Sample Solutions Tutorial 4 - Matrices (2)

Q1 $A^T = \begin{bmatrix} 1 & 7 & -3 \\ 6 & 8 & 0 \\ 9 & -12 & 1 \end{bmatrix}$ $\quad B^T = \begin{bmatrix} 1 & -3 & 0 \end{bmatrix}$ $\quad C^T = \begin{bmatrix} 3 & -3 \\ 4 & 0 \end{bmatrix}$

Q2 (i)
$$AA^T = \begin{bmatrix} 1 & 6 & 9 \\ 7 & 8 & -12 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 7 & -3 \\ 6 & 8 & 0 \\ 9 & -12 & 1 \end{bmatrix} = \begin{bmatrix} (1\times1)+(6\times6)+(9\times9) & (1\times7)+(6\times8)+(9\times-12) & (1\times-3)+(9\times1) \\ (7\times1)+(8\times6)+(-12\times9) & (7\times7)+(8\times8)+(-12\times-12) & (7\times-3)+(8\times0)+(-12 \\ (-3\times1)+(1\times9) & (-3\times7)+(0\times8)+(1\times-12) & (-3\times-3)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} (1+36+81) & (7+48-108) & (-3+9) \\ (7+48-108) & (49+64+144) & (-21+-12) \\ (-3+9) & (-21-12) & (9+1) \end{bmatrix}$$

$$= \begin{bmatrix} 118 & -53 & 12 \\ -53 & 257 & \cancel{-33}\,9 \\ 12 & \cancel{-33}\,9 & 10 \end{bmatrix}$$

(ii)
$$BB^T = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} (1\times1) & (1\times-3) & (1\times0) \\ (-3\times1) & (-3\times-3) & (-3\times0) \\ (0\times1) & (0\times-3) & (0\times0) \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iii)
$$CC^T = \begin{bmatrix} 3 & 4 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} (3\times3)+(4\times4) & (3\times-3)+(4\times0) \\ (-3\times3)+(0\times4) & (-3\times-3)+(0\times0) \end{bmatrix} = \begin{bmatrix} 25 & -9 \\ -9 & 9 \end{bmatrix}$$

They are all square symmetric matrices.

Q3. $2\times2: I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad 3\times3: I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q4 (i) $\det A = ad - bc = (1)(-1) - (2)(0) = -1$

(ii) $\det B = ad - bc = (3)(2) - (4)(5) = 6 - 20 = -14$

(iii) $\det C = ad - bc = (2)(2) - (-1)(3) = 4 + 3 = 7$ $\quad *7$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{14} & \frac{4}{14} \\ \frac{5}{14} & -\frac{3}{14} \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{5}{14} & -\frac{3}{14} \end{bmatrix}$$

$$C^{-1} = \frac{1}{\det C}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

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Q5 $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$a_{32} = 0$, therefore

$\det A = a_{31}C_{31} + a_{33}C_{33}$

For a_{31}: sub-matrix $= \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, $\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7$

$\Rightarrow M_{31} = 7$, $i + j = 3 + 1 = 4$ which is even

$\Rightarrow C_{31} = 7$.

For a_{33}: submatrix $= \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, $\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1 - 8 = -7$

$\Rightarrow M_{33} = -7$, $i + j = 3 + 3 = 6$ which is even

$\Rightarrow C_{33} = -7$.

$\det A = (6)(7) + (2)(-7) = 42 - 14 = 28.$

OR $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 1 & 5 & 4 & 1 & 5 \\ 6 & 0 & 2 & 6 & 0 & 2 \end{array}\right]$

$(1\times1\times2) + (2\times5\times6) + (3\times4\times0) = 62$

$(6\times1\times3) + (0\times5\times1) + (2\times4\times2) = 34\ -$

$\overline{28}$

Q6 (i) $A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{\det A} C^T$

a_{11}: $M_{11} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = 1 - 4 = -3$

$C_{11} = -3$

a_{12}: $M_{12} = \begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix} = -5 + 6 = -11$

$C_{12} = 11$

a_{13}: $M_{13} = \begin{vmatrix} 5 & -1 \\ 3 & 2 \end{vmatrix} = 10 + 3 = 13$

$C_{13} = 13$

$$\det A = (2)(-3) + (-3)(11) + (1)(13)$$
$$= -6 - 33 + 13 = -26$$

a_{21}: $M_{21} = \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} = 3 - 2 = 1$

$C_{21} = -1$

a_{22}: $M_{22} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2 - 3 = -5$ $\qquad C = \begin{bmatrix} -3 & 11 & 13 \\ -1 & -5 & -13 \\ -5 & 1 & 13 \end{bmatrix}$

$C_{22} = -5$

a_{23}: $M_{23} = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13$ $\qquad C^T = \begin{bmatrix} -3 & -1 & -5 \\ 11 & -5 & 1 \\ 13 & -13 & 13 \end{bmatrix}$

$C_{23} = -13$

a_{31}: $M_{31} = \begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} = -6 + 1 = -5$

$C_{31} = -5$

a_{32}: $M_{32} = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = 4 - 5 = -1$

$C_{32} = 1$

a_{33}: $M_{33} = \begin{vmatrix} 2 & -3 \\ 5 & -1 \end{vmatrix} = -2 + 15 = 13$

$C_{33} = 13$

$$A^{-1} = -\frac{1}{26}\begin{bmatrix} -3 & -1 & -5 \\ 11 & -5 & 1 \\ 13 & -13 & 13 \end{bmatrix} = \begin{bmatrix} 3/26 & 1/26 & 5/26 \\ -11/26 & 5/26 & -1/26 \\ -13/26 & 13/26 & -13/26 \end{bmatrix} = \begin{bmatrix} 3/26 & 1/26 & 5/26 \\ -11/26 & 5/26 & -1/26 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$$

$\underline{\underline{a6(ii)}}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -2 & -3 \\ 2 & 1 & -3 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{\det A} C^{T}$

$\underline{a_{11}}$: $m_{11} = \begin{vmatrix} -2 & -3 \\ 1 & -3 \end{vmatrix} = 6 + 3 = 9$

$c_{11} = 9$

$\underline{a_{12}}$: $m_{12} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -9 + 6 = -3$

$c_{12} = 3$

$\underline{a_{13}}$: $m_{13} = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 + 4 = 7$

$c_{13} = 7$

$\det A = (1)(9) + (1)(3) + (1)($
$= 9 + 3 + 7 = 19$

$\underline{a_{21}}$: $m_{21} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4$

$c_{21} = 4$

$\underline{a_{22}}$: $m_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$

$c_{22} = -5$

$\underline{a_{23}}$: $m_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

$c_{23} = 1$

$* \underline{a_{\cancel{33}}}$: $m_{31} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5$

$c_{31} = -5$

$\underline{a_{32}}$: $m_{32} = \begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = -3 - 3 = -6$

$c_{32} = 6$

$* \underline{a_{3\cancel{1}}}$: $m_{3\cancel{1}} = \begin{vmatrix} 1 & 1 \\ -\cancel{2} & -3 \end{vmatrix} = -3 + 2 = -1$

$c_{3\cancel{1}} = -1$

$C = \begin{bmatrix} 9 & +3 & 7 \\ +4 & -5 & 1 \\ -1 & 6 & -5 \end{bmatrix}$

$C^{T} = \begin{bmatrix} 9 & 4 & -1 \\ 3 & -5 & 6 \\ 7 & 1 & -5 \end{bmatrix}$

$A^{-1} = \frac{1}{19} \begin{bmatrix} 9 & 4 & -1 \\ 3 & -5 & 6 \\ 7 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 9/19 & 4/19 & -1/19 \\ 3/19 & -5/19 & 6/19 \\ 7/19 & 1/19 & -5/19 \end{bmatrix}$