(\mathrm{d}W(t))^3 + o\left((\,\mathrm{d}W(t))^3\right).
\end{aligned}
$$

Taylor's formula to second order is exact because $W(t)$ is a quadratic function. This yields the Itô formula

$$
\mathrm{d}f(W(t)) = f'(W(t))\mathrm{d}W(t) + \frac{1}{2}f''(W(t))\mathrm{d}t, \text{ as } \mathrm{d}t \to 0.
$$

2.4 Geometric Brownian Motion

Definition 2.2 (Geometric Brownian Motion)**.**

$$
S(t) = S(0)\exp\left\{\sigma W(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\},
$$

where μ and $\sigma > 0$ are constant.

We can apply the Itô's formula discussed above.
Define

$$
f(t,x) = S(0)\exp\left\{\sigma x + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\},
$$

so

$$
S(t) = f(t, W(t))
$$

Then

$$
f_t = \left(\mu - \frac{1}{2}\sigma^2\right)f, f_x = \sigma f, f_{xx} = \sigma^2 f.
$$

According to Itô's formula,

$$
\begin{aligned}
dS(t) &= df(t, W(t)) \\
&= f_t dt + f_x dW + \frac{1}{2}f_{xx}\underbrace{dWdW}_{dt} \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)fdt + \sigma f dW + \frac{1}{2}\sigma^2 f dt \\
&= \mu S(t)dt + \sigma S(t)dW(t)
\end{aligned}
$$

Thus, Geometric Brownian motion in differential form is

$$
dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2.1}
$$

and Geometric Brownian motion in integral form is

$$
S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t \sigma S(u)dW(u)
$$

3 Black-Scholes-Merton differential equation

Suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, by Itô's lemma and 2.1,

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \\ &= \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dW(t) \end{aligned} \tag{3.1}$$

The discrete versions of equations 2.1 and 3.1 are

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta W$$

where Δf and ΔS are the changes in f and S in a small time interval Δt. It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated. We assume the portfolio is: -1 derivative and $+k$: shares.[2]

The holder of this portfolio is short one derivative and long an amount k of shares. Define Π as the value of the portfolio. By definition $\Pi = -f + kS$. So we have

$$\begin{aligned} d\Pi &= -df + kdS \\ &= -\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - k\mu S\right)dt + \left(-\frac{\partial f}{\partial S}\sigma S + k\sigma S\right)dW(t). \end{aligned}$$

In order to hedge all risks, we can let $k = \frac{\partial f}{\partial S}$ to eliminate $dW(t)$ leading to

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{3.2}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\begin{aligned} \Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S}\Delta S \\ &= \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \end{aligned} \tag{3.3}$$

Because this equation does not involve ΔW, the portfolio must be riskless during time Δt. The non-arbitrage pricing assumption imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free securities. It follows that

$$\Delta\Pi = r\Pi\Delta t \tag{3.4}$$

where r is the risk-free interest rate. Substituting from equations 3.2 and 3.3 into 3.4, we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{3.5}$$

In the case of a European call option, the key boundary condition is $f = \max(S - K, 0)$, when $t = T$. By solving 3.5, we can obtain the option pricing formula.

4 Martingales and Measures

4.1 Asset dynamics

The market model contains two underlying securities.

- The risk-free asset, described by a deterministic function $dB_t = rB_t dt$, with $B_0 = 1$ (for convenience), where $r > 0$ is the risk-free rate. This is an ordinary differential equation with a unique solution: $B_t = e^{rt}$.
- The risky asset, thought of as a stock, is represented by an Itô process of the form:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{4.1}$$

with $S(0)$ given, where we call μ the drift, and $\sigma > 0$ the volatility of the stock price S.

4.2 risk-neutral probability

We consider the discounted stock price process $\widetilde{S}(t) = e^{-rt}S(t)$, which, by the definition of $S(t)$, becomes

$$\begin{aligned}\widetilde{S}(t) &= e^{-rt}S(0)\exp\left\{\sigma W(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\}\\ &= S(0)\exp\left\{(\mu - r)t - \frac{1}{2}\sigma^2 t + \sigma W(t)\right\}\end{aligned}$$

To explore situations where this becomes a martingale with respect to the probability P, we compute conditional expectations.

$$\begin{aligned}\mathbb{E}(\widetilde{S}(t)|\mathcal{F}(u)) &= S(0)\exp\left\{(\mu - r)t - \frac{1}{2}\sigma^2 t\right\}\exp\{\sigma W(u)\}\mathbb{E}\left(\exp\{\sigma[W(t) - W(u)]\}|\mathcal{F}(u)\right)\\ &= S(0)\exp\left\{(\mu - r)t - \frac{1}{2}\sigma^2 t\right\}\exp\{\sigma W(u)\}\exp\{\frac{1}{2}\sigma^2(t-u)\}\\ &= \widetilde{S}(u)\exp\{(\mu - r)(t-u)\}.\end{aligned}$$

Thus the process would be a martingale if the drift parameter μ were equal to the risk-free rate: $\mu = r$. While we cannot simply impose this condition on the expected values of the risky stock price, we may hope to make progress by changing the probability measure to a probability Q with the same null sets as P. (This restriction is relevant since the no-arbitrage condition concerns 'almost sure' properties of strategy values.) The question is how to find such a Q.[1]

It's easy to prove that $\exp\{-\frac{1}{2}\sigma^2 t + \sigma W(t)\}$ is a martingale where $W(t)$ is a Wiener process, so we try to write the discounted stock prices in a similar form.

$$\tilde{S}(t) = S(0)\exp\left\{-\frac{1}{2}\sigma^2 t + \sigma\left(\frac{\mu - r}{\sigma}t + W(t)\right)\right\}.$$

Setting

$$W_Q(t) = \frac{\mu - r}{\sigma}t + W(t)$$

we have

$$\tilde{S}(t) = S(0)\exp\left\{-\frac{1}{2}\sigma^2 t + \sigma W_Q(t)\right\}.$$

If we can construct a probability measure Q under which W_Q is a Wiener process, then $\tilde{S}(t)$ will be a martingale with respect to Q. To simplify notation, write $W_Q(t) = bt + W(t)$, where $b = \frac{\mu - r}{\sigma}$.

$$\begin{aligned}
\mathbb{E}_P\left(W_Q(t)\right) &= \int_{-\infty}^{\infty} (bt + x) f_{W(t)}(x) dx \\
&= \int_{-\infty}^{\infty} (bt + x) \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}} dx \\
&= bt.
\end{aligned}$$

In order to obtain zero it is sufficient to change the form of the density:

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} (bt + x) \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{(x+b)^2}{2t}} dx \\
&= \int_{-\infty}^{\infty} (bt + x) \mathrm{e}^{-\frac{1}{2}b^2 t - bx} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}} dx \\
&= \int_{-\infty}^{\infty} (bt + x) \mathrm{e}^{-\frac{1}{2}b^2 t - bx} f_{W(t)}(x) dx \\
&= \int_{\Omega} (bt + W(t)) \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)} dP,
\end{aligned}$$

For $A \subset \Omega, A \in \mathcal{F}$, we define a new measure Q by setting

$$Q(A) = \int_A \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)} dP$$

so that

$$\int_{\Omega} (bt + W(t)) \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)} dP = \int_{\Omega} (bt + W(t)) dQ.$$

Since $\mathbb{E}_P\left(W_Q(t) e^{-\frac{1}{2}b^2 t - bW(t)}\right) = \mathbb{E}_Q\left(W_Q(t)\right)$, the random variable

$$M(t) = \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)}$$

plays the role of a density of Q with respect to P. The process $M(t)$ is a martingale so it has constant expectation. For $t = 0, M(0) = 1$ so $\mathbb{E}_P(M(t)) = 1$, in other words

$$\int_{\Omega} \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)} dP = 1$$

for all t. Therefore all $M(t)$ meet the conditions for being a probability density function, and the measures P and Q are equivalent in the sense that their collections of null sets are the same.

Theorem 4.1 (Girsanov theorem - simple version)**.** *The process $W_Q(t) = bt + W(t)$ is a Wiener process under the probability Q defined by:*

$$Q(A) = \int_A \mathrm{e}^{-\frac{1}{2}b^2 t - bW(t)} dP$$

In the probability space $(\Omega, \mathcal{F}, Q)$ the stock price dynamics are said to be risk-neutral. This means that there is no drift when we work with discounted values, which turns the discounted stock price into a stochastic integral. In this situation,

$$S(t) = S(0) \exp\left\{\mu t - \frac{1}{2}\sigma^2 t + \sigma W(t)\right\}$$

$$= S(0)\exp\left\{rt - \frac{1}{2}\sigma^2 t + \sigma W_Q(t)\right\}$$

By Itô's lemma, we have

$$dS(t) = rS(t)dt + \sigma S(t)dW_Q(t) \tag{4.2}$$

which is a modification of equation 4.1 in a risk-neutral world.

Applying formula for the differential of the product in stochastic differential problem,

$$d(X_1X_2) = X_1dX_2 + X_2dX_1 + b_1b_2dt$$

where

$$dX_i = a_idt + b_idW, \quad i = 1, 2.$$

We take $X_1(t) = e^{-rt}, X_2 = S$, so that $dX_1(t) = -re^{-rt}dt$ and $b_1 = 0$, the discounted stock price process $\widetilde{S}(t) = e^{-rt}S(t)$ can be transformed.

$$\begin{aligned}
d\tilde{S}(t) &= d\left(\mathrm{e}^{-rt}S(t)\right) \\
&= -\mathrm{re}^{-rt}S(t)dt + \mathrm{e}^{-rt}dS(t) \\
&= (\mu - r)\tilde{S}(t)dt + \sigma\tilde{S}(t)dW(t) \\
&= \tilde{S}(t)\sigma dW_Q(t).
\end{aligned}$$

This means $\tilde{S}(t)$ is a martingale.

4.3 martingale

Options are referred to as in the money, at the money, or out of the money. If S is the stock price and K is the strike price, a call option is in the money when $S > K$, at the money when $S = K$, and out of the money when $S < K$. The intrinsic value of an option is defined as the value it would have if there were no time to maturity, so that the exercise decision had to be made immediately. For a call option, the intrinsic value is therefore $\max(S - K, 0)$.

The profit of a European call option at expiration date is $C(T) = (S(T) - K)^+$. Since $\widetilde{S}(t) = e^{-rt}S(t)$ is a martingale, $e^{-rt}C(t)$ is also a martingale.

$$\begin{aligned}
C(0) &= e^{-rT}\mathbb{E}_Q\left(C(T)|S(0) = S\right) \\
&= e^{-rT}\int_{-\infty}^{\infty}\left(S\exp\left[(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}\right] - K\right)^+ f_Z(z)dz \\
&= e^{-rT}\int_{z_0}^{\infty}\left(S\exp\left[(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}\right] - K\right) f_Z(z)dz
\end{aligned}$$

where $S\exp\left[(r - \frac{1}{2}\sigma^2)T + \sigma z_0\sqrt{T}\right] = K$, i.e. $z_0 = \frac{ln(\frac{K}{S}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

$$\begin{aligned}
C(0) &= e^{-rT}\int_{z_0}^{\infty}\left(S\exp\left[(r - \frac{1}{2}\sigma^2)T + \sigma z\sqrt{T}\right] - K\right) f_Z(z)dz \\
&= Se^{-\frac{1}{2}\sigma^2 T}\int_{z_0}^{\infty}\exp(\sigma z\sqrt{T})f_Z(z)dz - Ke^{-rT}\int_{z_0}^{\infty}f_Z(z)dz \\
&= SN(d_1) - Ke^{-rT}N(d_2).
\end{aligned}$$

where $d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$. This is the final Black-Scholes option pricing formula.

References

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