Path-based depth-first search for strong and biconnected components

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Outline

Characterastics of Gabow's Algorithms

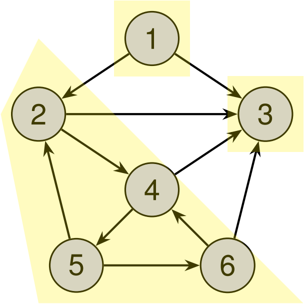
- One-pass algorithm. But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.
- Lower time and space complexity. This algorithm only use two stacks and an array, and do not employ a disjoint-set data structure.

Outline

Review: What have we learned from the textbook?

Concepts of Strong Components

- Two mutually reachable vertices are in the same *strong component*.



Review: What have we learned from the textbook?

Algorithms to Find Strong Components

- Idea: Run DFS twice. Once on the original graph G, once on its *tranposition* G^T.
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity: $O(V + E)$
- Proposed by S. Rao Kosaraju, known as the *Kosaraju's Algorithm*.

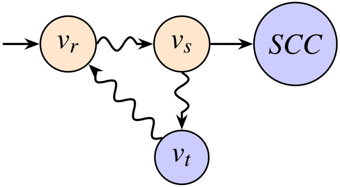
Outline

1. Introduction
2. Strong Components
   - Reviews
   - Purdom and Munro's High-Level Algorithm
   - Contribution
   - Discussion
3. Biconnected Components
   - Thinking about Biconnected Components
   - High-Level Algorithm
   - Gabow's Algorithms

Purdom and Munro's High-Level Algorithm: Plain text

- If H has no vertices stop. Otherwise *start a new path* P by choosing a vertex v and setting $P = (v)$. Continue by growing P as follows.
- To grow the path $P = (v_1, \cdots, v_k)$ choose an edge (v_k, w) directed from the last vertex of P and do the following:
  - If $w \notin P$, *add* w *to* P, making it the new last vertex of P. Continue growing P.
  - If $w \in P$, say $w = v_i$, contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P. P is now a path in the new graph H. Continue growing P.
  - If no edge leaves v_k, output v_k as a vertex of the strong component graph. Delete v_k from both H and P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

A simple example



- Assume the current node is v_s which has at least two adjacent nodes. The current path is $P = (\cdots, v_r, \cdots, v_s)$.
- For the node adjacent to v_s but also in the SCC, after running Sub-DFS() on this node, it will be removed with the SCC.
- For nodes in strong components $(v_r, \cdots, v_s, \cdots, v_t)$, they will be not deleted(but be contracted first) until the Sub-DFS() on v_r is finished.

Pseudo Code

Algorithm 1: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); `/* start a new path` $P = (v)$ `*/`

Pseudo Code

Algorithm 2: Strong components: Sub-DFS(v) (DFS callee)

add the v as the new last vertex of path P;
for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else /* $w = v_i$, and $v = v_k$ */
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;
if *no edge leaves* v **then**
  output v as a vertex of the strong component graph;
  delete v from both H and P;

Assessment

- Note that *contracting* means selecting one vertex as a representation and merging others rather than deleting them.
- Correctness: If no edge leaves v_k then v_k is a vertex of the finest acyclic contraction.

Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except this statement:

```
contract the cycle v_i, v_{i+1}, ···, v_k, both in H and in P;
```

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint-set merging is needed usually.

Outline

Gabow's Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.

Data Structure Used in Algorithm

- In DFS, the path P from root to each node is almost always significant. So it is in this algorithm.
- A stack S contains the sequence of vertices in P.
- A stack B contains the boundaries between contracted vertices.
- An array $I[1 \ldots n]$ is used to store stack indices corresponding to vertices.

Contraction Makes Much Difference

- When contraction is executed, some vertices merges into a set.
- It is possible that several elements in stack S are in the same vertex in path P. More formal,

$$v_i = S[j] : B[i] \leq j < B[i+1]$$

- By the way, the formal definition of $I[v]$ is

$$I[j] = \begin{cases} 0, & \text{if } v \text{ has never been in P;} \\ j, & \text{if } v \text{ is currently in P and } S[j] = v; \\ c, & \text{if the strong component containing } v \text{ has} \\ & \text{been deleted and numbered as } c. \end{cases}$$

where c counts from $n+1$.

New algorithm to discover strong components

Procedure 3: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
    $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
    if $I[v] = 0$ **then** /* vertex v has never been accessed yet */
        DFS(v);

New algorithm to discover strong components

Procedure 4: DFS(v)

```
PUSH(v, S);   I[v] = TOP(S);   PUSH(I[v], B);
/* add v to the end of P                                   */
for egdes(v, w) ∈ E do
    if I[w] = 0 then
        DFS(w);
    else /* contract if necessary                          */
        while I[w] < B[TOP(B)] do
            POP(B);
if I[v] = B[TOP(B)] then /* number vertices of the next
 strong component                                          */
    POP(B);
    c = c + 1;
    while I[v] ≤ TOP(S) do
        I[POP(S)] = c;
```

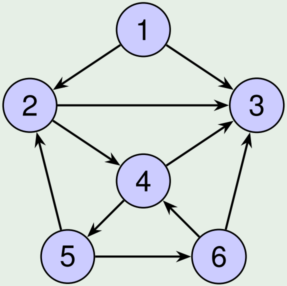
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.

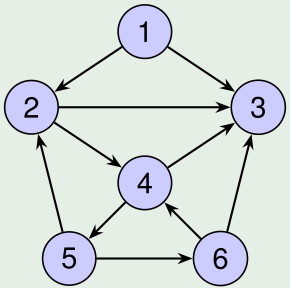
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)
- Code: `for edges(v,w)∈E do ...`
- $w = 2$.

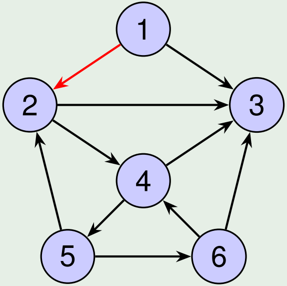
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `for edges(v,w)∈E do ...`
- $w = 3$.

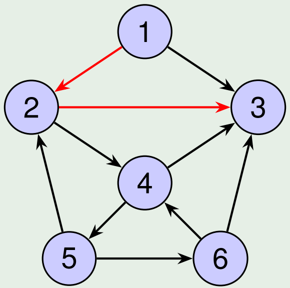
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	3	3	3
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(3)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

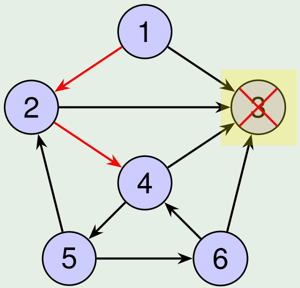
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
		4	3
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(4)
- Code: `for edges(v,w)∈E do ...`
- $w = 5$.

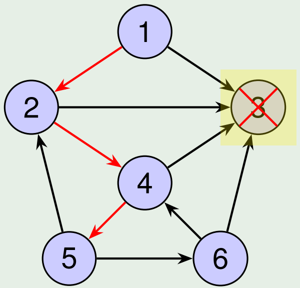
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: **for** edges(v,w)$\in$E **do** ...
- $w = 2$.

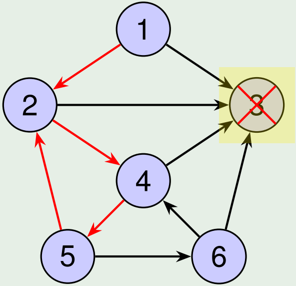
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

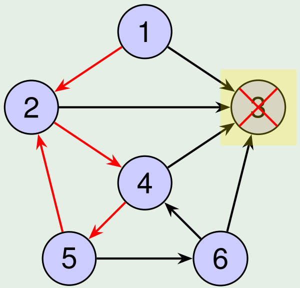
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

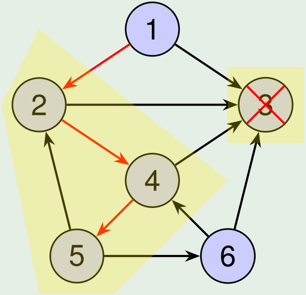
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \to$DFS(1)$\to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: **if** $I[w] = 0$ **then** DFS(w);
- $w = 6$.

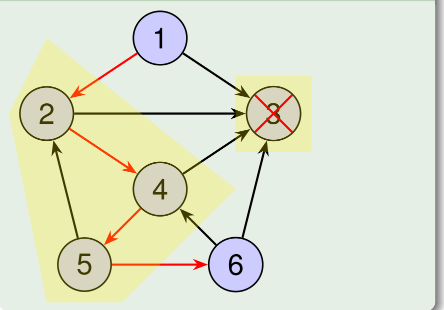
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `for edges(v,w)`$\in$`E do ...`
- $w = 4$.

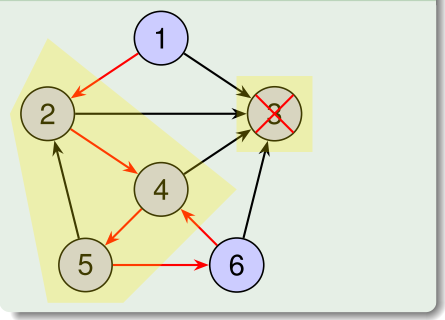
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)$\to$DFS(6)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 4$, contract!

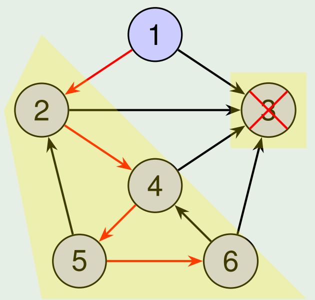
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

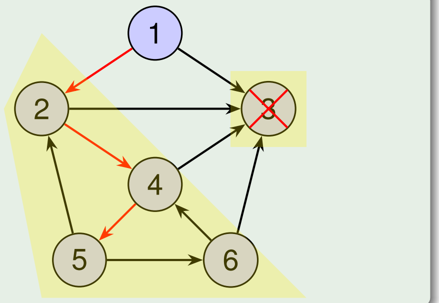
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \to$DFS(2)$\to$DFS(4)$\to$DFS(5)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

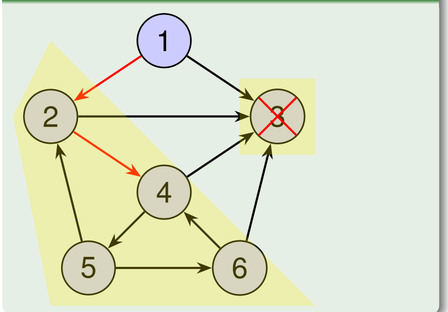
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \to$DFS(2)$\to$DFS(4)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

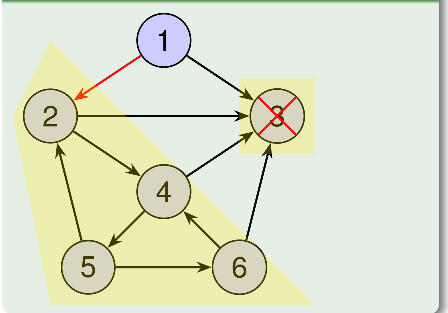
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: **if** `I[v]=B[TOP(B)]` **then** ...
- Go back. But this time, Condition in last line is satisfied!

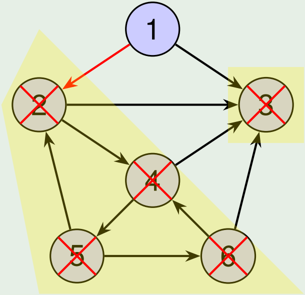
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.

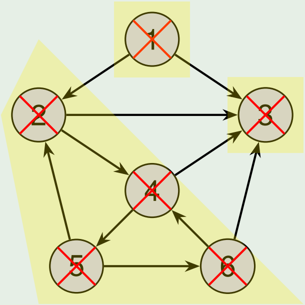
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()$\rightarrow$DFS(1)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop the last one both in B and in S. *Finished!!*

Outline

Correctness of Gabow's Strong Components Algorithm

Theorem (Correctness and Time Complexity)

When STRONG(G) halts each vertex $v \in V$ belongs to the strong component numbered $I[v]$. The time and space are both $O(V+E)$.

- The key of proof is to show that STRONG(G) is a valid implementation of the P&M's high-level algorithm.

Framework of STRONG(G)

Algorithm 5: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); `/* start a new path` $P = (v)$ `*/`

Procedure 6: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ **then** `/*` v `has never been accessed */`
    DFS(v);

Growing Path P

Algorithm 7: A Part of High-Level Algorithm

for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else /* $w = v_i$, and $v = v_k$ */
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;

Procedure 8: A Part of DFS(v)

for *egdes*$(v, w) \in E$ **do**
  if $I[w] = 0$ **then**
    DFS(w);
  else /* contract if necessary */
    while $I[w] < B[TOP(B)]$ **do**
      POP(B);

Having Found a Strong Components

Algorithm 9: A Part of High-Level Algorithm

if *no edge leaves* v **then**
  output v as a vertex of the strong component graph;
  delete v from both H and P;

Procedure 10: A Part of DFS(v)

if $I[v] = B[TOP(B)]$ **then** `/* number vertices of the next strong component */`
  POP(B);
  $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do**
    $I[\text{POP}(S)] = c$;

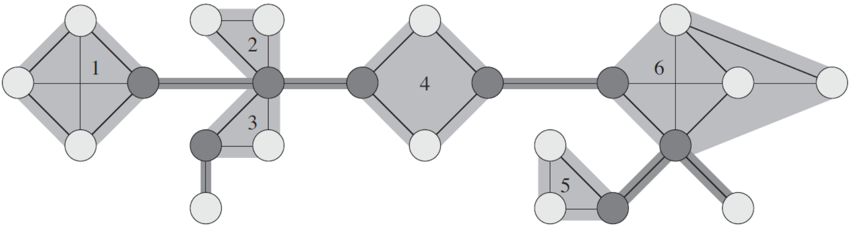
Time Complexity

- Every vertex is pushed onto and popped from each stack S, B exactly once. So the algorithm spends $O(1)$ time on each vertex or edge.
- Time Complexity: $O(V + E)$
- Intuitively, from another view, this algorithm is based on DFS, and no loop is executed on one vertex or one edge.

Outline

Concepts: Biconnected Component

- A *biconnected component* of G is a maximal set of edges such that any two edges in the set lie on a common simple cycle.



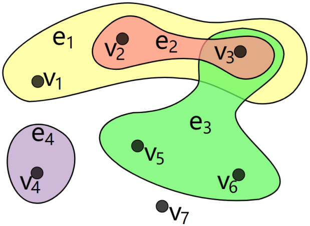
Outline

Concepts: Hypergraph

- A *hypergraph* $H = (V, E)$ is a generalization of a graph in which an edge can join any number of vertices.
- In the following hypergraph,

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

$$\begin{aligned} E &= \{e_1, e_2, e_3, e_4\} \\ &= \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\} \end{aligned}$$



Concepts: Hypergraph

- Therefore, we need redefine the *edge*, *path*, *cycle*, $\cdots$, and nearly all concepts as long as it is relative to edge.
- A *path* is a sequence $(v_1, e_1, \cdots, v_k, e_k)$ of distinct vertices v_i and distinct edges e_i, $1 \leq i \leq k$, with $v_1 \in e_1$ and $v_i \in e_{i-1} \cap e_i$ for every $1 < i \leq k$.
- An important property:

$$v_{i+1} \in e_i - v_i, \quad 1 \leq i < k$$

- Merging a set of edges is to replace old edges with the new one:

$$e_{new} = \bigcup_{i=1}^{k} e_i$$

Pseudo Code

Algorithm 11: Biconnected Components

$H = G$;
while H *still has an edge* $\{v, w\}$ **do**
  start a new path $P = (v, \{v, w\})$;
  while P *is not empty* **do** `/* Grows path` P `*/`
    if *the last vertex* v_k *of* P *has an edge* (v_k, w) **then**
      if $w \notin VP$ **then** `/*` $V\{P\}$ `is the set of all vertices in` P `*/`
        add $v, \{v, w\}$ to the end of P, as the new last vertex and edge of P;
      else `/*` $w \in e_i - v_{i+1}$`, but most likely` $w \neq v_i$ `*/`
        replace the cycle $w,\ e_i,\ v_{i+1},\ e_{i+1},\ \cdots,\ v_k,\ e_k,\ v,\ \{v,\ w\}$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P;
    else
      output v_k as a vertex of the strong component graph;
      delete v_k from both H and P;

Outline

Algorithms

Procedure 12: BICONN(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ *and* v *is not isolated* **then**
    DFS(v);

Algorithms

Procedure 13: DFS(v)

```
PUSH(v, S);   I[v] = TOP(S);
if I[v] > 1 then /* create a filled arrow on B                 */
|   PUSH(I[v], B);
for egdes{v, w} ∈ E do
|   if I[w] = 0 then /* create an open arrow on B               */
|   |   PUSH(I[v], B);   DFS(w);
|   else /* possible merge                                      */
|   |   while I[v] > 1 and I[w] < B[TOP(B) − 1] do
|   |   |   POP(B);   POP(B);
if I[v] = 1 then
|   I[POP(S)] = c;
else if I[v] = B[TOP(B)] then
|   POP(B);   POP(B);   c = c + 1;
|   while I[v] ≤ TOP(S) do I[POP(S)] = c;
```

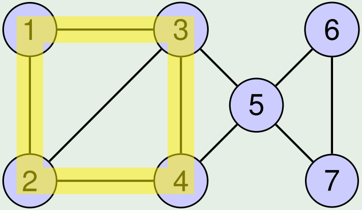
Demo: Gabow's biconnected components algorithm

Data in memory

Here is a memory map.

- Null.

Graph



Summary

- Gabow gave algorithms to find the strong components and biconnected components more effectively. They are one-pass algorithms while do not need a disjoint-set data structure.
- There is a close relationship between strong components and biconnected components, like two faces of a coin: one is directed graph, another is undirected graph.