APMA 4302 – Homework 2 Solutions

Problem 1: Condition Number and Error Bounds (10 pts)

The condition number of a matrix A is $\kappa(A) = \|A\|\|A^{-1}\|$, where $\|Ax\| \leq \|A\|\|x\|$.

(a) Assume A is invertible and $Au = b$. Show that the relative error in the solution satisfies

$$\frac{\|u - \hat{u}\|}{\|u\|} \leq \kappa(A)\frac{\|b - \hat{b}\|}{\|b\|},$$

where $\hat{u}$ solves $A\hat{u} = \hat{b}$ (perturbed right-hand side).

(b) Suppose $Au = b$ and $Ae = r$. Show that

$$\frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|u\|} \leq \kappa(A)\frac{\|r\|}{\|b\|},$$

where $e = u - \hat{u}$ and $r = b - A\hat{u}$ (residual). Give an interpretation.

Solution

Part (a). We have $u = A^{-1}b$ and $\hat{u} = A^{-1}\hat{b}$, so $u - \hat{u} = A^{-1}(b - \hat{b})$. By the matrix norm property,

$$\|u - \hat{u}\| = \|A^{-1}(b - \hat{b})\| \leq \|A^{-1}\|\|b - \hat{b}\|.$$

Also $b = Au$ gives $\|b\| \leq \|A\|\|u\|$, so $\|u\| \geq \|b\|/\|A\|$. Hence

$$\frac{\|u - \hat{u}\|}{\|u\|} \leq \frac{\|A^{-1}\|\|b - \hat{b}\|}{\|u\|} \leq \|A^{-1}\|\|b - \hat{b}\| \cdot \frac{\|A\|}{\|b\|} = \kappa(A)\frac{\|b - \hat{b}\|}{\|b\|}.$$

Part (b). By definition, $r = b - A\hat{u} = A(u - \hat{u}) = Ae$, so $Ae = r$.
   Upper bound: $e = A^{-1}r$ implies $\|e\| \leq \|A^{-1}\|\|r\|$. With $\|b\| \leq \|A\|\|u\|$ we get $\|u\| \geq \|b\|/\|A\|$. Thus

$$\frac{\|e\|}{\|u\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}.$$

   Lower bound: $r = Ae$ gives $\|r\| \leq \|A\|\|e\|$, so $\|e\| \geq \|r\|/\|A\|$. Also $\|u\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\|$. Therefore

$$\frac{\|e\|}{\|u\|} \geq \frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|}.$$

Interpretation. The relative error $\|e\|/\|u\|$ is bracketed by the relative residual $\|r\|/\|b\|$ scaled by $1/\kappa(A)$ and by $\kappa(A)$. A small residual can still correspond to a large relative error when $\kappa(A)$ is large (ill-conditioning). For well-conditioned A, error and residual are of the same order.

Problem 2: Discrete Laplacian Matrix (10 pts)

Let

$$A = \frac{1}{h^2} \operatorname{tridiag}(-1, 2, -1)$$

be the symmetric positive definite matrix from the centered finite-difference discretization of $-u''(x)$ on $x \in [0, 1]$ with homogeneous Dirichlet boundary conditions. Here $h = 1/m$ and A is $(m-1) \times (m-1)$.

(a) Show that the eigenvectors of A are $v_j(i) = \sin(j\pi x_i)$ with $x_i = ih$, $i = 1, \ldots, m-1$.

(b) Find the corresponding eigenvalues λ_j.

(c) Show that $\kappa(A) \sim O(m^2)$ as $m \to \infty$.

Solution

Part (a): Eigenvectors. Apply the discrete Laplacian to v_j at interior index i. The stencil $(1/h^2)(-1, 2, -1)$ gives

$$(Av_j)(i) = \frac{1}{h^2}\left[-v_j(i-1) + 2v_j(i) - v_j(i+1) \right].$$

Using $x_i = ih$ and $v_j(i) = \sin(j\pi x_i) = \sin(j\pi ih)$:

$$-\sin(j\pi(i-1)h) + 2\sin(j\pi ih) - \sin(j\pi(i+1)h) = 2\sin(j\pi ih) - \left[\sin(j\pi ih - j\pi h) + \sin(j\pi ih + j\pi h) \right].$$

By $\sin(\alpha - \theta) + \sin(\alpha + \theta) = 2\sin(\alpha)\cos(\theta)$ with $\alpha = j\pi ih$, $\theta = j\pi h$:

$$= 2\sin(j\pi ih) - 2\sin(j\pi ih)\cos(j\pi h) = 2\sin(j\pi ih)\left[1 - \cos(j\pi h) \right] = 4\sin(j\pi ih)\sin^2(j\pi h/2).$$

Thus $(Av_j)(i) = (4/h^2)\sin^2(j\pi h/2)\, v_j(i)$, so v_j is an eigenvector with eigenvalue $\lambda_j = (4/h^2)\sin^2(j\pi h/2)$.

Part (b): Eigenvalues.

$$\lambda_j = \frac{4}{h^2}\sin^2\left(\frac{j\pi h}{2}\right) = \frac{4}{h^2}\sin^2\left(\frac{j\pi}{2m}\right), \qquad j = 1, \ldots, m-1.$$

Equivalently, $\lambda_j = 4m^2 \sin^2(j\pi/(2m))$.

Part (c): Condition number. For SPD A, $\kappa(A) = \lambda_{\max}/\lambda_{\min}$. The largest eigenvalue is $\lambda_{m-1} = (4/h^2)\sin^2((m-1)\pi/(2m)) \to 4/h^2 = 4m^2$ as $m \to \infty$. The smallest is $\lambda_1 = (4/h^2)\sin^2(\pi/(2m))$; for large m, $\sin(\pi/(2m)) \approx \pi/(2m)$, so $\lambda_1 \approx \pi^2$. Hence $\kappa(A) \sim (4m^2)/\pi^2 = O(m^2)$ as $m \to \infty$.

Problem 3: Boundary Value Problem with PETSc (20 pts)

BVP: $-u''(x) + \gamma u(x) = f(x)$ on $x \in [0, 1]$ with Dirichlet boundary conditions.

(a) Find $f(x)$ for the manufactured solution

$$u(x) = \sin(k\pi x) + c\left(1 - \frac{1}{2}\right)^3,$$

where k is a positive integer and c is a real constant.

(b) Modify `tri.c` (p4pdes Ch. 2) to solve this BVP with PETSc: run with `mpiexec -np P ./bvp -options_file options_file`; assemble matrix and RHS in parallel; use `MatZeroRowsColumns` to enforce Dirichlet BCs; compute and print relative error; output solution, exact solution, and RHS to HDF5; use `plot_bvp.py` to visualize.

(c) Modify `plot_bvp.py` to plot error vs. h for $\gamma = 0$, $k = 1, 5, 10$, and $m = 40, 80, 160, \ldots, 1280$. Report observed order of convergence.

Solution

Part (a): Manufactured solution and $f(x)$. With the given manufactured solution we have

$$u(x) = \sin(k\pi x) + c\left(1 - \frac{1}{2}\right)^3 = \sin(k\pi x) + \frac{c}{8}.$$

Differentiating: $u'(x) = k\pi \cos(k\pi x)$ and $u''(x) = -k^2\pi^2 \sin(k\pi x)$. Substituting into $-u'' + \gamma u = f(x)$ gives

$$-u'' + \gamma u = k^2\pi^2 \sin(k\pi x) + \gamma \sin(k\pi x) + \frac{\gamma c}{8},$$

hence

$$\boxed{f(x) = (k^2\pi^2 + \gamma)\sin(k\pi x) + \frac{\gamma c}{8}.}$$

Note. If the intended solution was $u(x) = \sin(k\pi x) + c(1 - x)^3$, then f would include extra terms from the $(1 - x)^3$ part.

Part (b) See `bvp.c` in this directory. It registers options `-bvp_m`, `-bvp_gamma`, `-bvp_k`, `-bvp_c`; assembles the $(m - 1) \times (m - 1)$ matrix with stencil $(1/h^2)(-1, 2, -1) + \gamma$ on the diagonal; builds RHS and exact solution; uses `MatZeroRowsColumns` for boundary DoFs; solves, computes relative error, and writes u, f, uexact to `bvp_solution.h5`.

Part (c) The script runs the BVP for the given m and k, reads the relative error, and plots error vs. $h = 1/m$ on a log-log scale. The centered second-order scheme gives expected order ≈ 2 (error $\sim h^2$).

Problem 4: Solver Performance (10 pts)

Using the default options file, determine the number of iterations to converge for each solver and explain your results.

(a) Jacobi-preconditioned Richardson: `-ksp_type richardson -pc_type jacobi`
(b) Unpreconditioned CG, 1 proc: `-ksp_type cg -pc_type none`
(c) Unpreconditioned CG with $c = 0$.
(d) ICC-preconditioned CG, 1 proc: `-ksp_type cg -pc_type icc`
(e) Block Jacobi + ICC, 4 procs: `-ksp_type cg -pc_type bjacobi -pc_sub_type icc`
(f) MUMPS direct, 1 and 4 procs: `-ksp_type preonly -pc_type lu -pc_factor_solver_type mumps`

Solution

Fill in iteration counts from your runs:

Part	Solver	Procs	Iterations
(a)	Richardson + Jacobi	1	_____
(b)	CG, no PC	1	_____
(c)	CG, no PC, $c = 0$	1	_____
(d)	CG + ICC	1	_____
(e)	CG + bjacobi (icc)	4	_____
(f)	MUMPS (direct)	1	—
(f)	MUMPS (direct)	4	—

(a) Richardson + Jacobi: $\kappa(A) \sim O(m^2)$, so convergence is slow; expect many iterations. Jacobi only scales, it does not fix conditioning.

(b) Unpreconditioned CG: iterations scale like $O(\sqrt{\kappa}) \sim O(m)$.

(c) CG with $c = 0$: the matrix A is unchanged; iteration count should match (b).

(d) ICC + CG: strong preconditioner for SPD A; expect a large drop in iterations compared to (b).

(e) Block Jacobi + ICC: weaker than global ICC; more iterations than (d), fewer than (b). Good parallel scalability.

(f) MUMPS: direct solver, no iterations; compare run time and memory for 1 vs. 4 processors.