№ 19.07

Лист 1

Леденев Д. А
гр. ФФ3-24-2
Д/з № 2

② $f(z) = e^{-z^2}$, $a = \infty$

$\lim\limits_{z\to a} e^{-z^2} = \lim\limits_{z\to\infty} \frac{1}{e^{z^2}}$ – не существует $\Rightarrow$ $z = \infty$ – существ. особенная

③ $f(z) = \sin\frac{\pi}{z^2}$, $a = 0$

$\lim\limits_{z\to a} (\sin \pi/z^2)$ – не существует $\Rightarrow$ $z = 0$ – существенно особенная

№ 19.15

① $f(z) = \frac{(1+z^2)^2}{1-z^2} = \frac{(z+i)^2(z-i)^2}{(1-z)(1+z)}$

• $z = \pm i$ – нули II порядка, • $z = \infty$ – ~~ноль~~ полюс II порядка

• $z = 1 \longrightarrow z = 1 + \varepsilon$

$f(z) = \frac{(1+z^2)^2}{\varepsilon(2+\varepsilon)} = \frac{A + B\varepsilon + C\varepsilon^2 + \ldots}{\varepsilon(2+\varepsilon)} = \underbrace{\frac{1}{\varepsilon}}_{\text{даст сингулярн.}}(A + B\varepsilon + \ldots)$

$\Rightarrow$ нужен только линейный член $\Rightarrow$ можем убрать ε в степени выше 0

$\Rightarrow\ = \frac{1}{2\varepsilon} + O(1) = \frac{1}{2(z-1)} + O(1)$ $\Rightarrow$ $z = 1$ – полюс I порядка

Аналогично, $z = -1$ – полюс I порядка

• $z = \infty$

$\lim\limits_{z\to\infty} f(z) = \infty \Rightarrow$ полюс II порядка

Ответ:
$z = \pm i$ – нули II порядка
$z = \pm 1$ – полюс I порядка
$z = \infty$ – полюс II порядка

② $f(z) = ctg\, z = \frac{\cos z}{\sin z}$

~~$z = \pi k ... \frac{\pi}{2} + \pi n$~~

• $z = \pm\frac{\pi}{2} + \pi n$, $n = \dots -1, 0, 1 \dots$ – нули I порядка

• $\sin z = 0 \Rightarrow z = \pm\pi n \longrightarrow z = \pi n + \varepsilon$.

$$\frac{\cos z}{\sin(\pi n + \varepsilon)} = \frac{(-1)^n}{\sin\varepsilon} \cdot \underbrace{\cos(\pi n + \varepsilon)}_{\text{част рег. часть}} = \left\{ \begin{array}{l} \sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + \dots \\ \cos(\pi n + \varepsilon) = 1 - \frac{1}{2}\varepsilon^2 + \dots \end{array} \right\} =$$

$$= \frac{(-1)^n}{\left(\varepsilon - \frac{\varepsilon^3}{6} + \dots\right)} = \frac{(-1)^n}{\varepsilon\left(1 - \varepsilon^2/6 + \dots\right)} = \frac{(-1)^n}{\varepsilon\left(1 - 1(-\varepsilon^2/6)\right)} = \frac{(-1)^n}{\varepsilon\left(1 - \varepsilon^2/6\right)} =$$

$$\left\{ \left(1 - \frac{\varepsilon^2}{6}\right)^{-1} \simeq 1 + (-1)\left(-\frac{\varepsilon^2}{6}\right) \right\}$$

~~$= (-1)^n$~~ $\Rightarrow$ $z = \pm\pi n$ – полюс I порядка

③ $f(z) = z \cdot tg^2 z = z \frac{\sin^2 z}{\cos^2 z}$

• $z = \pm\pi n$ – нули II порядка

• $z = 0$ – нуль III порядка

• $z = \pm\frac{\pi}{2} + \pi n \longrightarrow z = \frac{\pi}{2} + \varepsilon$.

$$f\left(\frac{\pi}{2} + \varepsilon\right) = \left(\frac{\pi}{2} + \varepsilon\right) \frac{\sin^2\left(\frac{\pi}{2} + \varepsilon\right)}{\cos^2\left(\frac{\pi}{2} + \varepsilon\right)} = \left(\frac{\pi}{2} + \varepsilon\right) \frac{\cos^2\varepsilon}{\sin^2\varepsilon}$$

№ 20.01

① $\sum\limits_{n=-\infty}^{\infty} 2^{-|n|} \cdot z^n = \sum\limits_{n=-\infty}^{\infty} 2^{-|n|} (z-0)^n$

$$\sum_{n=1}^{\infty} 2^{-|n|} (z-0)^n$$

$$R = \lim_{n\to\infty} \left|\frac{C_n}{C_{n+1}}\right|$$

$$r = \lim_{n\to\infty} \left|\frac{C_{-n-1}}{C_{-n}}\right|$$

$$R = \lim_{n\to\infty} \left|\frac{2^{-|n|}}{2^{-|n+1|}}\right| = \lim_{n\to\infty} \left|\frac{2^{|n+1|}}{2^{|n|}}\right| = 2$$

$$r = \lim_{n\to\infty} \left|\frac{2^{|-n-1|}}{2^{|-n|}}\right| = \frac{1}{2}$$

$\Rightarrow$ кольцо сходимости:

$$\frac{1}{2} \leq |z| \leq 2$$

② $\sum\limits_{n=-\infty}^{\infty} \frac{1}{3^n+1} (z-0)^n \Rightarrow C_n = \frac{1}{3^n+1}$

$$R = \lim_{n\to\infty} \left|\frac{3^{n+1} \cancel{+1}}{3^n \cancel{+1}}\right| = 3.$$

$$r = \lim_{n\to\infty} \left|\frac{3^{-n}+1}{3^{-n-1}+1}\right| = 1$$

$\Rightarrow$ кольцо сходимости

$$1 \leq |z| \leq 3$$

④ $\sum\limits_{n=-\infty}^{\infty} 2^{-n^2} (z+1)^n, \quad C_n = 2^{-n^2}$

$$R = \lim_{n\to\infty} \left|\frac{2 \cdot 2^{(n\mp 1)^2}}{2^{n^2}}\right| = \infty$$

$$r = \lim_{n\to\infty} \left|\frac{2^{-(-n+1)^2}}{2^{-(-n)^2}}\right| = \lim_{n\to\infty} \left|\frac{2^{(-n)^2}}{2^{(-n+1)^2}}\right| = 0$$

$\Rightarrow$ кольцо сходимости

$$0 < |z+1| < \infty$$

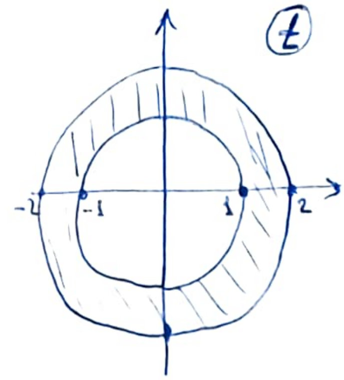
N 20.09

① $f(z) = \dfrac{1}{z(z-3)^2}$ $a=1$, $D: 1<|z-1|<2$

$t = z-1 \longrightarrow a_t = 0$

$z = t+1 \Rightarrow f(t) = \dfrac{1}{(t+1)(t-2)^2} = \dfrac{A}{t+1} + \dfrac{B}{(t-2)^2} + \dfrac{C}{(t-2)}$

$A(t-2)^2 + B(t+1) + C(t+1)(t-2) = 1 \Rightarrow \begin{cases} A = -1/9 \\ B = 1/9 \\ C = 1/9 \end{cases}$



$f(t) = -\dfrac{1}{9}\left(\dfrac{1}{t+1} - \dfrac{1}{(t-2)^2} + \dfrac{1}{t-2}\right)$

- $\dfrac{1}{t+1} = \dfrac{1}{t\left(\frac{1}{t}+1\right)} = \dfrac{1}{t}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{t^n} = \sum_{-\infty}^{-1}(-1)^{n-1}t^n$

- $\dfrac{1}{t-2} = \dfrac{1}{2\left(\frac{t}{2}-1\right)} = -\dfrac{1}{2\left(1-\frac{t}{2}\right)} = -\dfrac{1}{2}\sum_{n=0}^{\infty}\dfrac{(-1)^n t^n}{2^n} = \sum \dfrac{(-1)^{n+1} t^n}{2^{n+1}}$

$f(z) = \sum_{-\infty}^{-1}\dfrac{(-1)^{n-1}}{9}(z-1)^n + \sum_{n=0}^{\infty}\dfrac{(-1)^{n+2}}{9\cdot 2^{n+2}}(z-1)^n$

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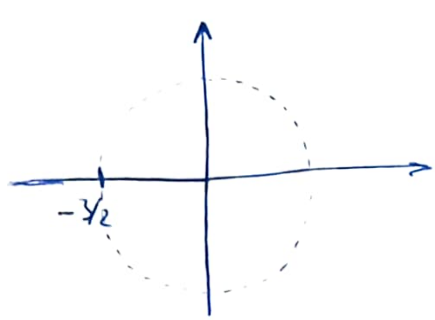
⑤ $f(z) = \dfrac{1}{z(z-1)(z-2)} =$ $a=0$ | $-3/2 \in D$



$= \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{(z-2)} =$

$A(z-1)(z-2) + B(z-2)\cdot z + C(z-1)z = 1$

$\begin{cases} B = -1 \\ C = 1/2 \\ A = 1/2 \end{cases}$

$= \dfrac{1}{2}\left(\dfrac{1}{z} - \dfrac{1}{2(z-1)} + \dfrac{1}{(z-2)}\right)$

лист 3

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^n}\,1}{z^{n+1}} = \sum_{-\infty}^{-1} \cancel{(-1)^n}\, z^n$$

$R = 1$

$$\frac{1}{z-2} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = +\sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^n}{2^{n+1}}$$

$R = 2$

$$\Rightarrow f(z) = -\sum_{n=-\infty}^{-2} z^n - \frac{1}{2 \cdot z} - \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} z^n$$

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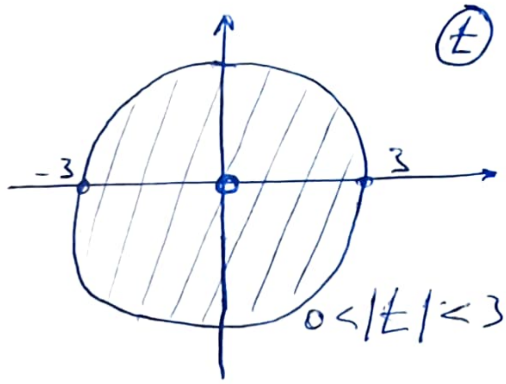
7 $f(z) = \dfrac{z^3}{(z+1)(z-2)}$ $\quad a = -1 \quad 0 < |z+1| < 3$



$t = z+1 \;, \quad z = t-1$

$$f(t) = \frac{(t-1)^3}{\cancel{t}(t-3)} = \frac{A}{t} + \frac{B}{t-3}$$

$$At - 3A + Bt = (t-1)^3 = t^3 - 3t^2 + 3t - 1$$

$$\left.\begin{aligned} A+B &= 3 \\ -3A &= -1 \end{aligned}\right\} \quad \left.\begin{aligned} B &= 8/3 \\ A &= 1/3 \end{aligned}\right] \quad \Rightarrow \quad f(t) = \frac{1}{3}\left(\frac{8}{t} + \frac{1}{t-3}\right)$$

$$\frac{1}{t-3} = \frac{1}{3(t/3-1)} = \frac{-1}{3(1-t/3)} = -\sum_{n=0}^{\infty} \frac{t^n}{3^{n+1}} \;, \quad \left|\frac{t}{3}\right| < 1 \Rightarrow |t| < 3$$

$$f(t) = \frac{1}{3} t^{-1} + \frac{8}{3} \sum_{n=} \frac{t^n}{3^{n+1}}$$

$$f(z) = \frac{1}{3}(z+1)^{-1} - \sum_{n=0}^{\infty} \frac{8}{3^{n+2}} (z+1)^n$$

⑧ $f(z) = \dfrac{1}{(z^2-1)(z^2+4)} =$ $\qquad a = 0$, $|z| > 2$

$$= \frac{1}{(z-1)(z+1)(z+2i)(z-2i)} = \frac{A}{z^2-1} + \frac{B}{z^2+4}$$

$$A(z^2+4) + B(z^2-1) = 1$$

$$\begin{cases} 4A - B = 1 \\ A + B = 0 \end{cases} \qquad \begin{cases} A = 1/5 \\ B = -A = -1/5 \end{cases}$$

$$z^2 = t \Rightarrow f(t) = \frac{1}{(t-1)(t+4)} = \frac{A}{t-1} + \frac{B}{t+4} = \frac{1}{5}\left(\frac{1}{t-1} - \frac{1}{t+4}\right)$$

$$\frac{1}{t-1} = \frac{1}{t(1-1/t)} = \sum_{n=0}^{\infty} \frac{1}{t^n} \qquad \left|\frac{1}{t}\right| < 1$$

$$|t| < 1 \quad - \not\subset D$$

$$\frac{1}{t+4} = \frac{1}{4(1+t/4)} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{4^n}$$

$$f(z) = \frac{1}{5} \sum_{+\infty}^{-1} \frac{(-1)^n}{4^{n+1}} z^{2n}$$