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лист 1

ТФКП ДЗ № 1

№ 1.04

① $\frac{1}{1-i} = \frac{(1+i)}{(1-i)(1+i)} = \frac{1+i}{2} = \overbrace{\frac{1}{2}}^{\operatorname{Re} z} + i\cdot\overbrace{\frac{1}{2}}^{\operatorname{Im} z}$

② $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{(1-i)(1+i)}{(1+i)(1-i)}\right)^3 = \left(\frac{-2i}{2}\right)^3 = (-i)^3 = i$

$\Rightarrow \begin{cases} \operatorname{Re} z = 0 \\ \operatorname{Im} z = 1 \end{cases}$

③ $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\cdot(-i)\frac{\sqrt{3}}{2} + 3\cdot\frac{1}{2}\left(-i\frac{\sqrt{3}}{2}\right)^2 - \left(-i\frac{\sqrt{3}}{2}\right)^3 =$

$= \frac{1}{8} \cancel{+ i\frac{3}{4}\frac{\sqrt{3}}{2}} - \frac{3}{2}\cdot\frac{3}{4} \cancel{- i\frac{3\sqrt{3}}{8}} = \frac{1}{8} - \frac{9}{8} = -1$

$\Rightarrow \begin{cases} \operatorname{Re} z = -1 \\ \operatorname{Im} z = 0 \end{cases}$

④ $\left(\frac{i^5+2}{i^{19}+1}\right)^2 = \left[\frac{(i^5+2)(i^{19}-1)}{(i^{19}+1)(i^{19}-1)}\right]^2 = \left[\frac{i^{24} - i^5 + 2i^{19} - 2}{i^{38}-1}\right]^2 =$

$= \begin{Bmatrix} i^{24} = -1 \\ i^5 = -i \\ i^{19} = i \end{Bmatrix} = \left[\frac{-1 + i + 2i - 2}{i^{38}-1}\right]^2 = \left[\frac{-3+i}{-2}\right]^2 =$

$= \left(\frac{3}{2} + \frac{i}{2}\right)^2 = \frac{9}{4} + 2\,\frac{3}{2}\,\frac{i}{2} - \frac{1}{4} = 2 + i\frac{3}{2}$

$\Rightarrow \begin{cases} \operatorname{Re} z = 2 \\ \operatorname{Im} z = +3/2 \end{cases}$

⑤ $\frac{(1+i)^5}{(1-i)^3} = \frac{(1+i)^5\cdot(1+i)^3}{(1-i)^3(1+i)^3} = \frac{(1+i)^8}{2^3} =$

$= \frac{16}{8} = 2 \Rightarrow \begin{cases} \operatorname{Re} z = 2 \\ \operatorname{Im} z = 0 \end{cases}$

N 1.06

(1) $z = i$

$$\left.\begin{array}{l} x = 0 \\ y = 1 \end{array}\right\} \Rightarrow |z| = 1, \quad \varphi = \arcsin 1 = \arccos 0 = \frac{\pi}{2} + \pi n$$

(2) $z = -3$

$$\left.\begin{array}{l} x = -3 \\ y = 0 \end{array}\right\} \Rightarrow |z| = 3, \quad \left.\begin{array}{l} \varphi = \arcsin 0 \\ \varphi = \arccos(-1) \end{array}\right\} \Rightarrow \varphi = 2\pi n + \pi$$

(3) $z = 1 + i^{123} = 1 - i$

$$\left.\begin{array}{l} x = 1 \\ y = -1 \end{array}\right\} \Rightarrow |z| = \sqrt{2}, \quad \begin{array}{l} \varphi = \arccos(-1/\sqrt{2}) \\ \varphi = \arcsin(1/\sqrt{2}) \end{array} \Rightarrow \varphi = -\frac{\pi}{4} + 2\pi n$$

(4) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\left.\begin{array}{l} x = -1/2 \\ y = \sqrt{3}/2 \end{array}\right\} \Rightarrow |z| = 1, \quad \begin{array}{l} \varphi = \arccos(-1/2) \\ \varphi = \arcsin(\sqrt{3}/2) \end{array} \Rightarrow \varphi = \frac{2\pi}{3} + 2\pi n$$

(5) $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = -i$

$$\left.\begin{array}{l} x = 0 \\ y = -1 \end{array}\right\} \Rightarrow |z| = 1 \quad \begin{array}{l} \varphi = \arccos(-1) \\ \varphi = \arcsin(0) \end{array} \Rightarrow \varphi = -\frac{\pi}{2} + 2\pi n$$

(6) $z = -\cos\frac{\pi}{7} + i \sin\frac{\pi}{7}$

$$\left.\begin{array}{l} x = -\cos\frac{\pi}{7} \\ y = \sin\frac{\pi}{7} \end{array}\right\} \quad |z| = \sqrt{\cos^2\frac{\pi}{7} + \sin^2\frac{\pi}{7}} = 1$$

$$\begin{array}{l} \varphi = \arccos\left(-\cos\frac{\pi}{7}\right) \\ \varphi = \arcsin\left(\sin\frac{\pi}{7}\right) \end{array} \Rightarrow \varphi = -\frac{\pi}{7} + 2\pi n$$

⑦ $(-4+3i)^3=(-4)^3+3\cdot(-4)^2\cdot 3i+3\cdot(-4)\cdot(-3i)^2+(3i)^3=$ | лист 2

$=44+117i$

$\left.\begin{matrix} x=44 \\ y=117 \end{matrix}\right\}$ $|z|=\sqrt{44^2+117^2}=\sqrt{15625}=125$

$\varphi=\arccos\frac{44}{125}$

$\varphi=\arcsin\frac{117}{125}$

⑧ $(1+i)^8(1-i\sqrt{3})^{-6}=16\cdot\frac{1}{64}=\frac{1}{4}$

$\left.\begin{matrix} x=1/4 \\ y=0 \end{matrix}\right\}\Rightarrow|z|=1/4$ $\quad\begin{matrix}\varphi=\operatorname{arc}\cancel{tg}\cos(1) \\ \varphi=\arcsin(0)\end{matrix}\Rightarrow\varphi=2\pi n$

⑨ $z=1+\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$

$\left.\begin{matrix} x=1+\cos\frac{\pi}{7} \\ y=\sin\pi/7 \end{matrix}\right\}$ $|z|=\sqrt{1+2\cos\frac{\pi}{7}+\cos^2\frac{\pi}{7}+\sin^2\frac{\pi}{7}}=$

$=\sqrt{2\left(1+\cos\frac{\pi}{7}\right)\frac{2}{2}}=\left\{\pi/7=2\alpha\Rightarrow\alpha=\frac{\pi}{14}\right\}=$

$=\sqrt{2\cdot\frac{1}{2}\left(1+\cos\pi/7\right)}=\sqrt{2}\sqrt{\cos^2\frac{\pi}{14}}=\sqrt{2}\cos\pi/14$

$\varphi=\arccos\frac{1+\cos\pi/7}{\sqrt{2}\cos\pi/14}=\operatorname{arc}\cancel{cos}$

$\varphi=\frac{\pi}{14}+2\pi n$

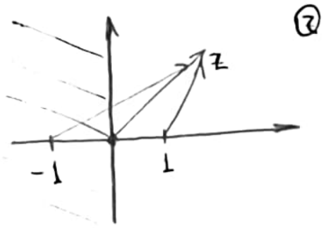
$\varphi=\arcsin\frac{\sin\pi/7}{\sqrt{2}\sin\pi/14}$

N 1.21

④ $|1+z| < |1-z|$

$|z+1| < |-(z-1)|$ $\qquad$ $|z+1| = |z-(-1)|$

$|z+1| < |z-1|$



$\Rightarrow$ левая полуплоскость $\qquad$ $A = (-1,0)$, $B = (1,0)$

① $|z-i| + |z+i| < 4$ $\qquad$ $z = x+iy$

$|x+iy-i| + |x+iy+i| < 4$ $\qquad$ $|x+i(y-1)| + |x+i(y+1)| < 4$

$\cancel{\sqrt{(x+iy-i)^2} + \sqrt{(x+iy+i)^2} < 4}$ $\qquad$ $\sqrt{x^2+(y-1)^2} + \sqrt{x^2+(y+1)^2} < 4$

$\cancel{\sqrt{(x+i(y-1))^2} < 4 - \sqrt{(x+(y+1))^2}}$ $\qquad$ $\sqrt{x^2+(y-1)^2} < 4 - \sqrt{x^2+(y+1)^2}$

$\cancel{x^2}+(y-1)^2 < 16 - 8\sqrt{x^2+(y+1)^2} + \cancel{x^2}+(y+1)^2$

$\cancel{y^2}-2y+\cancel{1} < 16 - 8\sqrt{x^2+(y+1)^2} + \cancel{y^2}+2y+\cancel{1}$ $\quad \Big| \cdot \frac{1}{4}$

$0 < 4 - 2\sqrt{x^2+(y+1)^2} + y$

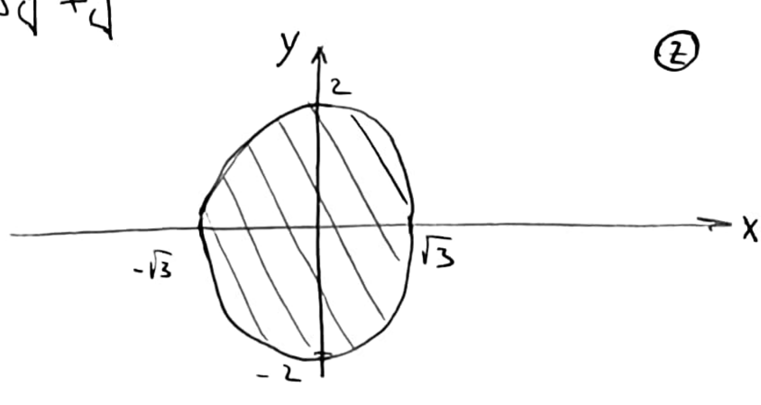
$2\sqrt{x^2+(y+1)^2} < 4+y$

$4(x^2+(y+1)^2) < (4+y)^2$

$4x^2+4y^2\cancel{+8y}+4 < 16\cancel{+8y}+y^2$

$4x^2+3y^2 < 12$

$\frac{x^2}{3} + \frac{y^2}{4} < 1$



N 8.51 | Лист 3

(1)

$\operatorname{Re} f = x^3+6x^2y-3xy^2-2y^3, \qquad f(0)=0$

Коши - Риман: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

$$\frac{\partial u}{\partial x} = 3x^2+12xy-3y^2 = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = 3x^2y+6xy^2-y^3+\psi(x)$$

$$\cancel{\frac{\partial u}{\partial y} = 6x^2-6xy-6y^2 = -\frac{\partial v}{\partial x}}$$

$$\frac{\partial v}{\partial x} = 6xy+6y^2+\psi'_x = -\frac{\partial u}{\partial y} = -6x^2+6xy+6y^2$$

$$\Rightarrow \psi'_x = -6x^2 \Rightarrow \psi(x) = -\int 6x^2dx = -2x^3+C$$

$$\Rightarrow v(x,y) = 3x^2y+6xy^2-y^3-2x^3+C$$

$$f(0)=0 \Rightarrow C=0 \Rightarrow v(x,y) = 3x^2y+6xy^2-y^3-2x^3$$

$$f = u+iv = x^3+6x^2y-3xy^2-2y^3+i3x^2y+6ixy^2-iy^3-i2x^3=$$

$$= (x+iy)^3-2i(x+iy)^3 = z^3(1-2i)$$

(2)

$$U(x,y) = e^x \cdot x\cos y - e^x y \sin y \qquad f(0) = 0$$

$$\frac{\partial U}{\partial x} = e^x \cdot x\cos y + e^x\cos y - e^x y\sin y = \frac{\partial V}{\partial y}$$

$$\Rightarrow V(x,y) = \int e^x \cancel{x}\,dx \cdot \cos y + \int e^x dx \cdot \cos y - \int e^x dx \cdot y\sin y$$

$$= (x-1)e^x \cdot \cos y + e^x\cos y - e^x y\sin y + \psi(x) =$$

$$= xe^x\cos y \cancel{- e^x\cos y + e^x\cos y} - e^x y\sin y + \psi(x) =$$

$$= e^x(x\cos y - y\sin y) + \psi(x)$$

$$\frac{\partial V}{\partial x} = e^x \cdot x\cos y + e^x\cos y - e^x y\sin y + \psi'(x) =$$

$$= -\frac{\partial U}{\partial y} = + e^x \cdot x\sin y + e^x\sin y + e^x y\cos y$$

$$\Rightarrow \psi'(x) = 0 \Rightarrow \psi(x) = \text{const}$$

$$f(z) = U(x,y) + i\,V(x,y) = e^x \cdot (x\cos y - y\sin y) +$$
$$+ i\,e^x(x\cos y - y\sin y) = z e^z$$

(5) $|f| = (x^2+y^2)\,e^x$

$w = \ln f \quad \Rightarrow \quad \operatorname{Re} w = \ln|f|$
$\qquad\qquad\qquad\quad \operatorname{Im} w = \arg f$

$$\operatorname{Re} w = \ln\left((x^2+y^2)\cdot e^x\right) = U(x,y) = x + \ln(x^2+y^2)$$

$$\frac{\partial U}{\partial x} = 1 + \frac{2x}{x^2+y^2} = \frac{\partial V}{\partial y} \Rightarrow V = \int dy + 2x\int \frac{dy}{x^2+y^2} =$$

$$= y + 2\cancel{x}\frac{1}{\cancel{x}}\operatorname{arctg}\frac{y}{x} + \psi(x)$$

N 8.51 (продолжение) лист 4

$$V(x,y) = y + 2\,\mathrm{arctg}\,\frac{y}{x} + \psi(x)$$

$$\frac{\partial V}{\partial x} = + \frac{2x^{-2}}{1+\frac{y^2}{x^2}} + \psi'(x) = -\frac{\partial U}{\partial y} = \frac{-2y}{x^2+y^2} = -\frac{2x^{-2}}{1+\frac{y^2}{x^2}}$$

$$\Rightarrow \psi'(x) = 0 \Rightarrow \psi(x) = \mathrm{const}$$

$$\Rightarrow V(x,y) = y + 2\,\mathrm{arctg}\,\frac{y}{x} + C$$

$$U = x + \ln(x^2+y^2)$$

$$w = U + iV = (x+iy) + 2\,\mathrm{arctg}\,\frac{y}{x} + i\ln(x^2+y^2) + C\cdot i$$

$$f = e^{w} = e^{(x+iy)} \cdot e^{i\alpha} \cdot (x^2 + 2ixy - y^2) = e^{z} \cdot e^{i\alpha} \cdot z^2$$