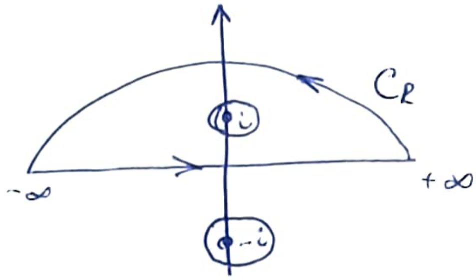
Леденев Д.
Д/з N 3
лист 1

N3.1(1)

$$\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx$$



$$f(z) = \frac{z^4}{1+z^6} \qquad z\to\infty \Rightarrow z^6+1 \simeq z^6$$

~~$z^6+1=0$~~
~~$z^6=-1$, $z=\sqrt[6]{-1}$~~
~~$(z-i)(z+i)^3=0$~~
~~$z=i$ – полюс порядка~~
~~$z=-i$ – полюс порядка~~

$$\oint \frac{z^4}{z^6}\,dz = \oint \frac{1}{z^2}\,dz$$

Замена: $z = Re^{i\varphi}$
$dz = R\,e^{i\varphi}\, i\,d\varphi$

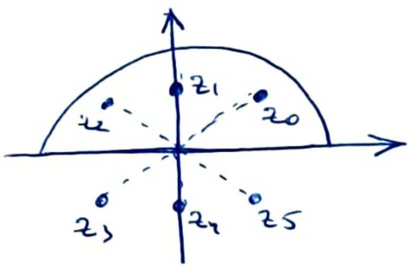
$$\oint \frac{1}{z^2}\,dz = \int_0^{\infty} \frac{Re^{i\varphi}\, i\,d\varphi}{R^2 e^{2i\varphi}} = \frac{1}{R}\int_0^{\infty} \longrightarrow 0, \quad R\to\infty$$

$$\Rightarrow \oint = \int_{-\infty}^{\infty} + \cancel{\int_{C_R}}$$

$z^6+1=0$

$$|z|^6 e^{6i\varphi} = e^{i(\pi+2\pi k)} = \cos(\pi+2\pi k) + i\sin(\pi+2\pi k) = -1$$

$$\Rightarrow \varphi = \frac{\pi}{6} + \frac{\pi k}{3}$$



$$\oint f(z)\,dz = 2\pi i \sum \operatorname*{Res}_{z=z_0} f(z) \;\cancel{= 2\pi i \operatorname*{Res}_{z=i} f(z)}$$

Особые точки: z_0, z_1, z_2 – входят в область
z_3, z_4, z_5 – нет

$$\Rightarrow \oint f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z)\right)$$

$$\begin{cases} z_0 = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt3 + i}{2} \\ z_1 = e^{i\pi/2} = i \\ z_2 = e^{i\frac{5\pi}{6}} = \frac{i-\sqrt3}{2} \end{cases}$$

– полюсы I порядка
$\Rightarrow$ можем по упрощ. формуле для вычетов

$$\underset{z_a}{\mathrm{res}}\, f(z) = \frac{h(z=a)}{g'(z=a)}$$

$$f(z) = \frac{z^4}{1+z^6} \Rightarrow \mathrm{res}\, f(z) = \frac{z^4}{6z^5} = \frac{1}{6z}$$

1) $z_0 = \frac{\sqrt{3}+i}{2} \Rightarrow \underset{z=z_0}{\mathrm{res}}\, f(z) = \frac{2}{6(\sqrt{3}+i)} = \frac{1}{3(\sqrt{3}+i)}$

2) $z_1 = i \Rightarrow \underset{z=z_1}{\mathrm{res}}\, f(z) = \frac{1}{6i}$

3) $z_2 = \frac{i-\sqrt{3}}{2} \Rightarrow \underset{z=z_2}{\mathrm{res}}\, f(z) = \frac{-1}{3(\sqrt{3}-i)}$

$$\Rightarrow \oint f(z)\,dz = \frac{1}{3} 2\pi i \left(\frac{1}{2i} + \frac{1}{\sqrt{3}+i} - \frac{1}{\sqrt{3}-i}\right) = \frac{1}{3} \cdot 2\pi i \left(\frac{1}{2i} + \frac{\cancel{\sqrt{3}}-i-\cancel{\sqrt{3}}-i}{4}\right) =$$

$$= \frac{1}{3} 2\pi i \cdot \frac{1}{i} = \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6+1}\,dx = \frac{2\pi}{3} \quad \text{- ответ.}$$

<u>N 3.1 (2)</u>

$$\int_0^{2\pi} \frac{\cos 2\theta}{\cos\theta + 2}\,d\theta$$

$\cos\theta + 2 = 0 \quad \varnothing$

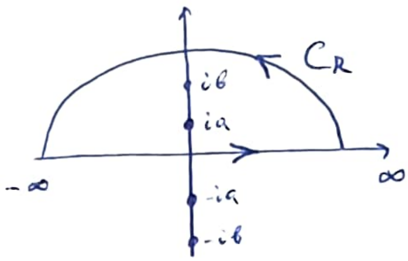
$\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} + \pi n$

$\theta = \frac{\pi}{4} + \frac{\pi}{2} n$

N 3.1(3)

лист 2

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$$



$$f(z) = \frac{\cancel{dz}\quad 1}{(z+ia)(z-ia)\,(z-ib)^2(z+ib)^2} = \frac{1}{(z^2+a^2)(z^2+b^2)^2}$$

$$f(z\to\infty) \simeq \frac{1}{z^6} \Rightarrow \begin{matrix} z = Re^{i\varphi} \\ dz = Re^{i\varphi} i\, d\varphi \end{matrix} \Rightarrow \oint_{C_R} \sim \frac{1}{R^5} \longrightarrow 0$$

$$\Rightarrow \oint = \int_{-\infty}^{+\infty} + \cancelto{0}{\oint_{C_R}}$$

$z = \pm ia$ – полюса I п.

$z = \pm ib$ – полюса II п.

$$\oint f(z)\,dz = 2\pi i \sum \operatorname*{res}_{z=ia} f(z)\,dz$$

$$\bullet \operatorname*{res}_{z=ia} f(z) = \left.\frac{h(z)}{g'(z)}\right|_{z=ia} = \frac{1}{(z+ia)(z^2+b^2)^2} = \frac{1}{2ia(b^2-a^2)^2}$$

$$\bullet \operatorname*{res}_{z=ib} f(z) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}} f(z)\cdot(z-ib)^2 = \frac{d}{dz}\left[\frac{1\cdot\cancel{(z-ib)^2}}{(z^2+a^2)\,\cancel{(z-ib)}(z+ib)^2}\right]$$

$n = 2$

$$= \frac{d}{dz}(z^2+a^2)^{-1}\cdot(z+ib)^{-2}\Big|_{z=ib} = \frac{1-2b^2}{4ib^3(a^2-b^2)}$$

$$\Rightarrow \oint f(z)\,dz = 2\pi i\cdot\frac{1}{2ia(b^2-a^2)^2} + 2\pi i\,\frac{1-2b^2}{4ib^3(b^2-a^2)} =$$

$$= \pi\,\frac{2b^3+a^3-ab^2-2b^2a^3+2ab^4}{2ab^3(a^2-b^2)^2}$$

N 3.2

$$f(z) = z^3 \cdot \cos\frac{1}{z-2}$$

$\operatorname*{res}_{z=\infty} f(z)$ — ?

$$\left((z-2)+2\right)^3 \cdot \left(1 - \frac{1}{2}(\cancel{z}-2)^{-2} + \frac{1}{4!}(z-2)^{-4} + \dots\right) =$$

$$= \left[8 + 3(z-2)^2 + 12(z-2) + (z-2)^3\right] \cdot \left[1 - \frac{1}{2}\frac{1}{(z-2)^2} + \frac{1}{24}\frac{1}{(z-2)^4} + \dots\right] =$$

$$= A_0 + B_1\frac{1}{(z-2)^4} + C\frac{1}{(z-2)^3} + D\frac{1}{(z-2)^2} + \frac{-6+1/24}{(z-2)^1} + \dots$$

$$\Rightarrow C_{-1} = \operatorname*{res}_{z=\infty} = -\left(-6+\frac{1}{24}\right) = +\frac{143}{24}$$

N 3.3

$$f(z) = \frac{1}{z^3 - z^5} =$$

$$= \frac{1}{z^3(1-z^2)} =$$

$$= \frac{1}{z^3(z-1)(z+1)}$$

a) $\operatorname*{res}_{z=-1} f(z)$ — ?

$\left[z = 0 \right.$ — полюс III пор.

$1 - z^2 = 0$

$z^2 = 1$

$|z|^2 e^{2i\varphi} = e^{2i\pi n} \Rightarrow \varphi =$

$\left[\begin{array}{l} z = 1 \text{ — полюс I порядка} \\ z = -1 \text{ — I пор} \end{array}\right.$

$$\operatorname*{res}_{z=-1} f(z) = \left.\frac{1}{3z^2 - 5z^4}\right|_{z=-1} = \frac{1}{3-5} = -\frac{1}{2}$$

$$\operatorname*{res}_{z=1} f(z) = \left.\frac{1}{3z^2 - 5z^4}\right|_{z=1} = \frac{1}{3-5} = -\frac{1}{2}$$

$$\operatorname*{res}_{z=0,\ n=3} f(z) = \frac{1}{2}\frac{d^2}{dz^2}\left.\frac{1}{1-z^2}\right|_{z=0} = \frac{1}{2}\cdot 2z\left.\frac{1}{(1-z^2)^2}\right|_{z=0} = 0.$$

$$f(z) = \frac{1}{z^3(z^2-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{(z-1)} + \frac{E}{(z+1)}$$ лист 3

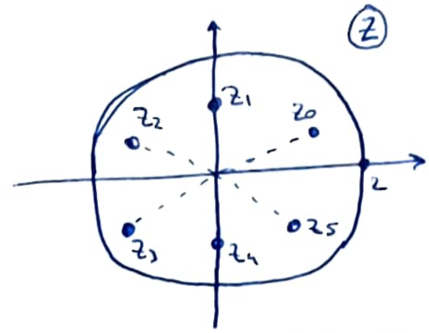
$$\left.\begin{aligned} A &= -1 \\ B &= 0 \\ C &= -1 \\ D &= 1/2 \\ E &= \tfrac{1}{2} \end{aligned}\right\} \Rightarrow f(z) = -\frac{1}{z} + \frac{1}{z^3} + \frac{1}{2}\left(\frac{z+1}{z^2-1} + \frac{z-1}{\cancel{z+1}}\right) \quad \frac{z}{z^2-1}$$

$$\operatorname*{res}_{z=\infty} f(z) = -C_{-1} = 1$$

№ 3.4

$$\oint_C \frac{z^5\,dz}{1+z^6} \qquad C: |z| \leq 2$$

$$\oint = \int_{-\infty}^{\infty} + \int_{C_R}$$



$z^6 + 1 = 0$; $z^6 = -1$

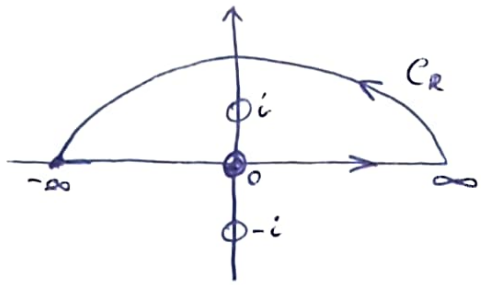
$|z|^6 e^{6i\varphi} = e^{i\pi + 2\pi n} \Rightarrow \varphi = \frac{\pi}{6} + \frac{\pi}{3} n$ при $|z| = 1$

$$\left[\begin{aligned} z_0 &= e^{\frac{i\pi}{6}} = \frac{\sqrt{3}+i}{2} \\ z_1 &= e^{\frac{i\pi}{2}} = i \\ z_2 &= e^{i\frac{5\pi}{6}} = \frac{i-\sqrt{3}}{2} \\ z_3 &= e^{i\frac{7\pi}{6}} = \\ z_4 &= e \quad = -i \\ z_5 &= e^{i\frac{11\pi}{6}} \end{aligned}\right.$$

полюса I порядка

N 3.5

$$\int_{-\infty}^{\infty} \frac{\sin^2 x \, dx}{x^2(x^2+1)}$$



$$f(z) = \frac{\sin^2 z \, dz}{z^2(z^2+1)} = \frac{\sin^2 z \, dz}{z^2(z+1)(z-1)} \qquad \oint = \int_{-\infty}^{\infty} + \cancel{\int_{C_R}}^{\,0}$$

$$z = Re^{i\varphi}$$
$$dz = Re^{i\varphi} i \, d\varphi \qquad f(z\to\infty) = \frac{\sin^2 z}{z^4} \Rightarrow \oint \sim \frac{1}{R^3} \to 0.$$

$$\oint = \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_n \underset{z_n}{\mathrm{res}}\, f(z)$$

$$\left[\begin{array}{l} z = 0 \text{ — полюс II порядка} \\ z = \pm i \text{ — полюс I порядка} \end{array}\right.$$

$$\underset{z=i}{\mathrm{res}} = \frac{\sin^2 z}{4z^3+2z}\Big|_{z=i} = \frac{\sin^2 i}{-4i+2i} = -\frac{\sin^2 i}{2i}$$

$$\underset{\substack{z=0\\ n=2}}{\mathrm{res}} = 1 \cdot \frac{d}{dz} \frac{\sin^2 z}{\cancel{z^2}(z^2+1)} \cdot \cancel{z^2} = \frac{d}{dz}\frac{\sin^2 z}{z^2+1} = \frac{d}{dz}\frac{1-\cos 2z}{2(z^2+1)} =$$

$$= \frac{+\sin 2z(z^2+1) - 2z(1-\cos 2z)}{(z^2+1)^2}\Big|_{z=0} = 0$$

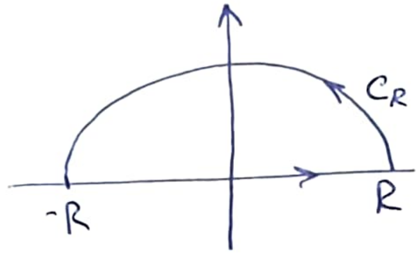
$$\Rightarrow \oint f(z)\,dz = 2\pi i \left[\underset{z=i}{\mathrm{res}}\, f(z) + \underset{z=0}{\mathrm{res}}\, f(z)\right] = -\cancel{2}\pi \cancel{i} \frac{\sin^2 i}{\cancel{2i}} =$$

$$= -\pi \sin^2 i$$

Лист 4

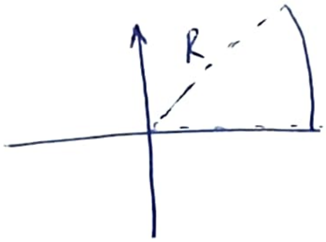
N 3.6

а) $\lim\limits_{R\to\infty}\int\limits_{C_R} e^{iz}dz$



Не существует.

б) $\lim\limits_{R\to\infty}\int\limits_{C_R} e^{iz^2}$



$z = Re^{i\varphi}$ $\quad \varphi\in[0, \pi/4]$

существует

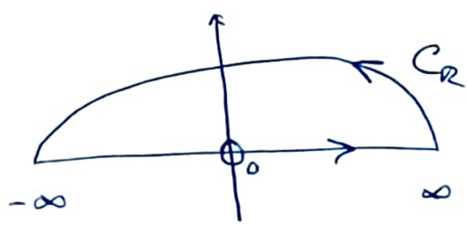
N 3.7

а) $\int\limits_0^\infty \frac{x-\sin x}{x^3}dx = \int\limits_0^\infty \frac{dx}{x^2} - \int\limits_0^\infty \frac{\sin x\, dx}{x^3} = I_1 + I_2 = I$

$I_1 = \int\limits_0^\infty \frac{\sin x\, dx}{x^3} = \frac{1}{2i}\int\limits_0^\infty \frac{e^{ix}}{x^3}dx - \frac{1}{2i}\int\limits_0^\infty \frac{e^{-ix}}{x^3}dx = \left\{ \begin{matrix} x = -z \\ dx = -dz \end{matrix} \right\} =$

$= \frac{1}{2i}\left[\int\limits_0^{+\infty} \frac{e^{+iz}}{z^3}dz - \int\limits_0^{-\infty} \frac{e^{iz}dz}{z^3}\right] = \frac{1}{2i}\int\limits_{-\infty}^{\infty}\frac{e^{iz}dz}{z^3}$

$f(z) = \frac{1}{2i}\frac{e^{iz}}{z^3}$ $\qquad \lambda = 1$



$\oint = \int\limits_{-\infty}^{\infty} + \cancel{\int\limits_{C_R}}^{\to 0} = 2\pi i \sum\limits_{z_k} \operatorname{res}_{z_k} f(z)$

$z = 0$ — полюс $\underline{III}$ порядка

$\Rightarrow \operatorname{res}_{z=0} = \frac{1}{2}\frac{d^2}{dz^2}\frac{1}{2i}\frac{e^{iz}\cancel{z^3}}{\cancel{z^3}} = \frac{1}{4i}\frac{d^2}{dz^2}e^{iz}\Big|_{z=0} = \frac{-1}{4i}$

$I_1 = \int\limits_0^\infty \frac{dx}{x^2} = -\frac{1}{x^2}\Big|_0^\infty$

$I = 2\pi \cancel{i}\cdot -\frac{1}{{}_2 4\cancel{i}} = -\frac{\pi}{2}$