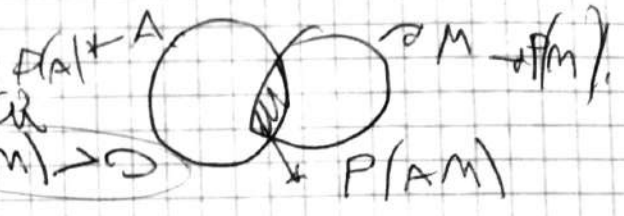
STOCHASTIC PROCESSES AN QUEUING THEORY MIDTERM - QUESTIONS 1-70

1) CONDITIONAL PROBABILITY

$$P(A|M) = \frac{P(AM)}{P(M)}$$

with $P(M) > 0$



2) TOTAL PROBABILITY THEOREM FOR EVENTS

Given $A_1, \ldots A_n$ events

① $A_i \cdot A_j = \emptyset \quad i \neq j$

(n mutually exclusive events)

② $A_1 + A_2 + \ldots + A_n = S$

(sum of events is S)

$$P(B) = P(B|A_1) \cdot P(A_1) + \ldots + P(B|A_2) \cdot P(A_2)$$

3) BAYES' FORMULA - 1st type

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

with $P(B) \neq 0$

4) INDEPENDENT EVENTS.

For independent events $A_1, A_2, A_3 \rightarrow$

$$P(A_1 \cdot A_2) = P(A_1) \cdot P(A_2)$$
$$P(A_2 \cdot A_3) = P(A_2) \cdot P(A_3)$$
$$P(A_1 \cdot A_3) = P(A_1) \cdot P(A_3)$$
$$P(A_1 \cdot A_2 \cdot A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

(for 2 events)

Events A and B are independent if

$$P(AB) = P(A) \cdot P(B)$$

$\Rightarrow P(A|B) = P(A)$ (B has no effect on A)

$P(B|A) = P(B)$ (A has no effect on B)

5) GENERAL EXPRESSION OF $P(A+B)$

$$P(A+B) = P(A) + P(B) - P(AB)$$

ADVANCED PROGRAMMING

6) PROPERTIES OF DISTRIBUTION FUNCTION $F_X(x)$

IDC - PROOF

1) $F_X(-\infty) = 0$

$F_X(+\infty) = 1$

(F: non decreasing)

2) $F_X(x)$ is a non-decreasing function

$F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

(continuous from the right)

3) $F_X(x)$ is continuous from the right

STAIRS $F_X(x^+) = F_X(x)$

4) $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

5) $P\{X = x\} = F_X(x) - F_X(x^-)$ (point)

6) if $F_X(x_0) = 0$, then $F_X(x) = 0 \quad \forall x \leq x_0$

7) $\lim_{x \to +\infty} F_X(x) = F_X(\infty^-) = F_X(\infty) = 1$

7) CLASSIFICATION OF 3 types of R.V and description of $F_X(x)$

- CONTINUOUS R.V:

A random variable X is <u>continuous</u> if $F_X(x)$ is continuous. The probability to be in a single point of $F_X(x)$ is 0. PROB. FROM LEFT = PROB. IN POINT.

$$F_X(x^-) = F_X(x) = P\{X = x\} = 0 \quad \forall x$$

- DISCRETE R.V.

A random variable X is <u>discrete</u> if $F_X(x)$ is staircase. In discontinuity points:

~~$F_X(x)$~~ $F_X(x_i) - F_X(x_i^-) = P\{X = x_i\} = p_i$

(step height)

There are discontinuity points

- MIXED R.V:

A random variable X is mixed if $F_X(x)$ is discontinuous but not staircase.

8) DEFINITION OF PROBABILITY DENSITY FUNCTION (PDF) AND PROPERTIES

~~and properties~~

DEFINITION

$$f_X(x) \triangleq \frac{d}{dx} F_X(x)$$

PNB (PN BALL)

PROPERTIES POSITIVE

1) ~~...~~ $f_X(x) \geq 0$ (since $F_X(x)$ is monotonic)

2) Integration of PDF $f_X(x)$ yields $F_X(x)$:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

3) NORMALIZATION

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

4) ~~BOUNDED ... FUNCTION~~

PDF of a bounded r.v.

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx$$

9) DEFINITION OF DELTA FUNCTION ($\delta(x)$) ASO

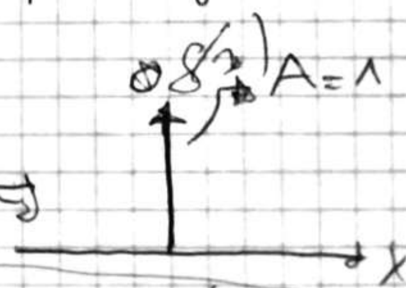
$$\delta(x) \triangleq \int_{-\infty}^{+\infty} \phi(x)\cdot\delta(x)\,dx = \phi(0)$$ $\forall \phi(x)$ continuous in 0

⇒ Function to describe the derivative in points of discontinuity.

ASO PROPERTIES

1) AREA OF SURE-DEFINED EVENT

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \Rightarrow$$



2) SAMPLING PROPERTY

$$\int_{-\infty}^{+\infty} \phi(x)\cdot\delta(x-x_0)\,dx = \phi(x_0)$$

3) DERIVATIVE OF UNITARY STEP

$$\delta(x) = \frac{d}{dx} u(x) \quad \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

10) PDF $f_X(x)$ of a Gaussian RANDOM VARIABLE, MEAN, VARIANCE:

For GAUSSIAN R.V. X:

$$f_X(x) = \frac{1}{\theta\sqrt{2\pi}} \cdot e^{-\frac{(x-\eta)^2}{2\theta^2}}$$

→ normalization

with parameters
η = mean
θ = S.D

MEAN: $\mu = E\{X\}$

VARIANCE: $\theta^2 = \theta_X^2 = E\{(X-\eta_X)^2\}$ (θ_X = standard deviation)

11) PDF OF EXPONENTIAL R.V.:

For exponential R.V. X:

$$f_X(x) = c \cdot e^{-c \cdot x} \, U(x)$$

where c is a parameter. $U(x)$ is the unitary step of x.

12) PDF OF ERLANG R.V. and relation with EXPONENTIAL R.V.:

for an Erlang R.V. X:

$$f_X(x) = \frac{c^n}{(n-1)!} \cdot x^{n-1} \cdot e^{-cx} \, U(x)$$

with c, n parameters and $U(x)$ unitary step.

If $n = 1$, Erlang PDF is exponential PDF.

$$f_X(x) = \frac{c^1}{(1-1)!} \cdot x^{1-1} \cdot e^{-cx} \, U(x)$$

$$f_X(x) = c \cdot e^{-cx} \cdot U(x)$$

13) CONDITIONAL DISTRIBUTION $F_X(x|M)$ → PROBABILITY!

$$F_X(x|M) = P\{X \le x | M\} = \frac{P\{X \le x, M\}}{P(M)}$$

with $P(M) > 0$

14) TOTAL PROBABILITY THEOREM FOR $F_X(x)$ given $A_1, \dots, A_n$ of S

If $A_1, \dots, A_n$ is a partition of S

$$F_X(x) = F_X(x|A_1)P(A_1) + \dots + F_X(x|A_n)P(A_n)$$

15) DERIVE $f_X(x|X \le a)$ in terms of $f_X(x)$

⇒ By definition of CONDITIONAL DISTRIBUTION

$F_X(x|M)$, where $M = \{X \le a\}$ (a is constant)

$$F_X\{x|X \le a\} = \frac{P\{X \le x, X \le a\}}{P\{X \le a\}} \quad \text{where } P\{X \le a\} \neq 0$$

$= P\{X \le x\}$

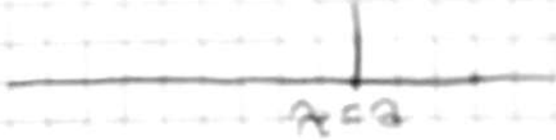
We have 3 cases (knowing $X \le a$)

1) $x > a$

⇒ $F_X(x|X \le a) = 1$

2) $x = a$

$F_X(x|X \le a) = 1$



3) $x < a$

$F_X(x|X \le a)$ is

$$\frac{F_X(x)}{F_X(a)}$$

$$f_X(x|X \le a) = \begin{cases} 0 & x \ge a \\ \dfrac{f_X(x)}{F_X(a)} & x < a \end{cases}$$

16) RELATIONSHIPS BETWEEN $F_X(x)$ and $f_X(x)$

(direct & inverse)

Given $F_X(x) \to f_X(x)$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Given $f_X(x) \to F_X(x)$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

17) MIXED BAYES FORMULA (2nd type Bayes)

$$P(A \mid X = x) = \frac{f_X(x \mid A) \cdot P(A)}{f_X(x)}$$

18) EXPECTED VALUE (mean) $E\{X\}$ in CONTINUOUS and DISCRETE CASE

CONTINUOUS:

$$E\{X\} = \int_{-\infty}^{+\infty} f_X(x)\, x\, dx = \eta_X \text{ (mean)}$$

CONTINUOUS, DISCRETE, MIXED

DISCRETE ONLY (weighted sum by prob with x_n any possible value of X.

$$E\{X\} = \sum_n x_n \cdot P\{X = x_n\} = \eta_X$$

19) EXPECTED VALUE of $Y = g(X)$ in terms of $f_X(x)$ (like 20)

For $Y = g(X)$:

$$E\{Y\} = E\{g(X)\} = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx$$

20) EXPECTED VALUE OF $E\{X \mid M\}$ (CONDITIONAL EXPECTED VALUE)

$$E\{X \mid M\} = \int_{-\infty}^{+\infty} x \cdot f_X(x \mid M)\, dx$$

21) ~~Definition of~~ VARIANCE and relation with $E\{X\}$ and $E\{X^2\}$

$$\sigma_x^2 \stackrel{D}{=} E\{(X-\eta_x)^2\} = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \cdot f_X(x)\, dx$$

RELATION with $E\{X\}$ and $E\{X^2\}$:

$$E\{X^2\} = \sigma_x^2 + E^2\{X\} \geq 0$$

(variance is generally positive) (η_x)

~~22) Telephone call in (ϕ, T). Sketch for "time of occurrence". F_t f_t~~

23) PROBABILITY OF A "POISSON-DISTRIBUTED" R.V., MEAN AND VARIANCE:

X poisson:

$$P\{X=K\} = e^{-\lambda t_0} \cdot \frac{(\lambda t_0)^K}{K!}$$

MEAN $E\{X\} = \eta$ VARIANCE $\sigma_x^2 = \eta$

24) PROBABILITY of K successes in n trials in a repeated-trials experiment:

$$\binom{n}{k} \cdot p^k \cdot q^{n-k} = \frac{n!}{k!(n-k)!} \cdot p^k \cdot q^{n-k}$$

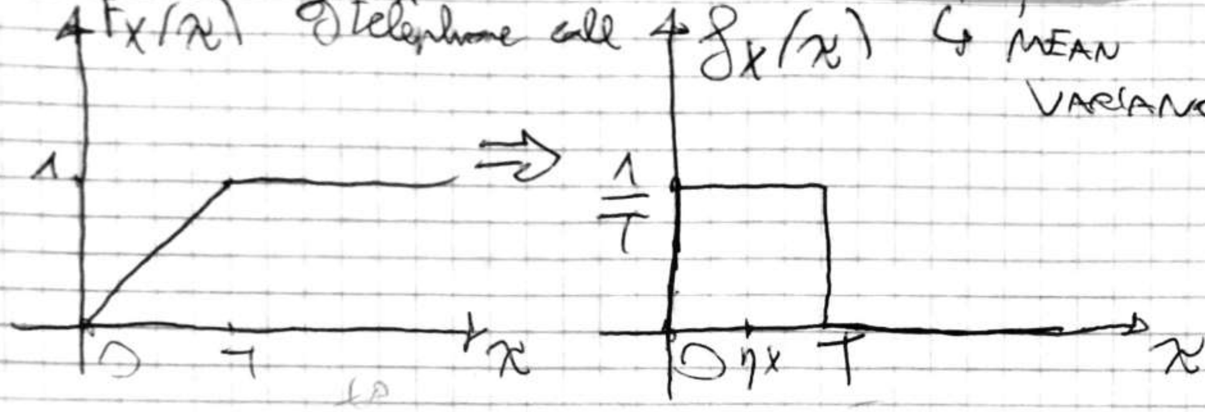
where $P =$ probability of success

$q = 1 - P$ (probability of failure)

22) DISTRIBUTION FUNCTION AND PDF OF R.V

"Time occurrence of a telephone call in a random interval $(0, T)$.

↳ MEAN
VARIANCE



$$f_X(x) = \begin{cases} \frac{1}{T-0} & \text{if } 0 \leq x \leq T \\ 0 & \text{otherwise.} \end{cases}$$

MEAN:

$$E\{X\} = M_X = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_{0}^{T} \frac{1}{T} x\,dx = \frac{x^2}{2T}\Big|_0^T$$

$$= \frac{T^{\cancel{2}}}{2\cancel{T}} - 0 = \frac{T}{2}$$

VARIANCE:

$$\cancel{\sigma_X^2 = E\{X^2\} = \int_{-\infty}^{+\infty} (x - \eta_x)^2}$$

$$\sigma_X^2 = E\{(X-\eta_x)^2\} = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \cdot f_X(x)\,dx$$

$$= \int_0^T \left(x - \frac{T}{2}\right)^2 \cdot \frac{1}{T}\,dx = \int_0^T \left(x^2 - 2x\frac{T}{2} + \frac{T^2}{4}\right) \cdot \frac{1}{T}\,dx$$

$$= \int_0^T \frac{x^2}{T} - \frac{\cancel{2}x\cancel{T}}{\cancel{2}\cancel{T}} + \frac{T^{\cancel{2}}}{4\cancel{T}} = \int_0^T \frac{x^2}{T} - x + \frac{T}{4}\,dx$$

$$= \Big|_0^T \frac{x^3}{3T} - \frac{x^2}{2} + x\frac{T}{4}\,dx = \frac{T^{\cancel{3}2}}{3\cancel{T}} - \frac{T^2}{2} + \frac{T \cdot T}{4}$$

$$= \frac{4T^2 - 6T^2 + 3T^2}{12} = \frac{T^2}{12}$$

17) Obtain the MIXED Bayes formula (Bayes 2nd Type)

1ST-TYPE BAYES.

$$P(A|M) = \frac{P(M|A) \cdot P(A)}{P(M)} \qquad P(M) > 0$$

If $P(A) > 0$ and $M = \{x_1 < X \leq x_2\}$

$$P(A|x_1 < X \leq x_2) = \frac{\overbrace{[F_X(x_2|A) - F_X(x_1|A)]}^{P(M|A)} \cdot P(A)}{\underbrace{F_X(x_2) - F_X(x_1)}_{P(M)}}$$

If we set:

$$x_2 = x_1 + \Delta x$$
$$\text{for } M = \{X = x\}$$

$$P(A|x_1 < X \leq x_1 + \Delta x)$$
$$= \frac{\frac{\Delta x}{\Delta x}\left[F_X(x_1 + \Delta x|A) - F_X(x_1|A)\right] P(A)}{F_X(x_1 + \Delta x) - F_X(x_1)}$$

$$\Rightarrow P(A|X = x) = \frac{f_X(x|A) \cdot P(A)}{f_X(x)}$$

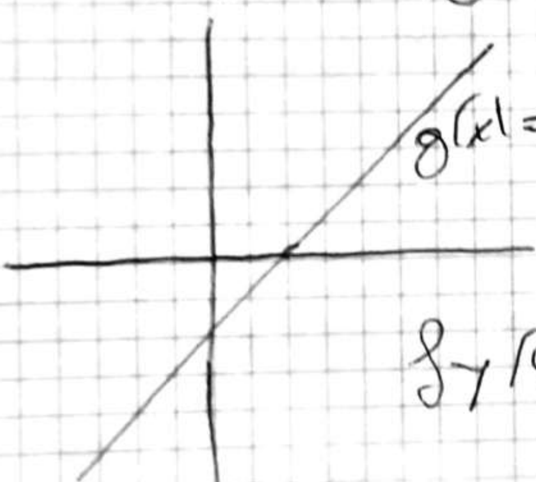
28) ~~Given~~ $y = g(x)$

Apply fundamental theorem to $g(x) = ax + b$

Need to solve equation: $y = g(x)$ to find
$x_1, x_2, x_3, \dots x_n$

$$y = ax + b \Rightarrow x = \frac{y-b}{a}$$

$$\Rightarrow y = g(x_1) = \dots = g(x_n)$$



$$y = g\left(\frac{y-b}{a}\right)$$

$$y = g(x)$$

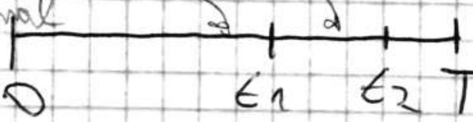
$$f_Y(y) = \frac{f_X(x)}{\left|\left[g\left(\frac{y-b}{a}\right)\right]'\right|} = f_X\left(\frac{y-b}{a}\right)\frac{1}{a}$$

25) n points in interval $(0, T)$.
Probability k points fall in subinterval (t_1, t_2)

REPEATED TRIALS problem with elementary event:
"Place a single point in $(0, T)$".

$A = \{$ single point falls in $(t_1, t_2)\}$

$P(A) = \frac{t_2 - t_1}{T} = p = \frac{t_a}{T}$ Prob. of falling in subinterval



where $t_a \triangleq t_2 - t_1$

If we repeat the experiment n times:
with $q = 1 - p$

$\Rightarrow P\{k$ points falling in $(t_1, t_2)\} = \binom{n}{k} \cdot p^k \cdot q^{n-k}$

26) For experiment in 25), we have:
$n \to +\infty$, $T \to +\infty$ $\lambda = \frac{n}{T}$ (infinite points in infinite time).
Probability that k points fall in interval of length t_a.

$P\{k$ points fall in $t_a\} = e^{-\lambda t_a} \cdot \frac{(\lambda t_a)^k}{k!}$

27) Given $y = g(x)$, describe FUNDAMENTAL THEOREM to obtain $f_Y(y)$ from $f_X(x)$

We need to solve the equation $y = g(x)$ to find its real roots:
(1) For the given y, find: $x_1, x_2, \ldots, x_n$

$\Rightarrow y = g(x_1) = g(x_2) = \ldots = g(x_n)$

Then: $f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \ldots + \frac{f_X(x_n)}{|g'(x_n)|}$

where $g'(x) = \frac{d}{dx} g(x)$

If, for a certain y the equation $y = g(x)$ has no real roots, then:

$$f_Y(y) = 0 \text{ for that } y$$

29) MEAN $E\{Y\}$ for $Y = g(X)$ in terms of $f_X(x)$

$$E\{Y\} = E\{g(X)\} = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx$$

30) CHARACTERISTIC FUNCTION OF A R.V. X:

$e^{j\omega} = \cos(\omega x) + j\sin(\omega x)$

For a R.V. X, the characteristic function Φ_X is the Fourier Transform of the PDF:

Fourier transform

$$\Phi_X(\omega) = \int_{-\infty}^{+\infty} e^{j\omega x} \cdot f_X(x)\, dx = F\{f_X(x)\}$$

+ ~~ISSO~~ TCHEBYCHEFF INEQUALITY:

$$P\{|X - \eta_X| \geq k\sigma_X\} \leq \frac{1}{k^2}$$

The probability that a R.V. is displaced from its average η_X more than k times the standard deviation σ_X is smaller than $\frac{1}{k^2}$.

31) JOINT DISTRIBUTION OF TWO R.V. X and Y:

$$F_{XY}(x,y) = P\{X \leq x, Y \leq y\}$$

32) RELATIONSHIP (direct and inverse) between $F_{XY}(x,y)$ and $f_{XY}(x,y)$:

$F_{XY}(x,y) \rightarrow f_{XY}(x,y)$ (assuming derivatives exist up to 2nd order)

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{XY}(x,y) \geq 0$$

2I) DIRECT METHOD: Get $f_Y(y)$ from $f_X(x)$.
⇒ Apply it to $g(x) = ax + b$

(1) Given y, determine set I_y of values of x where:
if $g(x) \le y \Rightarrow x \in I_y$

$\Rightarrow X \in I_y$ iff $g(x) \le y$

(2) ⇒ Events $\{Y \le y\}$ and $\{X \in I_y\}$
(3) are equal, as they have the same outcomes.

$F_Y(y) = P\{Y \le y\} = P\{X \in I_y\} \neq F_X(x)$

(4) $f_Y(y) = \frac{d}{dy} F_Y(y) \neq \frac{d}{dy} F_X(x)$

APPLICATION:
For $g(x) = ax + b$, we need to find set I_y ⇒ Solve $g(x) \le y$ for x.

$\Rightarrow ax + b \le y \Rightarrow x \le \frac{y-b}{a}$ for $a > 0$

$F_Y(y) = P\{X \le \frac{y-b}{a}\} = F_X\left(\frac{y-b}{a}\right) \rightarrow x$

$\Rightarrow f_Y(y) = \left[F_X\left(\frac{y-b}{a}\right)\right]' = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$

~~Application of fundamental theorem to $g(x) = ax+b$~~
~~$y = ax + b$~~
~~⇒~~

37) UNCORRELATED R.V.s

Two random variables X and Y are said to be UNCORRELATED if

covariance $C_{XY} = 0$

$\Rightarrow E\{XY\} = \eta_X \eta_Y$

where

$C_{XY} \triangleq E\{(X-\eta_X)\cdot(Y-\eta_Y)\}$

39) ~~write f_X~~ Write $f_{XY}(x,y)$ when X and Y are jointly normal (Gaussian).

$f_{XY}(x,y) = A \cdot e^{-(ax^2 + bxy + cy^2 + dx + ey)}$

with a, b, c, d, e parameters s.t.:

$ax^2 + bxy + cy^2 + dx + ey \geq 0 \quad \forall x \forall y$

And assuming $\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial y \partial x} F_{XY}(x,y)$

Sufficient condition $\rightleftarrows$ SCHWARZ THEOREM:

Continuity of $\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$ and $\frac{\partial^2}{\partial y \partial x} F_{XY}(x,y)$

$$f_{XY}(x,y) \rightarrow F_{XY}(x,y)$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$$

33) RELATIONSHIP between MARGINAL and JOINT DENSITY of two R.V. X and Y.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$ integrate in y to get $f_X(x)$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$ integrate in x to get $f_Y(y)$

34) Probability $P\{(X,Y) \in D\}$ in terms of $f_{XY}(x,y)$

$$P\{(X,Y) \in D\} = \iint_D f_{XY}(x,y)\,dx\,dy$$ D a set of points in (x,y)-plane

35) Expression $f_Y(y \mid X = x)$ in terms of $f_{XY}(x,y)$

$$f_Y(y \mid X = x) = \frac{f_{XY}(x,y)}{f_X(x)}$$ because:

$$f_{XY}(x,y) = f_Y(y \mid X = x) \cdot f_X(x)$$

36) RELATIONSHIP between $f_X(x|Y=y)$ and $f_Y(y|X=x)$ $\Rightarrow$

$$\cancel{f_Y(y|X=x) = \frac{f_X(x|Y=y)\, f_Y(y)}{f_X(x)}}$$

$$\cancel{f_{XY}(x,y) = f_Y(y|X=x)\, f_X(x) = f_X(x|Y=y)}$$

$$f_{XY}(x,y) = f_Y(y|X=x) \cdot f_X(x) = f_X(x|Y=y)\, f_Y(y)$$

(from this, the 3rd-type Bayes formula can be derived $f_Y(y|X=x)$)

37) TOTAL PROBABILITY THEOREM to obtain $f_Y(y)$ when another R.V. is defined over S. R.V. X defined over S:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_Y(y|X=x) \cdot f_X(x)\, dx$$

38) 2 INDEPENDENT R.V. X and Y

Two R.V.s X and Y are said to be independent if

$$P\{X \leq x, Y \leq y\} = P\{X \leq x\} \cdot P\{X \leq y\} \quad \forall x, \forall y$$

or, equivalently

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$$

or, equivalently:

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

40) Joint distribution (F) of n random variables

For example, take $n = 3$ random variables (X, Y, Z)

~~$F_{XYZ}(x, y, z) \triangleq P\{X \leq$~~

$$F_{XYZ}(a, b, c) \triangleq P\{X \leq a, Y \leq b, Z \leq c\}$$

41) Fundamental theorem to get $f_{y_1 \dots y_n}(y_1, \dots, y_n)$ from $f_{x_1, \dots, x_n}(x_1, \dots, x_n)$ when $y_1 = g_1(x_1, \dots x_n)$ ... $y_n = g_n(x_1, \dots x_n)$

① Find real roots of

$$\begin{cases} y_1 = g_1(x_1, \dots, x_n) \\ y_2 = g_2(x_1, \dots, x_n) \\ \vdots \\ y_n = g_n(x_1, \dots, x_n) \end{cases}$$

Assuming $[y_1, \dots y_n]$ is a given fixed vector.

IF there are no real roots

$$\Rightarrow f(y_1, \dots y_n) = 0$$

IF there are real roots $[x_1, \dots, x_n]_1$, $[x_1, \dots, x_n]_2$ for the given vector $[y_1, \dots, y_n]$: — REAL ROOTS

$$f(y_1, \dots, y_n) = \frac{f([x_1, \dots, x_n]_1)}{|J([x_1, \dots, x_n]_1)|} + \frac{f([x_1 \dots, x_n]_2)}{|J([x_1, \dots x_n]_2)|}$$

where $J =$

$$J(x_1,\dots,x_n) \triangleq \begin{bmatrix} \frac{\partial g_1(x_1,\dots,x_n)}{\partial x_1} & \frac{\partial g_1(x_1,\dots,x_n)}{\partial x_n} \\ \frac{\partial g_2(x_1,\dots,x_n)}{\partial x_1} & \frac{\partial g_2(x_1,\dots,x_n)}{\partial x_n} \\ \frac{\partial g_n(x_1,\dots,x_n)}{\partial x_1} & \frac{\partial g_n(x_1,\dots,x_n)}{\partial x_n} \end{bmatrix}$$

42) CHAIN RULE for a sequence of R.V. $X_1,\dots,X_n$

LEFT n + RIGHT x_{n-1}

$$f(x_1,\dots,x_n) = f(x_n|x_{n-1},\dots,x_1)\, f(x_{n-1}|x_{n-2},\dots,x_1) \cdot \dots \cdot f(x_2|x_1)\cdot f(x_1)$$

43) RULES TO REMOVE "LEFT" AND "RIGHT" R.Vs. FROM A CONDITIONAL DENSITY OF TYPE $f(x_1,x_2,x_3|x_4,x_5,x_6)$:

LEFT: Integrate with respect to R.V. you want to remove

Remove x_2, x_3:

$$f(x_1|x_4,x_5,x_6) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2,x_3|x_4,x_5,x_6)\,dx_2\,dx_3$$

RIGHT: Multiply by conditional density of R.V. you want to remove, given remaining R.V.s, and integrate with respect to it.

Remove x_4:

$$f(x_1,x_2,x_3|x_5,x_6) = \int_{-\infty}^{+\infty} f(x_1,x_2,x_3|x_4,x_5,x_6)\cdot f(x_4|x_5,x_6)\,dx_4$$

RIGHT. Given $f(x_1, x_2, x_3 | x_4, x_5, x_6)$. To remove a variable from the right, multiply by the conditional density of r.v. to remove and integral with respect to it.

$$f(x_1, x_2, x_3 | x_5, x_6) = \int_{-\infty}^{+\infty} f(x_1, x_2, x_3 | x_4, x_5, x_6) \cdot f(x_4 | x_5, x_6) \, dx_4$$

Remove x_4

IF want to remove 2 variables:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \quad x_4 \, x_5 .$$

44) CHAPMAN-KOLMOGOROFF EQUATION in general case for X_n, X_r, X_k.

$$f(x_1 | x_3) = \int_{-\infty}^{+\infty} f(x_1 | x_2, x_3) \cdot f(x_2 | x_3) \, dx_2$$

$\Rightarrow$ Same as rule for removing x_2 from the right. $x_1 = x_n$, $x_2 = x_r$, $x_3 = x_k$

$$f(x_n | x_k) = \int_{-\infty}^{+\infty} f(x_n | x_r, x_k) \cdot f(x_r | x_k) \, dx_r$$

45) RELATIONSHIP among CONDITIONAL DENSITIES when x_1 is independent of x_2, assuming x_3.

$$f(x_1, x_2 | x_3) = f(x_1 | x_3) \cdot f(x_2 | x_3)$$

(CONDITIONAL INDEPENDENCE).

46) DEFINITION OF MARKOFF SEQUENCE OF R.V.
(ordered random variables).

"An ordered sequence $X_1, \dots, X_n$ is called Markoff sequence if, for any n"

$$F(x_n \mid x_{n-1}, x_{n-2}, \dots x_1) = F(x_n \mid x_{n-1}) \ \forall n$$

→ holds for PDF as well
[it only dependent on x_{n-1} (future only depends on present)]

47) CHAIN RULE FOR MARKOFF SEQUENCES

~~$f(x_1, x_2, \dots x_n) = f(x_n \mid x_{n-1}) \cdot f(x_{n-1} \mid x_{n-2}) \cdot f(x_2 \mid x_1) \cdot f(x_1)$~~

$$f(x_1, x_2, \dots, x_n) = f(x_n \mid x_{n-1}) \, f(x_{n-1} \mid x_{n-2}) \cdot f(x_2 \mid x_1) \, f(x_1)$$

(future only depending on present)

48) 3 FUNDAMENTAL PROPERTIES OF MARKOFF SEQUENCES

1. A subsequence of a Markoff sequence is Markoff
2. The reverse of a Markoff sequence is also Markoff

$$f(x_n \mid x_{n+1}, \dots, x_{n+k}) = f(x_n \mid x_{n+1})$$

x_{n+1}, x_{n+1}

3. Future is independent of past, if present is given.

$g > r > s$ ~~…~~ g, [RECALL DEFINITION OF CONDITIONAL INDEPENDENCE]

g → future, r → present, s → past

$$\Rightarrow f(x_g, x_s \mid x_r) = f(x_g \mid x_r) \cdot f(x_s \mid x_r)$$

49) CHAPMAN-KOLMOGOROFF EQUATION FOR MARKOFF SEQUENCES.

- If $g > r > s$ → past
  - ↳ present
  - ↳ future

$f(x_g | x_r, x_s)$ ↓ ANY SUBSEQUENCE IS MARKOFF $f(x_g | x_r)$

$$f(x_g|x_s) = \int_{-\infty}^{+\infty} f(x_g|x_r) \cdot f(x_r|x_s)\, dx_r$$

"Any future given any past just depends on present".

- 50) CONDITION FOR A MARKOFF SEQUENCE TO BE HOMOGENEOUS.

A Markoff sequence is said to be HOMOGENEOUS if the "one-step" transitional densities do not depend on n. (if for the same value of everywhere, no matter n.

ONE-STEP TRANSITIONAL DENSITY:

$$f(x_n|x_{n-1})$$

$$\Rightarrow f_{X_n}(x|X_{n-1}=c_g) = f_{X_{n-1}}(x|x_{n-2}=c_g) \ \forall n$$

(all transitions have the same value of

51) ~~TRANSITIONAL~~ CONDITION FOR A MARKOFF SEQUENCE TO BE STATIONARY.

A Markoff sequence is said to be STATIONARY if it is HOMOGENEOUS and marginal densities do not depend on n.
↳ full row.

$$f_{X_n}(x) = f_{X_{n-1}}(x) \ \forall n$$

52) TRANSITION MATRIX for a Markoff Sequence (Chain)

All transition probabilities $\pi_{ij}[n_1, n_2]$ for a Markoff Chain can be arranged in a transition matrix from X_{n_1} to X_{n_2}:

$$\Pi[n_1, n_2] \triangleq \begin{bmatrix} \pi_{11}[n_1,n_2] & \pi_{12}[n_1,n_2], & \dots, & \pi_{1N}[n_1,n_2] \\ \pi_{21}[n_1,n_2] & \pi_{22}[n_1,n_2], & \dots, & \pi_{2N}[n_1,n_2] \\ \vdots & \vdots & & \\ \pi_{N1}[n_1,n_2] & \pi_{N2}[n_1,n_2], & \dots, & \pi_{NN}[n_1,n_2] \end{bmatrix}$$

π_{ij} for $N \times N$ matrix

row, column (square)

MARKOFF CHAINS (definition):

A Markoff sequence of DISCRETE R.V.s taking on a finite number N of values $a_1, a_2, \dots, a_N$ on the same alphabet is a Markoff chain. It satisfies:

$$P\{X_n = a_j | X_{n-1} = a_i, \dots, X_1 = a_k\} = P\{X_n = a_j | X_{n-1} = a_i\} \quad \forall n$$

STATE PROBABILITY: $p_i[n] \triangleq P\{X_n = a_i\}$

$(n_2 - n_1)$ STEP TRANSITION PROBABILITY: $\pi_{ij}[n_1, n_2] \triangleq P\{X_{n_2} = a_j | X_{n_1} = a_i\}$

PROPERTIES:

- NORMALIZATION: The marginal probability in n

$$\sum_i p_i[n] = 1 \quad \forall n$$

- NORMALIZATION - CONDITIONAL PROBABILITY

$$\sum \pi_{ij}[n_1, n_2] = 1 \quad \forall n_1, n_2$$

53) CHAPMAN-KOLMOGOROFF EQUATION for Markoff Chain in matrix form:

$$\Pi[n_1,n_3] = \begin{bmatrix} \pi_{11}[n_1,n_3] & \pi_{12}[n_1,n_3] & \dots & \pi_{1N}[n_1,n_3] \\ \pi_{21}[n_1,n_3] & \pi_{22}[n_1,n_3] & \dots & \pi_{2N}[n_1,n_3] \\ \vdots & & & \\ \pi_{N1}[n_1,n_3] & \pi_{N2}[n_1,n_3] & \dots & \pi_{NN}[n_1,n_3] \end{bmatrix}$$

where: matrix product of two transitional matrices non-commutative from n_1 to n_2 and n_2 to n_3.

$$\Pi[n_1,n_3] = \Pi[n_1,n_2]\cdot\Pi[n_2,n_3]$$

where $\Pi[n_1,n_2]$ is the one-step transitional probability from n_1 to n_2.

Corresponds to:

$$f(x_3|x_1) = \int f(x_3|x_2)\, f(x_2|x_1)\, dx_2$$

54) TOTAL PROBABILITY THEOREM for Markoff Chain in matrix form:

STATE PROBABILITY VECTOR:

For a Markoff chain, the state probability vector for X_{n_1} is the row vector given by

$$P[n_1] \triangleq [P_1[n_1], \dots, P_N[n_1]]$$

$P[n_1]$

TOTAL PROBABILITY THEOREM:

$$P[n_2] = P[n_1]\cdot\Pi[n_1,n_2]$$

pair-wise transition prob from n_1 to n_2.

It is the state probability vector of X_{n_2} starting from X_{n_1} written transitional probability given by $\Pi[n_1,n_2]$.

$$\Pi[n_1,n_2] = \begin{bmatrix} \pi_{11}[n_1,n_2], & \dots, & \pi_{1N}[n_1,n_2] \\ \pi_{21}[n_1,n_2], & \dots, & \pi_{2N}[n_1,n_2] \\ \vdots & & \\ \pi_{N1}[n_1,n_2], & \dots, & \pi_{NN}[n_1,n_2] \end{bmatrix}$$

S5) CHAPMAN - KOLMOGOROFF EQUATION for HOMOGENEOUS MARKOFF CHAINS IN MATRIX FORM

$$\Pi[n+k] = \Pi[n] \cdot \Pi[k]$$

A Markoff chain is said to be homogeneous if the transition probabilities depend only on the DIFFERENCE $n_2 - n_1 \triangleq m$, not individually on n_2 and n_1, so that:

$$\pi_{ij}[m] \triangleq P\{X_{n+m} = a_j \mid X_n = a_i\} \; \forall n$$

S6) RELATIONSHIP BETWEEN STATE VECTOR and TRANSITION MATRIX for a HOMOGENEOUS MARKOFF CHAIN.

Given a one-step transition matrix of an homogeneous Markoff chain:

$$\Pi[1] = \Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2N} \\ \vdots & & & \\ \pi_{N1} & \pi_{N2} & \dots & \pi_{NN} \end{bmatrix}$$

One can always select a STATE VECTOR (eigenvector of Π) P such that the resulting chain is STATIONARY (does not change with n).

$$P = [p_1, p_2, \dots, p_N]$$

~~S3). 5) WRITE THE CONDITIONS FOR A MARKOFF CHAIN TO BE HOMOGENEOUS: If the transition probabilities depend only on the difference $m \triangleq n_2 - n_1$~~

51.5) Obtain the condition of stationarity in matrix terms for a Markoff chain.

~~We call a sequence~~

A Markoff Chain is said to be stationary if it is HOMOGENEOUS and:

$$P[1] = P[2] = \ldots = P[n]$$ Proof.

~~Proof~~ Let's call the state vector for X_1:

$$P[1] = P$$

~~If the first two r.v. X_1 and X_2 of a homogeneous Markoff sequence have the same PDF, then the sequence is stationary.~~

$\Rightarrow$ If X_2 has same state vector as X_1:

$$P[1] = P[2] = P$$ (condition for stationarity)

Then, ~~homogeneous~~ the homogeneous chain is stationary, since we have:

$$P = P \cdot \Pi$$ (STATIONARITY)

This means that P is the eigenvector of Π.

$$P[3] = P[2]\Pi = P\Pi = P$$

and:

$$P[n] = P$$ (no change with n)

$$P[0] = P$$

$\Rightarrow$ The Markoff Chain is stationary, as ~~these vectors~~ all states have the same distribution of probabilities at different stage

Sol 5) Give a method to generate a Markoff sequence and demonstrate it

Given a sequence $X_1, X_2, \dots, X_n$ INDEPENDENT RVs with marginal densities $f_{X_n}(x)$, we form:

$$Y_1 = X_1 \qquad\qquad = X_1$$

$$Y_2 = X_1 + X_2 \qquad\qquad = Y_1 + X_2$$

$$Y_n = X_1 + X_2 + \dots + X_n \qquad\qquad = Y_{n-1} + X_n$$

The new sequence $\{Y_n\}$ is Markoff.

PROOF: We want to show that: $\{Y_n\}$ is Markoff

$\Rightarrow$ Find JOINT PDF of $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $\dots$, $Y_n = X_1 + \dots + X_n$.

$$f_{Y_1, Y_2, \dots, Y_n} = \frac{f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)}{|J(x_1, \dots, x_n)|}$$

Solving the system, given $y_1, y_2, \dots, y_n$:

$$\begin{cases} y_1 = x_1 \leftarrow g_1(x_1, \dots, x_n) \\ y_2 = x_1 + x_2 \leftarrow g_2(x_1, \dots, x_n) \\ \vdots \\ y_n = x_1 + x_2 + \dots + x_n \leftarrow g_n(x_1, \dots, x_n) \end{cases}$$

We have:

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 - y_1 \\ \vdots \\ x_n = y_n - y_{n-1} \end{cases}$$

REAL ROOT (one!)

Since $X_1, X_2, \dots, X_n$ are independent, we have:

$$f_{X_1 X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$$
$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \dots \cdot f_{X_n}(x_n)$$

$$J(x_1, \dots, x_n) \triangleq \begin{vmatrix} \frac{\partial g_1(x_1, \dots, x_n)}{\partial x_1} & \dots & \frac{\partial g_1(x_1, \dots, x_n)}{\partial x_n} \\ \frac{\partial g_2(x_1, \dots, x_n)}{\partial x_1} & \dots & \frac{\partial g_2(x_1, \dots, x_n)}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_n(x_1, \dots, x_n)}{\partial x_1} & \dots & \frac{\partial g_n(x_1, \dots, x_n)}{\partial x_n} \end{vmatrix}$$

TRIANGULAR DETERMINANT =

$\Downarrow$

Result = product of diagonal elements

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow f_{Y_1, Y_2, \dots, Y_n} = f_{X_1}(y_1) \cdot f_{X_2}(y_2 - y_1) \cdot f_{X_3}(y_3 - y_2) \dots f_{X_n}(y_n - y_{n-1})$$

FOR CONDITIONAL PDF:

$$f_{Y_n}(y_n | y_{n-1}, \dots, y_1) = \frac{f(y_1, y_2, \dots, y_n)}{f(y_1, y_2, \dots, y_{n-1})}$$

$$= f_{X_n}(y_n - y_{n-1}) = f_{Y_n}(y_n | y_{n-1})$$

$\Rightarrow$ It is Markov!

33.5) CONDITIONS FOR A MARKOFF CHAIN TO BE HOMOGENEOUS:

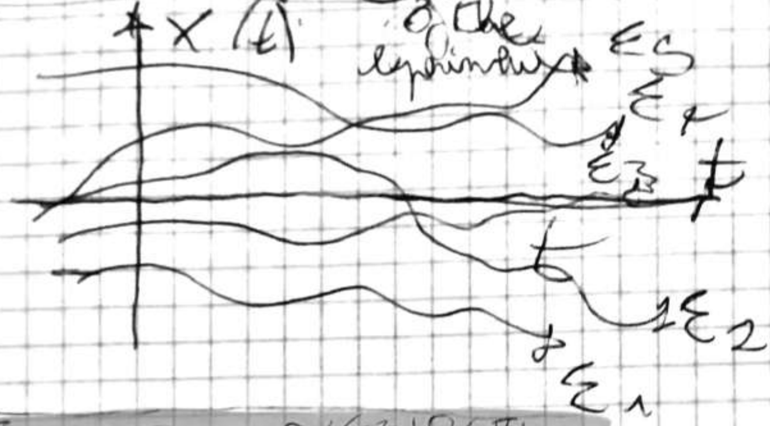
A Markoff Chain is said to be homogeneous if the transition probabilities depend ONLY on the DIFFERENCE ($m \triangleq n_2 - n_1$) not individually on n_2 and n_1, so that:

HOMOGENEOUS CHAIN

$$\Pi_{ij}[\underbrace{m}_{n_2-n_1}] \triangleq P\{x_{n+m} = a_j \mid x_n = a_i\} \quad \forall n$$

60) DEFINITION OF STOCHASTIC PROCESS:

- Given an experiment, a family of real or complex time functions $X(t, \mathcal{E})$ associated to all outcomes $\mathcal{E}$ of the experiment is a STOCHASTIC PROCESS



$X(t, \mathcal{E})$ function of two variables

DOMAIN of $\mathcal{E}$: space S

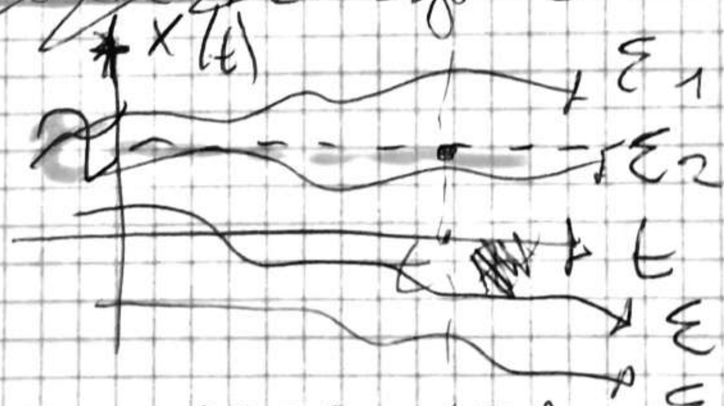
DOMAIN of t: real axis.

- CN FIRST-ORDER DISTRIBUTION FUNCTION AND FIRST-ORDER PDF for a STOCHASTIC PROCESS $X(t)$:

FIRST-ORDER DISTRIBUTION FUNCTION (F_x) for a stochastic process $X(t)$:

$$F_X(x; t) \triangleq P\{X(t) \le x\}$$

~~FIRST-ORDER PDF for a stochastic process $X(t)$~~



FIRST-ORDER PDF for a stochastic process $X(t)$:

$$f_X(x; t) \triangleq \frac{\partial}{\partial x} F_X(x, t)$$

6.1) Definition of mean, autocorrelation, autocovariance for a process $X(t)$:

- MEAN of a stochastic process $X(t)$:

The MEAN $\eta(t)$ of a real process $X(t)$ is the expected value of the R.V. $X(t)$.

$$\eta(t) = E\{X(t)\} = \int_{-\infty}^{+\infty} x \cdot f(x;t)\, dx$$

- AUTOCORRELATION of a stochastic process $X(t)$:

The AUTOCORRELATION $R(t_1, t_2)$ of a real process $X(t)$ is the EXPECTED VALUE of the product of two R.V. $X(t_1)$ and $X(t_2)$ $R(t_1,t_2) = E\{X(t_1) \cdot X(t_2)\}$
(X taken at 2 different time instants)

$$R(t_1,t_2) \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 \cdot f(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

- AUTOCOVARIANCE of a stochastic process $X(t)$.

The AUTOCOVARIANCE $C(t_1, t_2)$ of a real stochastic process $X(t)$ is:

$$C(t_1,t_2) \triangleq E\{\underbrace{[X(t_1) - \eta(t_1)] \cdot [X(t_2) - \eta(t_2)]}^{\text{variance}}\}$$

If we have: $t_2 > t_1$ [where t_2, t_1 generic instants]

$\Rightarrow X(t_2) - X(t_1)$ is a Poisson R.V. with parameter $\lambda(t_2 - t_1)$

$\Rightarrow E\{X(t_2) - X(t_1)\} = \lambda(t_2 - t_1)$

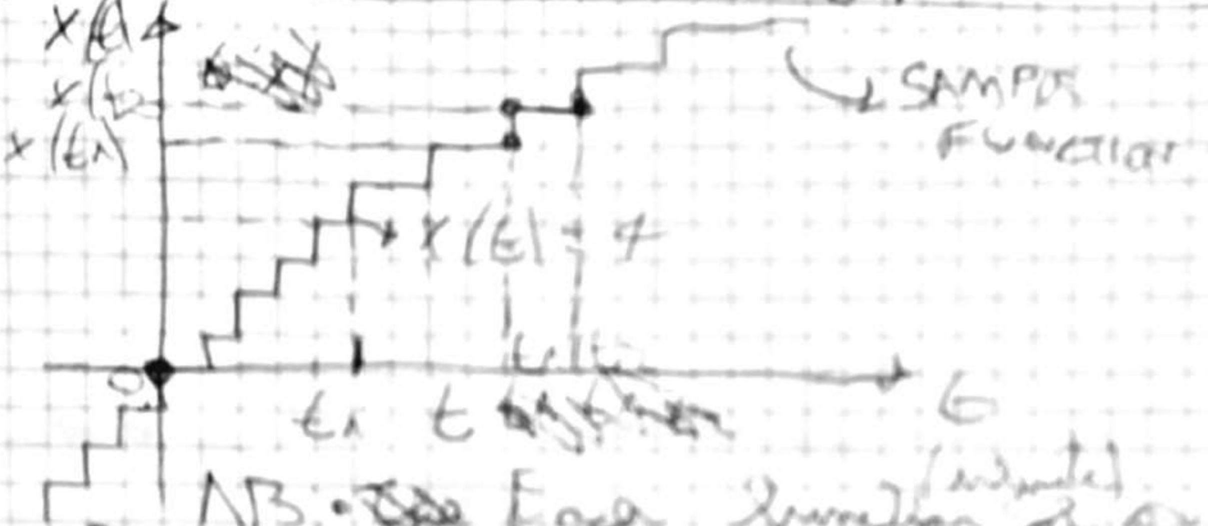
If $t > 0$

$\Rightarrow E\{X(t)\} = \lambda t$

Q3) DESCRIBE the POISSON (COUNTING) PROCESS, obtain its MEAN VALUE, sketch a sample function

A Poisson Counting Process is defined as

$$\boxed{\begin{array}{l} X(t) = 0 \quad \text{for } t = 0 \\ X(t_2) - X(t_1) = \text{number of points in } (t_1, t_2) \end{array}}$$



- NB: Each function of a Poisson process is STAIRCASE
- There are infinite points placed random in $(-\infty, +\infty)$. Interval being considered

For a given $t > 0$

$X(t) = \#$ points in $(0, t)$

For $T \to \infty$, parameter λt and indep., we have that $X(t)$ is Poisson distributed

$$P\{X(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

If we have $t_2 > t_1$,

$X(t_2) - X(t_1)$ Poisson RV with parameter $\lambda(t_2 - t_1)$

$$P\{X(t_2) - X(t_1) = k\} = e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_2 - t_1)]^k}{k!}$$

- $t_1 = 0$

$E\{X(t)\} = \lambda t \to$ mean!

54) Obtain the AUTOCORRELATION of a Poisson counting process.

~~$R(t_1,t_2) = E\{x(t_1)\cdot x(t_2)\} = \{\lambda^2 t_1 t_2$~~

for $(t_1, t_2 > 0)$

$$R_{xx}(t_1,t_2) = E\{x(t_1)\cdot x(t_2)\} = \begin{cases} \lambda^2 t_1 \cdot t_2 + t_2 & t_2 > t_1 \\ \lambda^2 t_1 \cdot t_2 + t_1 & t_2 < t_1 \end{cases}$$

$$\Rightarrow R(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$$

55) Describe the Poisson ~~counting~~ process increments, obtain its mean value and sketch a sample function.

Let $x(t)$ be a Poisson ~~process~~ counting process and $\varepsilon > 0$ a given constant. Then:

$$Y(t) = \frac{X(t+\varepsilon) - x(t)}{\varepsilon}$$

If k is the number of points in $(t, t+\varepsilon) \Rightarrow$

$$\Rightarrow Y(t) = \underset{\text{equals}}{\frac{k}{\varepsilon}}$$

So: (just use k as Poisson RV)

$$P\{Y(t) = \frac{k}{\varepsilon}\} = e^{-\lambda\varepsilon}\cdot\frac{(\lambda\varepsilon)^k}{k!}$$

MEAN:

$$E\{Y(t)\} = \frac{1}{\varepsilon} E\{x(t+\varepsilon) - X(t)\} = \frac{\lambda\varepsilon}{\varepsilon} = \lambda$$

(since $E\{x(t)\} = \lambda t$ → ε here) (constant mean)

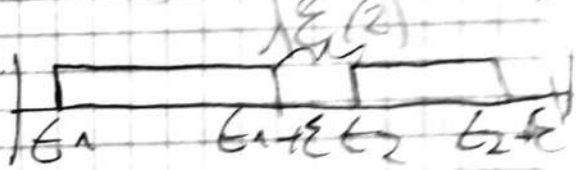
8) OBTAIN THE AUTOCORRELATION OF A POISSON INCREMENTS PROCESS:

$$R_{YY}(t_1,t_2) = E\{y(t_1)\cdot y(t_2)\}$$

$$R_{YY}(t_1,t_2) = E\left\{\frac{[x(t_1+\varepsilon)-x(t_1)]}{\varepsilon}\cdot\frac{[x(t_2+\varepsilon)-x(t_2)]}{\varepsilon}\right\}$$

$$= \frac{1}{\varepsilon^2}\cdot E\{\underbrace{[x(t_1+\varepsilon)-x(t_1)]}_{\lambda\varepsilon(1)}\underbrace{[x(t_2+\varepsilon)-x(t_2)]}_{\lambda\varepsilon(2)}\}$$

2 CASES:



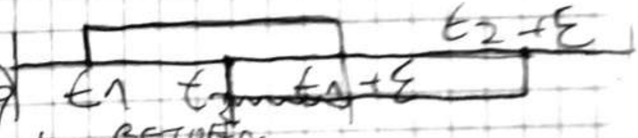
① NON-OVERLAPPING: (INDEPENDENT INTERVALS)

$t_2 > t_1+\varepsilon$ or $t_1 > t_2+\varepsilon$

$|t_1-t_2| > \varepsilon$ (non-overlapping intervals)

$$R_{YY}(t_1,t_2) = \frac{1}{\cancel{\varepsilon^2}}\cdot\lambda\varepsilon\cdot\lambda\varepsilon = \lambda^2$$

② OVERLAPPING INTERVALS:



$\Rightarrow$ $|t_1-t_2| < \varepsilon$. (overlap, no space in-between)

If $t_1 < t_2$:

$$R_{YY}(t_1,t_2) = \frac{1}{\varepsilon^2}\Big[\lambda^2\cdot(\cancel{t_2}+\varepsilon-\cancel{t_2})\cdot(\cancel{t_1}+\varepsilon-\cancel{t_1}) + \lambda(t_1+\varepsilon-t_2)\Big] = \lambda\frac{|t_1-t_2|}{\varepsilon}$$

If $t_1 > t_2$: =

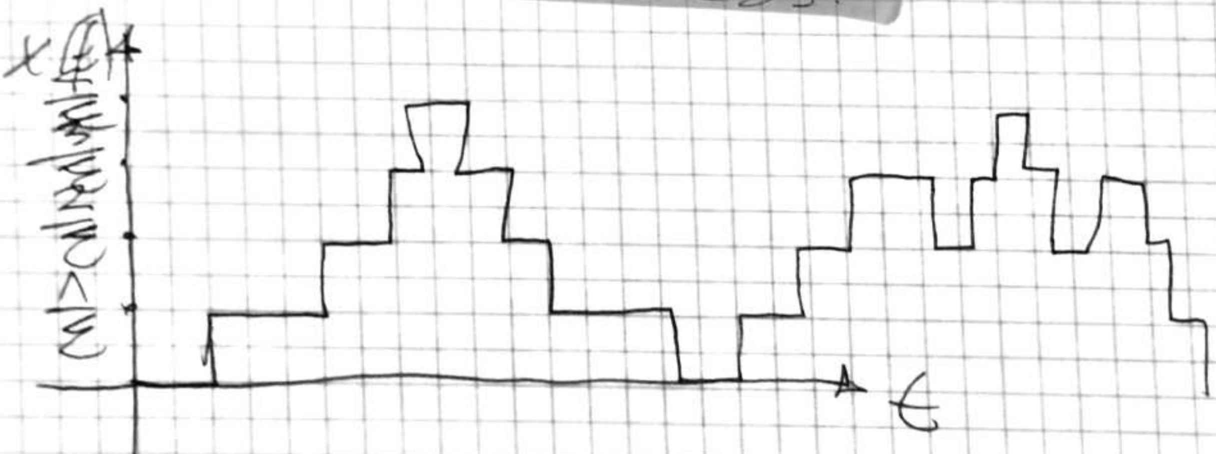
$$R_{YY}(t_1,t_2) = \frac{1}{\varepsilon^2}\Big[\lambda^2\cdot(\cancel{t_2}+\varepsilon-\cancel{t_2})(\cancel{t_1}+\varepsilon-\cancel{t_1}) + \lambda\cdot(t_2+\varepsilon-t_2)\Big] = \frac{\lambda}{\varepsilon}$$

IN GENERAL:

$$R_{YY}(t_1,t_2) = \lambda^2 + \frac{\lambda}{\varepsilon} - \frac{\lambda \cdot |t_1 - t_2|}{\varepsilon^2}$$

$$R_{YY}(t_1,t_2) = \begin{cases} \lambda^2 & |t_1 - t_2| > \varepsilon \\ \lambda^2 + \frac{\lambda}{\varepsilon} - \frac{\lambda |t_1 - t_2|}{\varepsilon^2} & |t_1 - t_2| < \varepsilon \end{cases}$$

65) SKETCH OF A POISSON INCREMENTS PROCESS.



6) Describe the POISSON IMPULSES PROCESS, obtain its mean value, its autocorrelation $R_{ZZ}(t_1, t_2)$ and sketch a sample function.

Consider POISSON COUNTING PROCESS $X(t)$

$\Rightarrow$ form STOCHASTIC PROCESS $Z(t)$:

$$Z(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[X(t+\varepsilon) - X(t)\right] = \frac{dX(t)}{dt}$$

$\Rightarrow$ With $\varepsilon \to 0$ in $E\{Y(t)\}$

MEAN: $E\{Z(t)\} = \lambda$

AUTOCORRELATION:

$$R_{ZZ}(t_1, t_2) = \lambda^2 + \lambda \cdot \delta(t_1 - t_2)$$

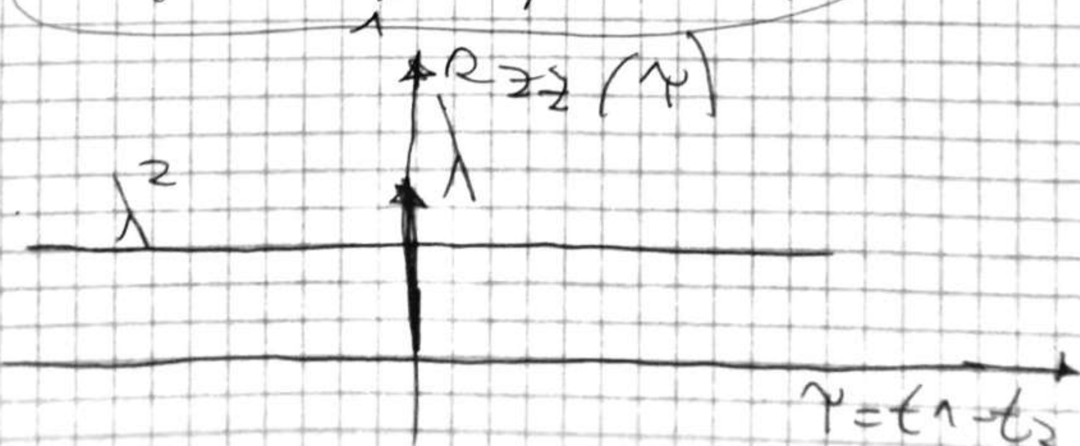
$\tau \triangleq t_1 - t_2$

$$R_{ZZ}(\tau) = \lambda^2 + \lambda \cdot \delta(\tau)$$

$t_i \Rightarrow$ time of arrival on time axis.

$Z(t) \Rightarrow$ derivative of Poisson process $X(t)$

$$Z(t) = \sum_i \delta(t - t_i)$$



5.8) Give 2 properties specifying a Poisson Point Process

1) # POINTS IN AN INTERVAL (t_1, t_2)

The amount of points ~~[illegible]~~ in an interval (t_1, t_2) is a POISSON R.V with parameter a

$$a = \int_{t_1}^{t_2} \lambda(t)\, dt$$

$$P\{n = k\} = e^{-a} \frac{a^k}{k!} \text{ in } (t_1, t_2)$$

2) INDEPENDENCE OF COUNTS IN NON OVERLAPPING INTERVALS

if $t_1 < t_2 \le t_3 \le t_4$, then R.Vs $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent

69) EXPECTED VALUE OF # POISSON POINTS IN A RANDOM INTERVAL

Suppose c is a R.V. FORM n_c from it:

$$n_c = n(t, t+c)$$

(expected value)

$\Rightarrow$ Determine statistics of n_c using:

$$\left[E\{Y\} = E_X\{E\{Y|X\}\} \right]$$

If $c = c$ constant, n_c is a Poisson R.V. with parameter λc.

$\Rightarrow$ The mean amount (value) of points n in an interval of length $n(t, t+c)$ is:

$$E\{n_c | c = c\} = \lambda c$$

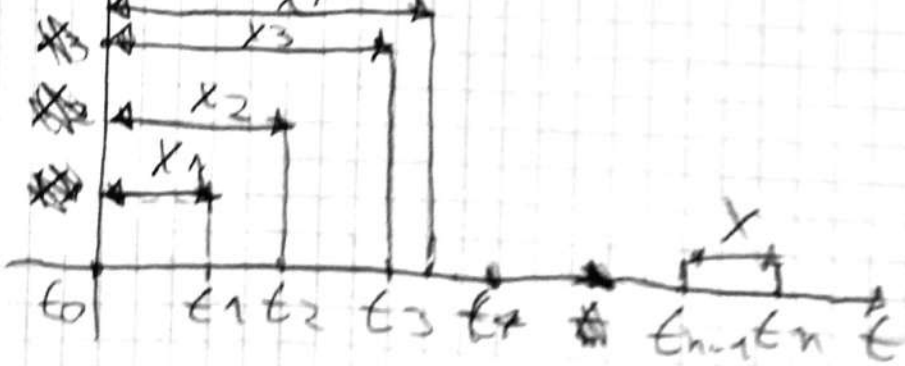
$$E\{n_c | c\} = \lambda c$$

$$\Rightarrow E\{n_c\} = E\{E\{n_c | c\}\} = E\{\lambda n_c\}$$

$\Rightarrow$ AVG. # points in a random interval is the AVG number of points per time unit multiplied by AVG. time duration.

70) DETERMINE PDF of DISTANCE of POISSON POINTS from a FIXED time instant

Consider R.V. $X = t_n - t_0$



We want to prove that
$x_n = t_n - t_0$ is an Erlang - distributed continuous R.V. with PDF:

$$f_{x_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \cdot e^{-\lambda x} U(x)$$

PROOF BY INDUCTION:

If $n=1$

$F_{x_1}(x) = P\{x_1 \le x\} = P\{\text{at least one point is in } (t_0, t_0+x]\}$
$= 1 - P\{0 \text{ points in } (t_0, t_0+x]\} = 1 - e^{-\lambda x} \cdot (\lambda x)^0 / 0!$

EXPONENTIAL

$f_{x_1}(x) = (1 - e^{-\lambda x})' = -\lambda \cdot (-e^{-\lambda x}) \cdot U(x)$
$= \lambda e^{-\lambda x} \cdot U(x)$

If $n = 2$

$F_{x_2}(x) = P\{x_2 \le x\} = P\{\text{at least two points in } (t_0, t_0+x]\}$
$= 1 - P\{0 \text{ points in } (t_0, t_0+x]\} - P\{1 \text{ point in } (t_0, t_0+x]\}$
$= 1 - \frac{e^{-\lambda x} \cdot (\lambda x)^0}{0!} - e^{-\lambda x} \frac{(\lambda x)^1}{1!} = (\lambda x)^{n-1}$

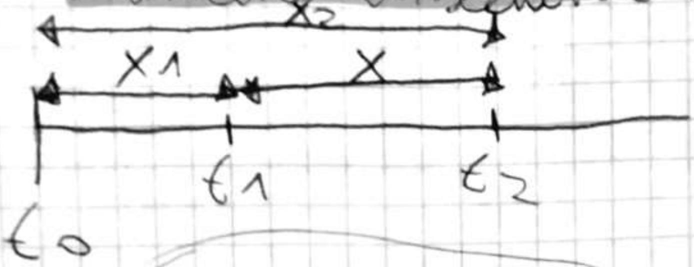
$= 1 - e^{-\lambda x} - e^{-\lambda x} \cdot (\lambda x)$

$f_{x_2}(x) = (1 - e^{-\lambda x} - e^{-\lambda x} \cdot \lambda x)'$
$= -(-\lambda \cdot e^{-\lambda x}) - [(-\lambda) \cdot e^{-\lambda x} \cdot \lambda x + e^{-\lambda x} \cdot \lambda]$
$= -(-\lambda e^{-\lambda x}) - [-\lambda^2 x \cdot e^{-\lambda x} + e^{-\lambda x} \cdot \lambda]$
$= \lambda \cdot e^{-\lambda x} + \lambda^2 x \cdot e^{-\lambda x} - e^{-\lambda x} \cdot \lambda$

$\Rightarrow$ By INDUCTION FOR A GENERIC n:

$$f_{x_n}(x) = \frac{\lambda^n \cdot x^{n-1} \cdot e^{-\lambda x}}{(n-1)!} \cdot U(x)$$

71) Determine the PDF of the distance between consecutive Poisson Points.



$$X = X_2 - X_1$$

We know that X_2 is Erlang-2.
X_1 is exponential.

$\Rightarrow$ X must be exponential and independent for X_1 to be Erlang.

$$F_X(x) = P\{X \le x\} = P\{X_2 - X_1 \le x\}$$

By THE TOT. PROB. THEOREM

$$P(B) = P(B|A_1)\cdot P(A_1) + \dots + P(B|A_n)\cdot P(A_n)$$

$$F_X(x) = \int P\{X_2 - X_1 \le x \mid X_1 = x_1\}\cdot f_{X_1}(x_1)\,dx_1$$

$$= \int \underbrace{P\{X_2 \le x + x_1 \mid X_1 = x_1\}}_{P\{\text{at least one point in } (x_1,\, x + x_1)\} \;=\; 1 - e^{-\lambda x} \text{ (exponential)}} \cdot f_{X_1}(x_1)\,dx_1$$

$$F_X(x) = \int_{-\infty}^{+\infty} \left[1 - e^{-\lambda x}\right]\cdot f_{X_1}(x_1)\,dx_1$$

$$= (1 - e^{-\lambda x}) \underbrace{\int f_{X_1}(x_1)\,dx_1}_{1} = 1 - e^{-\lambda x}$$

$$\Rightarrow f_X(x) = \lambda \cdot e^{-\lambda x}\, u(x)$$

72) DEMONSTRATE that the DISTANCES of Poisson Points from a given time instant are a Markoff Sequence

THEOREM: Distances X_n of Poisson points t_n from a fixed time t_0 are Markoff sequences.

PROOF: We know that

$X_1 \to$ Erlang - 1

⋮

$X_n \to$ Erlang - n

and that:

same density $\begin{cases} X_2 - X_1 & \text{is exponential } \lambda e^{-\lambda x} U(x) \\ x_n - x_{n-1} & \text{is exponential } \lambda e^{-\lambda x} U(x) \end{cases}$

$\Rightarrow$ $X_1, \dots, X_n$ must be INDEPENDENT for the sum to generate ERLANG distributed X_n.

$\Rightarrow$ Sequence X_n is Markoff since it is generated by summing n exponential R.V $\boxed{F}(x_n | x_{n-1}, \dots, x_1) = F(x_n | x_{n-1})$

$\Rightarrow$ Sequence X_n cannot be STATIONARY.

$\Rightarrow$ Prove that X_n is HOMOGENEOUS, since transition probabilities do not change with n.

TRANSITIONAL PDF for a Markoff Sequence $\{X_n\}$.

For $x > y$:

$F_x(x | X_{n-1} = y) = P\{X_n \leq x | X_{n-1} = y\} =$

P{at least 1 point in (

72) DEMONSTRATE THAT the distances of Poisson points from a given time instant are a Markoff sequence.

THEOREM Distances X_n of Poisson Points t_n from a fixed time t_0 are Markoff sequences.

PROOF. We know that:

X_1 is exponential (Erlang-1)

X_2 is ~~exponential~~ (Erlang-2)

$\vdots$

X_n is Erlang-n.

and that: $X_2 - X_1$ is exponential $\to \lambda e^{-\lambda x} U(x)$

$\vdots$

$X_n - X_{n-1}$ is exponential $\to \lambda e^{-\lambda x} U(x)$

All of them have the SAME DENSITY!

$X_1, \dots, X_n$ must be INDEPENDENT since they generate ERLANG-DISTRIBUTED X_n.

$\Rightarrow \{X_n\}$ is MARKOFF since it is generated by summing n i.i.d exponential R.V.s. (cumulative procedure)

$$F(x_n | x_{n-1}, \dots, x_1) = F(x_n | x_{n-1}) = P\{X_n \leq x_n | X_{n-1} = x_{n-1}\}$$

(depends only on last point t_{n-1} with $X_{n-1} = x_{n-1}$)

$\Rightarrow$ The sequence X_n cannot be STATIONARY, as its marginal PDFs are Erlang, dependent on n.

However, X_n is HOMOGENEOUS, as transition probabilities do not change with n.

$\Rightarrow$ TRANSITIONAL PDF for Markoff sequence $\{X_n\}$:

If $x > y$:

$F_X(x \mid X_{n-1} = y) = P\{X_n \leq x \mid X_{n-1} = y\} = P\{$at least one point is in $(t_0 + y, t_0 + x)\}$

$= P\{$at least one point in $(0, x - y)\} = 1 - e^{-\lambda(x-y)}$

ONE-STEP TRANSITIONAL PDF:

$$f_{X_n}(x \mid X_{n-1} = y) = \lambda \cdot e^{-\lambda(x-y)} \quad \text{for } x > y$$

(does not depend on n

$\Rightarrow X_n$ is homogeneous!

7.3) CONSTRUCTIVE DEFINITION OF A Poisson Point Process:

Given a sequence $w_1, w_2, \ldots, w_n$ i.i.d. R.Vs (independent, identically distributed) with PDF:

$$f(w) = \lambda \cdot e^{-\lambda w} U(w)$$

EXPONENTIAL

$\Rightarrow$ Form set of points t_n with $t=0$ arbitrary origin:

$$t_n = w_1 + w_2 + \ldots + w_n$$

t_n is Erlang - distributed.

$\Rightarrow$ We have constructed POISSON RANDOM POINTS (a Poisson Point process)

... FUNCTION OF A TELEGRAPH

7A) DESCRIBE THE SEMI-RANDOM TELEGRAPH SIGNAL, OBTAIN ITS MEAN VALUE,

A telegraph signal is a Poisson-related process $x(t)$ defined as:

$$x(t) = \begin{cases} 1 & \text{if \# Poisson Points in } (0,t) \text{ EVEN} \\ -1 & \text{if \# Poisson Points in } (0,t) \text{ ODD} \end{cases}$$

~~Since ... zero-count of Poisson points is treated as even.~~
~~We call it SEMI-RANDOM TELEGRAPH SIGNAL.~~
~~In other cases, if count $\neq 0$, we can freely choose~~

$$\text{\# POINTS in } (0,t) \begin{cases} \text{\# points} = 0 \Rightarrow \text{EVEN} \Rightarrow \text{SEMI-RANDOM VALUE TELEGRAPH SIGNAL} \\ \text{otherwise } (\text{\# points} > 0) \Rightarrow \text{EVEN or ODD VALUE} \Rightarrow \text{RANDOM TELEGRAPH SIGNAL} \end{cases}$$

$E\{x(t)\} = +1 \cdot P\{x(t) = +1\} - 1 \cdot P\{x(t) = -1\}$

We know:

$P\{k \text{ points in } (0,t)\} = \dfrac{e^{-\lambda t} (\lambda t)^k}{k!}$ POISSON POINTS

The events {k points in $(0,t)$ are mutually exclusive for different values of k, for $(t > 0)$}

EVEN

$P\{x(t) = +1\} = P\{\text{even number of points in } (0,t)\}$

$= P\{0 \text{ points in } (0,t)\} + P\{2 \text{ points in } (0,t)\} + P\{4 \text{ points in } (0,t)\} + \ldots$

$= e^{-\lambda t} + e^{-\lambda t} \cdot \dfrac{(\lambda t)^2}{} + e^{-\lambda t} \cdot \dfrac{(\lambda t)^4}{} + \ldots$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \dots\right] = e^{-\lambda t} \cosh(\lambda t)$$

Since

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

ODD:

$$P\{x(t) = -1\} = P\{\text{odd number in } (0,t)\} = \cancel{e^{-\lambda t} \sinh}$$

$$= e^{-\lambda t} \cdot \sinh \cdot (\lambda t)$$

↳ $t < 0$

$$\cancel{P\{x(t) = +1\} + P\{x(t) \dots\}}$$

$$P\{x(t) = +1\} + P\{x(t) = -1\} = 1$$

$$\Rightarrow E\{x(t)\} = +1 \cdot e^{-\lambda t} \cosh(\lambda t) - 1 \cdot e^{-\lambda t} \sinh(\lambda t)$$

$$= e^{-\lambda t} \cdot (\cosh(\lambda t) - \sinh(\lambda t))$$

$$= e^{-\lambda t} \cdot \left[\frac{e^{\lambda t} + e^{-\lambda t}}{2} - \frac{e^{\lambda t} - e^{-\lambda t}}{2}\right] = e^{-2\lambda t}$$

$$E\{x(t)\} = e^{-2\lambda |t|}$$

75) AUTOCORRELATION FUNCTION OF A TELEGRAPH SIGNAL

$$R(t_1, t_2) = E\{x(t_1) \cdot x(t_2)\}$$

We need the probabilities of possible values of

$$x(t_1) \cdot x(t) = \begin{cases} +1 & ① \\ -1 & ② \end{cases}$$

The different combinations yielding such values are, by the product sign rules.

~~$P\{x(t_1) \cdot x(t_2) = +1$~~

① $P\{x(t_1) \cdot x(t_2) = +1\} = P\{x(t_1) = +1, x(t_2) = +1\} + P\{x(t_1) = -1, x(t_2) = -1\}$ ① ③

② $P\{x(t_1) \cdot x(t_2) = -1\} = P\{x(t_1) = +1, x(t_2) = -1\} + P\{x(t_1) = -1, x(t_2) = +1\}$ ② ④

⇒ Compute the conditional probabilities for each case:

① $P\{x(t_1) = +1, x(t_2) = +1\} = \overbrace{P\{x(t_2) = +1 \mid x(t_1) = +1\}}^{\text{even number in } (t_1, t_2)} \cdot \underbrace{P\{x(t_1) = 1\}}_{\text{even number in } (0, t_1)}$

EVEN → cos ODD → sin

$$= e^{-\lambda t_1} \cosh(\lambda t_1) \, e^{-\lambda(t_2 - t_1)} \cdot \cosh[\lambda(t_2 - t_1)]$$

~~② $P\{x(t_1) = -1, x(t_2) = -1\} = P\{x(t_2) = -1 \mid x(t_1) = -1\} \cdot P\{x(t_1) = -1\}$~~

② $P\{x(t_1)=+1, x(t_2)=-1\}$
$= e^{-\lambda t_1} \cdot \cosh(\lambda t_1) \cdot e^{-\lambda(t_2-t_1)} \cdot \sinh[\lambda(t_2-t_1)]$

③ $P\{x(t_1)=-1, x(t_2)=+1\}$
$= e^{-\lambda t_1} \cdot \sinh(\lambda t_1) \cdot e^{-\lambda(t_2-t_1)} \sinh(\lambda(t_2-t_1))$

④ $P\{x(t_1)=-1, x(t_2)=-1\} =$
$= e^{-\lambda t_1} \cdot \sinh(\lambda t_1) \cdot e^{-\lambda(t_2-t_1)} \cdot \cosh[\lambda(t_2-t_1)]$

$\Rightarrow R_{xx}(t_1,t_2) = e^{-2\lambda(t_2-t_1)} =$ (for $t_2-t_1=\tau$)

$R_{xx}(t_1,t_2) = e^{-2\lambda|\tau|}$

76) DEFINITION OF STRICT-SENSE STATIONARY (SSS) PROCESS and GENERAL PROPERTY of its MEAN VALUE

The process $X(t)$ is STATIONARY in STRICT-SENSE (SSS) if its statistics are not affected by a shift in the time origin.
~~[illegible]~~ → Two processes have the same statistics for any t.

$$E\{X(t)\} = \eta = \text{constant}$$

77) DEFINITION OF WIDE-SENSE STATIONARY (WSS) PROCESS

A process $X(t)$ is STATIONARY in WIDE SENSE if its expected value is CONSTANT. ~~(STATIONARY)~~

$$E\{X(t)\} = \eta = \text{CONSTANT}$$

And the autocorrelation of $X(t)$ depends only on $\tau = t_1 - t_2$

$$E\{X(t+\tau) \cdot X(t)\} = R(\tau) = R(-\tau)$$

78) DEFINITION AND PROPERTIES of the AUTO-CORRELATION FUNCTION OF A STATIONARY PROCESS.

$E\{X(t)\} = \eta \Rightarrow R_{XX}(\tau) = E\{X(t+\tau) \cdot X^*(t)\}$

1) SYMMETRY: if $X(t)$ is real:

$$R_{XX}(\tau) = R_{XX}(-\tau) \Rightarrow \text{real even function.}$$

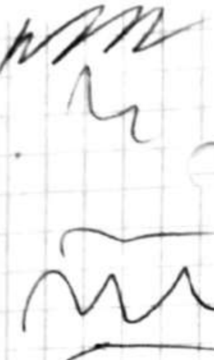
2) VIP PROPERTY:

$$R(0) = E\{X^2(t)\} \rightarrow \text{(normalized AVG. power of } X(t))$$

3) max in $R(0)$. For $x(t)$ real,

$R(\tau)$ has a maximum in the origin.

For a complex process $R(0) = E\{|x(t)|^2\} = E\{x(t) \cdot x^*(t)\}$



79) DEFINITION OF ERGODIC PROCESS.

An SSS process is ERGODIC if its ensemble averages equal appropriate time ~~…~~ averages of any single sample function $x(t, \xi)$.

(EXPLANATION: Any statistics of $x(t)$ can be determined with probability 1 from a single sample function $x(t, \xi)$).

80) NECESSARY CONDITION FOR A PROCESS TO BE MEAN-ERGODIC: (for real or complex $X(t)$).

$$\frac{1}{2T}\int_{-2T}^{2T} C_x(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau \to 0 \quad \text{for } T \to \infty.$$

can also be written as:

WIENER KHINCHIN - THEOREM (necessary & sufficient condition for a process to be mean-ergodic.)

$$\lim_{T \to +\infty} \frac{1}{2T}\int_{-T}^{+T} C_x(\tau)\, d\tau = 0$$

$X(t)$ can be real or complex.

If a process $x(t)$ is mean-ergodic, its statistics can be expressed as the expected value (mean) of appropriate functions.

19) CORRELATED and INDEPENDENT PROCESSES

~~...~~ $x(t)$ and $y(t)$.

~~...~~

~~...~~ • CROSS-CORRELATION

The cross-correlation of $\overbrace{x(t), y(t)}^{\text{Real}}$ is:

$$R_{xy}(\tau) = E\{x(t+\tau) \cdot y(t)\}$$

$t' = t + \tau$

$$= E\{y(t' + (-\tau)) \cdot x(t')\} = R_{yx}(-\tau)$$

~~...~~

• PRODUCT OF INDEPENDENT $x(t), y(t)$

↳ AUTOCORRELATION $z(t) = x(t) \cdot y(t)$

$$R_{zz}(\tau) = E\{x(t+\tau) \cdot y(t+\tau) \cdot x(t) \cdot y(t)\}$$
$$= R_{xx}(\tau) R_{yy}(\tau)$$

20) DESCRIBE THE SAMPLE FUNCTION OF A RANDOM WALK, obtain its MEAN ~~...~~ VALUE and MEAN SQUARE VALUE

Define an experiment:

"Tossing a fair coin an infinite number of times, with each tossing occurring every T seconds".

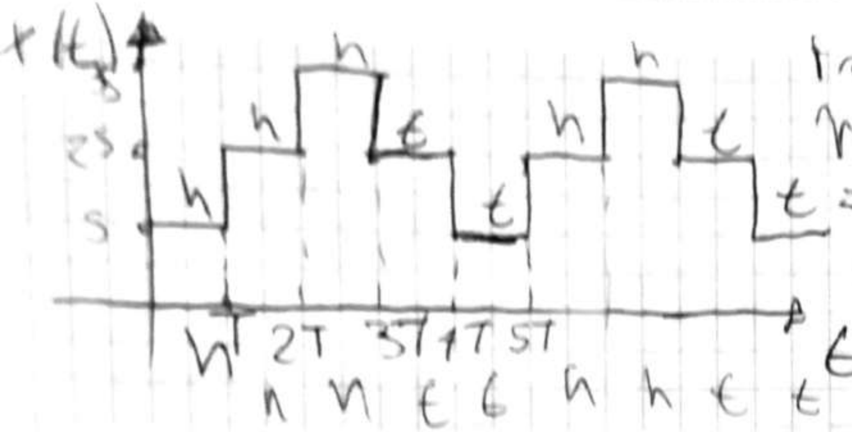
$P\{\text{heads}\} = P\{\text{tails}\} = p = q = \frac{1}{2}$ in each trial

In each tossing, we take an instantaneous step.

LEFT if head (H)

RIGHT if tail (T)

Our position "t" seconds after the first tossing is $x(t)$ ("RANDOM WALK").



For first n tossings, suppose k heads show.

$t = n\Delta t$ time

$(n-1)T < t \le nT$

we have taken

$\begin{cases} k \text{ steps to the left} \\ n-k \text{ steps to the right} \end{cases}$

~~[illegible]~~

MEAN VALUE

Denote by X_i a sequence of independent R.Vs with equally likely values $\pm s$ depending on the tossing outcome. Then, for:

$$(n-1)T \le t \le nT$$

$$x(t) = x_1 + x_2 + \dots + x_n$$

$$\Rightarrow E\{x(t)\} = 0$$

MEAN SQUARE VALUE:

$$\cancel{E\{x^2(t)\} = E\{x_1^2\} + E\{x_2^2\} + \dots + E\{x_n^2\}}$$

$$E\{x^2(t)\} = E\{x_1^2\} + E\{x_2^2\} + \dots + E\{x_n^2\} + E\{x_1 x_2\} + \dots = n s^2$$

\# tossings — values it takes

21) **DEFINE A WIENER-LEVY | BROWNIAN MOTION PROCESS, define its MEAN VALUE AND ITS MEAN SQUARE VALUE**

~~[illegible]~~ Limiting form of Random Walk process $x(t)$. For a Random Walk Process

$$E\{x(t)\} = 0 \qquad E\{x^2(t)\} = n s^2$$

Now, fixing the time, $t = nT$ and $n = \frac{t}{T}$

$n = \frac{t}{T}$ tossings

$$E\{x^2(t)\} = \left(\frac{t}{T}\right) \cdot s^2$$

Keep t constant, but let the step s and interval T tend to zero (continuously take small steps close to one another)

To keep up, we need more and more tossings to reach a certain position (n gets much larger).

$\Rightarrow$ Variance $\sigma_x^2(t) = E\{x^2(t)\} = \left(\frac{t}{T}\right) s^2$ will remain finite if $s \to 0$ as $\sqrt{T}$

$$\frac{s^2}{T} \to \alpha$$

$$w(t) = \lim x(t) \text{ for } \begin{cases} T \to 0 \\ s \to 0 \\ \frac{s^2}{T} \to \alpha \end{cases}$$

$\Rightarrow$ MEAN $E\{w(t)\} = 0$

MEAN SQUARE VALUE:

Keeping $(n-1)T < t \leq nT$:
from $E\{x^2(t)\} = ns^2$, we get at time $nT = t$

$$E\{x^2(t)\} = \frac{t}{T} s^2 \quad \left(\text{as } n = \frac{t}{T}\right)$$

$$\lim_{T \to 0} \frac{s^2}{T} = \alpha = E\{w^2(t)\} = \alpha t$$

22) WRITE IN ANALYTICAL TERMS THE REFLECTION THEOREM FOR WIENER-LEVY PROCESS AND EXPLAIN IT:

~~P{w(t) ≤ w | τ₁ ≤ t} = P{w(t) ≥ 2d - w}~~

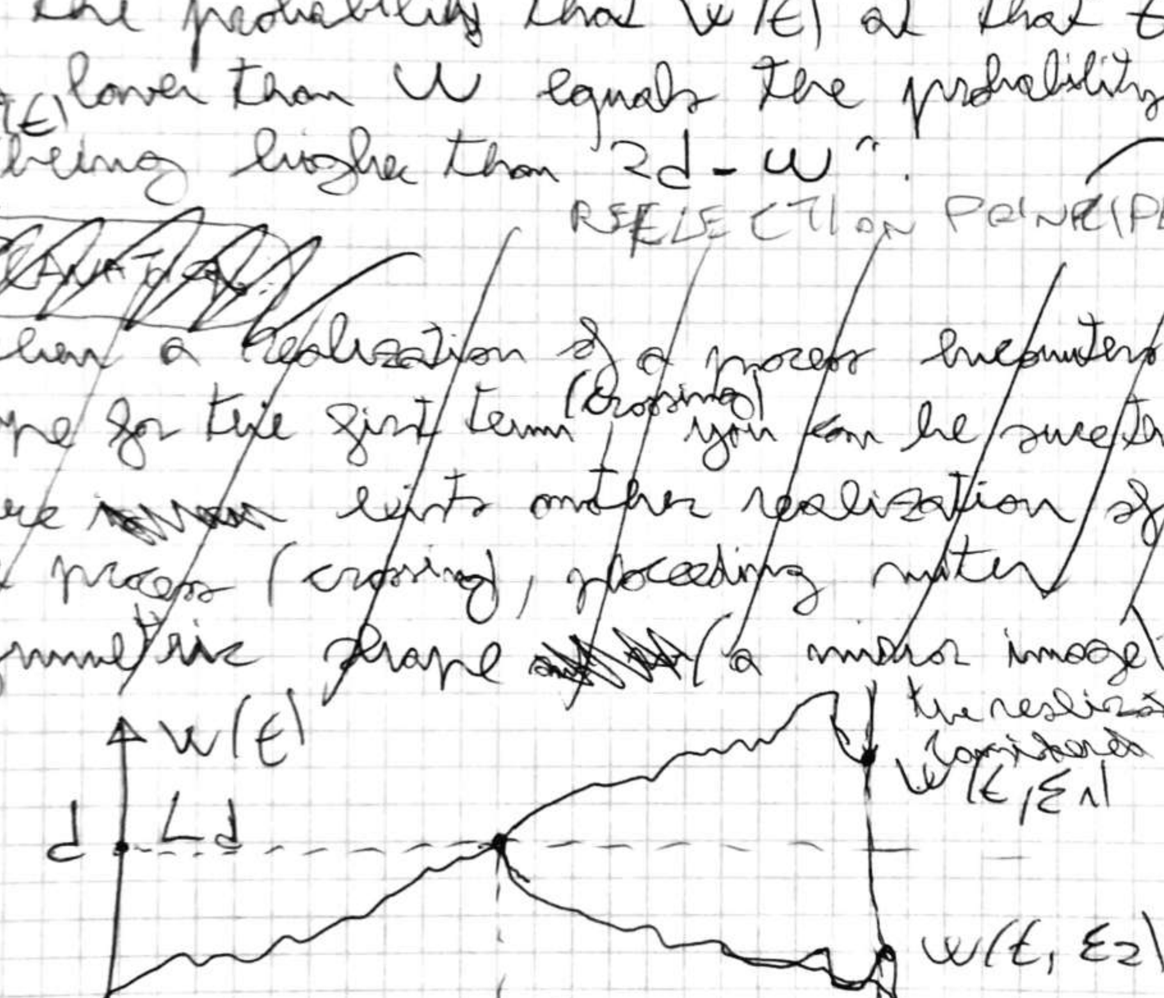
$$P\{w(t) \leq w \mid \tau_1 \leq t\} = P\{w(t) \geq 2d - w \mid \tau_1 \leq t\} \quad \forall w$$

"For those realizations that cross L_d before t, the probability that $w(t)$ at that t is lower than w equals the probability of $w(t)$ being higher than $2d - w$".

REFLECTION PRINCIPLE:

~~EXPLANATION~~

When a realization of a process encounters the shape for the first time (crossing) you can be sure that there exists another realization of the process (crossing), proceeding with a symmetric shape (a mirror image to the realization considered)

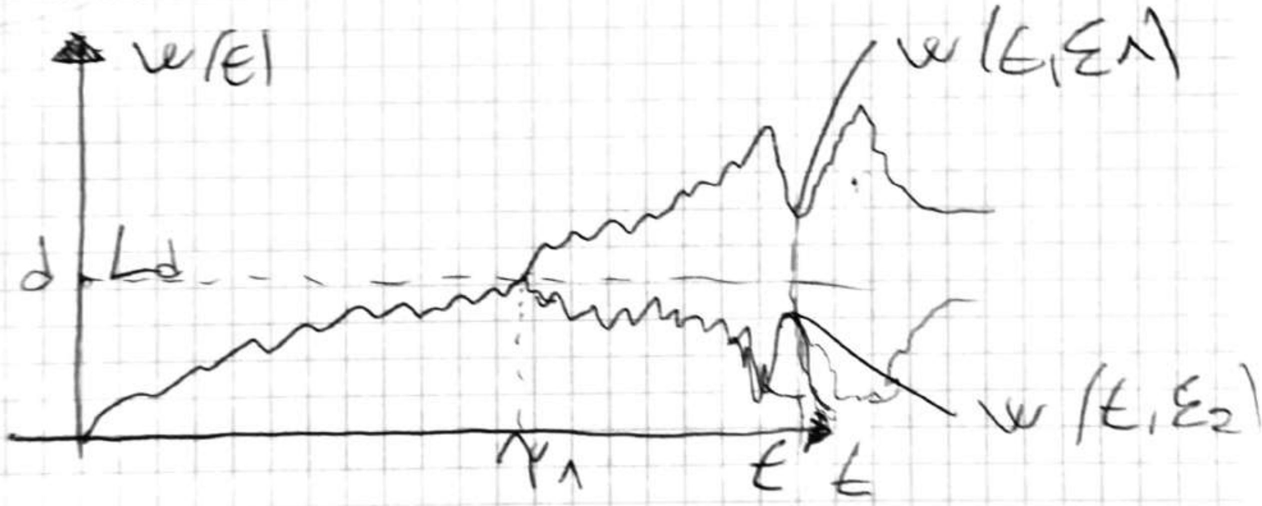


NB. Each process has an infinite number of reflections!

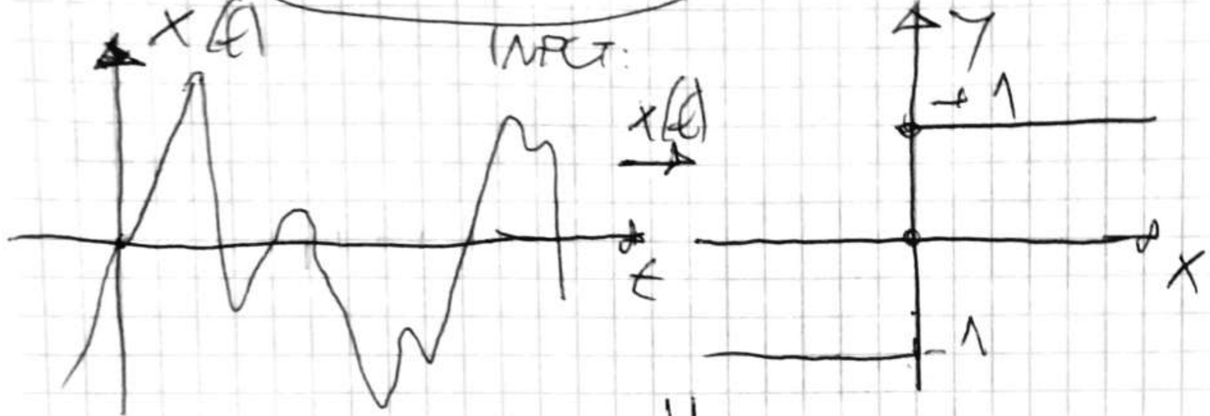
Any sample functions of a Wiener-Levy process that cross a line $w = d$ (L_d) continue also on symmetric paths.

REFLECTION PRINCIPLE FOR WIENER-LEVY EXPLAINED

If, for some experimental outcome $\mathcal{E}_1$ the curve $w(t, \mathcal{E}_1)$ crosses L_d at $t = \tau_1$ for the first time, then the process contains also another curve $w(t, \mathcal{E}_2)$ that coincides with $w(t, \mathcal{E}_1)$ ~~with respect~~ ~~to~~ for $t < \tau_1$ and is the mirror image of $w(t, \mathcal{E}_1)$ with respect to the line L_d for $t > \tau_1$.

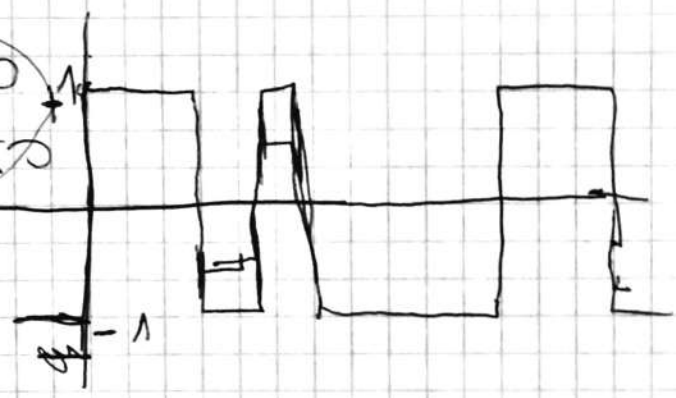


103) OBTAIN MEAN VALUE AND AUTOCORRELATION OF OUTPUT OF A "HARD LIMITER" if the input is a SSS process (QUANTIZATION)



$\Downarrow$

$y = g(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$



$x(t)$ is SSS $\Rightarrow$ MEAN VALUE.

$\cancel{E\{y(t)\}}$ $E\{y(t)\} = +1 \cdot P\{x(t) > 0\} - 1 \cdot P\{x(t) < 0\}$

$= [1 - F_x(0)] - F_x(0) = 1 - 2F_x(0)$

AUTOCORRELATION:

$R_y(\tau) = E\{y(t) \cdot y(t+\tau)\}$

$= +1 \cdot P\{y(t)$ and $y(t+\tau)$ same sign$\}$ +

$-1 \cdot P\{y(t)$ and $y(t+\tau)$ different sign$\}$

$= +1 \cdot P\{x(t) \cdot x(t+\tau) > 0\} - 1\, P\{x(t) \cdot x(t+\tau) < 0\}$

Now consider two cases:

(1) POSITIVE

$$P\{x(t)\cdot x(t+\tau)>0\} = P\{\underbrace{\overset{X_1}{x(t)}>0,\ \overset{X_2}{x(t+\tau)}>0}_{\text{both positive}}\} + P\{\underbrace{x(t)<0,\ x(t+\tau)<0}_{\text{both negative}}\}$$

positive

both positive $P\{X_1>0\}\ P\{X_2>0\}$ both negative

$$= 1 - F_x(0) - F_x(0) + F_x(0,0;\tau) + F_x(0,0;\tau)$$

$P\{X_1<0, X_2<0\}$

$$= 1 - 2F_x(0) + 2F_x(0,0;\tau)$$

~~(2) NEGATIVE~~

$$\cancel{P\{x(t)\cdot x(t+\tau)<0\} = 1 - P\{x(t)\cdot x(t+\tau)>0\}}$$

$$\cancel{= 2F_x(0) - 2F_x(0,0;\tau)}$$

$$\cancel{\Rightarrow R_Y(\tau) = 1 - 4F_x(0) + 4F_x(0,0;\tau)}$$

(2) NEGATIVE:

$$P\{x(t)\cdot x(t+\tau)<0\} = 1 - \overbrace{P\{x(t)\cdot x(t+\tau)>0\}}^{\text{BOTH POSITIVE}}$$

$$= 1 - [1 - 2F_x(0) + 2F_x(0,0,\tau)]$$

$$= \cancel{1} - \cancel{1} + 2F_x(0) - 2F_x(0,0,\tau)$$

$$= 2F_x(0) - 2F_x(0,0,\tau)$$

$$\Rightarrow R_Y(\tau) = +1\cdot P\{x(t)>0\} - 1\cdot P\{x(t)<0\}$$

$$= 1 - 2F_x(0) + 2F_x(0,0,\tau) - 2F_x(0) + 2F_x(0,0,\tau) = 1 - 4F_x(0) + 4F_x(0,0,\tau)$$

29) OBTAIN MEAN VALUE AND AUTOCORRELATION OF THE OUTPUT OF A "THRESHOLDING" DEVICE if the input is SSS

Consider an ON-THRESHOLDING DEVICE.

$$y(t) = \begin{cases} a & \text{if } x(t) \geq V_{ON} \\ 0 & \text{if } x(t) < V_{ON} \end{cases}$$

For input $x(t)$ SSS: MEAN VALUE

$$E\{y(t)\} = a \cdot P\{x(t) \geq V_{ON}\} = a\left[1 - F_x(V_{ON})\right]$$

$$R_y(\tau) = E\{y(t+\tau) \cdot y(t)\}$$ AUTOCORRELATION

$$= a^2 \cdot P\{x(t) \geq V_{ON}, x(t+\tau) \geq V_{ON}\}$$

$$= a^2 \cdot \left[1 - 2F_x(V_{ON}) + F_{xx}(V_{ON}, V_{ON}; \tau)\right]$$

10) How to obtain MEAN VALUE and the AUTOCORRELATION of the output of a linear time-invariant system with process SSS input

If input $x(t)$ is SSS $\Rightarrow$ output $y(t)$ is SSS

If input $x(t)$ is WSS $\Rightarrow$ output $y(t)$ is WSS

$$E\{y(t)\} = \eta_y = \text{constant} = \eta_x \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha$$

$$R_{yy}(t_1, t_2) = R_{yy}(t_1 - t_2)$$

10b) Obtain the mean value and the AUTOCORRELATION of a differentiator if input is SSS

A differentiator is a linear system whose output is the derivative of the input.

$$L[x(t)] = x'(t) = \frac{dx(t)}{dt}$$

MEAN:

$$\eta_{x'}(t) = E\{x'(t)\} = L[E\{x(t)\}] = \frac{dE\{x(t)\}}{dt}$$

$$= \frac{d\eta_x(t)}{dt}$$

AUTOCORRELATION

$$R_{xx'}(t_1, t_2) = \frac{\partial}{\partial t_2} R_{xx}(t_1, t_2)$$

$$R_{x'x'}(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_{xx}(t_1, t_2)$$

1) Give the definition of POWER SPECTRAL DENSITY and its main properties.

$$S(\omega) \triangleq \int_{-\infty}^{+\infty} R(\tau) \cdot e^{-j\omega\tau} d\tau = \mathcal{F}\{R(\tau)\}$$

The power spectrum density of a WSS process is the Fourier transform of the autocorrelation function.

PROPERTIES:

- INVERSION FORMULA:

$$R(\tau) = \mathcal{F}^{-1}\{S(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) e^{j\omega\tau} d\omega$$

- If $R(\tau)$ even:

$R(\tau) = R^*(-\tau)$ if $x(\tau)$ complex

$R(\tau) = R^*(-\tau)$ if $X(\tau)$ real

- If $x(t)$ real:

$R(\tau)$ is real and even.

Spectrum $\Rightarrow$ $S(\omega)$ also real and even.

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau) \cdot \cos(\omega\tau) d\tau$$

$$R(\tau) = \int_{-\infty}^{+\infty} S(\omega) \cdot \cos(\omega\tau) \frac{d\omega}{2\pi}$$

- If $x(t)$ complex, $S(\omega)$ is real, but not even.

$$S(\omega) \neq S(-\omega)$$

10a) Give a procedure to calculate the POWER SPECTRAL DENSITY of a process directly from the PDF

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau)\cdot e^{-j\omega\tau}\, d\tau$$ (By definition of PSD)

~~By the mean-value theorem~~

$$\cancel{S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega\tau} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, f_x(x_1, x_2; \tau)\, dx_1 dx_2}$$

~~By the mean-value theorem:~~

$$\cancel{S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega\tau} \iint}$$

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega\tau}\, E\{x(t+\tau)\cdot x(t)\}\, d\tau$$

By the mean-value theorem of a R.V.:

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-j\omega\tau} \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, f_x(x_1, x_2; \tau)\, dx_1 dx_2\right] d\tau$$

Interchange integrals

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 \underbrace{\left[\int_{-\infty}^{+\infty} e^{-j\omega\tau}\, f_x(x_1, x_2; \tau)\, d\tau\right]}\, dx_1 dx_2$$

Double-integrate $G(x_1, x_2; \omega) = \mathcal{F}\{f_x(x_1, x_2; \tau)\}$

the $\mathcal{F}$-Transform of 2nd-order PDF

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, G(x_1, x_2; \omega)\, dx_1 dx_2$$

112) Obtain the spectrum of the output of a "sliding window" smoothing device when the input is a SSS process.

OUTPUT SPECTRUM ($\mathcal{F}$-transform of $h(t)$).

$$H(\omega) = \frac{1}{2T}\int_{-T}^{+T} e^{-j\omega t}\,dt = \frac{1}{2T}\,\frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T}^{+T}$$

$$= \frac{1}{\omega T}\,\frac{e^{j\omega T} - e^{-j\omega T}}{2j}$$

Since $\dfrac{e^{x} - e^{-x}}{2} = \sinh(x)$

$$\Rightarrow H(\omega) = \frac{\sin(\omega T)}{\omega T}$$

$$E\Big\{\sum_{k=1}^{N(t)} a_k\Big\} = E\{N(t)\}\cdot E\{a_k\}$$

$$E\Big\{\Big(\sum_{k=1} a_k\Big)^2\Big\} \le E\{N(t)^2\}\cdot E\{a_k^2\} < \infty$$

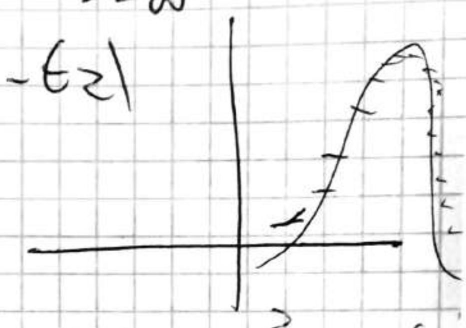
PROOF: $E\{x^2\} = M$

$$E\{S^2\} = E\Big\{\Big|\sum_{k=1}^{n} x_k\Big|^2\Big\}$$

LINEAR, TIME-INVARIANT-

$$E\{y(t)\} = \eta_Y = \eta_X \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha$$

$$C_{YY}(t_1,t_2) = C_{YY}(t_1-t_2)$$



$$L[x(t)] = \frac{d(x(t))}{dt}$$

$$R_{YY}(t_1,t_2) = \frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t\, \partial t'}$$

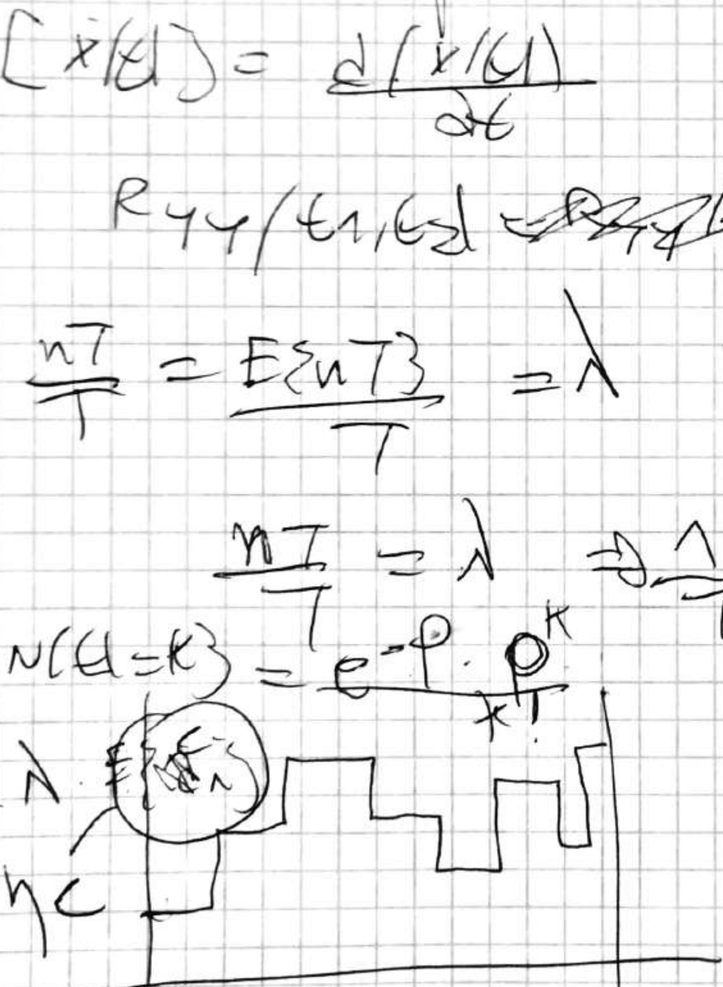
$$\frac{n_T}{T} = \frac{E\{n_T\}}{T} = \lambda$$

$$\frac{n_T}{T} = \lambda \Rightarrow \frac{1}{T} = \frac{\lambda}{n_T}$$

$$P\{N(t)=k\} = e^{-\rho} \cdot \frac{\rho^k}{k!}$$

$$\rho = \lambda E\{c_1\}$$

η_C



$E\{a_i\}$

$$\lim \frac{1}{T}\int_0^T N(t)\,dt = \frac{\lambda}{n_T}\int_0^T N(t)\,dt$$

$a_i = c_i$

$\forall_i$ $a_i = \cancel{b_i} + c_i$ service time

system time

waiting time