Test 2 Review

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October 17, 2023

Preface

I'm writing this in order for me not to fail this fucking Test. Best of luck to everyone. If I make a mistake please contact me on discord (user: `dnwmn`) and I appreciate any feedback on these. I'm using the James Stewart Calculus 8th edition for my theory just so you guys know

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Theory

1 Partial Derivatives

In general, if f is a function of two variables x and y supposed we take a derivative of x while keeping y constant, this is considered to be a partial derivative of f with respect to x.

The same applies to y, this section is pretty straight forward it's just derivatives and keep the rest of the stuff constant.

Notation:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_1 = D_1 f = D_y f$$

To compute partial derivatives all we have to do is remember that a partial derivative with respect to x is just the normal derivative of f with respect to x while keeping y constant.

1.1 Interpretation

Sure, but what does this all mean?

if $z = f(x,y)$, f_x represents the rate of change of z with respect to x when y is fixed (and vice versa)

1.2 Examples

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

$$f_x = 3x^3 + 2xy^3$$
$$f_y = 3x^2y^2 - 4y$$

1.3 Higher Derivatives

Just do it again, but now we can point out that the order of the derivatives does not matter due to claireauts theorem

Claireauts theorem only applies if functions are continuous

$$f_{xy} = f_{yx}$$

2 Tangent Planes and Linear Approximations

2.1 Tangent Planes

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x -_0) + B(y - y_0) + C(z - z_0) = 0$$

by dividing this by C and letting $a = \frac{-A}{C}$ and $b = \frac{-B}{C}$ we can write it in the following form

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

blah blah algebra basically $z - z_0 = a(x - x_0)$ is the equation of point slope of a line of slope a, and we know the slope of a tangent is f_x so we can say $a = f_x(x_0, y_0)$, this way we can simplify the formula to the following

$$z - z_0 = f_x\big|_{(x_0,y_0)}(x - x_0) + f_y\big|_{(x_0,y_0)}(y - y_0)$$

where $f_x\big|_{(x_0,y_0)}$ is just f_x with (x_0, y_0) plugged in (applies to the rest as well)

2.2 Linear Approximations

In general, we know that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x\big|_{(a,b)}(x - a) + f_y\big|_{(a,b)}(y - b)$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x\big|_{(a,b)}(x - a) + f_y\big|_{(a,b)}(y - b)$$

is called the linearization of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x\big|_{(a,b)}(x - a) + f_y\big|_{(a,b)}(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b)

2.3 Total Differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

way easier to calculate this versus the increment lmfao

2.4 Three or more variables

same conventions as before nothing too crazy

3 Chain Rule

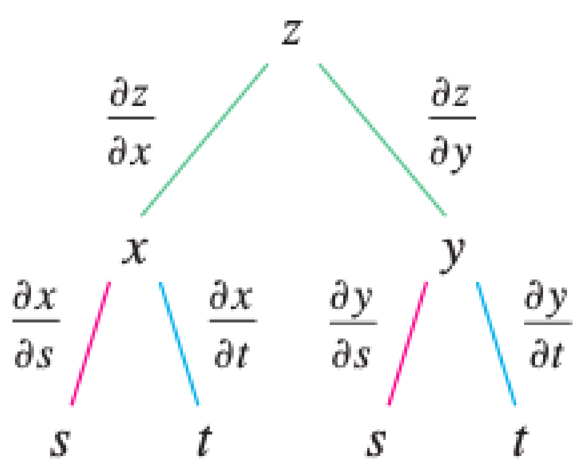
Recall that the chain rule for functions of a single variable gives the rule for differentiating a composite function, if $y = f(x)$ and $x = g(t)$ where f and g are dfferentiable functions, then y is indirectly a function of t and can be written as

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

the easiest way to do this is to draw a dependency tree, and then follow the following rules:

- Draw a dependency tree with each function as a node and then each child of this m-way tree is the corresponding parameters of that function
- for each factor m there will be n terms determined by doing an inorder search for variable t that is where t is the term being differentiated with respect to and n is the depth of the target node
- each appearance of the node t will correspond directly to the amount of factor m in the resulting formula.

Example: if $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$ then we can draw a dependency tree as follows to find $\frac{\partial z}{\partial s}$



as such, following the inorder traversal we get the following formula for $\frac{\partial z}{\partial s}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

so we can see, we performed a search for the term s. m is the number of factors that corresponds to the number of appearances of s, so in this case we have 2, and each factor has n terms, in this case each factor had 2 terms since the depth of the target node s in each case was 2

4 Directional Derivatives and the Gradient Vector

4.1 Directional Derivative

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b\rangle$ and

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

this is the same as saying

$$D_u f(x,y) = \nabla f \cdot \vec{u}$$

where ∇f is

$$\nabla f = \langle f_x, f_y\rangle$$

If the unit vector $\vec{u}$ makes an angle θ then we can write $\vec{u} = \langle cos\theta, sin\theta\rangle$

4.2 Gradient Vector

Formal Definition:

$$\nabla f(x,y) = \langle f_x, f_y\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

In what direction does f change the fastest and what is the maximum rate of change? the answers are provided by the following theorem:

Suppose f is a differentiable fuction of two or three variables. The maximum value of the directional derivative $D_u f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

5 Shapes of Functions !!TODO!!

6 Maximum and Minimum Values

Theorem:

> if f has a local maximum or minium at (a, b) and the first-order partial derivative of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

A point (a, b) is called a **critical point** (or stationary point i guess) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives do not exist. However, not all critical points give a maxima or minima, at a critical point, a function could have a local mximum or a local minimum or neither. (Saddle points!)

We need a way to classify these critical points, and that's why we have the second derivative test.

$$D = f_{xx}(a, b)f_{yy}(a, b) + (f_{xy}(a, b))^2$$

we can now classify a point (a, b) as follows:

- If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum
- If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum
- If $D < 0$ then (a, b) is a Saddle point
- If $D = 0$ then the test is inconclusive

6.1 Absolute Maximum and Minimum Values

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest is the absolute minimum value.

7 Double Integrals over Rectangles

8 Double Integrals over General Regions

9 Double Integrals over Polar Coordinates

10 How to Graph Regions when Taking Integrals !!TODO!!

11 Applications of Double Integrals

12 Surface Area

13 Review Sheet Problems and Solutions

14 Formula Sheet