Test 3 Notes

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Contents

1 Change of Variables (Jacobians)

In one-dimensional calculus we often use a change of variable to simplify an integrals.

A change of variables can also be useful in double integrals. We have already seen one example of this: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

and the change of variables formula can we written as

$$\iint\limits_R f(x,y)\mathrm{d}A = \iint\limits_S f(r\cos\theta, r\sin\theta)r\ dr\ d\theta$$

Where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane

More generally, we consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$x = g(u,v) \qquad y = h(u,v)$$

or, as we sometimes write,

$$x = x(u,v) \qquad y = y(u,v)$$

We usually assume that T is a $\mathbf{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1.y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1). If no two points have the same image, T is called **one-to-one**.

If T is a one-to-one transformation, then it has an **inverse transformation** $\mathbf{T^-1}$ from the xy-plane to the uv-plane and it may be possible to solve

$$x = x(u,v) \qquad y = y(u,v)$$

for u and v in terms of x and y:

$$u = G(x,y) \qquad v = H(x,y)$$

1.1 Jacobian

The **Jacobian** of the transformation T is given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

1.2 Change of Variables in a Double Integrals

Supopse that T is a C^1 tranformation whose Jacobian is nonzero and that T maps a region S in the uv- plane onto a region R in the xy-plane. Supposed that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_R f(x,y)\ \mathrm{d}A = \iint\limits_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

1.2.1 Triple Integrals

Lets use the definition of the Jacobian, extend it to three dimensions and find the formula for a triple integral and use it to derive the formula for spherical coordinates.

The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

this gives us the MASSIVE formula lol:

$$\iiint\limits_R f(x,y,z)\,\mathrm{d}V = \iiint\limits_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

now lets use this to find the formula for triple intrgration in spherical coordinates!!!! (i'm losing my fucking mind)

$$x = \rho \sin\phi \cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

lets compute this absolute unit of a jacobian

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin(\phi) \end{vmatrix} \\ &= \cos\phi(-\rho^2\sin\phi\cos\phi\sin^2\theta - p^2\sin\phi\cos\phi\cos^2\theta) - \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\ &= \text{(This reduces all the way to) } p^2\sin\phi \ \ \text{(lol)}\end{aligned}$$

anyways, putting this back into our equation would give us

$$\iiint\limits_R f(x,y,z)\ \mathrm{d}V = \iiint\limits_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\sin\phi)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

lets goo!!!!!!

2 Vector Fields

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$ in three dimensions) is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y). Of course it's impossible to do this for all points (x, y), but we can gain a reasonable impression of $\mathbf{F}$ by doing it for a few representative points in D. since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = \langle P(x, y), Q(x, y)\rangle$$

or, for short, $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j}$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Let E be a subset of $\mathbb{R}^3$. a **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

2.1 Gradient Fields

if f is a scalar function of two variables, recall that ∇f is defined by $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$

Therefore, ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Likewise, if f is a scalar function of three variables (it extends but im too lazy to type this out).

The length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph.

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. in this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative though!!

3 Line Integrals

4 The Fundamental Theorem of Line Integrals

5 Green's Theorem

6 Curl and Divergence

7 Parametric Surfaces

8 Surface Integrals