Numerical Methods

LU decomposition

Contents

- Topics
  - Eigenvalues and eigenvectors
  - LU decomposition
  - Decomposition with pivot

❖ Objectives

- ➢Understanding the mathematical definition of eigenvalues and eigenvectors
- ➢Understanding the physical interpretation of eigenvalues and eigenvectors within the context of engineering systems that vibrate or oscillate

MATHEMATICAL BACKGROUND

a homogeneous linear algebraic system has a right-hand side equal to zero

$$\boxed{[A]\{x\} = 0}$$

eigenvalue problems associated with engineering are typically of the general form

$$\boxed{[[A] - \lambda[I]]\,\{x\} = 0}$$

where the parameter λ is the eigenvalue

for nontrivial solutions to be possible, the determinant of the matrix must equal zero

$$|[A] - \lambda[I]| = 0$$

Expanding the determinant yields a polynomial in λ, which is called the characteristic polynomial. The roots of this polynomial are the solutions for the eigenvalues

❖ Case Study

Two-dimensional matrix example

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial of A

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix},$$
$$= 3 - 4\lambda + \lambda^2.$$

Setting the characteristic polynomial equal to zero, it has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Case Study

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For λ = 1, we have that equation

$$(A - I)v_{\lambda=1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any non-zero vector with $v_1 = -v_2$ solves this equation. Therefore, $v_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

is an eigenvector of A corresponding to λ = 1, as is any scalar multiple of this vector

For λ = 3, we have that equation

$$(A - 3I)v_{\lambda=3} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of A corresponding to λ = 3, as is any scalar multiple of this vector

LU decomposition

- **Topics**
  - Eigenvalues and eigenvectors
  - LU decomposition (or LU factorization)
  - Decomposition with pivot

When is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Table. Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$	**3.28**	**25.83**	**250.8**	**2501**

Objectives

- Another way of solving a system of equations is by using a factorization technique for matrices called LU decomposition
- Understanding that LU factorization involves decomposing the coefficient matrix into two triangular matrices that can then be used to efficiently evaluate different right-hand-side vectors.
- Knowing how to express Gauss elimination as an LU factorization

Applications

- Solving linear equations
- Inverting a matrix
- Computing the determinant

❖ Definition

➢ In numerical analysis and linear algebra, LU decomposition (where 'LU' stands for 'lower upper', and also called LU factorization) factors a matrix as the product of a lower triangular matrix and an upper triangular matrix

$$[A]\{x\} = \{b\}$$

$$\Downarrow$$

$$[A] = [L][U]$$

Lower triangular matrix

Upper triangular matrix

LU decomposition

- LU decomposition was originally derived as a decomposition of quadratic and bilinear forms. Lagrange, in the very first paper in his collected works (1759) derives the algorithm we call Gaussian elimination. Later Turing introduced the *LU* decomposition of a matrix in 1948 that is used to solve the system of linear equation
- Let *A* be a *m* × *m* with nonsingular square matrix. Then there exists two matrices *L* and *U* such that, where *L* is a lower triangular matrix and *U* is an upper triangular matrix.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{mm} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix}$$

Where, $A = LU$

LU decomposition

- Systems of linear algebraic equations $$[A]\{x\} = \{b\}$$

- Equation can be rearranged to give

$$[A]\{x\} - \{b\} = 0 \quad \Rightarrow \quad [L]\{[U]\{x\} - \{d\}\} = [A]\{x\} - \{b\}$$

- Suppose that Equation could be expressed as an upper triangular system. For example, for a 3 × 3 system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

- Equation can also be expressed in matrix notation and rearranged to give

$$\Rightarrow \quad [U]\{x\} - \{d\} = 0$$

LU decomposition

$$[U]\{x\} - \{d\} = 0 \quad (*)$$

- Lower diagonal matrix with 1's on the diagonal

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix} \quad \text{where} \quad f_{21} = \frac{a_{21}}{a_{11}} \quad f_{31} = \frac{a_{31}}{a_{11}} \quad f_{32} = \frac{a'_{32}}{a'_{22}}$$

- when Eq. (*) is premultiplied by [*L*]

$$[L]\{[U]\{x\} - \{d\}\} = [A]\{x\} - \{b\}$$

- it follows from the rules for matrix multiplication that

$$[L][U] = [A] \quad \text{and} \quad [L]\{d\} = \{b\}$$

LU decomposition

- A two-step strategy for obtaining solutions

1. *LU factorization step*

   [*A*] is factored or "decomposed" into lower [*L*] and upper [*U*] triangular matrices

$$[A] = [L][U]$$

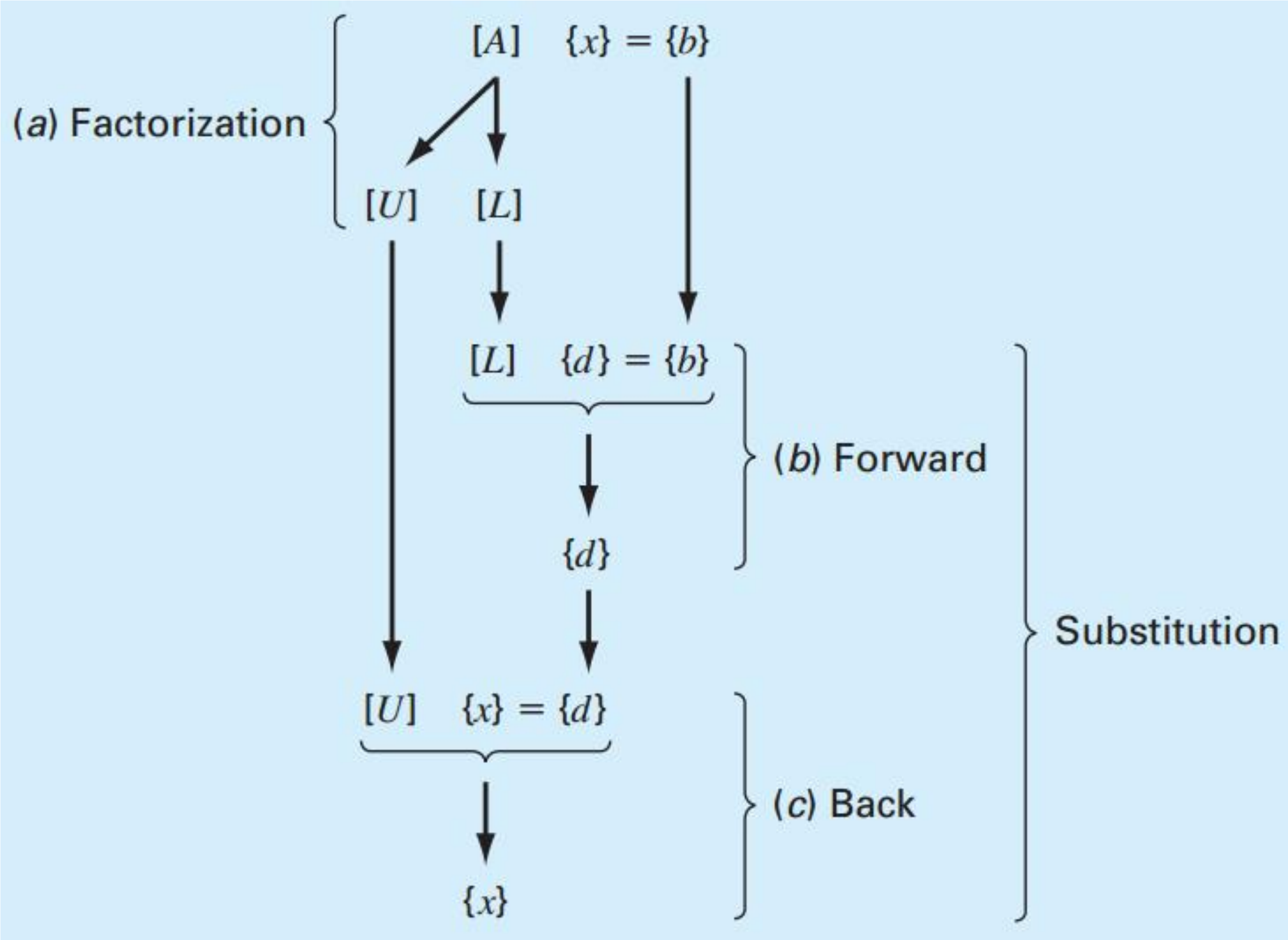
2. Substitution step

   [*L*] and [*U*] are used to determine a solution {*x*} for a right-hand side {*b*}. This step itself consists of two steps.

   a. First, generate an intermediate vector {*d*} by forward substitution

   b. Then, the result is substituted into equation which can be solved by back substitution for {*x*}

❖ The steps in LU factorization



1. LU decomposition step

2. Substitution step

Case study

Factorize the following 2-by-2 matrix

$$[A] = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} \Rightarrow [L] \text{ and } [U]\ ?$$

One way to find the LU decomposition of this simple matrix would be to simply solve the linear equations by inspection. Expanding the matrix multiplication gives

$$\begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

The forward-substitution phase is implemented

$$\begin{aligned} l_{11} \cdot u_{11} + 0 \cdot 0 &= 4 \\ l_{11} \cdot u_{12} + 0 \cdot u_{22} &= 3 \\ l_{21} \cdot u_{11} + l_{22} \cdot 0 &= 6 \\ l_{21} \cdot u_{12} + l_{22} \cdot u_{22} &= 3 \end{aligned}$$

❖ Case study

the lower triangular matrix L is a unit triangular matrix. Then the system of equations has the following solution

$$\begin{aligned} l_{21} &= 1.5 \\ u_{11} &= 4 \\ u_{12} &= 3 \\ u_{22} &= -1.5 \end{aligned}$$

Substituting these values into the LU decomposition above yields

$$\begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix}$$

Method: [A] Decompose to [L] and [U]

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1:

$$\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1:

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Step 2:

$$\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [*L*] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}= \ ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 10$$
$$2.56 z_1 + z_2 = 177.2$$
$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for $[Z]$

$$
\begin{aligned}
z_1 &= 106.8 \\
z_2 &= 177.2 - 2.56 z_1 \\
&= 177.2 - 2.56(106.8) \\
&= -96.2 \\
z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\
&= 279.2 - 5.76(106.8) - 3.5(-96.21) \\
&= 0.735
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}
$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$\begin{aligned} 25a_1 + 5a_2 + a_3 &= 106.8 \\ -4.8a_2 - 1.56a_3 &= -96.21 \\ 0.7a_3 &= 0.735 \end{aligned}$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned} a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5(19.70) - 1.050}{25} \\ &= 0.2900 \end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Case study

use LU to solve a set of linear algebraic equations

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

Case study

use LU to solve a set of linear algebraic equations

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

the forward-elimination phase of conventional Gauss elimination resulted in

$$[U] \Leftarrow \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

The forward-substitution phase is implemented to determine [L]

Case study

forward-elimination

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix} \Rightarrow \begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 7.00333x_2 - 0.293333x_3 &= -19.5617 \\ -0.190000x_2 + 10.0200x_3 &= 70.6150 \end{aligned}$$

$$f_{21} = \frac{0.1}{3} = 0.0333333 \qquad f_{31} = \frac{0.3}{3} = 0.1000000 \qquad f_{32} = \frac{-0.19}{7.00333} = -0.0271300$$

The forward-substitution phase is implemented

$$[L] \Leftarrow \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

❖ Case study

Consequently, the LU factorization is

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

This result can be verified by performing the multiplication of [L][U] to give

$$[L][U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.0999999 & 7 & -0.3 \\ 0.3 & -0.2 & 9.99996 \end{bmatrix} \qquad [A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

where the minor discrepancies are due to roundoff

Case study

We can solve the first equation for $d_1 = 7.85$

which can be substituted into the second equation to solve for

$$d_2 = -19.3 - 0.0333333(7.85) = -19.5617$$

Both d_1 and d_2 can be substituted into the third equation to give

$$d_3 = 71.4 - 0.1(7.85) + 0.02713(-19.5617) = 70.0843$$

Thus

$$\{d\} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

This result can then be substituted into Eq $[U]\{x\} = \{d\}$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix} \Rightarrow \{x\} = \begin{Bmatrix} 3 \\ -2.5 \\ 7.00003 \end{Bmatrix}$$

Finding the inverse of a square matrix

The inverse [B] of a square matrix [A] is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

$$[A][B] = [I] = [B][A]$$

How can LU Decomposition be used to find the inverse?

Assume the first column of [B] to be $[b_{11}\ \ b_{12}\ \ \ldots\ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of [B]

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of [B]

$$[A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in [B] can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix [*A*]

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the [*L*] and [*U*] matrices are found to be

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of [*B*] requires two steps

1) Solve [*L*] [*Z*] = [*C*] for [*Z*]

2) Solve [*U*] [*X*] = [*Z*] for [*X*]

Step 1:

$$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

This generates the equations:

$$z_1=1$$

$$2.56z_1+z_2=0$$

$$5.76z_1+3.5z_2+z_3=0$$

Example: Inverse of a Matrix

Solving for [*Z*]

$$
\begin{aligned}
z_1 &= 1 \\
z_2 &= 0 - 2.56 z_1 \\
&= 0 - 2.56(1) \\
&= -2.56 \\
z_3 &= 0 - 5.76 z_1 - 3.5 z_2 \\
&= 0 - 5.76(1) - 3.5(-2.56) \\
&= 3.2
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}
$$

Example: Inverse of a Matrix

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$
$$-4.8b_{21} - 1.56b_{31} = -2.56$$
$$0.7b_{31} = 3.2$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$b_{21} = \frac{-2.56 + 1.560 b_{31}}{-4.8}$$

$$= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524$$

$$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25}$$

$$= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762$$

So the first column of the inverse of [*A*] is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of [*A*] is

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

❖ LU decomposition with pivoting

- Just as for standard Gauss elimination, partial pivoting is necessary to obtain reliable solutions with LU factorization
- One way to do this involves using a permutation matrix. The approach consists of the following steps:
  - Step 1- Elimination. The *LU* factorization with pivoting of a matrix [A] can be represented in matrix form as

$$[P][A]=[L][U]$$

The upper triangular matrix, [U], is generated by elimination with partial pivoting, while storing the multiplier factors in [L] and employing the permutation matrix, [P], to keep track of the row switches

❖ LU decomposition with pivoting

➢ Step 2 - Forward substitution

The matrices [L] and [P] are used to perform the elimination step with pivoting on {b} in order to generate the intermediate right-hand-side vector, {d}

This step can be represented concisely as the solution of the following matrix formulation:

$$[L]\{d\}=[P]\{b\}$$

LU decomposition with pivoting

Step 3 – Back substitution

The final solution is generated in the same fashion as done previously for Gauss elimination. This step can also be represented concisely as the solution of the matrix formulation:

$$[U]\{x\}=\{d\}$$

Case study

used LU to solve a set of linear algebraic equations

$$\begin{bmatrix} 0.0003 & 3.0000 \\ 1.0000 & 1.0000 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2.0001 \\ 1.0000 \end{Bmatrix}$$

Before elimination, we set up the initial permutation matrix

$$[P] = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

We immediately see that pivoting is necessary, so prior to elimination we switch the rows

$$[A] = \begin{bmatrix} 0.0003 & 3.0000 \\ 1.0000 & 1.0000 \end{bmatrix} \Longrightarrow [A] = \begin{bmatrix} 1.0000 & 1.0000 \\ 0.0003 & 3.0000 \end{bmatrix}$$

Case study

use LU to solve a set of linear algebraic equations

$$\begin{bmatrix} 0.0003 & 3.0000 \\ 1.0000 & 1.0000 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2.0001 \\ 1.0000 \end{Bmatrix}$$

At the same time, we keep track of the pivot by switching the rows of the permutation matrix:

$$[P] = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} \Rightarrow [P] = \begin{bmatrix} 0.0000 & 1.0000 \\ 1.0000 & 0.0000 \end{bmatrix}$$

the elimination step is complete with the result

$$[U] = \begin{bmatrix} 1 & 1 \\ 0 & 2.9997 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 \\ 0.0003 & 1 \end{bmatrix}$$

Case study

Before implementing forward substitution, the permutation matrix is used to reorder the right-hand-side vector to reflect the pivots as in

$$[P]\{b\} = \begin{bmatrix} 0.0000 & 1.0000 \\ 1.0000 & 0.0000 \end{bmatrix} \begin{Bmatrix} 2.0001 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.0001 \end{Bmatrix}$$

Then, forward substitution is applied as in

$$\begin{bmatrix} 1 & 0 \\ 0.0003 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2.0001 \end{Bmatrix}$$

which can be solved for $d_1 = 1$ and $d_2 = 2.0001 - 0.0003(1) = 1.9998$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2.9997 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.9998 \end{Bmatrix} \Rightarrow$$

$$x_2 = \frac{1.9998}{2.9997} = 0.66667$$

$$x_1 = \frac{1 - 1(0.66667)}{1} = 0.33333$$

Case study

- The *LU* factorization algorithm requires the same total flops as for Gauss elimination

- The only difference is that a little less effort is expended in the factorization phase since the operations are not applied to the right-hand side

- Conversely, the substitution phase takes a little more effort

Which is better, Gauss Elimination or LU Decomposition?

When is LU Decomposition better than Gaussian Elimination?

To solve $[A][X] = [B]$

Table. Time taken by methods

Gaussian Elimination	LU Decomposition
$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$	$T\left(\frac{8n^3}{3}+12n^2+\frac{4n}{3}\right)$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

To find inverse of [A]

Time taken by Gaussian Elimination

$$= n\left(CT|_{FE} + CT|_{BS}\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

Time taken by LU Decomposition

$$= CT|_{LU} + n \times CT|_{FS} + n \times CT|_{BS}$$

$$= T\left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3}\right)$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\|_{\text{inverse GE}} / CT\|_{\text{inverse LU}}$	**3.28**	**25.83**	**250.8**	**2501**

Exercises

Use the LU factorization to find the solution

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 3x_3 &= -6 \\ 2x_1 + 3x_2 + x_3 &= 7 \end{aligned}$$

Exercises

Use the LU factorization to find the solution

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}. \qquad [U] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 13/3 \end{bmatrix}.$$

$$[x] = \{1 \quad 2 \quad -1\}$$

Method	Procedure	Remedies
Gauss elimination	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vert & c_1 \\ a_{21} & a_{22} & a_{23} & \vert & c_2 \\ a_{31} & a_{32} & a_{33} & \vert & c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vert & c_1 \\ & a'_{22} & a'_{23} & \vert & c'_2 \\ & & a''_{33} & \vert & c''_3 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = c''_3/a''_{33} \\ x_2 = (c'_2 - a'_{23}x_3)/a'_{22} \\ x_1 = (c_1 - a_{12}x_1 - a_{13}x_3)/a_{11} \end{matrix}$	**Problems:** Ill conditioning Round-off Division by zero **Remedies:** Higher precision Partial pivoting
LU decomposition	Decomposition; Back Substitution $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ Forward Substitution	**Problems:** Ill conditioning Round-off Division by zero **Remedies:** Higher precision Partial pivoting
Gauss-Seidel method	$\left.\begin{matrix} x_1^j = (c_1 - a_{12}x_2^{j-1} - a_{13}x_3^{j-1})/a_{11} \\ x_2^j = (c_2 - a_{21}x_1^j - a_{23}x_3^{j-1})/a_{22} \\ x_3^j = (c_3 - a_{31}x_1^j - a_{32}x_2^j)/a_{33} \end{matrix}\right\}$ continue iteratively until $\left\vert \frac{x_i^j - x_i^{j-1}}{x_i^j} \right\vert 100\% < \epsilon_s$ for all x_i's	**Problems:** Divergent or converges slowly **Remedies:** Diagonal dominance Relaxation

Matlab

Eigenvalues and Eigenvectors With MATLAB

Use MATLAB to determine all the eigenvalues and eigenvectors for the system

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix}$$

The matrix can be entered as

```
>> A = [40 -20 0;-20 40 -20;0 -20 40];
```

If we just desire the eigenvalues, we can enter

```
>> e = eig(A)
```

```
e =
11.7157
40.0000
68.2843
```

Eigenvalues and Eigenvectors With MATLAB

Use MATLAB to determine all the eigenvalues and eigenvectors for the system

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix}$$

If we want both the eigenvalues and eigenvectors, we can enter

```
>> [v,d] = eig(A)
```

```
v =
    0.5000   -0.7071   -0.5000
    0.7071   -0.0000    0.7071
    0.5000    0.7071   -0.5000
```

```
d =
   11.7157         0         0
         0   40.0000         0
         0         0    8.2843
```

full matrix whose columns are the corresponding eigenvectors

SOLVING LINEAR ALGEBRAIC EQUATIONS WITH MATLAB

Use MATLAB to compute the LU factorization and find the solution

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

The coefficient matrix and the right-hand-side vector can be entered in standard fashion as

```
>> A = [3 -.1 -.2;.1 7 -.3;.3 -.2 10];
>> b = [7.85; -19.3; 71.4];
```

Next, the *LU* factorization can be computed with

```
>> [L,U] = lu(A)
```

```
L =
    1.0000         0         0
    0.0333    1.0000         0
    0.1000   -0.0271    1.0000
```

We can test that it is correct by computing the original matrix as

```
>> L*U
```

To generate the solution, we first compute
```
>> d = L\b
```

And then use this result to compute the solution
```
>> x = U\d
```

```
x =     3.0000
       -2.5000
        7.0000
```

Exercises

```
function [L,U,P] = mylup(A)
[n,m] = size(A);
C = zeros(n,n+1);
% initialize
C(1:n,1:n) = A;
C(1:n,n+1) = (1:n)';
% main algorithm
for j=1:n-1
    if (sum(C(1:n,j))==0)
        error('mylup:input','The solution
is not determined.');
    end
    [~,i]= max(abs(C(1:n,j)));
    if (i~= j)
        tt = C(i,1:n+1);
        C(i,1:n+1) = C(j,1:n+1);
        C(j,1:n+1) = tt;
    end
    for  i = j+1:n
         t = C(i,j)/C(j,j);
         C(i,j) = t;
         C(i,j+1:n) = C(i,j+1:n)
- t*C(j, j+1:n);
    end
end
% refine the results
PP = C(1:n,n+1);
C = C(1:n,1:n);
L = tril(C,-1);
U = triu(C);
P = zeros(n,n);
for i=1:n
    L(i,i)=1;
    P(i, PP(i))=1;
end
end
```