Numerical Methods

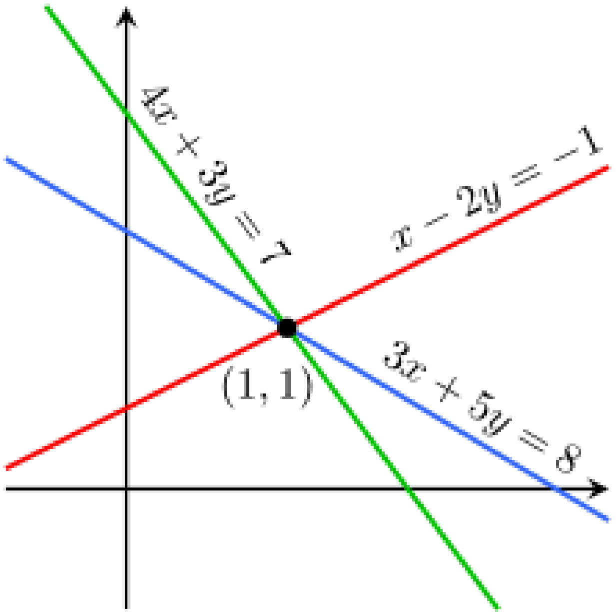
Linear Programming

Contents

Linear Programming

- System of linear equations

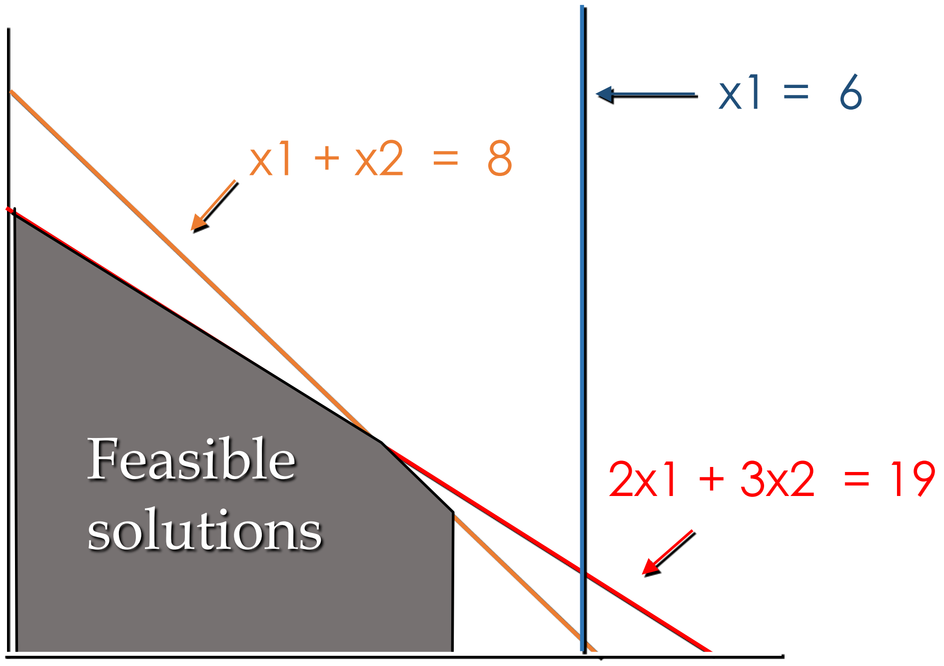
$$
\begin{aligned}
x - 2y &= -1 \\
3x + 5y &= 8 \\
4x + 3y &= 7
\end{aligned}
$$



The solution is the single point (1, 1)

- System of linear inequalities

$$
\begin{aligned}
x1 &< 6 \\
2x1 + 3x2 &< 19 \\
x1 + x2 &< 8
\end{aligned}
$$

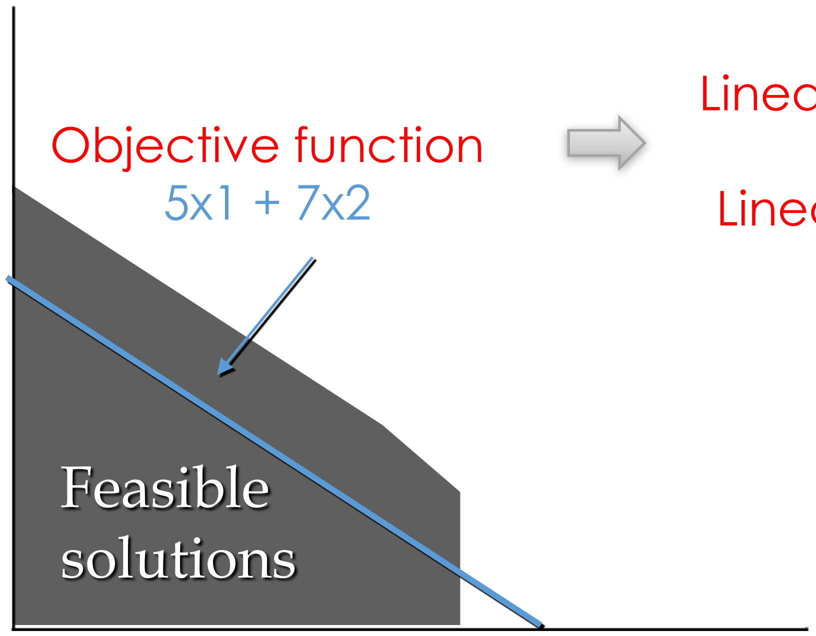


Linear Programming

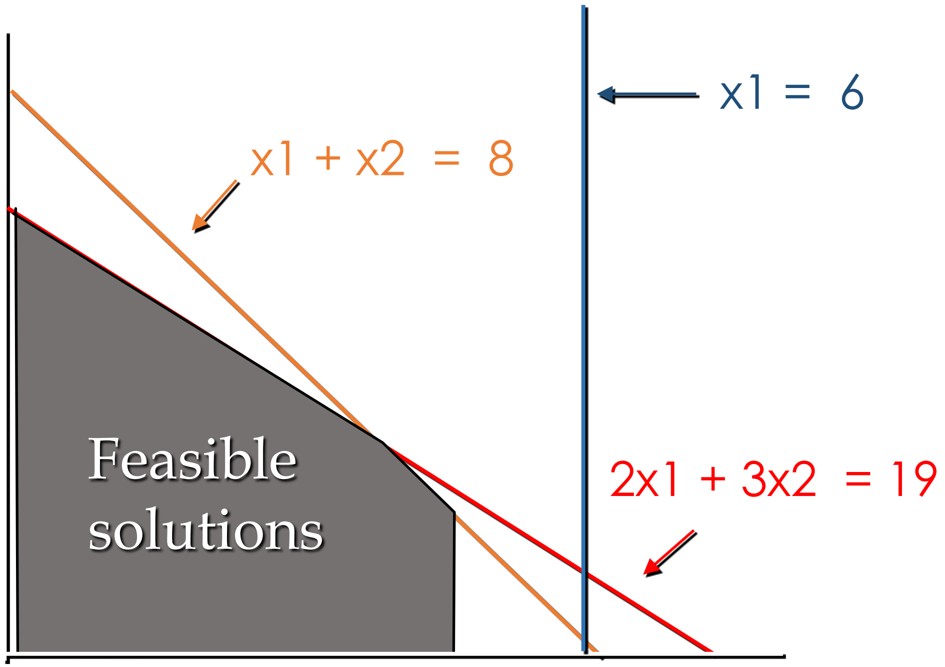
Maximize: $F = 5x_1 + 7x_2$?

- System of linear inequalities (constraints)

$$
\begin{aligned}
x_1 &< 6 \\
2x_1 + 3x_2 &< 19 \\
x_1 + x_2 &< 8
\end{aligned}
$$



Linear programming
Or
Linear optimization



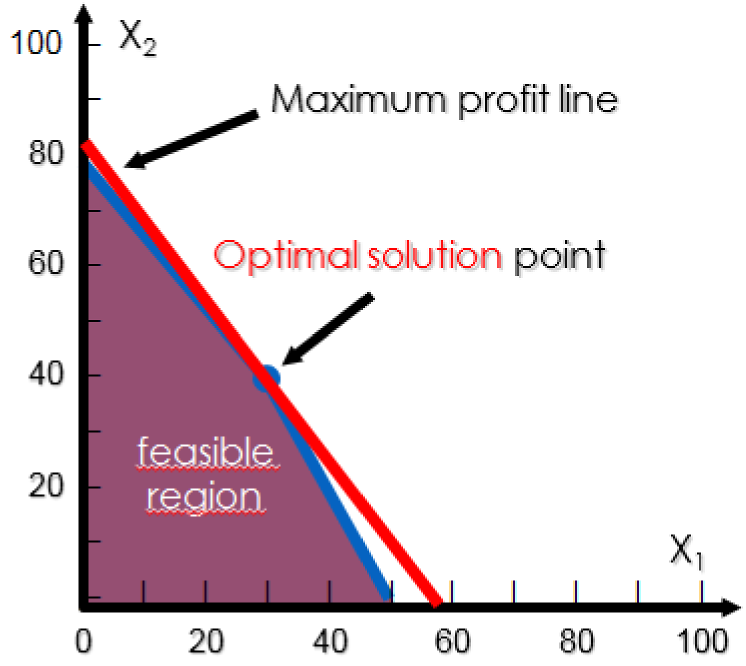
A Simple Maximization Problem

Linear Programming

- Linear programming
  - Standard forms
  - Simplex method

Linear Programming

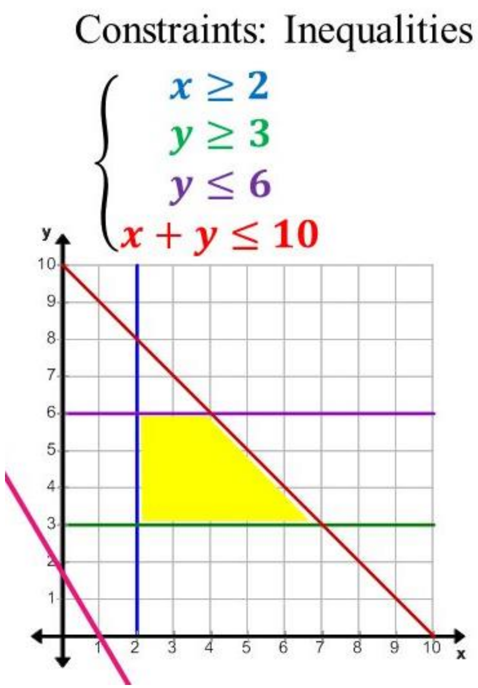
- Notion
  - Linear programming (LP) is an optimization approach that deals with meeting a desired objective such as maximizing profit or minimizing cost in the presence of constraints such as limited resources.

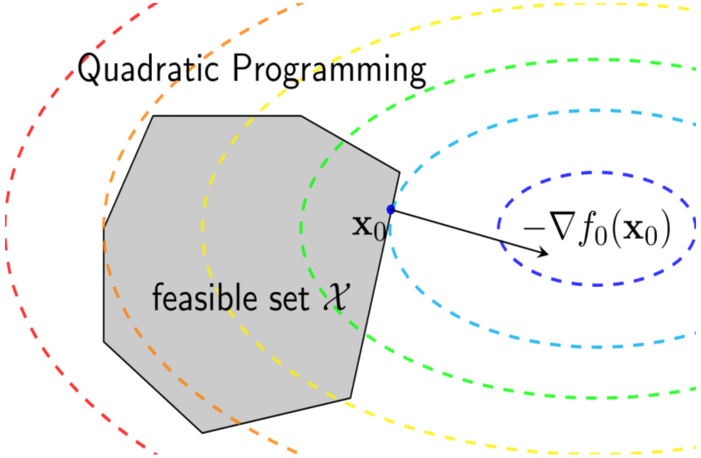


Linear Programming

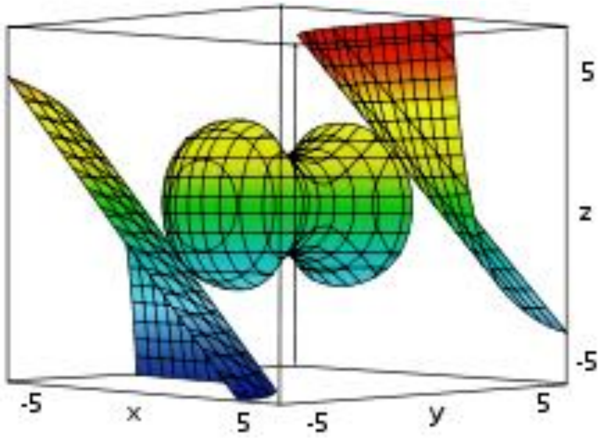
❖ Notion

➢ The term linear connotes that the mathematical functions representing both the objective and the constraints are linear. Its feasible region is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality





objective : quadratic
constraints : linear



Nonlinear Programming

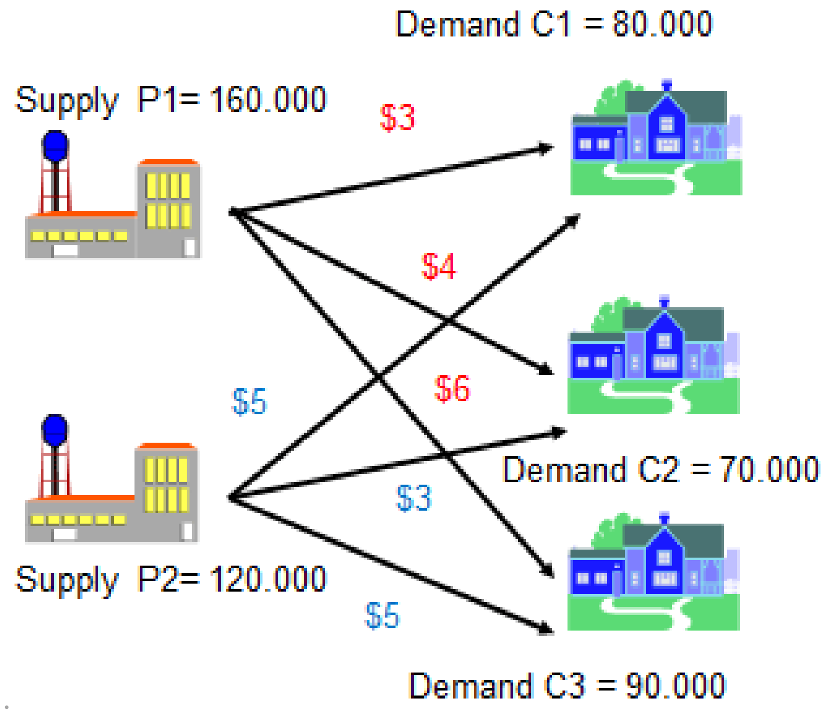
objective : nonlinear
constraints : nonlinear

Linear Programming

❖ Notion

➢ LP can be applied for engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing.

⇨ cost optimization



Standard Form

the basic linear programming problem consists of two major parts: the objective function and a set of constraints. For a maximization problem, the objective function is generally expressed as

$$Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

c_j = payoff of each unit of the j^{th} activity that is undertaken
x_j = magnitude of the j^{th} activity.
the value of the objective function, Z, is the total payoff due to the total number of activities, n.
The constraints can be represented generally as

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$$

a_{ij} = amount of the i^{th} resource that is consumed for each unit of the j^{th} activity
b_i = amount of the i^{th} resource that is available.
That is, the resources are limited.

Standard Form

the second general type of constraint specifies that all activities must have a positive value,

$$x_i \geq 0$$

Together, the objective function and the constraints specify the linear programming problem.
They say that we are trying to maximize the payoff for a number of activities under the constraint that these activities utilize finite amounts of resources

Standard Form

Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following three parts:

- A linear function to be maximized $f(x_1, x_2) = c_1x_1 + c_2x_2$
- Problem constraints of the following form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\le b_1 \\ a_{21}x_1 + a_{22}x_2 &\le b_2 \\ a_{31}x_1 + a_{32}x_2 &\le b_3 \end{aligned}$$

- Non-negative variables

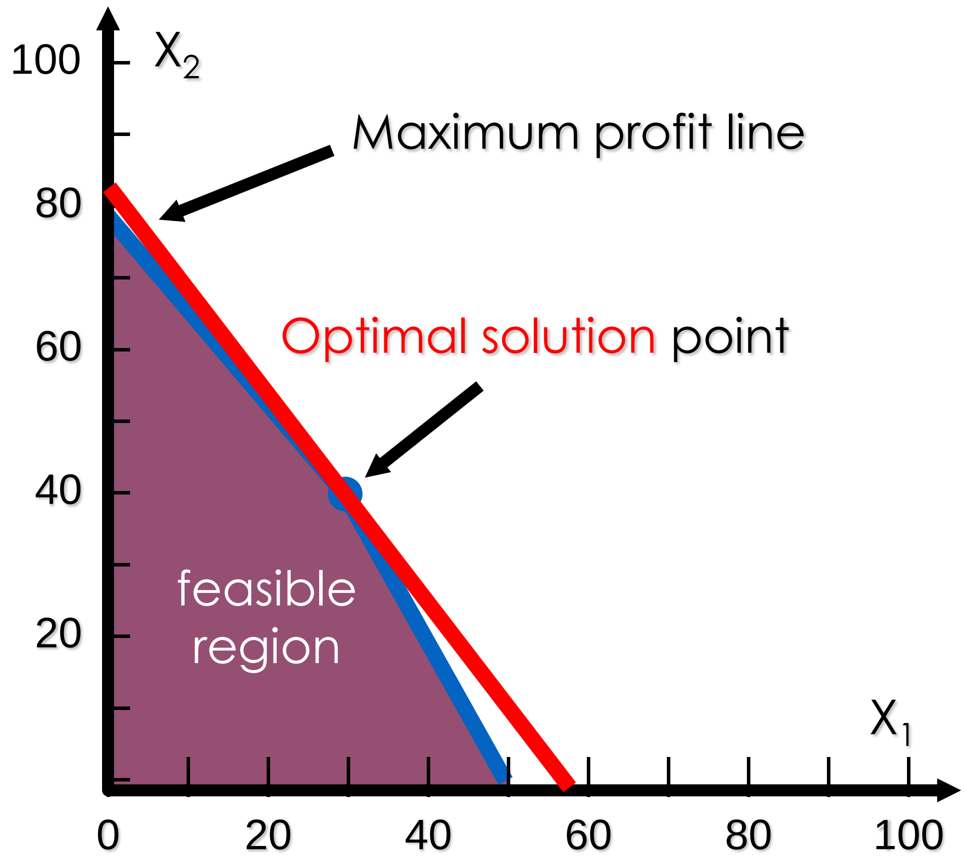
$$\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

The problem is usually expressed in matrix form, and then becomes:

$$\max\{\mathbf{c}^{\mathrm{T}}\mathbf{x} \mid A\mathbf{x} \le \mathbf{b} \land \mathbf{x} \ge 0\}$$

Standard Form

- A feasible solution or feasible region satisfies all the problem's constraints.
- An optimal solution is a feasible solution that results in the largest possible objective function value when maximizing (or smallest when minimizing).
- A graphical solution method can be used to solve a linear program



Example 1: A Simple Maximization Problem

$$\begin{aligned} \text{Max} \quad & 5x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 \le 6 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \\ & x_1 \ge 0 \text{ and } x_2 \ge 0 \end{aligned}$$

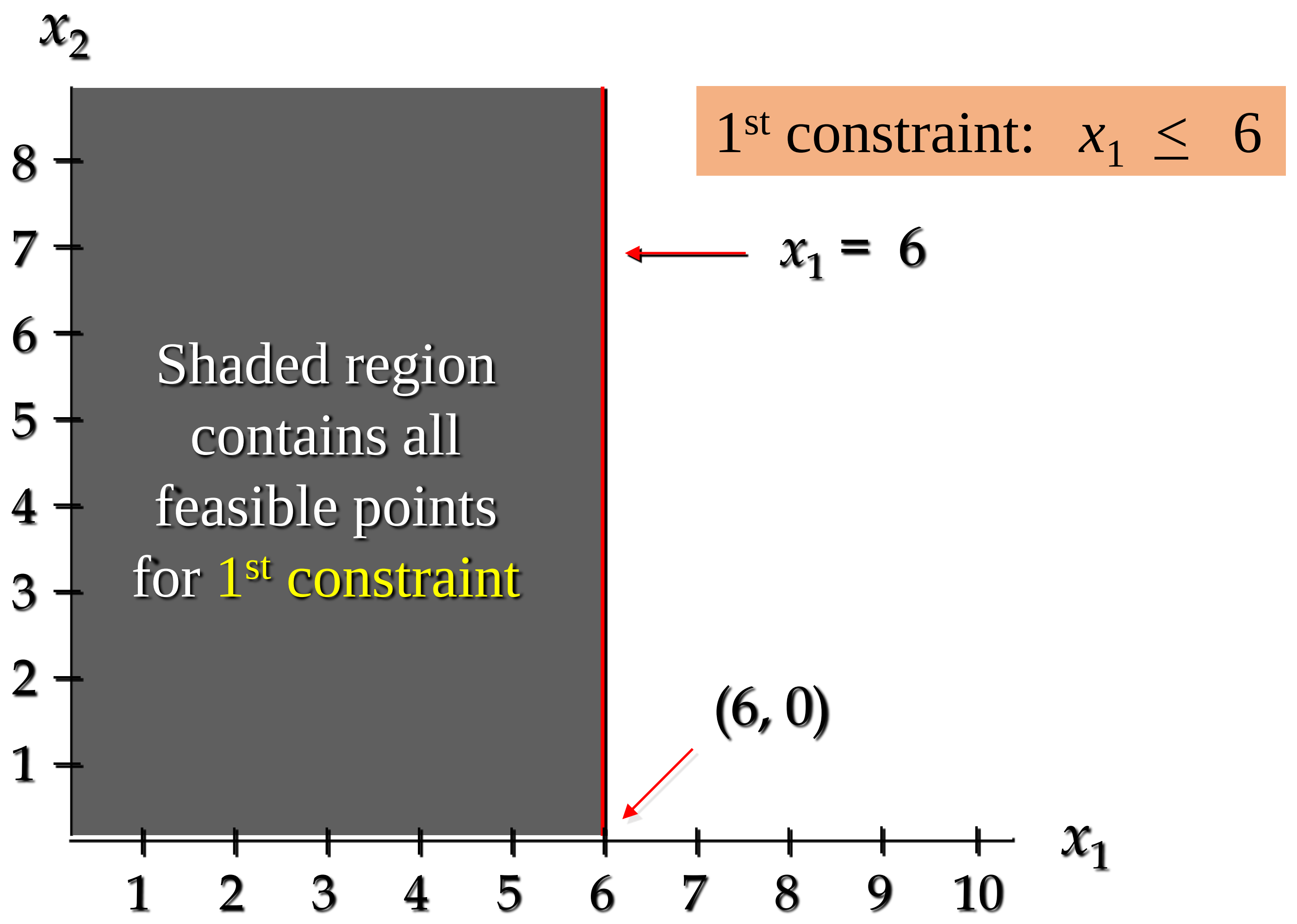
Objective Function

"Regular" Constraints

Non-negativity Constraints

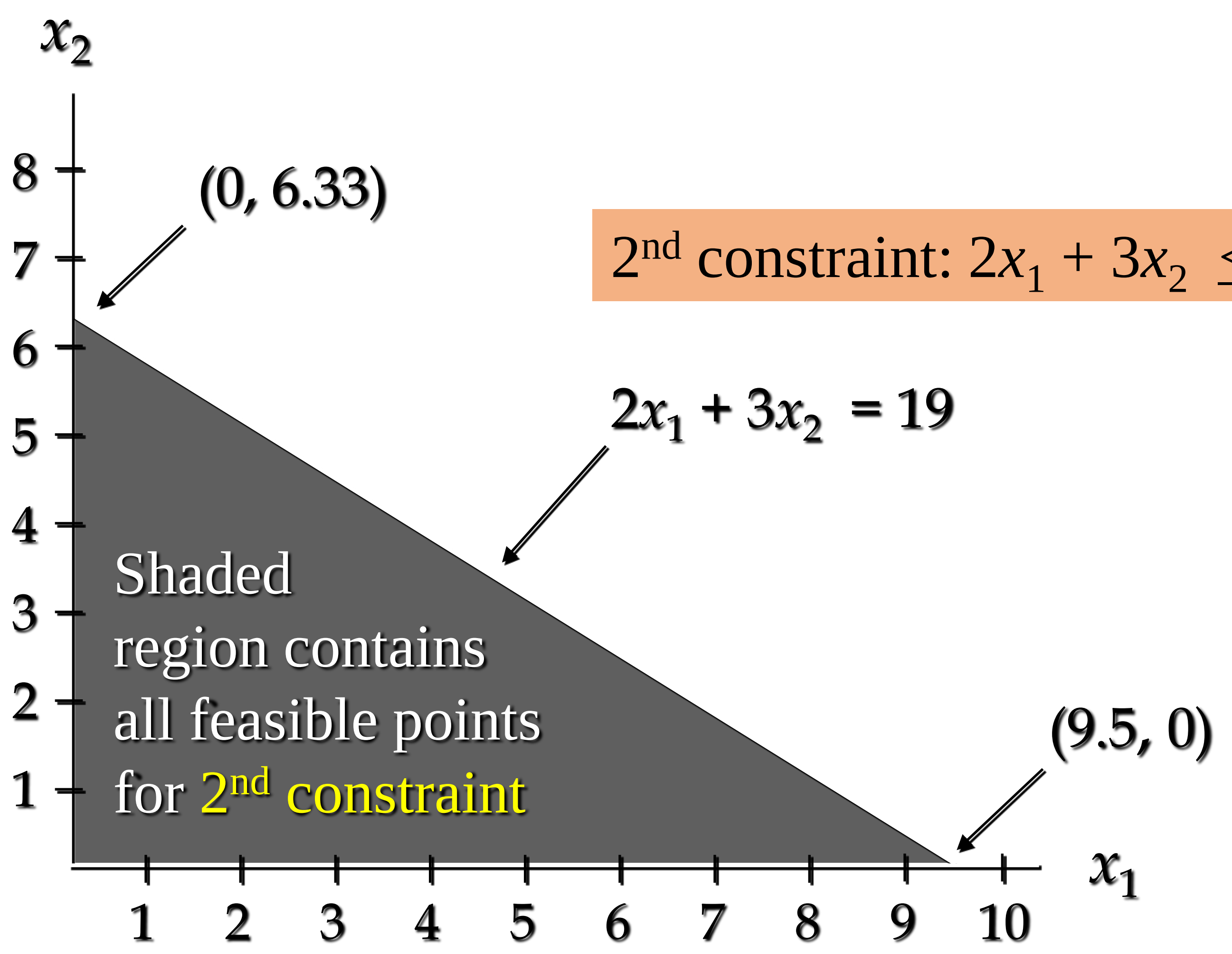
Example 1: Graphical Solution

- First Constraint Graphed



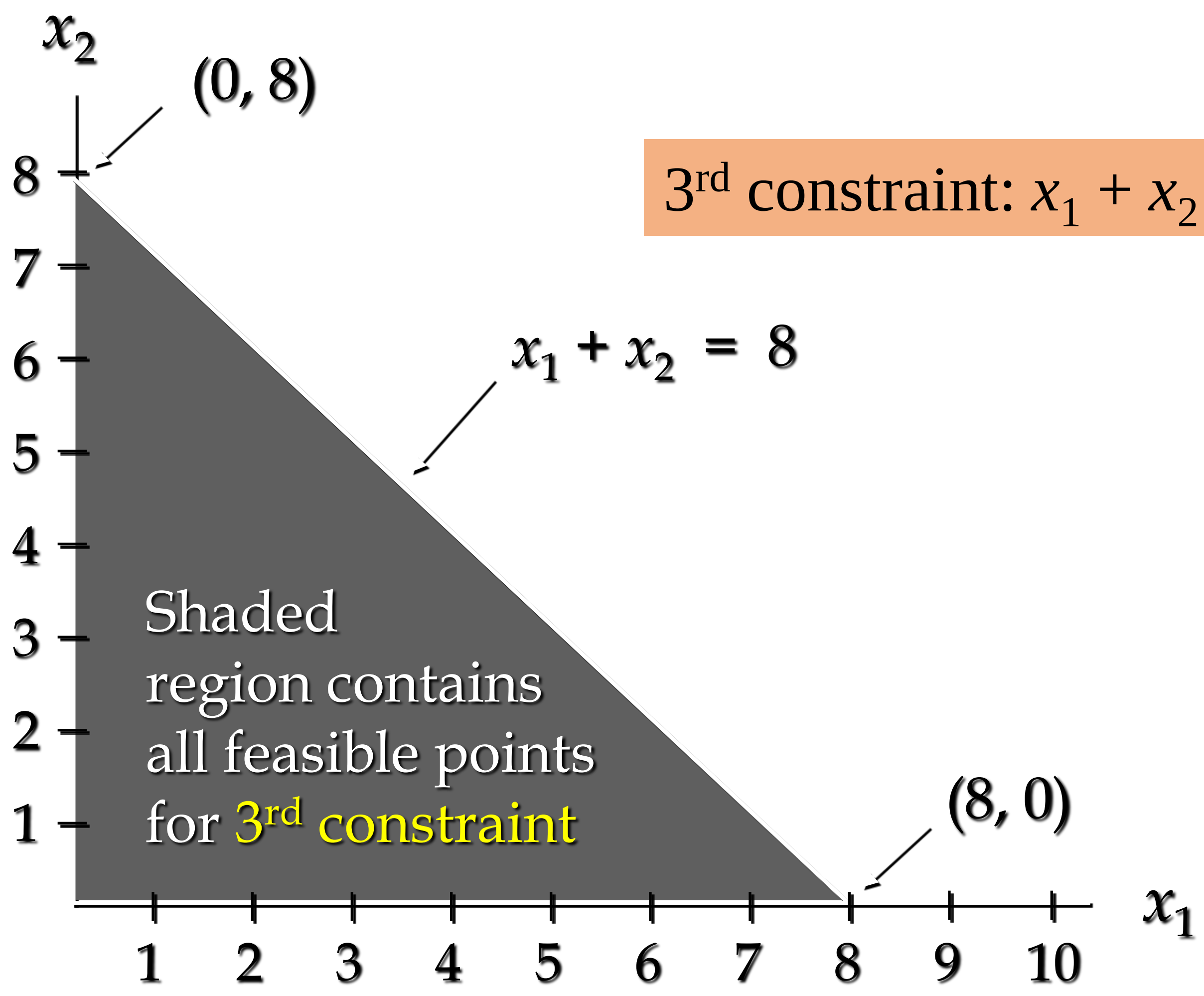
Example 1: Graphical Solution

- Second Constraint Graphed



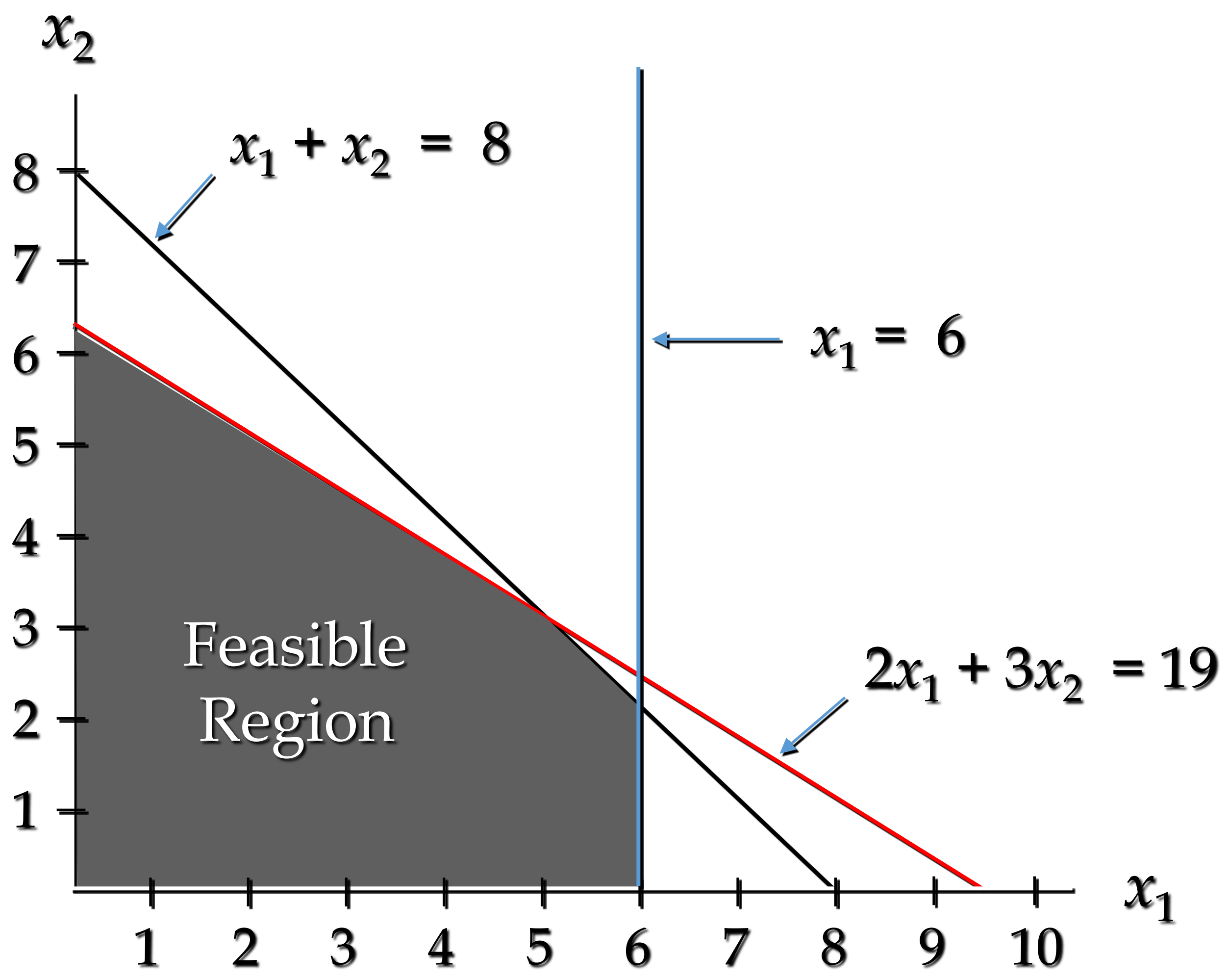
Example 1: Graphical Solution

- Third Constraint Graphed



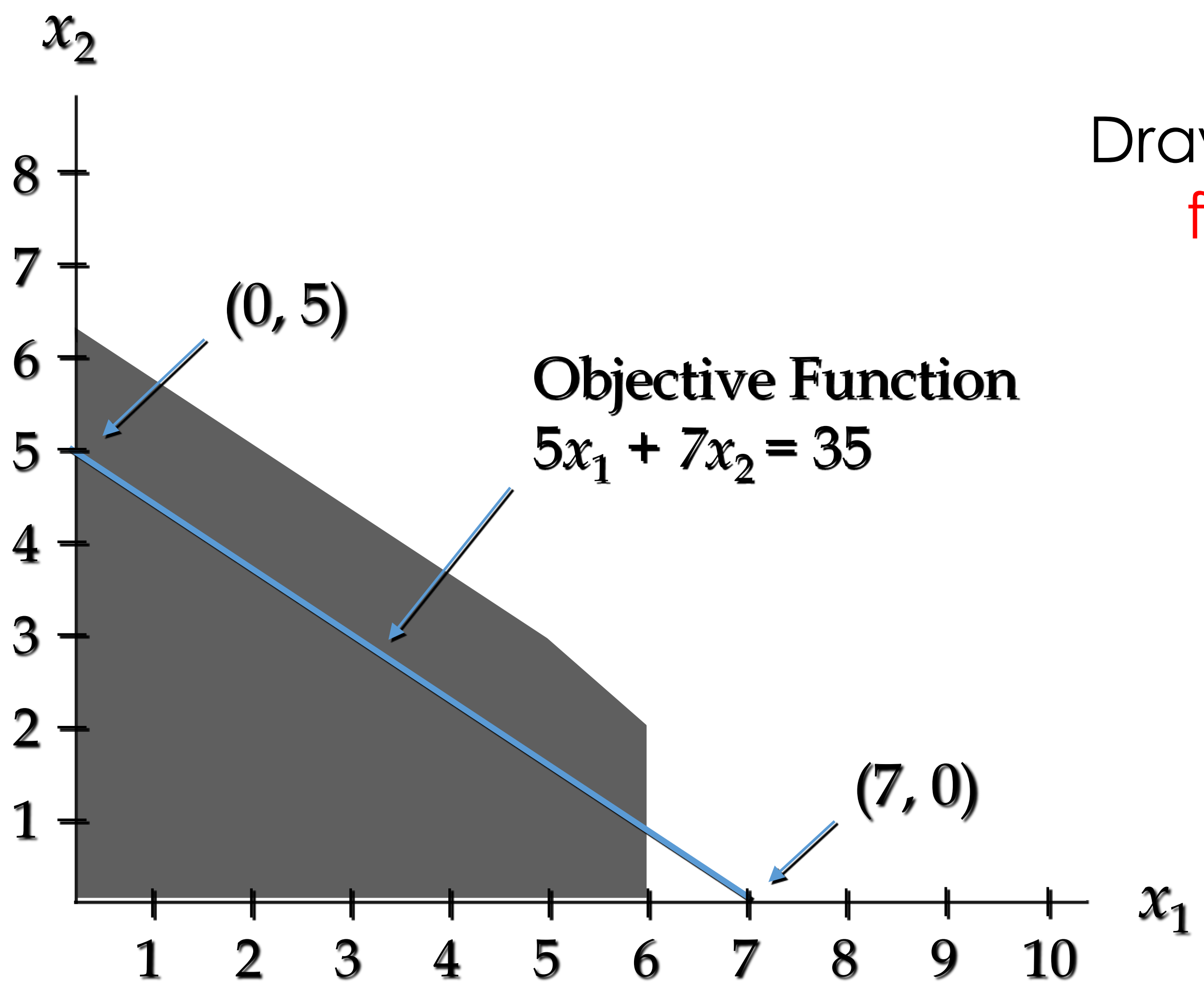
Example 1: Graphical Solution

- Combined-Constraint Graph Showing Feasible Region



Example 1: Graphical Solution

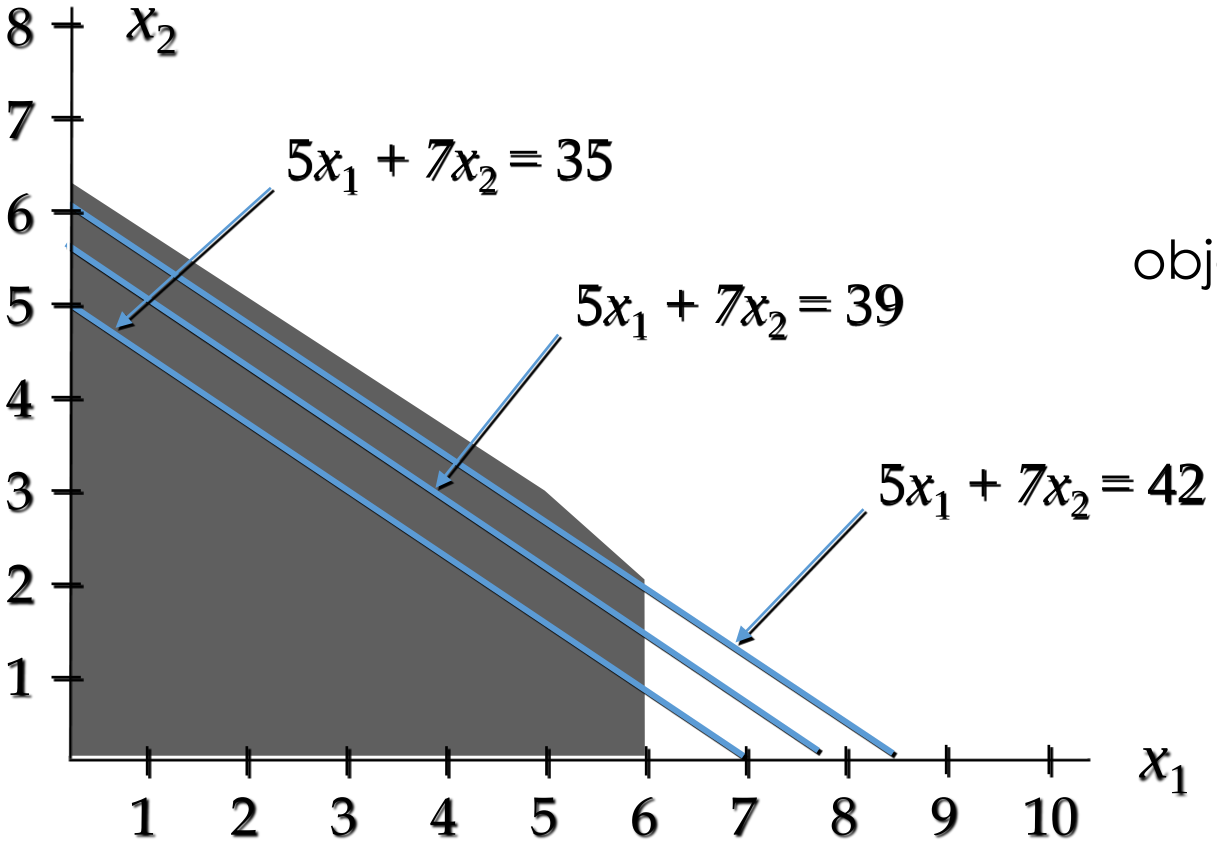
- Objective Function Line



Draw an objective function line

Example 1: Graphical Solution

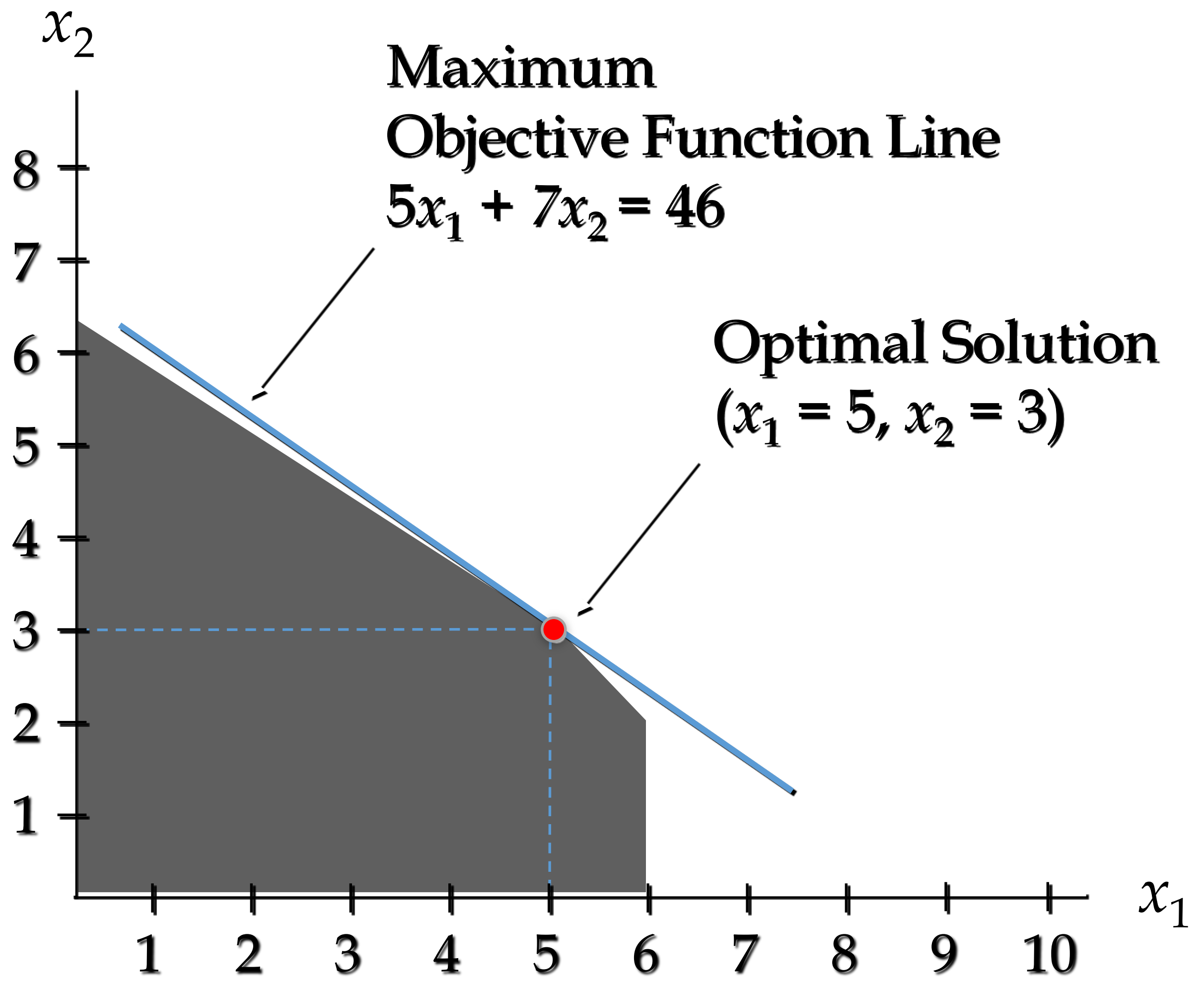
- Selected Objective Function Lines



Move parallel
objective function lines

Example 1: Graphical Solution

- Optimal Solution



largest value is an optimal solution

- Summary of the Graphical Solution Procedure for Maximization Problems
  - Prepare a graph of the feasible solutions for each of the constraints.
  - Determine the feasible region that satisfies all the constraints simultaneously.
  - Draw an objective function line.
  - Move parallel objective function lines toward larger objective function values without entirely leaving the feasible region.
  - Any feasible solution on the objective function line with the largest value is an optimal solution.

Case Study

- Objective function: maximum profit
  - Wheat:
    1. area x_1 km^2, profit: S_1/km^2
    2. Fertilizer: F_1 kg , insecticide P_1 kg
  - Barley:
    1. area x_2 km^2, profit: S_2/km^2
    2. Fertilizer: F_2 kg , insecticide P_2 kg

- Constraints:
  - Area : L km^2
  - Fertilizer: F kilograms,
  - Insecticide: P kilograms

Case Study

Suppose that a farmer has a piece of farm land, say L km^2, to be planted with either wheat or barley or some combination of the two.

- The farmer has a limited amount of fertilizer, F kilograms, and insecticide, P kilograms.
- Every square kilometer of wheat requires F1 kilograms of fertilizer and P1 kilograms of insecticide, while every square kilometer of barley requires F2 kilograms of fertilizer and P2 kilograms of insecticide.
- Let S1 be the selling price of wheat per square kilometer, and S2 be the selling price of barley.
- If we denote the area of land planted with wheat and barley by x1 and x2 respectively, then profit can be maximized by choosing optimal values for x1 and x2.

Case Study

- Objective function: maximum profit
  - Wheat:
    1. area x_1 km^2, profit: S_1/km^2
    2. Fertilizer: F_1 kg , insecticide P_1 kg
  - Barley:
    1. area x_2 km^2, profit: S_2/km^2
    2. Fertilizer: F_2 kg , insecticide P_2 kg

Profit

$$S_1 . x_1 + S_2 . x_2$$

- Constraints:
  - Area : *L* km^2
  - Fertilizer: F kilograms,
  - Insecticide: P kilograms

$$x_1 + x_2 \leq L$$

$$F_1 . x_1 + F_2 . x_2 \leq F$$

$$P_1 . x_1 + P_2 . x_2 \leq P$$

❖ Case Study

Maximize $S_1 . x_1 + S_2 . x_2$ (maximize the revenue—revenue is the "objective function")

Subject to

$x_1 + x_2 \le L$ (limit on total area)

$F_1 . x_1 + F_2 . x_2 \le F$ (limit on fertilizer)

$P_1 . x_1 + P_2 . x_2 \le P$ (limit on insecticide)

$x_1, x_2 \ge 0$ (cannot plant a negative area)

Which in matrix form becomes:

maximize $$\begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to $$\begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ P_1 & P_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} L \\ F \\ P \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

❖ Case Study

Suppose that a gas-processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas, regular and premium quality. These grades of gas are in high demand (that is, they are guaranteed to sell) and yield different profits to the company. However, their production involves both time and on-site storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hr/week. Further, there is limited on-site storage for each of the products. All these factors are listed below (note that a metric ton, or tonne, is equal to 1000 kg):

	Product		
Resource	**Regular**	**Premium**	**Resource Availability**
Raw gas	7 m^3/tonne	11 m^3/tonne	77 m^3/week
Production time	10 hr/tonne	8 hr/tonne	80 hr/week
Storage	9 tonnes	6 tonnes	
Profit	150/tonne	175/tonne	

Develop a linear programming formulation to maximize the profits for this operation?

Case Study

- Objective function: maximum profit
- Constraints:
  - Materials : $77m^3$
  - Production time: 80hr
  - Storage: 9 tonnes and 6tonnes

	Product		
Resource	**Regular**	**Premium**	**Resource Availability**
Raw gas	7 m^3/tonne	11 m^3/tonne	77 m^3/week
Production time	10 hr/tonne	8 hr/tonne	80 hr/week
Storage	9 tonnes	6 tonnes	
Profit	150/tonne	175/tonne	

Solution

The engineer operating this plant must decide how much of each gas to produce to maximize profits. If the amounts of regular and premium produced weekly are designated as x_1 and x_2, respectively, the total weekly profit can be calculated as

$$\text{Total profit} = 150x_1 + 175x_2$$

or written as a linear programming objective function,

$$\text{Maximize } Z = 150x_1 + 175x_2$$

The constraints can be developed in a similar fashion. For example, the total raw gas used can be computed as

$$\text{Total gas used} = 7x_1 + 11x_2$$

	Product		
Resource	**Regular**	**Premium**	**Resource Availability**
Raw gas	7 m^3/tonne	11 m^3/tonne	77 m^3/week
Production time	10 hr/tonne	8 hr/tonne	80 hr/week
Storage	9 tonnes	6 tonnes	
Profit	150/tonne	175/tonne	

Maximize $Z = 150x_1 + 175x_2$

$7x_1 + 11x_2 \leq 77$ (1) (material constraint)

$10x_1 + 8x_2 \leq 80$ (2) (time constraint)

$x_1 \leq 9$ (3) ("regular" storage constraint)

$x_2 \leq 6$ (4) ("premium" storage constraint)

$x_1, x_2 \geq 0$ (5) (positivity constraints)

equations constitute the total LP formulation

graphical solution

First, the constraints can be plotted on the solution space. For example, the first constraint can be reformulated as a line by replacing the inequality by an equal sign and solving for x_2:

$$x_2 = -\frac{7}{11}x_1 + 7 \qquad (1)$$

$10x_1 + 8x_2 \leq 80$ (2)

$x_1 \leq 9$ (3)

$x_2 \leq 6$ (4)

$x_1, x_2 \geq 0$ (5,6)

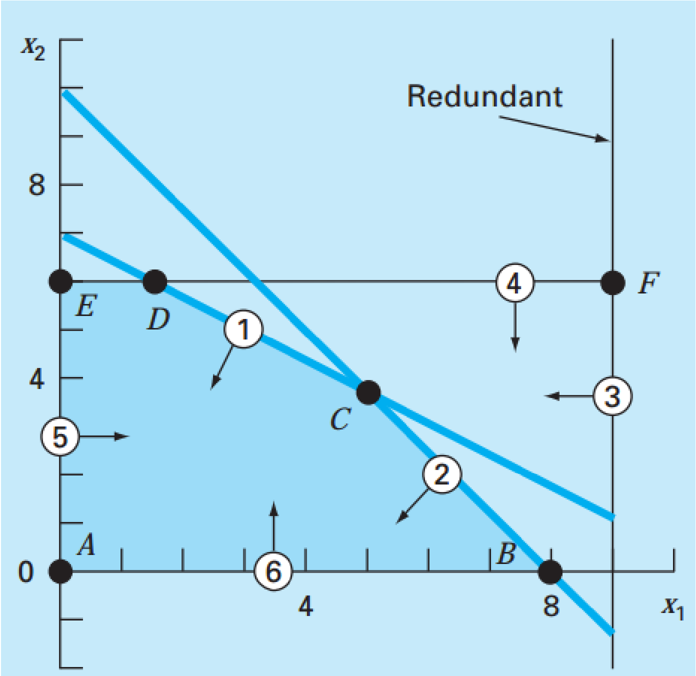


Fig a: The constraints define a feasible solution space

graphical solution

Aside from defining the feasible space, Fig. a also provides additional insight. In particular, we can see that constraint 3 (storage of regular gas) is “redundant.” That is, the feasible solution space is unaffected if it were deleted

Next, the objective function can be added to the plot. To do this, a value of Z must be chosen. For example, for Z = 0, the objective function becomes

$$0 = 150x_1 + 175x_2$$

or, solving for x2, we derive the line

$$x_2 = -\frac{150}{175}x_1$$

graphical solution

As displayed in Fig. b, this represents a dashed line intersecting the origin. Now, since we are interested in maximizing Z, we can increase it to say, 600, and the objective function is

$$x_2 = \frac{600}{175} - \frac{150}{175} x_1$$

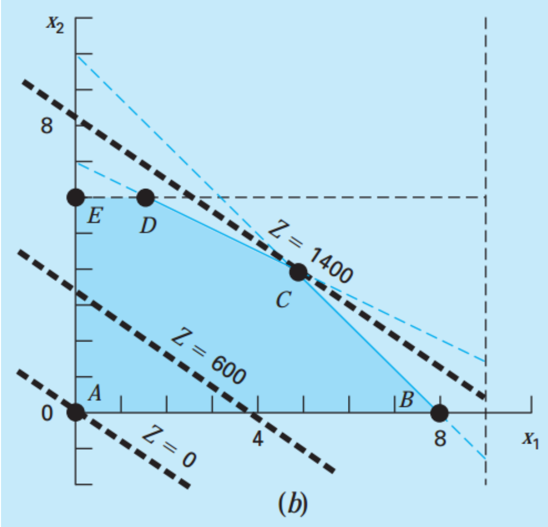


Fig b: The objective function can be increased until it reaches the highest value that obeys all constraints

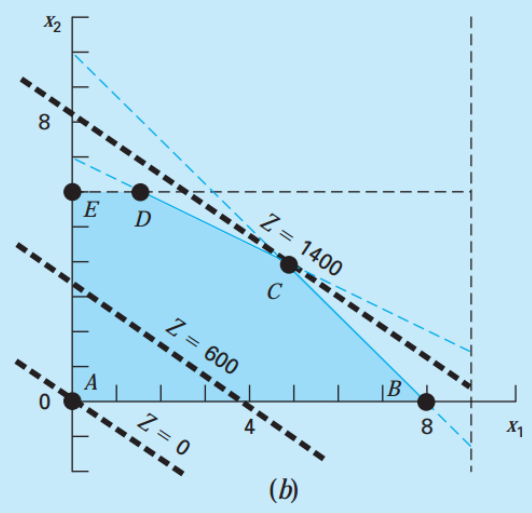
- increasing the value of the objective function moves the line away from the origin.

- the maximum value of Z corresponds to approximately 1400. At this point, x1 and x2 are equal to approximately 4.9 and 3.9, respectively.

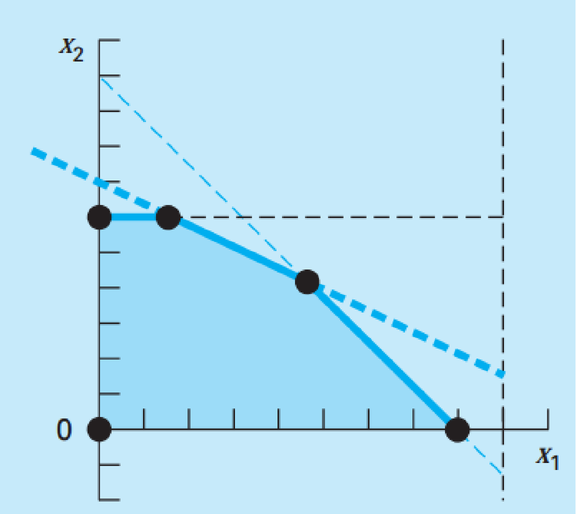
if we produce these quantities of regular and premium, we will reap a maximum profit of about 1400.

❖ Four possible outcomes that can be generally obtained in a linear programming problem

- Unique solution. As in the example, the maximum objective function intersects a single point

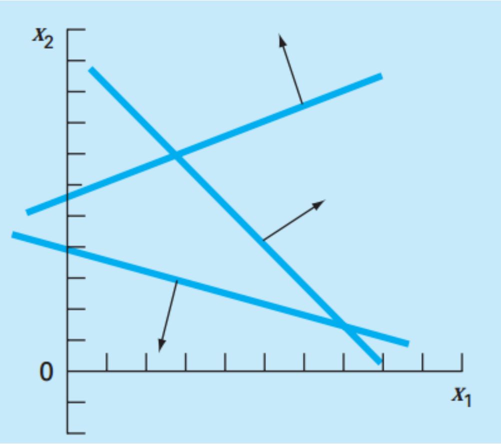


- Alternate solutions. Suppose that the objective function in the example had coefficients so that it was precisely parallel to one of the constraints. Then, rather than a single point, the problem would have an infinite number of optima corresponding to a line segment



- No feasible solution

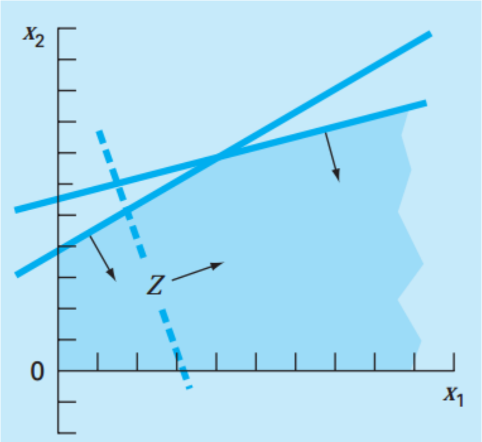
it is possible that the problem is set up so that there is no feasible solution. This can be due to dealing with an unsolvable problem or due to errors in setting up the problem. The latter can result if the problem is overconstrained to the point that no solution can satisfy all the constraints



no feasible solution

- Unbounded problems

this usually means that the problem is underconstrained and therefore open-ended. As with the no-feasible-solution case, it can
often arise from errors committed during problem specification.



an unbounded result

- Topics
  - Standard forms and duality
  - Simplex method

- Simplex method
  - The simplex method is predicated on the assumption that the optimal solution will be an extreme point.
  - Thus, the approach must be able to discern whether during problem solution an extreme point occurs. To do this, the constraint equations are reformulated as equalities by introducing what are called slack variables
  - Standard form is attained by adding slack variables to "less than or equal to" constraints, and by subtracting surplus variables from "greater than or equal to" constraints

Augmented form (slack form)

LP problems can be converted into an augmented form in order to apply the common form of the simplex algorithm. This form introduces non-negative slack variables to replace inequalities with equalities in the constraints.

- maximize

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \qquad \mathbf{x}, \mathbf{s} \geq 0$$

- where s are the newly introduced slack variables, and z is the variable to be maximized.
- Slack and surplus variables represent the difference between the left and right sides of the constraints
- Slack and surplus variables have objective function coefficients equal to 0

Slack Variables

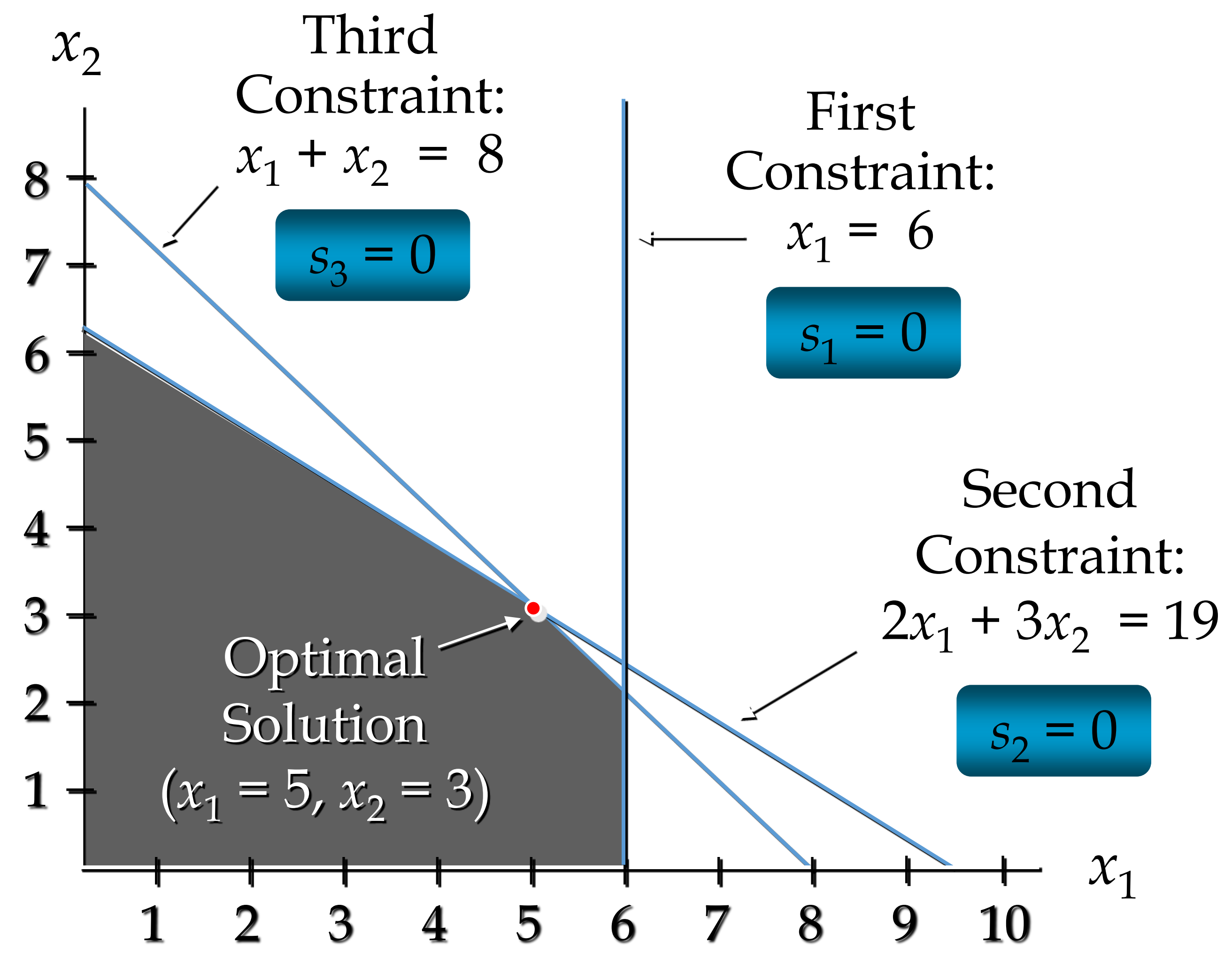
- Example

$$
\begin{array}{ll}
\text{Max} & 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t.} & x_1 + s_1 = 6 \\
 & 2x_1 + 3x_2 + s_2 = 19 \\
 & x_1 + x_2 + s_3 = 8 \\
 & x_1, x_2, s_1, s_2, s_3 \geq 0
\end{array}
$$

s_1 , s_2 , and s_3 are slack variables

Slack Variables

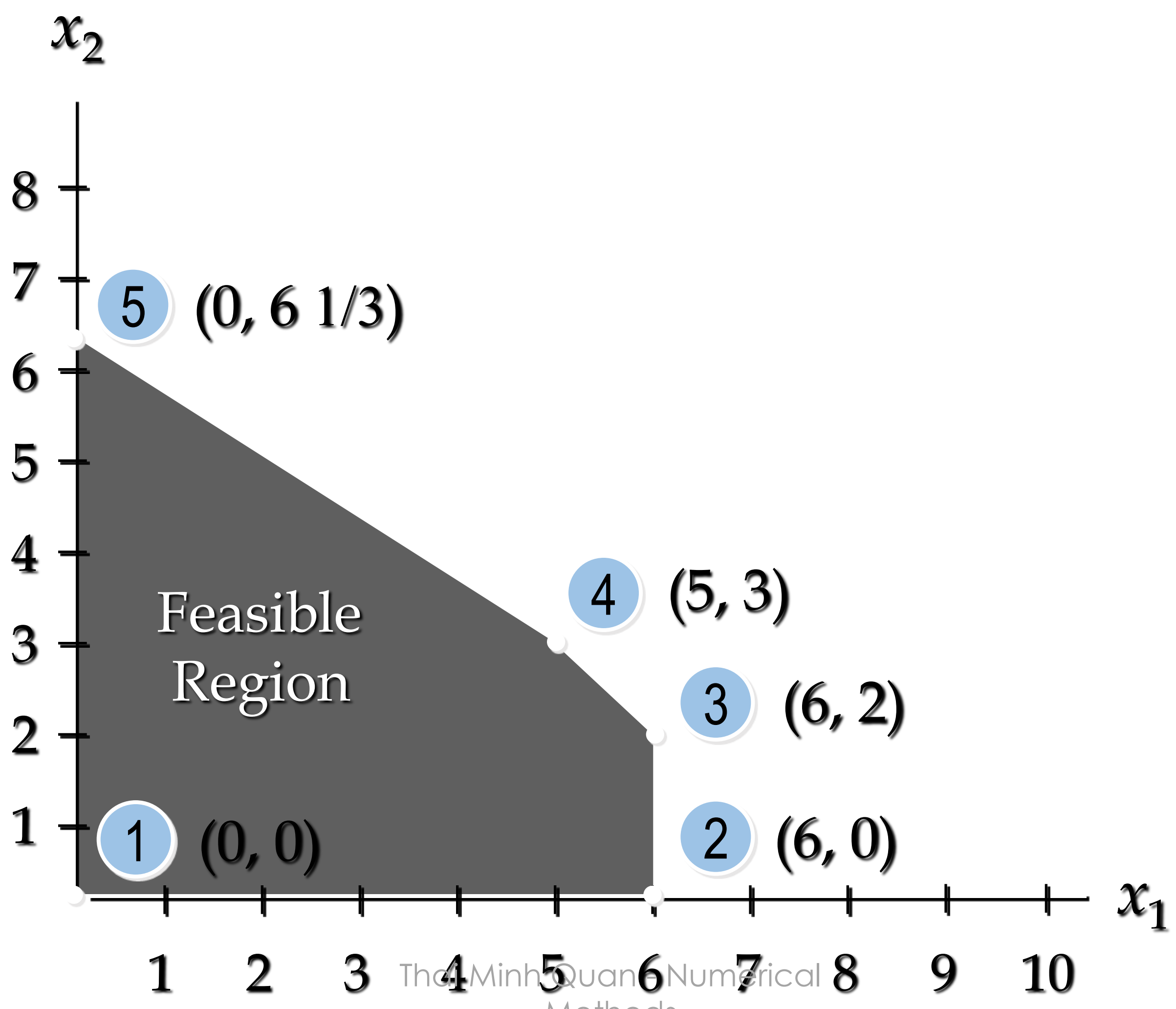
- Optimal Solution



Extreme Points and the Optimal Solution

- The corners or vertices of the feasible region are referred to as the <u>extreme points</u>.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.

Example 1: Extreme Points



Augmented form (slack form)

Example

The example above is converted into the following augmented form:

Maximize	$S_1 . x_1 + S_2 . x_2$	→	$S_1 . x_1 + S_2 . x_2$
Subject to	$x_1 + x_2 \leq L$		$x_1 + x_2 + x_3 = L$
	$F_1 . x_1 + F_2 . x_2 \leq F$		$F_1 . x_1 + F_2 . x_2 + x_4 = F$
	$P_1 . x_1 + P_2 . x_2 \leq P$		$P_1 . x_1 + P_2 . x_2 + x_5 = P$
	$x_1, x_2 \geq 0$		$x_1, x_2, x_3, x_4, x_5 \geq 0$

where x_1, x_2, x_3, x_4, x_5 are (non-negative) slack variables

Augmented form (slack form)

Matrix form:

Maximize $S_1 . x_1 + S_2 . x_2$

Subject to

$x_1 + x_2 + x_3 = L$

$F_1 . x_1 + F_2 . x_2 + x_4 = F$

$P_1 . x_1 + P_2 . x_2 + x_5 = P$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Maximize z

$$\begin{bmatrix} 1 & -S_1 & -S_2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & F_1 & F_2 & 0 & 1 & 0 \\ 0 & P_1 & P_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ L \\ F \\ P \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \geq 0.$$

Exercises: Minimization Problem

- LP Formulation

$$
\begin{array}{lrcl}
\text{Min} & 5x_1 + 2x_2 & & \\
\text{s.t.} & 2x_1 + 5x_2 & \geq & 10 \\
 & 4x_1 - x_2 & \geq & 12 \\
 & x_1 + x_2 & \geq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

Graphical Solution

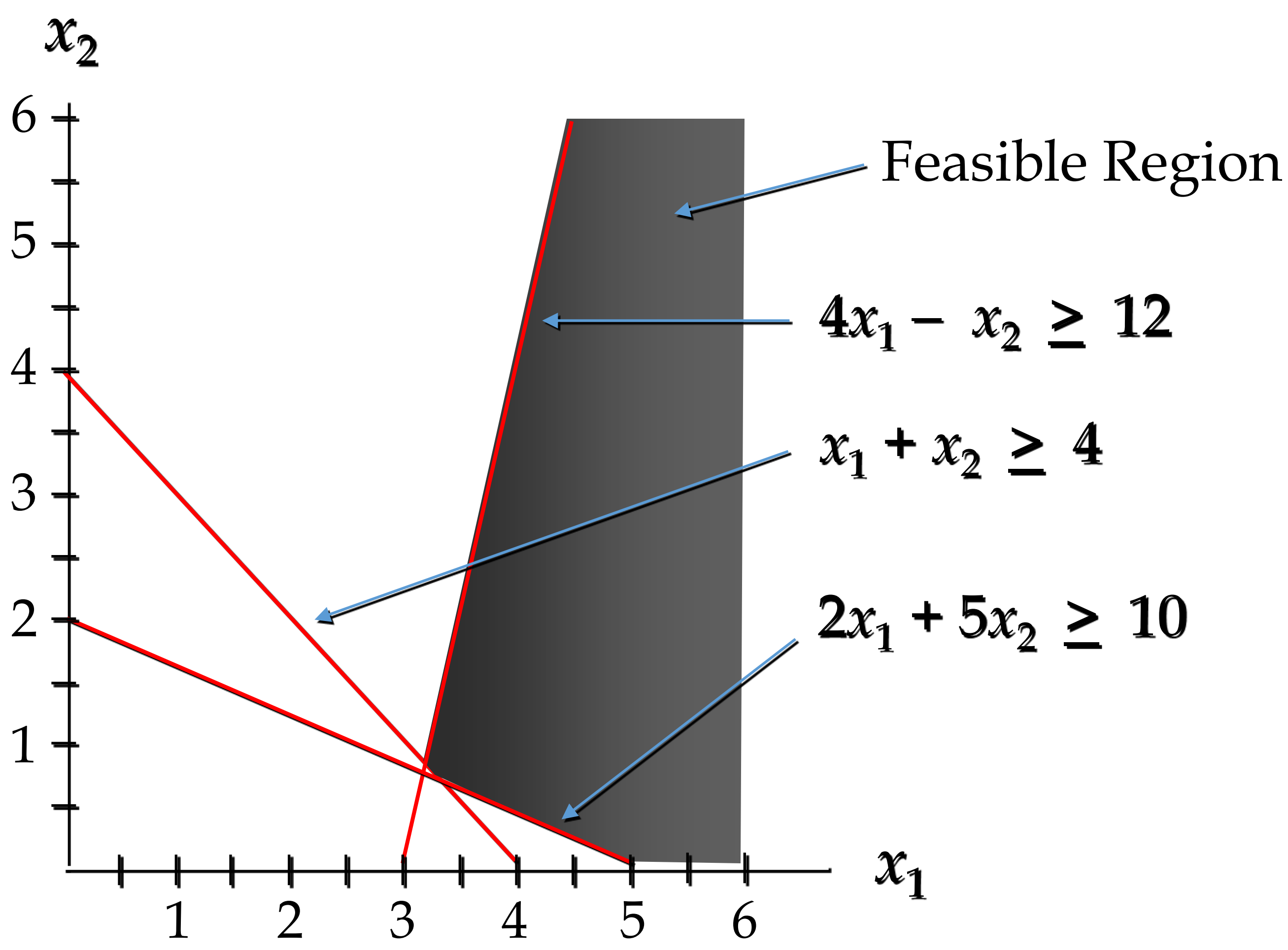
- Graph the Constraints

  Constraint 1: When $x_1 = 0$, then $x_2 = 2$; when $x_2 = 0$, then $x_1 = 5$. Connect (5,0) and (0,2). The ">" side is above this line.

  Constraint 2: When $x_2 = 0$, then $x_1 = 3$. But setting x_1 to 0 will yield $x_2 = -12$, which is not on the graph. Thus, to get a second point on this line, set x_1 to any number larger than 3 and solve for x_2: when $x_1 = 5$, then $x_2 = 8$. Connect (3,0) and (5,8). The ">" side is to the right.

  Constraint 3: When $x_1 = 0$, then $x_2 = 4$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,4). The ">" side is above this line.

- Constraints Graphed



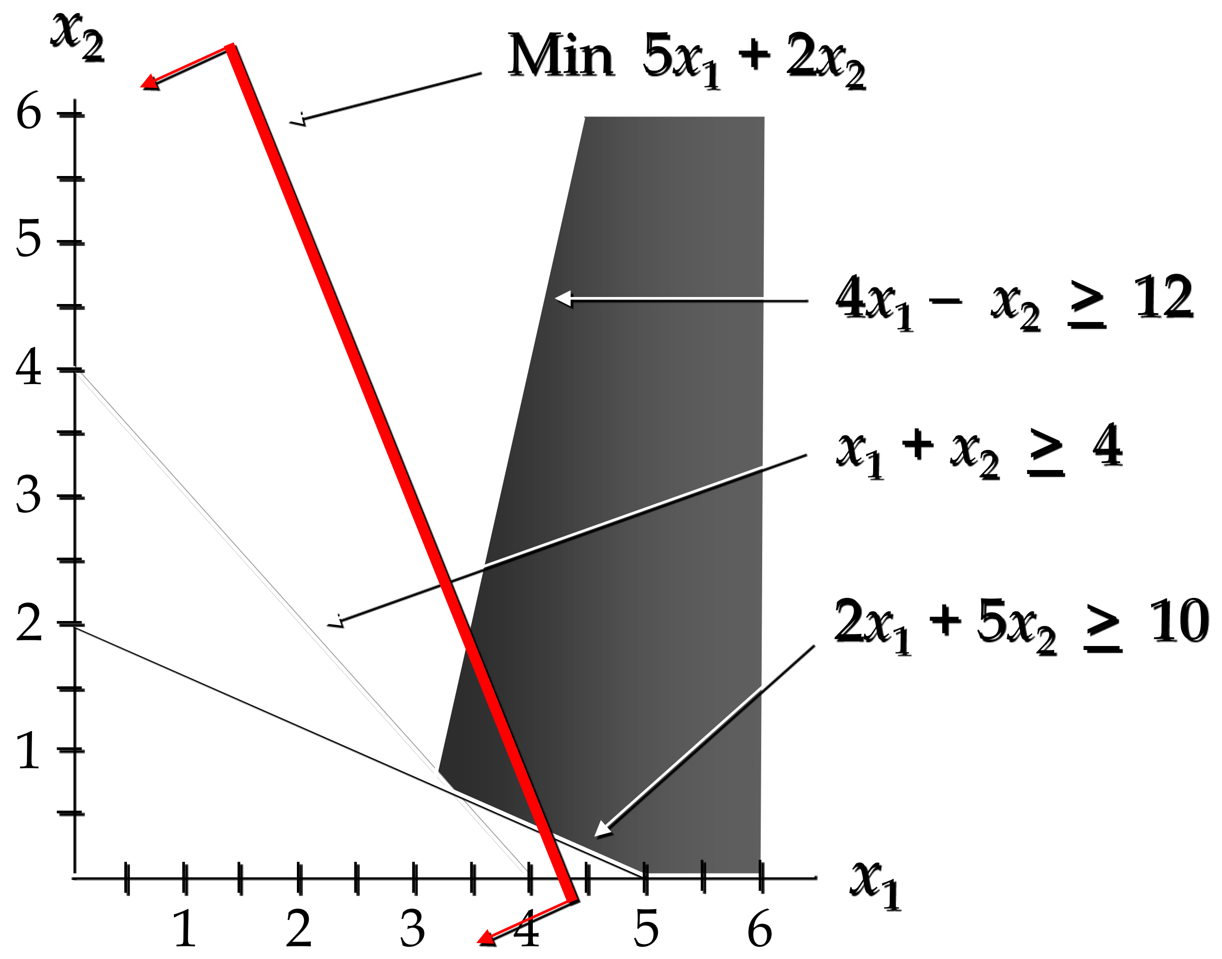
- Graph the Objective Function

  Set the objective function equal to an arbitrary constant (say 20) and graph it. For $5x_1 + 2x_2 = 20$, when $x_1 = 0$, then $x_2 = 10$; when $x_2 = 0$, then $x_1 = 4$. Connect (4,0) and (0,10).

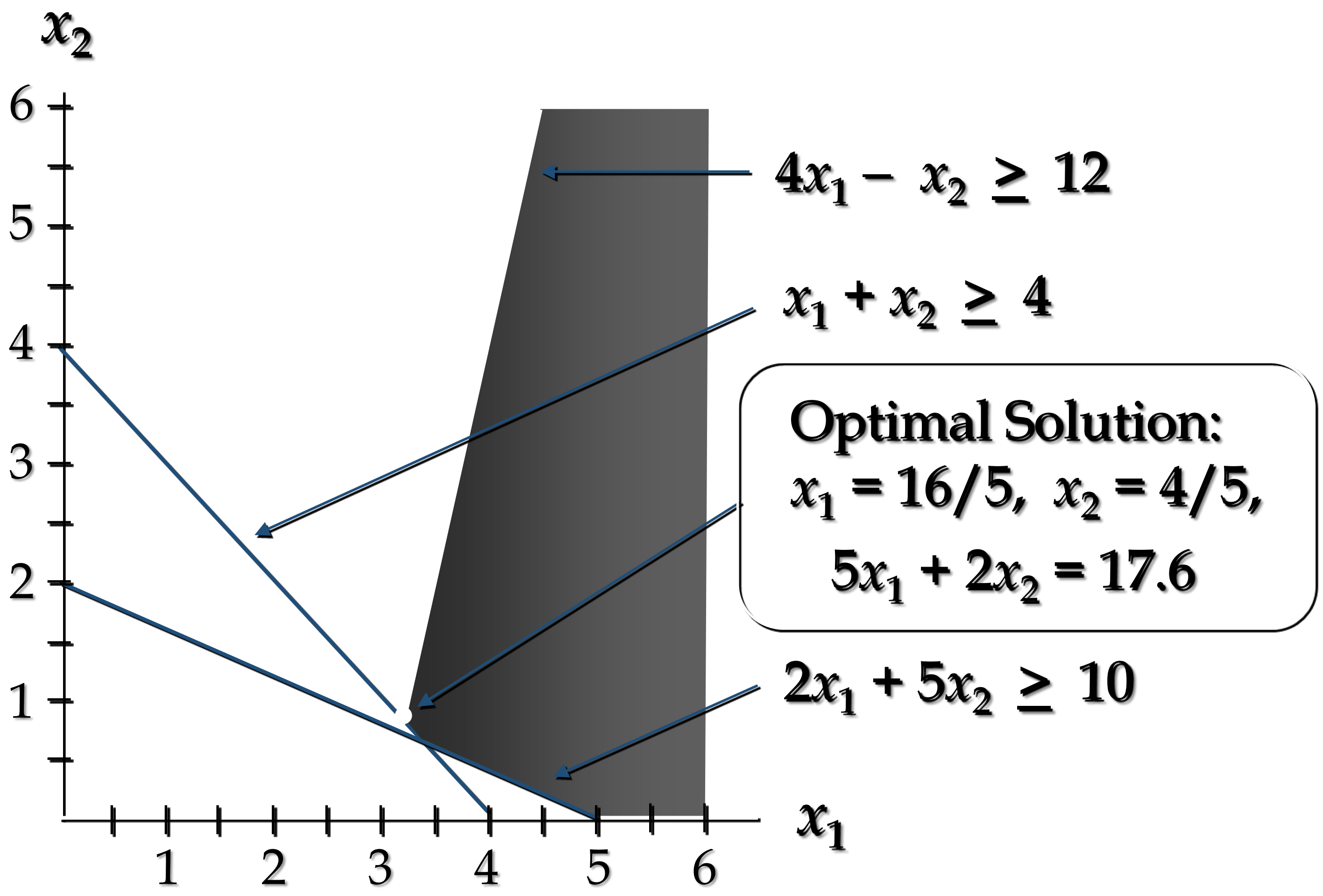
- Move the Objective Function Line Toward Optimality

  Move it in the direction which lowers its value (down), since we are minimizing, until it touches the last point of the feasible region, determined by the last two constraints.

- Objective Function Graphed



- Optimal Solution



linprog

Matlab

Graphical method

TABLE 15.1 MATLAB functions to implement optimization.

Function	Description
fminbnd	Minimize function of one variable with bound constraints
fminsearch	Minimize function of several variables

Matlab

- Find minimum of single-variable function on fixed interval
- fminbnd is a one-dimensional minimizer that finds a minimum for a problem specified by

$$\min_{x} f(x) \text{ such that } x_1 < x < x_2.$$

x = fminbnd(fun,x1,x2)

Matlab

Graphical method

```
% Demonstrate with graphical solution
x = linspace(0, 6);
y1 = (4*x - 12);
y2 = (4 - x);
y3 = ((10-2*x)/5);
%ytop = min([y1; y2; y3]);
[u, v] = meshgrid(linspace(0,6),
linspace(0,6));
plot(x, y1, 'r', 'LineWidth', 2)
hold on;
     plot(x, y2, 'r', 'LineWidth', 2);
     plot(x, y3, 'r', 'LineWidth', 2);
     contour(u,v, 5*u + 2*v, 25);
     axis([0 6 0 6]);
hold off;
```

Matlab

Find x that minimizes

subject to

$$f(x) = -5x_1 - 4x_2 - 6x_3,$$

$$\begin{aligned} x_1 - x_2 + x_3 &\le 20 \\ 3x_1 + 2x_2 + 4x_3 &\le 42 \\ 3x_1 + 2x_2 &\le 30 \\ 0 \le x_1, 0 \le x_2&, 0 \le x_3. \end{aligned}$$

linprog

Linear Programming

First, enter the coefficients

```
f = [-5; -4; -6];
A =  [1 -1   1
      3  2  4
      3  2  0];
b = [20; 42; 30];
lb = zeros(3,1);
```

Next, call a linear programming routine

```
[x,fval,exitflag,output,lambda] = linprog(f,A,b,[],[],lb);
```