

# Path-based depth-first search for strong and biconnected components

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# Outline

- 1 Introduction
- 2 Strong Components
  - Reviews
  - Purdom and Munro's High-Level Algorithm
  - Contribution
  - Discussion
- 3 Biconnected Components
  - Review
  - High-Level Algorithm
  - Gabow's Algorithms



# Characterastics of Gabow's Algorithms

- **One-pass algorithm.** But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.
- **Lower time and space complexity.** This algorithm only use two stacks and an array, and do not employ a disjoint-set data structure.



# Outline

## 1 Introduction

## 2 Strong Components

- **Reviews**
- Purdom and Munro's High-Level Algorithm
- Contribution
- Discussion

## 3 Biconnected Components

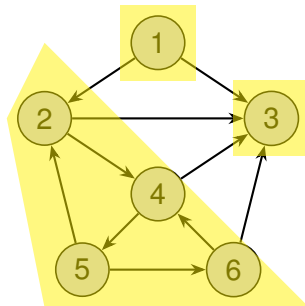
- Review
- High-Level Algorithm
- Gabow's Algorithms



# Review: What have we learned from the textbook?

## Concepts of Strong Components

- Two **mutually reachable** vertices are in the same *strong component*.
- It is a **equivalence relation**.



# Review: What have we learned from the textbook?

## Algorithms to Find Strong Components

- Idea: Run DFS twice: Once on the original graph  $G$ , once on its *transpose*  $G^T$ .
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity:  $O(V + E)$
- Proposed by S. Rao Kosaraju, known as the *Kosaraju's Algorithm*.



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  - **Purdom and Munro's High-Level Algorithm**
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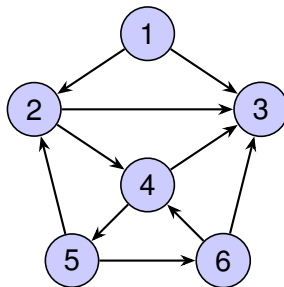
# Purdom and Munro's High-Level Algorithm: Plain text

- Initially  $H$  is the given graph  $G$ . If  $H$  has no vertices stop. Otherwise *start a new path  $P$*  by choosing a vertex  $v$  and setting  $P = (v)$ . Continue by growing  $P$  as follows.
- To grow the path  $P = (v_1, \dots, v_k)$  choose an edge  $(v_k, w)$  directed from the last vertex of  $P$  and do the following:
  - If  $w \notin P$ , *add  $w$  to  $P$* , making it the new last vertex of  $P$ . Continue growing  $P$ .
  - If  $w \in P$ , say  $w = v_i$ , contract the cycle  $v_i, v_{i+1}, \dots, v_k$ , both in  $H$  and in  $P$ .  $P$  is now a path in the new graph  $H$ . Continue growing  $P$ .
  - If no edge leaves  $v_k$ , output  $v_k$  as a vertex of the strong component graph. Delete  $v_k$  from both  $H$  and  $P$ . If  $P$  is now nonempty continue growing  $P$ . Otherwise try to start a new path  $P$ .





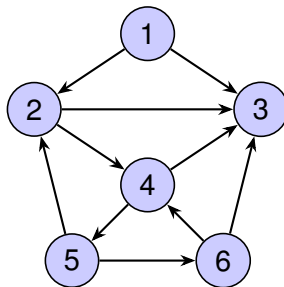
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \}$
- Initially,  $H = G$ .



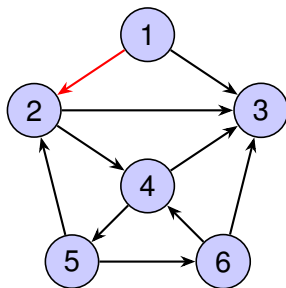
# Demo: P&M's High-Level Algorithm



- Path  $P = \{\{1\}\}$
- Grow  $P$  by adding  $v_1$ .



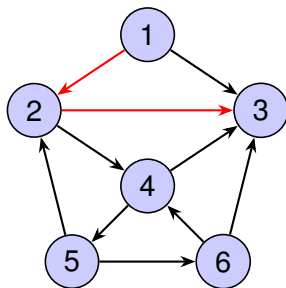
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2\} \}$
- Grow  $P$  by adding  $v_2$ .



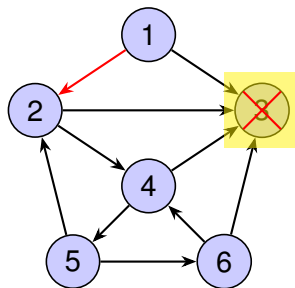
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2\}, \{3\} \}$
- Grow  $P$  by adding  $v_3$ .



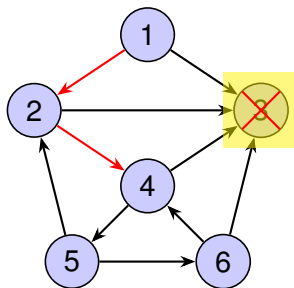
# Demo: P&M's High-Level Algorithm



- Path  $P = \{\{1\}, \{2\}\}$
- As  $v_3$  is isolated, no edge leaves from  $v_3$ , so just **delete** it.



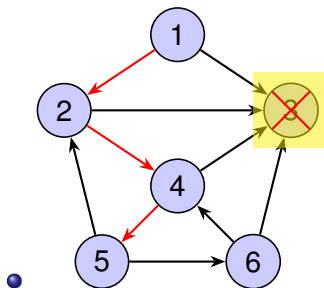
## Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2\}, \{4\} \}$
- Grow  $P$  by adding  $v_4$ .



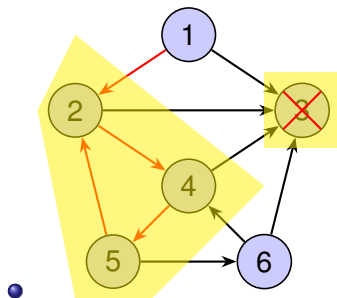
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2\}, \{4\}, \{5\} \}$
- Grow  $P$  by adding  $v_5$ .



## Demo: P&M's High-Level Algorithm

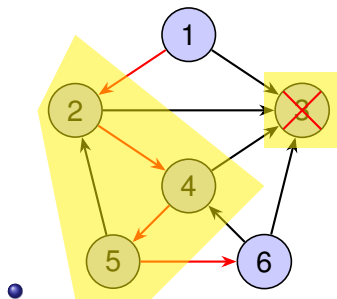


- Path  $P = \{ \{1\}, \{2, 4, 5\} \}$
- The cycle  $v_2, v_4, v_5$  in  $P$  is detected. Contract this cycle.





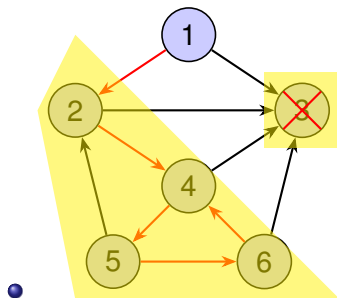
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2, 4, 5\}, 6 \}$
- Grow  $P$  by adding  $v_6$ .



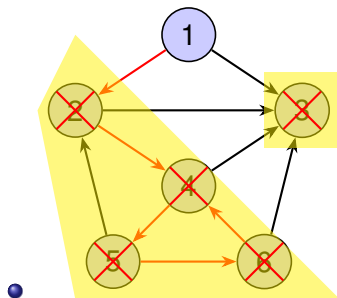
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2, 4, 5, 6\} \}$
- The cycle  $\{v_2, v_4, v_5\}, v_6, v_4$  in  $P$  is detected. **Contract** this cycle.



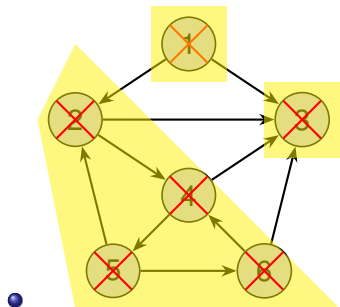
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\}, \{2, 4, 5, 6\} \}$
- No edge leaves from  $\{v_2, v_4, v_5, v_6\}$ , so we **delete** it.



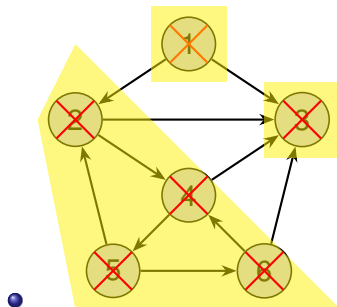
# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \{1\} \}$
- No edge leaves from  $\{v_1\}$ , so we **delete** it.



# Demo: P&M's High-Level Algorithm



- Path  $P = \{ \}$
- Now graph  $H$  is **empty**, which has no vertex.

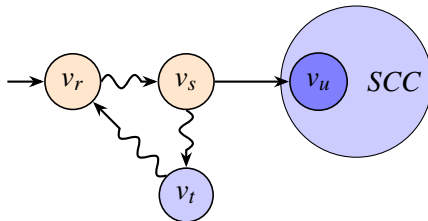


# Correctness

- Correctness: If no edge leaves  $v_k$  then  $v_k$  is a vertex of the finest acyclic contraction.
- Easy to prove by contradiction: Assume there is an edge leaves  $v_k \dots$



# How to implement by DFS?



- Assume the current node is  $v_s$  which has two adjacent nodes. The current path is  $P = (\dots, v_r, \dots, v_s)$ .
- For the node  $v_u$  incident from  $v_s$  but also in the  $SCC$ , after running Sub-DFS() on this node,  $v_u$  will be removed with the  $SCC$ .



# Pseudo Code

---

**Algorithm 1:** Strong components: Main-DFS( $G$ ) (DFS caller)

---

$H = G$ ;

**while**  $H$  still has a vertex  $v$  **do**

    | Sub-DFS( $v$ ); /\* start a new path  $P = (v)$  \*/

---





# Pseudo Code

---

## Algorithm 2: Strong components: Sub-DFS( $v$ ) (DFS callee)

---

add the  $v$  as the new last vertex of path  $P$ ;

**for**  $w \in \{\text{vertices adjacent to } v\}$  **do**

**if**  $w \notin P$  **then**

        Sub-DFS( $w$ );

**else**  $\star w = v_i, \text{ and } v = v_k \star /$

        contract the cycle  $v_i, v_{i+1}, \dots, v_k$ , both in  $H$  and in  $P$ ;

**if** *no edge leaves  $v$  and  $v$  is the last DFS-finished vertex in a SCC* **then**

    output  $v$  as a vertex of the strong component graph;

    delete  $v$  from both  $H$  and  $P$ ;

---



# Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except this statement:

contract the cycle  $v_i, v_{i+1}, \dots, v_k$ , both in  $H$  and in  $P$ ;

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint-set merging is needed usually.



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# Gabow's Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.



# Data Structure Used in Algorithm

- In DFS, the **path**  $P$  from root to each node is almost always significant. So it is in this algorithm.
- A **stack**  $S$  contains the sequence of vertices in  $P$ .
- A **stack**  $B$  contains the boundaries between contracted vertices.
- An array  $I[1 \dots n]$  is used to store stack  $S$  indices corresponding to vertices.



# Contraction Makes Much Difference

- When contraction is executed, some vertices merges into a set.
- It is possible that several elements in stack  $S$  are in the same vertex in path  $P$ . More formal,

$$v_i = \{S[j] : B[i] \leq j < B[i + 1]\}$$

- By the way, the formal definition of  $I[v]$  is

$$I[v] = \begin{cases} 0, & \text{if } v \text{ has never been in } P; \\ j, & \text{if } v \text{ is currently in } P \text{ and } S[j] = v; \\ c, & \text{if the strong component containing } v \text{ has} \\ & \text{been deleted and numbered as } c. \end{cases}$$

where  $c$  counts from  $n + 1$ .



# New Algorithm to Discover Strong Components

---

## Procedure 3: STRONG( $G$ )

---

empty stacks  $S$  and  $B$ ;

**for**  $v \in V$  **do**

$I[v] = 0$ ;

$c = n$ ;

**for**  $v \in V$  **do**

**if**  $I[v] = 0$  **then** /\* vertex  $v$  has never been  
         accessed yet

        DFS( $v$ );

\*/

---



# New Algorithm to Discover Strong Components

---

## Procedure 4: DFS( $v$ )

---

```

PUSH( $v, S$ );  $I[v] = \text{TOP}(S)$ ; PUSH( $I[v], B$ );
/* add  $v$  to the end of  $P$  */
for  $\text{edges}(v, w) \in E$  do
    if  $I[w] = 0$  then
        | DFS( $w$ );
    else /* contract if necessary */
        | while  $I[w] < B[\text{TOP}(B)]$  do
        | | POP( $B$ );
if  $I[v] = B[\text{TOP}(B)]$  then /* number vertices of the next
    strong component */
    POP( $B$ );
     $c = c + 1$ ;
    while  $I[v] \leq \text{TOP}(S)$  do
        |  $I[\text{POP}(S)] = c$ ;
    
```

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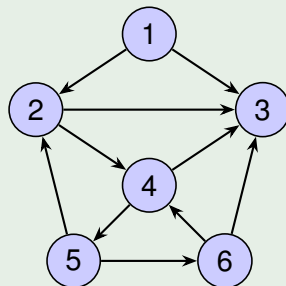
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

## Graph H



- Call stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.



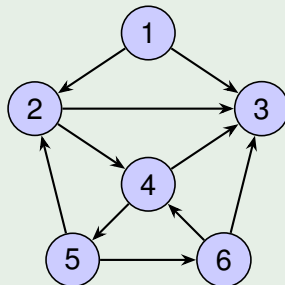
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
		2
		3
		4
		5
		6

## Graph H



- Call stack: **STRONG()** → **DFS(1)**
- Code: **for** edges  $(v, w) \in E$  **do** ...
- $w = 2$ .



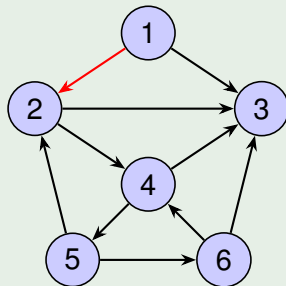
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
				3
				4
				5
				6

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2)$
- Code: **for** edges  $(v, w) \in E$  **do** ...
- $w = 3$ .



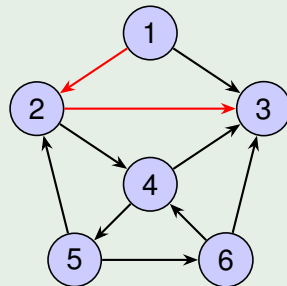
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
3	→	3		3
				4
				5
				6

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(3)$
- Code: **if**  $I[v] = B[\text{TOP}(B)]$  **then** ...
- Go back.



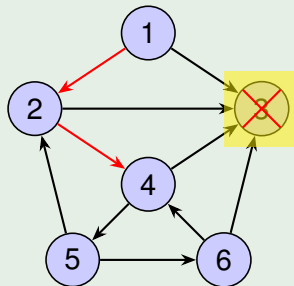
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
3	→	4		3
				7
				3
				0
				0

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4)$
- Code: **for** edges  $(v, w) \in E$  **do** ...
- $w = 5$ .



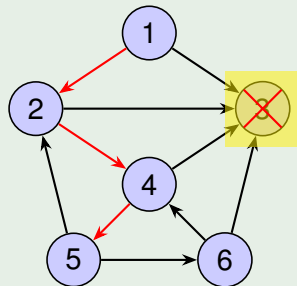
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
3	→	4		3
4	→	5		4
				5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5)$
- Code: **for** edges  $(v, w) \in E$  **do**  $\dots$
- $w = 2$ .



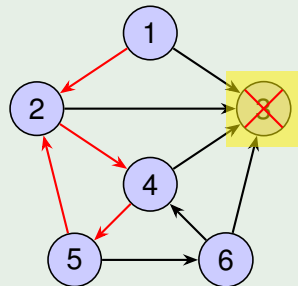
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
3	→	4		3
4	→	5		4
				5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5)$
- Code: **while**  $I[w] < B[\text{TOP}(B)]$  **do**  $\text{POP}(B)$  ;
- Now,  $w = 2$ , contract!



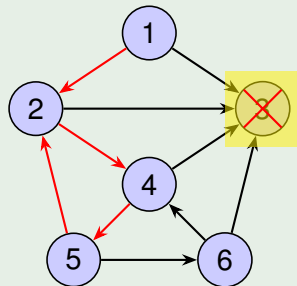
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
				5
				6
				0

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5)$
- Code: **while**  $I[w] < B[\text{TOP}(B)]$  **do**  $\text{POP}(B)$  ;
- Now,  $w = 2$ , contract!





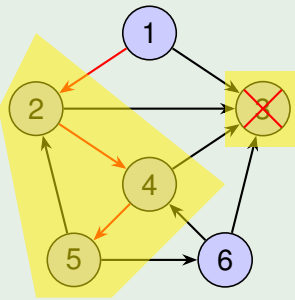
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## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
				5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5)$
- Code: **if**  $I[w] = 0$  **then**  $\text{DFS}(w)$  ;
- $w = 6$ .



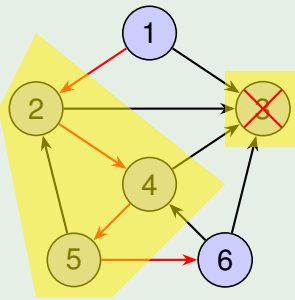
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## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
5		4		3
		5		4
	→	6		5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5) \rightarrow \text{DFS}(6)$
- Code: **for** edges  $(v, w) \in E$  **do**  $\dots$
- $w = 4.$



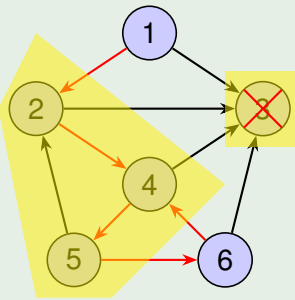
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
5		4		3
		5		4
	→	6		5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5) \rightarrow \text{DFS}(6)$
- Code: **while**  $I[w] < B[\text{TOP}(B)]$  **do**  $\text{POP}(B)$  ;
- Now,  $w = 4$ , contract!



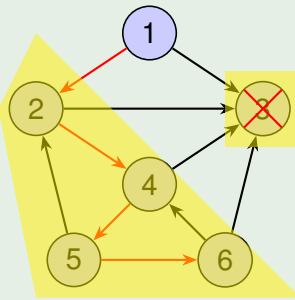
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## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
		6		5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5) \rightarrow \text{DFS}(6)$
- Code: **if**  $I[v] = B[\text{TOP}(B)]$  **then**  $\dots$
- Go back.



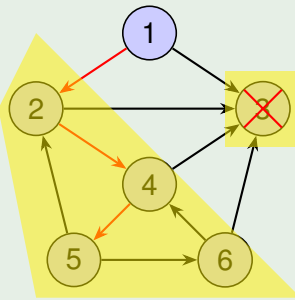
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## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
		6		5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4) \rightarrow \text{DFS}(5)$
- Code: **if**  $I[v] = B[\text{TOP}(B)]$  **then**  $\dots$
- Go back.



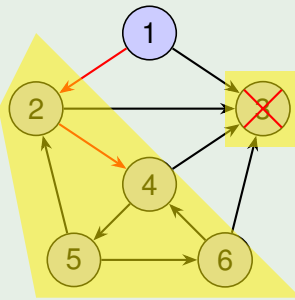
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
		6		5
				6

## Graph H



- Call stack:  $\dots \rightarrow \text{DFS}(2) \rightarrow \text{DFS}(4)$
- Code: **if**  $I[v] = B[\text{TOP}(B)]$  **then**  $\dots$
- Go back.



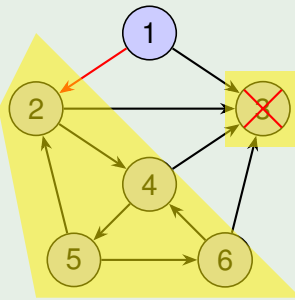
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## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
		4		3
		5		4
		6		5
				6

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2)$
- Code: **if**  $I[v] = B[\text{TOP}(B)]$  **then** ...
- Go back. But this time, **Condition in last line is satisfied!**



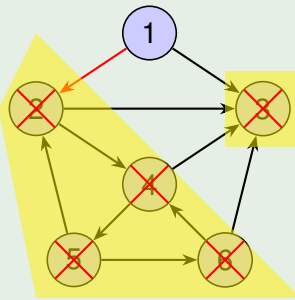
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
		2
		3
		4
		5
		6

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1) \rightarrow \text{DFS}(2)$
- Code: **while**  $I[v] \leq \text{TOP}(S)$  **do**  $I[\text{POP}(S)] = c$ ;
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.



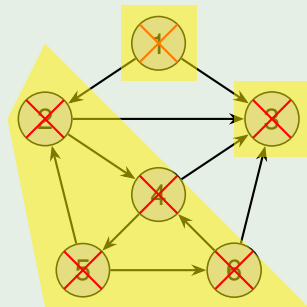
# Demo: Gabow's Strong Components Algorithm

## Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	8
		3	7
		4	8
		5	8
		6	8

## Graph H



- Call stack:  $\text{STRONG}() \rightarrow \text{DFS}(1)$
- Code: **while**  $I[v] \leq \text{TOP}(S)$  **do**  $I[\text{POP}(S)] = c$ ;
- Pop the last one both in B and in S. *Finished!!*

# Outline

- 1 Introduction
- 2 **Strong Components**
  - Reviews
  - Purdom and Munro's High-Level Algorithm
  - Contribution
  - **Discussion**
- 3 Biconnected Components
  - Review
  - High-Level Algorithm
  - Gabow's Algorithms



# Correctness of Gabow's Strong Components Algorithm

## Theorem (Correctness and complexity)

*When  $STRONG(G)$  halts each vertex  $v \in V$  belongs to the strong component numbered  $I[v]$ . The time and space are both  $O(V + E)$ .*

- The key of proof is to show that  $STRONG(G)$  is a valid implementation of the P&M's high-level algorithm.



# Framework of STRONG(G)

---

**Algorithm 5:** Strong components: Main-DFS(G) (DFS caller)

---

$H = G$ ;

**while**  $H$  still has a vertex  $v$  **do**

    | Sub-DFS( $v$ ); /\* start a new path  $P = (v)$  \*/

---



---

**Procedure 6:** STRONG(G)

---

empty stacks  $S$  and  $B$ ;

**for**  $v \in V$  **do**

    |  $I[v] = 0$ ;

$c = n$ ;

**for**  $v \in V$  **do**

    | **if**  $I[v] = 0$  **then** /\*  $v$  has never been accessed \*/  
         | DFS( $v$ );

---



# Growing Path P

---

## Algorithm 7: A Part of High-Level Algorithm

---

```

for  $w \in \{\text{vertices adjacent to } v\}$  do
    if  $w \notin P$  then
        | Sub-DFS( $w$ );
    else /*  $w = v_i$ , and  $v = v_k$  */
        | contract the cycle  $v_i, v_{i+1}, \dots, v_k$ , both in  $H$  and in  $P$ ;
    */
    
```

---



---

## Procedure 8: A Part of DFS( $v$ )

---

```

for  $edges(v, w) \in E$  do
    if  $I[w] = 0$  then
        | DFS( $w$ );
    else /* contract if necessary */
        | while  $I[w] < B[TOP(B)]$  do
            | POP( $B$ );
    */
    
```

---



# Having Found a Strong Components

---

## Algorithm 9: A Part of High-Level Algorithm

---

**if** *no edge leaves  $v$  and  $v$  is the last DFS-finished vertex in a SCC*  
**then**  
    output  $v$  as a vertex of the strong component graph;  
    delete  $v$  from both  $H$  and  $P$ ;

---

---

## Procedure 10: A Part of DFS( $v$ )

---

**if**  $I[v] = B[TOP(B)]$  **then** /\* number vertices of the next  
    strong component \*/  
    POP( $B$ );  
     $c = c + 1$ ;  
    **while**  $I[v] \leq TOP(S)$  **do**  
        |  $I[POP(S)] = c$ ;

---



# Time Complexity

- Every vertex is pushed onto and popped from each stack  $S$ ,  $B$  exactly once. So the algorithm spends  $O(1)$  time on each vertex or edge.
- Time complexity:  $O(V + E)$
- Intuitively, from another view, this algorithm is based on DFS, and no loop is executed on one vertex or one edge.



# Outline

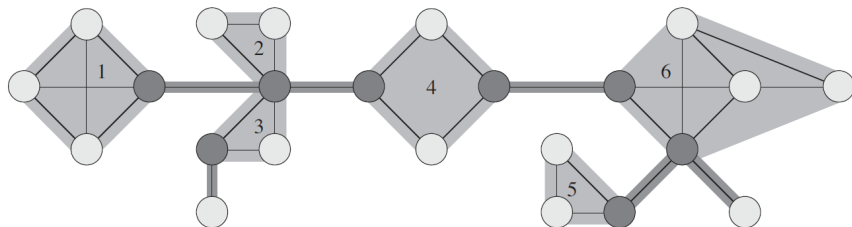
- 1 Introduction
- 2 Strong Components
  - Reviews
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  - Contribution
  - Discussion
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  - **Review**
  - High-Level Algorithm
  - Gabow's Algorithms





# Review: Biconnected Component

- A *biconnected component* of  $G$  is a maximal set of edges such that any two edges in the set lie on a common simple cycle.



# Outline

- 1 Introduction
- 2 Strong Components
  - Reviews
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  - Gabow's Algorithms



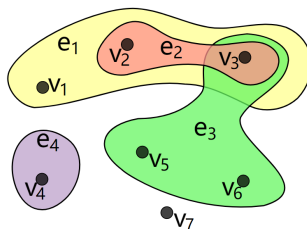
# Concepts: Hypergraph

- A *hypergraph*  $H = (V, E)$  is a generalization of a graph in which an edge can join any number of vertices.
- In the following hypergraph,

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

$$E = \{e_1, e_2, e_3, e_4\}$$

$$= \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\}$$



# Concepts: Hypergraph

- Therefore, we need redefine the *edge*, *path*, *cycle*,  $\dots$ , and nearly all concepts as long as it is relative to edge.
- A *path* is a sequence  $(v_1, e_1, \dots, v_k, e_k)$  of distinct vertices  $v_i$  and distinct edges  $e_i$ ,  $1 \leq i \leq k$ , with  $v_1 \in e_1$  and  $v_i \in e_{i-1} \cap e_i$  for every  $1 < i \leq k$ .
- An important property:

$$v_{i+1} \in e_i - \{v_i\}, \text{ or } v_i \in e_i - \{v_{i+1}\}, \quad 1 \leq i < k$$

- Merging a set of edges is to replace old edges with the new one:

$$e_{\text{new}} = \bigcup_{i=1}^k e_i$$



## Additional Concepts We Need

- The *block hypergraph*  $H$  of  $G$  is the hypergraph formed by merging the edges of each biconnected component of  $G$ .
- The set of *all vertices in edges of*  $P$  is denoted

$$V(P) = \bigcup_{i=1}^k e_i$$



# High-Level Algorithm in Plain Text

- Initially  $H$  is the given graph  $G$ . If  $H$  has no edges stop. Otherwise start a new path  $P$  by choosing an edge  $\{v, w\}$  and setting  $P = (v, \{v, w\})$ . Continue by growing  $P$ .
- To grow the path  $P = (v_1, e_1, \dots, v_k, e_k)$  choose an edge  $\{v, w\} \neq e_k$  with  $v \in e_k - \{v_k\}$  and do:
  - If  $w \notin V(P)$ , add  $v, \{v, w\}$  to the end of  $P$ . Continue growing  $P$ .
  - If  $w \in V(P)$ , say  $w \in e_i - \{v_{i+1}\}$ , merge the edges of the cycle  $w, e_i, v_{i+1}, e_{i+1}, \dots, v_k, e_k, v, \{v, w\}$  to a new edge  $e = \bigcup_{j=i}^k e_j$ , both in  $H$  and in  $P$ . Continue growing  $P$ .
  - If no edge leaves  $e_k - \{v_k\}$ , output  $e_k$  as an edge of the block hypergraph. Delete  $e_k$  from  $H$  and delete  $(v_k, e_k)$  from  $P$ . If  $P$  is now nonempty continue growing  $P$ . Otherwise try to start a new path  $P$ .



# Pseudo Code

---

**Algorithm 11:** Biconnected Components: Main-DFS (DFS caller)

---

$H = G;$

**while**  $H$  still has an edge  $\{v, w\}$  **do**

    | Sub-DFS( $v$ ); /\* start a new path  $P = (v, \{v, w\})$       \*/

---



# Pseudo Code

---

## Algorithm 12: Biconnected Components: Sub-DFS (DFS callee)

---

```

add the  $v$  as the new last vertex of path  $P$ ;
for  $w \in \{\text{vertices adjacent to } v\}$  do /* Grows path  $P$  */
    if  $w \notin V\{P\}$  then
        add  $\{v, w\}$  to the end of  $P$ , as the new last edge of  $P$ ;
        Sub-DFS( $w$ );
        remove the edge  $\{v, w\}$  if necessary;
    else /*  $w \in e_i - v_{i+1}$ , but most likely  $w \neq v_i$  */
        replace the cycle  $w, e_i, v_{i+1}, e_{i+1}, \dots, v_k, e_k, v$  to a new edge
             $e = \bigcup_{j=i}^k e_j$ , both in  $H$  and in  $P$ ;
if no edge leaves  $e_k - v_k$  then
    output  $e_k$  as an edge of the block hypergraph;
    delete  $e_k$  from  $H$  and delete  $(v_k, e_k)$  from  $P$ ;
    
```

---





# Correctness

- When  $v$ ,  $\{v, w\}$  is added to  $P$  the result is a valid path, by the condition  $v \in e_k - v_k$ . When edges are merged they form a valid cycle, by the condition  $\{v, w\} \neq e_k$ .
- The algorithm correctly forms the finest acyclic merging of  $G$ , it finds the block hypergraph as desired.



# Outline

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  - **Gabow's Algorithms**



# Data Structure Used in Algorithm

- A **stack  $S$**  contains the vertices  $V(P)$ .
- A **stack  $B$**  contains the boundaries between edges of  $P$ , *two vertices per boundary*.
- An array  $I[1 \dots n]$  is used to store stack indices corresponding to vertices.
- All of above are similar (*but not the same*) to these in strong components.



# Algorithms

---

## Procedure 13: BICONN(G)

---

empty stacks  $S$  and  $B$ ;

**for**  $v \in V$  **do**

$I[v] = 0$ ;

$c = n$ ;

**for**  $v \in V$  **do**

**if**  $I[v] = 0$  *and*  $v$  *is not isolated* **then**  
        DFS( $v$ );

---



# Marked Arrows

- *Open arrows*: They point to the vertices  $v_i$  of  $P$ .

$$v_i = S[B[2i - 1]], \quad i = 1, \dots, k$$

*Filled arrows*: They demarcate the sets  $e_i - v_i$ ; these sets are the "nonfirst" vertices of edges  $e_i$  of  $P$ .

$$e_i - v_i = \{S[j] : B[2i] \leq j < B[2i + 2]\}, \quad i = 1, \dots, k$$



# Algorithms

---

## Procedure 14: DFS(v)

---

```

PUSH( $v, S$ );  $I[v] = \text{TOP}(S)$ ;
if  $I[v] > 1$  then /* create a filled arrow on  $B$  */
|   PUSH( $I[v], B$ );
for  $\text{edges}\{v, w\} \in E$  do
|   if  $I[w] = 0$  then /* create an open arrow on  $B$  */
|   |   PUSH( $I[v], B$ ); DFS( $w$ );
|   else /* possible merge */
|   |   while  $I[v] > 1$  and  $I[w] < B[\text{TOP}(B) - 1]$  do
|   |   |   POP( $B$ ); POP( $B$ );
if  $I[v] = 1$  then
|    $I[\text{POP}(S)] = c$ ;
else if  $I[v] = B[\text{TOP}(B)]$  then
|   POP( $B$ ); POP( $B$ );  $c = c + 1$ ;
|   while  $I[v] \leq \text{TOP}(S)$  do  $I[\text{POP}(S)] = c$ ;

```

---



# Demo: Gabow's biconnected components algorithm

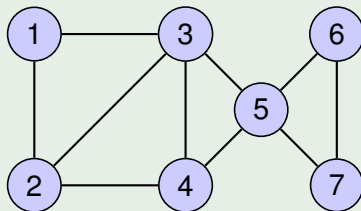
## Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0
		7	0

- Current procedure:  
BICONN(G)

## Graph



# Demo: Gabow's biconnected components algorithm

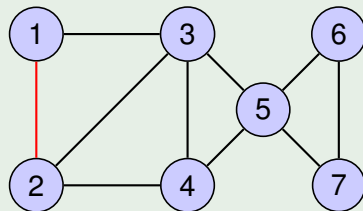
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
		2
		3
		4
		5
		6
		7

- Current procedure:  
DFS(1): w=2

## Graph





# Demo: Gabow's biconnected components algorithm

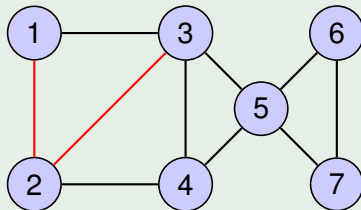
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
2		3
		4
		5
		6
		7

- Current procedure:  
DFS(2):  $w=3$

## Graph



# Demo: Gabow's biconnected components algorithm

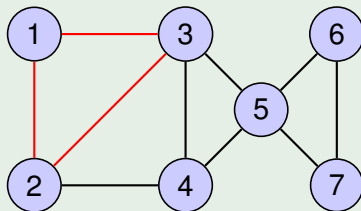
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
		4
		5
		6
		7

- Current procedure:  
DFS(3):  $w=1$

## Graph



# Demo: Gabow's biconnected components algorithm

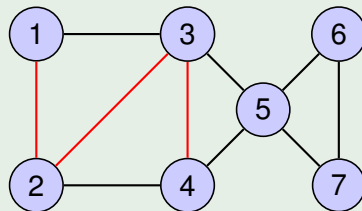
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
3	3	3
		4
		5
		6
		7

- Current procedure:  
DFS(3):  $w=4$

## Graph



# Demo: Gabow's biconnected components algorithm

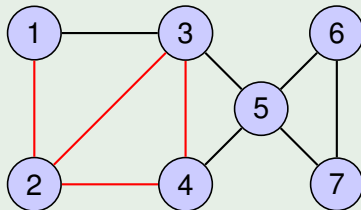
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
		5
		6
		7

- Current procedure:  
DFS(4): w=2

## Graph



# Demo: Gabow's biconnected components algorithm

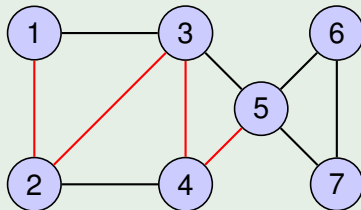
## Data in memory

B and S: stack. I: array.

B		S		I
1	→	1		1
2	→	2		2
4		3		3
		4		4
	→			5
				6
				7

- Current procedure:  
DFS(4):  $w=5$

## Graph



# Demo: Gabow's biconnected components algorithm

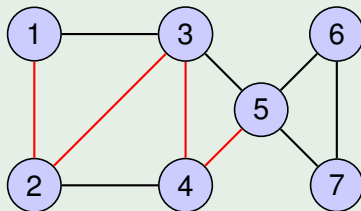
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
	5	5
		6
		7

- Current procedure:  
DFS(5):  $w=3$

## Graph



# Demo: Gabow's biconnected components algorithm

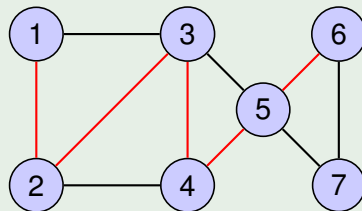
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
5	3	3
	4	4
	5	5
		6
		7

- Current procedure:  
DFS(5): w=6

## Graph



# Demo: Gabow's biconnected components algorithm

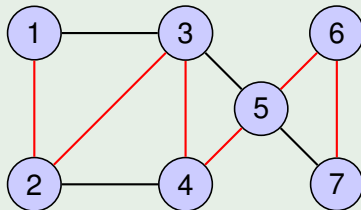
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
5	3	3
6	4	4
6	5	5
	6	6
		7
		0

- Current procedure:  
DFS(6):  $w=7$

## Graph





# Demo: Gabow's biconnected components algorithm

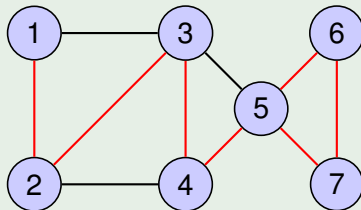
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
5	3	3
6	4	4
	5	5
	6	6
	7	7

- Current procedure:  
DFS(7):  $w=5$

## Graph



# Demo: Gabow's biconnected components algorithm

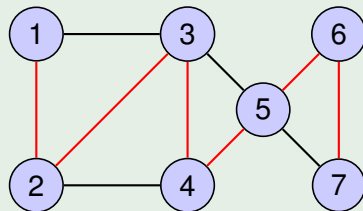
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
5	3	3
6	4	4
	5	5
	6	6
	7	7

- Current procedure:  
DFS(7): End

## Graph



# Demo: Gabow's biconnected components algorithm

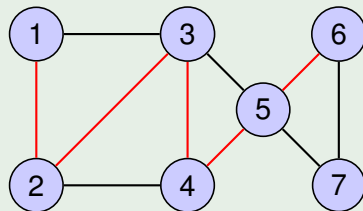
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
	5	5
		6
		7

- Current procedure:  
DFS(6): End

## Graph



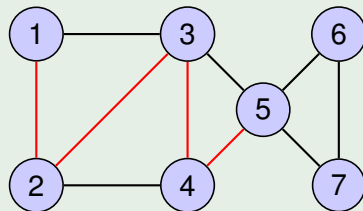
# Demo: Gabow's biconnected components algorithm

## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
	5	5
		6
		7

## Graph



- Current procedure:  
DFS(5): End(No operation when  $w=7$ )



# Demo: Gabow's biconnected components algorithm

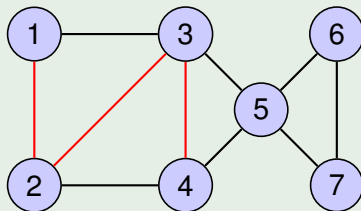
## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
	5	5
		6
		7
		8

- Current procedure:  
DFS(4): End

## Graph



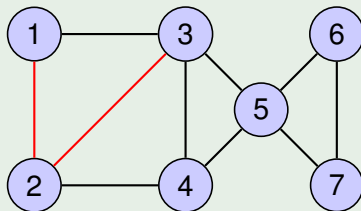
# Demo: Gabow's biconnected components algorithm

## Data in memory

B and S: stack. I: array.

B	S	I
1	1	1
2	2	2
	3	3
	4	4
	5	5
		6
		7

## Graph



- Current procedure:  
DFS(3): End(No operation when  $w=5$ )



# Demo: Gabow's biconnected components algorithm

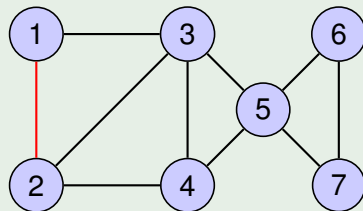
## Data in memory

B and S: stack. I: array.

B	S	I
	1	1
		2
		3
		4
		5
		6
		7

- Current procedure:  
DFS(2): End

## Graph



# Demo: Gabow's biconnected components algorithm

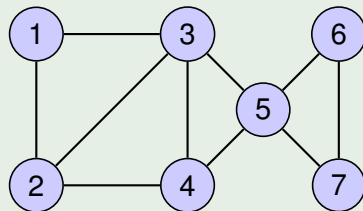
## Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	9
		3	9
		4	9
		5	9
		6	8
		7	8

- Current procedure:  
DFS(1): End

## Graph





# Correctness

- In order to keep the completeness, the correctness is given as follow.

## Theorem (Correctness and complexity)

*When  $BICONN(G)$  halts any edge  $\{v, w\} \in E$  belongs to the biconnected component numbered  $\min\{I[v], I[w]\}$ . The time and space are both  $O(V + E)$ .*



# Summary

- Gabow gave algorithms to find the strong components and biconnected components more effectively. They are one-pass algorithms while do not need a disjoint-set data structure.
- There is a close relationship between strong components and biconnected components, like two faces of a coin: one is directed graph, another is undirected graph.

