Topic 2:

Solution of Nonlinear Equations

Lectures 5-11:

Lecture 5

Solution of Nonlinear Equations

(Root Finding Problems)

- Definitions
- Classification of Methods
  - Analytical Solutions
  - Graphical Methods
  - Numerical Methods
    - Bracketing Methods
    - Open Methods
- Convergence Notations

Root Finding Problems

Many problems in Science and Engineering are expressed as:

Given a continuous function *f(x)*,

find the value r such that $f(r) = 0$

These problems are called root finding problems.

Roots of Equations

A number *r* that satisfies an equation is called a root of the equation.

The equation: $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots: $-2,\ 3,\ 3,\ and\ -1$.

i.e., $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x+2)(x-3)^2(x+1)$

The equation has two simple roots (-1 and -2)

and a repeated root (3) with multiplicity $= 2$.

Zeros of a Function

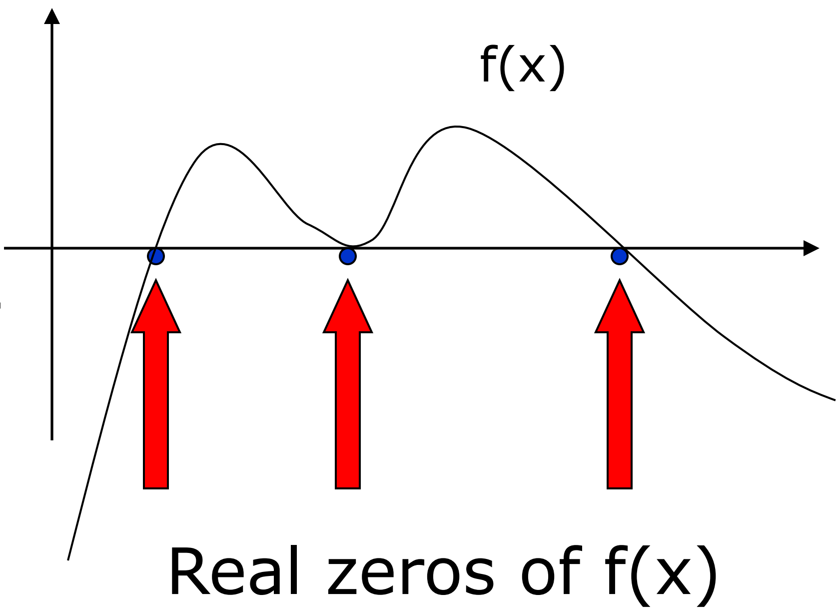
Let $f(x)$ be a real-valued function of a real variable. Any number r for which $f(r)=0$ is called a zero of the function.

Examples:

2 and 3 are zeros of the function $f(x) = (x-2)(x-3)$.

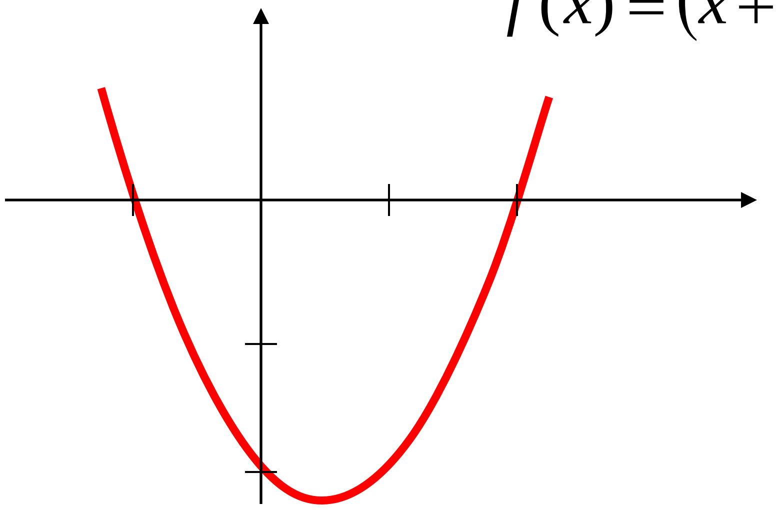
Graphical Interpretation of Zeros

- The real zeros of a function ***f(x)*** are the values of ***x*** at which the graph of the function crosses (or touches) the x-axis.



Simple Zeros

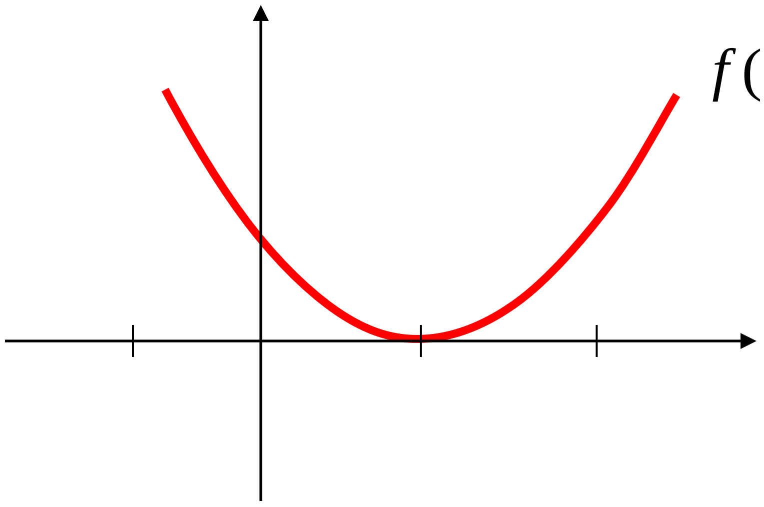
$$f(x) = (x+1)(x-2)$$



$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at $x = 2$ and one at $x = -1$)

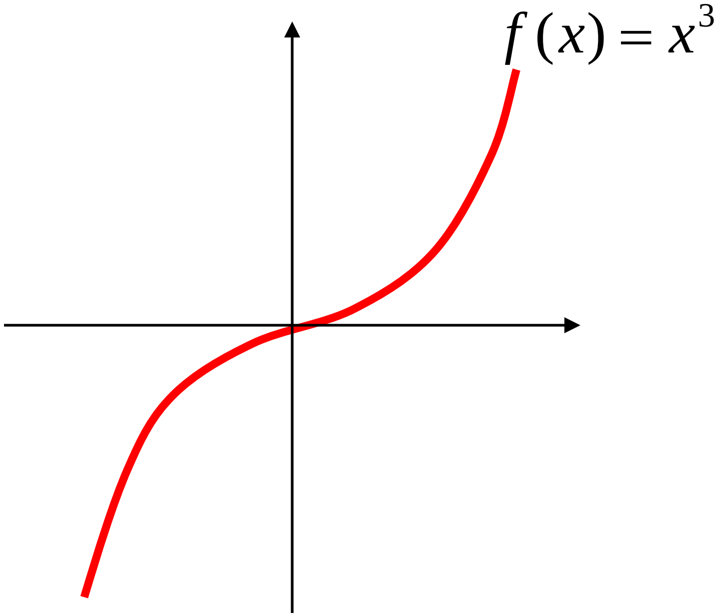
Multiple Zeros



$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with muliplicity $= 2$) at $x = 1$

Multiple Zeros



$$f(x) = x^3$$

has a zero with muliplicity $= 3$ at $x = 0$

Facts

- Any n^{th} order polynomial has exactly n zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at $\boldsymbol{x=r}$ with multiplicity $\boldsymbol{m}$ then the function and its first $\boldsymbol{(m-1)}$ derivatives are zero at $\boldsymbol{x=r}$ *and the* $\boldsymbol{m}^{th}$ *derivative at* $\boldsymbol{r}$ *is not zero*.

Roots of Equations & Zeros of Function

Given the equation:

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation:

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define $f(x)$ as:

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of $f(x)$ are the same as the roots of the equation $f(x) = 0$

(Which are -2, 3, 3, and -1)

Solution Methods

Several ways to solve nonlinear equations are possible:

- Analytical Solutions
  - Possible for special equations only
- Graphical Solutions
  - Useful for providing initial guesses for other methods
- Numerical Solutions
  - Open methods
  - Bracketing methods

Analytical Methods

Analytical Solutions are available for special equations only.

Analytical solution of : $a\,x^2 + b\,x + c = 0$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for: $x - e^{-x} = 0$

Graphical Methods

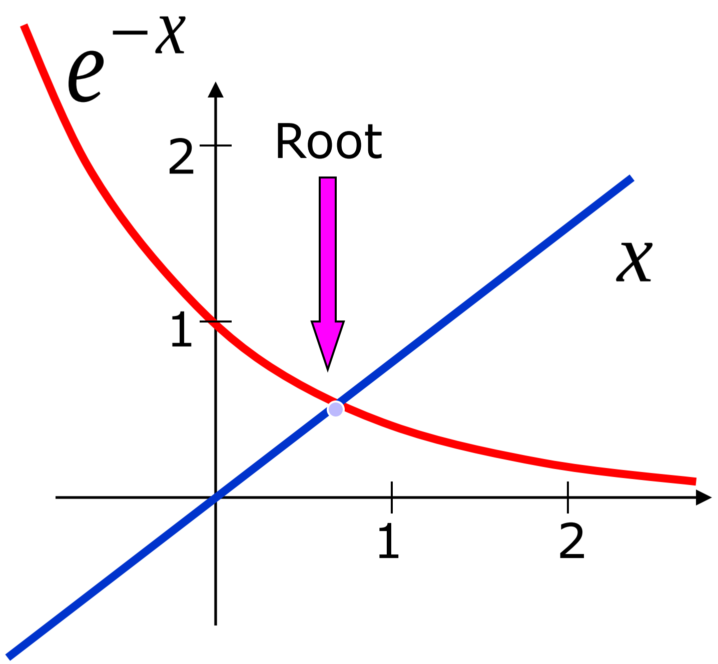
- Graphical methods are useful to provide an initial guess to be used by other methods.

$$Solve$$

$$x = e^{-x}$$

$$The\ \ root \in [0,1]$$

$$root \approx 0.6$$



Numerical Methods

Many methods are available to solve nonlinear equations:

- Bisection Method
- Newton’s Method
- Secant Method

These will be covered in CSC 427

- False position Method
- Muller’s Method
- Bairstow’s Method
- Fixed point iterations
- ……….

Bracketing Methods

- In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.

- Examples of bracketing methods:
  - Bisection method
  - False position method

Open Methods

- In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- Open methods are usually more efficient than bracketing methods.
- They may not converge to a root.

Convergence Notation

A sequence $x_1, x_2, \ldots, x_n, \ldots$ is said to **converge** to x if to every $\varepsilon > 0$ there exists N such that:

$$\left| x_n - x \right| < \varepsilon \quad \forall n > N$$

Convergence Notation

Let $x_1, x_2, \ldots.,$ converge to x.

Linear Convergence : $$\frac{|x_{n+1} - x|}{|x_n - x|} \le C$$

Quadratic Convergence : $$\frac{|x_{n+1} - x|}{|x_n - x|^2} \le C$$

Convergence of order P : $$\frac{|x_{n+1} - x|}{|x_n - x|^p} \le C$$

Speed of Convergence

- We can compare different methods in terms of their convergence rate.
- Quadratic convergence is faster than linear convergence.
- A method with convergence order q converges faster than a method with convergence order p if $q>p$.
- Methods of convergence order $p>1$ are said to have super linear convergence.

Lectures 6-7
Bisection Method

- The Bisection Algorithm
- Convergence Analysis of Bisection Method
- Examples

Introduction

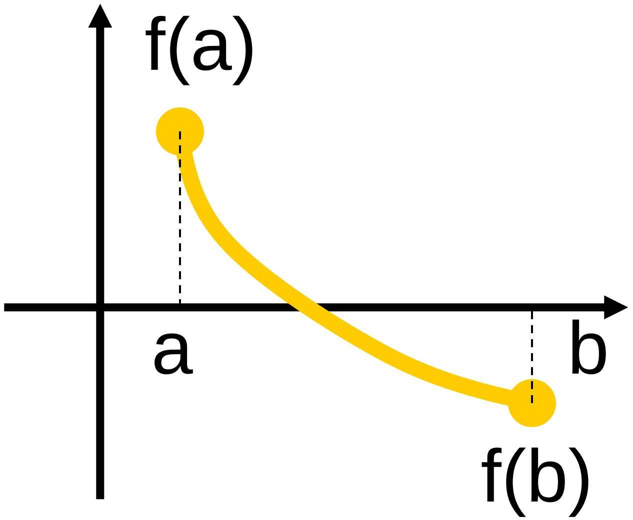
- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

Intermediate Value Theorem

- Let f(x) be defined on the interval [a,b].

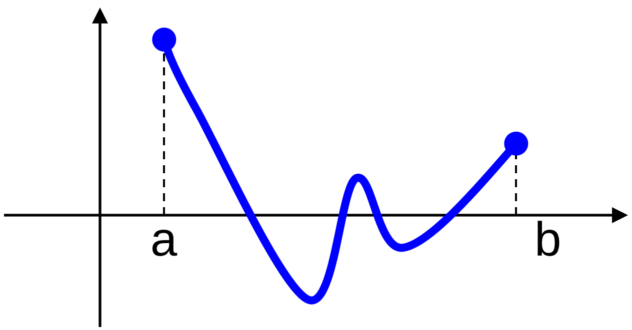
- Intermediate value theorem:

  if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].



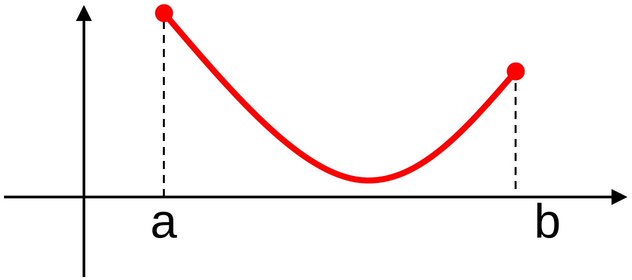
Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].



The function has four real zeros

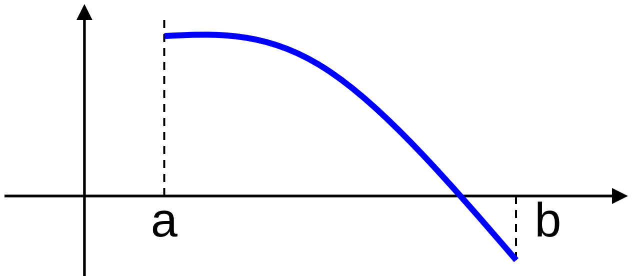
- Bisection method can not be used in these cases.



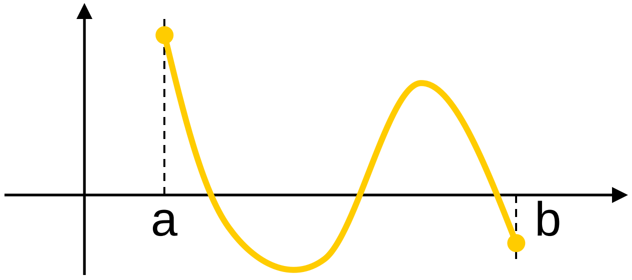
The function has no real zeros

Two More Examples

- If f(a) and f(b) have different signs, the function has at least one real zero.
- Bisection method can be used to find one of the zeros.



The function has one real zero



The function has three real zeros

Bisection Method

- If the function is continuous on [a,b] and f(a) and f(b) have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.

- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

Bisection Method

Assumptions:

Given an interval [a,b]

f(x) is continuous on [a,b]

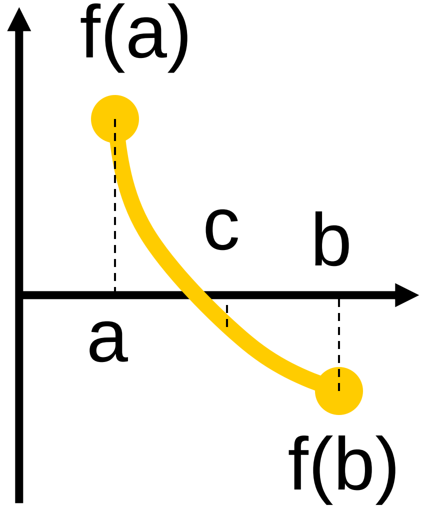
f(a) and f(b) have opposite signs.

These assumptions ensure the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm

Assumptions:

- f(x) is continuous on [a,b]
- f(a) f(b) < 0



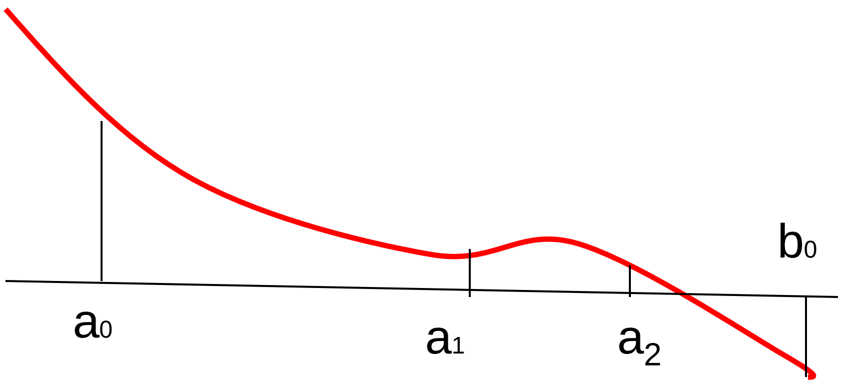
Algorithm:

Loop

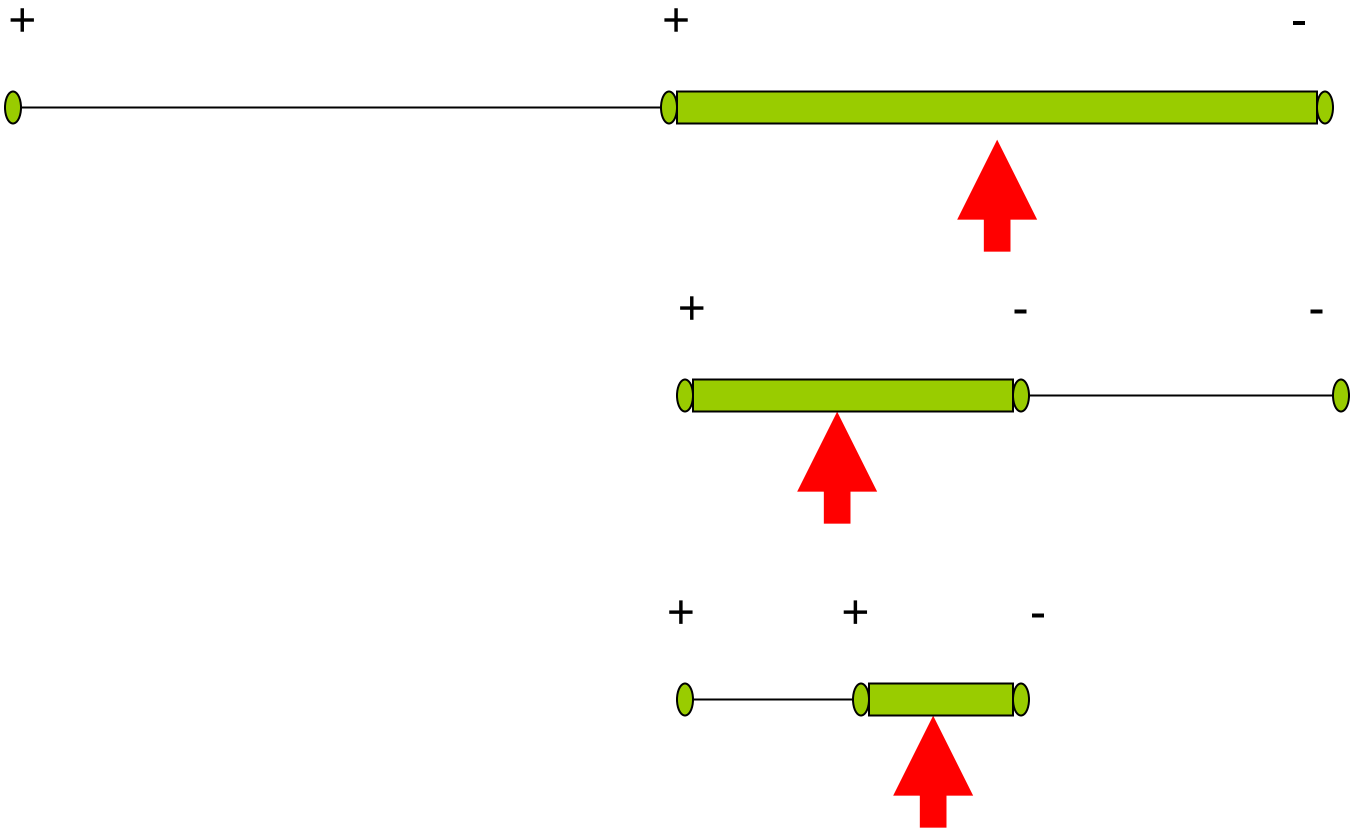
1. Compute the mid point c=(a+b)/2
2. Evaluate f(c)
3. If   f(a) f(c) < 0 then new interval [a, c]
   If   f(a) f(c) > 0 then new interval [c, b]

End loop

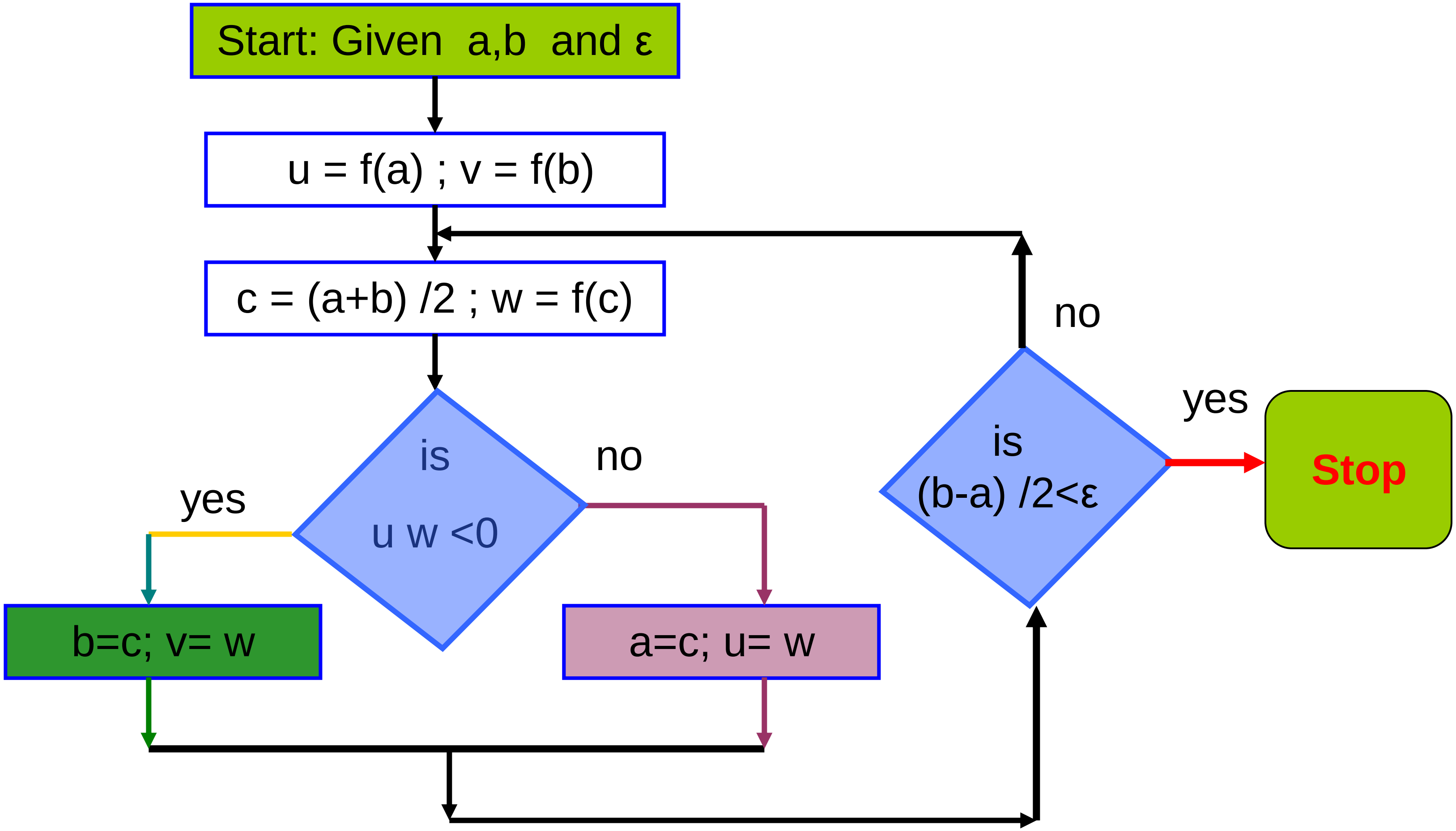
Bisection Method



Example



Flow Chart of Bisection Method



Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,2]?

Answer:

$f(x)$ is continuous on $[0,2]$

and $f(0) * f(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

Example

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,1]$?

Answer:

$f(x)$ is continuous on $[0,1]$

and $\mathrm{f}(0) * \mathrm{f}(1) = (1)(-1) = -1 < 0$

$\Rightarrow$ Assumptions are satisfied

$\Rightarrow$ Bisection method can be used

Best Estimate and Error Level

Bisection method obtains an interval that is guaranteed to contain a zero of the function.

Questions:

- What is the best estimate of the zero of ***f(x)***?
- What is the error level in the obtained estimate?

Best Estimate and Error Level

The best estimate of the zero of the function ***f(x)*** after the first iteration of the Bisection method is the mid point of the initial interval:

$$Estimate\ of\ the\ zero:\ \ r = \frac{b+a}{2}$$

$$Error \le \frac{b-a}{2}$$

Stopping Criteria

Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

Stopping Criteria

c_n : is the midpoint of the interval at the n$^{\text{th}}$ iteration

(c_n is usually used as the estimate of the root).

r : is the zero of the function.

After n iterations:

$$|error| = |r - c_n| \le E_a^n = \frac{b-a}{2^n} = \frac{\Delta x^0}{2^n}$$

Convergence Analysis

$Given \quad f(x),\ a,\ b,\ and\ \ \varepsilon$

How many iterations are needed such that : $|x - r| \leq \varepsilon$

where *r* is the zero of *f(x)* and *x* is the bisection estimate (i.e., $x = c_k$)?

$$\boxed{n \geq \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}}$$

Convergence Analysis – Alternative Form

$$n \geq \frac{\log(b-a) - \log(\varepsilon)}{\log(2)}$$

$$n \geq \log_2\left(\frac{\text{width of initial interval}}{\text{desired error}}\right) = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

Example

$a = 6,\ b = 7,\ \varepsilon = 0.0005$

How many iterations are needed such that : $|x - r| \leq \varepsilon$?

$$n \geq \frac{\log(b-a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$$\Rightarrow n \geq 11$$

Example

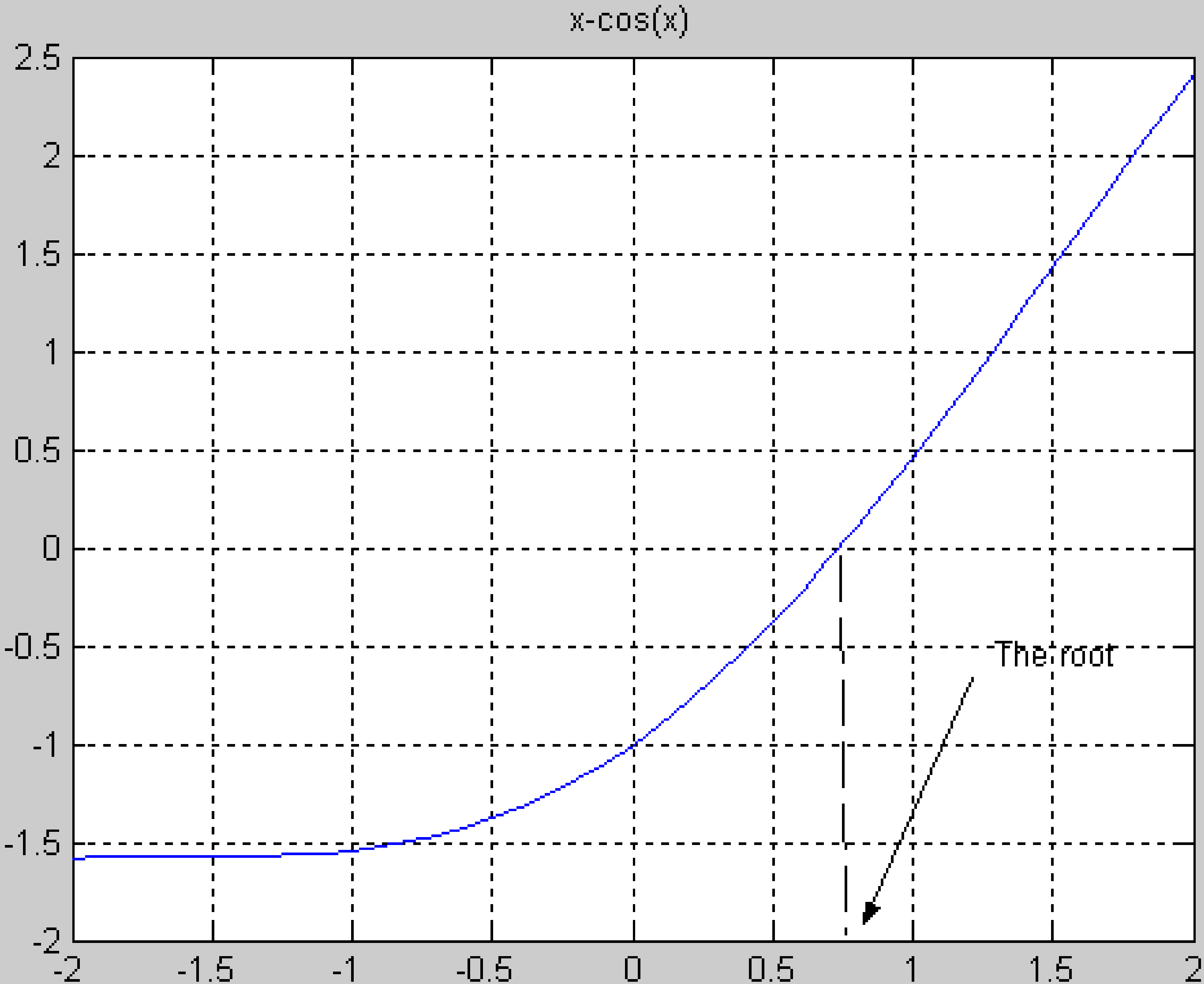
- Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error <0.02 (assume the initial interval [0.5, 0.9])

Question 1: What is $f(x)$?

Question 2: Are the assumptions satisfied ?

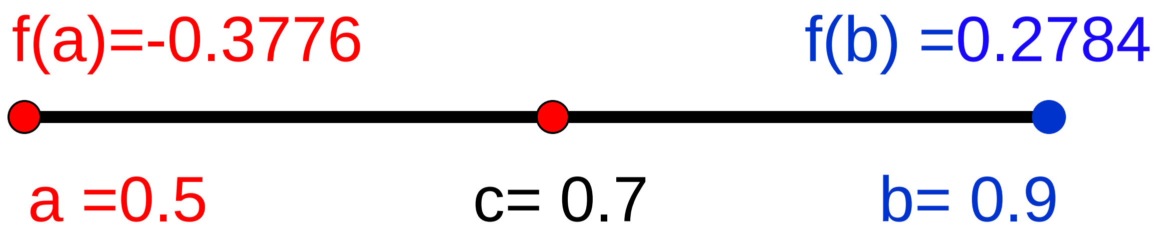
Question 3: How many iterations are needed ?

Question 4: How to compute the new estimate ?


x-cos(x)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
The root

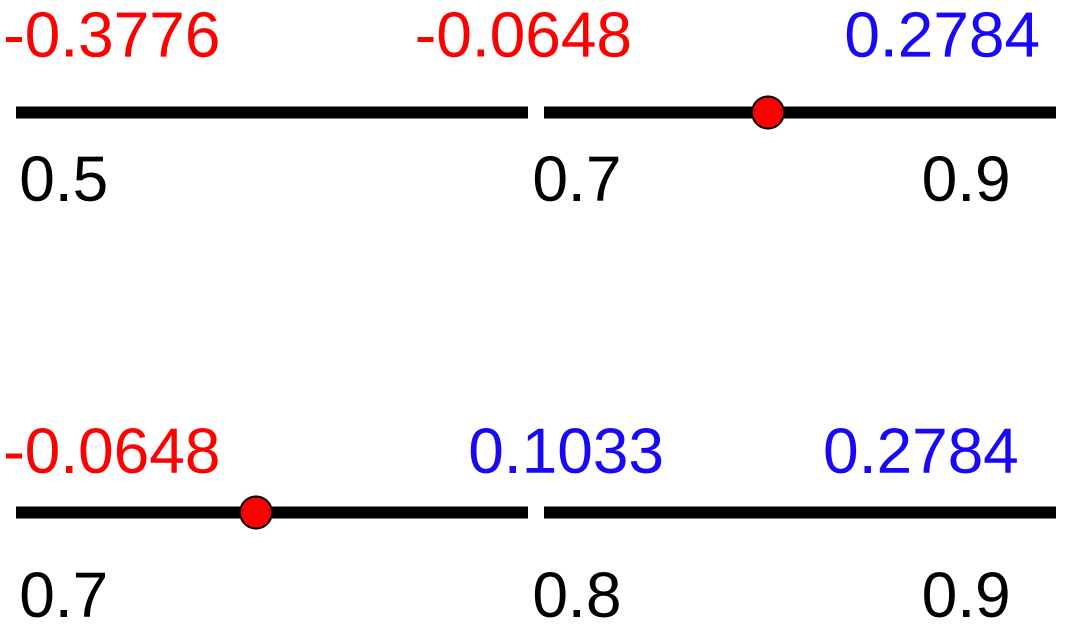
Bisection Method

Initial Interval



Error < 0.2

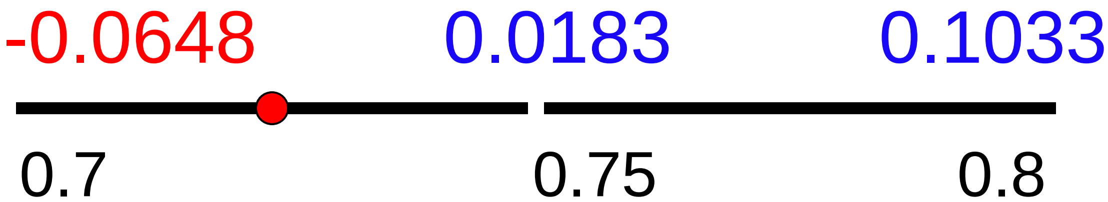
Bisection Method



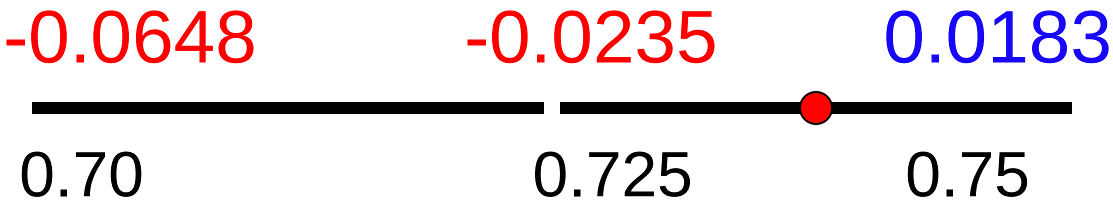
Error < 0.1

Error < 0.05

Bisection Method



Error < 0.025



Error < .0125

Summary

- Initial interval containing the root: [0.5,0.9]

- After 5 iterations:
  - Interval containing the root: [0.725, 0.75]
  - Best estimate of the root is 0.7375
  - | Error | < 0.0125

A Matlab Program of Bisection Method

```
a=.5; b=.9;
u=a-cos(a);
v=b-cos(b);
    for i=1:5
        c=(a+b)/2
        fc=c-cos(c)
        if u*fc<0
           b=c ; v=fc;
        else
           a=c; u=fc;
        end
    end
```

```
c =
    0.7000
fc =
   -0.0648
c =
    0.8000
fc =
    0.1033
c =
    0.7500
fc =
    0.0183
c =
    0.7250
fc =
  -0.0235
```

Example

Find the root of:

$$f(x) = x^3 - 3x + 1 \text{ in the interval} : [0,1]$$

$* \quad f(x) \text{ is continuous}$

$* \quad f(0) = 1, \ f(1) = -1 \Rightarrow f(a)\ f(b) < 0$

$\Rightarrow$ Bisection method can be used to find the root

Example

Iteration	a	b	c= $\frac{(a+b)}{2}$	f(c)	$\frac{(b-a)}{2}$
1	0	1	0.5	-0.375	0.5
2	0	0.5	0.25	0.266	0.25
3	0.25	0.5	.375	-7.23E-3	0.125
4	0.25	0.375	0.3125	9.30E-2	0.0625
5	0.3125	0.375	0.34375	9.37E-3	0.03125

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded

Lecture 8-9
Newton-Raphson Method

- Assumptions
- Interpretation
- Examples
- Convergence Analysis

Newton-Raphson Method

(Also known as Newton's Method)

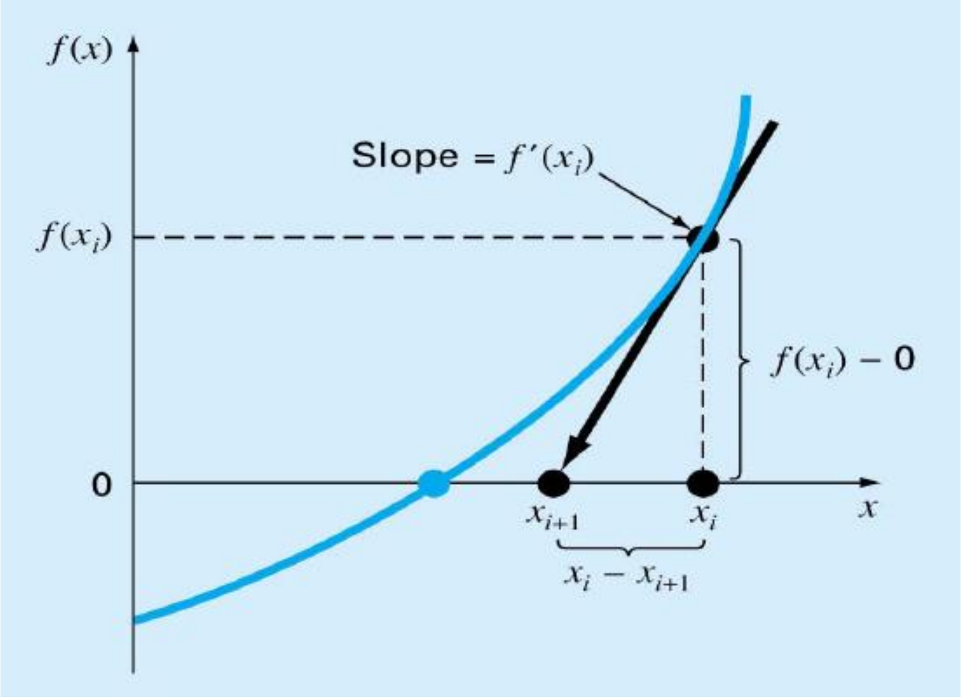
Given an initial guess of the root x_0, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

Assumptions:

- $f(x)$ is continuous and the first derivative is known
- An initial guess x_0 such that $f'(x_0) \neq 0$ is given

Newton Raphson Method
- Graphical Depiction -

- If the initial guess at the root is x_i, then a tangent to the function of x_i that is $f'(x_i)$ is extrapolated down to the *x-axis* to provide an estimate of the root at $x_i + 1$.



Derivation of Newton's Method

Given: x_i an initial guess of the root of $f(x)=0$

Question: How do we obtain a better estimate x_{i+1}?

Taylor Therorem: $f(x+h) \approx f(x) + f'(x)h$

Find h such that $f(x+h)=0$.

$$\Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

Newton – Raphson Formula

A new guess of the root: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$

$$Assumption \quad f'(x_0) \neq 0$$

____

$$for \quad i = 0{:}n$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$end$$

```
C     FORTRAN PROGRAM
      F(X) = X**3-3*X**2+1
      FP(X) = 3*X**2-6*X
      X = 4
      DO 10 I = 1,5
       X = X - F(X)/FP(X)
       PRINT *, X
10     CONTINUE
      STOP
      END
```

Newton's Method

$$Given \quad f(x),\ f'(x),\ x_0$$

$$Assumption \quad f'(x_0) \neq 0$$

$$for \quad i = 0:n$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$end$$

F.m

```
function [F] = F(X)
    F = X^3 - 3*X^2 + 1
```

FP.m

```
function [FP] = FP(X)
    FP = 3*X^2 - 6*X
```

```
% MATLAB PROGRAM
X = 4
for i = 1:5
    X = X - F(X)/FP(X)
end
```

Example

Find a zero of the function $f(x) = x^3 - 2x^2 + x - 3$, $x_0 = 4$

$$f'(x) = 3x^2 - 4x + 1$$

Iteration 1: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 - \frac{33}{33} = 3$

Iteration 2: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{9}{16} = 2.4375$

Iteration 3: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.4375 - \frac{2.0369}{9.0742} = 2.2130$

Example

k (Iteration)	x_k	$f(x_k)$	$f'(x_k)$	x_{k+1}	$\lvert x_{k+1} - x_k \rvert$
0	**4**	**33**	**33**	**3**	1
1	**3**	**9**	**16**	**2.4375**	0.5625
2	**2.4375**	**2.0369**	**9.0742**	**2.2130**	0.2245
3	**2.2130**	**0.2564**	**6.8404**	**2.1756**	0.0384
4	**2.1756**	**0.0065**	**6.4969**	**2.1746**	0.0010

Convergence Analysis

Theorem :

Let $f(x), f'(x)$ *and* $f''(x)$ be continuous at $x \approx r$

where $f(r) = 0$. *If* $f'(r) \neq 0$ then there exists $\delta > 0$

such that $|x_0 - r| \leq \delta \Rightarrow \dfrac{|x_{k+1} - r|}{|x_k - r|^2} \leq C$

$$C = \frac{1}{2} \frac{\displaystyle\max_{|x_0 - r| \leq \delta} |f''(x)|}{\displaystyle\min_{|x_0 - r| \leq \delta} |f'(x)|}$$

Convergence Analysis
Remarks

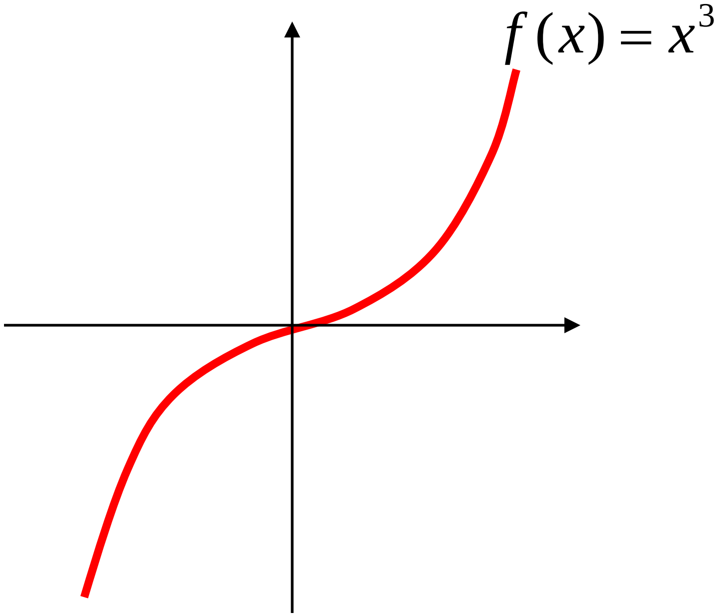
When the guess is close enough to a simple root of the function then Newton's method is guaranteed to converge quadratically.

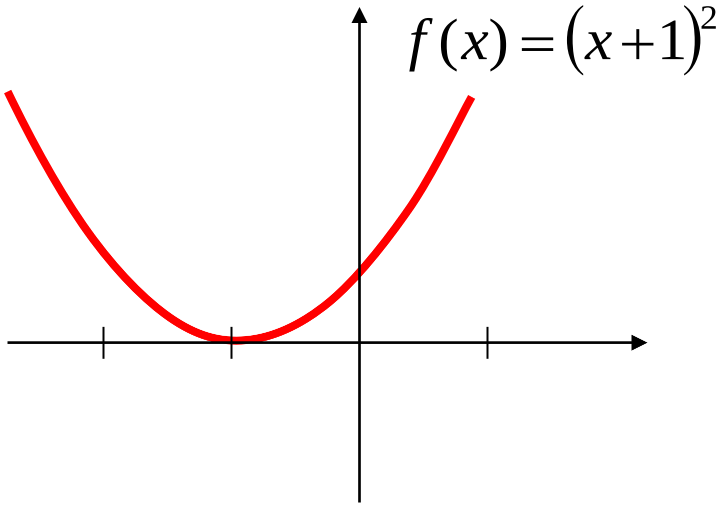
Quadratic convergence means that the number of correct digits is nearly doubled at each iteration.

Problems with Newton's Method

- If the initial guess of the root is far from the root the method may not converge.
- Newton's method converges linearly near multiple zeros { $f(r) = f'(r) = 0$ }. In such a case, modified algorithms can be used to regain the quadratic convergence.

Multiple Roots

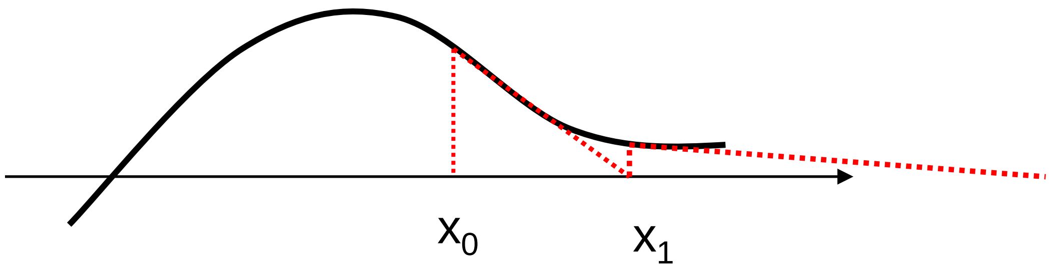




$f(x)$ has three zeros at $x = 0$

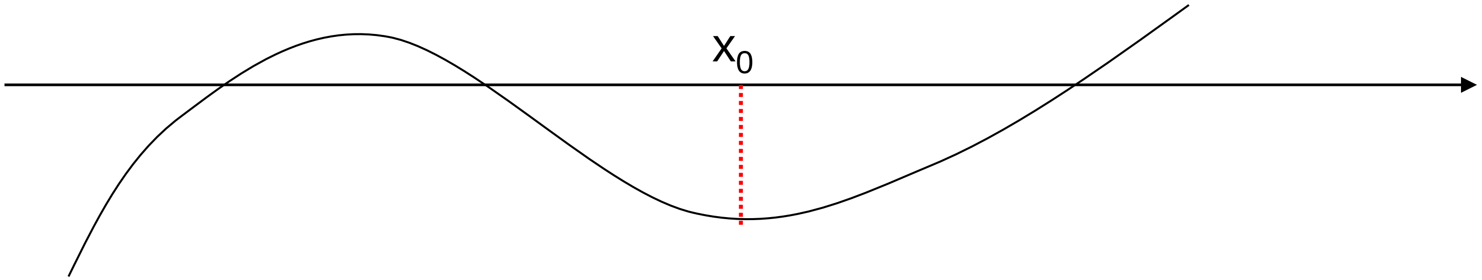
$f(x)$ has two zeros at $x = -1$

Problems with Newton's Method
- Runaway -



The estimates of the root is going away from the root.

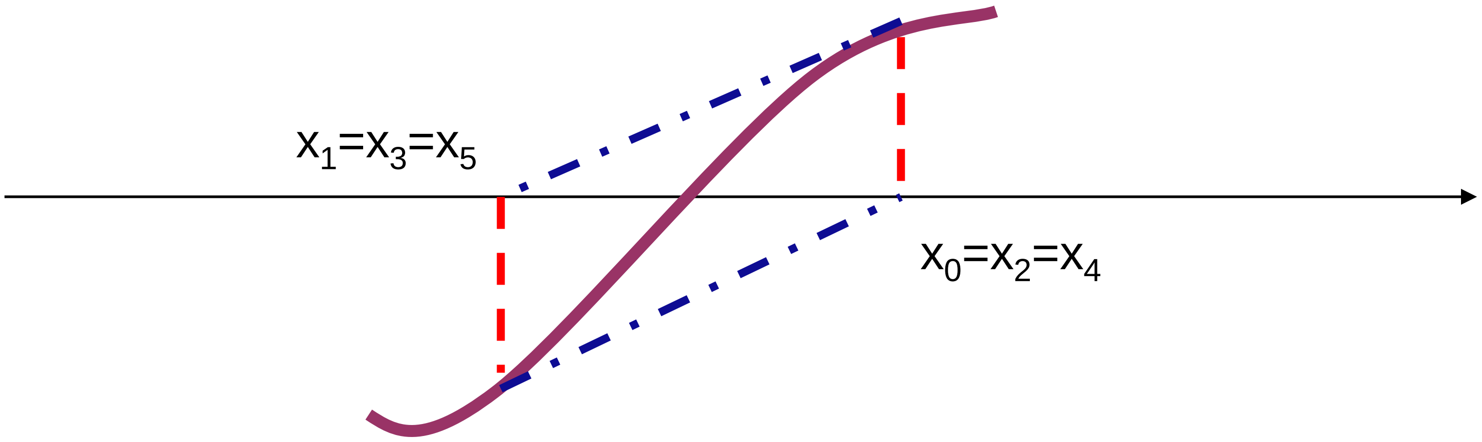
Problems with Newton's Method
- Flat Spot -



The value of ***f'(x)*** is zero, the algorithm fails.

If ***f '(x)*** is very small then $\boldsymbol{x_1}$ will be very far from $\boldsymbol{x_0}$.

Problems with Newton's Method
- Cycle -



The algorithm cycles between two values x_0 and x_1

Newton's Method for Systems of Non Linear Equations

Given: X_0 an initial guess of the root of $F(x)=0$

Newton's Iteration

$$X_{k+1} = X_k - \left[F'(X_k)\right]^{-1} F(X_k)$$

$$F(X) = \begin{bmatrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \\ \vdots \end{bmatrix}, \quad F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \\ \vdots & & \end{bmatrix}$$

Example

- Solve the following system of equations:

$$y + x^2 - 0.5 - x = 0$$

$$x^2 - 5xy - y = 0$$

$$\text{Initial guess } \; x = 1, \;\; y = 0$$

$$F = \begin{bmatrix} y + x^2 - 0.5 - x \\ x^2 - 5xy - y \end{bmatrix}, \;\; F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x - 5y & -5x - 1 \end{bmatrix}, \; X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution Using Newton's Method

Iteration1:

$$F=\begin{bmatrix} y+x^2-0.5-x \\ x^2-5xy-y \end{bmatrix}=\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}=,\quad F'=\begin{bmatrix} 2x-1 & 1 \\ 2x-5y & -5x-1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}$$

$$X_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}-\begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix}^{-1}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}=\begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}$$

Iteration 2:

$$F=\begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}=,\quad F'=\begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}$$

$$X_2=\begin{bmatrix} 1.25 \\ 0.25 \end{bmatrix}-\begin{bmatrix} 1.5 & 1 \\ 1.25 & -7.25 \end{bmatrix}^{-1}\begin{bmatrix} 0.0625 \\ -0.25 \end{bmatrix}=\begin{bmatrix} 1.2332 \\ 0.2126 \end{bmatrix}$$

Example

Try this

- Solve the following system of equations:

$$y + x^2 - 1 - x = 0$$

$$x^2 - 2y^2 - y = 0$$

$$\text{Initial guess } \; x = 0, \;\; y = 0$$

$$F = \begin{bmatrix} y + x^2 - 1 - x \\ x^2 - 2y^2 - y \end{bmatrix}, \; F' = \begin{bmatrix} 2x - 1 & 1 \\ 2x & -4y - 1 \end{bmatrix}, \; X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

Solution

$Iteration$	0	1	2	3	4	5
X_k	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} -0.5287 \\ 0.1969 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$	$\begin{bmatrix} -0.5257 \\ 0.1980 \end{bmatrix}$

Lectures 10

Secant Method

- Secant Method
- Examples
- Convergence Analysis

Newton's Method (Review)

$$Assumptions: \quad f(x),\ f'(x),\ x_0 \quad are\ available,$$

$$f'(x_0) \neq 0$$

$$Newton's\ Method\ new\ estimate:$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem:

$f'(x_i)$ is not available,

or difficult to obtain analytically.

Secant Method

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

if x_i *and* x_{i-1} are two initial points:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})}} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

Secant Method

$$\text{Assumptions:}$$

$$\text{Two initial points } x_i \text{ and } x_{i-1}$$

$$\textit{such that } f(x_i) \neq f(x_{i-1})$$

$$\text{New estimate (Secant Method):}$$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method

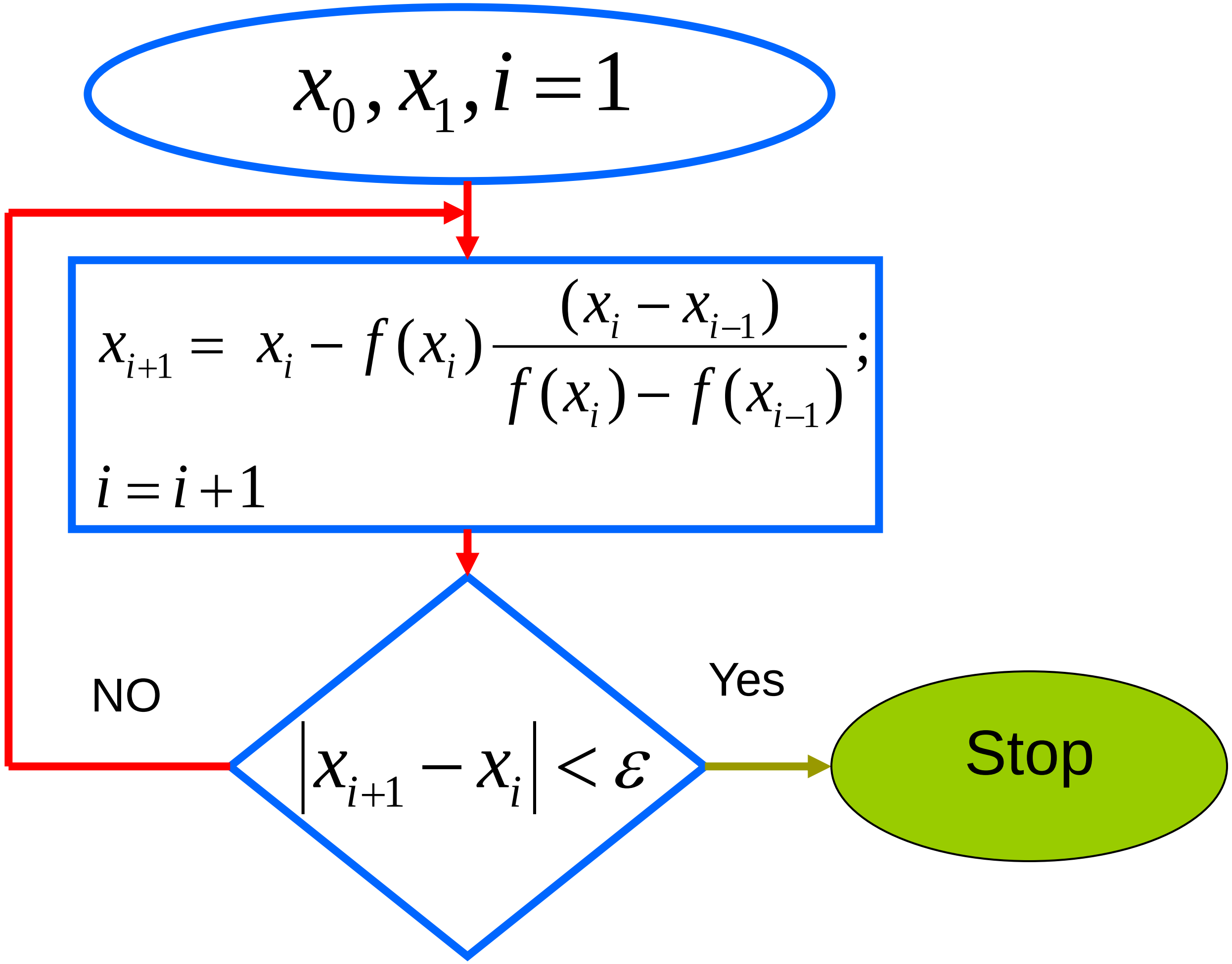
$$f(x) = x^2 - 2x + 0.5$$

$$x_0 = 0$$

$$x_1 = 1$$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method - Flowchart



Modified Secant Method

In this modified Secant method, only one initial guess is needed:

$$f'(x_i) \approx \frac{f(x_i + \delta_i x_i) - f(x_i)}{\delta_i x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_i + \delta_i x_i) - f(x_i)}{\delta_i x_i}} = \quad x_i - \frac{\delta_i x_i f(x_i)}{f(x_i + \delta_i x_i) - f(x_i)}$$

Problem: How to select δ_i ?

If not selected properly, the method may diverge.

Example

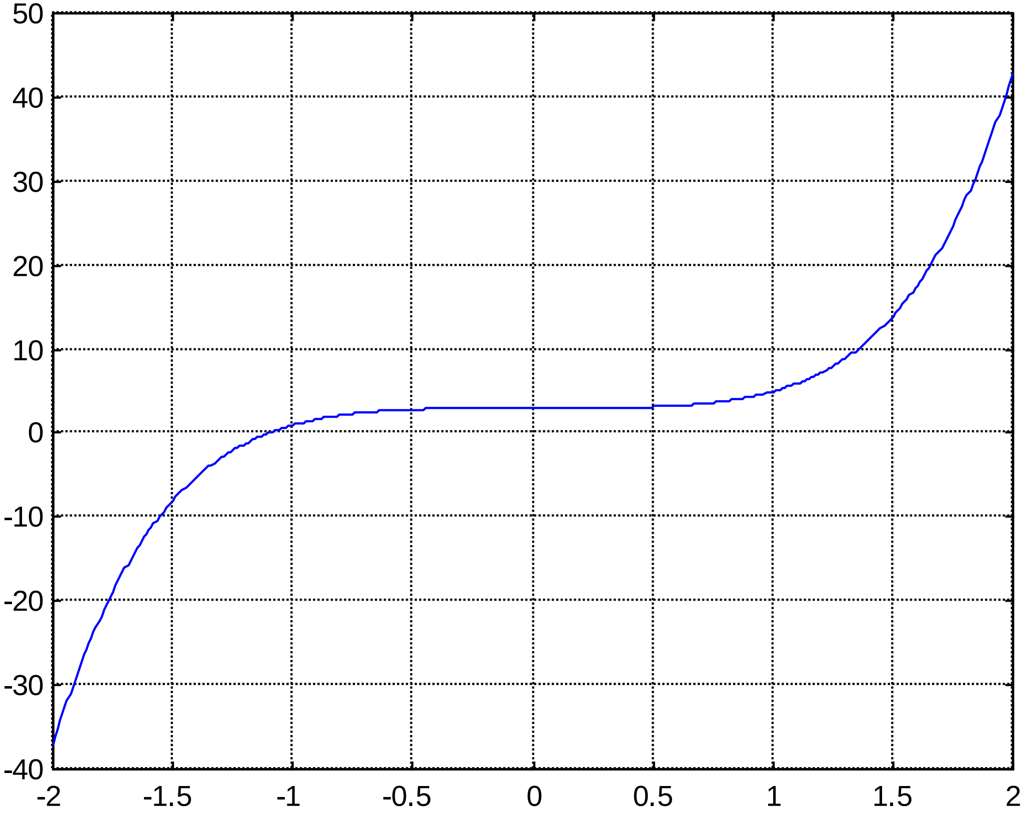
Find the roots of :

$$f(x) = x^5 + x^3 + 3$$

Initial points

$$x_0 = -1 \; and \; x_1 = -1.1$$

$with \;\; error < 0.001$



Example

x(i)	f(x(i))	x(i+1)	|x(i+1)-x(i)|
-1.0000	1.0000	-1.1000	0.1000
-1.1000	0.0585	-1.1062	0. 0062
-1.1062	0.0102	-1.1052	0.0009
-1.1052	0.0001	-1.1052	0.0000

Convergence Analysis

- The rate of convergence of the Secant method is super linear:

$$\frac{|x_{i+1}-r|}{|x_i - r|^{\alpha}} \leq C, \qquad \alpha \approx 1.62$$

$$r : root \qquad x_i : \text{estimate of the root at the i}^{\text{th}} \text{ iteration.}$$

- It is better than Bisection method but not as good as Newton's method.

Lectures 11

Comparison of Root Finding Methods

- Advantages/disadvantages
- Examples

Summary

Method	Pros	Cons
Bisection	- Easy, Reliable, Convergent - One function evaluation per iteration - No knowledge of derivative is needed	- Slow - Needs an interval [a,b] containing the root, i.e., $f(a)f(b)<0$
Newton	- Fast (if near the root) - Two function evaluations per iteration	- May diverge - Needs derivative and an initial guess x_0 such that $f'(x_0)$ is nonzero
Secant	- Fast (slower than Newton) - One function evaluation per iteration - No knowledge of derivative is needed	- May diverge - Needs two initial points guess x_0, x_1 such that $f(x_0)- f(x_1)$ is nonzero

Example

Use Secant method to find the root of :

$$f(x) = x^6 - x - 1$$

Two initial points $x_0 = 1$ *and* $x_1 = 1.5$

$$x_{i+1} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Solution

k	x_k	$f(x_k)$
0	1.0000	-1.0000
1	1.5000	8.8906
2	1.0506	-0.7062
3	1.0836	-0.4645
4	1.1472	0.1321
5	1.1331	-0.0165
6	1.1347	-0.0005

Example

Use Newton's Method to find a root of:

$$f(x) = x^3 - x - 1$$

Use the initial point: $x_0 = 1$.

Stop after three iterations, or

if $\left| x_{k+1} - x_k \right| < 0.001$, or

if $\left| f(x_k) \right| < 0.0001$.

Five Iterations of the Solution

k	x_k	$f(x_k)$	$f'(x_k)$	ERROR
0	1.0000	-1.0000	2.0000	
1	1.5000	0.8750	5.7500	0.1522
2	1.3478	0.1007	4.4499	0.0226
3	1.3252	0.0021	4.2685	0.0005
4	1.3247	0.0000	4.2646	0.0000
5	1.3247	0.0000	4.2646	0.0000

Example

Use Newton's Method to find a root of:

$$f(x) = e^{-x} - x$$

Use the initial point: $x_0 = 1$.

Stop after three iterations, or

if $\left|x_{k+1} - x_k\right| < 0.001$, or

if $\left|f(x_k)\right| < 0.0001$.

Example

Use Newton's Method to find a root of:

$$f(x) = e^{-x} - x, \qquad f'(x) = -e^{-x} - 1$$

x_k	$f(x_k)$	$f'(x_k)$	$\dfrac{f(x_k)}{f'(x_k)}$
1.0000	-0.6321	-1.3679	0.4621
0.5379	0.0461	-1.5840	-0.0291
0.5670	0.0002	-1.5672	-0.0002
0.5671	0.0000	-1.5671	-0.0000

Example

Estimates of the root of: x-cos(x)=0.

0.60000000000000	Initial guess
0.74401731944598	1 correct digit
0.73909047688624	4 correct digits
0.73908513322147	10 correct digits
0.73908513321516	14 correct digits

Example

In estimating the root of: ***x-cos(x)=0***, to get more than 13 correct digits:

- 4 iterations of Newton ($x_0=0.8$)
- 43 iterations of Bisection method (initial interval [0.6, 0.8])
- 5 iterations of Secant method ($x_0=0.6$, $x_1=0.8$)