Briefing Document: Numerical Methods

Overall Themes:

- **Solving Equations:** A core focus is on techniques for finding solutions to both linear and non-linear equations, including systems of equations.
- **Approximation:** Numerical methods often rely on approximation techniques when exact solutions are difficult or impossible to obtain. Iterative methods are a common theme.
- **Error Analysis:** Understanding and controlling errors (e.g., round-off error, truncation error) is crucial in numerical methods.
- **Ordinary Differential Equations (ODEs):** Several sources address techniques for solving initial value problems involving ODEs, using methods like Taylor series, Euler's method (and its modifications), and Runge-Kutta methods.
- **Interpolation:** Approximating function values between known data points using various techniques like Newton's interpolation and Lagrange interpolation.
- **Linear Algebra:** Solving systems of linear equations using Gaussian elimination, forward/backward substitution, and addressing the issue of unique/infinite solutions depending on matrix properties.

Key Ideas and Facts from Sources:

1. Non-Linear Equations and Systems:

- **Newton's Method:** This iterative method is used to find roots of non-linear equations or systems of equations. The iteration formula for a system is: " Xk+1 = Xk - [F'(Xk)]-1F(Xk)". An initial guess is required.
- **Example of Newton's Method:** The document provides a concrete example of solving a system of two non-linear equations using Newton's method with an initial guess.
- **Secant Method:** An alternative to Newton's method when the derivative is not available. It approximates the derivative using a difference quotient: "xi+1 = xi-1 - (xi - xi-1) * xf / (xf - xf)". It requires two initial points.
- **Limitation of Newton's Method:** "Problem: Newton’s Methods new estimate is xi+1 = xi - xf / xf not available, if xf = 0."

2. Linear Equations and Systems:

- **Vectors and Matrices:** Introduction of vectors and matrices as fundamental components for solving systems of linear equations. Examples of column and row vectors are provided.
- **Gaussian Elimination:** A method to solve linear systems by transforming the system's matrix into an upper triangular form.
- **Forward Elimination:** Eliminating variables systematically. "To eliminate x2 in equations 2, 3, ... n aij <- aij - ai2/a22 * a2j."
- **Backward Substitution:** After forward elimination, solving for the variables starting from the last equation and working backward.
- **Condition for Unique/Infinite Solutions:** "Equations AX=B Have? No Unique solution Infinitesolution =≠ if det(A)0det(A)0det(A)" This indicates the role of the determinant in determining solution uniqueness.
- **Pivoting:** Strategies (like scaled partial pivoting) to improve the stability and accuracy of Gaussian elimination by selecting the best pivot element (equation) to use for elimination.
- **Tridiagonal Systems:** Efficient methods exist for solving systems with tridiagonal matrices.
- **Gauss-Jordan Method:** An extension of Gaussian elimination that further reduces the matrix to a diagonal form, directly yielding the solution.
- **Remarks for solving Linear Equations:** "We use index vector to avoid the need to move the rows which may not be practical for large problems...Elements in the super diagonal are not affected...Elements in the main diagonal, and B need updating."

3. Curve Fitting and Interpolation:

- **Least Squares Method:** Finding the best fit line (or curve) to a set of data points by minimizing the sum of the squares of the residuals.
- **Normal Equations:** Equations derived from the least squares method to solve for the coefficients of the fitting function (e.g., a line: $y = ax + b$).
- **Newton's Bivariate Interpolation for Equispaced Points:** Focuses on interpolating over two variables x, y using Newton's forward difference method.
- **Lagrange Interpolation:** A method to construct a polynomial that passes through a given set of points.

4. Ordinary Differential Equations (ODEs):

- **Taylor Series Method:** Approximating the solution of an ODE using its Taylor series expansion. The document contains examples on how to apply the Taylor's series method and find solutions at different values of x.
- **Euler's Method:** A first-order numerical method for approximating the solution of an ODE.
- **Modified Euler's Method:** An improvement over the standard Euler's method. An iterative approach to refine the estimate of y using a mean slope.
- **Runge-Kutta Methods:** A family of higher-order methods for solving ODEs. The fourth-order Runge-Kutta method is a popular choice.

5. Problem set snippets

- **Gothenburg Univ., Sweden, BIT 8 (1968), 138) Solution Choose the permutation matrix as:** Demonstrates how to select a permutation matrix and what it does.
- **Lund Univ., Sweden, BIT 27(1987), 285) Solution (a) Using y0 = 1, h = 0.2, we obtain:** Demonstrates an approach to solving problem sets for ODE with example.

6. Additional snippets from sources

- Mention of Galerkin method for a minimum condition.
- Mention of Newton-Raphson method for solving φ(s) = 0 is given by s(k+1) = s(k) – φ φ

Quotes:

- "Newton's Methods new estimate is xi+1 = xi - xf / xf not available, if xf = 0." (Highlighting a limitation)
- "We use index vector to avoid the need to move the rows which may not be practical for large problems." (Practical consideration in linear solvers)

Implications:

These sources provide a foundation for understanding and applying various numerical methods. The methods cover a range of problem types. A recurring theme is the balance between accuracy and computational cost, which influences the choice of method and parameters like step size. The emphasis on error analysis shows the importance of validating the results obtained from numerical approximations.