MEC Day 6 Assignment Results

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May 27, 2018

Q1 Max flow using Gurobi

Here are the results:

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[1] "Gurobi achieved optimation in 0.0579999999999998s"
[1] "With Gurobi, maximum flow = 130.028922623099"
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Q2 Max flow using Ford-Fulkerson Algorithm

Q3 Social welfare for traffic congestion problem

Cost per unit through arc a is:

$$c(\mu_a) = 1 + \mu_a^{\frac{2}{3}}$$

Social Planner's problem is:

$$\min_{\mu \geq 0} \sum_a \mu_a c(\mu_a)$$

$$\text{s.t. } \nabla^T \mu = s$$

$=$

$$\max_p (\sum_z p_z s_z + \min_{\mu \geq 0} \sum_a \mu_a c(\mu_a) - \sum_z p_z (\nabla^T \mu)_z)$$

hence,

$$\forall a \in \mathscr{A}, \mu_a > 0 \Rightarrow c(\mu_a) + \mu_a c'(\mu_a) = (\nabla p)_a$$

$\Rightarrow$

$$\mu_a = [\frac{3}{5}(\nabla P - 1)^+]^{\frac{3}{2}}$$

The original problem now is simplified as:

$$\max_p \sum_z p_z s_z - \sum_a \frac{2}{3}[\frac{3}{5}((\nabla P)_a - 1)^+]^{\frac{5}{2}}$$

where p_z is the multiplier of the constraint z . The gradient function is:

$$s - \nabla^T [\frac{3}{5}[(\nabla P - 1)^+]^{\frac{3}{2}} = s - \nabla^T \mu$$

In our NYC subway travel case,

$$(\nabla P)_{xy} = P_y - P_x$$

Normalize $p_0 = 0$, using gradient method in optimization, we can get following results:

```
[1] "Gradient method in optimization fomulation converged in 0.0579999999999998s."
[1] "The minimum cost of social planner problem is = 7.97981961547729"
```

Q4 Social welfare for Nash equilibrium with congestion, i.e., negative externality

Wardrop equilibrium solves the following question:

$$\min_{\mu \geq 0} \sum_a (\mu_a + \frac{3}{5}\mu_a^{\frac{5}{3}})$$

$$\text{s.t. } \nabla^T \mu = s$$

$=$

$$\max_p (\sum_z p_z s_z + \min_{\mu \geq 0} \sum_a (\mu_a + \frac{3}{5}\mu_a^{\frac{5}{3}}) - \sum_z p_z (\nabla^T \mu)_z)$$

hence,

$$\forall a \in \mathscr{A}, \mu_a > 0 \Rightarrow 1 + \mu_a^{\frac{2}{3}} = (\nabla p)_a$$

$\rightarrow$

$$\mu = [(\nabla P - 1)^+]^{\frac{3}{2}}$$

The original problem now is simplified as:

$$\max_p \sum_z p_z s_z - \sum_a \frac{2}{5}[((\nabla P)_a - 1)^+]^{\frac{5}{2}}$$

and the gradient function is:

$$s - \nabla^T \mu$$

Normalize $p_0 = 0$, using gradient method in optimization, we can get following results:

```
[1] "Gradient method in optimization fomulation converged in 0.704999999999998s."
[1] "The social cost of Wardrop equilibrium (NE) is = 8.00028433408372"
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Not surprisingly, the social cost in Wardrop equilibrium (i.e., NE equilibrium here) is larger than the social optimal one. However, the negative externality of congestion doesn't seem very large in this case, the extra social cost is only 2.5 ‰.

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[1] "In social optimal solution,"
[1] "Number of arcs with positive flow is 13"
[1] "The average flow of an arc is 0.313331014630152"
[1] "In Wardrop equilibrium,"
[1] "Number of arcs with positive flow is 20"
[1] "The average flow of an arc is 0.200015346303584"
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