Path-based depth-first search for strong and biconnected components

Author of the paper: Harold N. Gabow

Reported by: T.T. Liu D.P. Xu B.Y. Chen

May 27, 2017

Outline

1. Introduction
2. Strong Components
   - Thinking about Strong Components
   - Purdom and Munro's high-level algorithm
   - Contribution

Characterastics of Gabow's Algorithms

- One-pass algorithm. But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.

Several Questions

- LOWPOINT?
- Ear decomposition?
- Compele version?
- Robbin's Theorem?

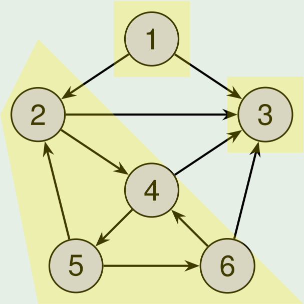
Outline

Review: What have we learned from the textbook?

Concepts of Strong Components

- Two mutually reachable vertices are in the same *strong component*.

Example



Review: What have we learned from the textbook?

Algorithms to Find Strong Components

- Idea: Run DFS twice. Once on the original graph G, once on the *tranposition* of G^T.
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity: $O(V + E)$

Outline

1. Introduction

2. Strong Components
   - Thinking about Strong Components
   - Purdom and Munro's high-level algorithm
   - Contribution

Pseudo-Code

```
H = G;
while H still has a vertex v
   start a new path P = (v);
   while P is not empty
      if the last vertex v_k of P has an edge (v_k, w)
         if w belongs to P
            contract the cycle v_i(w), ... , v_k, both
                in H and in P; /* w and v_i are
                identical. */
         else
            add w to P, as the new last vertex of P;
         end if
```

- Note that *contracting* means selecting one vertex as a representation and merging others rather than deleting them.

Pseudo-Code (Continued)

```
        else
            output v_k as a vertex of the strong component
                graph;
            delete v_k from both H and P;
        end if
    end
end
```

Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except the statement in line 7:

```
7 contract the cycle v_i(w), ... , v_k, both in H and in
      P; /* w and v_i are identical. */
```

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint set merging is needed.

Outline

1. Introduction
2. Strong Components
   - Thinking about Strong Components
   - Purdom and Munro's high-level algorithm
   - Contribution

His Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.

Data Structure Used in Algorithm

- In DFS, the path P from root to each node is almost always significant. So it is in this algorithm.
- A stack S contains the sequence of vertices in P.
- A stack B contains the boundaries between contracted vertices.
- An array I$[1 \dots n]$ is used to store stack indices corresponding to vertices.

Contraction Makes Much Difference

- When contraction is excuted, some vertices merges into a set.
- It is possible that several elements in stack S are in the same vertex in path P. More formal,

$$v_i = S[j] \,:\, B[i] \le j < B[i+1]$$

- By the way, the formal definition of $I[v]$ is

$$I[j] = \begin{cases} 0, & \text{if} \\ j, & \text{if} \\ c, & \text{if} \end{cases}$$

New algorithm to discover strong components

Procedure 1: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ **then**
    DFS(v);

New algorithm to discover strong components

Procedure 2: DFS(v)

PUSH(v,S); I[v]=TOP(S); PUSH(I[v],B);
`/* add v to the end of P */`
for *egdes*$(v, w) \in E$ **do**
  if $I[v] = 0$ **then**
    DFS(w);
  else `/* contract if necessary */`
    while $I[w] < B[TOP[B]]$ **do**
      POP(B);
  if $I[v] = B[TOP(B)]$ **then**
    `/* number vertices of the next strong component */`
    POP(B);
    increase c by 1;
    while $I[v] \leq TOP[S]$ **do**
      I[POP(S)]=c;

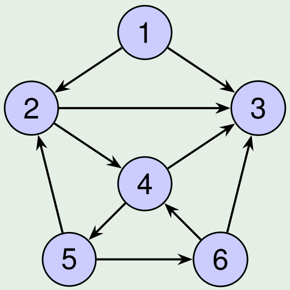
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.

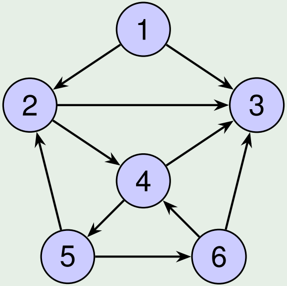
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)
- Code: `for edges(v,w)∈E do ...`
- $w = 2$.

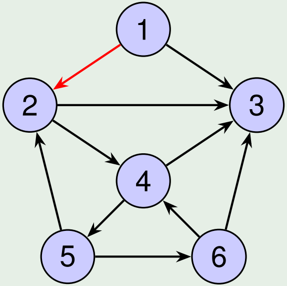
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `for edges(v,w)∈E do ...`
- $w = 3$.

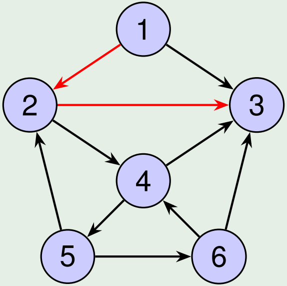
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	3	3	3
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(3)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

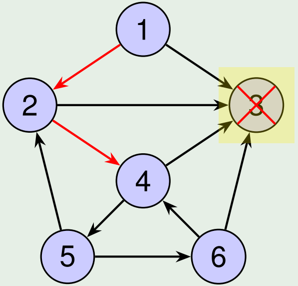
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
		4	3
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(4)
- Code: `for edges(v,w)∈E do ...`
- $w = 5$.

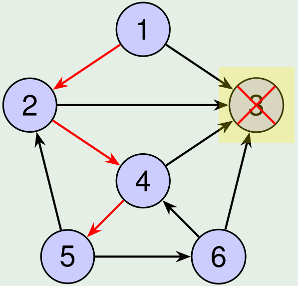
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `for edges (v,w)∈E do ...`
- $w = 2$.

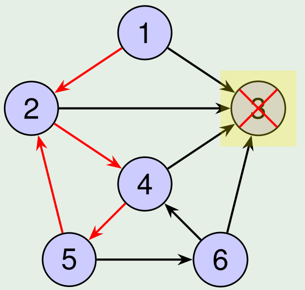
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

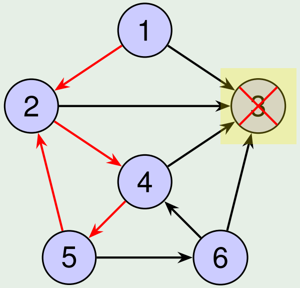
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

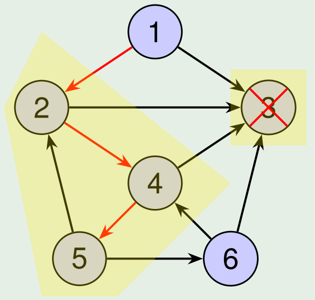
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: **if** `I[w]` $= 0$ **then** `DFS(w);`
- $w = 6$.

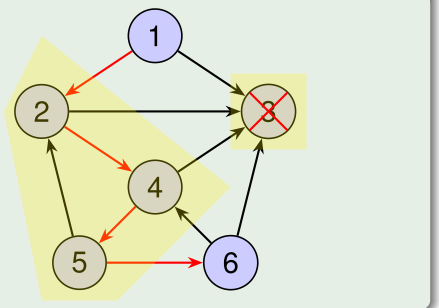
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `for edges(v,w)`$\in$`E do ...`
- $w = 4$.

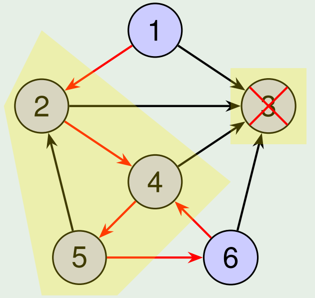
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: **while** `I[w]<B[TOP(B)]` **do** `POP(B);`
- Now, $w = 4$, contract!

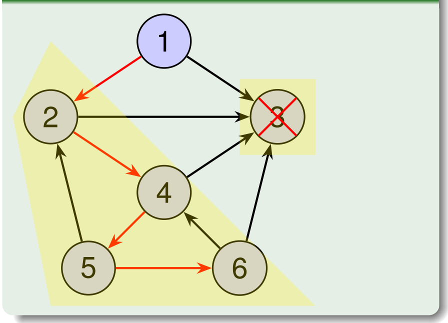
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

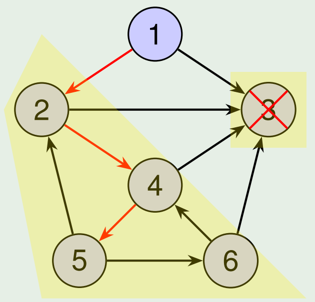
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

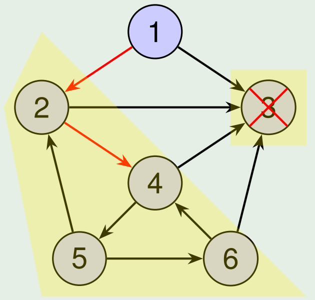
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

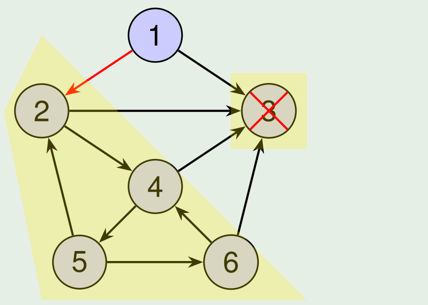
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back. But this time, Condition in last line is satisfied!

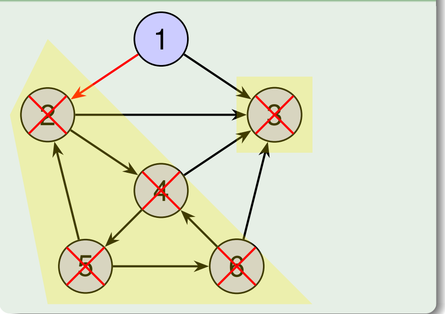
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: **while** `I[v]`$\leq$`TOP(S)` **do** `I[POP(S)]=c;`
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.

Demo: Gabow's strong components algorithm

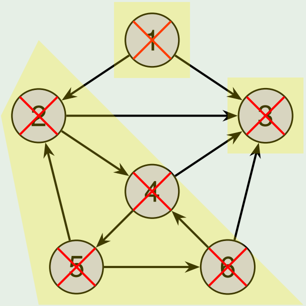
Data in memory

B and S: stack. I: array.

B S

	I
1	9
2	8
3	7
4	8
5	8
6	8

Graph H



- Call Stack: STRONG()$\rightarrow$DFS(1)
- Code: **while** I[v]$\leq$TOP(S) **do** I[POP(S)]=c;
- Pop the last one both in B and in S. *Finished!!*

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal of This and That, 2(1):50–100, 2000.