Path-based depth-first search for strong and biconnected components

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Outline

Characterastics of Gabow's Algorithms

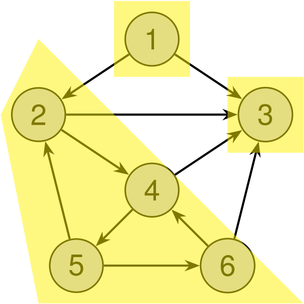
- One-pass algorithm. But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.
- Lower time and space complexity. This algorithm only use two stacks and an array, and do not employ a disjoint-set data structure.

Outline

Review: What have we learned from the textbook?

Concepts of Strong Components

- Two mutually reachable vertices are in the same *strong component*.
- It is a equivalence relation.



Review: What have we learned from the textbook?

Algorithms to Find Strong Components

- Idea: Run DFS twice: Once on the original graph G, once on its *transpose* G^T.
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity: $O(V+E)$
- Proposed by S. Rao Kosaraju, known as the *Kosaraju's Algorithm*.

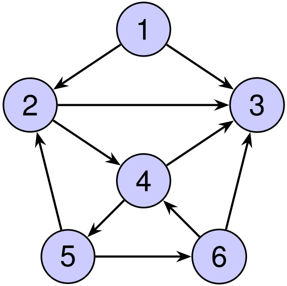
Outline

1. Introduction
2. Strong Components
   - Reviews
   - **Purdom and Munro's High-Level Algorithm**
   - Contribution
   - Discussion
3. Biconnected Components
   - Review
   - High-Level Algorithm
   - Gabow's Algorithms

Purdom and Munro's High-Level Algorithm: Plain text

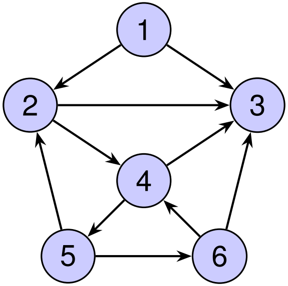
- Initially H is the given graph G. If H has no vertices stop. Otherwise *start a new path* P by choosing a vertex v and setting $P = (v)$. Continue by growing P as follows.
- To grow the path $P = (v_1, \cdots, v_k)$ choose an edge (v_k, w) directed from the last vertex of P and do the following:
  - If $w \notin P$, *add* w *to* P, making it the new last vertex of P. Continue growing P.
  - If $w \in P$, say $w = v_i$, contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P. P is now a path in the new graph H. Continue growing P.
  - If no edge leaves v_k, output v_k as a vertex of the strong component graph. Delete v_k from both H and P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

Demo: P&M's High-Level Algorithm



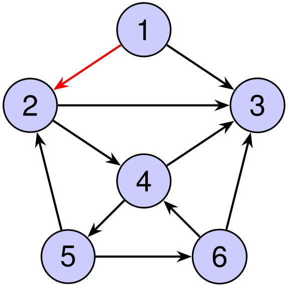
- Path $P = \{\ \}$
- Initially, $H = G$.

Demo: P&M's High-Level Algorithm



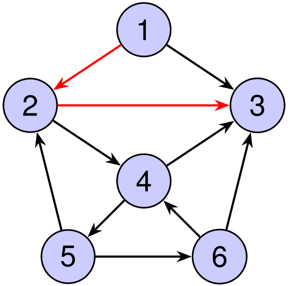
- Path $P = \{\,\{1\}\,\}$
- Grow P by adding v_1.

Demo: P&M's High-Level Algorithm



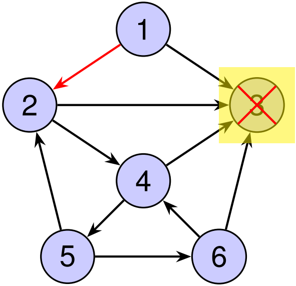
- Path $P = \{\, \{1\},\, \{2\}\, \}$
- Grow P by adding v_2.

Demo: P&M's High-Level Algorithm



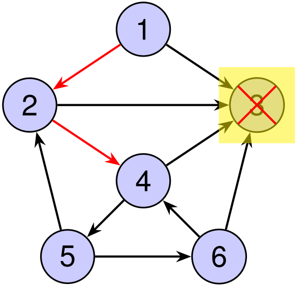
- Path $P = \{\,\{1\},\,\{2\},\,\{3\}\,\}$
- Grow P by adding v_3.

Demo: P&M's High-Level Algorithm



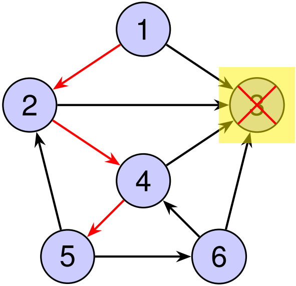
- Path $P = \{\,\{1\}, \{2\}\,\} \quad \{3\}$
- As v_3 is isolated, no edge leaves from v_3, so just delete it.

Demo: P&M's High-Level Algorithm



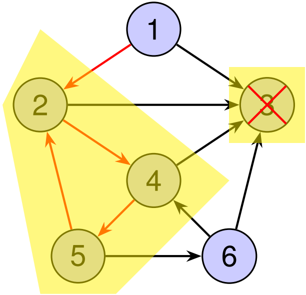
- Path $P = \{\,\{1\}, \{2\}, \{4\}\,\} \quad \{3\}$
- Grow P by adding v_4.

Demo: P&M's High-Level Algorithm



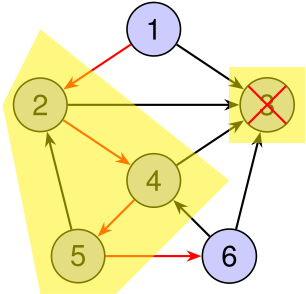
- Path $P = \{\,\{1\}, \{2\}, \{4\}, \{5\}\,\} \quad \{3\}$
- Grow P by adding v_5.

Demo: P&M's High-Level Algorithm



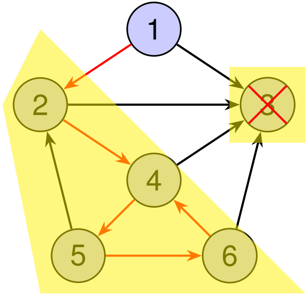
- Path $P = \{\,\{1\},\,\{2,\,4,\,5\}\,\}\quad\{3\}$
- The cycle $v_2,\ v_4,\ v_5$ in P is detected. Contract this cycle.

Demo: P&M's High-Level Algorithm



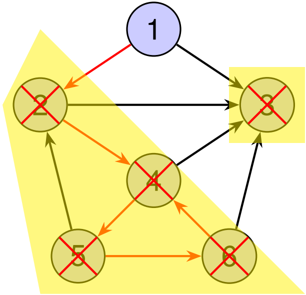
- Path $P = \{\, \{1\}, \{2, 4, 5\}, \{6\} \,\} \quad \{3\}$
- Grow P by adding v_6.

Demo: P&M's High-Level Algorithm



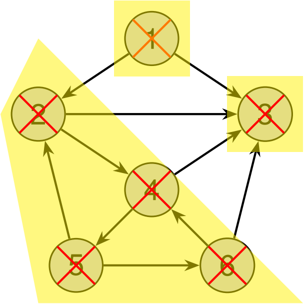
- Path $P = \{\, \{1\}, \{2, 4, 5, 6\}\,\} \quad \{3\}$
- The cycle $\{v_2, v_4, v_5\}, v_6, v_4$ in P is detected. Contract this cycle.

Demo: P&M's High-Level Algorithm



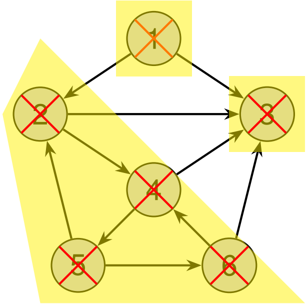
- Path $P = \{\,\{1\}\,\} \quad \{2,\,4,\,5,\,6\},\,\{3\}$
- No edge leaves from $\{v_2,\,v_4,\,v_5,\,v_6\}$, so we delete it.

Demo: P&M's High-Level Algorithm



- Path $P = \{\ \}$ $\{1\}, \{2, 4, 5, 6\}, \{3\}$
- No edge leaves from $\{v_1\}$, so we delete it.

Demo: P&M's High-Level Algorithm



- Path $P = \{\ \}$ $\{1\}, \{2, 4, 5, 6\}, \{3\}$
- Now graph H is empty, which has no vertex.

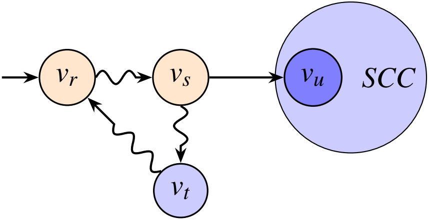
Correctness

- Correctness: If no edge leaves v_k then v_k is a vertex of the finest acyclic contraction.
- Easy to prove by contradiction: If no edge leaves v_k, but v_k is not a vertex of the finest acyclic contraction. That is to say, v_k is a part of some strong component S', so there is a vertex $v' \in S'$, which satisfies $v_k \neq v'$ while v_k and v' are mutually reachable. Therefore, one edge at least leaving v_k must be existent.

Three Cases When Growing Path

- To grow the path $P = (v_1, \cdots, v_k)$ choose an edge (v_k, w) directed from the last vertex of P and do the following:
  1. If $w \notin P$, *add* w *to* P, making it the new last vertex of P. Continue growing P.
  2. If $w \in P$, say $w = v_i$, contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P. P is now a path in the new graph H. Continue growing P.
  3. If no edge leaves v_k, output v_k as a vertex of the strong component graph. Delete v_k from both H and P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

How to implement by DFS?



- Assume the current node is v_s which has two adjacent nodes. The current path is $P = (\cdots, v_r, \cdots, v_s)$.
- For the node v_u incident from v_s but also in the SCC, after running Sub-DFS() on this node, v_u will be removed with the SCC.

Pseudo Code

Algorithm 1: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); `/* start a new path` $P = (v)$ `*/`

Pseudo Code

Algorithm 2: Strong components: Sub-DFS(v) (DFS callee)

add the v as the new last vertex of path P;
for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else `/*` $w = v_i$, `and` $v = v_k$ `*/`
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;
if *no edge leaves* v **then**
  output v as a vertex of the strong component graph;
  delete v from both H and P;

Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except this statement:

`contract the cycle` $v_i, v_{i+1}, \cdots, v_k$`, both in` H `and in` P`;`

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint-set merging is needed usually.

Outline

Gabow's Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.

Data Structure Used in Algorithm

- In DFS, the path P from root to each node is almost always significant. So it is in this algorithm.
- A stack S contains the sequence of vertices in P.
- A stack B contains the boundaries between contracted vertices.
- An array $I[1 \ldots n]$ is used to store stack S indices corresponding to vertices.

Contraction Makes Much Difference

- S and B correspond to $P = (v_1, \cdots, v_k)$ where $k = TOP(B)$ and for $i = 1, \cdots, k$.
- When contraction is executed, some vertices merges into a set.
- It is possible that several elements in stack S are in the same vertex in path P. More formal,

$$v_i = \{S[j] | B[i] \leq j < B[i+1]\}$$

Contraction Makes Much Difference

- By the way, the formal definition of $I[v]$ is

$$I[v] = \begin{cases} 0, & \text{if } v \text{ has never been in P;} \\ j, & \text{if } v \text{ is currently in P and } S[j] = v; \\ c, & \text{if the strong component containing } v \text{ has} \\ & \text{been deleted and numbered as } c. \end{cases}$$

where c counts from $n+1$.

New Algorithm to Discover Strong Components

Procedure 3: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
    $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
    if $I[v] = 0$ **then** `/* vertex v has never been accessed yet */`
        DFS(v);

New Algorithm to Discover Strong Components

Procedure 4: DFS(v)

PUSH(v, S); $I[v]$ =TOP(S); PUSH$(I[v], B)$;
/* add v to the end of P */
for *egdes*$(v, w) \in E$ **do**
  if $I[w] = 0$ **then**
    DFS(w);
  else /* contract if necessary */
    while $I[w] < B[TOP(B)]$ **do**
      POP(B);
if $I[v] = B[TOP(B)]$ **then** /* number vertices of the next strong component */
  POP(B);
  $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do**
    $I[$POP$(S)] = c$;

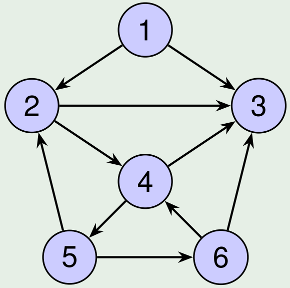
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.

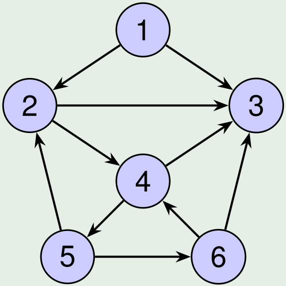
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call stack: STRONG()→DFS(1)
- Code: **for** `edges(v,w)` $\in$ `E` **do** $\cdots$
- $w = 2$.

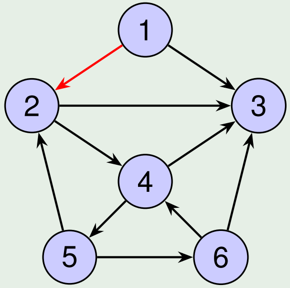
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
			3	0
			4	0
			5	0
			6	0

Graph H



- Call stack: STRONG()→DFS(1)→DFS(2)
- Code: `for edges(v,w)∈E do ···`
- $w = 3$.

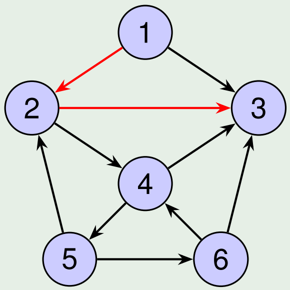
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
3	→	3	3	3
			4	0
			5	0
			6	0

Graph H



- Call stack: STRONG()→DFS(1)→DFS(2)→DFS(3)
- Code: `if I[v]=B[TOP(B)] then ···`
- Go back.

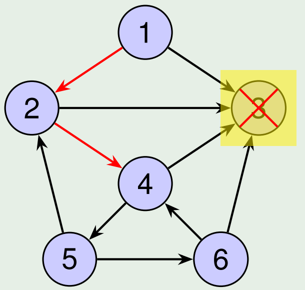
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
		4	3
		5	0
		6	0

Graph H



- Call stack: STRONG()→DFS(1)→DFS(2)→DFS(4)
- Code: `for edges(v,w)∈E do ···`
- $w = 5$.

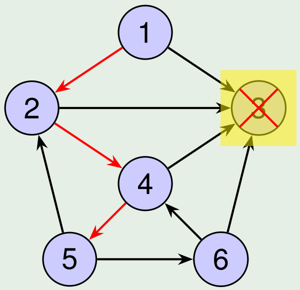
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
3	→	4	3	7
4	→	5	4	3
			5	4
			6	0

Graph H



- Call stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: **for** `edges(v,w)`$\in$`E` **do** $\cdots$
- $w = 2$.

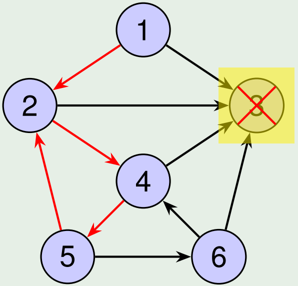
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
3	→	4	3	7
4	→	5	4	3
			5	4
			6	0

Graph H



- Call stack: $\cdots\rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

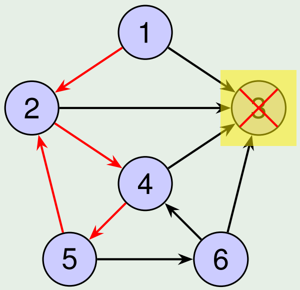
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
		4	3	7
		5	4	3
			5	4
			6	0

Graph H



- Call stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

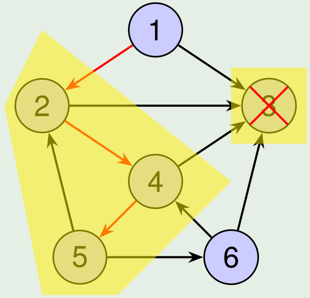
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→ 1		1	1
2	→ 2		2	2
	4		3	7
	5		4	3
			5	4
			6	0

Graph H



- Call stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: **if** `I[w]=0` **then** `DFS(w);`
- $w = 6$.

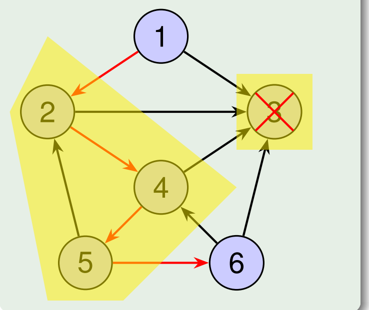
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
5		4	3	7
		5	4	3
	→	6	5	4
			6	5

Graph H



- Call stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)$\to$DFS(6)
- Code: `for edges(v,w)∈E do ...`
- $w = 4$.

Demo: Gabow's Strong Components Algorithm

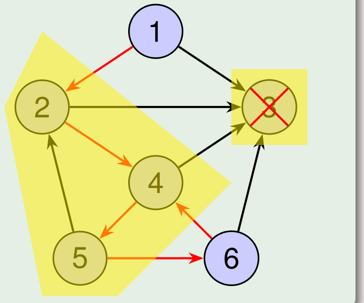
Data in memory

B and S: stack. I: array.

B		S
1	→	1
2	→	2
5		4
		5
	→	6

	I
1	1
2	2
3	7
4	3
5	4
6	5

Graph H



- Call stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)$\to$DFS(6)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 4$, contract!

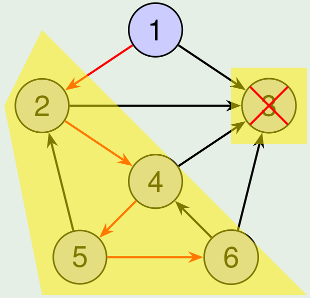
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
		4	3	7
		5	4	3
		6	5	4
			6	5

Graph H



- Call stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)$\to$DFS(6)
- Code: **`if`** `I[v]=B[TOP(B)]` **`then`** `...`
- Go back.

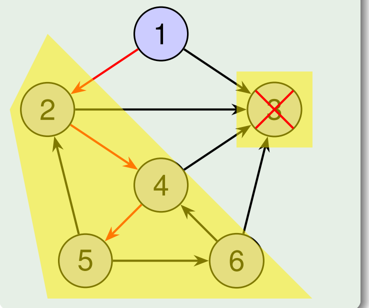
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
		4	3	7
		5	4	3
		6	5	4
			6	5

Graph H



- Call stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

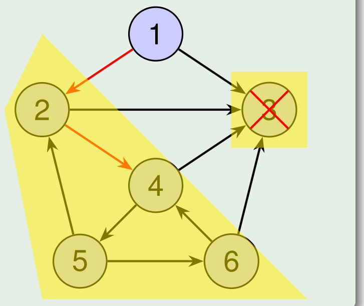
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
		4	3	7
		5	4	3
		6	5	4
			6	5

Graph H



- Call stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

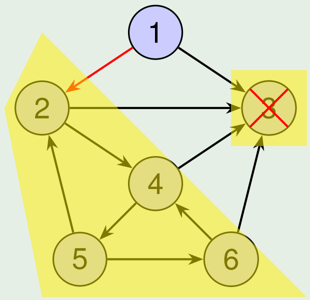
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
2	→	2	2	2
		4	3	7
		5	4	3
		6	5	4
			6	5

Graph H



- Call stack: STRONG()→DFS(1)→DFS(2)
- Code: **if** `I[v]=B[TOP(B)]` **then** ...
- Go back. But this time, Condition in last line is satisfied!

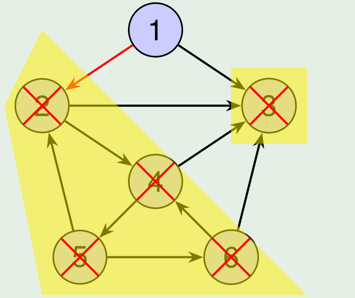
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B		S		I
1	→	1	1	1
			2	8
			3	7
			4	8
			5	8
			6	8

Graph H



- Call stack: STRONG()→DFS(1)→DFS(2)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.

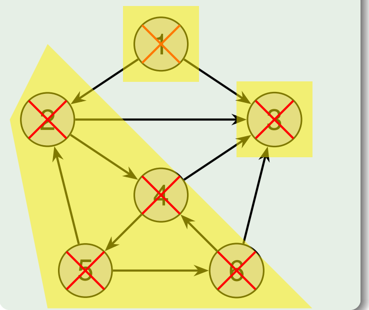
Demo: Gabow's Strong Components Algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call stack: STRONG()→DFS(1)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop the last one both in B and in S. *Finished!!*

Outline

Correctness of Gabow's Strong Components Algorithm

Theorem (Correctness and complexity)

When STRONG(G) halts each vertex $v \in V$ belongs to the strong component numbered $I[v]$. The time and space are both $O(V+E)$.

- The key of proof is to show that STRONG(G) is a valid implementation of the P&M's high-level algorithm.
- Tip: Use induction as a tool.

Framework of STRONG(G)

Algorithm 5: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); `/* start a new path` $P = (v)$ `*/`

Procedure 6: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ **then** `/*` v `has never been accessed */`
    DFS(v);

Growing Path P

Algorithm 7: A Part of High-Level Algorithm

for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else /* $w = v_i$, and $v = v_k$ */
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;

Procedure 8: A Part of DFS(v)

for *egdes*$(v, w) \in E$ **do**
  if $I[w] = 0$ **then**
    DFS(w);
  else /* contract if necessary */
    while $I[w] < B[TOP(B)]$ **do**
      POP(B);

Having Found a Strong Components

Algorithm 9: A Part of High-Level Algorithm

if *no edge leaves* v **then**
  output v as a vertex of the strong component graph;
  delete v from both H and P;

Procedure 10: A Part of DFS(v)

if $I[v] = B[TOP(B)]$ **then** `/* number vertices of the next strong component */`
  POP(B);
  $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do**
    $I[\text{POP}(S)] = c$;

Time Complexity

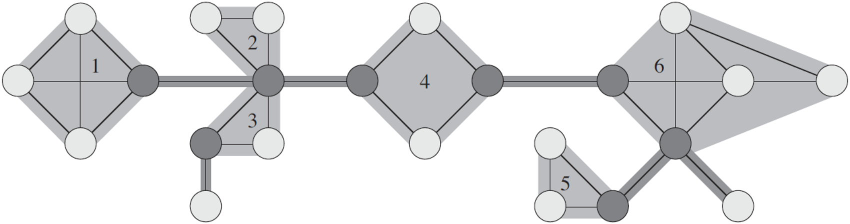
- Every vertex is pushed onto and popped from each stack S, B exactly once. So the algorithm spends $O(1)$ time on each vertex or edge.
- Time complexity: $O(V + E)$
- Intuitively, from another view, this algorithm is based on DFS, and no loop is executed on one vertex or one edge.

Outline

1. Introduction
2. Strong Components
   - Reviews
   - Purdom and Munro's High-Level Algorithm
   - Contribution
   - Discussion
3. Biconnected Components
   - Review
   - High-Level Algorithm
   - Gabow's Algorithms

Review: Biconnected Component

- A *biconnected component* of G is a maximal set of edges such that any two edges in the set lie on a common simple cycle.



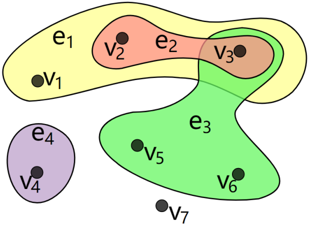
Outline

Concepts: Hypergraph

- A *hypergraph* $H = (V, E)$ is a generalization of a graph in which an edge can join any number of vertices.
- In the following hypergraph,

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

$$\begin{aligned} E &= \{e_1, e_2, e_3, e_4\} \\ &= \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\} \end{aligned}$$



Concepts: Hypergraph

- Therefore, we need redefine the *edge*, *path*, *cycle*, $\cdots$, and nearly all concepts as long as it is relative to edge.
- A *path* is a sequence $(v_1, e_1, \cdots, e_{i-1}, v_i, e_i, v_{i+1}, \cdots, v_k, e_k)$ of distinct vertices v_i and distinct edges e_i, $1 \le i \le k$, with $v_1 \in e_1$ and $v_i \in e_{i-1} \cap e_i$ for every $1 < i \le k$.
- An important property:

$$v_{i+1} \in e_i - \{v_i\}, \text{ or } v_i \in e_i - \{v_{i+1}\}, \quad 1 \le i < k$$

- To merge a set of edges is to replace old edges with the new one:

$$e_{new} = \bigcup_{i=1}^{k} e_i$$

Addtional Concepts We Need

- The *block hypergraph* H of G is the hypergraph formed by merging the *edges of each biconnected component* of G.
- The set of *all vertices in edges of* P is denoted

$$V(P) = \bigcup_{i=1}^{k} e_i$$

High-Level Algorithm in Plain Text

- Initially H is the given graph G. If H has no edges stop. Otherwise start a new path P by choosing an edge $\{v, w\}$ and setting $P = (v, \{v, w\})$. Continue by growing P.
- To grow the path $P = (v_1, e_1, \cdots, v_k, e_k)$ choose an edge $\{v, w\} \neq e_k$ with $v \in e_k - \{v_k\}$ and do:
  - If $w \notin V(P)$, add $v, \{v, w\}$ to the end of P. Continue growing P.
  - If $w \in V(P)$, say $w \in e_i - \{v_{i+1}\}$, merge the edges of the cycle $w, e_i, v_{i+1}, e_{i+1}, \cdots, v_k, e_k, v, \{v, w\}$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P. Continue growing P.
  - If no edge leaves $e_k - \{v_k\}$, output e_k as an edge of the block hypergraph. Delete e_k from H and delete (v_k, e_k) from P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

Pseudo Code

Algorithm 11: Biconnected Components: Main-DFS (DFS caller)

$H = G$;
while H *still has an edge* $\{v, w\}$ **do**
  Sub-DFS(v); `/* start a new path` $P = (v, \{v, w\})$ `*/`

Pseudo Code

Algorithm 12: Biconnected Components: Sub-DFS (DFS callee)

add the v as the new last vertex of path P;
for $w \in \{$*vertices adjacent to* $v\}$ **do** `/* Grows path` P `*/`
    if $w \notin V\{P\}$ **then**
        add $\{v, w\}$ to the end of P, as the new last edge of P;
        Sub-DFS(w);
        remove the edge $\{v, w\}$ if necessary;
    else `/*` $w \in e_i - \{v_{i+1}\}$`, but most likely` $w \neq v_i$ `*/`
        replace the cycle $w, e_i, v_{i+1}, e_{i+1}, \cdots, v_k, e_k, v$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P;
if *no edge leaves* $e_k - \{v_k\}$ **then**
    output e_k as an edge of the block hypergraph;
    delete e_k from H and delete (v_k, e_k) from P;

Correctness: Validity of Path and Cycle

- Pseudo code: "To grow the path $P = (v_1, e_1, \cdots, v_k, e_k)$ choose an edge $\{v, w\} \neq e_k$ with $v \in e_k - \{v_k\}$ and do:"
- *Distinct vertices*: When v, $\{v, w\}$ is added to P the result is a valid path, by the condition $v \in e_k - \{v_k\}$.
- *Distinct edges*: When edges are merged they form a valid cycle, by the condition $\{v, w\} \neq e_k$.

Correctness: Finest Acyclic Merging

- Pseudo code:
  - If $w \notin V(P)$, add v, $\{v, w\}$ to the end of P. Continue growing P.
  - If $w \in V(P)$, say $w \in e_i - \{v_{i+1}\}$, merge the edges of the cycle $w, e_i, v_{i+1}, e_{i+1}, \cdots, v_k, e_k, v, \{v, w\}$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P. Continue growing P.
  - If no edge leaves $e_k - \{v_k\}$, output e_k as an edge of the block hypergraph. Delete e_k from H and delete (v_k, e_k) from P. If P is now nonempty continue growing P. Otherwise try to start a new path P.
- *Finest*: The algorithm correctly forms the finest acyclic merging of G, which finds the block hypergraph as desired.
- The proof employs the contradiction very similar to that in strong components.

Outline

Data Structure Used in Algorithm

- A stack S contains the vertices $V(P)$.
- A stack B contains the boundaries between edges of P, *two vertices per boundary*.
- An array $I[1 \ldots n]$ is used to store stack indices corresponding to vertices.
- All of above are similar (*but not the same*) to these in strong components.

Algorithms

Procedure 13: BICONN(G)

empty stacks S and B;
for $v \in V$ **do**
    $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
    if $I[v] = 0$ *and* v *is not isolated* **then**
        DFS(v);

Algorithms

Procedure 14: DFS(v)

PUSH(v, S); $I[v]$ =TOP(S);
if $I[v] > 1$ **then** `/* create a filled arrow on` B `*/`
  PUSH$(I[v], B)$;
for *egdes*$\{v, w\} \in E$ **do**
  if $I[w] = 0$ **then** `/* create an open arrow on` B `*/`
    PUSH$(I[v], B)$;   DFS(w);
  else `/* possible merge */`
    while $I[v] > 1$ *and* $I[w] < B[TOP(B) - 1]$ **do**
      POP(B);   POP(B);
if $I[v] = 1$ **then**
  $I[\text{POP}(S)] = c$;
else if $I[v] = B[TOP(B)]$ **then**
  POP(B);   POP(B); $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do** $I[\text{POP}(S)] = c$;

Representations

- *Open arrows*: They point to the vertices v_i of P.

$$v_i = S[B[2i-1]], \quad i = 1, \cdots, k$$

Filled arrows: They demarcate the sets $e_i - \{v_i\}$; these sets are the "nonfirst" vertices of edges e_i of P.

$$e_i - \{v_i\} = \{S[j] | B[2i] \leq j < B[2i+2]\}, \quad i = 1, \cdots, k$$

- *Biconnected components*: Each edge $\{v, w\}$ belongs to the biconnected component with number $\min\{I[v], I[w]\}$.

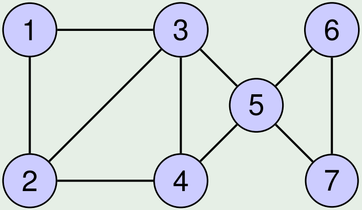
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0
		7	0

Graph



- Procedure: BICONN(G)
- $P = \{\ \}$

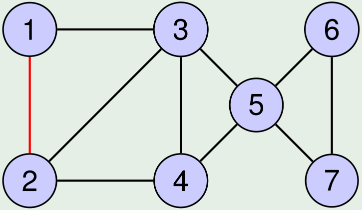
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1 →	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(1): w=2
- $P = \{v_1, \{v_1, v_2\}\}$

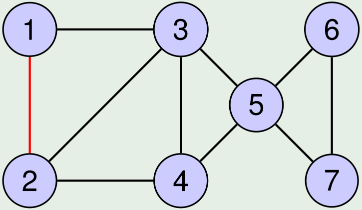
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
		3	0
		4	0
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(2)
- $P = \{ v_1, \{v_1, v_2\} \}$

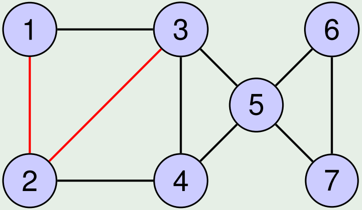
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
2		3	0
		4	0
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(2): w=3
- $P = \{\, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\} \,\}$

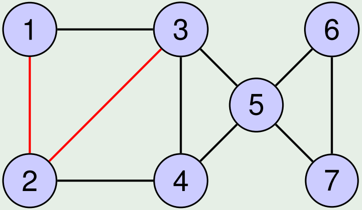
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
2	3	3	3
3		4	0
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(3)
- $P = \{\, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\} \,\}$

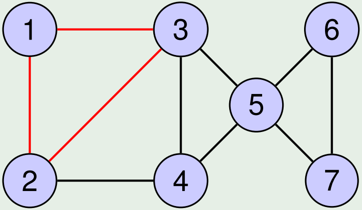
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
		4	0
		5	0
		6	0
		7	0

Graph

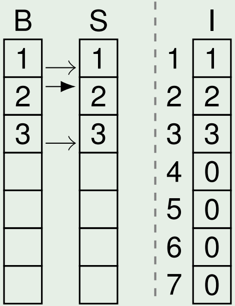


- Procedure: DFS(3): w=1
- $P = \{ v_1, \{v_1, v_2, v_3\} \}$

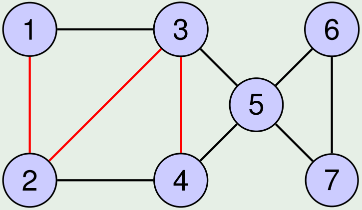
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



- Procedure: DFS(3): w=4
- $P = \{\, v_1, \{v_1, v_2, v_3\}, v_3, \{v_3, v_4\} \,\}$

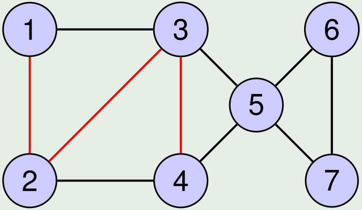
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(4)
- $P = \{\, v_1,\ \{v_1,\ v_2,\ v_3\},\ v_3,\ \{v_3,\ v_4\}\,\}$

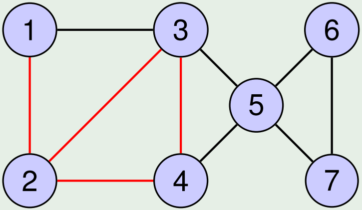
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(4): w=2
- $P = \{ v_1, \{v_1, v_2, v_3, v_4\} \}$

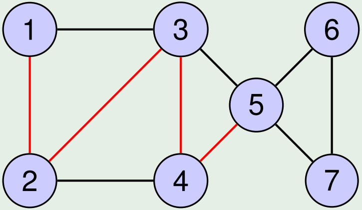
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
4	3	3	3
	4	4	4
		5	0
		6	0
		7	0

Graph



- Procedure: DFS(4): w=5
- $P = \{ v_1, \{v_1, v_2, v_3, v_4\}, v_4, \{v_4, v_5\} \}$

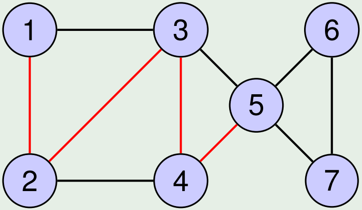
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
4	3	3	3
5	4	4	4
	5	5	5
		6	0
		7	0

Graph



- Procedure: DFS(5)
- $P = \{ v_1, \{v_1, v_2, v_3, v_4\}, v_4, \{v_4, v_5\} \}$

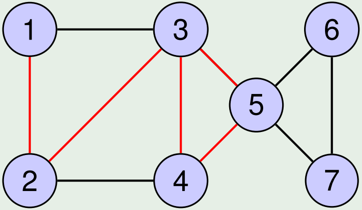
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
	5	5	5
		6	0
		7	0

Graph



- Procedure: DFS(5): w=3
- $P = \{\, v_1,\ \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}\ \}$

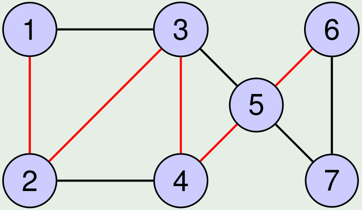
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
	4	4	4
	5	5	5
		6	0
		7	0

Graph



- Procedure: DFS(5): w=6
- $P = \{\, v_1,\ \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\},\ v_5,\ \{v_5,\ v_6\}\, \}$

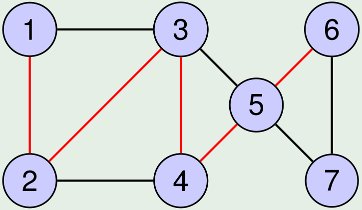
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
	5	5	5
	6	6	6
		7	0

Graph



- Procedure: DFS(6)
- $P = \{\, v_1, \{v_1, v_2, v_3, v_4, v_5\}, v_5, \{v_5, v_6\} \,\}$

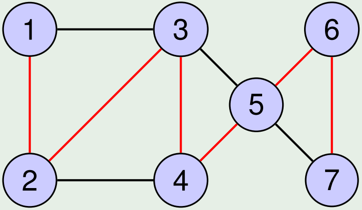
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
6	5	5	5
	6	6	6
		7	0

Graph



- Procedure: DFS(6): w=7
- $P = \{ v_1, \{v_1, v_2, v_3, v_4, v_5\}, v_5, \{v_5, v_6\}, v_6, \{v_6, v_7\} \}$

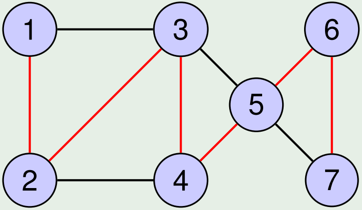
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
6	5	5	5
7	6	6	6
	7	7	7

Graph



- Procedure: DFS(7)
- $P = \{ v_1, \{v_1, v_2, v_3, v_4, v_5\}, v_5, \{v_5, v_6\}, v_6, \{v_6, v_7\} \}$

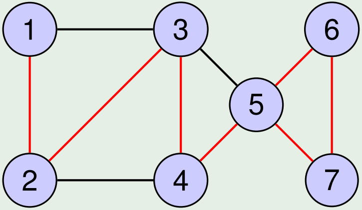
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
	5	5	5
	6	6	6
	7	7	7

Graph



- Procedure: DFS(7): w=5
- $P = \{\, v_1, \{v_1, v_2, v_3, v_4, v_5\}, v_5, \{v_5, v_6, v_7\} \,\}$

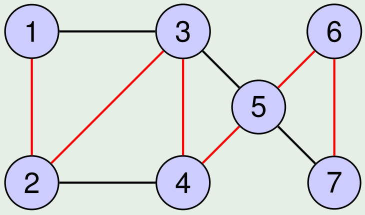
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
	5	5	5
	6	6	6
	7	7	7

Graph



- Procedure: DFS(7): End
- $P = \{v_1, \{v_1, v_2, v_3, v_4, v_5\}, v_5, \{v_5, v_6, v_7\}\}$

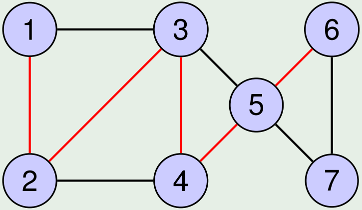
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
	5	5	5
		6	8
		7	8

Graph



- Procedure: DFS(6): End
- $P = \{ v_1, \{v_1, v_2, v_3, v_4, v_5\} \} \quad \{v_5, v_6, v_7\}$

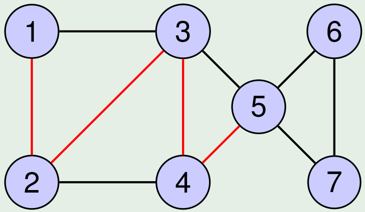
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
	5	5	5
		6	8
		7	8

Graph



- Procedure: DFS(5): End(No operation when w=7)
- $P = \{\, v_1,\ \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}\,\} \quad \{v_5,\ v_6,\ v_7\}$

Demo: Gabow's biconnected components algorithm

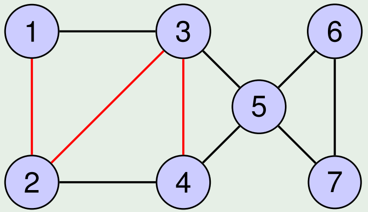
Data in memory

B and S: stack. I: array.

B	S
1	1
2	2
	3
	4
	5

	I
1	1
2	2
3	3
4	4
5	5
6	8
7	8

Graph



- Procedure: DFS(4): End
- $P = \{ v_1, \{v_1, v_2, v_3, v_4, v_5\} \} \quad \{v_5, v_6, v_7\}$

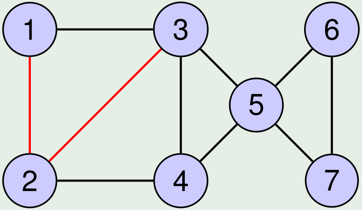
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
	5	5	5
		6	8
		7	8

Graph



- Procedure: DFS(3): End(No operation when w=5)
- $P = \{ v_1, \{v_1, v_2, v_3, v_4, v_5\} \} \quad \{v_5, v_6, v_7\}$

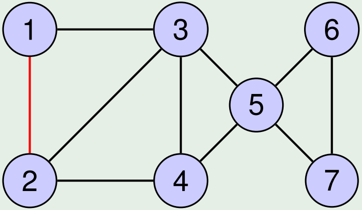
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
	1	1	1
		2	9
		3	9
		4	9
		5	9
		6	8
		7	8

Graph



- Procedure: DFS(2): End
- $P = \{\ \}\quad \{v_1, v_2, v_3, v_4, v_5\}\quad \{v_5, v_6, v_7\}$

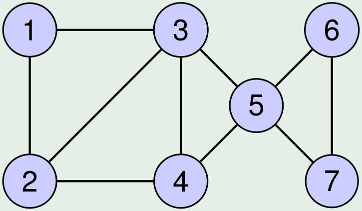
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	9
		3	9
		4	9
		5	9
		6	8
		7	8

Graph



- Procedure: DFS(1): End
- $P = \{\ \}\quad \{v_1, v_2, v_3, v_4, v_5\}\quad \{v_5, v_6, v_7\}$

Correctness

- In order to keep the completeness, the correctness is given as follow.

Theorem (Correctness and complexity)

When BICONN(G) halts any edge $\{v, w\} \in E$ belongs to the biconnected component numbered $\min\{I[v], I[w]\}$. The time and space are both $O(V + E)$.