CSC 432 Principles of Programming Languages II

https://runestone.academy/runestone/books/published/pythonds/BasicDS/InfixPrefixandPostfixExpressions.html

Expression Notations

Infix, Prefix and Postfix Expressions

When you write an arithmetic expression such as B * C, the form of the expression provides you with information so that you can interpret it correctly. In this case we know that the variable B is being multiplied by the variable C since the multiplication operator * appears between them in the expression. This type of notation is referred to as **infix** since the operator is *in between* the two operands that it is working on.

Consider another infix example, A + B * C. The operators + and * still appear between the operands, but there is a problem. Which operands do they work on? Does the + work on A and B or does the * take B and C? The expression seems ambiguous.

In fact, you have been reading and writing these types of expressions for a long time and they do not cause you any problem. The reason for this is that you know something about the operators + and *. Each operator has a **precedence** level. Operators of higher precedence are used before operators of lower precedence. The only thing that can change that order is the presence of parentheses. The precedence order for arithmetic operators places multiplication and division above addition and subtraction. If two operators of equal precedence appear, then a left-to-right ordering or associativity is used.

Let's interpret the troublesome expression A + B * C using operator precedence. B and C are multiplied first, and A is then added to that result. (A + B) * C would force the addition of A and B to be done first before the multiplication. In expression A + B + C, by precedence (via associativity), the leftmost + would be done first.

Although all this may be obvious to you, remember that computers need to know exactly what operators to perform and in what order. One way to write an expression that guarantees there will be no confusion with respect to the order of operations is to create what is called a **fully**

parenthesized expression. This type of expression uses one pair of parentheses for each operator. The parentheses dictate the order of operations; there is no ambiguity. There is also no need to remember any precedence rules.

The expression A + B * C + D can be rewritten as ((A + (B * C)) + D) to show that the multiplication happens first, followed by the leftmost addition. A + B + C + D can be written as (((A + B) + C) + D) since the addition operations associate from left to right.

There are two other very important expression formats that may not seem obvious to you at first. Consider the infix expression A + B. What would happen if we moved the operator before the two operands? The resulting expression would be + A B. Likewise, we could move the operator to the end. We would get A B +. These look a bit strange.

These changes to the position of the operator with respect to the operands create two new expression formats, **prefix** and **postfix**. Prefix expression notation requires that all operators precede the two operands that they work on. Postfix, on the other hand, requires that its operators come after the corresponding operands. A few more examples should help to make this a bit clearer (see Table 2).

A + B * C would be written as + A * B C in prefix. The multiplication operator comes immediately before the operands B and C, denoting that * has precedence over +. The addition operator then appears before the A and the result of the multiplication.

In postfix, the expression would be A B C * +. Again, the order of operations is preserved since the * appears immediately after the B and the C, denoting that * has precedence, with + coming after. Although the operators moved and now appear either before or after their respective operands, the order of the operands stayed exactly the same relative to one another.

Table 2: Examples of Infix, Prefix, and Postfix

Infix Expression	Prefix Expression	Postfix Expression
A + B	+ A B	A B +
A + B * C	+ A * B C	A B C * +

Now consider the infix expression (A + B) * C. Recall that in this case, infix requires the parentheses to force the performance of the addition before the multiplication. However, when A + B was written in prefix, the addition operator was simply moved before the operands, + A B. The result of this operation becomes the first operand for the multiplication. The multiplication operator is moved in front of the entire expression, giving us * + A B C. Likewise, in postfix A B + forces the addition to happen first. The multiplication can be done to that result and the remaining operand C. The proper postfix expression is then A B + C *.

Consider these three expressions again (see Table 3). Something very important has happened. Where did the parentheses go? Why don't we need them in prefix and postfix? The answer is that the operators are no longer ambiguous with respect to the operands that they work on. Only infix notation requires the additional symbols. The order of operations within prefix and postfix expressions is completely determined by the position of the operator and nothing else. In many ways, this makes infix the least desirable notation to use.

Table 3: An Expression with Parentheses

Infix Expression	Prefix Expression	Postfix Expression
(A + B) * C	* + A B C	A B + C *

Table 4 shows some additional examples of infix expressions and the equivalent prefix and postfix expressions. Be sure that you understand how they are equivalent in terms of the order of the operations being performed.

Table 4: Additional Examples of Infix, Prefix, and Postfix

Infix Expression	Prefix Expression	Postfix Expression
A + B * C + D	+ + A * B C D	A B C * + D +
(A + B) * (C + D)	* + A B + C D	A B + C D + *

Table 4: Additional Examples of Infix, Prefix, and Postfix

Infix Expression	Prefix Expression	Postfix Expression
A * B + C * D	+ * A B * C D	A B * C D * +
A + B + C + D	+ + + A B C D	A B + C + D +

Conversion of Infix Expressions to Prefix and Postfix

So far, we have used ad hoc methods to convert between infix expressions and the equivalent prefix and postfix expression notations. As you might expect, there are algorithmic ways to perform the conversion that allow any expression of any complexity to be correctly transformed.

The first technique that we will consider uses the notion of a fully parenthesized expression that was discussed earlier. Recall that A + B * C can be written as (A + (B * C)) to show explicitly that the multiplication has precedence over the addition. On closer observation, however, you can see that each parenthesis pair also denotes the beginning and the end of an operand pair with the corresponding operator in the middle.

Look at the right parenthesis in the subexpression (B * C) above. If we were to move the multiplication symbol to that position and remove the matching left parenthesis, giving us B C *, we would in effect have converted the subexpression to postfix notation. If the addition operator were also moved to its corresponding right parenthesis position and the matching left parenthesis were removed, the complete postfix expression would result (see Figure 6).

Figure 6: Moving Operators to the Right for Postfix Notation

If we do the same thing but instead of moving the symbol to the position of the right parenthesis, we move it to the left, we get prefix notation (see Figure 7). The position of the parenthesis pair is actually a clue to the final position of the enclosed operator.

Figure 7: Moving Operators to the Left for Prefix Notation

So in order to convert an expression, no matter how complex, to either prefix or postfix notation, fully parenthesize the expression using the order of operations. Then move the enclosed operator to the position of either the left or the right parenthesis depending on whether you want prefix or postfix notation.

Here is a more complex expression: (A + B) * C - (D - E) * (F + G). Figure 8 shows the conversion to postfix and prefix notations.

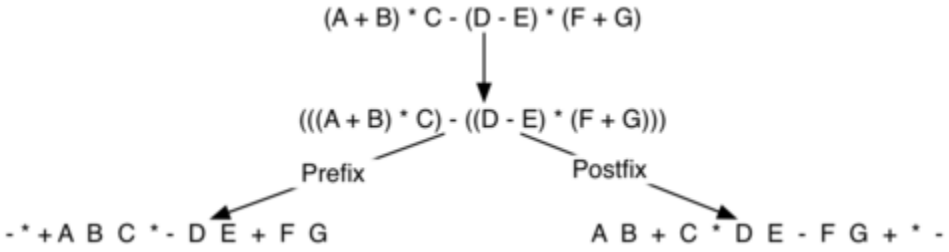


Figure 8: Converting a Complex Expression to Prefix and Postfix Notations

General Infix-to-Postfix Conversion

We need to develop an algorithm to convert any infix expression to a postfix expression. To do this we will look closer at the conversion process.

Consider once again the expression A + B * C. As shown above, A B C * + is the postfix equivalent. We have already noted that the operands A, B, and C stay in their relative positions. It is only the operators that change position. Let's look again at the operators in the infix expression. The first operator that appears from left to right is +. However, in the postfix expression, + is at the end since the next operator, *, has precedence over addition. The order of the operators in the original expression is reversed in the resulting postfix expression.

As we process the expression, the operators have to be saved somewhere since their corresponding right operands are not seen yet. Also, the order of these saved operators may need to be reversed due to their precedence. This is the case with the addition and the multiplication in this example. Since the addition operator comes before the multiplication operator and has lower precedence, it needs to appear after the multiplication operator is used. Because of this reversal of order, it makes sense to consider using a stack to keep the operators until they are needed.

What about (A + B) * C? Recall that A B + C * is the postfix equivalent. Again, processing this infix expression from left to right, we see + first. In this case, when we see *, + has already been placed in the result expression because it has precedence over * by virtue of the parentheses. We can now start to see how the conversion algorithm will work. When we see a left parenthesis, we will save it to denote that another operator of high precedence will be coming. That operator will need to wait until the corresponding right parenthesis appears to denote its position (recall the fully parenthesized technique). When that right parenthesis does appear, the operator can be popped from the stack.

As we scan the infix expression from left to right, we will use a stack to keep the operators. This will provide the reversal that we noted in the first example. The top of the stack will always be the most recently saved operator. Whenever we read a new operator, we will need to consider how that operator compares in precedence with the operators, if any, already on the stack.

Assume the infix expression is a string of tokens delimited by spaces. The operator tokens are *, /, +, and -, along with the left and right parentheses, (and). The operand tokens are the single-character identifiers A, B, C, and so on. The following steps will produce a string of tokens in postfix order.

1. Create an empty stack called opstack for keeping operators. Create an empty list for output.
2. Convert the input infix string to a list by using the string method split.
3. Scan the token list from left to right.
    - If the token is an operand, append it to the end of the output list.
    - If the token is a left parenthesis, push it on the opstack.
    - If the token is a right parenthesis, pop the opstack until the corresponding left parenthesis is removed. Append each operator to the end of the output list.

- If the token is an operator, *, /, +, or -, push it on the opstack. However, first remove any operators already on the opstack that have higher or equal precedence and append them to the output list.

4. When the input expression has been completely processed, check the opstack. Any operators still on the stack can be removed and appended to the end of the output list.

   Figure 9 shows the conversion algorithm working on the expression A * B + C * D. Note that the first * operator is removed upon seeing the + operator. Also, + stays on the stack when the second * occurs, since multiplication has precedence over addition. At the end of the infix expression the stack is popped twice, removing both operators and placing + as the last operator in the postfix expression.

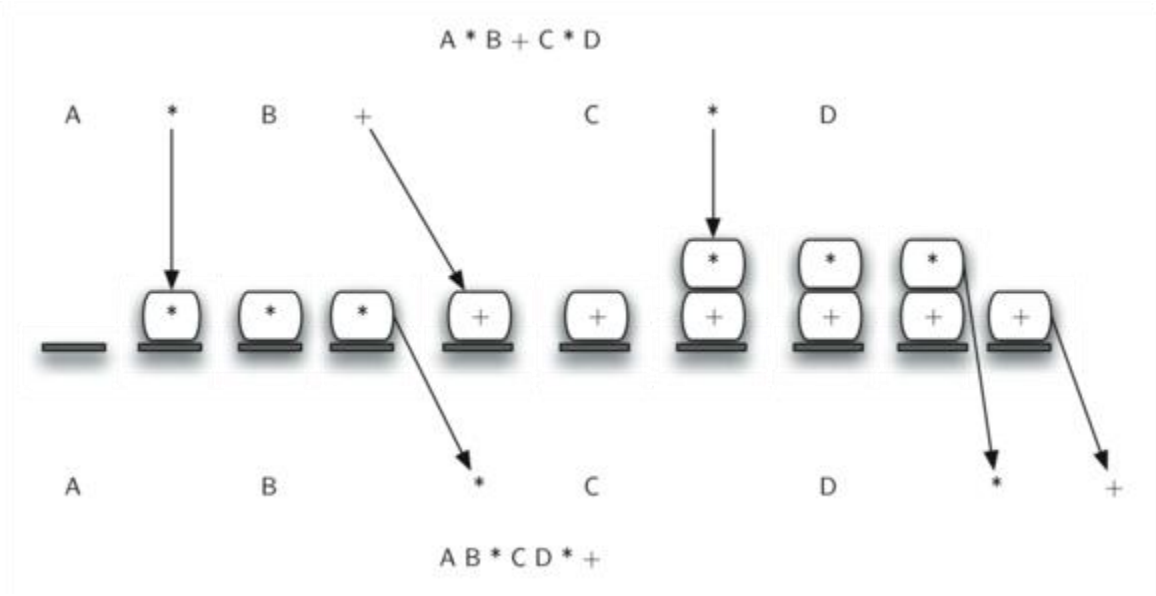


Figure 9: Converting A * B + C * D to Postfix Notation

In order to code the algorithm in Python, we will use a dictionary called prec to hold the precedence values for the operators. This dictionary will map each operator to an integer that can be compared against the precedence levels of other operators (we have arbitrarily used the integers 3, 2, and 1). The left parenthesis will receive the lowest value possible. This way any operator that is compared against it will have higher precedence and will be placed on top of it. Line 15 defines the operands to be any upper-case character or digit. The complete conversion function is shown in ActiveCode 1.

A few more examples of execution in the Python shell are shown below.

```
>>> infixtopostfix("( A + B ) * ( C + D )")
'A B + C D + *'
>>> infixtopostfix("( A + B ) * C")
'A B + C *'
>>> infixtopostfix("A + B * C")
'A B C * +'
>>>
```

Postfix Evaluation

As a final stack example, we will consider the evaluation of an expression that is already in postfix notation. In this case, a stack is again the data structure of choice. However, as you scan the postfix expression, it is the operands that must wait, not the operators as in the conversion algorithm above. Another way to think about the solution is that whenever an operator is seen on the input, the two most recent operands will be used in the evaluation.

To see this in more detail, consider the postfix expression `4 5 6 * +`. As you scan the expression from left to right, you first encounter the operands 4 and 5. At this point, you are still unsure what to do with them until you see the next symbol. Placing each on the stack ensures that they are available if an operator comes next.

In this case, the next symbol is another operand. So, as before, push it and check the next symbol. Now we see an operator, *. This means that the two most recent operands need to be used in a multiplication operation. By popping the stack twice, we can get the proper operands and then perform the multiplication (in this case getting the result 30).

We can now handle this result by placing it back on the stack so that it can be used as an operand for the later operators in the expression. When the final operator is processed, there will be only one value left on the stack. Pop and return it as the result of the expression. Figure 10 shows the stack contents as this entire example expression is being processed.

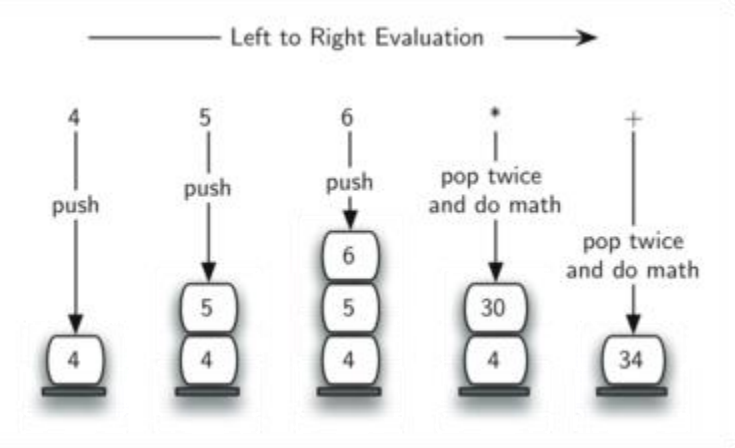


Figure 10: Stack Contents During Evaluation

Figure 11 shows a slightly more complex example, 7 8 + 3 2 + /. There are two things to note in this example. First, the stack size grows, shrinks, and then grows again as the subexpressions are evaluated. Second, the division operation needs to be handled carefully. Recall that the operands in the postfix expression are in their original order since postfix changes only the placement of operators. When the operands for the division are popped from the stack, they are reversed. Since division is *not* a commutative operator, in other words $15/5$ 15/5 is not the same as $5/15$ 5/15, we must be sure that the order of the operands is not switched.

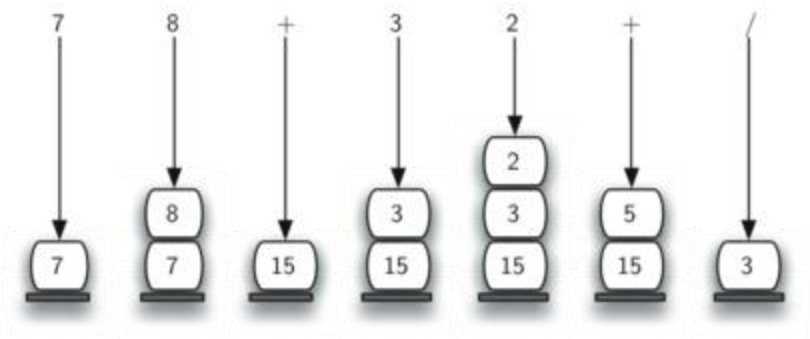


Figure 11: A More Complex Example of Evaluation

Assume the postfix expression is a string of tokens delimited by spaces. The operators are *, /, +, and - and the operands are assumed to be single-digit integer values. The output will be an integer result.

1. Create an empty stack called operandStack.
2. Convert the string to a list by using the string method split.
3. Scan the token list from left to right.
    - If the token is an operand, convert it from a string to an integer and push the value onto the operandStack.
    - If the token is an operator, *, /, +, or -, it will need two operands. Pop the operandStack twice. The first pop is the second operand and the second pop is the first operand. Perform the arithmetic operation. Push the result back on the operandStack.
4. When the input expression has been completely processed, the result is on the stack. Pop the operandStack and return the value.

   The complete function for the evaluation of postfix expressions is shown in ActiveCode 2. To assist with the arithmetic, a helper function doMath is defined that will take two operands and an operator and then perform the proper arithmetic operation.

Run Load History

```
from pythonds.basic import Stack

def postfixEval(postfixExpr):
    operandStack = Stack()
    tokenList = postfixExpr.split()

    for token in tokenList:
        if token in "0123456789":
            operandStack.push(int(token))
        else:
            operand2 = operandStack.pop()
            operand1 = operandStack.pop()
            result = doMath(token,operand1,operand2)
            operandStack.push(result)
    return operandStack.pop()
```

Activity: 1 -- Postfix Evaluation (postfixeval)

Postfix Evaluation (postfixeval)

It is important to note that in both the postfix conversion and the postfix evaluation programs we assumed that there were no errors in the input expression. Using these programs as a starting point, you can easily see how error detection and reporting can be included. We leave this as an exercise at the end of the chapter.

Q-1: Without using the activecode infixToPostfix function, convert the following expression to postfix `10 + 3 * 5 / (16 - 4)` .

Q-2: What is the result of evaluating the following: `17 10 + 3 * 9 / ==` ?

https://runestone.academy/runestone/books/published/pythonds/BasicDS/InfixPrefixandPostfixExpressions.html

Infix, Postfix and Prefix

Infix, Postfix and Prefix notations are three different but equivalent ways of writing expressions. It is easiest to demonstrate the differences by looking at examples of operators that take two operands.

Infix notation: X + Y

> Operators are written in-between their operands. This is the usual way we write expressions. An expression such as `A * ( B + C ) / D` is usually taken to mean something like: "First add B and C together, then multiply the result by A, then divide by D to give the final answer."
>
> Infix notation needs extra information to make the order of evaluation of the operators clear: rules built into the language about operator precedence and associativity, and brackets `( )` to allow users to override these rules. For example, the usual rules for associativity say that we perform operations from left to right, so the multiplication by A is assumed to come before the division by D. Similarly, the usual rules for precedence say that we perform multiplication and division before we perform addition and subtraction. (see CS2121 lecture).

Postfix notation (also known as "Reverse Polish notation"): X Y +

> Operators are written after their operands. The infix expression given above is equivalent to `A B C + * D /`
> The order of evaluation of operators is always left-to-right, and brackets cannot be used to change this order. Because the "+" is to the left of the "*" in the example above, the addition must be performed before the multiplication. Operators act on values immediately to the left of them. For example, the "+" above uses the "B" and "C". We can add (totally unnecessary) brackets to make this explicit:
> `( (A (B C +) *) D /)`
> Thus, the "*" uses the two values immediately preceding: "A", and the result of the addition. Similarly, the "/" uses the result of the multiplication and the "D".

Prefix notation (also known as "Polish notation"): + X Y

Operators are written before their operands. The expressions given above are equivalent to `/ * A + B C D`
As for Postfix, operators are evaluated left-to-right and brackets are superfluous. Operators act on the two nearest values on the right. I have again added (totally unnecessary) brackets to make this clear: `(/ (* A (+ B C) ) D)`

Although Prefix "operators are evaluated left-to-right", they use values to their right, and if these values themselves involve computations then this changes the order that the operators have to be evaluated in. In the example above, although the division is the first operator on the left, it acts on the result of the multiplication, and so the multiplication has to happen before the division (and similarly the addition has to happen before the multiplication). Because Postfix operators use values to their left, any values involving computations will already have been calculated as we go left-to-right, and so the order of evaluation of the operators is not disrupted in the same way as in Prefix expressions.

In all three versions, the operands occur in the same order, and just the operators have to be moved to keep the meaning correct. (This is particularly important for asymmetric operators like subtraction and division: `A - B` does not mean the same as `B - A`; the former is equivalent to `A B -` or `- A B`, the latter to `B A -` or `- B A`).

Examples:

Infix	**Postfix**	**Prefix**	**Notes**
`A * B + C / D`	`A B * C D / +`	`+ * A B / C D`	multiply A and B, divide C by D, add the results
`A * (B + C) / D`	`A B C + * D /`	`/ * A + B C D`	add B and C, multiply by A, divide by D
`A * (B + C / D)`	`A B C D / + *`	`* A + B / C D`	divide C by D, add B, multiply by A

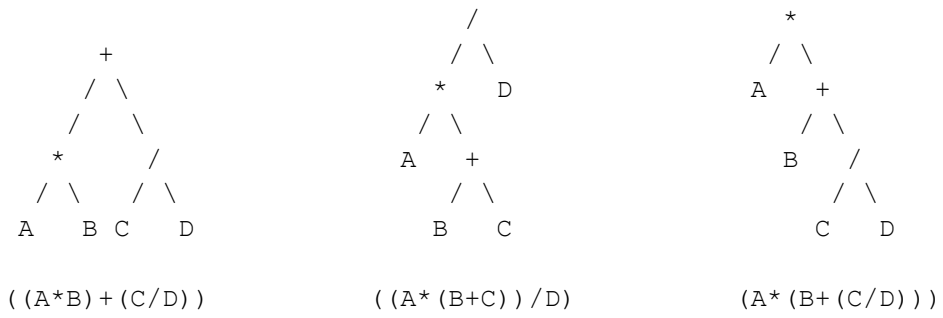
Converting between these notations

The most straightforward method is to start by inserting all the implicit brackets that show the order of evaluation e.g.:

Infix	Postfix	Prefix
((A * B) + (C / D))	((A B *) (C D /) +)	(+ (* A B) (/ C D))
((A * (B + C)) / D)	((A (B C +) *) D /)	(/ (* A (+ B C)) D)
(A * (B + (C / D)))	(A (B (C D /) +) *)	(* A (+ B (/ C D)))

You can convert directly between these bracketed forms simply by moving the operator within the brackets e.g. `(X + Y)` or `(X Y +)` or `(+ X Y)`. Repeat this for all the operators in an expression, and finally remove any superfluous brackets.

You can use a similar trick to convert to and from parse trees - each bracketed triplet of an operator and its two operands (or sub-expressions) corresponds to a node of the tree. The corresponding parse trees are:



Computer Languages

Because Infix is so common in mathematics, it is much easier to read, and so is used in most computer languages (e.g. a simple Infix calculator). However, Prefix is often used for operators that take a single operand (e.g. negation) and function calls.

Although Postfix and Prefix notations have similar complexity, Postfix is slightly easier to evaluate in simple circumstances, such as in some calculators (e.g. a simple Postfix calculator), as the operators really are evaluated strictly left-to-right (see note above).

For lots more information about notations for expressions,

Abstract syntax tree

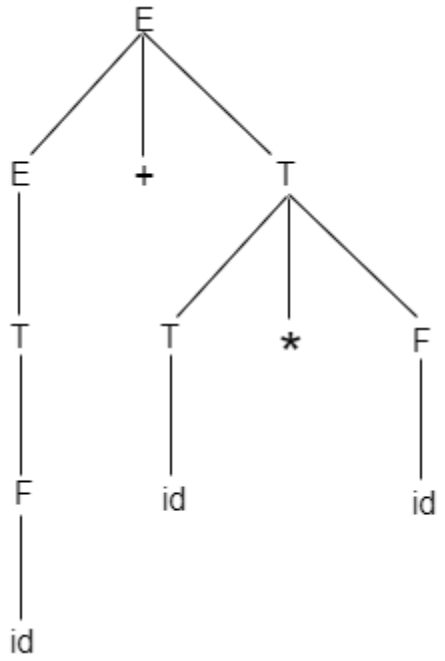
In computer science, an **abstract syntax tree** (**AST**), or just **syntax tree**, is a tree representation of the abstract syntactic structure of source code written in a programming language. Each node of the tree denotes a construct occurring in the source code.

The syntax is "abstract" in the sense that it does not represent every detail appearing in the real syntax, but rather just the structural or content-related details. For instance, grouping parentheses are implicit in the tree structure, so these do not have to be represented as separate nodes. Likewise, a syntactic construct like an if-condition-then expression may be denoted by means of a single node with three branches.

This distinguishes abstract syntax trees from concrete syntax trees, traditionally designated parse trees. Parse trees are typically built by a parser during the source code translation and compiling process. Once built, additional information is added to the AST by means of subsequent processing, e.g., contextual analysis.

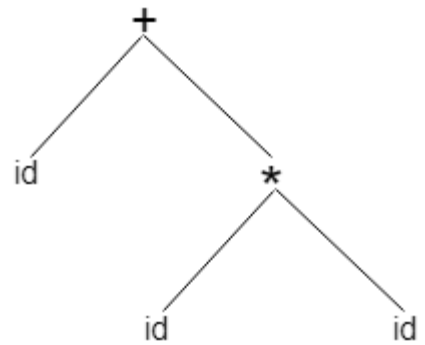
Abstract syntax trees are also used in program analysis and program transformation systems.

When you create a parse tree then it contains more details than actually needed. So, it is very difficult to compiler to parse the parse tree. Take the following parse tree as an example:

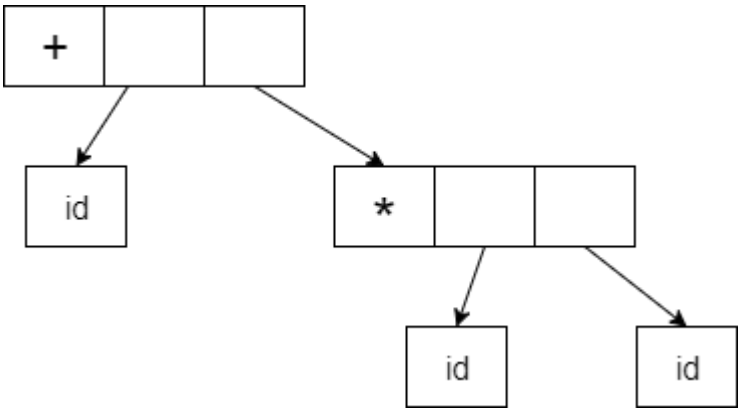


- In the parse tree, most of the leaf nodes are single child to their parent nodes.
- In the syntax tree, we can eliminate this extra information.
- Syntax tree is a variant of parse tree. In the syntax tree, interior nodes are operators and leaves are operands.
- Syntax tree is usually used when represent a program in a tree structure.

A sentence **id + id * id** would have the following syntax tree:

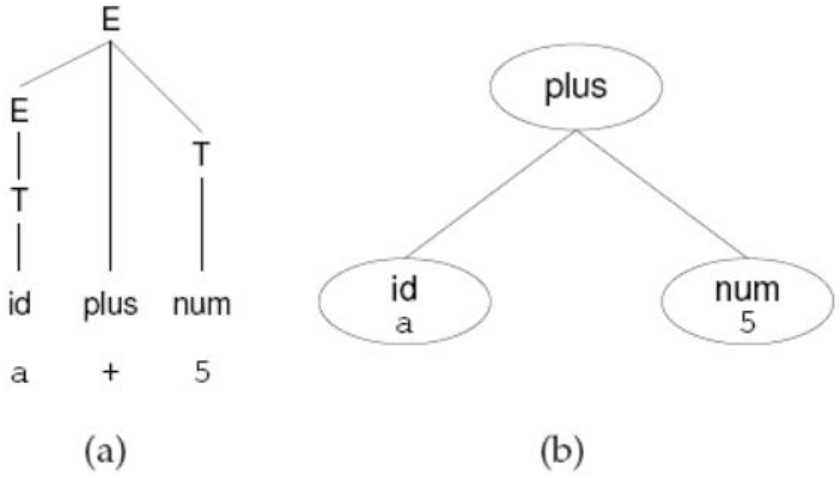


Abstract syntax tree can be represented as:

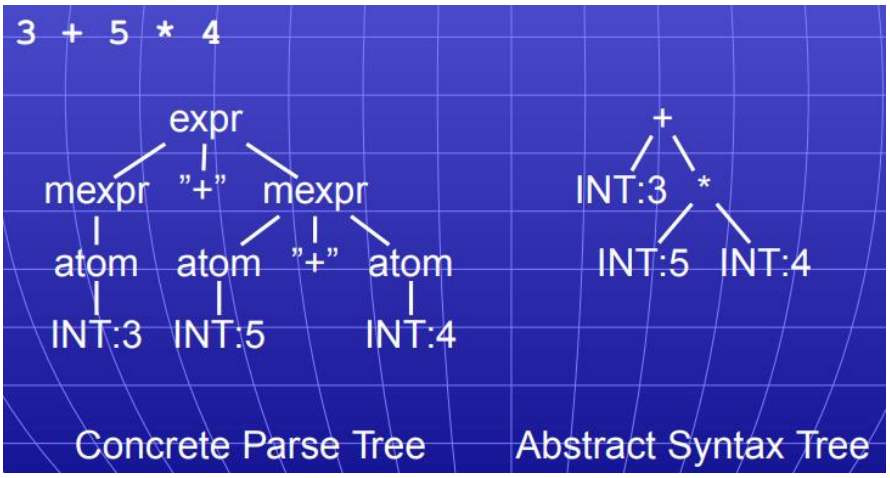


Abstract syntax trees are important data structures in a compiler. It contains the least unnecessary information.

Abstract syntax trees are more compact than a parse tree and can be easily used by a compiler.



: (a) Derivation of a + 5 from E;
(b) Abstract representation of a + 5.


3 + 5 * 4
expr
mexpr
"+"
mexpr
atom
atom
"+"
atom
INT:3
INT:5
INT:4
Concrete Parse Tree
+
INT:3
*
INT:5
INT:4
Abstract Syntax Tree