

Test 3 Notes

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1 Change of Variables (Jacobians)

In one-dimensional calculus we often use a change of variable to simplify an integrals.

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r \cos \theta \quad y = r \sin \theta$$

and the change of variables formula can be written as

$$\iint_R f(x, y) dA = \iint_S f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

Where S is the region in the $r\theta$ -plane that corresponds to the region R in the xy -plane

More generally, we consider a change of variables that is given by a **transformation** T from the uv -plane to the xy -plane:

$$x = g(u, v) \quad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \quad y = y(u, v)$$

We usually assume that T is a **\mathbf{C}^1 transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of \mathbb{R}^2 . If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1) . If no two points have the same image, T is called **one-to-one**.

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy -plane to the uv -plane and it may be possible to solve

$$x = x(u, v) \quad y = y(u, v)$$

for u and v in terms of x and y :

$$u = G(x, y) \quad v = H(x, y)$$

1.1 Jacobian

The **Jacobian** of the transformation T is given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

1.2 Change of Variables in a Double Integrals

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv - plane onto a region R in the xy -plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S . Then

$$\iint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$$

1.2.1 Triple Integrals

Lets use the definition of the Jacobian, extend it to three dimensions and find the formula for a triple integral and use it to derive the formula for spherical coordinates.

The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

this gives us the MASSIVE formula lol:

$$\iiint_R f(x, y, z) \, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw$$

now lets use this to find the formula for triple integration in spherical coordinates!!!! (i'm losing my fucking mind)

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

lets compute this absolute unit of a jacobian

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi (-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta) - \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \\ &= (\text{This reduces all the way to}) \rho^2 \sin \phi \quad (\text{lol}) \end{aligned}$$

anyways, putting this back into our equation would give us

$$\iiint_R f(x, y, z) \, dV = \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

lets goo!!!!!!

2 Vector Fields

3 Line Integrals

4 The Fundamental Theorem of Line Integrals

5 Green's Theorem

6 Curl and Divergence

7 Parametric Surfaces

8 Surface Integrals