113) Define a "Quadrature Filter" for a process.

We call "Quadrature filter" a system with the following transfer function.

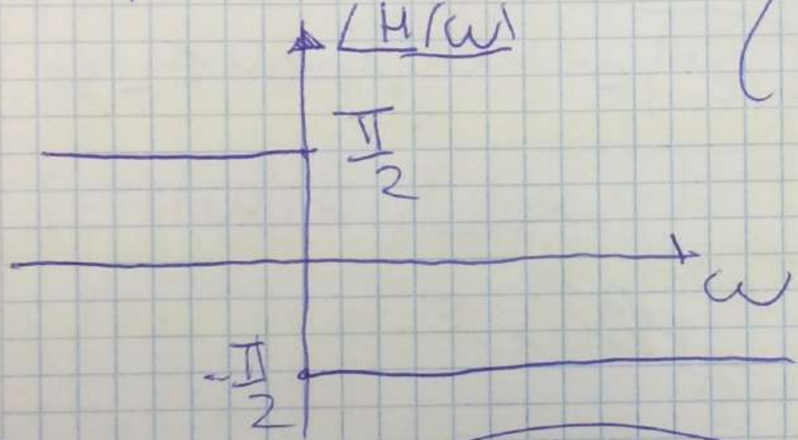
$$H(\omega) = \begin{cases} +j & \text{if } \omega < 0 \\ -j & \text{if } \omega > 0 \end{cases} = -j\,\text{sign}(\omega)$$

$$H(\omega) = e^{j\varphi(\omega)}$$ → phase of ω

MODULUS $|H(\omega)| = 1 \quad \forall \omega$

PHASE:

$$\varphi(\omega) \triangleq \angle H(\omega) = \begin{cases} -\frac{\pi}{2} & \text{if } \omega > 0 \\ \frac{\pi}{2} & \text{if } \omega < 0 \end{cases}$$



111) Obtain the spectrum of the output of a differentiator when the input is a SSS process.

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow x'(t) = y(t)$$

SPECTRUM: $H(\omega) = j\omega$

PROOF:

$$\cancel{Y(\omega) = \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \left[\frac{x(t+\varepsilon) - x(t)}{\varepsilon} \right] e^{-j\omega t} dt}$$

$$Y(\omega) = \int_{-\infty}^{+\infty} y(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \left[\frac{x(t+\varepsilon) - x(t)}{\varepsilon} \right] e^{-j\omega t} dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \int_{-\infty}^{+\infty} \left[x(t+\varepsilon) - x(t) \right] \cdot e^{-j\omega t} dt$$

$$t' = t + \varepsilon$$

$$\cancel{= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \left[x(t+\varepsilon) - x(t) \right] \cdot e^{-j\omega t} dt}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\underbrace{\int_{-\infty}^{+\infty} x(t') e^{-j\omega t'} dt'}_{X(\omega)} e^{j\omega\varepsilon} - X(\omega) \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(e^{j\omega\varepsilon} - 1 \right) X(\omega)$$

$$\lim_{\varepsilon \to 0} \frac{e^{j\omega\varepsilon} - 1}{\varepsilon} \cdot X(\omega) = \frac{0}{0} \text{ (indefinite.}$$

BY DE L'HOSPITAL RULE:

$$\lim_{\varepsilon \to 0} \frac{e^{j\omega\varepsilon} \cdot j\omega}{1} \cdot X(\omega) = j\omega \cdot X(\omega) \cdot 1$$

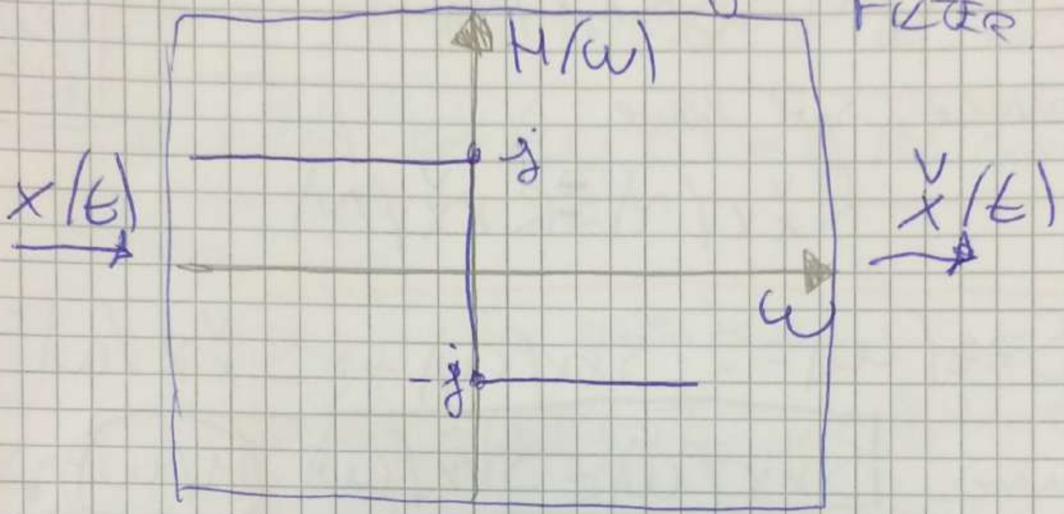
$$\Rightarrow H(\omega) = j\omega$$

113) DEFINE HILBERT TRANSFORM of a Process:

INPUT: REAL WSS process $X(t)$

$$\Rightarrow \check{X}(t) \triangleq x(t) \circledast \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x(\lambda)}{t-\lambda} d\lambda$$

$\check{X}(t)$ is the OUTPUT of a QUADRATURE FILTER



114) DEFINITION of "ANALYTIC SIGNAL" associated to a process $X(t)$ and DERIVE the relationship between its spectrum and that of the process $X(t)$.

DEFINITION OF ANALYTIC SIGNAL:

$x(t)$ and $\check{x}(t)$ real and WSS processes, we form COMPLEX PROCESS:

$$z(t) \triangleq x(t) + j\check{x}(t)$$

analytic signal associated with $X(t)$.

RELationship between SPECTRUM of $Z(t)$ and SPECTRUM of $X(t)$.

Derive spectrum of $z(t)$. (1) Compute $R_{zz}(\tau)$

$$R_{zz}(\tau) = E\{z(t+\tau)\, z^*(t)\}$$

Because $z(t) = x(t) + j\check{x}(t)$

$$= E\{x(t+\tau)\cdot x(t)\} + E\{\check{x}(t+\tau)\cdot \check{x}(t)\}$$
$$- j\cdot E\{x(t+\tau)\, \check{x}(t)\} + j\cdot E\{\check{x}(t+\tau)\cdot x(t)\}$$
$$= R_{xx}(\tau) + R_{\check{x}\check{x}}(\tau) + j\left[-R_{x\check{x}}(\tau) + R_{\check{x}x}(\tau)\right]$$
$$= 2R_{xx}(\tau) + 2jR_{\check{x}x}(\tau)$$

$$\boxed{R_{xx}(\tau) = R_{\check{x}\check{x}}(\tau)}$$

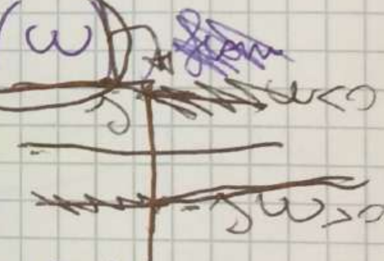
Since we have

$$R_{\check{x}x}(\tau) = -R_{x\check{x}}(\tau)$$

$$\Rightarrow S_{zz}(\omega) = 2\left[S_{xx}(\omega) + jS_{\check{x}x}(\omega)\right]$$

Since $\boxed{S_{\check{x}x}(\omega) = S_{xx}(\omega)\cdot H(\omega)}$

From $R_{\check{x}x}(\tau) = R_{xx}(\tau) \circledast h(\tau)$



$$S_{zz}(\omega) = \begin{cases} 2S_{xx}(\omega) + 2S_{xx}(\omega)(j)(-j) & \omega > 0 \\ 2S_{xx}(\omega) + 2S_{xx}(\omega)(j)(+j) & \omega < 0 \end{cases}$$

$$S_{zz}(\omega) = \begin{cases} 4S_{xx}(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$

$\Rightarrow$ IN GENERAL:

$$S_{zz}(\omega) = 4S_{xx}(\omega)\cdot U(\omega)$$

4) Write the DEFINITION of RICE REPRESENTATION OF A SSS process $X(t)$.
Give the RELATIONSHIP between its spectrum and that of its I-Q component.

DEFINITION OF RICE REPRESENTATION:

Given a WSS real process $X(t)$ with ZERO MEAN, we can always write $X(t)$ as:

$$\boxed{X(t) = i(t) \cdot \cos(\omega_0 t) - q(t) \cdot \sin(\omega_0 t)}$$

where:

ω_0 is arbitrary

$$i(t) = \text{Real}\{z(t) \cdot e^{-j\omega_0 t}\}$$

$$q(t) = \text{Imag}\{z(t) \cdot e^{-j\omega_0 t}\}$$

$$z(t) e^{-j\omega_0 t} = i(t) + j\,q(t)$$

$z(t)$ is the ANALYTIC SIGNAL associated to $X(t)$.

RELATIONSHIP between SPECTRA:

$$\boxed{S_{ii}(\omega) = S_{qq}(\omega) = S_{xx}(\omega + \omega_0)\, U(\omega + \omega_0) + S_{xx}(\omega - \omega_0)\, U(-\omega + \omega_0)}$$

4o/ DEFINITION OF "RICE REPRESENTATION" of a SSS process $x(t)$. Give the relationship between its spectrum and that of ~~...~~ ~~...~~ its I-Q components

DEFINITION OF RICE REPRESENTATION:

Given a WSS Real process $x(t)$ with ZERO MEAN. We can always write $x(t)$ as:

$$x(t) = i(t) \cdot \cos(\omega_0 t) - q(t) \cdot \sin(\omega_0 t)$$

RELATIONSHIP:

$$R_{ww}(\tau) = E\{w(t+\tau) \cdot w^*(t)\} =$$

$$= E\{[i(t+\tau) + j\, q(t+\tau)] \cdot [i(t) - j q(t)]\}$$

$$= R_{ii}(\tau) + R_{qq}(\tau) + j R_{qi}(\tau) - j R_{iq}(\tau)$$

$$= \underbrace{2 R_{ii}(\tau)}_{\text{REAL}} + \underbrace{2 j R_{qi}(\tau)}_{\text{REAL}}$$

ω_0 arbitrary

$$\boxed{\begin{aligned} i(t) &= \text{Real}\{z(t) \cdot e^{-j\omega_0 t}\} \\ q(t) &= \text{Imag}\{z(t) \cdot e^{-j\omega_0 t}\} \end{aligned}}$$

Because:

$$\Rightarrow R_{ii}(\tau) = R_{qq}(\tau)$$

$$R_{qi}(\tau) = -R_{iq}(\tau)$$ ~~...~~

$$R_{ii}(\tau) = \frac{1}{2} \text{Real}\{R_{ww}(\tau)\}$$

$z(t)$

$$R_{qi}(\tau) = \frac{1}{2} \text{Imaginary}\{R_{ww}(\tau)\}$$

At the end of the story:

$$S_{ii}(\omega) = S_{qq}(\omega) = S_{xx}(\omega + \omega_0) \cdot U(\omega + \omega_0) + S_{xx}(\omega - \omega_0) \cdot U(-\omega + \omega_0)$$

43) DEFINITION OF MEAN-SQUARE PERIODIC PROCESS.

A WSS process $X(t)$ is MEAN-SQUARE (M.S.) PERIODIC if its autocorrelation $R_{XX}(\tau)$ is PERIODIC.

$$R_{XX}(\tau + T) = R_{XX}(\tau) \quad \forall \tau \text{ for a certain } T$$

44) DEFINITION OF FOURIER EXPANSION REPRESENTATION OF A SSS PROCESS $X(t)$.

Given a WSS PROCESS $x(t)$, the sum:

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{j n \omega_0 t}$$

where:

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) \cdot e^{-j n \omega_0 t} \, dt$$

with: $\omega_0 = \frac{2\pi}{T}$

If we have T arbitrary,

$\hat{x}(t)$ is the Fourier series representation of process $x(t)$ over $\left(-\frac{T}{2}, \frac{T}{2}\right)$

SOLUTION: KARHUNEN - LOÈVE CONTINUATION

$\varphi_n(t)$ are the EIGENFUNCTIONS of the integral equation:

SAME FUNCTION scaled by λ

$$\boxed{\int_0^T R_{xx}(t_1, t_2) \cdot \varphi(t_2)\, dt_2 = \lambda \cdot \varphi(t_1) \quad 0 < t_1 < T}$$

where the solutions $\varphi_n(t)$ are ORTHONORMAL and satisfy the identity:

$$\boxed{R_{xx}(t,t) = \sum_{n=1}^{+\infty} \lambda_n \, |\varphi_n(t)|^2 \quad 0<t<T}$$

where λ_n are the corresponding EIGENVALUES

for each orthonormal function

$\Rightarrow$ This generates:

$\Rightarrow$ 0 variance between $x(t)$ and $\hat{x}(t)$ over $(0, T)$

$$\boxed{E\left\{|x(t) - \hat{x}(t)|^2\right\} = 0}$$

$$\boxed{E\{c_n \cdot c_m^*\} = \lambda_n \cdot \delta[n-m]}$$

4.3) DEFINITION OF KARHUNEN-LOÈVE EXPANSION.

In order to represent $X(t)$ over an interval $0 < t < T$, the Fourier Series is a a special case of expansion of a more general form.

$$\hat{X}(t) = \sum_{n=1}^{+\infty} c_n \cdot \varphi_n(t) \quad \text{in } 0 < t < T$$

Where $\varphi_n(t)$ is a set of ORTHONORMAL FUNCTIONS in the interval $(0, T)$.

$$\int_0^T \varphi_n(t) \cdot \varphi_m^*(t)\, dt = \delta[n-m]$$

And coefficients c_n are R.V.s given by:

$$c_n = \int_0^T X(t) \cdot \varphi_n^*(t)\, dt$$

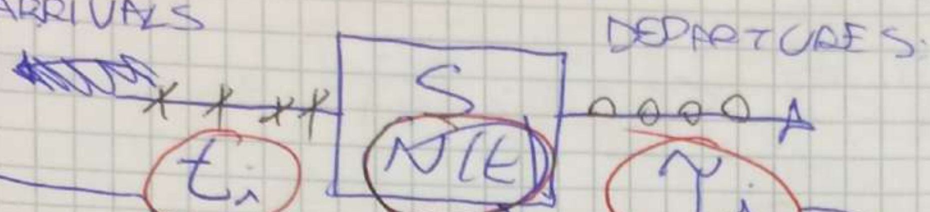
$X(t)$ very general.

Not necessarily WSS, not M.S. periodic.

$$\delta[n-m] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

HOW TO QUEUING THEORY – INTRODUCTION:

Queueing theory describes a large set of phenomena, concerning arrivals, waiting and servicing.



t_i → Arrival time

⇓

Arriving units are identified by their arrival time t_i

τ_i → Departure time

⇓

Time at what the system releases a certain arriving unit. Identified by τ_i

$N(t)$ ⇒ STATE OF THE SYSTEM.

It describes the # units inside the system S (being processed by S or queueing) at time t.

FUNDAMENTAL TIMES IN A SYSTEM.

- SYSTEM TIME: a_i $\boxed{a_i = \underbrace{\tau_i}_{\text{departure}} - \underbrace{t_i}_{\text{arrival}}}$

Time spent by the i-th unit inside the system S (how long it stays in S).

- WAITING TIME: * b_i $\boxed{b_i = \underbrace{\tau_{i-1}}_{\text{departure previous}} - \underbrace{t_i}_{\text{arrival}}}$

Time spent by the i-th unit in queue, before entering the system S.

SERVICE TIME (c_i) : $\boxed{c_i = \gamma_i - \gamma_{i-1}}$

Time spent by the unit ~~being~~ to be processed by the system. (serviced unit)

RELATION AMONG 3 TIMES

$$\boxed{a_i = b_i + c_i} \Rightarrow \text{R.Vs!!}$$

SYSTEM TIME — WAITING TIME — SERVICE TIME

The system time is the sum of the waiting time and the service time.

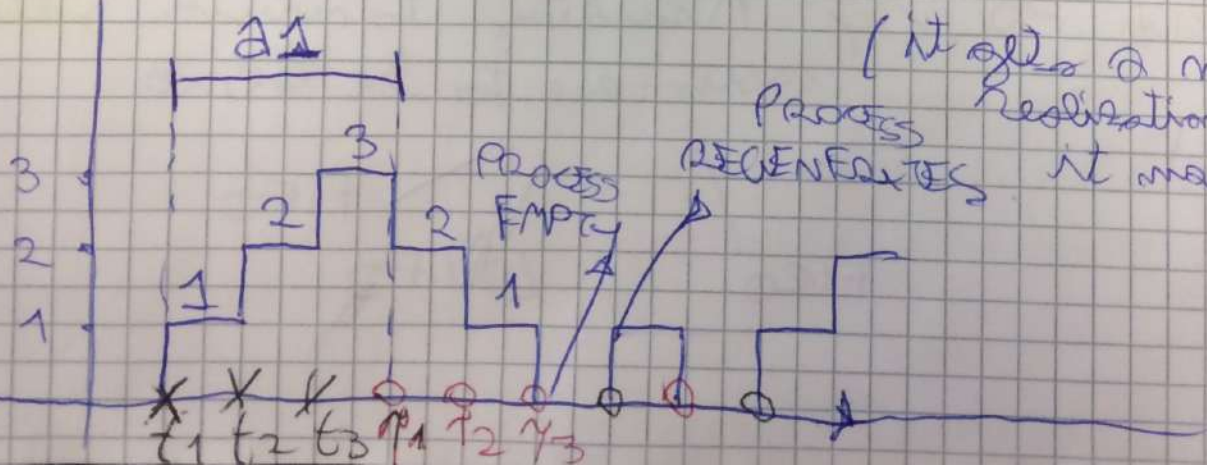
REGENERATING PROCESS (REGENERATIVE)

If $N(t)$ is the state of the system, counting the # units inside the system at time t, and if we consider a STATIONARY PROCESS $X(t)$, where a queue has already been existing!

$N(t)$ is a DISCRETE-STATE CONTINUOUS TIME ~~Domain~~ continuous process:

- increasing by 1 at t_i (arrival)
- decreasing by 1 at γ_1 (departure)

(it offers a new realization after it was empty)



KENDALL'S NOTATION: Field of 3/6 pieces of information with statistics about the process.

STATISTICS ABOUT: (| | |) (| | |)

① → ARRIVAL TIMES

② → SERVICE TIMES

③ → # SERVERS

④ → MAX. CAPACITY

⑤ → # USERS

⑥ → SERVER POLICY

① TYPE OF INPUT PROCESS (ARRIVAL TIMES)
- → M: Memoryless / Poisson (Markov)
- → G: General. Arbitrary arrival times
- → E_K: Erlang-K

② DISTRIBUTION OF SERVICE TIME
- → M: " "
- → G: " "
- → D: Deterministic service time. (NO R.V.), constant.

③ # SERVERS AVAILABLE
- → 1 Server
- → ∞ Servers
  - ↳ Immediate service

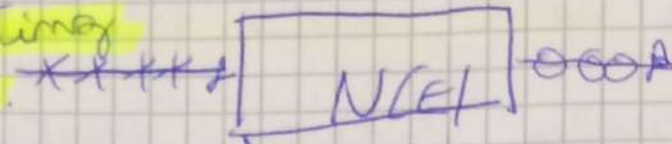
④ CAPACITY: + EXTRA ~~Max # users in the system~~ (How many people can wait in queue)

⑤ MAX # USERS (POPULATION): How many users can be serviced at the same time. (∞ capacity ⇒ ∞ users)

⑥ SERVER POLICY: According to which policy users are to be served.
- FIRO
- LIFO
- FIFO

N(t)
↓
STATE

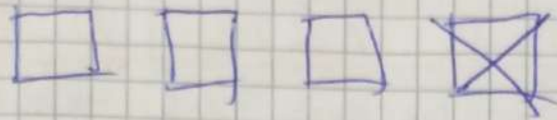
QUEUE # units waiting to be served



Ex: $\underline{N(t)} = 4$ (if capacity = 1)

STATE

⇒ 3 units in queue
1 unit serviced



ASSUMPTIONS:

- Always consider Poisson arrival times. (Poisson Point Process), with intensity: $\lambda = \frac{\# \text{ units}}{\text{time}}$
- Distribution function of service time $F_c(c)$, $\forall i$.
- Arrival times and service times are INDEPENDENT R.Vs.
- N(t) is a SSS process. → Strict-Sense Stationary
  ⇒ N(t) not affected by a shift of the time origin.

WRITE AND DERIVE LITTLE'S FORMULA:

Formula holding without any conditions about interarrival times (not just Poisson), independently of the type of the system or properties of a_i.

IDEA: Little's Formula relates # units at any time with the intensity and the system time.

ASSUME: SSS process $x(t)$

Suppose point processes t_i and a_i are mean-ergodic.

INTENSITY of points over T.

random # points t_i in $(0,T)$

$$\boxed{\lim_{T\to\infty} \frac{n_T}{T} \xrightarrow{2} \boxed{\lambda = \frac{E\{n_T\}}{T}}}$$

sum of all arrival times

$$\boxed{\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = E\{a_i\}} \rightarrow \text{SYSTEM TIME}$$

$a_i = b_i + c_i$

The amount of units in system = INT $\times$ EXPECTED VALUE OF ARRIVAL TIME

$$\boxed{E\{N(t)\} = \lambda \cdot E\{a_i\}} \leftarrow \text{LITTLE'S FORMULA.}$$

intensity

Expected value of (arrival time)

PROOF:

We will show that $N(t)$ is also mean-ergodic:

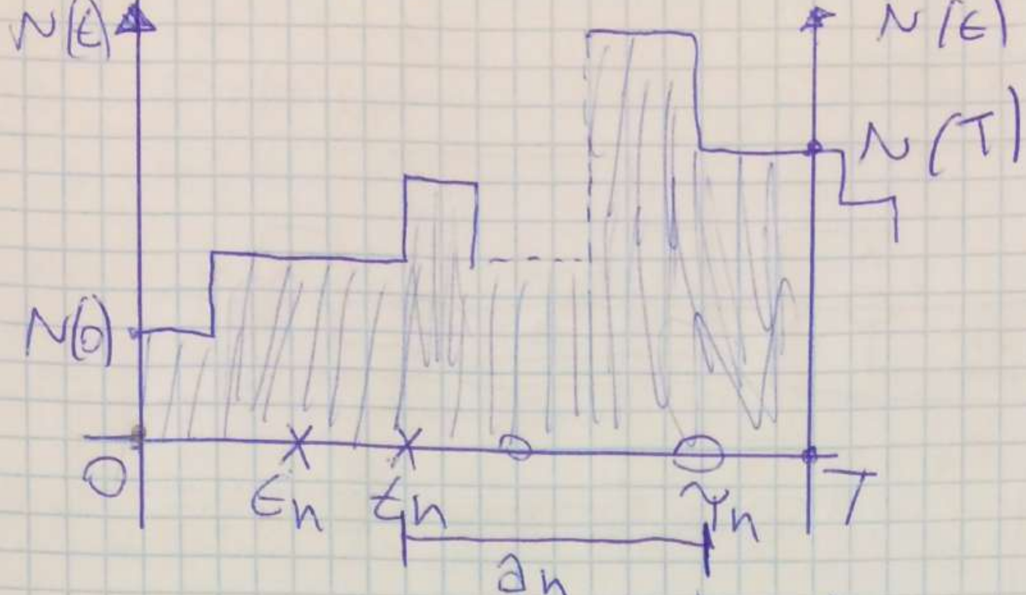
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T N(\tau)\,dt = \lambda \cdot E\{a_i\} = E\{N(t)\}$$

Consider a time interval $(0,T]$:

RECALL: MEAN-ERGODICITY DEFINITION:

For a SSS $x(t)$:

$$E\{x(t)\} = \eta_x$$



↳ system time, time a unit n stays inside the system.

We have to prove that:

$$\boxed{-\underbrace{\sum_{r=1}^{N(T)} a_r}_{\text{(1) NEGATIVE}}^{\text{END}} = \underbrace{\int_0^T N(t)\,dt}_{(2)} - \underbrace{\sum_{n=1}^{n_T} a_n}_{(3)}^{\#\text{ points arriving in } (0,T)} = \underbrace{\sum_{i=1}^{N(0)} a_i}_{(4)}^{\text{BEGINNING}}}$$

1) Sum of times that $N(t)$ units ~~at N(t) are~~ still in S at $T=t$ remain in S (also beyond T).

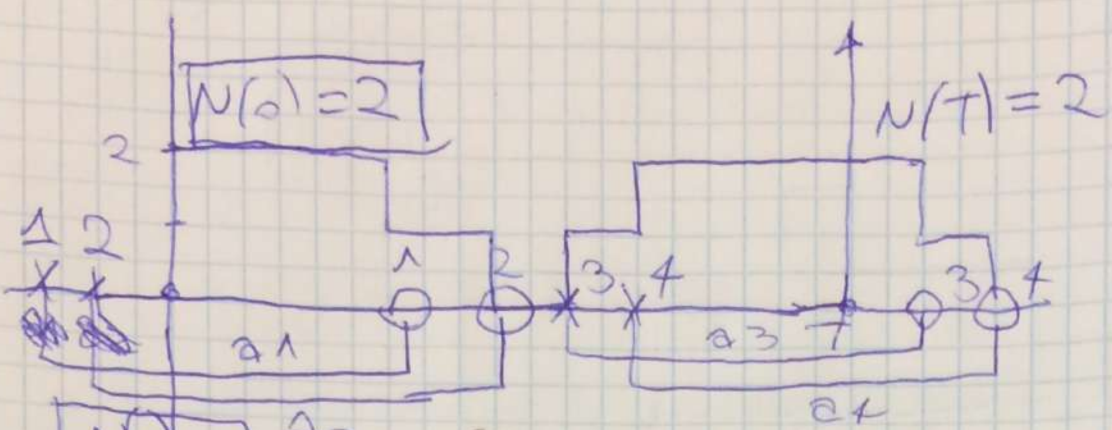
2) Overall system time of units in S in time (0, T)

3) Sum of times that units arriving in (0, T) stay in S (also beyond T)

4) Sum of times that N(0) units that are in S at $t=0$ remain in S (also beyond T).

PROOF: (4) $\sum_{i=1}^{N(0)} a_i$ is an area with a height 1 and base given by: $a_1 + a_2 + \ldots + a_{N(0)}$

because.



$\boxed{\sum_{i=1}^{N(0)} a_i}$ → a_1, a_2 ⇒ Height = 1, base = $a_1 + a_2 + .. + a_{N(0)}$

2-SIDES PROOF:

(1) RIGHT-HAND INEQUALITY

(2) LEFT-HAND INEQUALITY

$\boxed{1}$ RIGHT-HAND INEQUALITY

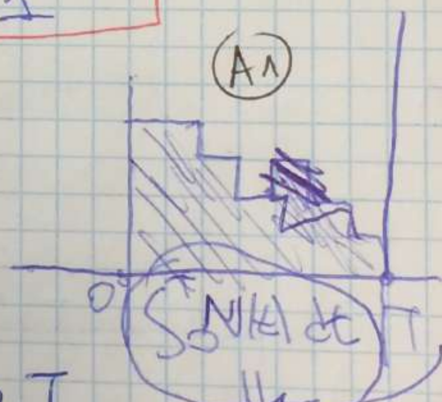
Prove that:

$$\boxed{\int_0^T N(t)\,dt - \sum_{n=1}^{n_T} a_n \le \sum_{i=1}^{N(T)} a_i}$$

~~Two sub-cases~~

A) No units enter (0,T)

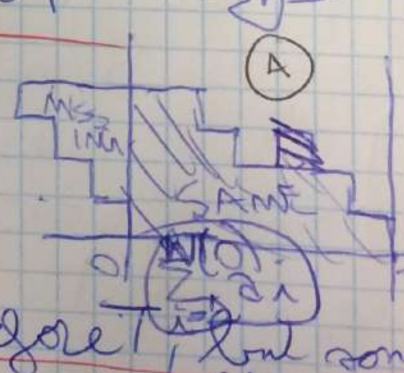
⇒ $n_T = 0$



A1) All units exit S before T

⇒ Surely

$$\int_0^T N(t)\,dt \le \sum_{i=1}^{N(0)} a_i$$



A2) Not all units exit S before T, but some exit after T:

⇒ Situation even more in favour of $\sum_{i=1}^{N(0)} a_i$.

$\int_0^T N(t)\,dt$ is just a portion of $\sum_{i=1}^{N(0)} a_i$

(B) If some n_T units enter S during $(0,T)$.

$\Rightarrow n_T \neq 0$.

Both (A1) and (A2) have

$\int_0^T N(t)\,dt$ subtracted by positive quantity $\sum_{n=1}^{n_T} a_n$

This difference is always less than or equal to $\int_0^T N(t)\,dt$ for $n_T = 0$.

So we have proved ~~that~~ the right-hand side

$$\boxed{\underbrace{\int_0^T N(t)\,dt - \sum_{n=1}^{n_T} a_n}_{\text{may be negative}} \le \sum_{i=1}^{N(0)} a_i} \rightarrow \text{unchanged.}$$

2 LEFT-HAND INEQUALITY: $-\sum_{r=1}^{N(t)} a_r \le \int_0^T N(t)\,dt - \sum_{n=1}^{n_T} a_n$

We now have to prove that

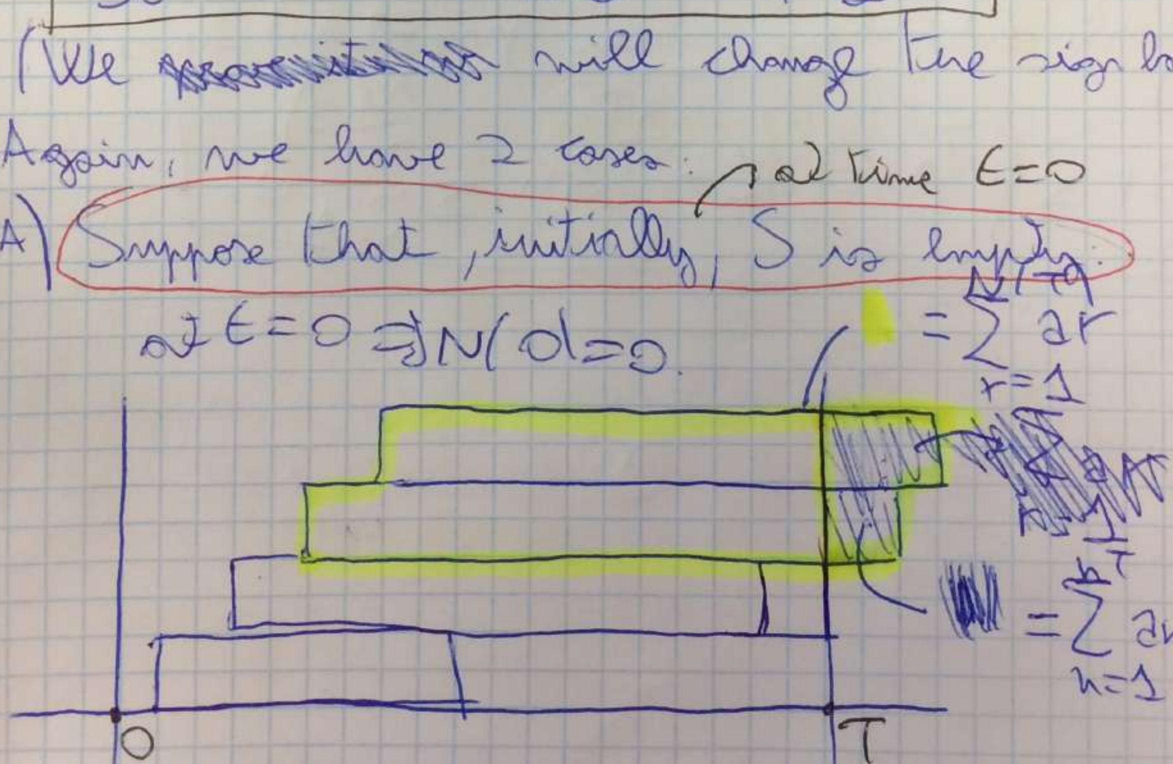
$$\boxed{-\int_0^T N(t)\,dt + \sum_{n=1}^{n_T} a_n \le \sum_{r=1}^{N(t)} a_r}$$

(We ~~[illegible]~~ will change the sign later on).

Again, we have 2 cases:

A) Suppose that, initially (at time $t=0$), S is empty

at $t=0 \Rightarrow N(0)=0$.



$= \sum_{r=1}^{N(T)} a_r$

$= \sum_{n=1}^{n_T} a_n - \int_0^T N(t)\,dt$

Quantity ~~[illegible]~~

$$\boxed{\sum_{n=1}^{n_T} a_n - \int_0^T N(t)\,dt < \sum_{r=1}^{N(T)} a_r}$$

part of hashed area — continued part

B) If $N(0) \neq 0$, then (S not empty at $t=0$)

$\sum_{n=1}^{n_T} a_n - \int_0^T N(t)\,dt$ was smaller than $\sum_{r=1}^{N(T)} a_r$ became:

$\int_0^T N(t)\,dt$ increases, whereas n_T does not change.

$\sum_{r=1}^{N(T)} a_r$ does not change or increases (if some units alive at $t=0$ die before T or after T).

$$\Rightarrow \boxed{\sum_{n=1}^{n_T} a_n - \int_0^T N(t)\,dt \leq \sum_{r=1}^{N(T)} a_r}$$

$\Rightarrow$ Both the left-hand side and the right-hand side have been proved and hence:

$$\boxed{-\sum_{r=1}^{N(T)} a_r \leq \int_0^T N(t)\,dt - \sum_{n=1}^{n_T} a_n \leq \sum_{i=1}^{N(0)} a_i}$$

Continuing with the derivation of Little's Formula from what we just proved:

$$-\sum_{r=1}^{N(T)} a_r \leq \int_0^T N(t)\,dt - \sum_{n=1}^{NT} a_n \leq \sum_{i=1}^{N(0)} a_i$$

We know that the RANDOM SUM at any given time instant t is:

MEAN SQUARE VALUE

$$\boxed{E\left\{\left(\sum_{k=1}^{N(t)} a_k\right)^2\right\} \leq \underbrace{E\{N^2(t)\}}_{\text{R.V. Assumed limited/finite}} \cdot \underbrace{E\{a_k^2\}}_{\text{R.V. Assumed finite/limited}} < \infty}$$

If the M.S. value of a R.V. is finite, then the R.V. takes the ∞ value with zero probability $\Rightarrow$ It is a FINITE R.V.

We want to show that this inequality holds because we do not want a unit to stay in S for an infinite time.

RSENCO

PROOF:

(1) Consider the RANDOM SUM of X_k:

$$\boxed{S = \sum_{k=1}^{N} X_k}$$

(2) Suppose $X_k \geq 0$ and identically distributed, ~~...~~ but NOT independent and ~~...~~ the mean square value of X_k is M:

ASSUME: $\boxed{E\{X_k^2\} = M}$

③ MEAN SQUARE VALUE OF S^2

$$E\{s^2\} = E\left\{\left(\sum_{k=1}^{n} x_k\right)^2\right\} = E\left\{\sum_{k=1}^{n}\sum_{m=1}^{n} x_k x_m\right\}$$

④ Fix $n = n$

PROVED LATER

$$E\{s^2 | n = n\} = \underbrace{\sum_{k=1}^{n}\sum_{m=1}^{n} E\{x_k \cdot x_m\}}\leq n^2 M$$

The double summation contains n terms with $k = m$ and ~~terms~~ $n^2 - n$ terms with $k \neq m$ where:

$$\boxed{E\{x_k \cdot x_m\} \leq M} \text{ and } x_k \perp 0$$

⇒ ⑤ CONCLUDE THAT:

$$\boxed{E\{s^2\} = E\{E\{s^2 | n\}\} \leq E\{n^2\} \cdot E\{x_i^2\}}$$

(which corresponds to what we wanted to prove)

⑥ NOW PROVE THAT:

$$E\{x_k \cdot x_m\} \leq M$$

In our case, for $k \neq m$, we have:

$$\boxed{E^2\{x_k \cdot x_m\} \leq \overbrace{E\{x_k^2\}}^{M} \cdot \overbrace{E\{x_m^2\}}^{M}}$$

$\Rightarrow E\{x_k \cdot x_m\} \leq M$ Because in general:

$$\boxed{E^2\{xy\} \leq E\{x^2\} \cdot E\{y^2\}}$$

In fact, consider the general case:

Consider $E\{(ax-y)^2\} = a^2 \underbrace{E\{x^2\}}_{a} - 2a\underbrace{E\{xy\}}_{b} + \underbrace{E\{y^2\}}_{c}$

Quadratic in a and non-negative in $-2a$.

$\Rightarrow$ $b^2 - 4ac \leq 0$ (discriminant non-positive)

$\Delta \leq 0$.

$$\Rightarrow E^2\{xy\} - E\{x^2\} \cdot E\{y^2\} \leq 0$$

Consider now again

$$\boxed{-\sum_{r=1}^{N(T)} a_r \leq \int_0^T N(t)\,dt - \sum_{n=1}^{n_T} a_n \leq \sum_{i=1}^{N(0)} a_i}$$

Multiply it by $\frac{1}{T}$ & write $T \to \infty$

$$\underbrace{-\frac{1}{T} \sum_{r=1}^{N(T)} a_r}_{0} \leq \frac{1}{T}\int_0^T N(t)\,dt - \frac{1}{T}\sum_{n=1}^{n_T} a_n \leq \underbrace{\frac{1}{T}\sum_{i=1}^{N(0)} a_i}_{0}$$

$$0 \leq \frac{1}{T}\int_0^T N(t)\,dt - \frac{1}{T}\sum_{n=1}^{n_T} a_n \leq 0$$

$$\boxed{\frac{1}{T}\int_0^T N(t)\,dt \approx \frac{1}{T}\sum_{n=1}^{n_T} a_n}$$

2 1

By assumption of MEAN-ERGODICITY:

$\boxed{n_T \approx \lambda T} \Rightarrow \frac{1}{T} = \frac{\lambda}{n_T}$

$\Rightarrow$ Replace $\frac{1}{T}$ by $\frac{\lambda}{n_T}$ in $\frac{1}{T} \sum_{n=1}^{n_T} a_n$

$$\cancel{\frac{1}{n_T}\int_0^T N(t)\,dt \approx \frac{1}{n_T}\sum_{n=1}^{n_T} a_n}$$

(1)

$$\Rightarrow \frac{1}{T}\sum_{n=1}^{n_T} a_n \approx \frac{\lambda}{n_T}\sum_{n=1}^{n_T} a_n \approx \lambda \cdot E\{a_n\}$$

$= E\{a_n\}$

Now replace $\frac{1}{T}$ by $\frac{\lambda}{n_T}$ in $\frac{1}{T}\int_0^T N(t)\,dt$.

(2)

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T \underbrace{N(t)}_{\text{R.V.}}\,dt = \cancel{\lambda}\; E\{a_i\} \rightarrow \text{constant}$$

(with $N(t)$ stationary)

$\Rightarrow$ Take $E(\;)$ of both ~~terms~~ members.

$$\boxed{E\{N(T)\} = \lambda \cdot E\{a_i\}}$$

PROVED!

Since both terms are equal to $\lambda \cdot E\{a_i\}$:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T N(t)\,dt = E\{N(t)\}$$

$\Rightarrow$ $\boxed{N(t) \text{ is ergodic}}$.

45) Calculate the distribution function of the state $N(t)$ in a $\boxed{M/G/\infty \text{ queue}}$

We want to prove the following theorem:
"The state $N(t)$ of a $M/G/\infty$ system (is) is Poisson distributed:

$$\boxed{P\{N(t)=k\} = e^{-\rho} \cdot \frac{\rho^k}{k!}}$$ ← Regarding the distribution $\forall t$

Where ρ is the traffic intensity / offered load:

$$\boxed{\rho = \lambda \cdot E\{\tau_n\} = \lambda \cdot \eta_\tau}$$

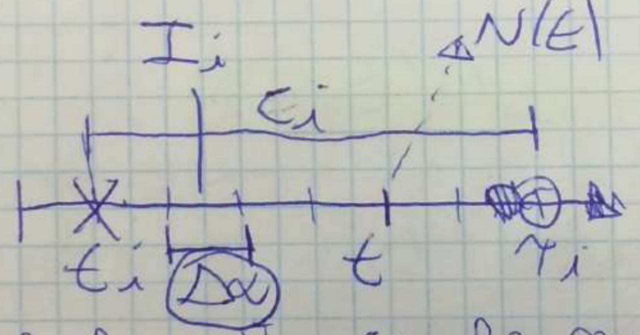
→ mean time per service

Amount of units arriving to S per unit time.

service time

intensity of arrivals

PROOF: Study $N(t)$:



Because each unit can be served immediately, the service time is IMMEDIATE.

⇒ In IMMEDIATE SERVICE, each service time τ_i does not affect the other units.

1) ARRIVALS IN I_i: The time axis preceding t is divided in small subintervals I_i with duration $\Delta\alpha$. Δn is the amount of arrivals in I_i, which can be only 0 or 1 (0 units or 1 unit) and is Poisson distributed

Formally

$\boxed{\Delta n_i = \#\text{ arrivals in } I_i}$

Probability of j points in $\Delta\alpha$ time:

$(0 \text{ units}, 1 \text{ unit})$

$$\boxed{P\{\Delta n_i = j\} = e^{-\lambda \cdot \Delta\alpha} \cdot \frac{(\lambda \cdot \Delta\alpha)^j}{j!}}$$

We can have 3 cases:

- $P\{\Delta n_i = 1\} = \underbrace{e^{-\lambda \cdot \Delta\alpha}}_{1} \cdot \frac{(\lambda \cdot \Delta\alpha)^1}{1!} = \lambda \cdot \Delta\alpha$; 1 Poisson Point in $\Delta\alpha$

- $P\{\Delta n_i > 1\} = e^{-\lambda \cdot \Delta\alpha} \cdot \frac{(\lambda \cdot \Delta\alpha)^2}{2!} + e^{-\lambda\Delta\alpha} \frac{(\lambda \Delta\alpha)^3}{3!} + \dots$

$\simeq 0$

- $P\{\Delta n_i = 0\} = 1 - P\{\Delta n_i \geq 1\} = 1 - \lambda \Delta\alpha$

2) DENOTE BY $\Delta N(t, \alpha_i)$ (α_i: time distance of I_i from t) the contribution to the state $N(t)$ of S at time t due to the arrivals Δn_i in I_i (Δn_i can only be 0 or 1).

We then have:

$$\boxed{N(t) = \sum_i \Delta N(t, \alpha_i)}$$

$N(t)$ is the sum of the contributions of all sub-intervals.

(3) Study the probability of:

$$\boxed{\Delta N(t, \alpha_i)}$$

($\Delta n_i = \#$ points in I_i)

ΔN_i can only be 0 or 1.

$$\Delta N_i = \begin{cases} 0 \\ 1 \end{cases}$$

• If $\Delta n_i = 0 \Rightarrow$ No must contribute.

• If $\Delta n_i = 1$ ~~we may have a contribution depending on:~~
$\Downarrow$

$\Delta N(t, \alpha_i) = \begin{cases} 1 \\ 0 \end{cases}$ Contribution Depending on:

if $\Delta n_i = 1 \Rightarrow$ $\boxed{\Delta N(t, \alpha_i) = \begin{cases} 1 & \text{if service time } \boxed{C_i > \alpha_i} \text{ (exceeding the duration of interval)} \\ 0 & \text{if } C_i \text{ completed before } t \ \boxed{C_i < \alpha_i} \end{cases}}$

Summing up:

* if $\Delta n = 1 \Rightarrow \Delta N(t, \alpha_i) = \begin{cases} 1 & \text{if } C_i > \alpha_i \\ 0 & \text{if } C_i \leq \alpha_i \end{cases}$

~~Therefore that pro~~
~~$P\{\Delta N(t, \alpha_i) = 1\}$~~

Therefore, the probabity that there is a contribution depends on the distribution function of $C_i > \alpha_i$

$$\boxed{P\{\underbrace{\Delta N(t, \alpha_i) = 1}\,|\,\Delta n_i = 1\} = P\{C_i > \alpha_i\} = 1 - F_C(\alpha_i)}$$

By the TOTAL PROBABILITY THEOREM:

$$P\{\Delta N(t, \alpha_i) = 1\} = P\{\Delta N(t, \alpha_i) = 1 \,|\, \Delta n_i = 1\} \cdot P\{\Delta n_i = 1\} + 0$$

other case: zero contribution $\quad P\{\Delta N(t, \alpha_i) = 1 \,|\, \Delta n_i = 0\} = 0$

$$= \boxed{[1 - F_C(\alpha_i)] \cdot \lambda \Delta \alpha}$$

(Poisson counts over non-overlapping intervals)

$\Rightarrow$ R.V. $\Delta N(t, \alpha_i)$ are 0-1 and independent because R.V. Δn_i are independent.

4) Now studying $\boxed{N(t) = \sum \Delta N(t, \alpha_i)}$ as a sum of 0-1 R.Vs. We need the following theorem:

THEOREM

Suppose R.Vs. X_i are independent and each one takes on the values 1 and 0.

$\boxed{\text{0-1 R.V.}} \longrightarrow \boxed{p_i = P\{x_i = 1\}}\ \boxed{q_i = P\{x_i = 0\} = 1 - p_i}$

if $Z = x_1 + x_2 + \dots + x_n$, then: (we would like Z to be Poisson distributed)

WHAT WE WANT: $P\{Z = k\} = \frac{a^k}{k!} \cdot e^{-a}$ for $n \to \infty$,

with $a = p_1 + p_2 + \dots + p_n$

for $p_i \to 0$ for $n \to \infty$

PROOF:

1) Consider the CHARACTERISTIC FUNCTION of X_i

$$\boxed{\Phi_{x_i}(\omega)} = E\{e^{j\omega x_i}\} = p_i \cdot e^{j\omega \cdot 1} + q_i \cdot e^{j\omega 0}$$

$$= p_i \cdot e^{j\omega} + q_i = e^{j\omega} + (1 - p_i)$$

$$= \boxed{1 + p_i \cdot (e^{j\omega} - 1)}$$

For $Z = x_1 + x_2 + \dots + x_n$, we have the sum of n independent R.Vs.

2) Char. function of Z:

$$\boxed{\begin{aligned}\Phi_Z(\omega) &= E\{e^{j\omega(x_1 + x_2 + \dots x_n)}\} \\ &= \Phi_{x_1}(\omega) \cdot \Phi_{x_2}(\omega) \cdot \dots \cdot \Phi_{x_n}(\omega)\end{aligned}}$$

$$\Rightarrow \Phi_Z(\omega) = \left\{[1 + p_1(e^{j\omega} - 1)] \cdot \dots \cdot [1 + p_n(e^{j\omega} - 1)]\right\}$$

Now take the TAYLOR EXPANSION.

③
TAYLOR EXPANSION:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots$$

Set $x = a e^{j\omega}$

(recall: $a \to p_1 + p_2 + \dots + p_n$ for $p_i \to 0$

$$\Rightarrow e^{p_i(e^{j\omega}-1)} \approx \phi_{x_i}(\omega) = 1 + p_i(e^{j\omega}-1)$$

for $p_i \ll 1$ (truncated to linear term)

~~that~~, we have

$\Rightarrow$ The product of $\phi_{x_1}(\omega), \dots, \phi_{x_n}(\omega)$ (product of exponentials) is approximately $\phi_z(\omega)$. Finally: ④

$$\boxed{\phi_z(\omega) = e^{p_1(e^{j\omega}-1)} \cdot e^{p_2(e^{j\omega}-1)} \cdot \dots \cdot e^{p_n(e^{j\omega}-1)}}$$

If $p_i \to 0$, $p_1 + p_2 + \dots + p_n \to a$, $n \to \infty$

$$\boxed{\phi_z(\omega) = e^{a \cdot (e^{j\omega}-1)}}$$

$\hookrightarrow$ Since this is the characteristic function of a Poisson-distributed R.V., R.V. Z is Poisson Distributed

$$P\{Z=k\} = \frac{a^k}{k!} e^{-a} \text{ with } Z = x_1 + x_2 + \dots + x_n$$

PROVED it is Poisson-distributed

$\Rightarrow$ Therefore the R.V.

$$N(t) = \sum \Delta N(t, \alpha_i)$$ is Poisson-distributed

with parameter (a) where

A POISSON + DISTRIBUTED:

Consider only intervals with contributions

$$a = \underbrace{\lim_{\Delta\alpha \to 0} \sum_{i=1} P\{\Delta N(t, \alpha_i) = 1\}}$$

$$= \lim_{\Delta\alpha \to 0} \sum_{i=1}^{+\infty} [1 - F_c(\alpha_i)] \cdot \lambda \cdot \Delta\alpha$$

$$= \underbrace{\lambda \cdot \int_0^{+\infty} [1 - F_c(\alpha)] \, d\alpha}$$

THEOREM: Since we want to show that:

$$E\{N(t)\} = \lambda \cdot E\{c\} = a$$

$$\boxed{\int_0^{+\infty} [1 - F_c(\alpha)] \, d\alpha = E\{c\}} \text{ with } c > 0$$

PROOF: In general we have, for $X \geq 0$:

$$\int_0^{+\infty} [1 - F_X(x)] \, dx = \int_0^{+\infty} P\{X \geq x\} \, dx = E\{X\}$$

because:

$$\int_0^{+\infty} P\{X \geq x\} \, dx = \int_0^{+\infty} \left(\int_0^{+\infty} f_X(u) \, du \right) dx$$

$$= \int_0^{+\infty} \left(\int_0^{u} f_X(u) \, dx \right) du = \int_0^{+\infty} f_X(u) \int_0^{u} dx \, du$$

$$= \int_0^{+\infty} f_X(u) \, u \, du = E\{X\}$$

by definition of mean value of $N(t)$?

POISSON PARAMETER

$$\Rightarrow \boxed{a = \lambda \cdot E\{c\} = \lambda \cdot \eta_c}$$

$$a = \lambda \cdot E\{c\} = \lambda \cdot \eta_c$$

This coincides with Little's Formula:

$$E\{N(t)\} = a = \lambda \cdot \eta_c = \rho$$

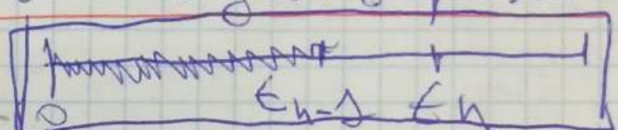
$$\boxed{P\{N(t) = k\} = e^{-\rho} \cdot \frac{\rho^k}{k!}}$$

→ Distribution of units for M | G | ∞ system.

(11) Write the definition of a Markoff process and specify it for a continuous-time discrete-state Markoff Chain.

DEFINITION OF MARKOFF PROCESS:

A Markoff process is a stochastic process $x(t)$ where the past has no influence on the future if its present is specified.



If $\overset{\text{past}}{t_{n-1}} < \overset{\text{present}}{t_n}$ (generic instants)

given present

$$\boxed{\begin{aligned} &P\{x(t_n) \leq x \mid x(t) \text{ given for } t \leq t_{n-1}\} \\ &= P\{x(t_n) \leq x \mid x(t_{n-1})\} \end{aligned}}$$

given past

where time has continuous values.

ANALYTIC DEFINITION FOR CONTINUOUS-TIME (CT) DISCRETE-STATE Markoff Chain.

A continuous-Time Markoff Chain is defined as:

~~Continuous-Time Markoff Chain definition~~

~~$P\{x(t_{n+1}) = x_{n+1} \mid x(t_n) = x_n, \ldots, x(t_1) = x_1\}$~~

Continuous-Time (CT) Markoff Chain definition:

$$\boxed{\begin{aligned} &P\{x(t_{n+1}) = x_{n+1} \mid x(t_n) = x_n, \ldots, x(t_1) = x_1\} \\ &\quad = P\{x(t_{n+1}) = x_{n+1} \mid x(t_n) = x_n\} \end{aligned}}$$

where $\{x_n\}$ belong to the set of DISCRETE VALUES $\{a_1, a_2, \ldots, a_N\}$ and $t_1 < t_2 < \ldots < t_n < t_{n+1}$ are GENERIC INSTANTS

12. Definition and properties of the transition probability rates for a continuous Time Discrete-State Markoff Chain

NO! WRONG.

Markoff Chain

TRANSITION PROBABILITY DEFINITION for CT-DSMC

(from i-th state at time t_1 to j-th state at time t_2)

$$\boxed{\pi_{ij}[t_1, t_2] \triangleq P\{x(t_2) = a_j \mid x(t_1) = a_i\}}$$

PROPERTIES:

STATE PROBABILITY:

$$p_i(t) \triangleq P\{x(t) = a_i\}$$

1) NORMALIZATION:

$$\boxed{\sum_j \pi_{ij}(t_1, t_2) = 1}$$

2) TOTAL PROBABILITY THEOREM:

$$\boxed{\sum_i p_i(t_1)\, \pi_{ij}(t_1, t_2) = p_j(t_2)}$$

(12) Give the definition & properties of the transition probability rates for a Continuous-Time Discrete-State Markoff Chain

We know that $\Pi(\tau)$ is a transition probability matrix and it satisfies the differential equation:

$$\boxed{\Pi'(\tau) = \Pi(\tau)\,\Lambda}$$

$\Rightarrow \Pi(\tau)$ can be determined as:

$$\Pi(\tau) = e^{\Lambda\tau} \Rightarrow \text{KOLMOGOROFF EQUATION}$$

In matrix form, $\Lambda \triangleq \Pi'(0^+)$

$$\Rightarrow \Lambda \triangleq \Pi'(0^+) = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & & \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{bmatrix}$$

Matrix of numbers

where the elements λ_{ij} (numbers are):

$$\boxed{\lambda_{ij} = \pi'_{ij}(0^+) = \frac{d}{d\tau}\pi_{ij}(\tau)\Big|_{\tau=0^+}}$$

derivative from the right of π_{ij} at 0.

(Assuming π_{ij} differentiable in 0^+)

↳ holds if transition probability (π_{ij}) from a_i to a_j ~~...~~ over small interval $\Delta\tau$ is in the order of $\Delta\tau$.

So, we must have: CONDITION.

$$\pi_{ij}(\Delta\tau) = P\{x(\tau+\Delta\tau) = a_j \mid x(\tau) = a_i\}$$
$$= \begin{cases} 1 + \lambda_{ii}\Delta t & \text{if } j = i \\ \lambda_{ij}\,\Delta\tau & \text{if } j \neq i \end{cases}$$

So, at $\tau = 0$,
we consider how elements $\pi_{ij}(\tau)$ change,

$$\Pi(0) = \begin{vmatrix} 1 & 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & 0 & & 1 \end{vmatrix}$$

when you start moving forward in time from:

$$\boxed{\tau = 0 \to \tau = 0^+}$$

$$\boxed{\frac{d}{d\tau}\Pi(\tau)\Big|_{\tau=0} = \frac{d}{d\tau}\Pi(\tau)\Big|_{\tau=0^+}}$$

⇒ Derivatives λ_{ij} are the TRANSITION PROBABILITY RATES of $x(t)$.

⇒ λ_{ij} is the rate at which the process departs from a_j to a_i, because:

$\boxed{\lambda_{ij}\Delta t}$ = P{ a system at time τ in state a_i transitions to any other state a_j } during a small interval $(\tau, \tau + \Delta t)$

PROPERTIES

- $\boxed{\sum_j \lambda_{ij} = 0}$ $\quad \sum_i \pi_{ij} = 1$
- $\boxed{-\underbrace{\lambda_{ii}}_{\text{POSITIVE}} = \sum_{j} \lambda_{ij}}$ ⇒ λ_{ii} ~~negative~~ missing term for j
- For $i = j$

$$\boxed{\lambda_{ii} = \frac{d\pi_{ii}(\tau)}{d\tau}\Big|_{0^+} \leq 0}$$

3) CHAPMAN - KOLMOGOROFF

$$\boxed{\pi_{ij}(t_1, t_3) = \sum_r \pi_{ir}(t_1, t_2) \cdot \pi_{rj}(t_2, t_3)}$$

$$\Pi[t_1, t_3] = \Pi[t_1, t_2] \cdot \Pi[t_2, t_3]$$

with $t_1 \leq t_2 < t_3$

$\Rightarrow$ Assume $x(t)$ is HOMOGENEOUS

$\Rightarrow$ (Transition probability depends on difference)

$$\pi_{ij}(t_1, t_2) = \pi_{ij}(t_2 - t_1) = \pi_{ij}(\tau)$$

$$= P\{x(t+\tau) = a_j \mid x(t) = a_i\}$$

CHAPMAN - KOLMOGOROFF EQUATION FOR CT MARKOFF CHAINS.

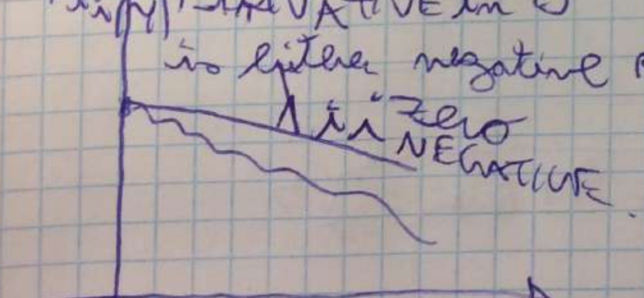
$$\boxed{\Pi(\tau + \alpha) = \Pi(\tau) \cdot \Pi(\alpha)}$$

where

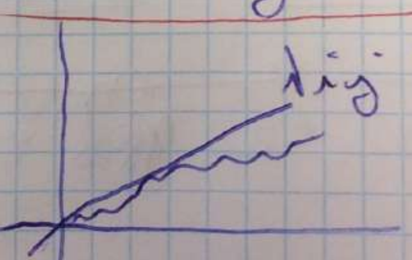
$\Pi(\tau)$ is the transition matrix with elements $\pi_{ij}(\tau)$, function of τ.

$$\Pi[n_1, n_2] = \begin{bmatrix} \pi_{11}[n_1, n_2] & \pi_{12}[n_1, n_2] & \cdots & \pi_{1N}[n_1, n_2] \\ \vdots & & & \\ \pi_{N1}[n_1, n_2] & \pi_{N2}[n_1, n_2] & \cdots & \pi_{NN}[n_1, n_2] \end{bmatrix}$$

$\pi_{ii}(\tau)$ DERIVATIVE in 0^+ is either negative or zero



$$\boxed{-\lambda_{ii} = \sum_j \lambda_{ij} \geq 0}$$



(13) Describe the Birth-Death Process and give its transition probability rate matrix:

BIRTH-DEATH PROCESS

Suppose a CT homogeneous Markoff chain takes on values 0, 1, 2, ... and only discontinuities equal to +1 or -1 are allowed.

⇒ $X(t)$ is called a BIRTH-DEATH process.

A Birth-Death process is a Markoff process:

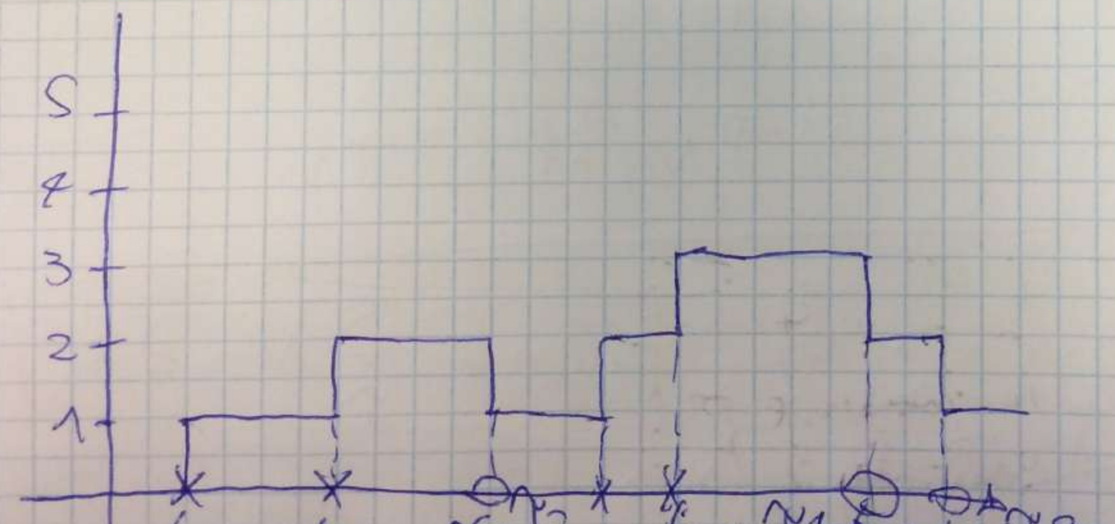
POSSIBLE TRANSITION RATES $\lambda_{ij} \neq 0$ ONLY IF

$$\begin{cases} \boxed{i = j} \text{ or (STEADY)} \\ \boxed{i = j-1} \text{ or (DOWN)} \\ \boxed{i = j+1} \text{ (UP)} \end{cases}$$

only changes of value allowed

⇒ Only corresponding transitions from t to $t + \Delta t$ are allowed [can only go UP or DOWN at transition times], but can also remain steady.

⇒ Intertransition times are exponential.



For two generic time instants t and $t+\Delta t$, there can be 3 cases for transition rates.

1. $n \to n$
2. $n \to n+1$ $\Rightarrow$ TRI-DIAGONAL MATRIX
3. $n \to n-1$

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 & \cdots & 0 \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \cdots & 0 \\ 0 & \lambda_{32} & \lambda_{33} & \lambda_{34} & 0 \\ \vdots & 0 & \lambda_{43} & \lambda_{44} & \lambda_{54} \\ 0 & \cdots & 0 & \lambda_{n\,n-1} & \lambda_{nn} \end{bmatrix}$$

(big Lambda)

Where we may have 2 transition rates:

UP $\boxed{\alpha_i \triangleq \lambda_{i(i+1)}}$ (transition rate increases by 1)

DOWN $\boxed{\beta_i \triangleq \lambda_{i(i-1)}}$ (transition rate decreases by 1)

MAIN DIAGONAL

$$\boxed{-\lambda_{ii} = \alpha_i + \beta_i} \quad \text{(with zero sum)}$$

$$\boxed{\lambda_{ii} = -(\alpha_i + \beta_i)}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & \alpha_1 & 0 & \cdots & 0 \\ \beta_2 & \lambda_{22} & \alpha_2 & & 0 \\ 0 & \beta_3 & \lambda_{33} & \alpha_3 & 0 \\ \vdots & 0 & \beta_4 & \lambda_{44} & \alpha_4 \\ 0 & \cdots & 0 & \beta_n & \lambda_{nn} \end{bmatrix}$$

TRANSITION PROBABILITY RATES MATRIX

We know that transition probabilities are:

$$\boxed{P\{x(t+\Delta t)=a_j \mid x(t)=a_i\} = \begin{cases} 1+\lambda_{ii}\,\Delta t & \text{if } i=j \\ \lambda_{ij}\,\Delta t & \text{if } i\neq j \end{cases}}$$

$\Rightarrow$ BIRTH-DEATH EQUATIONS:

$$P\{x(t+\Delta t)=n \mid x(t)=n-1\} = \alpha_{n-1}\cdot\Delta t$$ DOWN

$$P\{x(t+\Delta t)=n \mid x(t)=n\} = 1-(\alpha_n+\beta_n)\,\Delta t$$ STEADY

$$P\{x(t+\Delta t)=n \mid x(t)=n+1\} = \beta_{n+1}\,\Delta t$$ UP

(14) Determine ~~relation~~ relationship between STATE PROBABILITIES and TRANSITION PROBABILITY (EQUILIBRIUM EQUATION) for a BIRTH-DEATH PROCESS:

Given the BIRTH-DEATH EQUATIONS (describe TRANSITION PROBABILITIES from t to Δt)

$$\boxed{\begin{aligned} &(1)\ P\{x(t+\Delta t)=n \mid x(t)=n-1\} = \alpha_{n-1}\Delta t \quad \text{DOWN} \\ &(2)\ P\{x(t+\Delta t)=n \mid x(t)=n\} = 1-(\alpha_n+\beta_n)\Delta t \quad \text{STEADY} \\ &(3)\ P\{x(t+\Delta t)=n \mid x(t)=n+1\} = \beta_{n+1}\Delta t \quad \text{UP} \end{aligned}}$$

Birth-Death equations describe all possible behaviours of the QUEUE.

$\Rightarrow$ To Obtain the EQUILIBRIUM EQUATION:

Suppose: $x(t)$ is STATIONARY

$$\Rightarrow P\{x(t+\Delta t)=n\} = P\{x(t)=n\} \triangleq \boxed{p_n} \quad n=0,1,2,\dots \ \forall t$$

$\Rightarrow$ We can apply the TOTAL PROBABILITY THEOREM to $\boxed{p_n}$ $\forall \Delta t$. For $n>0$.

$$\begin{aligned} p_n = {} & \underset{(1)}{P}\{x(t+\Delta t)=n \mid x(t)=n-1\} \cdot \underbrace{P\{x(t)=n-1\}}_{p_{n-1}} \\ & + \underset{(2)}{P}\{x(t+\Delta t)=n \mid x(t)=n\} \cdot \underbrace{P\{x(t)=n\}}_{p_n} \\ & + \underset{(3)}{P}\{x(t+\Delta t)=n \mid x(t)=n+1\} \cdot \underbrace{P\{x(t)=n+1\}}_{p_{n+1}} \end{aligned}$$

(B and A are written above the conditioning events and conditions.)

$\Rightarrow$ By (1), (2), (3) equations:

$$p_n = \underbrace{\alpha_{n-1}\Delta t}_{(1)} \cdot p_{n-1} + \underbrace{[1-(\alpha_n+\beta_n)\Delta t]}_{(2)} \cdot p_n + \underbrace{[\beta_{n+1}\Delta t]}_{(3)} \cdot p_{n+1}$$

By Bayes Formula in the application of the TOTAL PROBABILITY THEOREM:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\underbrace{P(A|B) \cdot P(B)}_{NEW} = \underbrace{P(B|A) \cdot P(A)}_{OLD}$$

$$p_n =_{①} P\{x(t) = n-1 \mid x(t+\Delta t) = n\} \cdot \underbrace{P\{x(t+\Delta t) = n\}}_{p_n}$$

$$② + P\{x(t) = n \mid x(t+\Delta t) = n\} \cdot \underbrace{P\{x(t+\Delta t) = n\}}_{p_n}$$

$$③ + P\{x(t) = n+1 \mid x(t+\Delta t) = n\} \cdot \underbrace{P\{x(t+\Delta t) = n\}}_{p_n}$$

$\Rightarrow$ This is a BACKWARDS MOVEMENT.

$*$ = backwards movement

We hence have:

$$\boxed{\begin{aligned} P\{x(t) = n-1 \mid x(t+\Delta t) = n\} &= \beta_n^* \Delta t \\ P\{x(t) = n+1 \mid x(t+\Delta t) = n\} &= \alpha_n^* \Delta t \\ P\{x(t) = n \mid x(t+\Delta t) = n\} &= 1 - (\alpha_n^* + \beta_n^*)\Delta t \end{aligned}}$$

And p_n is now, by substituting in ①, ②, ③

$$p_n = \beta_n^* \Delta t \cdot p_n + [1 - (\alpha_n^* + \beta_n^*)] p_n + \alpha_n^* \Delta t \, p_n$$

Since by stationarity we have that:

$$p_n = P\{x(t) = n\} = P\{x(t+\Delta t) = n\}$$

$\Rightarrow$ By BAYES Formula:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\underbrace{P\{x(t) = n \mid x(t+\Delta t) = n\}}_{1-(\alpha_n^*+\beta_n^*)\Delta t} = \underbrace{P\{x(t+\Delta t) = n \mid x(t) = n\}}_{1-(\alpha_n+\beta_n)\Delta t}$$

$$\boxed{\alpha_n^* + \beta_n^* = \alpha_n + \beta_n}$$

Now we want to get α_n^* and β_n^*:

From the Birth-Death Equations:

$$P\{\underbrace{x(t+\Delta t)=n}_{A} \mid \underbrace{x(t)=n+1}_{B}\} = \beta_{n+1}\cdot\Delta t$$

and from the backwards equation:

$$P\{\underbrace{x(t)=n+1}_{B} \mid \underbrace{x(t+\Delta t)=n}_{A}\} = \alpha_n^*\cdot\Delta t$$

$\Rightarrow$ By Bayes Theorem:

$$\underbrace{P(A|B)}_{\beta_{n+1}\cdot\Delta t} = \frac{P(B|A)\cdot P(A)}{P(B)}$$

$$\beta_{n+1}\Delta t = \frac{\overbrace{P\{x(t)=n+1 \mid x(t+\Delta t)=n\}}^{\alpha_n^*\cdot\Delta t}\cdot\overbrace{P\{x(t+\Delta t)=n\}}^{p_n}}{\underbrace{P\{x(t)=n+1\}}_{p_{n+1}}}$$

$$\Rightarrow \beta_{n+1}\cdot\cancel{\Delta t} = \alpha_n^*\cdot\cancel{\Delta t}\cdot\frac{p_n}{p_{n+1}}$$

$$\Rightarrow \boxed{\alpha_n^* = \beta_{n+1}\cdot\frac{p_{n+1}}{p_n}}$$

Repeating the procedure:

$$P\{\underbrace{x(t+\Delta t)=n}_{A} \mid \underbrace{x(t)=n-1}_{B}\} = \overbrace{\alpha_{n-1}\cdot\Delta t}^{P(A|B)}$$

$$P\{\underbrace{x(t)=n-1}_{B} \mid \underbrace{x(t+\Delta t)=n}_{A}\} = \underbrace{\beta_n^*\cdot\Delta t}_{P(B|A)}$$

$$\Rightarrow P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)} \qquad \cancel{\Rightarrow \alpha_{n-1}\Delta t = \beta_n^*\cdot\Delta t\,\frac{p_n}{p_{n-1}}}$$

$$\cancel{\alpha_{n-1}\Delta t = P\{x(t+\Delta t)=n\}}$$

$$\alpha_{n-1}\cancel{\Delta t} = \beta_n^*\cancel{\Delta t}\,\frac{p_n}{p_{n-1}}$$

$$\Rightarrow \boxed{\beta_n^* = \alpha_{n-1} \frac{p_{n-1}}{p_n}}$$

We can now substitute α_n^* and β_n^* in:

$$\boxed{\alpha_n^* + \beta_n^* = \alpha_n + \beta_n}$$

After substitution:

$$\Rightarrow \beta_{n+1} \cdot \frac{p_{n+1}}{p_n} + \alpha_{n-1} \cdot \frac{p_{n-1}}{p_n} = \alpha_n + \beta_n$$

$$\frac{\beta_{n+1} \cdot p_{n+1} + \alpha_{n-1} \cdot p_{n-1}}{p_n} = \alpha_n + \beta_n$$

$$\boxed{\beta_{n+1} \cdot p_{n+1} + \alpha_{n-1} \cdot p_{n-1} = (\alpha_n + \beta_n) \cdot p_n}$$

BIRTH-DEATH EQUILIBRIUM EQUATION for $n > 0$.

If $n = 0$

$$\Rightarrow \boxed{\beta_1 \cdot p_1 - \alpha_0 \cdot p_0 = 0}$$

15 Prove that consecutive transitions in a M/M/1 queue are exponentially spaced.

We want to show that:

$\boxed{\text{Consecutive transitions in a M/M/1 queue are exponentially space}}$

Assume:

- Arrival times t_i are PPP with intensity λ.
- Service time c_i is exponentially distributed and independent of arrival times t_i:

ASSUMES

$$\boxed{f_{c_i}(c) = \mu \cdot e^{-\mu c} \cdot U(c)}$$ "it can keep up"

- $\mu > \lambda$ (SYSTEM is fast enough). SERVICE TIME, ARRIVAL TIME

$\Rightarrow \dfrac{1}{\mu} < \dfrac{1}{\lambda}$ it can keep up with arrivals

MEAN SERVICE TIME; MEAN INTERARRIVAL TIME

$\boxed{\frac{1}{\mu} < 1}$

PROOF: We want to show that the Birth-Death equations are SATISFIED and I am able to find p_n, $P\{x(t)=n\}$

- Assume that in Δt, only one transition can occur (multiple transitions have probability of infinitesimal order).
- By the PPP property, the probability that one units arrives in $(t, t+\Delta t)$ is:

$$\boxed{\lambda \cdot \Delta t}$$

$\Rightarrow P\{1 \text{ unit in } (t, t+\Delta t)\} = \lambda \cdot \Delta t$

So, the UP transition would be:

$$\boxed{P\{N(t+\Delta t) = n+1 \mid N(t)=n\} = \lambda \cdot \Delta t}$$

- We want to show that, for a DOWN transition

$$\boxed{P\{N(t+\Delta t) = n-1 \mid N(t)=n\} = \mu \cdot \Delta t} \quad \forall n \ \forall t$$

↳ We want to find probability that service c_i will be such that:

[one departure in $[t, t+\Delta t]$ when $\tau_{i-1} < t$] ~~$t < \tau_i < t+\Delta t$ / a departure in~~ $[t, t+\Delta t]$

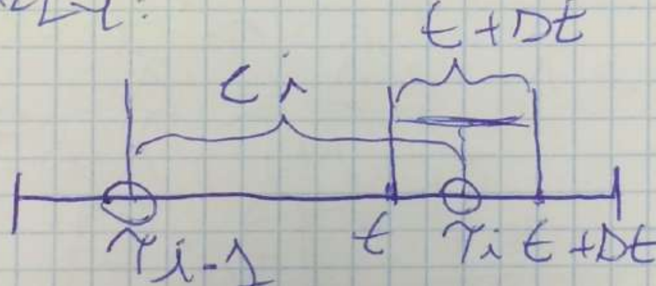
$$\boxed{P\{c_i : t < \tau_i < t+\Delta t \mid \tau_{i-1} < t\}}$$

c_i ↳ service for ith-unit

τ_i → first departure after t

τ_{i-1} → last departure before t.

GRAPHICALLY:



⇓

Use a

FAILURE APPROACH: Probability that ith unit fails in $[t+\Delta t]$ when we know it is still alive in t:

(THESIS) $\boxed{P\{\text{ith unit fails in } [t+\Delta t] \mid \text{is alive at } t\}}$

⇒ When life z_i of ith unit is known to be exponentially distributed, the probability of failure in $[t, t+\Delta t]$ does not depend on t when at t ith unit is still alive

↳ Same "failure" probability at any time when t is alive

PROOF:

PROOF that "Probability of failure of a unit does not depend on t, while unit is still alive.

SYSTEM FAILURE PROOF.

let t_f = time of failure of S

R.V. $X = t_f$, where t_f is the random time of failure of S.

$P\{\text{failure before } t\}$

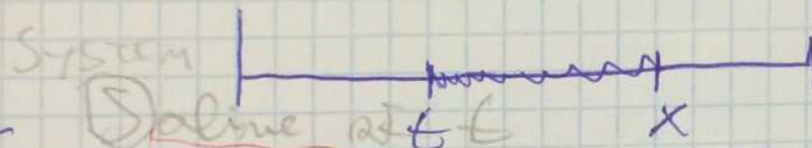
$$= P\{X \le t\} = F_X(t)$$

$P\{S \text{ alive at } t\}$ [surviving after t].

$$= P\{X > t\} = 1 - F_X(t)$$

RECALL CONDITIONAL DISTR:

$$F_X(x|M) = \frac{P\{X \le x, M\}}{P\{M\}}$$



For $M = \{X > t\}$, we know:

CONDITIONAL DISTRIBUTION OF S alive at t:

$$F_X(x|X>t) = \frac{P\{X \le x, X > t\}}{P\{X > t\}} = \begin{cases} \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & (\text{if } x \ge t) \\ 0 & \text{if } x < t \end{cases}$$

$$f_X(x|X>t) = \begin{cases} \dfrac{f_X(x)}{1 - F_X(t)} & \text{if } x \ge t \\ 0 & \text{if } x < t \end{cases}$$

$$\boxed{f_X(x|X>t) = \begin{cases} \dfrac{f_X(x)}{\int_t^{+\infty} f_X(x)\,dx} & \text{if } x \ge t \\ 0 & \text{if } x < t. \end{cases}}$$

$$\boxed{P\{x < X \leq x + dx \mid X > t\} = f_x(x \mid X > t)\, dx} \quad \text{for } x \geq t$$

For $\boxed{x \equiv t}$, consider:

CONDITIONAL FAILURE RATE (CFR)

$\beta(t) \cdot dt$ = Prob. that system fails in the interval $(t, t+dt)$ assuming it did NOT fail prior to t.

$$\beta(t)\, dt \triangleq P\{\text{failure in } (t, t+dt) \mid \text{alive until } t\}$$

$$= P\{t < X \leq t + dt \mid X > t\}$$

$$\boxed{\beta(t)\, dt = f_x(t \mid X > t)\, dt}$$

NB: $\beta(t)$ is a function of t only.

For EXPONENTIAL CASE (our case)

$$\boxed{f_x(x) = \mu \cdot e^{-\mu x} \cdot U(x)}$$

because: $f_x(x \mid X > t) = \dfrac{f_x(x)}{1 - F_x(t)}$

$$\Rightarrow \beta(t) = \frac{\mu \cdot e^{-\mu t}}{1 - (1 - e^{-\mu t})} = \mu = f_x(0)$$

(Probability that life ends in t does not depend on t).

~~REMEMBER~~

~~$\Rightarrow$ We can conclude that:~~

~~$P\{$one unit departs in $(t, t+\Delta t)\}$~~

~~$= P\{\exists i : t < T_i < t + \Delta t \mid T_i < t\}$~~

~~$= P\{$ith unit fails in $(t, t+\Delta t) \mid$ ith unit alive in $t\}$~~

~~$= P\{t < T_i < t + \Delta t \mid T_i > t\}$~~

~~$= P\{N(t + \Delta t) = n - 1 \mid N(t) = n\}$~~ DOWN

CONCLUSION:

$\Rightarrow$ We can conclude that: | ith unit alive

$P\{$ one unit departs in $(t, t+\Delta t) \}$

$= P\{ \tau_i : t < \tau_i < t + \Delta t \mid \tau_{i-1} < t \}$

and

$= P\{$ ith unit ~~departs~~ fails in $(t, t+\Delta t \mid$ ith unit still alive in $t \}$

$= P\{ t' < \tau_i < t' + \Delta t \mid \tau_i \geq t' \}$

$\Rightarrow = \boxed{P\{ N(t+\Delta t) = n-1 \mid N(t) = n \}}$ (Pow[illegible]

We also know that:

$$\boxed{\beta(t) = f_X(t \mid X > t)} = \mu$$

$$= P\{ t < X \leq t + dt \mid X > t \}$$

So, we can conclude that:

$$P\{ t < \tau_i < t + \Delta t \mid \tau_i \geq t \} = \overbrace{f_{\tau_i}(t \mid \tau_i \geq t)}^{\Delta t}$$

$= \beta(t) \cdot \Delta t$

$= \mu \cdot \Delta t$

$\Rightarrow$ For exponential PDF of τ_i, finally:

$$\boxed{P\{ N(t+\Delta t) = n-1 \mid N(t) = n \} = \mu \cdot \Delta t}$$

We have hence shown that: Q.E.D $\forall n$

$$P\{ N(t+\Delta t) = n \mid N(t) = n-1 \} = \lambda \cdot \Delta t \quad \forall t$$

$$P\{ N(t+\Delta t) = n-1 \mid N(t) = n \} = \mu \cdot \Delta t.$$

~~⇒ Since $\boxed{\beta(t) = P\{t < x \le t + dt \mid x > t\}}$~~

~~$= f_x(t \mid x > t)\, dt$ ... $\beta(t)$~~

~~⇒ Then: $P\{t < c_i < t + \Delta t \mid c_i \ge t\} = f_{c_i}(t \mid c_i \ge t)\, \Delta t$~~

~~$= \underbrace{\beta(t)}_{\mu} \Delta t = \boxed{\mu \Delta t}$~~

~~⇒ $\beta(t) = \mu$~~

~~FINALLY, for exponential PDF of c_i:~~

~~$\boxed{P\{N(t+\Delta t) = n-1 \mid N(t) = n\} = \mu \cdot \Delta t}$~~

~~⇒ We have shown what we wanted to prove~~ Q.E.D

16) Calculate the probability of "EMPTY SYSTEM" in a M/M/1 queue.

In a M/M/1 queue, we have:

avg arrival rate   avg system time

$\alpha_n = \lambda$ $\beta_n = \mu$

The EQUILIBRIUM EQUATION ~~[illegible]~~ to be solved becomes:

FROM: $\alpha_{n-1} \cdot p_{n-1} + \beta_{n+1} p_{n+1} = (\beta_n + \alpha_n) \cdot p_n$

TO: $\boxed{\lambda \cdot p_{n-1} + \mu \cdot p_{n+1} = (\lambda + \mu)\, p_n}$

DIFFERENCE EQUATION

From $p_n = p_0 \prod_{k=1}^{n} \frac{\alpha_{k-1}}{\beta_k}$ STATE PROBABILITY (Prob. that queue length is $n-1$)

⇒ $\boxed{p_n = P\{N(t) = n\}} = \boxed{p_0 \cdot \left(\frac{\lambda}{\mu}\right)^n}$ $\left(\frac{\lambda}{\mu}\right) = \rho$

where p_0 = Probability of empty system.

~~... If we want to obtain it~~

~~where $\lambda < \mu$ we know $\sum p_n = 1$~~

~~... system service to keep up with arrivals~~

$$\boxed{0 \leq \rho \triangleq \frac{\lambda}{\mu} < 1.}$$

Since $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$

From NORMALIZATION:

$$\cancel{\sum_n} p_n = p_0 \sum_{n=0}^{+\infty} \left(\frac{\lambda}{\mu}\right)^n \Rightarrow \text{Because: } \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow \sum_n p_n = p_0 \cdot \frac{1}{1-\frac{\lambda}{\mu}} = 1$$

$\frac{\lambda}{\mu} \to \frac{\lambda}{\mu} < 1$

We can then get p_0:

$$\Rightarrow \boxed{p_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho}$$

(11) Calculate the mean number of units in the server, the mean system time and the mean waiting time in a M/M/1 queue.

MEAN NUMBER OF UNITS IN S at a time t:

$$\cancel{E\{N(t)\} = \sum_{k=0}^{+\infty} k\, p_k = (1-\rho) \sum_{k=0}^{} k \cdot \rho^k}$$

$$\cancel{E\{N(t)\} = (1-\rho)\cdot \rho^n} \qquad p_n = (1-\rho)\rho^n$$

$$\boxed{E\{N(t)\} = \sum_{n=0}^{+\infty} n \cdot p_n} = \sum_{n=0}^{+\infty} n \cdot \underbrace{(1-\rho)}_{p_0} \cdot \rho^n$$

$$= (1-\rho) \sum_{n=0}^{+\infty} n \cdot \rho^n = (1-\rho) \cdot \rho \cdot \sum_{n=0}^{+\infty} \frac{d}{d\rho} \rho^n$$

$$= (1-\rho) \cdot \rho \, \frac{d}{d\rho} \sum_{n=0}^{+\infty} \rho^n$$

We know that:

$$\sum_{n=0}^{+\infty} \rho^n = \frac{1}{1-\rho}$$

$$\Rightarrow (1-\rho)\cdot \rho \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) = \cancel{(1-\rho)}\cdot \rho \frac{(-1)\cdot(-1)}{(1-\rho)^{\cancel{2}}} = \frac{\rho}{1-\rho}$$

$$\Rightarrow \boxed{E\{N(t)\} = \frac{\rho}{1-\rho}}$$

$$\left(\frac{1}{1-x}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2} = \frac{-(1)\cdot(-1)}{(1-\rho)^2}$$

MEAN SYSTEM TIME

AVG. length of queue at time t:

N=h!

$$E\{N(t)\} - 1 = \frac{\rho}{1-\rho} - 1 = \frac{\rho - 1 + \rho}{1-\rho} = \frac{2\rho - 1}{1-\rho}$$

$E\{N(t)\}$

$$\rho = \frac{\lambda}{\mu}$$

From Little's Formula:

$a_i = b_i + c_i$

$$E\{N(t)\} = \lambda \cdot \boxed{E\{a_i\}}$$

$$\boxed{E\{a_i\} = \frac{E\{N(t)\}}{\lambda}}$$

$$E\{a_i\} = \frac{1}{\lambda} \underbrace{\frac{\rho}{1-\rho}}_{E\{N(t)\}} = \frac{1}{\mu(1-\rho)}$$

Mean interarrival time.

$$\Rightarrow \boxed{E\{a_i\} = \frac{1}{\mu - \lambda}}$$

MEAN SYSTEM TIME

$$\boxed{E\{a_i\} = \frac{1}{\mu(1-\rho)}} = \frac{1}{\mu\cdot\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu\cdot\left(\frac{\mu-\lambda}{\mu}\right)} = \frac{1}{\mu-\lambda}$$

MEAN WAITING TIME:

system time — $a_i = b_i + c_i$ — service time

b_i: waiting time → We want this!

$$\boxed{E\{b_i\} = E\{a_i\} - E\{c_i\}}$$

We found that: And we know:

MEAN SYSTEM TIME $E\{a_i\} = \frac{1}{\mu \cdot (1-\rho)}$ $E\{c_i\} = \frac{1}{\mu}$ MEAN SERVICE TIME.

$$\Rightarrow E\{b_i\} = \underbrace{\frac{1}{\mu \cdot (1-\rho)}}_{E\{a_i\}} - \underbrace{\frac{1}{\mu}}_{E\{c_i\}} = \frac{1-(1-\rho)}{\mu \cdot (1-\rho)}$$

$$= \underbrace{\frac{1}{\mu}}_{E\{c_i\}} \cdot \underbrace{\frac{\rho}{(1-\rho)}}_{E\{N(t)\}} = E\{c_i\} \cdot E\{N(t)\} = \frac{\rho}{\mu \cdot (1-\rho)}$$

$$\Rightarrow \boxed{E\{b_i\} = E\{c_i\} \cdot E\{N(t)\}}$$

MEAN SERVICE TIME $= \frac{1}{\mu} \cdot \frac{\rho}{1-\rho}$

$$E\{b_i\} = \frac{\rho}{\mu \cdot (1-\rho)}$$

BEGINNING OF 18:

$$\boxed{q_i \triangleq N(\tau_i^-) = n_{a_i}}$$

~~$N(t) = n_{a_i}$~~ # units in S after iter-unit is out

1a) Determine the ONE-STEP TRANSITION MATRIX for the embedded Markoff Chain in a M/G/1 queue.

Denote by n_t the # arrivals in $[0,t]$:

$$\boxed{n_t = n(t) = n_t(0,t)}$$

Denote by $\boxed{N(t)}$ the # units in S at time t

⇒ Assume at $t=0$, no units

$$\boxed{n_0 = N(0) = 0}$$

Service times of $\boxed{\text{units}}$

Units: $1, 2, 3, \dots, n, \dots$

⇓

Service times: $c_1, c_2, c_3, \dots, c_n, \dots$ are R.Vs that are i.i.d. with: $\boxed{f_c(c)}$ and $\boxed{E\{c_i\} = \eta_c}$ KNOWN.

Now, we know $\boxed{q_i = N(\tau_i^+)}$ = π units in S (imaginary) after i-th unit has left.

THEOREM

R.Vs $\{q_i\}$, with $i \geq 1$ are a Markoff Chain with one-step transition matrix $\Pi[1] = \Pi$.

$$\Pi = \begin{bmatrix} \pi_{00}[1] & \pi_{01}[1] & \dots \\ \pi_{10}[1] & \pi_{11}[1] & \dots \\ \pi_{20}[1] & \pi_{21}[1] & \dots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots \\ 0 & \alpha_{21} & \alpha_{22} & \dots \\ 0 & 0 & \alpha_{32} & \dots \\ 0 & 0 & 0 & \end{bmatrix}$$

FIRST-ORDER TRANSITIONS

Where: $\alpha_{j,j+k} \triangleq P\{q_i = j+k \mid q_{i-1} = j\}$

$$\alpha_{j,j+k} = \begin{cases} \alpha_{j,j+k} = \int_0^{+\infty} e^{-\lambda u} \cdot \frac{(\lambda u)^{k+1}}{(k+1)!} f_C(u)\,du \\ \alpha_{0,k} = \int_0^{+\infty} e^{-\lambda u} \cdot \frac{(\lambda u)^{k}}{k!} f_C(u)\,du \end{cases}$$

TRANSITION PROBABILITY

PROOF of TRANSITIONS:

$P\{q_{i+1} = j \mid q_i, q_{i-1}, \dots, 0\} = P\{q_{i+1} = j \mid q_i\}$

Because $\{q_i\}$ is a SUBSEQUENCE of Markoff embedded Chain.

Π JUSTIFICATION:

Certain elements of Π are 0 ($\pi_{2,0}[1]$) like because we cannot have a decrease of 2 among consecutive departures, but only a decrease of 1!

PROOF of $\boxed{\alpha_{j,j+k}}$:

Assuming a HOMOGENEOUS CHAIN, we know:

$\boxed{q_i = N(\tau_i^+)}$ $\boxed{q_{i+1} = N(\tau_{i+1}^+)}$

$$\Rightarrow \boxed{\begin{aligned} \alpha_{j,j+k} &= P\{q_{i+1} = j+k \mid q_i = j\} \\ \alpha_{0,k} &= P\{q_{i+1} = k \mid q_i = 0\} \end{aligned}} \quad \forall j \neq 0$$

⇒ For CONSECUTIVE ~~[illegible]~~ TRANSITIONS

(time before next unit arrives)

$$\begin{cases} c_{i+1} = \tau^+_{i+1} - \tau_i & \text{if } q_i \geq 1 \\ c_{i+1} = \tau^+_{i+1} - t_{i+1} & \text{if } q_i = 0 \end{cases}$$

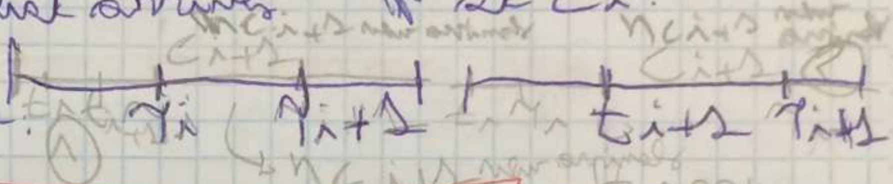
(time unit stays in S)

$n(\tau^+_{i+1}) = n_{c_{i+1}}$

$$\Rightarrow \boxed{\begin{cases} q_{i+1} = q_i - 1 + n_{c_{i+1}} & \text{for } q_i \geq 1 \\ q_{i+1} = q_i + n_{c_{i+1}} & \text{for } q_i = 0 \end{cases}}$$

"What you get at τ^+_{i+1} = what you get at τ_i^+ + what arrives at c_i"

So, for $j \geq 1$:



$$\boxed{\alpha_{j,j+k} = P\{n_{c_{i+1}} = k+1\}}$$

⇒ # ARRIVALS does not depend on i

BY TOT PROB. THEOREM

$$\Rightarrow \alpha_{j,j+k} = \int_0^{+\infty} P\{n_{c_{i+1}} = k+1 \mid c_{i+1} = c\} \cdot f_c(c)\,dc$$

⇒ Poisson with param $k+1$

$$\boxed{\alpha_{j,j+k} = \int_0^{+\infty} e^{-\lambda c} \frac{(\lambda c)^{k+1}}{(k+1)!} \cdot f_c(c)\,dc}$$

For $j = 0$ ~~[illegible]~~:

$$\alpha_{j,j+k} = \int_0^{+\infty} e^{-\lambda c} \cdot \frac{(\lambda c)^k}{k!} \cdot f_c(c)\,dc$$

Q.E.D

(One-step transition matrix, Embedded Markoff Chain, one-step transition probability)

~~RELATIONSHIP between q_i and q_{i+1}~~.

INITIAL SITUATION SKETCH

(1) if $N(\tau_{i-1}^+) \geq 1$

$$q_{i-1} \geq 1$$

$= N(\tau_{i-1}^+) + n_{ci}$

(item-unit just exited)

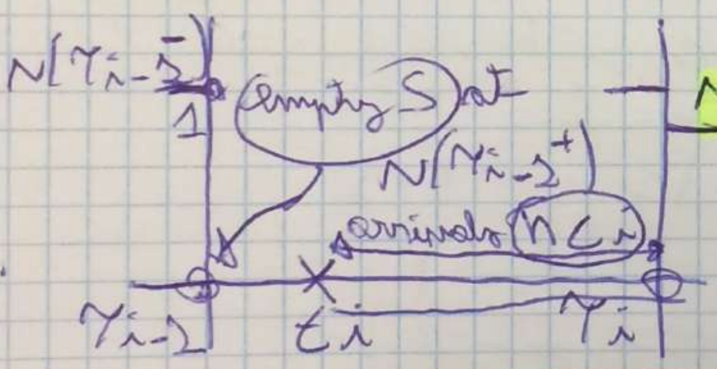
$$\Rightarrow \boxed{N(\tau_i^+) = N(\tau_{i-1}^+) + n_{ci} - 1}$$

Since the statistics of n_c do not depend on i -

$$\boxed{q_i = q_{i-1} + n_c - 1} \text{ for } q_i \geq 1$$

(2) if $N(\tau_{i-1}^+) = 0$

$$q_{i-1} = 0$$



$N(\tau_i^+) = n_c$

$$\boxed{N(\tau_i^+) = n_{ci}}$$

or:

$$\boxed{q_i = n_c} \text{ for } q_{i-1} = 0$$

ERLANG PDF: $\dfrac{c^n \cdot t^{n-1} \cdot e^{-ct}}{(n-1)!} \cdot U(t)$

(parameter c)

(18) Obtain the Probability Density Function (PDF) of system time in a M/M/1 queue.

We know that the MEAN WAITING TIME is:

$$\boxed{E\{b_i\} = E\left\{\sum_{i=1}^{N(t)} c_i\right\}}$$ from $\rightarrow \lambda$, μ

KNOWN

$$\boxed{E\{b_i\} = E\{c_i\} \cdot E\{N(t)\}} \rightarrow \frac{\rho}{1-\rho}$$

Now consider the PDF of system time (a_i).

$\Rightarrow$ After a new arrival, there are $n+1$ units in S.

$\Rightarrow$ a_i = sum of $n+1$ RVs.

$$a_i = \begin{cases} U + c_{i-(n-1)} + c_{i-(n-2)} + \dots + c_{i-1} + c_i & \text{for } n \geq 1 \\ c_i \text{ if } n=0 \end{cases}$$

$\hookrightarrow$ residual (service time) of customer served at arrival instant.

$U \Rightarrow$ exponential PDF with param μ (SYSTEM FAILURE)

$c_i \Rightarrow$ independent and exponential PDF

$\Rightarrow a_i$ = sum of $n+1$ independent exponential R.V.

$\Rightarrow P\{\#\text{ units in S at iter-arrival} = n\}$

= STATE EQUILIBRIUM PROBABILITY.

$P\{N(t)=n\} \Rightarrow P_n = (1-\rho)\cdot\rho^n$

$$\Rightarrow \boxed{f_{a_i}(t) = \sum_{n=0}^{+\infty} (1-\rho)\rho^n \cdot \underbrace{[f_c(t) \otimes f_c(t) \otimes \dots \otimes f_c(t)]}_{n+1 \text{ times}}} \quad t \geq 0$$

where $f_c(t) = \mu \cdot e^{-\mu t}$, $t \geq 0$ (exponential PDF)

$\Rightarrow n+1$ CONVOLUTION of exponential PDF $\Rightarrow$ $n+1$ Erlang PDF with param μ

⇒ Erlang - PDF with parameter μ:

$$\frac{\mu \cdot e^{-\mu t} \cdot (\mu t)^n}{n!}$$

(we substituted n with n+1 in Erlang formula)

$$\Rightarrow f_{a_i}(t) = \sum_{n=0}^{+\infty} (1-\rho) \cdot \rho^n \cdot \mu \cdot e^{-\mu t} \cdot \frac{(\mu t)^n}{n!}$$

$$\Rightarrow f_{a_i}(t) = (1-\rho) \cdot \mu \cdot e^{-\mu t} \cdot \sum_{n=0}^{+\infty} \frac{(\rho \mu t)^n}{n!}$$

$$\rho = \frac{\lambda}{\mu}$$

$$f_{a_i}(t) = \left(1 - \frac{\lambda}{\mu}\right) \cdot \mu \cdot e^{-\mu t} \cdot \sum_{n=0}^{+\infty} \frac{\left(\frac{\lambda}{\mu} \cdot \mu t\right)^n}{n!}$$

$$= \left(\frac{\mu - \lambda}{\cancel{\mu}}\right) \cancel{\mu} \cdot e^{-\mu t} \cdot \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!}$$

$$= (\mu - \lambda) \cdot e^{-\mu t} \cdot \sum_{n=0}^{+\infty} \frac{(\lambda t)^n}{n!} \rightarrow e^{\lambda t}$$

$$= (\mu - \lambda) \cdot e^{-\mu t} \cdot e^{\lambda t}$$

$$\boxed{f_{a_i}(t) = (\mu - \lambda) \cdot e^{-(\mu - \lambda) t}} \quad , t > 0$$

⇒ (a_i is exponential R.V. with parameter $(\mu - \lambda)$)

(19) PROBABILITY DENSITY FUNCTION OF WAITING TIME:

$\boxed{a_i = b_i + c_i}$ (c_i independent of b_i)

⇒ Take CHARACTERISTIC FUNCTION for all terms:

$$\boxed{\Phi_{a_i}(\omega) = \Phi_{b_i}(\omega) \cdot \Phi_{c_i}(\omega)}$$

We know that:

- a_i exponential with parameter $\mu - \lambda$.

$$\Rightarrow \boxed{\phi_{a_i}(\omega) = \frac{\mu \cdot (1-\rho)}{\mu \cdot (1-\rho) - j\omega}}$$

- c_i exponential with parameter μ:

$$\boxed{\phi_{c_i}(\omega) = \frac{\mu}{\mu - j\omega}}$$

$$\cancel{\phi_{b_i}(\omega) = \phi_{a_i}(\omega) \cdot \phi_{c_i}(\omega)}$$

$$\cancel{\Rightarrow \phi_{b_i}(\omega) = \frac{\mu \cdot (1-\rho)}{\mu \cdot (1-\rho) - j\omega} \cdot \frac{\mu}{\mu - j\omega}}$$

$$a_i = b_i + c_i$$

$$\Rightarrow \boxed{\phi_{a_i}(\omega) = \phi_{b_i}(\omega) \cdot \phi_{c_i}(\omega)}$$

$$\Rightarrow \boxed{\phi_{b_i}(\omega) = \frac{\phi_{a_i}(\omega)}{\phi_{c_i}(\omega)}}$$

$$\Rightarrow \phi_{b_i}(\omega) = \frac{\cancel{\mu} \cdot (1-\rho)}{\mu \cdot (1-\rho) - j\omega} \cdot \frac{(\mu - j\omega)}{\mu}$$

$$= \frac{\mu \cdot (1-\rho)}{\mu} - \frac{\rho (1-\rho) \cdot \mu}{j\omega - \mu \cdot (1-\rho)}$$

because:

$$\frac{\mu \cdot (1-\rho)}{\mu \cdot (1-\rho) - j\omega} \cdot \frac{(\mu - j\omega)}{\mu}$$

$$\frac{\mu(1-\rho)}{\mu}\cdot\frac{j\omega-\mu}{j\omega-\mu(1-\rho)}$$

$$=\frac{\mu(1-\rho)}{\mu}\cdot\frac{j\omega-\mu(1-\rho)+\mu(1-\rho)-\mu}{j\omega-\mu(1-\rho)}$$

$$=\frac{\mu(1-\rho)}{\mu}\cdot\left[\cancel{\frac{j\omega-\mu(1-\rho)}{j\omega-\mu(1-\rho)}}+\frac{\mu(1-\rho)-\mu}{j\omega-\mu(1-\rho)}\right]$$

$$=\frac{\mu(1-\rho)}{\mu}\cdot\left[1+\frac{\mu(1-\rho)-\mu}{j\omega-\mu(1-\rho)}\right]$$

$$=\frac{\cancel{\mu}(1-\rho)}{\cancel{\mu}}+\frac{[\mu(1-\rho)]\,[\mu(1-\rho)-\mu]}{\mu\,[j\omega-\mu(1-\rho)]}$$

$$=(1-\rho)+\frac{[\cancel{\mu}(1-\rho)]\cdot[\cancel{\mu}-\mu\rho-\cancel{\mu}]}{\cancel{\mu}\,[j\omega-\mu(1-\rho)]}$$

$$=(1-\rho)+\frac{(1-\rho)(-\mu\rho)}{j\omega-\mu(1-\rho)}$$

$$=(1-\rho)+\rho\,\frac{(1-\rho)\cdot\mu}{j\omega-\mu(1-\rho)}$$

$$\Phi_{b_i}(\omega)=\underbrace{(1-\rho)\cdot 1}_{t=0}+\underbrace{\frac{\rho(1-\rho)\cdot\mu}{\mu(1-\rho)-j\omega}}_{t\geq 0}$$

$\Rightarrow$ b_i is of MIXED TYPE

PROBABILITY DENSITY FUNCTION OF WAITING TIME

$t=0$ with $p=1-\rho$

$$\Rightarrow f_{b_i}(t)=\begin{cases}(1-\rho)\,\delta(t) & \text{if } t=0\\ \rho\cdot\mu(1-\rho)\cdot e^{-\mu(1-\rho)t} & \text{if } t>0\end{cases}$$

EXP. DISTRIB. WITH PARAMETER $\mu(1-\rho)$

$f_{bi}(t) \Rightarrow$ Using the reverse transform of $\Phi_{bi}(\omega)$, we get:

$$\cancel{f_{bi}(t) = (1-\rho)\cdot\delta(t) + \frac{\rho\cdot\mu\cdot(1-\rho)\cdot\mu}{\mu(1-\rho)-j\omega}}$$

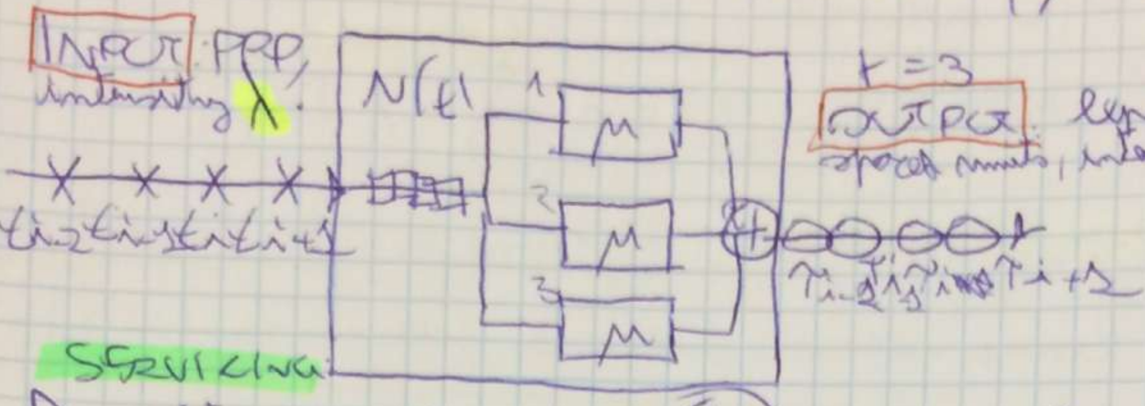
$1 \longrightarrow \delta(t)$

$$f_{bi}(t) = \underbrace{(1-\rho)\cdot\delta(t)}_{t=0} + \underbrace{\rho\cdot\mu(1-\rho)\cdot e^{-\mu(1-\rho)t}}_{t\geq 1}$$

$$\Rightarrow \boxed{f_{bi}(t) = \begin{cases} (1-\rho)\cdot\delta(t) & \text{if } t=0 \\ \rho\cdot\mu\cdot(1-\rho)\cdot e^{-\mu(1-\rho)t} & \text{if } t\geq 1 \end{cases}}$$

(20) Describe an M/M/r queue.

An M/M/r queue is a queue containing r identical servers operating in parallel with i.i.d exponential service time (μ parameter)



SERVICING

A unit arriving in t_i is served if there is at least one free server (or a busy server gets freed)

m = # busy servers. r = # servers.

if $\boxed{m < r}$ ⇒ immediate service

if $\boxed{m = r}$ ⇒ queue is formed

DEPARTING:

Each server releases units exponentially spaced in time with intensity μ.

⇒ Departure process is generated by merging independent PPPs, each with intensity μ.

if $\boxed{m < r}$ ⇒ intensity $= m \cdot \mu$

if $\boxed{m = r}$ ⇒ intensity $= r \cdot \mu$

Time between completions has parameter $m\mu$, $m \leq r$.

TRANSITIONS:

Studying the transition probabilities for $N(t)$ in S ⇒ Birth-death equations:

① $P\{N(t+\Delta t) = n \mid N(t) = n-1\} = ?$

② $P\{N(t+\Delta t) = n \mid N(t) = n+1\} = ?$

① In a small interval Δt (assuming $2 \geq$ transitions have negligible probability) because PPP arrivals:

UP TRANSITION

$$P\{N(t+\Delta t) = n \mid N(t) = n-1\} = \lambda \cdot \Delta t$$

② DOWN TRANSITION: (because of PPP departures)

$$P\{\text{one busy server becomes free in } (t, t+\Delta t)\} = \mu \cdot \Delta t$$

↳ reduces value of $N(t)$ by 1.

$\Rightarrow P\{$NO busy server gets free in $(t, t+\Delta t)\}$

$= (1-\mu \cdot \Delta t)^m$

$\Rightarrow P\{$at least one server gets free in $(t, t+\Delta t)\}$

$= 1-(1-\mu\Delta t)^m$

(but consider max. one server gets free)

because $\Rightarrow P\{$multiple server get free in $(t, t+\Delta t)\}$ is negligible.

DOWN TRANSITION

$$\Rightarrow \boxed{P\{N(t+\Delta t) = n \mid N(t) = n+1\} = 1-(1-\mu\Delta t)^m}$$

$$\Rightarrow \boxed{P\{N(t+\Delta t) = n \mid N(t) = n+1\} \approx m \cdot \mu \cdot \Delta t}, \; m \leq r$$

$\Rightarrow N(t)$ is a birth-death process with constant birth (arrivals) rate and departure based on S (# active servers in $(t, t+\Delta t)$) → m.

$$\alpha_{n-1} = \lambda$$

$$\beta_{n+1} = \begin{cases} m \cdot \mu & \text{if } m < r, \; m = n \\ r \cdot \mu & \text{if } m = r \end{cases}$$

19) Calculate the probability of empty system in a $M/G/1$ queue.

$\boxed{q_i = N(\tau_i^+)}$ ← # points in S after iter-unit has left S

$$\begin{cases} ① \; q_i = q_{i-1} + n_c - 1 & \text{for } q_{i-1} \geq 1 \\ ② \; q_i = n_c & \text{for } q_{i-1} = 0 \end{cases}$$

In compact form:

$\Rightarrow$ $\boxed{q_i = \bar{q}_{i-1} + n_c}$ for a MARKOVIAN EMBEDDED CHAIN

Where: $\bar{q}_i = \begin{cases} q_i - 1 & \text{if } q_i \geq 1 \\ 0 & \text{if } q_i = 0 \end{cases}$

$\boxed{P_k = P\{N(\tau_i^+) = k\} = P\{q_i = k\}}$

→ Probability that S contains k units at τ_i^+, where $q_i = N(\tau_i^+)$ = # units in S when the unit has just left.

$\Rightarrow$ We are interested in empty system $(k=0)$ $\Rightarrow$ Find P_0 for $k=0$

$P_0 = P\{S \text{ empty at } \tau_i^+\}$

$\boxed{P_0 = P\{N(\tau_i^+) = 0\} = P\{q_i = 0\}}$

We know that:

$q_i = \bar{q}_{i-1} + n_c$ = # points in random interval (random sum)

$\Rightarrow$ Take $E(\;)$

$\boxed{E\{q_i\} = E\{\bar{q}_{i-1}\} + E\{n_c\}}$

By stationarity of $\bar{q}_i$:

$$\boxed{E\{\bar{q}_i\} = E\{\bar{q}_{i-1}\}}$$

$\Rightarrow$ So:

$$E\{q_i\} = E\{\bar{q}_i\} + E\{n_c\}$$

We know that: ~~[illegible]~~

~~For p_0~~: By definition of $E\{\bar{q}_i\}$

$$\boxed{E\{\bar{q}_{i-1}\}} = \boxed{E\{\bar{q}_i\}} = 0\cdot\cancel{P\{q_i=0\}} + \boxed{\sum_{k=1}^{+\infty}(k-1)\cdot p_k}$$

By definition of $E\{q_i\}$:

$$\boxed{E\{q_i\}} \triangleq \boxed{\eta_q} \triangleq \sum_{k=0}^{+\infty} k\cdot p_k = \boxed{\sum_{k=1}^{+\infty} k\cdot p_k}$$

~~We know~~:

$\Rightarrow$ Since $\bar{q}_i = \begin{cases} q_i - 1 & q_i \geq 1 \\ 0 & \end{cases}$

Then:

$\Rightarrow$ $\cancel{E\{\bar{q}_{i-1}\} = \sum_{k=1}^{+\infty} p_k = 1 - p_0}$

$\Rightarrow$ $\boxed{E\{\bar{q}_{i-1}\} = \eta_q - (1 - p_0)}$

$\Rightarrow$ ~~[illegible]~~ We have:

$$E\{q_i\} = E\{\bar{q}_{i-1}\} + E\{n_c\} \qquad \lambda\eta_c = \rho$$

$$\Rightarrow \eta_q = \eta_q - (1 - p_0) + \lambda\,\eta_c$$

$$\Rightarrow 0 = -1 + p_0 + \lambda\cdot\eta_c \Rightarrow \boxed{p_0 = 1 - \underbrace{\lambda\eta_c}_{\rho}}$$

$$* \; E\{\bar{q}_i - 1\} = E\{q_i\} - E\{n_c\} = \eta_q - (\lambda \eta_c) = \eta_q - \rho = \eta_q - (1-p_0)$$

Then, we know from:

$$E\{\bar{q}_i - 1\} = \sum_{k=1}^{+\infty} (k-1)\, p_k$$

and from

$$\bar{q}_i = \begin{cases} q_i - 1 & q_i \geq 1 \\ 0 & \end{cases}$$

$$\Rightarrow \boxed{E\{\bar{q}_i - 1\} = \overset{E\{q_i\}}{\eta_q} - \underbrace{(1-p_0)}_{\sum_{k=1}^{+\infty} p_k}} \quad \rho \; *$$

$$\boxed{= E\{\bar{q}_i\}}$$

* From:

$$\boxed{\underset{\eta_q}{E\{q_i\}} = \underset{\eta_q - (1-p_0)}{E\{\bar{q}_i - 1\}} + \underset{\lambda \cdot \eta_c}{E\{n_c\}}}$$

$$\Rightarrow \cancel{\eta_q} = \cancel{\eta_q} - (1-p_0) + \lambda \cdot n_c$$

$$0 = -1 + p_0 + \lambda n_c$$

$$\Rightarrow p_0 = 1 - \lambda \eta_c \quad \rho$$

$$\boxed{p_0 = 1 - \rho}$$

where

$$\rho = \lambda \cdot \eta_c$$

and $\lambda \cdot \eta_c < 1$

$$\Rightarrow \eta_c < \frac{1}{\lambda}$$

20) Demonstrate the Pollaczek - Kinchin formula for the M/G/1 queue ⇒

We know that: ~~[illegible]~~

$$\boxed{\bar{q}_i = \begin{cases} q_i - 1 & \text{if } q_i \geq 1 \\ 0 & \text{if } q_i = 0 \end{cases}}$$

(at least 1 arrival before τ_i and after t_i)

(no arrivals after t_i and before τ_i).

$$\boxed{q_i = \bar{q}_{i-1} + n_{ci}}$$

Where:

- q_i = new units in (t_i, τ_i) of duration a_i — system time
- $\bar{q}_{i-1}$ = new units in b_i waiting time (queuing)
- n_{ci} = new units in c_i service time.

1 Now take the MEAN SQUARE VALUE of q_i

From → $\boxed{q_i = \bar{q}_{i-1} + n_{ci}}$

and $\boxed{\bar{q}_i = \begin{cases} q_i - 1 & \text{if } q_i \geq 1 \\ 0 & \text{if } q_i = 0 \end{cases}}$

We want to show

$$\Rightarrow \boxed{E\{q_i^2\} = E\{\bar{q}_i^2\} + E\{n_{ci}^2\} + 2E\{\bar{q}_i\} \cdot E\{n_{ci}\}}$$

PROOF: $E\{(q_i)^2\} = E\{(\bar{q}_{i-1} + n_{ci})^2\}$

~~$E\{q_i\} \quad E\{q_i^2\}$~~

From $q_i = \begin{cases} q_{i-1} - 1 + n_{ci} & \text{if } q_{i-1} \geq 1 \\ n_{ci} & \text{if } q_{i-1} = 0 \end{cases}$

$$\Rightarrow q_i = \bar{q}_{i-1} + n_{ci}$$

BEGINNING OF 20:

We are now interested in finding:

$\eta_q = E\{N(\tau_i^+)\}$ (length of queue at τ_i^+)

$\Rightarrow$ Derive $\boxed{\text{POLLACZEK-KINCHIN FORMULA}}$

By independence and stationarity, then:

$$E\{q_i^2\} = \underbrace{E\{\bar{q}_i^2\}}_{=E\{\bar{q}_{i-1}^2\}} + E\{n_{c_i}^2\} + 2\underbrace{E\{\bar{q}_i\}}_{=E\{\bar{q}_{i-1}\}} E\{n_{c_i}\}$$

(2) Now take the ~~...~~ MEAN SQUARE VALUE of $\bar{q}_i$:

From $\bar{q}_i = \begin{cases} q_i - 1 & \text{if } q_i \geq 1 \\ 0 & \text{if } q_i = 0 \end{cases}$

$\Rightarrow$ We want to show that:

$$\boxed{E\{\bar{q}_i^2\} = E\{q_i^2\} - 2\eta_q + \rho}$$

$E\{\bar{q}_{i-1}\} = E\{\bar{q}_i\} = \sum_{k=0}^{+\infty} (k-1) \cdot p_k$

PROOF:

$$\Rightarrow \boxed{E\{\bar{q}_i^2\} = \sum_{k=0}^{+\infty} (k-1)^2 \cdot p_k} \qquad p_k = P\{q_i = k\}$$

$$= \sum_{k=1}^{+\infty} p_k \cdot (k-1)^2 + p_0 = \sum_{k=1}^{+\infty} P\{q_i = k\} + \cancel{0^2 \cdot P\{q_i = 0\}}$$

$$= \sum_{k=1}^{+\infty} p_k \cdot (k^2 - 2k + 1) =$$

$$= \underbrace{\sum_{k=0}^{+\infty} k^2 \cdot p_k}_{=E\{q_i^2\}} - 2\underbrace{\sum_{k=0}^{+\infty} k \cdot p_k}_{=\eta_q} + \underbrace{\sum_{k=1}^{+\infty} p_k}_{= 1 - p_0}$$

$$\Rightarrow E\{\bar{q}_i^2\} = E\{q_i^2\} - 2\eta_q + 1 - p_0$$

⇒ We know:

$$E\{q_i^2\} = E\{\bar{q}_i^2\} + E\{n_{c_i}^2\} + 2E\{\bar{q}_i\}\cdot E\{n_{c_i}\}$$

and we have found that: and $\boxed{E\{\bar{q}_i\} = \eta_q - (1-p_0)}$

$$E\{\bar{q}_i^2\} = E\{q_i^2\} - 2\eta_q + 1 - p_0$$

⇒ ~~If we equate~~ $\cancel{\boxed{E\{q_i^2\} = E\{\bar{q}_i^2\}}}$

If we substitute $E\{\bar{q}_i^2\}$ in the first equation:

$$\cancel{E\{q_i^2\}} = \cancel{E\{q_i^2\}} - 2\eta_q + 1 - p_0 + E\{n_{c_i}^2\} + 2E\{\bar{q}_i\}\,E\{n_{c_i}\}$$

$$0 = -2\eta_q + 1 - p_0 + E\{n_{c_i}^2\} + 2\cdot\left[\eta_q - (1-p_0)\right]\cdot E\{n_{c_i}\}$$

where $1 - p_0 = \rho$

$$0 = -2\eta_q + \rho + E\{n_{c_i}^2\} + 2\left[\eta_q - \rho\right]E\{n_{c_i}\}$$

We know: $\boxed{E\{n_{c_i}\} = \lambda\cdot\eta_c = \rho}$

$$\cancel{0 = -2\eta_q\cdot(2}\quad E\{n_{c_i}^2\} = \lambda\eta_c + \lambda^2\cdot E\{c^2\}$$

$$\Rightarrow \boxed{E\{n_{c_i}^2\} = \rho + \lambda^2\cdot E\{c^2\}}$$

$$\Rightarrow 0 = -2\eta_q + \rho + \rho + \lambda^2\cdot E\{c^2\} + 2\cdot\left[\eta_q\cdot\rho - \rho^2\right]$$

$$0 = -2\eta_q + 2\rho + \lambda^2\cdot E\{c^2\} + 2\eta_q\cdot\rho - 2\rho^2$$

$$0 = -\eta_q\cdot\left[2 - 2\rho\right] + 2\rho + \lambda^2\cdot E\{c^2\} - 2\rho^2$$

$$\eta_q = \frac{2\rho\cdot(1-\rho) + \lambda^2\cdot E\{c^2\}}{2\cdot(1-\rho)}$$ ⇒ Solution 2.

$$\Rightarrow \eta_q = \frac{\cancel{2}\rho\,\cancel{(1-\rho)}}{\cancel{2}\cdot\cancel{(1-\rho)}} + \frac{\lambda^2 \cdot E\{C^2\}}{2\cdot(1-\rho)}$$

$$\boxed{\underset{E\{N(\tau_i^+)\}}{\underset{\parallel}{\eta_q}} = \rho + \frac{\lambda^2 \cdot E\{C^2\}}{2\cdot(1-\rho)}}$$

POLLACZEK-KINCHIN FORMULA.

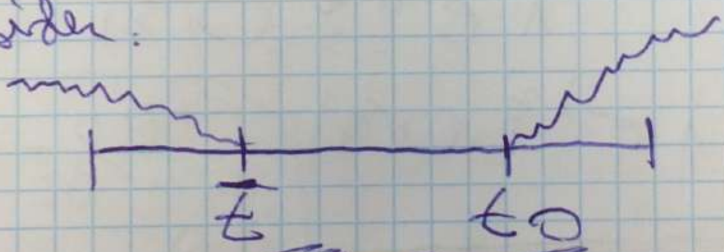
(21) Demonstrate that the idling time [empty server] in a M/G/1 queue is exponentially spaced.

EMPTY PERIOD DEFINITION.

A period of time ~~[illegible]~~ is called EMPTY PERIOD of S when $N(t)=0$.

The EMPTY PERIOD is a R.V. X.

Consider:



$\cancel{\boxed{X = t_0 - \bar{t}}}$

X = duration of empty period

t_0 = time of first arrival

$\bar{t}$ = Fixed origin (beginning of empty period)

time instant at that the empty period begins.

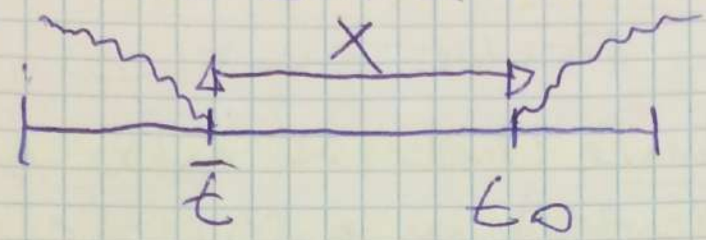
We know: X exp. distributed when origin $\bar{t}$ is fixed.

$$\boxed{f_X(x \mid \bar{t} \text{ given}) = \lambda \cdot e^{-\lambda x} \cdot U(x)} \rightarrow \text{EXPLANATION}$$

It is the distance of first Poisson point from a given point $\bar{t}$.

⇒ If starting point $\bar{t}$ of an empty period is a R.V. (departure time of last unit in S), this r.v. depends on what happened before $\bar{t}$ and is independent of first arrival t_0 after $\bar{t}$.

⇒ NON-OVERLAPPING INTERVALS!



$$\Rightarrow \boxed{X = t_0 - \bar{t}}$$

So, for any $f_{\bar{t}}(t)$, by the TOTAL PROBABILITY THEOREM:

$$f_X(x) = \int_{-\infty}^{+\infty} \underbrace{f_X(x \mid \bar{t} = t)}_{\lambda \cdot e^{-\lambda x} \cdot u(x)} \cdot f_{\bar{t}}(t)\, d\bar{t}$$

$$\Rightarrow f_X(x) = \lambda \cdot e^{-\lambda x} \cdot u(x) \cdot \underbrace{\int_{-\infty}^{+\infty} f_{\bar{t}}(t)\, d\bar{t}}_{=1}$$

$$\Rightarrow \boxed{f_X(x) = \lambda \cdot e^{-\lambda x} \cdot u(x)}$$

⇒ We have shown that the time interval X of first arrival after an empty period has EXPONENTIAL PDF with param λ even for $\bar{t}$ random.

$\Rightarrow \eta_X = \frac{1}{\lambda}$ (MEAN EMPTY TIME).

22) Calculate the "mean - system time" and the "mean waiting time" in a M | G | 1 queue.

MEAN - SYSTEM TIME

We defined:

$$q_i \triangleq n_{a_i}$$

$$E\{q_i\} \triangleq E\{n_{a_i}\} = \lambda \cdot E\{a_i\}$$ = MEAN # points in random interval a_i.

From Pollaczek formula:

$$\boxed{\eta_q = \underset{E\{N(\gamma_i^+)\}}{E\{q_i\}} = \rho + \frac{\lambda^2 \cdot E\{c_i^2\}}{2 \cdot (1-\rho)}}$$

$\Rightarrow E\{a_i\}$, dividing Pollaczek Formula by λ because of Little' Formula:

~~$E\{N(t)\} = \lambda \cdot E\{N(t)\}$~~

$$E\{N(t)\} = \lambda \cdot E\{a_i\}$$

$$\Rightarrow \boxed{E\{a_i\} = \underbrace{E\{N(t)\}}_{\text{POLLACZEK'S FORMULA}} \cdot \frac{1}{\lambda}}$$

$$E\{a_i\} = \frac{\rho}{\lambda} + \frac{\lambda^2 \cdot E\{c_i^2\}}{\lambda \cdot 2 \cdot \underbrace{(1-\rho)}_{p_0}}$$

$$\Rightarrow \boxed{E\{a_i\} = \frac{\rho}{\lambda} + \frac{\lambda \cdot E\{c_i^2\}}{2 \cdot p_0}}$$

where $\boxed{\rho = \lambda \cdot \eta_c}$

MEAN WAITING TIME.

$$a_i = b_i + c_i$$

$$\Rightarrow E\{b_i\} = E\{a_i\} - E\{c_i\}$$

$$\Rightarrow E\{b\} = E\{a\} - E\{c\} \qquad \rho = \lambda \cdot \eta c$$

$$\boxed{\frac{\rho}{\lambda} = \eta c}$$

We know that:

$$\boxed{E\{c\} = \eta c} \qquad \boxed{E\{a\} = \eta c + \frac{\lambda \cdot E\{c^2\}}{2 \cdot (1 - \lambda \eta c)}}$$

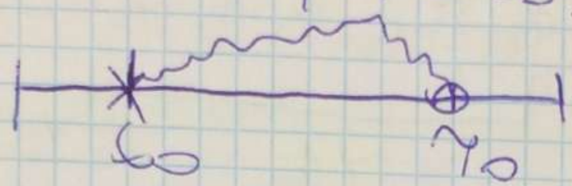
$$\Rightarrow E\{b\} = \cancel{\frac{\lambda \eta c}{\lambda}} + \frac{\lambda \cdot E\{c^2\}}{2 (1 - \lambda \eta c)} - \cancel{\eta c}$$

$$\Rightarrow \boxed{E\{b\} = \frac{\lambda \cdot E\{c^2\}}{2 \cdot (1 - \lambda \cdot \eta c)}}$$

(23) Calculate the "mean busy time" and the "mean number of units served" in a $M/G/1$ queue.

~~BUSY TIME~~: MEAN BUSY TIME (Y)

Consider a BUSY PERIOD Y: ↳ Want to find $E\{Y\}$



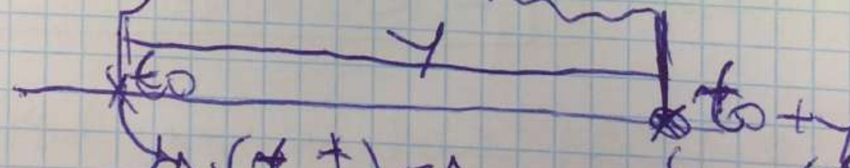
$$c_0 = \tau_0 - t_0 = \text{SERVICE TIME}$$

SERVICE INTERVAL n_0 ~~n_c~~ = ~~# arrivals~~ # arrivals in interval (t_0, τ_0) of duration c_0

Y = BUSY PERIOD = Interval where we have at least 1 unit in S (from first arr)

So, for:

$N(t_0^+) = 1$ (first arrival in empty server) $N(t_0^+) = n_c$ (# arrivals units left in S)



$N(t_0^+) = 1$ (first arrival)

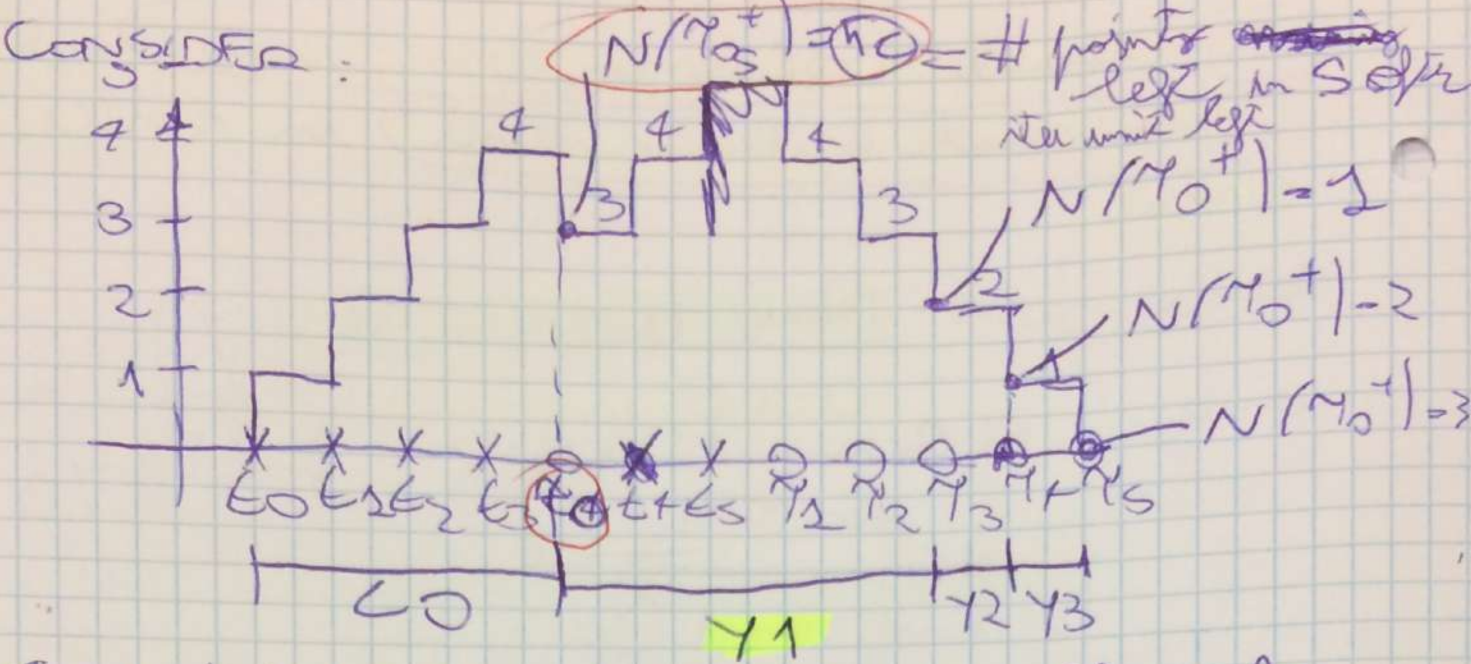
↳ $N(t_0^+) - 1 = 0$ (EMPTY AGAIN / last departure)

BUSY PERIOD Y goes from:

$(t_0, \underbrace{t_0 + Y}_{t}]$ where

$$\boxed{N(t) = N(t_0^+) - 1 = 0}$$

CONSIDER:



Y_1 = Interval from τ_0 to a new τ, where $N(\tau^+) = N(\tau_0) - 1$
($N(t)$ has decreased by 1 for the first time at τ^+)
→ SUB-BUSY PERIOD
→ (emptying the server if only one unit present ⇒ $n_c = 1$ in (t_0, τ_0)).

If n_c units arrive in (t_0, τ_0)
⇒ $N(t) = 0$ if it decreased by 1 n_c times.

NB: Sub-busy periods will be shown to have same behaviour as busy periods.

BUSY PERIOD $\boxed{Y = C_0 + Y_1 + Y_2 + \dots + Y_{n_c}}$

If $N(\tau_0^+) \geq 1$, the time Y_1 does not depend on $N(\tau_0^+)$ (initial value), but only on the sequence of arrivals and departures after τ_0^+.

(1) ⇒ $\boxed{Y_1, Y_2, \dots, Y_n}$ are independent because of events occurring in non-overlapping intervals.

→ from 1 to 0, also starting with C_i

(2) Each sub-busy period behaves as a busy period

departures = # arrivals + 1

③ Sub-busy periods y_i and the busy period y are identically distributed:

$$\boxed{y = c_0 + y_1 + y_2 + \dots + y_{n_c}}$$

TOTAL BUSY PERIOD

⇒ Take $E\{y\}$ and of all components to compute MEAN BUSY TIME. (η_y)

$$\boxed{\eta_y = \underbrace{E\{y\}}_{\eta_y} = \underbrace{E\{c\}}_{\eta_c} + E\left\{\sum_{i=1}^{n_c} y_i\right\}}$$

We know that:

$$\cancel{E\left\{\sum_{i=1}^{n_c} y_i\right\} = E\{n_c\}\cdot\eta_y}$$

$$E\left\{\sum_{i=1}^{+\infty} y_i\right\} = \overbrace{E\{n_c\}}^{\lambda\cdot\eta_c}\cdot\eta_y$$

⇒ $E\left\{\sum_{i=1}^{+\infty} y_i\right\} = \lambda\cdot\eta_c\cdot\eta_y$

and that: $E\{y\} = \eta_y$, $E\{c\} = \eta_c$

⇒ Substituting in initial equation:

$$\eta_y = \overset{E\{c\}}{\eta_c} + E\{n_c\}\cdot\eta_y$$

and we know $E\{n_c\} = \lambda\cdot\eta_c$

⇒ $\eta_y = \eta_c + (\lambda\cdot\eta_c)\,\eta_y$

$0 = -\eta_y + \eta_c + (\lambda\eta_c)\,\eta_y$

⇒ $-\eta_y\cdot(+1 - \lambda\eta_c) + \eta_c$

$$\Rightarrow \boxed{\underset{\underset{E\{Y\}}{\parallel}}{\eta_Y} = \frac{\eta_C}{1-\lambda\eta_C}} \quad \text{For } \rho = \lambda\eta_C \Rightarrow \boxed{\eta_Y = \frac{\eta_C}{1-\rho}}$$

MEAN NUMBER OF UNITS SERVED: (during busy time):

$$\boxed{\text{MEAN \# UNITS SERVED IN A BUSY TIME } Y} = \boxed{\text{MEAN \# ARRIVALS IN RANDOM INTERVAL } Y}$$

Want: $E\{\overset{\parallel}{\eta_Y}\}$

(All units arriving in busy time must be served!)

$$\Rightarrow E\{\eta_Y\} = \lambda \cdot E\{Y\} = \lambda \cdot \eta_Y$$

And we know:

$$E\{Y\} = \eta_Y = \frac{\eta_C}{1-\lambda\eta_C}$$

$$\Rightarrow E\{\eta_Y\} = \frac{\lambda \cdot \eta_C}{1-\rho} = \frac{\rho}{\underbrace{1-\rho}_{= p_0}} = \frac{\rho}{p_0}$$

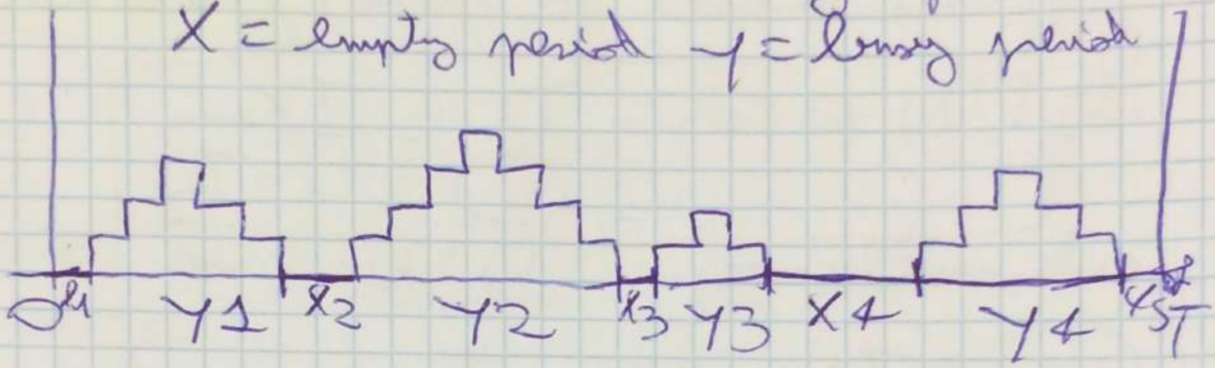
$$\Rightarrow \boxed{E\{\eta_Y\} = \frac{\lambda\eta_C}{1-\lambda\eta_C}}$$

~~• λ_i is exponential with parameter $(\mu-\lambda)$~~

~~[illegible]~~

(28) Obtain the probability of empty system at time t:

Consider an interval $(0, T]$, where there are both empty periods and busy periods:



$\boxed{T_X}$ = Time spent in all empty periods in $(0, T]$

$\boxed{T_Y}$ = Time spent in all busy periods in $(0, T)$

$$\boxed{T = T_X + T_Y}$$

$\eta_X = E\{X_i\}$ = MEAN OF empty periods X_i

$\eta_Y = E\{Y_i\}$ = MEAN OF # busy periods Y_i...

Assuming T is large and $\boxed{n_X = \text{\# empty periods},\ n_Y = \text{\# busy periods}}$

$n = n_X + n_Y$ = # overall intervals in $(0, T]$.

$\Rightarrow$ By MEAN-ERGODICITY:

$\boxed{\eta_X \approx \frac{T_X}{n_X}}$ $\boxed{\eta_Y \approx \frac{T_Y}{n_Y}}$

And assuming

$$\boxed{P\{N(t)=0\} = \frac{T_x}{T}}$$

$P\{\text{empty system}\} = \frac{\text{Time spent in empty period}}{\text{overall time}}$

And also assuming that:

$$\eta_x \approx \eta_y$$

$\Rightarrow$ We want to find $P\{N(t)=0\}$

$$\Rightarrow \boxed{P\{N(t)=0\} = \frac{T_x}{T} = \frac{T_x}{T_x+T_y}}$$

$\hookrightarrow$ We know: $T = T_x + T_y$

and we know:

$$\eta_x \approx \frac{T_x}{n_x} \qquad \eta_y = \frac{T_y}{n_y}$$

$$\Rightarrow \boxed{T_x = \eta_x \cdot n_x} \quad \boxed{T_y = \eta_y \cdot n_y}$$

$$\Rightarrow P\{N(t)=0\} = \frac{\eta_x \cdot n_x}{\eta_x \cdot n_x + \eta_y \cdot n_y}$$

We know that:

$$\boxed{\eta_x \approx \eta_y} \Rightarrow \frac{\cancel{\eta_x} \cdot n_x}{\cancel{\eta_x}(n_x + n_y)} = \frac{n_x}{n_x+n_y}$$

We also know that:

$$\eta_x = \frac{1}{\lambda} \quad \text{and} \quad \eta_y = \frac{n_c}{1-\rho}$$

$\Rightarrow$

We know that:

$$\boxed{n_x = n_y}$$

$$\Rightarrow P\{N(t)=0\} = \frac{\eta_x \cdot n_x}{\eta_x \cdot n_x + \eta_y \cdot n_y} = \frac{\eta_x \cdot \cancel{n_x}}{\cancel{n_x} \cdot (\eta_x + \eta_y)}$$

$\Rightarrow \dfrac{\eta_x}{\eta_x + \eta_y}$ and we know that: $\boxed{\eta_x = \frac{1}{\lambda}}$ and $\boxed{\eta_y = \frac{n_c}{1-\rho}}$

AVG. DURATION OF EMPTY PERIOD

AVG. DURATION OF BUSY PERIOD

$\lambda \cdot n_c$

$$\Rightarrow \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \frac{n_c}{1-\rho}} = \frac{\frac{1}{\cancel{\lambda}} \cdot \cancel{\lambda}}{\lambda \cdot \left(\frac{1}{\lambda} + \frac{n_c}{1-\rho}\right)} = \frac{1}{\left(1 + \frac{\lambda n_c}{1-\rho}\right)}$$

$$= \frac{1}{\left(1 + \frac{\rho}{1-\rho}\right)} = \frac{1}{\frac{1-\cancel{\rho}+\cancel{\rho}}{1-\rho}} = \frac{1}{\frac{1}{1-\rho}} = 1-\rho$$

$$\Rightarrow 1-\rho = P\{N(\tau_i^+)=0\} = p_0.$$

We also want to have that

$$\boxed{E\{N(t)\} = E\{N(\tau_i^+)\}}$$

$\Rightarrow$ Knowing:

$$N(\tau_i^+) = n_{a_i}$$

$$\Rightarrow E\{N(\tau_i^+)\} = E\{n_{a_i}\}$$

And we know: $E\{N(\tau_i^+)\} = E\{n a_i\}$

$$E\{n a_i\} = E\{a_i\} \cdot \lambda \; \cancel{\Rightarrow E\{N(\tau_i^+)\} = E}$$

$$\Rightarrow \boxed{E\{N(\tau_i^+)\} = E\{a_i\} \cdot \lambda}$$

By LITTLE'S FORMULA:

$$E\{N(t)\} = \lambda \cdot E\{a_i\}$$

$$\Rightarrow E\{N(t)\} = E\{N(\tau_i^+)\}$$

Q.E.D