Test 3 Notes

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1 Change of Variables (Jacobians)

In one-dimensional calculus we often use a change of variable to simplify an integrals.

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta$$
 $y = r\sin\theta$

and the change of variables formula can we written as

$$\iint\limits_{R} f(x,y) dA = \iint\limits_{S} f(r\cos\theta, r\sin\theta) r \ dr \ d\theta$$

Where S is the region in the $r\theta$ -plane that corresponds to the region R in the xy-plane

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$x = g(u, v)$$
 $y = h(u, v)$

or, as we sometimes write,

$$x = x(u, v)$$
 $y = y(u, v)$

We usually assume that T is a \mathbb{C}^1 transformation, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of \mathbb{R}^2 . If $T(u_1, v_1) = (x_1.y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1) . If no two points have the same image, T is called **one-to-one**

If T is a one-to-one transformation, then it has an **inverse transformation** \mathbf{T}^{-1} from the xy-plane to the uv-plane and it may be possible to solve

$$x = x(u, v)$$
 $y = y(u, v)$

for u and v in terms of x and y:

$$u = G(x, y)$$
 $v = H(x, y)$

1.1 Jacobian

The **Jacobian** of the transformation T is given by x = g(u, v) and y = h(u, v) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

1.2 Change of Variables in a Double Integrals

Suppose that T is a C^1 tranformation whose Jacobian is nonzero and that T maps a region S in the uv- plane onto a region R in the xy-plane. Supposed that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_{R} f(x,y) \, dA = \iint\limits_{S} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv$$

1.2.1 Triple Integrals

Lets use the definition of the Jacobian, extend it to three dimensions and find the formula for a triple integral and use it to derive the formula for spherical coordinates.

The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial w} \end{vmatrix}$$

this gives us the MASSIVE formula lol:

$$\iiint\limits_R f(x,y,z) \; \mathrm{d}V = \iiint\limits_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \; dv \; dw$$

now lets use this to find the formula for triple intrgration in spherical coordinates!!!! (i'm losing my fucking mind)

$$x = \rho \sin \phi \cos \theta$$
 $y = \rho \sin \phi \sin \theta$ $z = \rho \cos \phi$

lets compute this absolute unit of a jacobian

$$\begin{split} \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin(\phi) \end{vmatrix} \\ &= \cos\phi(-\rho^2\sin\phi\cos\phi\sin^2\theta - p^2\sin\phi\cos\phi\cos^2\theta) - \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\ &= (\text{This reduces all the way to}) \ p^2\sin\phi \ \ (\text{lol}) \end{split}$$

anyways, putting this back into our equation would give us

$$\iiint\limits_R f(x,y,z) \ dV = \iiint\limits_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \sin \phi) \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

lets goo!!!!!!

2 Vector Fields

In general, a vector field is a function whose domain is a set of points in \mathbb{R}^2 (or \mathbb{R}^3 in three dimensions) is a function \mathbf{F} that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

The best way to picture a vector field is to draw the arrow represending the vector $\mathbf{F}(x,y)$ starting at the point (x,y). Of course it's impossible to do this for all points (x,y), but we can gain a reasonable impression of \mathbf{F} by doing it for a few representative points in D. since $\mathbf{F}(x,y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j} = \langle P(x,y), Q(x,y) \rangle$$

or, for short, $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Let E be a subset of \mathbb{R}^3 . a **vector field on** \mathbb{R}^3 is a function \mathbf{F} that assigns each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

2.1 Gradient Fields

if f is a scalar function of two variables, recall that ∇f is defined by $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$

Therefore, ∇f is really a vector field on \mathbb{R}^2 and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables (it extends but im too lazy to type this out).

The length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph.

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. in this situation f is called a **potential function** for \mathbf{F} .

Not all vector fields are conservative though!!

3 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval [a, b], we integrate over a curve C. Such integrals are called *line integrals* (although curve integrals would honestly be a better term imo).

Lets start with a plane curve C given by the parametric equations

$$x = x(t)$$
 $y = y(t)$ $a \le t \le b$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, and we assume that C is a smooth curve

nerd shit incoming BUT this means that \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq 0$.

so if we divide the parameter interval [a, b] into n subintervals $[t_i - 1, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. we can do the blah blah riemann sum this is like the 8th time we've seen it so basically we get the limit of that and we get the following

if f is defined on a smooth curve C given by $x=x(t), y=y(t), a \le t \le b$, then the **line integral of f along c** is

$$\int_C f(x,y) \ ds = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if the limit exists but also its this

$$\int_{C} f(x,y) \ ds = \int_{a}^{b} f(x(t), y(t)) \ \sqrt{(\frac{dx}{dt})^{2} + (\frac{dy}{dt})^{2}} \ dt$$

when we want to distinguis the original line integral $\int_C f(x,y) ds$ from others, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: x = x(t), y = y(t), dx = x'(t)dt, dy = y'(t)dt.

$$\int_C f(x,y) \ dx = \int_a^b f(x(t), y(t)) \ x'(t) dt$$

$$\int_C f(x,y) \ dy = \int_a^b f(x(t),y(t)) \ y'(t) dt$$

it frequently happens that line integrals with respect to x and y occur together, when this happens it's customary to abbreviate by writing

$$\int_C P(x,y) \ dx + \int_C Q(x,y) \ dy = \int_C P(x,y) \ dx + Q(x,y) \ dy$$

when we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given. In particular, we often need to parameterize a line segment so its useful to remember that a vector representation of a line segment that starts at \mathbf{r}_0 and ends at \mathbf{r}_1 is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

3.1 Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t)$$
 $y = y(t)$ $z = z(t)$ $a < t < b$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. if f is a function of three variables that is continuous on some region containing C, then we define the **line integral of f along C** with respect to arc length in a similar manner to that for plane curves:

$$\int_{C} f(x, y, z) \ ds = \lim_{x \to \infty} \sum_{i=1}^{n} f(x_{i}^{*}, y_{i}^{*}, z_{i}^{*}) \ \Delta s_{i}$$

and we then evaluate it as follows

$$\int_{C} f(x, y, z) \ ds = \int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^{2} + \left(\frac{dy}{dt}\right)^{2} + \left(\frac{dz}{dt}\right)^{2}} \ dt$$

although, observe that all of these formulas can all be written in a more compact vector notation

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| dt$$

for the special case f(x, y, z) = 1, we get

$$\int_C dz = \int_a^b |\mathbf{r}'(t)| \ dt = L$$

where L is the length of the curve C

Line integrals along C with respect to x, y, and z can also be defined. Therefore, as with line integrals in the plane, we evaluate line integrals fo the form

$$\int_C P(x,y,z) \ dx + Q(x,y,z) \ dy + R(x,y,z) \ dz$$

by expressing everything (x, y, z, d, dy, dz) in terms of the parameter t.

3.2 Line Integrals of Vector Fields

We define the **work** W done by the force field ${\bf F}$ as the limit of the Riemann sums, namely

$$W = \int_{C} \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \ dx = \int_{C} \mathbf{F} \cdot \mathbf{T} \ ds$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C.

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, so using the equation from before we can rewrite it as

$$W = \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

this integral is often appreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$, therefore we make the following definition for the line integral of any continuous vector field.

Let **F** be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{C} \mathbf{F} \cdot \mathbf{T} ds$$

Notice, we can formally write $d\mathbf{r} = \mathbf{r}'(t) dt$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field \mathbf{F} on \mathbb{R}^3 is given in component for by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. we use the definition from before to compute its line integral along C:

$$\begin{split} \int_{C} \mathbf{F} \cdot d\mathbf{r} &= \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_{a}^{b} (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(y)\mathbf{k}) \ dt \\ &= \int_{a}^{b} [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] \ dt \end{split}$$

but importantly, notice this is actually the same integral from earlier, therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \ dx + Q \ dy + R \ dz \qquad \text{where } \mathbf{F} = P \ \mathbf{i} + Q \ \mathbf{j} + R \ \mathbf{k}$$

4 The Fundamental Theorem of Line Integrals

5 Green's Theorem

6 Curl and Divergence

7 Parametric Surfaces

8 Surface Integrals