Path-based depth-first search for strong and biconnected components

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Outline

1. Introduction

2. Strong Components
   - Thinking about Strong Components
   - Purdom and Munro's high-level algorithm
   - Contribution
   - Discussion

Characterastics of Gabow's Algorithms

- One-pass algorithm. But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.

Several Questions

- LOWPOINT?
- Ear decomposition?
- Compele version?
- Robbin's Theorem?

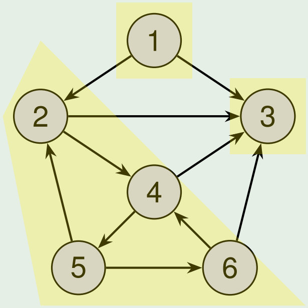
Outline

Review: What have we learned from the textbook?

Concepts of Strong Components

- Two mutually reachable vertices are in the same *strong component*.

Example



Review: What have we learned from the textbook?

Algorithms to Find Strong Components

- Idea: Run DFS twice. Once on the original graph G, once on the *tranposition* of G^T.
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity: $O(V + E)$

Outline

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Pseudo-Code

```
H = G;
while H still has a vertex v
   start a new path P = (v);
   while P is not empty
      if the last vertex v_k of P has an edge (v_k, w)
         if w belongs to P
            contract the cycle v_i(w), ... , v_k, both
                in H and in P; /* w and v_i are
                identical. */
         else
            add w to P, as the new last vertex of P;
         end if
```

- Note that *contracting* means selecting one vertex as a representation and merging others rather than deleting them.

Pseudo-Code (Continued)

```
        else
            output v_k as a vertex of the strong component
                 graph;
            delete v_k from both H and P;
        end if
    end
end
```

Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except the statement in line 7:

```
7 contract the cycle v_i(w), ... , v_k, both in H and in
      P; /* w and v_i are identical. */
```

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint set merging is needed.

Outline

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2. Strong Components
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His Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.

Data Structure Used in Algorithm

- In DFS, the path P from root to each node is almost always significant. So it is in this algorithm.
- A stack S contains the sequence of vertices in P.
- A stack B contains the boundaries between contracted vertices.
- An array $I[1 \ldots n]$ is used to store stack indices corresponding to vertices.

Contraction Makes Much Difference

- When contraction is executed, some vertices merges into a set.
- It is possible that several elements in stack S are in the same vertex in path P. More formal,

$$v_i = S[j] \,:\, B[i] \leq j < B[i+1]$$

- By the way, the formal definition of $I[v]$ is

$$I[j] = \begin{cases} 0, & \text{if } v \text{ has never been in P;} \\ j, & \text{if } v \text{ is currently in P and } S[j] = v; \\ c, & \text{if the strong component containing } v \text{ has} \\ & \text{been deleted and numbered as } c. \end{cases}$$

where c counts from $n+1$.

New algorithm to discover strong components

Procedure 1: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
    $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
    if $I[v] = 0$ **then**
        DFS(v);

New algorithm to discover strong components

Procedure 2: DFS(v)

```
PUSH(v, S);  I[v] = TOP(S);  PUSH([v], B);
/* add v to the end of P                                    */
for egdes(v, w) ∈ E do
    if I[v] = 0 then
        DFS(w);
    else /* contract if necessary                           */
        while I[w] < B[TOP[B]] do
            POP(B);
    if I[v] = B[TOP(B)] then
        /* number vertices of the next strong component     */
        POP(B);
        c = c + 1;
        while I[v] ≤ TOP[S] do
            I[POP(S)] = c;
```

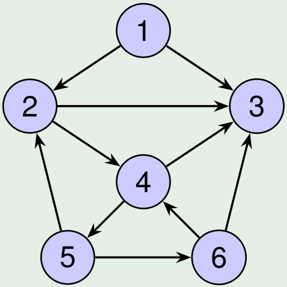
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.

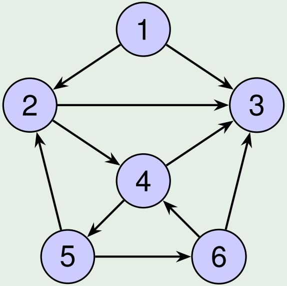
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()$\rightarrow$DFS(1)
- Code: **for** edges(v,w)$\in$E **do** ...
- $w = 2$.

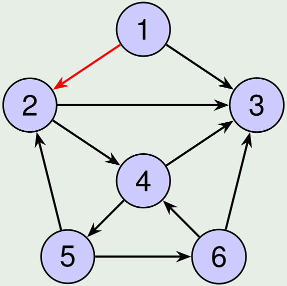
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `for edges(v,w)∈E do ...`
- $w = 3$.

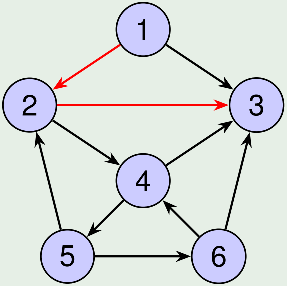
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	3	3	3
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(3)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

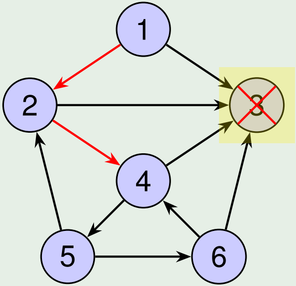
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
		4	3
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(4)
- Code: `for edges(v,w)∈E do ...`
- $w = 5$.

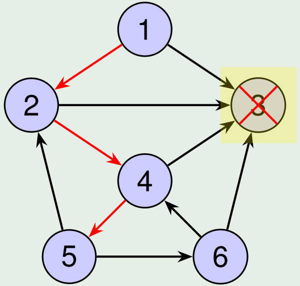
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `for edges (v,w)∈E do ...`
- $w = 2$.

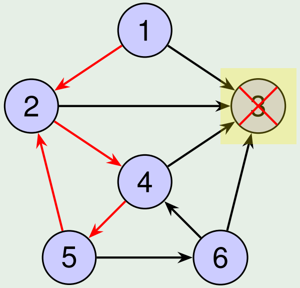
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

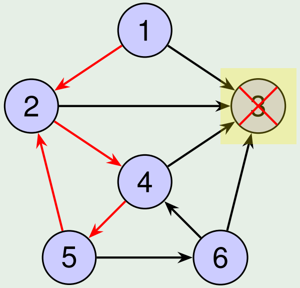
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: **while** `I[w]<B[TOP(B)]` **do** `POP(B);`
- Now, $w = 2$, contract!

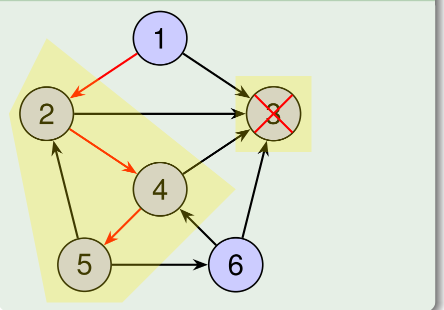
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: **if** `I[w]` $= 0$ **then** `DFS(w);`
- $w = 6$.

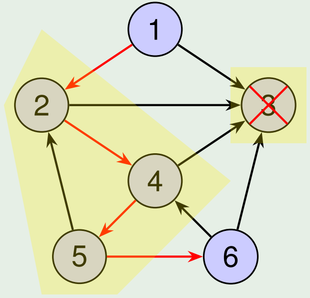
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `for edges(v,w)∈E do ...`
- $w = 4$.

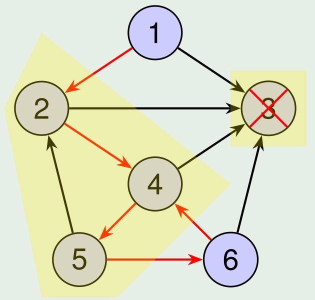
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)$\to$DFS(6)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 4$, contract!

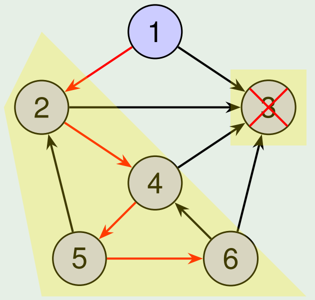
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

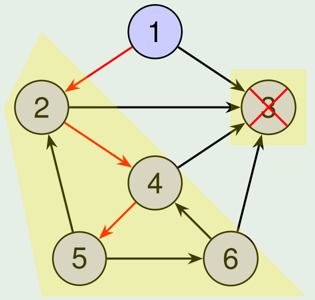
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: **if** `I[v]=B[TOP(B)]` **then** ...
- Go back.

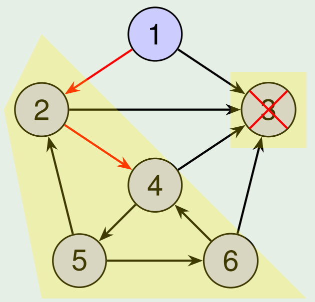
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

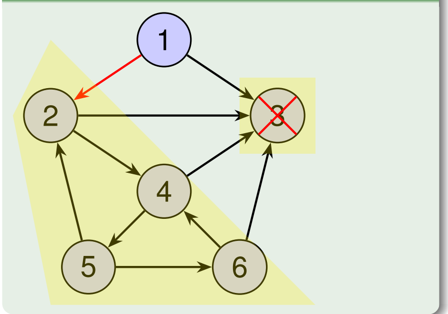
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back. But this time, Condition in last line is satisfied!

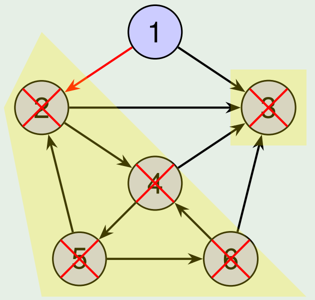
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.

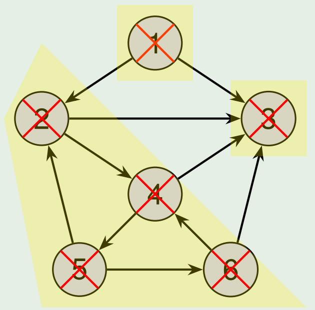
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()$\rightarrow$DFS(1)
- Code: **while** `I[v]`$\leq$`TOP(S)` **do** `I[POP(S)]=c;`
- Pop the last one both in B and in S. *Finished!!*

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

Outline

Correctness

- Null

Time Complexity

- Null.

For Further Reading I

- A. Author.
  Handbook of Everything.
  Some Press, 1990.
- S. Someone.
  On this and that.
  Journal of This and That, 2(1):50–100, 2000.