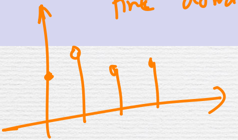
time - domain



FT

freq - domain

The z-Transform

z - Trans

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March 22, 2021

z - domain

Lesson Objectives

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2. understand basic properties of the z-Transform

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At the end of this lesson, you should be able to

1. calculate the z-Transform of any given signals/systems
2. understand basic properties of the z-Transform
3. find any signals knowing its z-Transform

The Direct z-Transform

Definition

The z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable.

Fourier Trans: $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \cdot e^{-jn\omega}$

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where z is a complex variable.

Notations

The z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z[x(n)]$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

Region Of Convergence (ROC)

Definition

ROC of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

$f(x)$ D_f

z-Trs: $\begin{cases} X(z) \\ ROC \end{cases}$

Region Of Convergence (ROC)

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Determine the z-transforms and the corresponding ROCs of the following signals:

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Determine the z-transforms and the corresponding ROCs of the following signals:

1. $x_1(n) = [\underset{\uparrow}{0}, 0, 3, 1, 6]$

$$\Rightarrow X_1(z) = \sum_{-\infty}^{\infty} x(n) \cdot z^{-n} = x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + \dots$$

$$= 3 \cdot z^{-2} + z^{-3} + 6 \cdot z^{-4}$$

ROC: $z \neq 0$ $\quad \mathbb{C} \setminus \{0\}$

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Exercises

Determine the z-transforms and the corresponding ROCs of the following signals:

1. $x_1(n) = [\underset{\uparrow}{0}, 0, 3, 1, 6]$
2. $x_2(n) = [1, 2, 5, 0, \underset{\uparrow}{0}]$ $\rightarrow$ $\mathbb{C} \setminus \{\infty\}$ $z \neq \infty$

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3. $x_3(n) = [2, 1, \underset{\uparrow}{2}, 5]$

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ROC of *finite-duration* signals

Entire z-plane, except possibly the point $z = 0$ and/or $z = \infty$

Region Of Convergence (ROC)

$1 + q + q^2 + q^3 + \dots + q^n = \frac{1-q^{n+1}}{1-q}$

$\begin{cases} 1 + q + q^2 + \dots + \dots = \frac{1}{1-q} \\ |q| < 1 \end{cases}$

What about *infinite-duration* signals?

Determine the z-transform of the signal:

$$x(n) = \left[\underset{\uparrow}{1}, \left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \dots, \left(\frac{1}{2}\right)^n, \dots\right]$$

$$X(z) = \sum x(n) \cdot z^{-n} = 1 + \frac{1}{2} \cdot z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \dots$$

$$= 1 + \frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \left(\frac{1}{2z}\right)^3 + \dots$$

$$= \frac{1}{1 - \frac{1}{2z}} \quad \left(\text{if } \left|\frac{1}{2z}\right| < 1\right)$$

$$X(z) = \frac{2z}{2z-1}$$

$$\text{ROC}: |z| > \frac{1}{2}$$

Region Of Convergence (ROC)

What about *infinite-duration* signals?

Determine the z-transform of the signal:

$$x(n) = \left[\underset{\uparrow}{1}, \left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right]$$

Solution

$$X(z) = \frac{1}{a - \frac{1}{2}z^{-1}}, \text{ ROC: } |z| > \frac{1}{2}$$

Region Of Convergence (ROC)

What about *infinite-duration* signals?

Determine the z-transform of the signal:

$$x(n) = \left[\underset{\uparrow}{1}, \left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right]$$

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$$X(z) = \frac{1}{a - \frac{1}{2}z^{-1}}, \text{ ROC: } |z| > \frac{1}{2}$$

Plot the ROC?

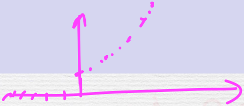
ROC Outside a Circle

Exercise

Determine the z-transform of the signal:

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

ROC Outside a Circle



Exercise

Determine the z-transform of the signal:

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Solution

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \text{ ROC: } |z| > |\alpha|$$

ROC Inside a Circle

Exercise

Determine the z-transform of the signal:

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & \text{if } n \geq 0 \\ -\alpha^n & \text{if } n < 0. \end{cases}$$

ROC Inside a Circle

Exercise

Determine the z-transform of the signal:

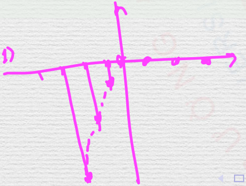
$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & \text{if } n \geq 0 \\ -\alpha^n & \text{if } n < 0. \end{cases}$$

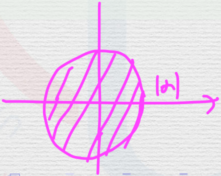
Solution

$$X(z) = +\frac{1}{1-\alpha z^{-1}}, \quad \text{ROC: } |z| < |\alpha|$$

$\alpha = 2$

$x(n) = -2^n \cdot u(-n-1)$





ROC

Exercise

Determine the z-transform of:

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1)$$

ROC

Exercise

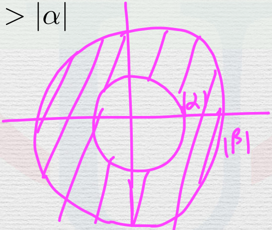
Determine the z-transform of:

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1)$$

Solution

There are two cases:

- $|\beta| < |\alpha|$: two ROCs do not overlap, $X(z)$ does not exist
- $|\beta| > |\alpha|$: ROC of $X(z)$ is $|\beta| > |z| > |\alpha|$



ROC Summary

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Objectives

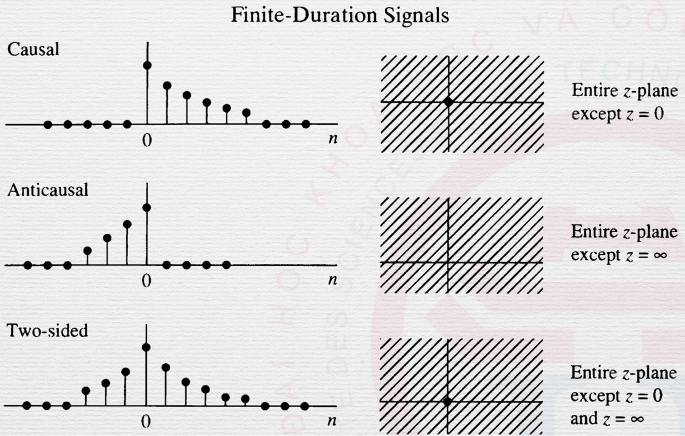
The z-Transform

The Direct z-Transform

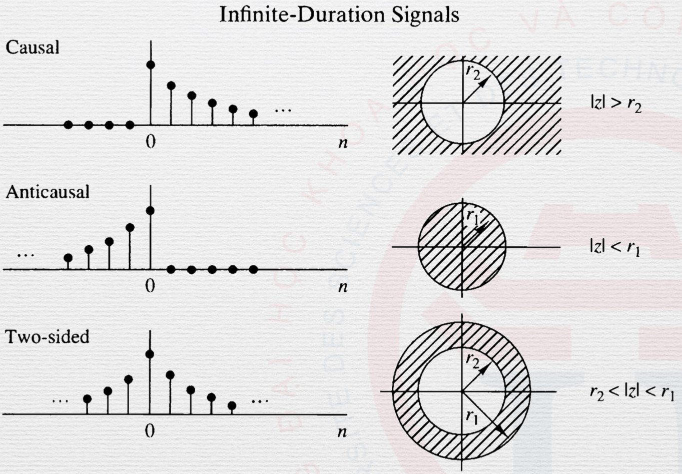
Region Of Convergence (ROC)

Properties of the z-Transform

Inverse z-Transform



ROC Summary


Infinite-Duration Signals
Causal
0
n
|z| > r2
Anticausal
0
n
|z| < r1
Two-sided
0
n
r2 < |z| < r1

Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$

Peroperties

The z-transform is a very powerful tool for the study of discrete-time signals thanks to some of its very important properties:

- Linearity
- Time shifting
- Scaling in the z-domain
- Time reversal
- Differentiation in the z-domain
- Convolution $x_1(n) * x(n)$ | $X_1(z) \cdot X_2(z)$

Properties

Linearity

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z}$$

$a_1 X_1(z) + a_2 . X_2(z)$

Linearity

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

$$X(z) = \sum x(n) \cdot z^{-n}$$

$$= \sum (a_1 \ldots + a_2 \ldots) z^{-n}$$

$$= a_1 \cdot \sum \ldots z^{-n} + a_2 \sum \ldots z^{-n}$$

$$= a_1 \cdot X_1(z) + a_2 X_2(z)$$

Properties

Linearity

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$x(n) = a_1x_1(n) + a_2x_2(n) \xleftrightarrow{z} X(z) = a_1X_1(z) + a_2X_2(z)$$

Time Shifting

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$x(n-k) \xleftrightarrow{z}$$

Properties

Linearity

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

Time Shifting

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

$x(n) \rightarrow \boxed{z^{-1}} \rightarrow x(n-1)$

then

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k} X(z)$$

Properties

Scaling in the z-domain

For any constant a, if

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC : r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z}$$

Scaling in the z-domain

For any constant a, if

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z), \qquad ROC: |a|r_1 < |z| < |a|r_2$$

$$X\left(\frac{z}{a}\right)$$

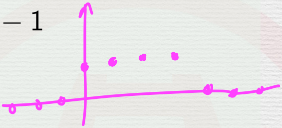
Exercise

$$1 + q + q^2 + \dots + q^{N-1} = \frac{1 - q^N}{1 - q}$$

Exercise

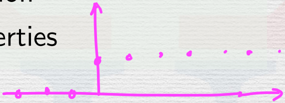
Determine the z-transform of the signal:

$$x(n) = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{elsewhere} \end{cases}$$



using

- Direct approach: z-transform definition
- Indirect approach: z-transform properties



Properties

Time Reversal

If

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

$$x(-n) \overset{z}{\longleftrightarrow}$$

Properties

Time Reversal

If

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

$$x(-n) \xleftrightarrow{z} X(z^{-1}), \qquad ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Properties

Differentiation in the z-domain

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

Convolution Property

Recall: Discrete Convolution

$$y[n] \equiv (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

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Convolution of Two Sequences

If $x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$ then

$$x(n) = x_1(n) * x_2(n) \stackrel{z}{\longleftrightarrow}$$

Convolution Property

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Recall: Discrete Convolution

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Convolution of Two Sequences

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

Computing the Convolution using z-Transform

Step-by-step

1. Compute the z-transform of the two signals
2. Multiply the two z-transfroms $X(z) = X_1(z)X_2(z)$
3. Find the inverse z-transform of $X(z)$

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Step-by-step

1. Compute the z-transform of the two signals
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Exercise

Compute the convolution $x(n)$ of the signals:

$$x_1(n) = [1, -2, 1]$$

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

Computing the Convolution using z-Transform

Partial-Fraction Expansion

1. Express $X(z)$ as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z)$$

where $X_k(z)$ are in the table of z-Transform pairs.

2. Use the table and get

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n)$$

Partial-Fraction Expansion

Example

Determine the partial-fraction expansion of:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

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Hint

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

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Example

Determine the partial-fraction expansion of:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Hint

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

Attention: Be careful with the ROC