Numerical Methods

ODE

Contents

❖ Overview

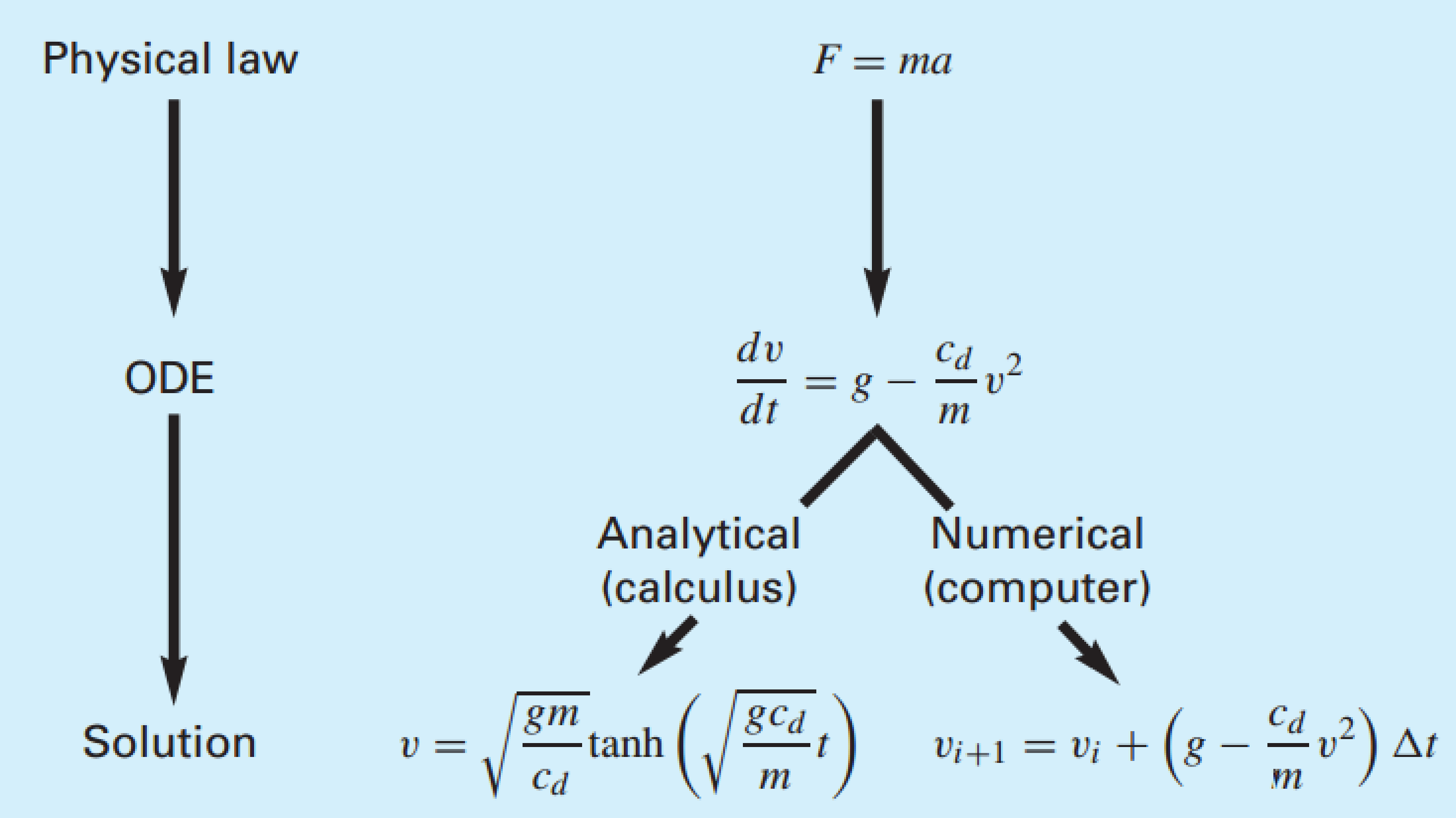

Physical law
$F = ma$
ODE
$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$
Analytical (calculus)
Numerical (computer)
Solution
$v = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$
$v_{i+1} = v_i + \left(g - \frac{c_d}{m} v^2\right) \Delta t$

TABLE 21.1 The one-dimensional forms of some constitutive laws commonly used in engineering and science.

Law	Equation	Physical Area	Gradient	Flux	Proportionality
Fourier's law	$q = -k\dfrac{dT}{dx}$	Heat conduction	Temperature	Heat flux	Thermal Conductivity
Fick's law	$J = -D\dfrac{dc}{dx}$	Mass diffusion	Concentration	Mass flux	Diffusivity
Darcy's law	$q = -k\dfrac{dh}{dx}$	Flow through porous media	Head	Flow flux	Hydraulic Conductivity
Ohm's law	$J = -\sigma\dfrac{dV}{dx}$	Current flow	Voltage	Current flux	Electrical Conductivity
Newton's viscosity law	$\tau = \mu\dfrac{du}{dx}$	Fluids	Velocity	Shear Stress	Dynamic Viscosity
Hooke's law	$\sigma = E\dfrac{\Delta L}{L}$	Elasticity	Deformation	Stress	Young's Modulus

Overview

- The fundamental laws of physics, mechanics, electricity, and thermodynamics are usually based on empirical observations that explain variations in physical properties and states of systems. Rather than describing the state of physical systems directly, the laws are usually couched in terms of spatial and temporal changes.
- Such equations, which are composed of an unknown function and its derivatives, are called differential equations

$$\frac{dv}{dt} = g - \frac{c}{m} v$$

Overview

ODE

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

- g is the gravitational constant, m is the mass, and c is a drag coefficient
- v: the dependent variable.
- t: the independent variable

Overview

- When the function involves one independent variable, the equation is called an ordinary differential equation (or ODE)
- This is in contrast to a partial differential equation (or PDE) that involves two or more independent variables

$$m\frac{d^2x}{dt^2} + kx = 0$$

ODE

$$\frac{\partial u}{\partial t} - a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

PDE

Overview

- Differential equations are also classified as to their *order*

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Eq. above is called a first-order equation because the highest derivative is a first derivative

A second order equation would include a second derivative

❖ Overview

TABLE PT7.1 Examples of fundamental laws that are written in terms of the rate of change of variables (t = time and x = position).

Law	**Mathematical Expression**	**Variables and Parameters**
Newton's second law of motion	$\frac{dv}{dt} = \frac{F}{m}$	Velocity (v), force (F), and mass (m)
Fourier's heat law	$q = -k' \frac{dT}{dx}$	Heat flux (q), thermal conductivity (k') and temperature (T)
Fick's law of diffusion	$J = -D \frac{dc}{dx}$	Mass flux (J), diffusion coefficient (D), and concentration (c)
Faraday's law (voltage drop across an inductor)	$\Delta V_L = L \frac{di}{dt}$	Voltage drop (ΔV_L), inductance (L), and current (i)

❖ Overview

- This part is devoted to solving ordinary differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

the method was of the general form

$$\text{New value} = \text{old value} + \text{slope} \times \text{step size}$$

or, in mathematical terms,

$$y_{i+1} = y_i + \phi h$$

where the slope ϕ is called an increment function

Euler's method

❖ Euler's method $\frac{dy}{dt} = f(t, y)$

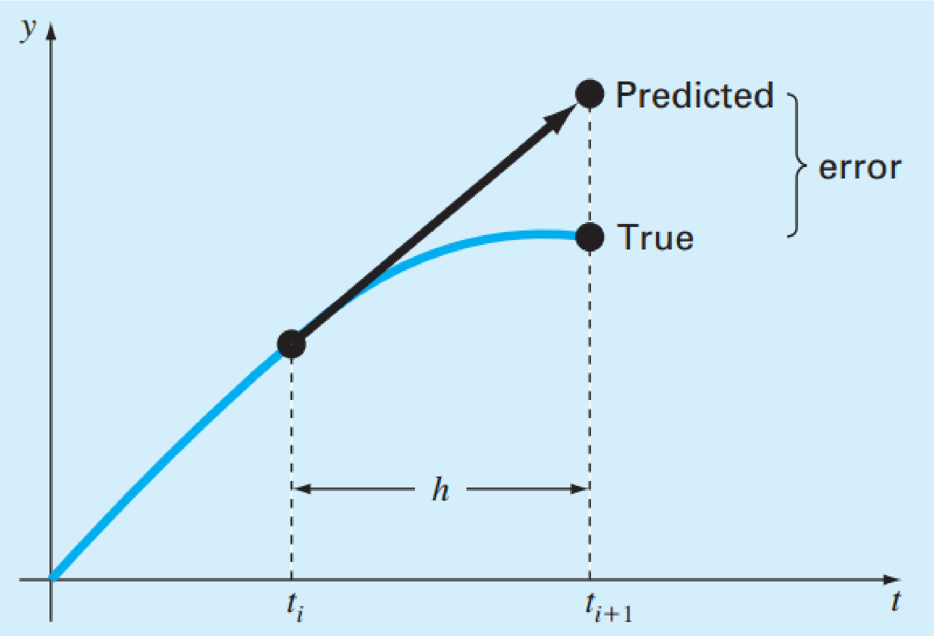
- The first derivative provides a direct estimate of the slope at t_i

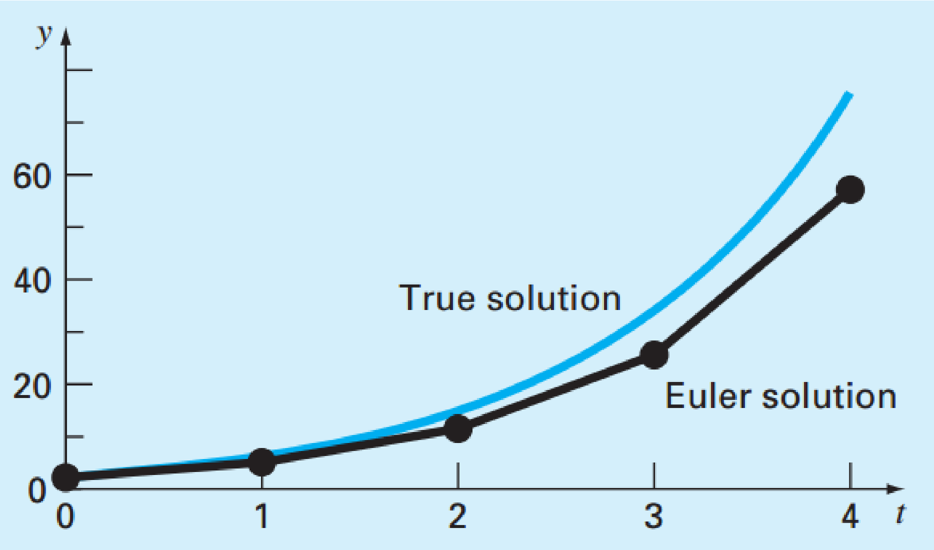
$$\phi = f(t_i, y_i)$$

where $f(t_i, y_i)$ is the differential equation evaluated at t_i and y_i . This estimate can be substituted into

$$y_{i+1} = y_i + f(t_i, y_i)h$$

$$y_{i+1} = y_i + \phi h$$





Example

Problem Statement. Use Euler's method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Note that the exact solution can be determined analytically as

$$y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$$

Hint: Euler's method

$$y_{i+1} = y_i + f(t_i, y_i)h$$

Euler's method

❖Example

Solution. Equation (22.5) can be used to implement Euler's method:

$$y(1) = y(0) + f(0, 2)(1)$$

where $y(0) = 2$ and the slope estimate at $t = 0$ is

$$f(0, 2) = 4e^0 - 0.5(2) = 3$$

Therefore,

$$y(1) = 2 + 3(1) = 5$$

The true solution at $t = 1$ is

$$y = \frac{4}{1.3}\left(e^{0.8(1)} - e^{-0.5(1)}\right) + 2e^{-0.5(1)} = 6.19463$$

Thus, the percent relative error is

$$\varepsilon_t = \left|\frac{6.19463 - 5}{6.19463}\right| \times 100\% = 19.28\%$$

Example

For the second step:

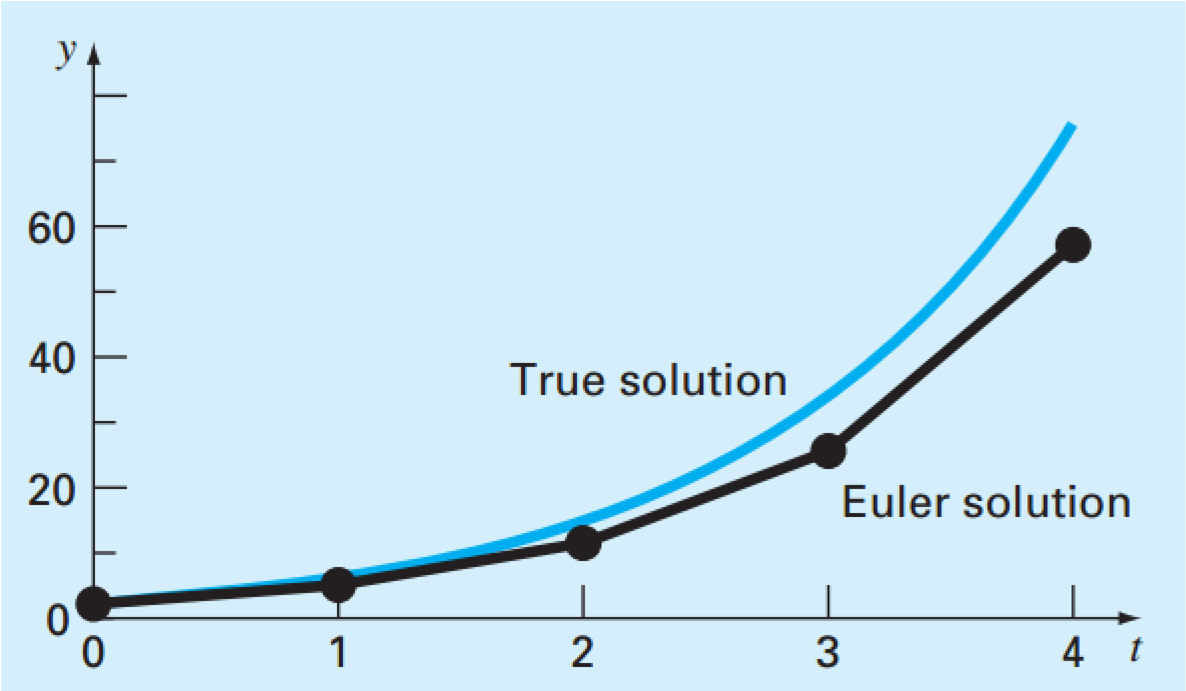
$$y(2) = y(1) + f(1, 5)(1)$$
$$= 5 + \left[4e^{0.8(1)} - 0.5(5)\right](1) = 11.40216$$

The true solution at t = 2 .0 is 14.84392 and, therefore, the true percent relative error is 23.19%. The computation is repeated

Euler's method

❖ Example

t	y_{true}	y_{Euler}	$\lvert\varepsilon_t\rvert$ (%)
0	2.00000	2.00000	
1	6.19463	5.00000	19.28
2	14.84392	11.40216	23.19
3	33.67717	25.51321	24.24
4	75.33896	56.84931	24.54



this error can be reduced by using a smaller step size.

❖Error Analysis for Euler's Method

1. Truncation, or discretization, errors caused by the nature of the techniques employed to approximate values of y.

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \cdots + \frac{y_i^{(n)}}{n!} h^n + R_n$$

2. Round off errors caused by the limited numbers of significant digits that can be retained by a computer.

The sum of the two is the total error. It is referred to as the global truncation error.

❖Stability of Euler's Method

- The truncation error of Euler's method depends on the step size in a predictable way based on the Taylor series.
- A numerical solution is said to be unstable if errors grow exponentially for a problem for which there is a bounded solution.
- The stability of a particular application can depend on three factors:
  - ✓ the differential equation,
  - ✓ the numerical method,
  - ✓ the step size

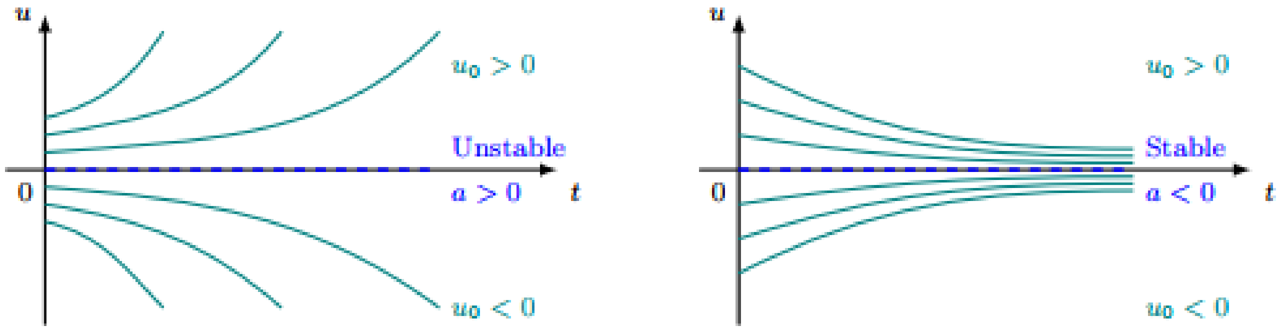


FIGURE 40. The graph of the functions $u(t) = u(0)\, e^{at}$ for $a > 0$ and $a < 0$.

❖Stability of Euler's Method

- Insight into the step size required for stability can be examined by studying a very simple ODE

$$\frac{dy}{dt} = -ay$$

If *y (0)* = *y*0, calculus can be used to determine the solution as

$$y = y_0 e^{-at}$$

Thus, the solution starts at y0 and asymptotically approaches zero

Stability of Euler's Method

- Now suppose that we use Euler's method to solve the same problem numerically:

$$y_{i+1} = y_i + \frac{dy_i}{dt} h$$

We have

$$y_{i+1} = y_i - a y_i h$$

or

$$y_{i+1} = y_i (1 - ah)$$

The parenthetical quantity (1 − *ah)* is called an *amplification factor*. If its absolute value is greater than unity, the solution will grow in an unbounded fashion

❖Stability of Euler's Method

- So clearly, the stability depends on the step size h, That is, |1 − ah| must be less than 1

$\Longrightarrow$ if $h > 2/a$, $|y_i| \to \infty$ as $i \to \infty$ $\Longrightarrow$ h<2/a

Euler's method is said to be conditionally stable: h<2/a.

❖Explicit and Implicit

- Explicit

$$\boxed{y_{i+1} = y_i + \frac{dy_i}{dt}h}$$

❖Explicit and Implicit

- Implicit: An implicit form of Euler's method can be developed by evaluating the derivative at the future time

$$\boxed{y_{i+1} = y_i + \frac{dy_{i+1}}{dt}h}$$

This is called the backward, or implicit, Euler's method

$$\Rightarrow \quad y_{i+1} = y_i - ay_{i+1}h \qquad \Rightarrow \quad y_{i+1} = \frac{y_i}{1 + ah}$$

For this case, regardless of the size of the step, $|y_i| \to 0$ as $i \to \infty$. Hence, the approach is called unconditionally stable.

Example

Use both the explicit and implicit Euler methods to solve

$$\frac{dy}{dt} = -1000y + 3000 - 2000e^{-t}$$

where y(0) = 0.

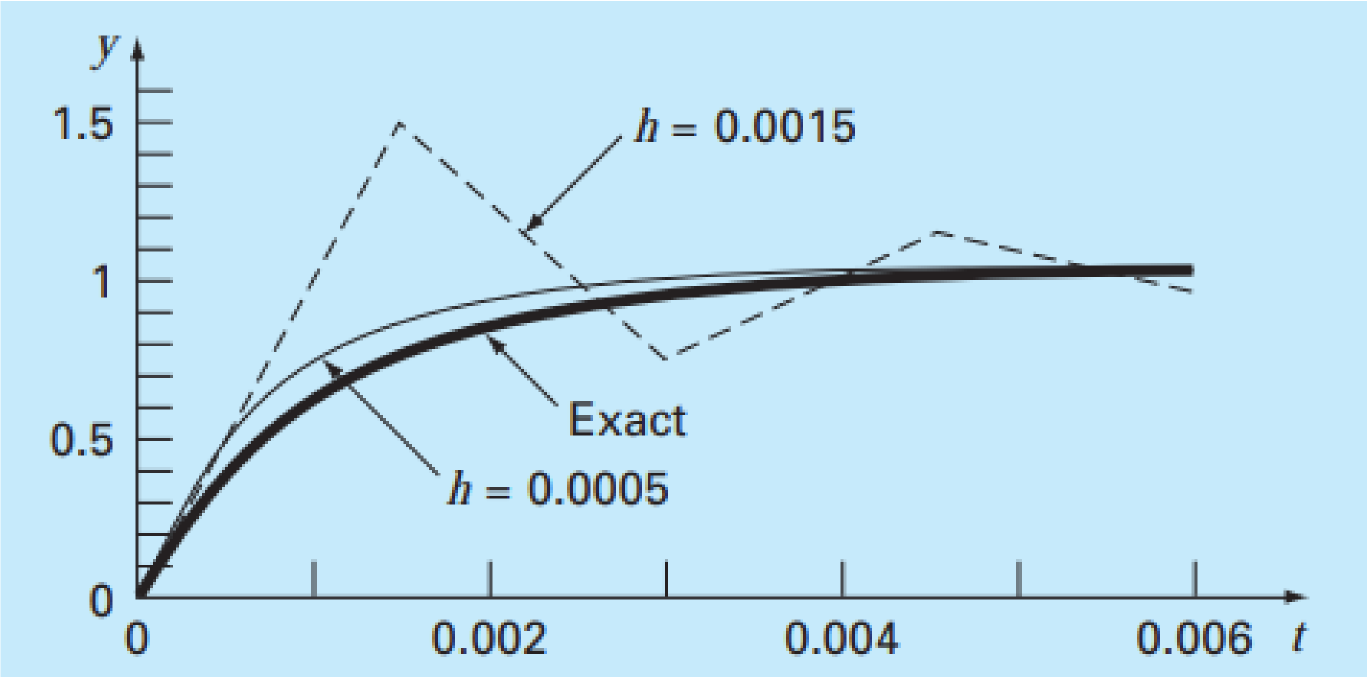
(a) Use the explicit Euler with step sizes of 0.0005 and 0.0015 to solve for *y* between *t* = 0 and 0.006.

(b) Use the implicit Euler with a step size of 0.05 to solve for y between 0 and 0.4.

❖Example

For this problem, the explicit Euler's method is

$$y_{i+1} = y_i + (-1000y_i + 3000 - 2000e^{-t_i})h$$



❖Example

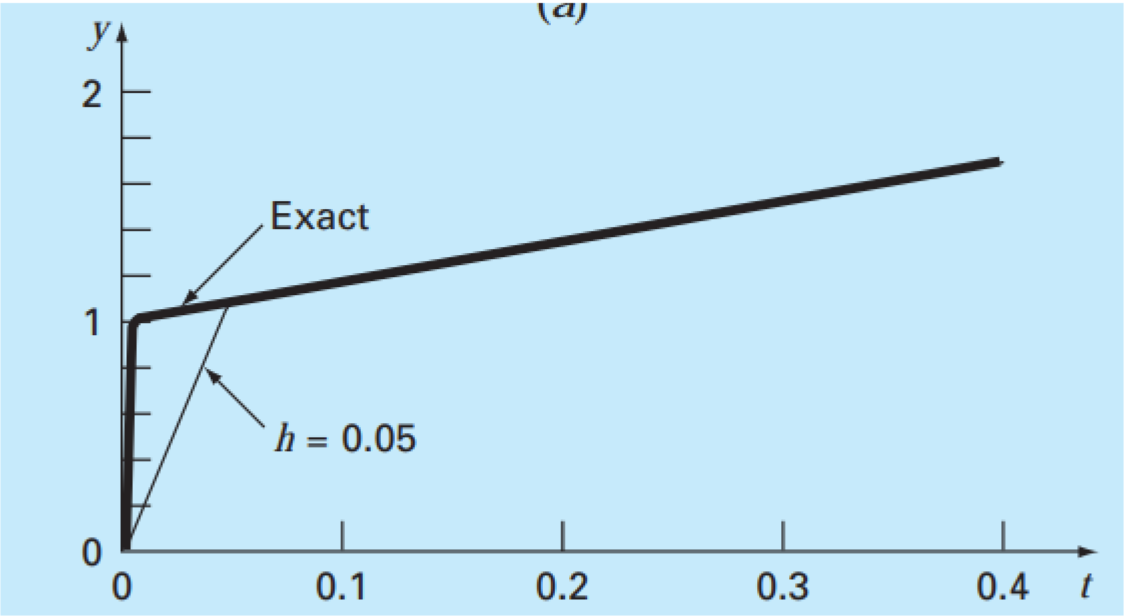
For this problem, the implicit Euler's method is

$$y_{i+1} = y_i + (-1000y_{i+1} + 3000 - 2000e^{-t_{i+1}})h$$

$$\Rightarrow \quad y_{i+1} = \frac{y_i + 3000h - 2000he^{-t_{i+1}}}{1 + 1000h}$$

Example

For this problem, the implicit Euler's method is



Project

- **Introduction**

The objective of this project is to determine the Heat Transfer through a wall by conduction using the heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

where: ρ = mass density *(kg/m³)*

c_p = specific heat capacity, *J/(kg. °C)*

λ = thermal conductivity of material, *W/(m.°C)*

$\partial T / \partial t$ = is the rate of change of temperature at a point over time

$\frac{\partial^2 T}{\partial x^2}; \frac{\partial^2 T}{\partial y^2}; \frac{\partial^2 T}{\partial z^2}$ = the second spatial derivatives (thermal conductions)

of temperature in the x, y, and z directions, respectively

T = temperature in the material *(°C)*

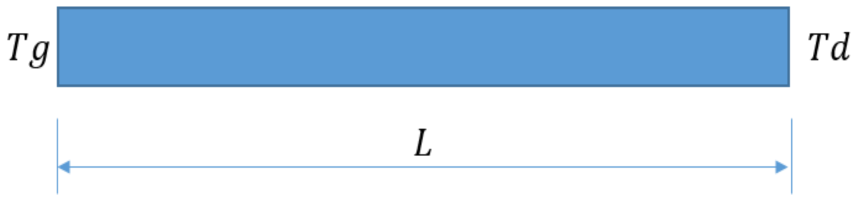
x = distance *(m)*

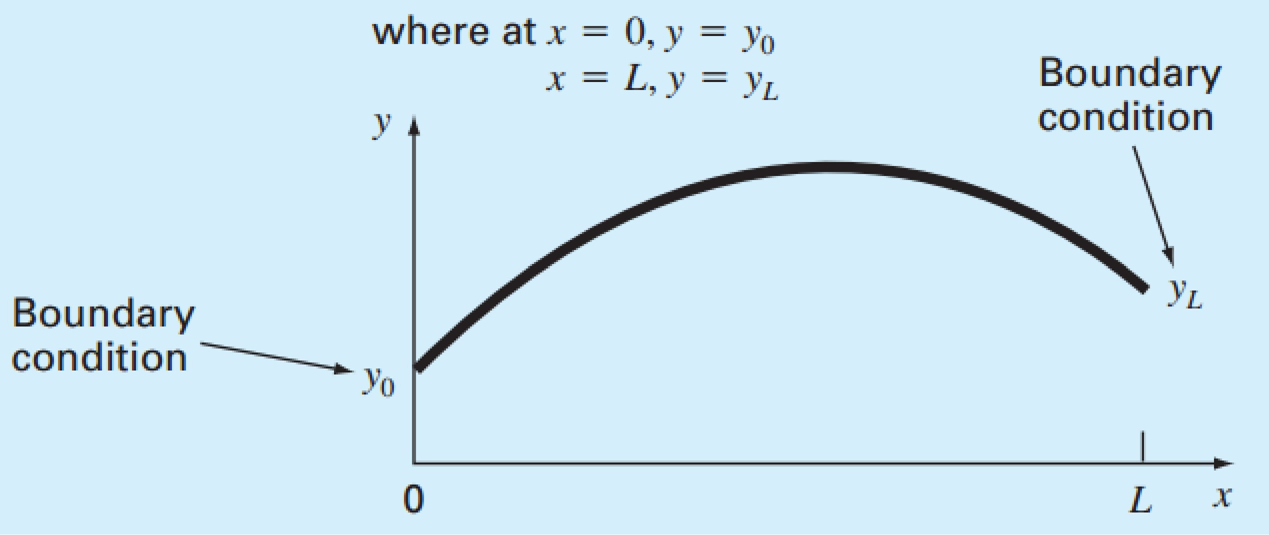
t = time *(s)*

Project

- **Project**

Let a bar of length L, consisting of a homogeneous and isotropic material. We suppose that The bar is perfectly insulated with the exception of the ends. The thermal properties of material will be taken constant





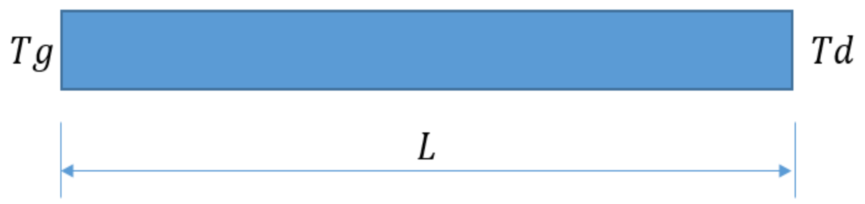
The 1D-equation describe the thermal transfers in this bar:

$$\rho c_p \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} \quad \text{or} \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (*)$$

Where $\alpha = \lambda / \rho c_p$ is called the thermal diffusivity.

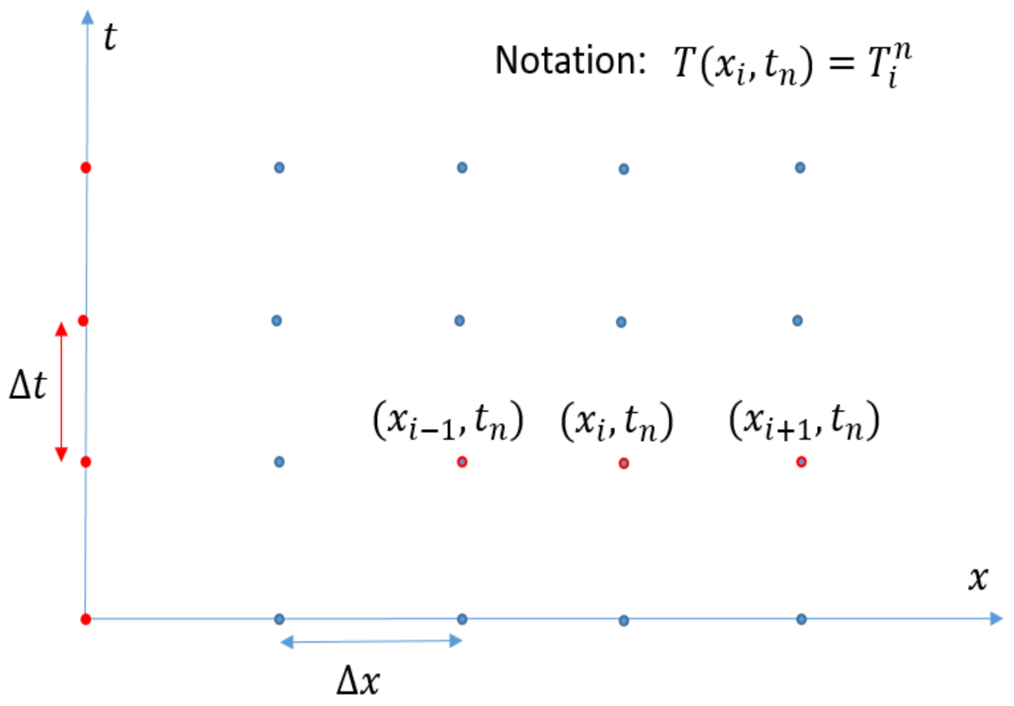
Project

To obtain the temperature filed in the bar, we define the following mesh: the bar is "cut" in M intervals of length Δx. Temperatures are calculated at different separate times by an interval Δt. The temperature $T(x_i, t_n)$ is denoted . We choose to define the coordinates spatial and temporal as well:



$x_i = (i-1)\Delta x$, $i = 0, \ldots M+1$

$t_n = (n-1)\Delta t$, $n = 0,\ldots N$



Project

Questions

1. Use the forward difference approximation for the first derivative

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

To express the left term of equation (*): $\dfrac{\partial T}{\partial t}$?

2. Use the centered difference approximation for the second derivative

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

To express the right term of equation (*): $\alpha \dfrac{\partial^2 T}{\partial x^2}$?

Matlab

- Matlab

❖Matlab

Problem Statement. Use Euler's method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Note that the exact solution can be determined analytically as

$$y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$$

Euler's method

❖ Matlab

Problem Statement. Use Euler's method to integrate $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Note that the exact solution can be determined analytically as

$$y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$$

```
dydt=@(t,y) 4*exp(0.8*t) - 0.5*y;

[t,y] = eulode(dydt,[0 4],2,1);

disp([t,y])

plot([t,y])
```

Euler's method

- Matlab

t	y_{true}	y_{Euler}	$\lvert\varepsilon_t\rvert$ (%)
0	2.00000	2.00000	
1	6.19463	5.00000	19.28
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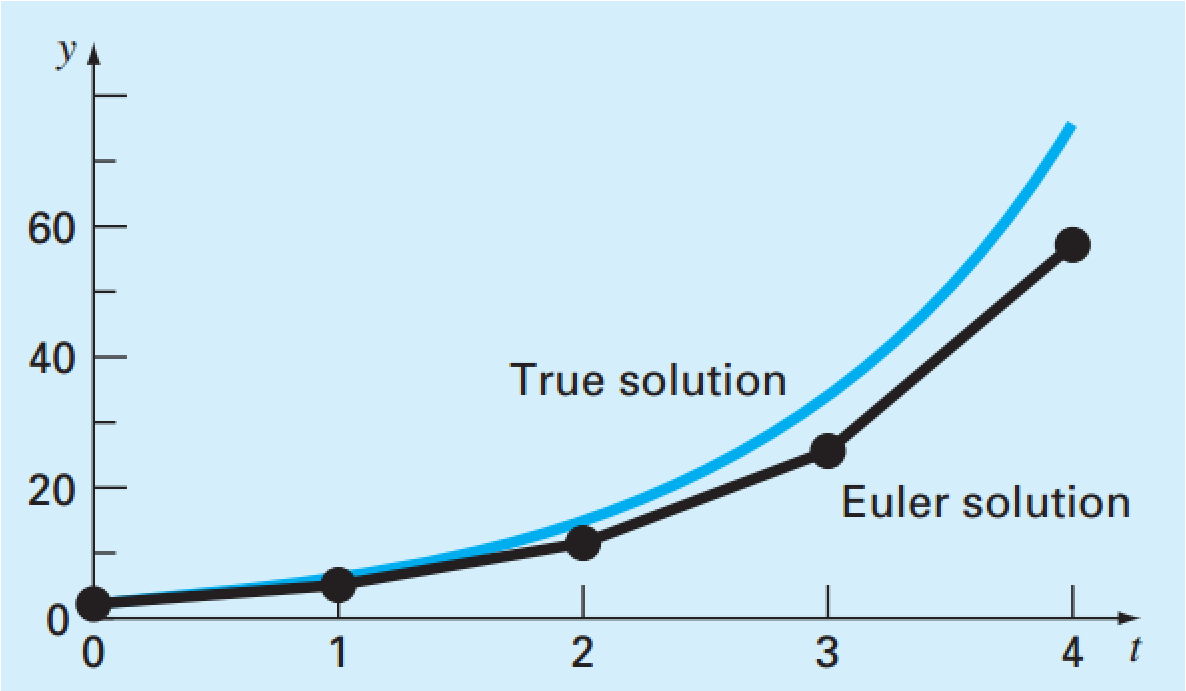

this error can be reduced by using a smaller step size.

Matlab

ODE Solvers in Matlab

Solver	Problem Type	Order of Accuracy	When to use
ode45	Nonstiff	Medium	Most of the time. This should be the first solver tried
ode23	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
ode113	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems
ode15s	Stiff	Low to medium	If ods45 is slow because the problem is stiff
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant
ode23t	Moderately stiff	Low	For moderately stiff problems is you need a solution without numerical damping
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems

Matlab

Solve the ODE:

$$y' = 2t$$

Use a time interval of [0,5] and the initial condition y0 = 0.

Matlab

```
tspan = [0 5];
y0 = 0;
[t,y] = ode45(@(t,y) 2*t, tspan, y0);
plot(t,y,'-o')
```