The Discrete Fourier Transform (DFT)

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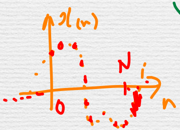
z-Transform, DTFT, DFT

time

$x(n)$

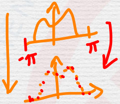


$$X(z) = \sum_{-\infty}^{+\infty} x(n) \cdot z^{-n}$$

$z = e^{j\omega}$

DTFT

$$X(e^{j\omega}) = \sum x(n) \cdot e^{-jn\omega}$$



N points

$$\frac{2\pi}{N}$$

DFT

$$X(k) \triangleq X\left(e^{j\frac{2\pi}{N}k}\right)$$

$$\text{FFT} = \sum x(n) \cdot e^{-j\frac{2\pi}{N}kn}$$

The Discrete Fourier Transform (DFT)

Definition

The Discrete Fourier Transform of an N-point sequence $\{\mathrm{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$$

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$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$$

Inverse DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}$$

1 The Discrete Fourier Transform
- As a Linear Transformation

2 Fast Fourier Transform (FFT)

The Discrete Fourier Transform (DFT)

$X(1) = \sum_{n=0}^{N-1} x(n) \cdot W_N^n$

Definition

$e^{-j\frac{2\pi}{N}nk} \rightsquigarrow W_N^{k.n}$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}} = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{n=0}^{N-1} X(k) W_N^{-kn}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^{\cdots} & W_N^{\cdots} & \cdots & W_N^{\cdots} \\ \vdots & & ? & & \\ 1 & & & W^{\cdots} & \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

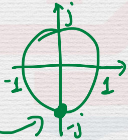
$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Example

Compute the DFT of $x(n) = [0, 1, 2, 3]$ $\quad N = 4$

$$W_4 = e^{-j\frac{2\pi}{4}} = -j$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ & & & \\ & & & \end{bmatrix}$$



Example

Example

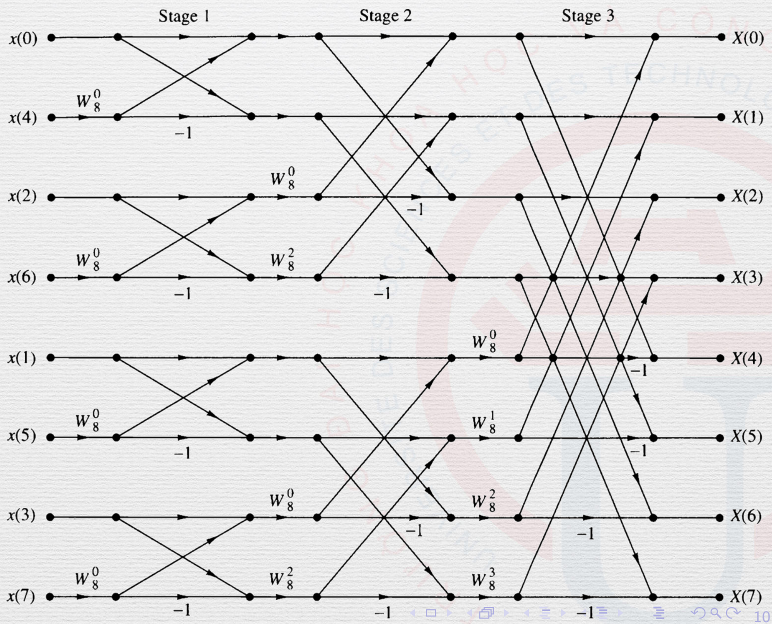
Compute the DFT of $x(n) = [0, 1, 2, 3]$

Answer: $X(k) = [6, -2+2j, -2, -2-2j]$

1 The Discrete Fourier Transform

2 Fast Fourier Transform (FFT)

8-point DFT


Stage 1
Stage 2
Stage 3
x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)
W_8^0
W_8^2
W_8^1
W_8^3
−1

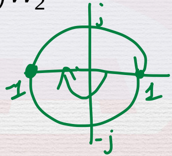
2-point DFT

length $= 2^n$

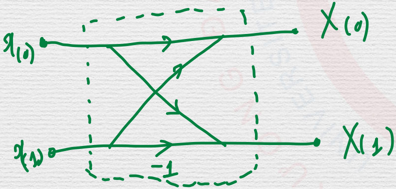
Definition

$$X(k) = \sum_{n=0}^{1} x(n) W_2^{kn} = x(0)W_2^0 + x(1)W_2^k$$

where $W_2 = e^{-j\pi}$



$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} = \begin{bmatrix} x(0) + x(1) \\ x(0) - x(1) \end{bmatrix}$$

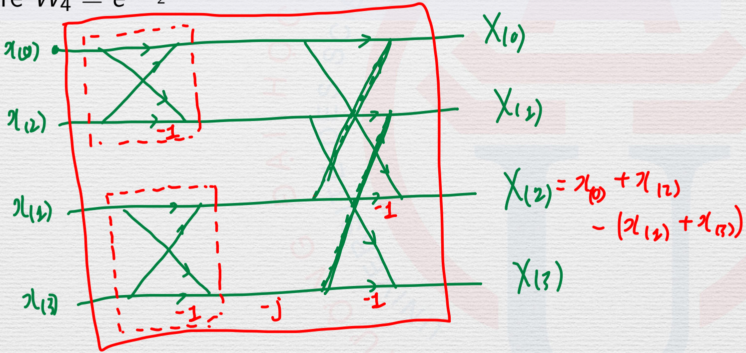


4-point DFT

Definition

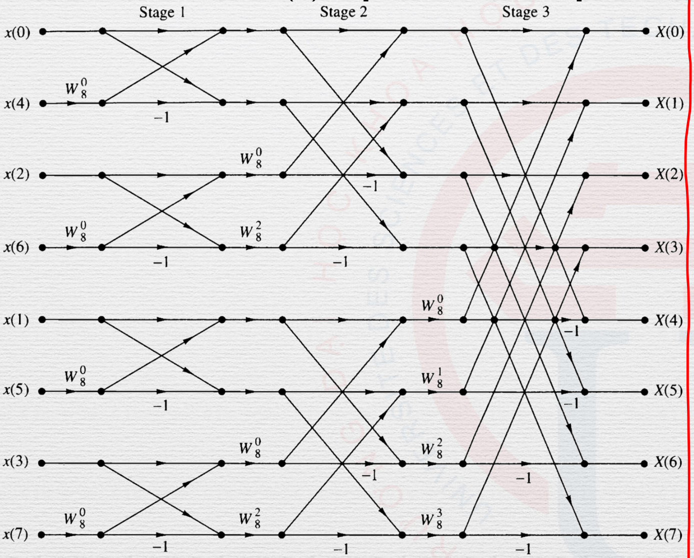
$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}$$

where $W_4 = e^{-j\frac{\pi}{2}}$



8-point DFT

Calculate the FFT of $x(n) = [1, 1, 1, 0, 0, 0, 0, 0]$



0 000

1 001

2 010

3 011

4 100

5 101

6 110

7 111