

# STA4724: Big Data Analytics Methods

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# 1 Definitions of Matrices and Vectors

## Matrix

- a matrix is an arrangement of numbers in rectangular form
- a  $J \times K$  matrix has  $J$  rows and  $k$  columns
- a Square matrix is of order  $(2, 2)$  as a special case
- Vectors are subcategories of matrices that have either one row or one column

$(1, k)$  is one row, and multiple columns, e.g.  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$(k, 1)$  is one column, and multiple rows, e.g.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- a matrix with one row and one column is the same as a scalar.  $a = 5 \Leftrightarrow a = \begin{bmatrix} 5 \end{bmatrix}$

## 2 Addition, Subtraction, Multiplication

- $A + B = C$
- $A + B \Leftrightarrow B + A$
- $(A + B) + C \Leftrightarrow A + (B + C)$

**Transposition** An order  $(j, j)$  matrix is said to be symmetric if  $A = A^T$

- $(A^T)^T \Leftrightarrow A$
- $(kA)^T \Leftrightarrow kA^T$  where  $k$  is a scalar
- $(A + B)^T \Leftrightarrow A^T + B^T$
- $kA \Rightarrow k \cdot \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \Rightarrow \begin{bmatrix} k \cdot a_1 & k \cdot a_2 & k \cdot a_3 \end{bmatrix}$
- Given matrix A of order  $(m, n)$  and matrix B of order  $(n, r)$   
 $C = A \cdot B$  is of order  $(m, r) = \begin{bmatrix} C_{mr} \end{bmatrix}$  where  $C_{mr} = \sum_{i=1}^n A_{mi} \cdot B_{ir}$

**Example 1.** Given the matrices  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$ , find

$$C = A \cdot B$$

$$\begin{aligned}
C_{11} &= 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 = 58 \\
C_{12} &= 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 = 64 \\
C_{21} &= 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 = 139 \\
C_{22} &= 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 = 154
\end{aligned}$$

Therefore,  $C = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$

### Properties

- $AB \neq BA$
- $A(BC) \Leftrightarrow (AB)C$
- $A(B + C) \Leftrightarrow AB + AC$
- $(AB)^T \Leftrightarrow B^T A^T$
- $A^n \Leftrightarrow A_0 \cdot A_1 \cdot \dots \cdot A_{n-1}$

## 3 Diagonal and Identity Matrices

**Diagonal matrix** A diagonal matrix is a square matrix with zero entries except possible on the main diagonal

**Example 2.**  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is a diagonal matrix. note that they dont need to be 1s, they can be any number, including zero.

In general, a diagonal matrix is given by  $D_{mn} = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$

### Echelon Form

1. row echeleon form (ref)

The first non-zero element in each row is called the leading entry, is always 1

Each leading entry is in a column to the right of the leading entry in the previous row (if any)

Rows with all zero elements are below rows with non-zero elements (if any)

2. reduced row echelon form (rref)

any ref with the leading entry in each row is the only non-zero entry in its column.

### Properties of Diagonal Matrices

- A diagonal matrix  $D$  is invertible if and only if all the diagonal elements are non zero.

**Example 3.** given  $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \dots & 0 \\ 0 & 1/d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/d_n \end{bmatrix}$   
 so  $DD^{-1} = \begin{bmatrix} d_1 \cdot 1/d_1 & 0 & \dots & 0 \\ 0 & d_2 \cdot 1/d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \cdot 1/d_n \end{bmatrix} \rightarrow I$  which is the identity matrix

**Identity Matrix** The identity matrix is a square matrix, consisting of ones along the diagonal and zeros elsewhere. Typically,  $I$  is used to denote the identity matrix.

**Example 4.**

$$I_{nn} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

### Properties of Identity Matrices

- $AI = IA = A$

**Zero Matrix** a zero matrix consists of all zero elements.

## 4 Determinant and Eigenstructure

**Determinant** Determinants are defined only for square matrices and scalars.

**Example 5.** let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $\det(A) = ad - cb$

the determinant of a matrix is denoted by  $|A|$  or  $\det(A)$  and is a number which encodes a lot of information about the matrix.

In general, we need to first define the cofactor  $Q_{r,s}$  of each element of  $A$ ,  $a_{r,s}$ . The cofactor of  $a_{r,s}$  is  $Q_{r,s} = -1^{r+s}|A_{r,s}|$  (where  $|A_{r,s}|$  is the determinant of the matrix obtained by deleting the  $r$ -th row and  $s$ -th column of  $A$ ).

The last step is to define the determinant of the matrix  $A$  as

$$|A| = \sum_{j=1}^n a_{ij}Q_{ij}$$

or

$$|A| = \sum_{i=1}^n a_{ij}Q_{ij}$$

### Properties of Determinants

- $|I| = 1$
- if exchanging two rows of a matrix, we only need to reverse the sign of its determinant
- If we multiply one row of a matrix by a scalar  $k$ , the determinant is also multiplied by  $k$ .
- The determinant behaves like a linear function on the rows of the matrix

**Lemma 6.** *The geometric multiplicity of an eigenvalue is at most its algebraic multiplicity.*

Characteristic equation  $\det(A - \lambda I) = 0$

### The Geometric Multiplicity of Eigenvalues

- It is the dimension of the linear space of its associated eigenvectors.

Let  $A$  be a  $k \times k$  matrix,  $\lambda_k$  be one of the eigenvalues of  $A$  and denote its associated eigenspace by  $E_k$ . Then the dimension of  $E_k$  is called the geometric multiplicity of this eigenvalue  $\lambda_k$ .

*Proof.* Suppose that the geometric multiplicity of  $\lambda_k$  is equal to  $g$ , so that there are  $g$  linearly independent eigenvectors.  $x_1, \dots, x_g$  associated to  $\lambda_k$ . Randomly choose  $k - g$  factors  $x_{g+1} \dots x_k$ , all having dimension  $k \times l$  and such that the  $k$  column vectors  $x_1, \dots, x_k$  are linearly independent.

Define the  $k \times k$  matrix

$$x = [x_1, \dots, x_k]$$

for any  $g$ , denoted by  $b_g$  the vector that solves  $xb_g = Ax_g = \lambda x_g$

Define the  $k \times (k - g)$  matrix

$$B = [b_g + 1, \dots, b_k]$$

and denote by  $C$  its upper  $g \times (k - g)$  block, and denote by  $D$  its lower  $(k - g) \times (k - g)$  block

$$B = \begin{bmatrix} C \\ D \end{bmatrix}$$

Denote by  $I$  the  $k \times k$  identity matrix. for any scalar  $\lambda$ , we have that  $(A - \lambda I)X = 0$  to find  $x$  for  $\lambda_k$

□

I dont understand anything she wrote after this so sorry that there isnt anything here

## 5 Inverses and Singularity

Suppose  $A$  is a square matrix. we look for an Inverse Matrix,  $A^{-1}$  of the same size, such that  $AA^{-1} \Rightarrow I$  (does nothing to a vector)

Thus,  $AA^{-1}x = x$

Multiplying  $Ax = b$  by  $A^{-1}$  gives  $A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$

If the determinant of  $A$  is non-zero, then  $A^{-1}$  exists, thus it is invertible.

## 6 Systems of Equations

## 7 Singular Value Decomposition (SVD)

## 8 Spectral Decomposition

## 9 Properties and Derivations of Matrix Traces

## 10 Projection and Isometry

## 11 Variance-Covariance Matrix

## 12 Multivariate Normal Distribution