


Topic 3:

Solution of Systems of Linear Equations

Lectures 12-17:

Lecture 12

Vector, Matrices, and Linear Equations



VECTORS

Vector: a one dimensional array of numbers

Examples :

row vector $[1 \quad 4 \quad 2]$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Identity vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATRICES

Matrix : a two dimensional array of numbers

Examples :

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

diagonal $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$,

Tridiagonal $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

MATRICES

Examples :

symmetric $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}$, upper triangular

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of a MATRICES

Defined for square matrices only

Examples :

$$\det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2 \begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} + 1 \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$
$$= 2(-25) - 1(12 + 5) + 1(15 - 0) = -82$$

Adding and Multiplying Matrices

The addition of two matrices A and B

- * Defined only if they have the same size
- * $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices A($n \times m$) and B($p \times q$)

- * The product $C = AB$ is defined only if $m = p$
- * $C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^m a_{ik} b_{kj} \quad \forall i, j$

Systems of Linear Equations

A system of linear equations can be presented in different forms

$$\left. \begin{array}{l} 2x_1 + 4x_2 - 3x_3 = 3 \\ 2.5x_1 - x_2 + 3x_3 = 5 \\ x_1 - 6x_3 = 7 \end{array} \right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form

Matrix form

Solutions of Linear Equations

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution to the following equations :

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$

Solutions of Linear Equations

- A set of equations is **inconsistent** if there exists no solution to the system of equations:

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 5$$

These equations are inconsistent

Solutions of Linear Equations

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

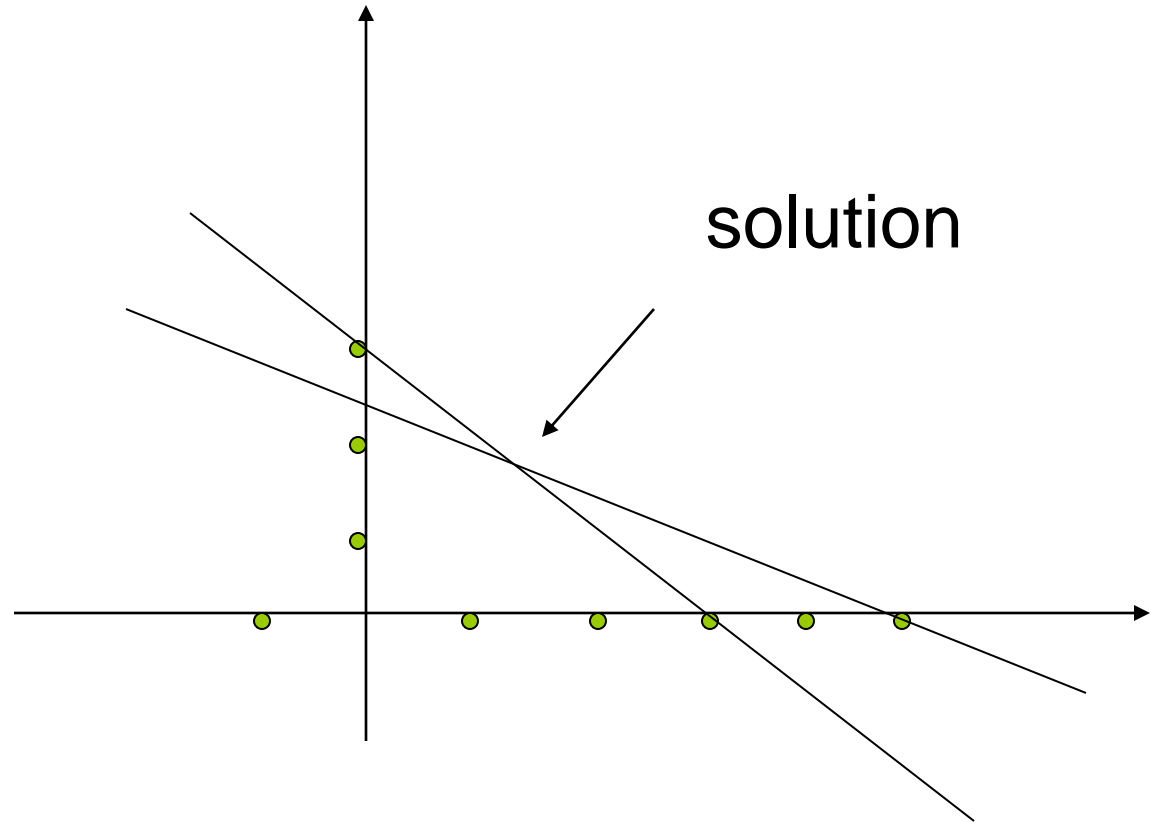
have infinite number of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0.5(3-a) \end{bmatrix} \text{ is a solution for all } a$$

Graphical Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$



Cramer's Rule is Not Practical

Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

A super computer needs 10^{17} years to solve a 30 by 30 system.

It can be used if the determinants are computed in efficient way

Lecture 13

Naive Gaussian Elimination

- Naive Gaussian Elimination
- Examples

Naive Gaussian Elimination

- The method consists of two steps:
 - **Forward Elimination:** the system is reduced to **upper triangular form**. A sequence of **elementary operations** is used.
 - **Backward Substitution:** Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Elementary Row Operations

- ▣ Adding a multiple of one row to another
- ▣ Multiply any row by a non-zero constant

Example

Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1 : Forward Elimination

Step1 : Eliminate x_1 from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Example

Forward Elimination

Step2: Eliminate x_2 from equations 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

Step3: Eliminate x_3 from equation 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Forward Elimination

Summary of the Forward Elimination :

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Backward Substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for x_4 , then solve for x_3 ,... solve for x_1

$$x_4 = \frac{-3}{-3} = 1,$$

$$x_3 = \frac{-9 + 5}{2} = -2$$

$$x_2 = \frac{-6 - 2(-2) - 2(1)}{-4} = 1, \quad x_1 = \frac{16 + 2(1) - 2(-2) - 4(1)}{6} = 3$$

Forward Elimination

To eliminate x_1

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}} \right) a_{1j} & (1 \leq j \leq n) \\ b_j &\leftarrow b_j - \left(\frac{a_{i1}}{a_{11}} \right) b_1 \end{aligned} \right\} 2 \leq i \leq n$$

To eliminate x_2

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}} \right) a_{2j} & (2 \leq j \leq n) \\ b_j &\leftarrow b_j - \left(\frac{a_{i2}}{a_{22}} \right) b_2 \end{aligned} \right\} 3 \leq i \leq n$$

Forward Elimination

To eliminate x_m

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{im}}{a_{mm}} \right) a_{mj} & (m \leq j \leq n) \\ b_j &\leftarrow b_j - \left(\frac{a_{im}}{a_{mm}} \right) b_m \end{aligned} \right\} m+1 \leq i \leq n$$

Continue until x_{n-1} is eliminated.

Backward Substitution

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}}$$

$$x_m = \frac{b_m - \sum_{j=m+1}^n a_{m,j}x_j}{a_{m,m}}$$

Lecture 14

Naive Gaussian Elimination

- Summary of the Naive Gaussian Elimination
- Example
- How to check a solution
- Problems with Naive Gaussian Elimination
 - Failure due to zero pivot element
 - Error

Naive Gaussian Elimination

- o The method consists of two steps
 - o **Forward Elimination:** the system is reduced to **upper triangular form**. A sequence of **elementary operations** is used.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

- o **Backward Substitution:** Solve the system starting from the last variable. Solve for x_n, x_{n-1}, \dots, x_1 .

Example 1

Solve using Naive Gaussian Elimination :

Part 1: Forward Elimination ____ Step 1: Eliminate x_1 from equations 2, 3

$$x_1 + 2x_2 + 3x_3 = 8 \quad \text{eq1 unchanged (pivot equation)}$$

$$2x_1 + 3x_2 + 2x_3 = 10 \quad \text{eq2} \leftarrow \text{eq2} - \left(\frac{2}{1}\right)\text{eq1}$$

$$3x_1 + x_2 + 2x_3 = 7 \quad \text{eq3} \leftarrow \text{eq3} - \left(\frac{3}{1}\right)\text{eq1}$$

$$x_1 + 2x_2 + 3x_3 = 8$$

$$-x_2 - 4x_3 = -6$$

$$-5x_2 - 7x_3 = -17$$

Example 1

Part 1: Forward Elimination Step 2: Eliminate x_2 from equation 3

$$x_1 + 2x_2 + 3x_3 = 8 \quad \text{eq1 unchanged}$$

$$-x_2 - 4x_3 = -6 \quad \text{eq2 unchanged (pivot equation)}$$

$$-5x_2 - 7x_3 = -17 \quad \text{eq3} \leftarrow \text{eq3} - \left(\frac{-5}{-1} \right) \text{eq2}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 8 \\ -x_2 - 4x_3 = -6 \\ 13x_3 = 13 \end{cases}$$

Example 1

Backward Substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

The solution is
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

How Do We Know If a Solution is Good or Not

❖ Given $AX=B$

- ❖ X is a solution if $AX-B=0$
- ❖ Due to computation error $AX-B$ may not be zero
- ❖ Compute the residuals $R=|AX-B|$
- ❖ One possible test is ?????

The solution is acceptable if $\max_i |r_i| \leq \varepsilon$

Determinant

The elementary operations do not affect the determinant

Example :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Elementary operations}} A' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = \det(A') = -13$$

How Many Solutions Does a System of Equations $AX=B$ Have?

Unique

$$\det(A) \neq 0$$

reduced matrix

has no zero rows

No solution

$$\det(A) = 0$$

reduced matrix

has one or more
zero rows

corresponding B
elements $\neq 0$

Infinite

$$\det(A) = 0$$

reduced matrix

has one or more
zero rows

corresponding B
elements $= 0$

Examples

Unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

solution:

$$X = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

No solution

0 = -1 impossible

infinte # of solutions

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Infinite# solutions

$$X = \begin{bmatrix} \alpha \\ 1 - .5\alpha \end{bmatrix}$$

Lectures 15-16:

Gaussian Elimination with Scaled Partial Pivoting

- ▣ Problems with Naive Gaussian Elimination
- ▣ Definitions and Initial step
- ▣ Forward Elimination
- ▣ Backward substitution
- ▣ Example

Problems with Naive Gaussian Elimination

- The Naive Gaussian Elimination may fail for very simple cases. (**The pivoting element is zero**).

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Very small pivoting element may result in serious computation errors

$$\begin{bmatrix} 10^{-10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting :

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 2

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector:

disregard sign

find largest in
magnitude in
each row

Scale vector $S = [2 \quad 4 \quad 8 \quad 5]$

Index Vector $L = [1 \quad 2 \quad 3 \quad 4]$

Why Index Vector?

- ❑ Index vectors are used because it is much easier to exchange a single index element compared to exchanging the values of a complete row.
- ❑ In practical problems with very large N , exchanging the contents of rows may not be practical since they could be stored at different locations.

Example 2

Forward Elimination-- Step 1: eliminate x_1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 & 4 & 8 & 5] \\ L = [1 & 2 & 3 & 4] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \mid i = 1, 2, 3, 4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \frac{|4|}{5} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 and l_1

$$L = [4 \ 2 \ 3 \ 1]$$

Example 2

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ \boxed{4} & \boxed{2} & \boxed{5} & \boxed{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

First pivot equation



$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ \boxed{4} & \boxed{2} & \boxed{5} & \boxed{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ \boxed{-1} \end{bmatrix}$$

Example 2

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$\text{Ratios: } \left\{ \frac{0.5}{4}, \frac{5.5}{8}, \frac{1.5}{2} \right\}$$

$$L = [4 \ 1 \ 3 \ 2]$$

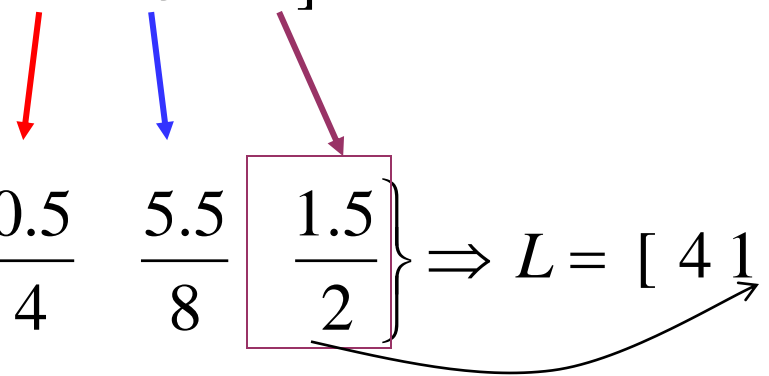
Example 2

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 2 \ 3 \ 1]$$

$$\text{Ratios: } \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \mid i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \frac{1.5}{2} \right\} \Rightarrow L = [\ 4 \ 1 \ 3 \ 2]$$


Example 2

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 2.5 & 0.16667 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 6.8333 \\ -1 \end{bmatrix}$$

**Third pivot
equation**



$$L = \begin{bmatrix} 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix}$$

Example 2

Backward Substitution

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix} \quad L = [4 \ 1 \ 2 \ 3]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{9}{2} = 4.5, \quad x_3 = \frac{b_{l_3} - a_{l_3,4}x_4}{a_{l_3,3}} = \frac{2.1667 - 1.8333x_4}{-2.5} = 2.4327$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4}x_4 - a_{l_2,3}x_3}{a_{l_2,2}} = \frac{1.25 - 0.25x_4 - 0.75x_3}{-1.5} = 1.1333$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4}x_4 - a_{l_1,3}x_3 - a_{l_1,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -7.2333$$

Example 3

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 3

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector $S = [2 \quad 4 \quad 8 \quad 5]$

Index Vector $L = [1 \quad 2 \quad 3 \quad 4]$

Example 3

Forward Elimination-- Step 1: eliminate x_1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 & 4 & 8 & 5] \\ L = [1 & 2 & 3 & 4] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \mid i = 1, 2, 3, 4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \frac{|4|}{5} \right\} \Rightarrow \max \text{ corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 and l_1

$$L = [4 \ 2 \ 3 \ 1]$$

Example 3

Forward Elimination-- Step 1: eliminate x1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 2: eliminate x₂

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$\text{Ratios: } \left\{ \left| \frac{0.5}{4} \right|, \left| \frac{-10.5}{8} \right|, \left| \frac{-1.5}{2} \right| \right\} = \left\{ \frac{0.5}{4}, \frac{10.5}{8}, \frac{1.5}{2} \right\}$$

$$L = [4 \ 3 \ 1 \ 2]$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 2 \ 3 \ 1]$$

$$\text{Ratios: } \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \mid i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{10.5}{8} \quad \frac{1.5}{2} \right\} \Rightarrow L = [\ 4 \ 3 \ 1 \ 2]$$

Example 3

Forward Elimination-- Step 2: eliminate x2

Updating A and B

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = [\begin{matrix} & 4 & 3 & 1 & 2 \end{matrix}]$$

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 3: eliminate x3

Selection of the third pivot equation

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 3 \ 2 \ 1]$$

$$\text{Ratios: } \left\{ \frac{|a_{l_i,3}|}{S_{l_i}} \mid i = 3,4 \right\} = \left\{ \frac{2.7619}{4}, \frac{0.7857}{2} \right\} \Rightarrow L = [\ 4 \ 3 \ 2 \ 1]$$

Example 3

Forward Elimination-- Step 3: eliminate x_3

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Backward Substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix} \quad L = [4 \ 3 \ 2 \ 1]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{1.4569}{0.8448} = 1.7245, \quad x_3 = \frac{b_{l_3} - a_{l_3,4}x_4}{a_{l_3,3}} = \frac{1.8571 - 1.7143x_4}{-2.7619} = 0.3980$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4}x_4 - a_{l_2,3}x_3}{a_{l_2,2}} = -0.3469$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4}x_4 - a_{l_1,3}x_3 - a_{l_1,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -1.8673$$

How Good is the Solution?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{solution} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.8673 \\ -0.3469 \\ 0.3980 \\ 1.7245 \end{bmatrix}$$

$$\text{Residues: } R = \begin{bmatrix} 0.005 \\ 0.002 \\ 0.003 \\ 0.001 \end{bmatrix}$$

Remarks:

- ❑ We use index vector to avoid the need to move the rows which may not be practical for large problems.
- ❑ If we order the equation as in the last value of the index vector, we have a triangular form.
- ❑ Scale vector is formed by taking maximum in magnitude in each row.
- ❑ Scale vector do not change.
- ❑ The original matrices A and B are used in checking the residuals.

Lecture 17

Tridiagonal & Banded Systems and Gauss-Jordan Method

- Tridiagonal Systems
- Diagonal Dominance
- Tridiagonal Algorithm
- Examples
- Gauss-Jordan Algorithm

Tridiagonal Systems

Tridiagonal Systems:

- The non-zero elements are in the **main diagonal**, **super diagonal** and **subdiagonal**.

- $a_{ij}=0$ if $|i-j| > 1$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Tridiagonal Systems

- ❑ Occur in many applications
- ❑ Needs less storage ($4n-2$ compared to $n^2 + n$ for the general cases)
- ❑ Selection of pivoting rows is unnecessary
(under some conditions)
- ❑ Efficiently solved by Gaussian elimination

Algorithm to Solve Tridiagonal Systems

- ❑ Based on Naive Gaussian elimination.
- ❑ As in previous Gaussian elimination algorithms
 - Forward elimination step
 - Backward substitution step
- ❑ Elements in the **super diagonal** are not affected.
- ❑ Elements in the **main diagonal**, and **B** need updating

Tridiagonal System

We need to update d and a , d and b

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 & c_1 & & & \\ & d_2 & c_2 & & \\ & & d_3 & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Diagonal Dominance

A matrix A is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^n |a_{ij}| \quad \text{for } (1 \leq i \leq n)$$

The magnitude of each diagonal element is larger than the sum of elements in the corresponding row.

Diagonal Dominance

Examples :

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 6 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

Diagonally dominant

$$\begin{bmatrix} -3 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Not Diagonally dominant

Diagonally Dominant Tridiagonal System

- ▣ A tridiagonal system is diagonally dominant if

$$|d_i| > |c_i| + |a_{i-1}| \quad (1 \leq i \leq n)$$

- ▣ Forward Elimination preserves diagonal dominance

Solving Tridiagonal System

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \quad 2 \leq i \leq n$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i} (b_i - c_i x_{i+1}) \quad \text{for } i = n-1, n-2, \dots, 1$$

Example

Solve

$$\begin{bmatrix} 5 & 2 & & \\ 1 & 5 & 2 & \\ & 1 & 5 & 2 \\ & & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}, \quad b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \quad 2 \leq i \leq 4$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}, \quad x_i = \frac{1}{d_i} (b_i - c_i x_{i+1}) \quad \text{for } i = 3, 2, 1$$

Example

$$D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_2 = d_2 - \left(\frac{a_1}{d_1} \right) c_1 = 5 - \frac{1 \times 2}{5} = 4.6, \quad b_2 = b_2 - \left(\frac{a_1}{d_1} \right) b_1 = 9 - \frac{1 \times 12}{5} = 6.6$$

$$d_3 = d_3 - \left(\frac{a_2}{d_2} \right) c_2 = 5 - \frac{1 \times 2}{4.6} = 4.5652, \quad b_3 = b_3 - \left(\frac{a_2}{d_2} \right) b_2 = 8 - \frac{1 \times 6.6}{4.6} = 6.5652$$

$$d_4 = d_4 - \left(\frac{a_3}{d_3} \right) c_3 = 5 - \frac{1 \times 2}{4.5652} = 4.5619, \quad b_4 = b_4 - \left(\frac{a_3}{d_3} \right) b_3 = 6 - \frac{1 \times 6.5652}{4.5652} = 4.5619$$

Example

Backward Substitution

□ After the Forward Elimination:

$$D^T = [5 \quad 4.6 \quad 4.5652 \quad 4.5619], B^T = [12 \quad 6.6 \quad 6.5652 \quad 4.5619]$$

□ Backward Substitution:

$$x_4 = \frac{b_4}{d_4} = \frac{4.5619}{4.5619} = 1,$$

$$x_3 = \frac{b_3 - c_3 x_4}{d_3} = \frac{6.5652 - 2 \times 1}{4.5652} = 1$$

$$x_2 = \frac{b_2 - c_2 x_3}{d_2} = \frac{6.6 - 2 \times 1}{4.6} = 1$$

$$x_1 = \frac{b_1 - c_1 x_2}{d_1} = \frac{12 - 2 \times 1}{5} = 2$$

Gauss-Jordan Method

- ❑ The method reduces the general system of equations $AX=B$ to $IX=B$ where I is an identity matrix.
- ❑ Only Forward elimination is done and no substitution is needed.
- ❑ It has the same problems as Naive Gaussian elimination and can be modified to do partial scaled pivoting.
- ❑ It takes 50% more time than Naive Gaussian method.

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 1 Eliminate x_1 from *equations 2 and 3*

$$\left. \begin{array}{l} eq1 \leftarrow eq1 / 2 \\ eq2 \leftarrow eq2 - \left(\frac{4}{1}\right)eq1 \\ eq3 \leftarrow eq3 - \left(\frac{2}{1}\right)eq1 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 2 Eliminate x_2 from *equations 1 and 3*

$$\left. \begin{array}{l} eq2 \leftarrow eq2 / 6 \\ eq1 \leftarrow eq1 - \left(\frac{-1}{1} \right) eq2 \\ eq3 \leftarrow eq3 - \left(\frac{0}{1} \right) eq2 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Step 3 Eliminate x_3 from *equations 1 and 2*

$$\left. \begin{array}{l} eq3 \leftarrow eq3 / 2 \\ eq1 \leftarrow eq1 - \left(\frac{0.1667}{1} \right) eq3 \\ eq2 \leftarrow eq2 - \left(\frac{-0.8333}{1} \right) eq3 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

is transformed to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \text{solution is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$