Path-based depth-first search for strong and biconnected components

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Outline

Characterastics of Gabow's Algorithms

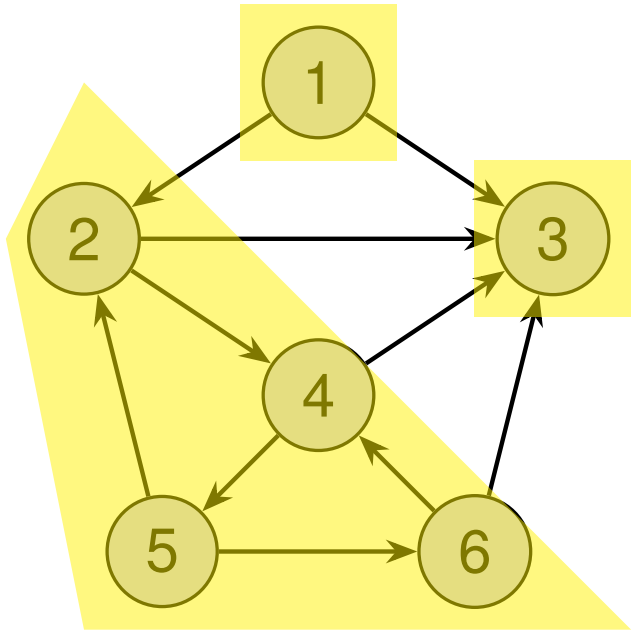
- One-pass algorithm. But for the algorithm of strong components, what we have learned from the textbook is a two-pass algorithm, by which we must traverse the whole graph twice.
- Lower time and space complexity. This algorithm only use two stacks and an array, and do not employ a disjoint-set data structure.

Outline

Review: What have we learned from the textbook?

Concepts of Strong Components

- Two mutually reachable vertices are in the same *strong component*.



Review: What have we learned from the textbook?

Algorithms to Find Strong Components

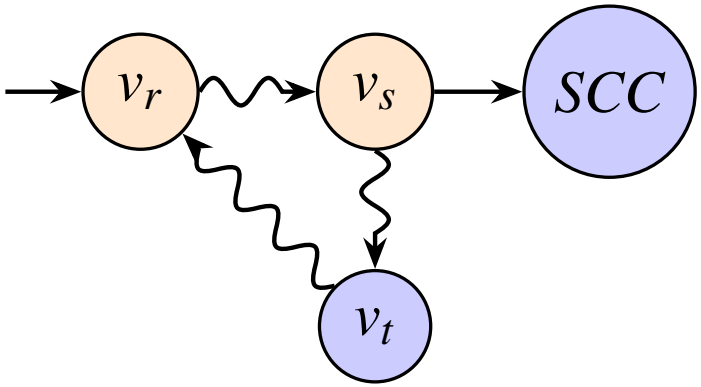
- Idea: Run DFS twice. Once on the original graph G, once on its *tranposition* G^T.
- Trick: Using *finishing times* of each vertex computed by the first DFS.
- Linear time complexity: $O(V + E)$
- Proposed by S. Rao Kosaraju, known as the *Kosaraju's Algorithm*.

Outline

Purdom and Munro's High-Level Algorithm: Plain text

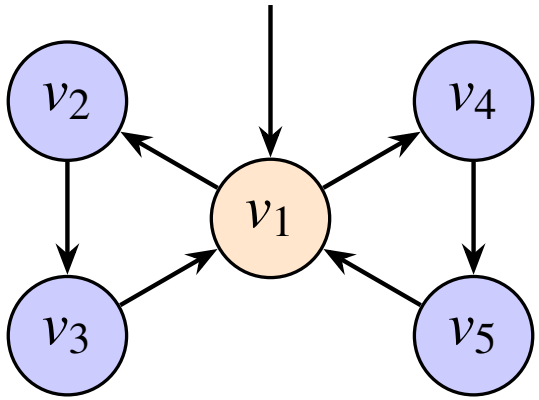
- Initially H is the given graph G. If H has no vertices stop. Otherwise *start a new path* P by choosing a vertex v and setting $P = (v)$. Continue by growing P as follows.
- To grow the path $P = (v_1, \cdots, v_k)$ choose an edge (v_k, w) directed from the last vertex of P and do the following:
  - If $w \notin P$, *add* w *to* P, making it the new last vertex of P. Continue growing P.
  - If $w \in P$, say $w = v_i$, contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P. P is now a path in the new graph H. Continue growing P.
  - If no edge leaves v_k, output v_k as a vertex of the strong component graph. Delete v_k from both H and P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

How to implement by DFS?



- Assume the current node is v_s which has at least two adjacent nodes. The current path is $P = (\cdots, v_r, \cdots, v_s)$.
- For the node adjacent to v_s but also in the SCC, after running Sub-DFS() on this node, it will be removed with the SCC.

How to implement by DFS?



- For nodes in strong components $(v_r, \cdots, v_s, \cdots, v_t)$, they cannot be deleted(but be contracted first) until the Sub-DFS() on the last vertex v_r in this SCC is finished.

Pseudo Code

Algorithm 1: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); /* start a new path $P = (v)$ */

Pseudo Code

Algorithm 2: Strong components: Sub-DFS(v) (DFS callee)

add the v as the new last vertex of path P;
for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else /* $w = v_i$, and $v = v_k$ */
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;
if *no edge leaves* v *and* v *is the last DFS-finished vertex in a SCC* **then**
  output v as a vertex of the strong component graph;
  delete v from both H and P;

Assessment

- Note that *contracting* means selecting one vertex as a representation and merging others rather than deleting them.
- Correctness: If no edge leaves v_k then v_k is a vertex of the finest acyclic contraction.

Assessment

- The time consumption of each statement in the pseudo-code is clear. Total time complexity is linear. except this statement:

`contract the cycle` $v_i, v_{i+1}, \cdots, v_k$, `both in` H `and in` P;

- Problem is how to merge in linear time while keeping the next time accessing this vertex still in constant time.
- Therefore, a good data structure for disjoint-set merging is needed usually.

Outline

Gabow's Contribution

- He gave a simple list-based implementation that achieves linear time.
- Use only stacks and arrays as data structure.
- Do not need a disjoint set merging data structure.

Data Structure Used in Algorithm

- In DFS, the path P from root to each node is almost always significant. So it is in this algorithm.
- A stack S contains the sequence of vertices in P.
- A stack B contains the boundaries between contracted vertices.
- An array $I[1 \ldots n]$ is used to store stack indices corresponding to vertices.

Contraction Makes Much Difference

- When contraction is executed, some vertices merges into a set.
- It is possible that several elements in stack S are in the same vertex in path P. More formal,

$$v_i = S[j] : B[i] \leq j < B[i+1]$$

- By the way, the formal definition of $I[v]$ is

$$I[j] = \begin{cases} 0, & \text{if } v \text{ has never been in P;} \\ j, & \text{if } v \text{ is currently in P and } S[j] = v; \\ c, & \text{if the strong component containing } v \text{ has} \\ & \text{been deleted and numbered as } c. \end{cases}$$

where c counts from $n+1$.

New algorithm to discover strong components

Procedure 3: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ **then** `/* vertex v has never been accessed yet */`
    DFS(v);

New algorithm to discover strong components

Procedure 4: DFS(v)

PUSH(v, S); $I[v] =$TOP(S); PUSH$(I[v], B)$;
/* add v to the end of P */
for *egdes*$(v, w) \in E$ **do**
  if $I[w] = 0$ **then**
    DFS(w);
  else /* contract if necessary */
    while $I[w] < B[TOP(B)]$ **do**
      POP(B);
if $I[v] = B[TOP(B)]$ **then** /* number vertices of the next strong component */
  POP(B);
  $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do**
    $I[$POP$(S)] = c$;

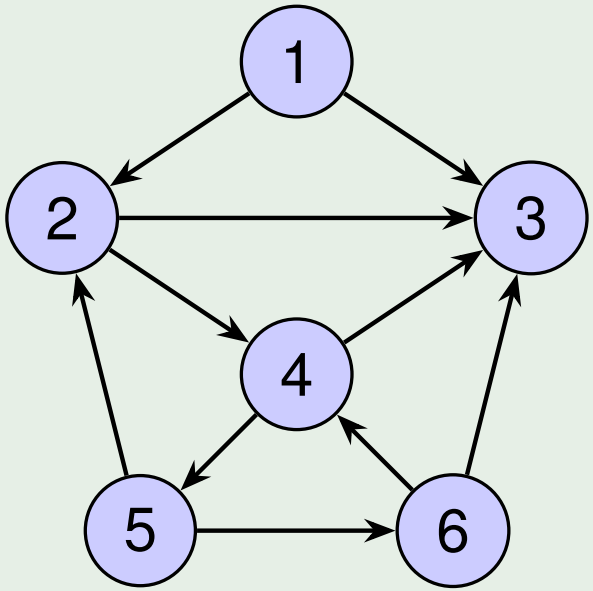
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	0
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()
- This state is the first after initialized. DFS(1) is going to be called.

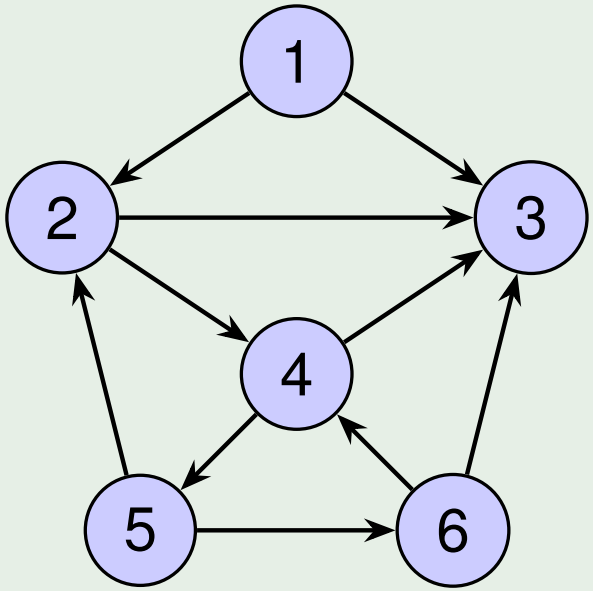
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
		2	0
		3	0
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)
- Code: `for edges(v,w)∈E do ...`
- $w = 2$.

Demo: Gabow's strong components algorithm

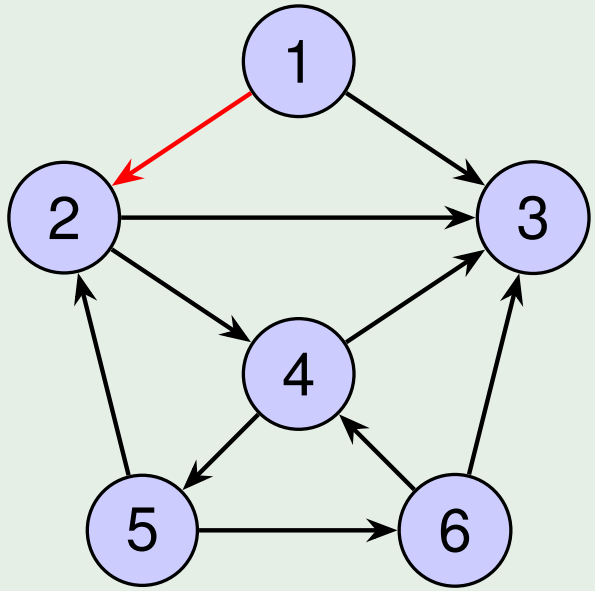
Data in memory

B and S: stack. I: array.

B	S
1	1
2	2

	I
1	1
2	2
3	0
4	0
5	0
6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `for edges(v,w)∈E do ...`
- $w = 3$.

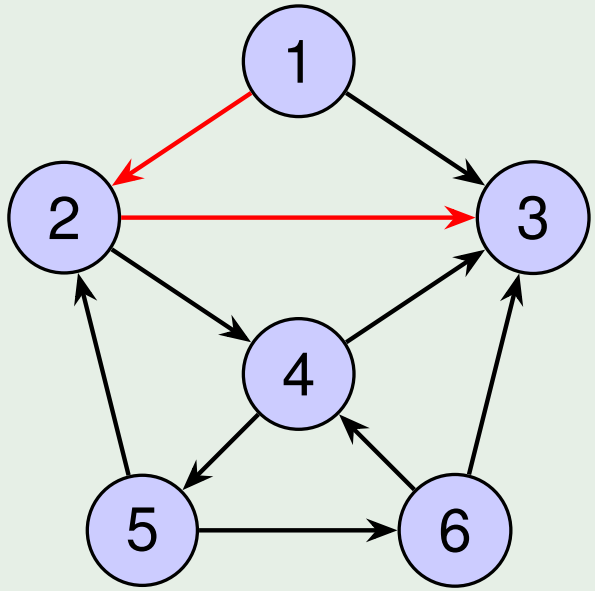
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	3	3	3
		4	0
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(3)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

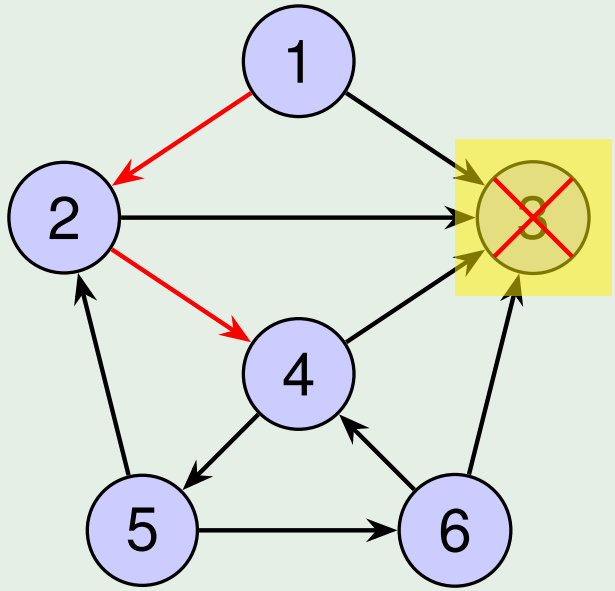
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
		4	3
		5	0
		6	0

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)→DFS(4)
- Code: `for edges(v,w)∈E do ...`
- $w = 5$.

Demo: Gabow's strong components algorithm

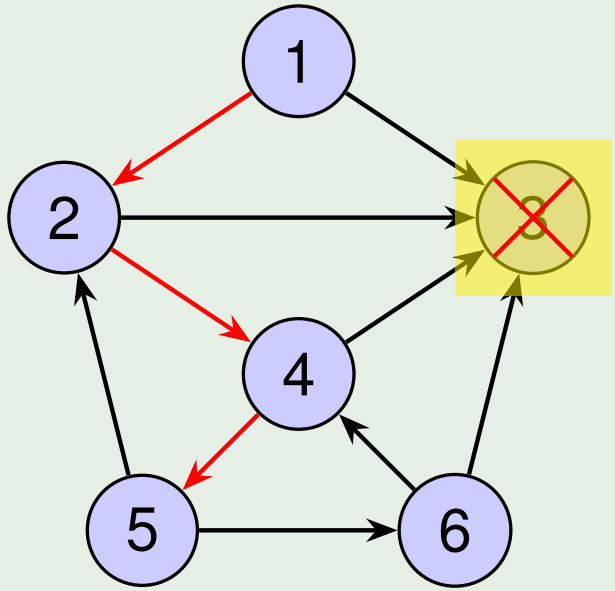
Data in memory

B and S: stack. I: array.

B	S
1	1
2	2
3	4
4	5

	I
1	1
2	2
3	7
4	3
5	4
6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `for edges(v,w)∈E do ...`
- $w = 2$.

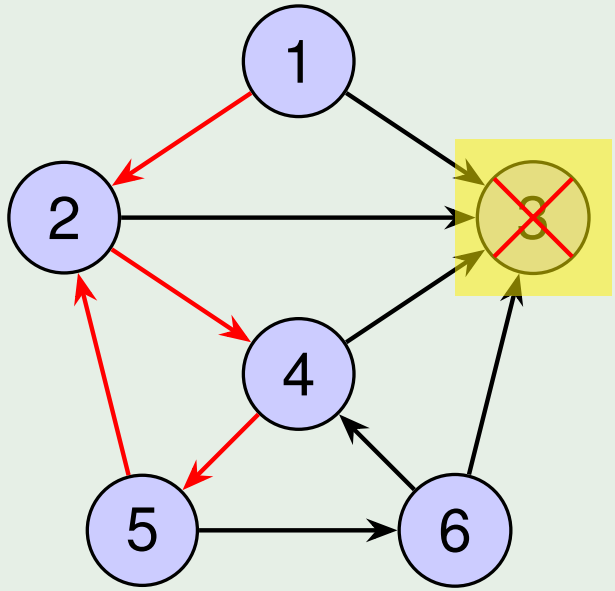
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
3	4	3	7
4	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

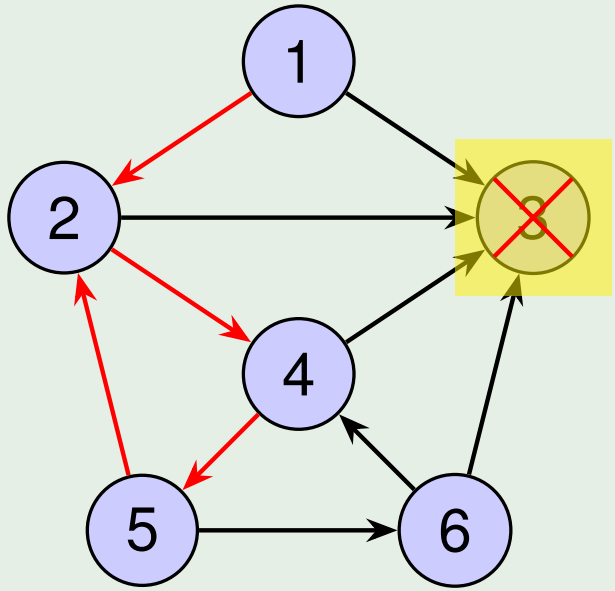
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 2$, contract!

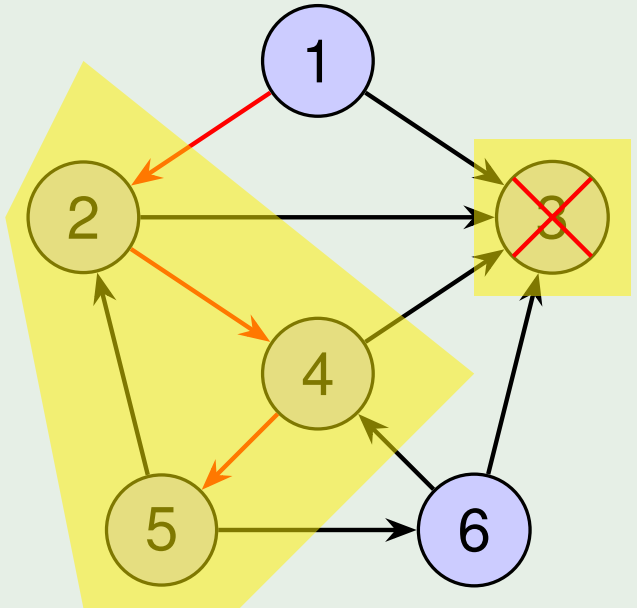
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
		5	4
		6	0

Graph H



- Call Stack: $\cdots \rightarrow$DFS(1)$\rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `if I[w]=0 then DFS(w);`
- $w = 6$.

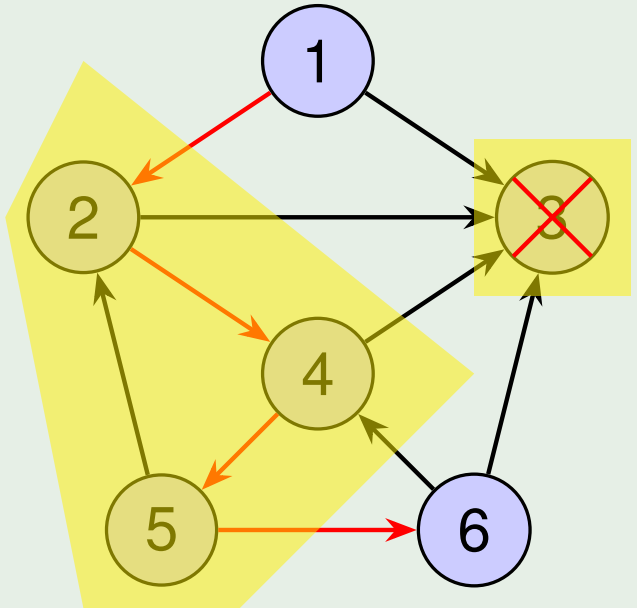
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `for edges(v,w)∈E do ...`
- $w = 4$.

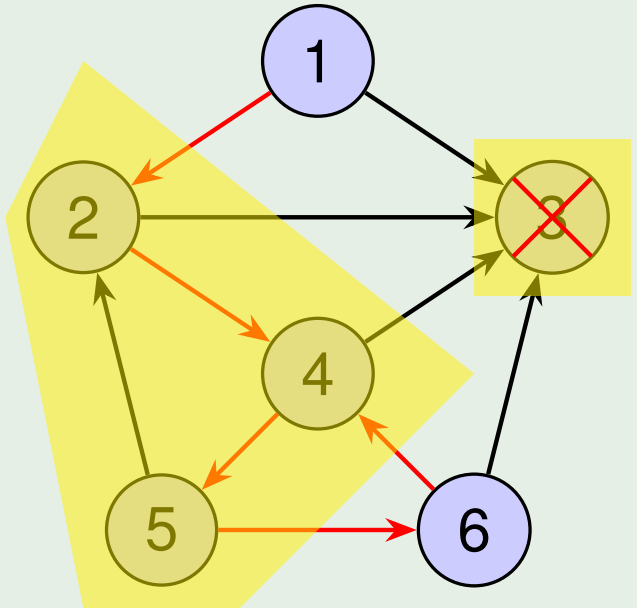
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `while I[w]<B[TOP(B)] do POP(B);`
- Now, $w = 4$, contract!

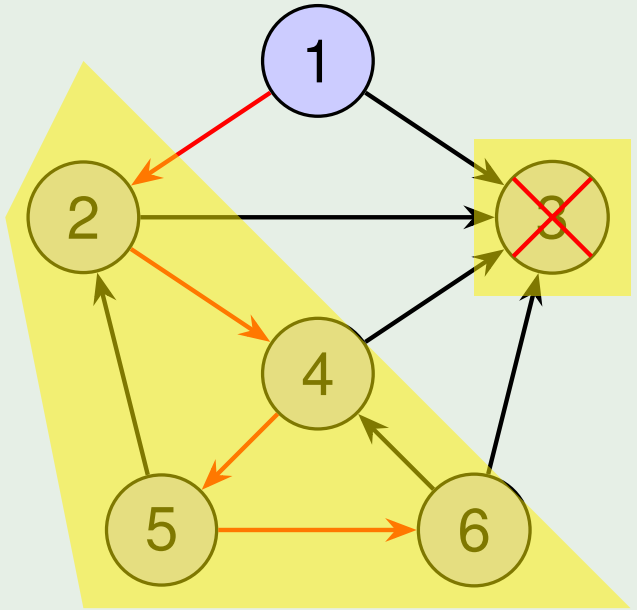
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)$\rightarrow$DFS(6)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

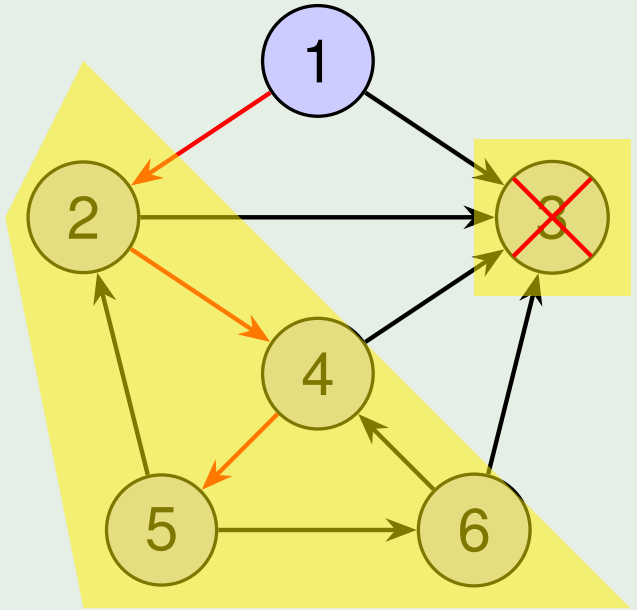
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)$\rightarrow$DFS(5)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

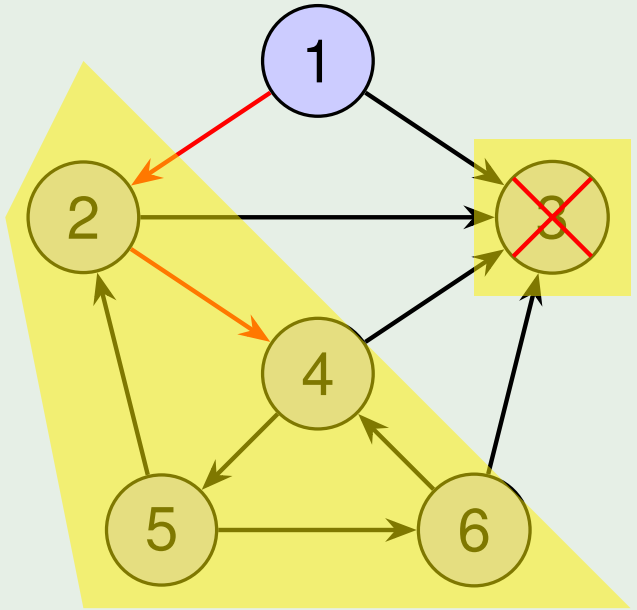
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: $\cdots \rightarrow$DFS(2)$\rightarrow$DFS(4)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back.

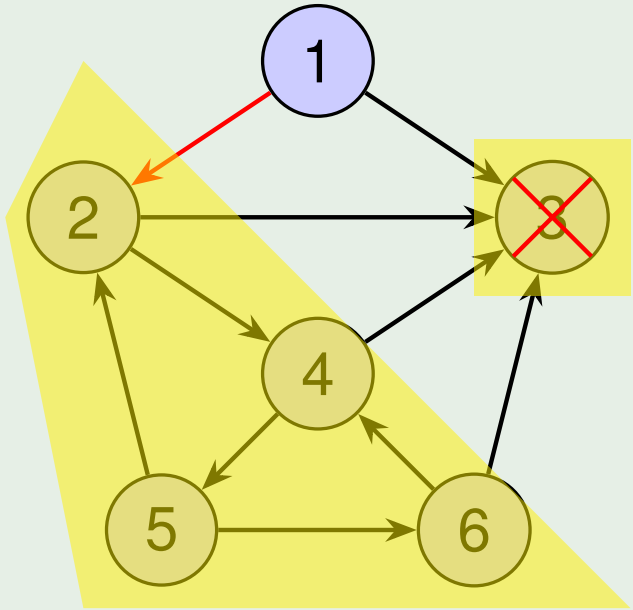
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	4	3	7
	5	4	3
	6	5	4
		6	5

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `if I[v]=B[TOP(B)] then ...`
- Go back. But this time, Condition in last line is satisfied!

Demo: Gabow's strong components algorithm

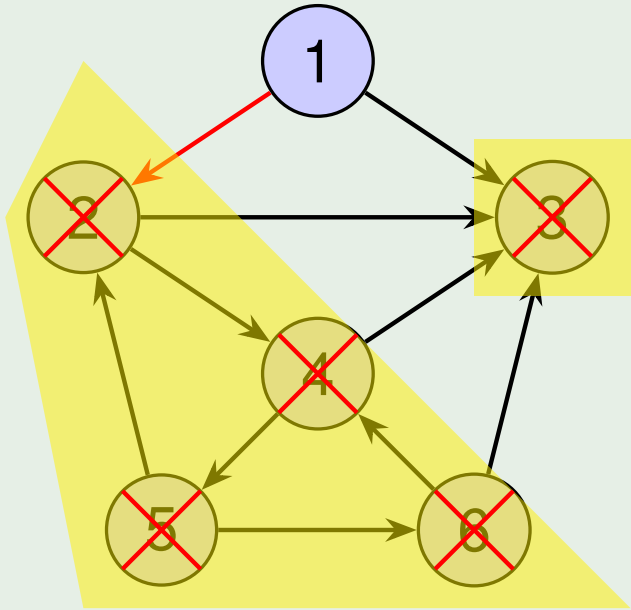
Data in memory

B and S: stack. I: array.

B	S
1	1

	I
1	1
2	8
3	7
4	8
5	8
6	8

Graph H



- Call Stack: STRONG()→DFS(1)→DFS(2)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop 2 from B, while 2, 4, 5, 6 in S are also popped.

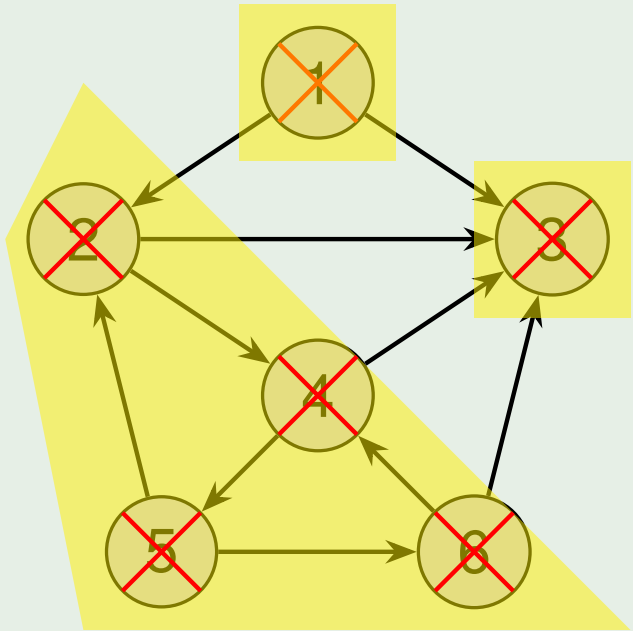
Demo: Gabow's strong components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	8
		3	7
		4	8
		5	8
		6	8

Graph H



- Call Stack: STRONG()→DFS(1)
- Code: `while I[v]≤TOP(S) do I[POP(S)]=c;`
- Pop the last one both in B and in S. *Finished!!*

Outline

Correctness of Gabow's Strong Components Algorithm

Theorem (Correctness and Time Complexity)

When STRONG(G) halts each vertex $v \in V$ belongs to the strong component numbered $I[v]$. The time and space are both $O(V + E)$.

- The key of proof is to show that STRONG(G) is a valid implementation of the P&M's high-level algorithm.

Framework of STRONG(G)

Algorithm 5: Strong components: Main-DFS(G) (DFS caller)

$H = G$;
while H *still has a vertex* v **do**
  Sub-DFS(v); /* start a new path $P = (v)$ */

Procedure 6: STRONG(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ **then** /* v has never been accessed */
    DFS(v);

Growing Path P

Algorithm 7: A Part of High-Level Algorithm

for $w \in \{$*vertices adjacent to* $v\}$ **do**
  if $w \notin P$ **then**
    Sub-DFS(w);
  else /* $w = v_i$, and $v = v_k$ */
    contract the cycle $v_i, v_{i+1}, \cdots, v_k$, both in H and in P;

Procedure 8: A Part of DFS(v)

for *egdes*$(v, w) \in E$ **do**
  if $I[w] = 0$ **then**
    DFS(w);
  else /* contract if necessary */
    while $I[w] < B[TOP(B)]$ **do**
      POP(B);

Having Found a Strong Components

Algorithm 9: A Part of High-Level Algorithm

if *no edge leaves* v *and* v *is the last DFS-finished vertex in a SCC* **then**

    output v as a vertex of the strong component graph;
    delete v from both H and P;

Procedure 10: A Part of DFS(v)

if $I[v] = B[TOP(B)]$ **then** `/* number vertices of the next strong component */`

    POP(B);
    $c = c + 1$;
    while $I[v] \leq TOP(S)$ **do**
        $I[\text{POP}(S)] = c$;

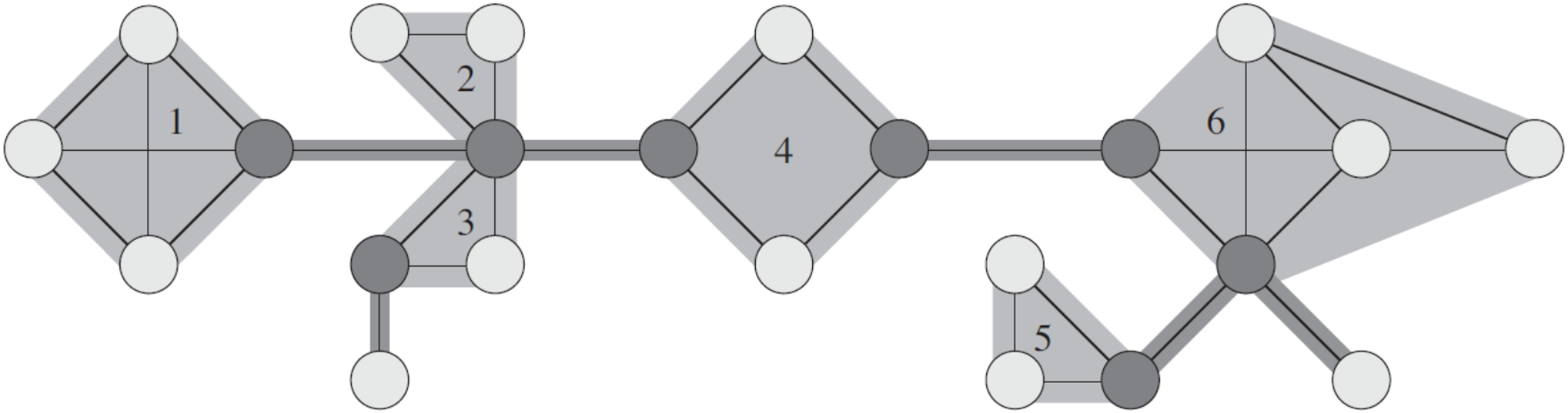
Time Complexity

- Every vertex is pushed onto and popped from each stack S, B exactly once. So the algorithm spends $O(1)$ time on each vertex or edge.
- Time Complexity: $O(V + E)$
- Intuitively, from another view, this algorithm is based on DFS, and no loop is executed on one vertex or one edge.

Outline

Review: Biconnected Component

- A *biconnected component* of G is a maximal set of edges such that any two edges in the set lie on a common simple cycle.



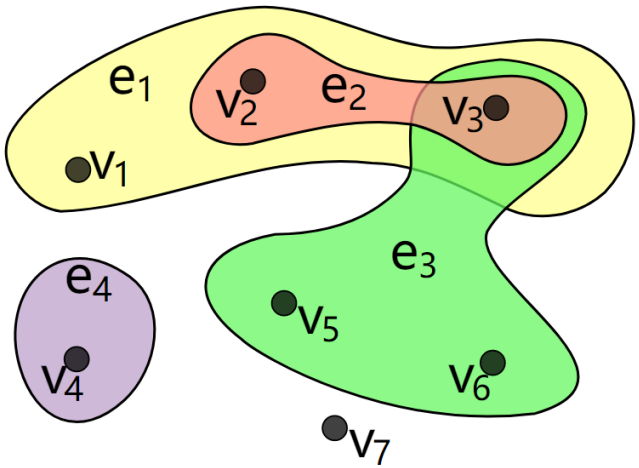
Outline

Concepts: Hypergraph

- A *hypergraph* $H = (V, E)$ is a generalization of a graph in which an edge can join any number of vertices.
- In the following hypergraph,

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

$$\begin{aligned} E &= \{e_1, e_2, e_3, e_4\} \\ &= \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\} \end{aligned}$$



Concepts: Hypergraph

- Therefore, we need redefine the *edge*, *path*, *cycle*, $\cdots$, and nearly all concepts as long as it is relative to edge.
- A *path* is a sequence $(v_1, e_1, \cdots, v_k, e_k)$ of distinct vertices v_i and distinct edges e_i, $1 \leq i \leq k$, with $v_1 \in e_1$ and $v_i \in e_{i-1} \cap e_i$ for every $1 < i \leq k$.
- An important property:

$$v_{i+1} \in e_i - v_i, \quad 1 \leq i < k$$

- Merging a set of edges is to replace old edges with the new one:

$$e_{new} = \bigcup_{i=1}^{k} e_i$$

Addtional Concepts We Need

- The *block hypergraph* H of G is the hypergraph formed by merging the edges of each biconnected component of G.
- The set of all vertices in edges of P is denoted

$$V(P) = \bigcup_{i=1}^{k} e_i$$

High-Level Algorithm in Plain Text

- Initially H is the given graph G. If H has no edges stop. Otherwise start a new path P by choosing an edge $\{v, w\}$ and setting $P = (v, \{v, w\})$. Continue by growing P.
- To grow the path $P = (v_1, e_1, \cdots, v_k, e_k)$ choose an edge $\{v, w\} \neq e_k$ with $v \in e_k - v_k$ and do:
  - If $w \notin V(P)$, add $v, \{v, w\}$ to the end of P. Continue growing P.
  - If $w \in V(P)$, say $w \in e_i - v_{i+1}$, merge the edges of the cycle $w, e_i, v_{i+1}, e_{i+1}, \cdots, v_k, e_k, v, \{v, w\}$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P. Continue growing P.
  - If no edge leaves $e_k - v_k$, output e_k as an edge of the block hypergraph. Delete e_k from H and delete (v_k, e_k) from P. If P is now nonempty continue growing P. Otherwise try to start a new path P.

Pseudo Code

Algorithm 11: Biconnected Components

$H = G$;
while H *still has an edge* $\{v, w\}$ **do**
    start a new path $P = (v, \{v, w\})$;
    while P *is not empty* **do** /* Grows path P */
        if *the last vertex* v_k *of* P *has an edge* (v_k, w) **then**
            if $w \notin VP$ **then** /* $V\{P\}$ is the set of all vertices in P */
                add $v, \{v, w\}$ to the end of P, as the new last vertex and edge of P;
            else /* $w \in e_i - v_{i+1}$, but most likely $w \neq v_i$ */
                replace the cycle w, e_i, v_{i+1}, e_{i+1}, $\cdots$, v_k, e_k, v, $\{v, w\}$ to a new edge $e = \bigcup_{j=i}^{k} e_j$, both in H and in P;
        else
            output v_k as a vertex of the strong component graph;
            delete v_k from both H and P;

Outline

Algorithms

Procedure 12: BICONN(G)

empty stacks S and B;
for $v \in V$ **do**
  $I[v] = 0$;
$c = n$;
for $v \in V$ **do**
  if $I[v] = 0$ *and* v *is not isolated* **then**
    DFS(v);

Algorithms

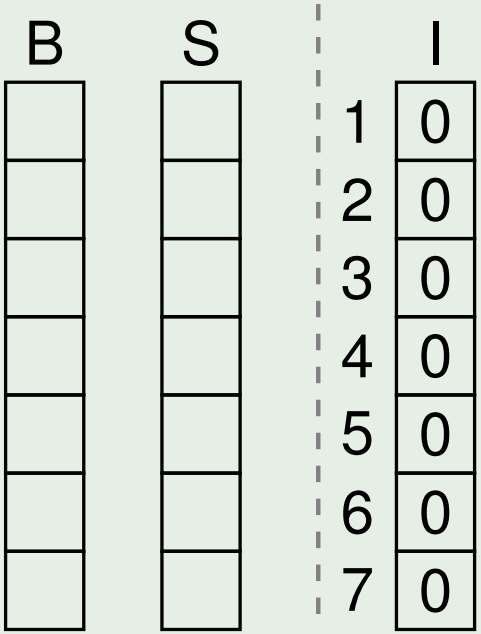
Procedure 13: DFS(v)

PUSH(v, S); $I[v] =$TOP(S);
if $I[v] > 1$ **then** /* create a filled arrow on B */
  PUSH$(I[v], B)$;
for *egdes*$\{v, w\} \in E$ **do**
  if $I[w] = 0$ **then** /* create an open arrow on B */
    PUSH$(I[v], B)$; DFS(w);
  else /* possible merge */
    while $I[v] > 1$ *and* $I[w] < B[TOP(B) - 1]$ **do**
      POP(B); POP(B);
if $I[v] = 1$ **then**
  $I[$POP$(S)] = c$;
else if $I[v] = B[TOP(B)]$ **then**
  POP(B); POP(B); $c = c + 1$;
  while $I[v] \leq TOP(S)$ **do** $I[$POP$(S)] = c$;

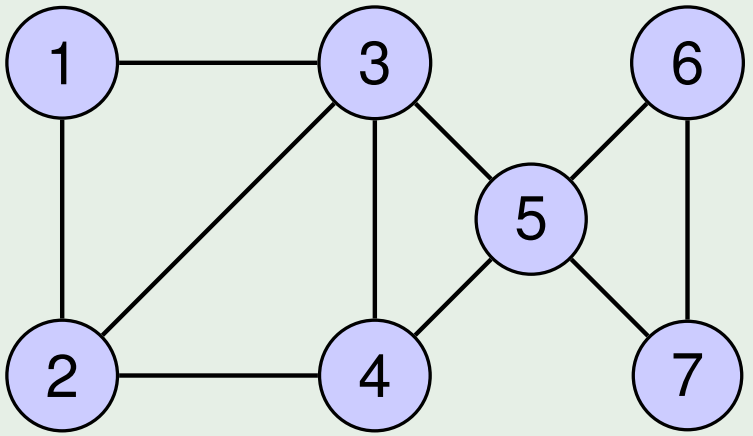
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

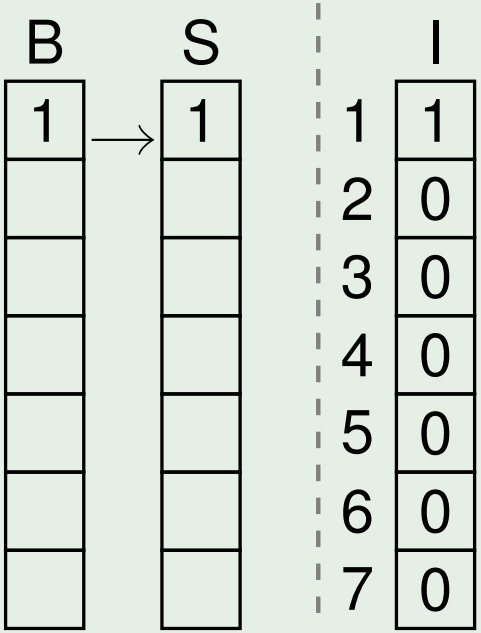


- Current procedure: BICONN(G)

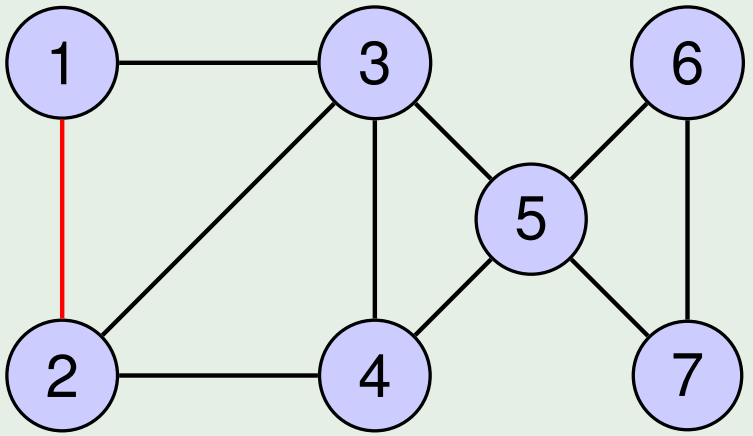
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

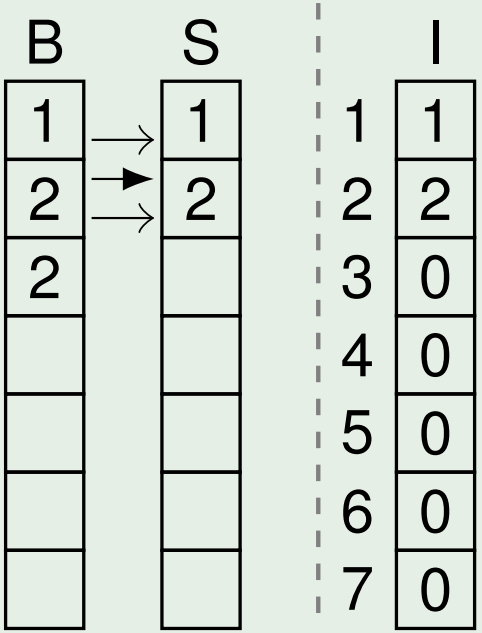


- Current procedure: DFS(1): w=2

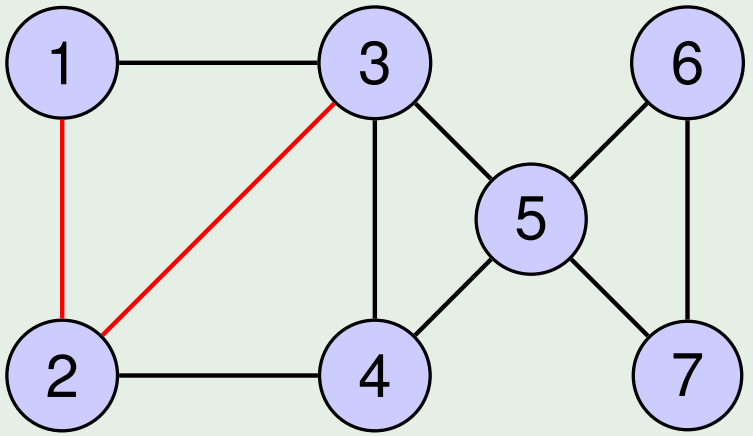
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

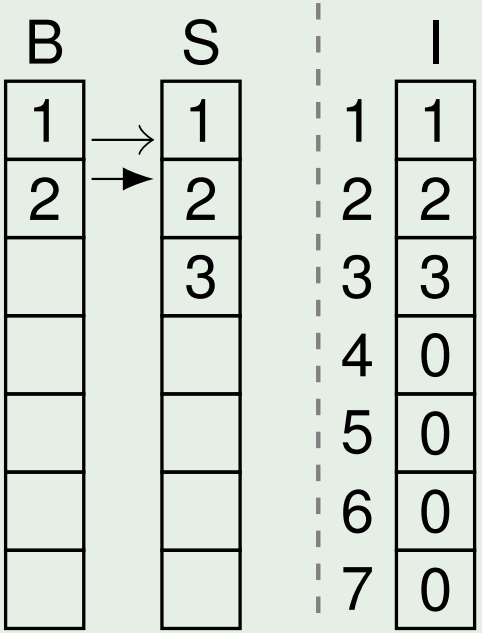


- Current procedure: DFS(2): w=3

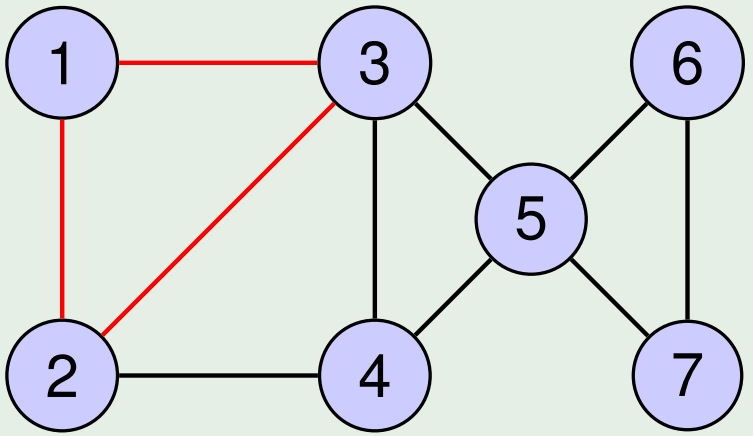
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

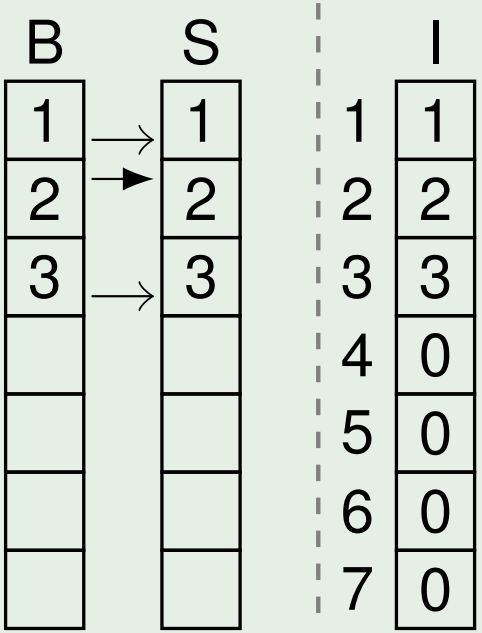


- Current procedure: DFS(3): w=1

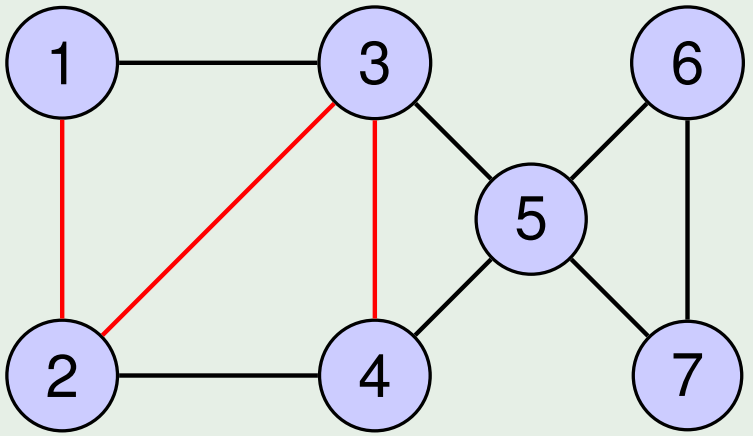
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



- Current procedure: DFS(3): w=4

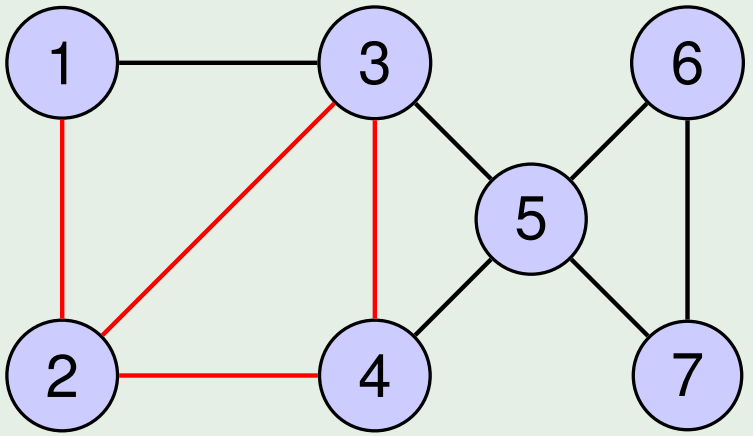
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
		5	0
		6	0
		7	0

Graph

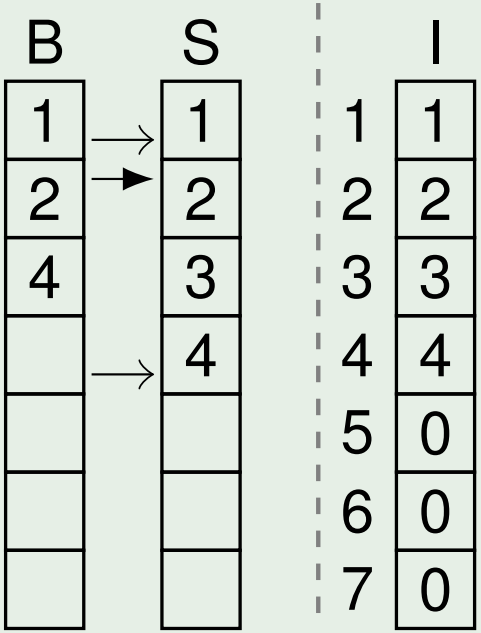


- Current procedure: DFS(4): w=2

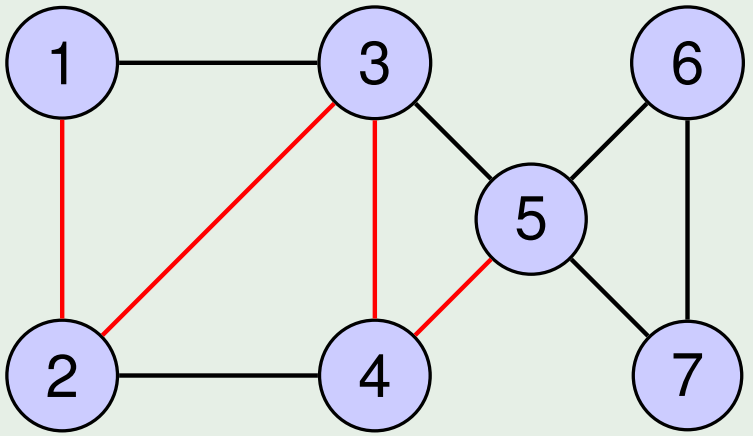
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



- Current procedure: DFS(4): w=5

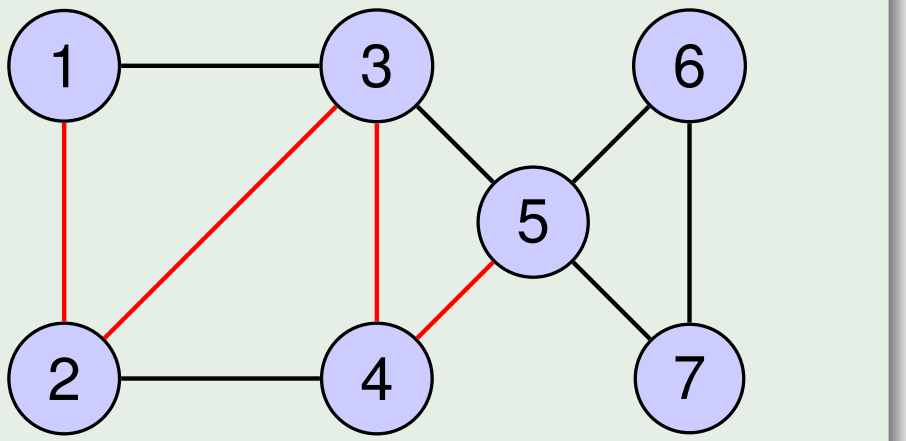
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
	3	3	3
	4	4	4
	5	5	5
		6	0
		7	0

Graph

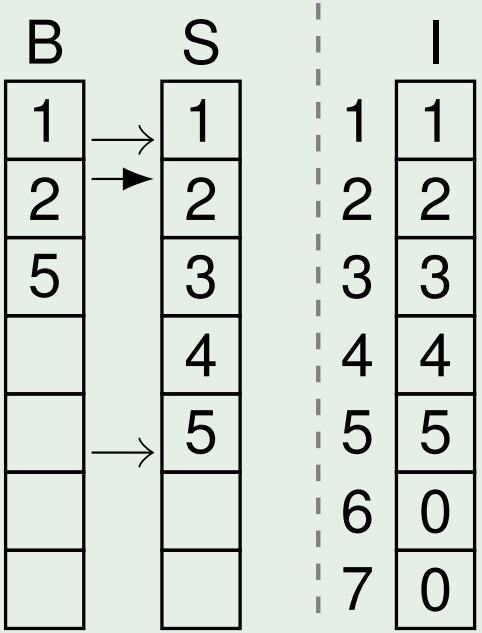


- Current procedure: DFS(5): w=3

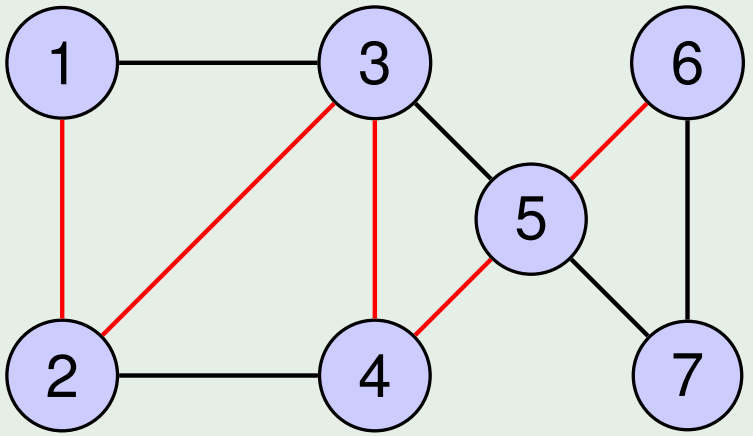
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



- Current procedure: DFS(5): w=6

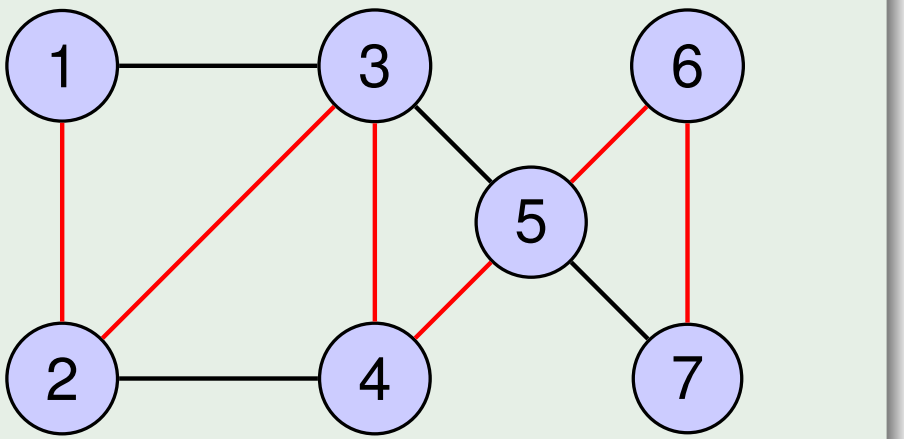
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
1	1	1	1
2	2	2	2
5	3	3	3
6	4	4	4
6	5	5	5
	6	6	6
		7	0

Graph

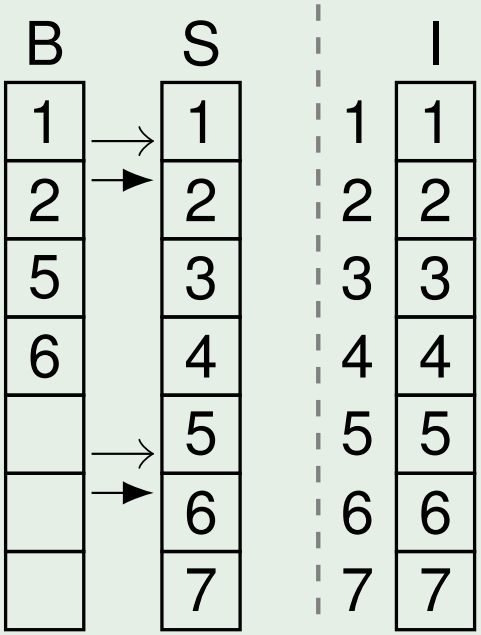


- Current procedure: DFS(6): w=7

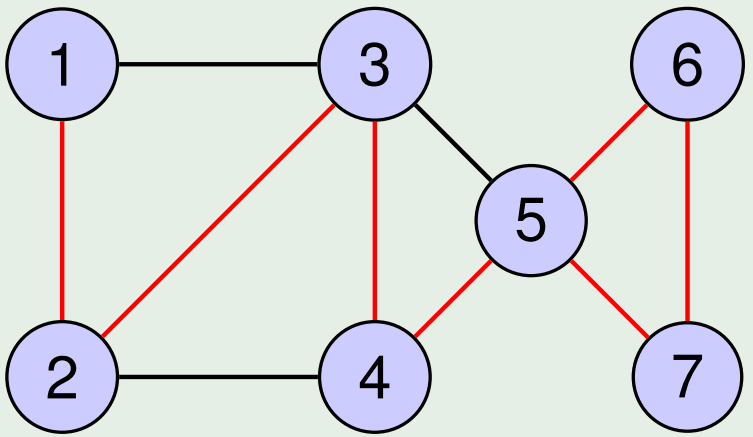
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

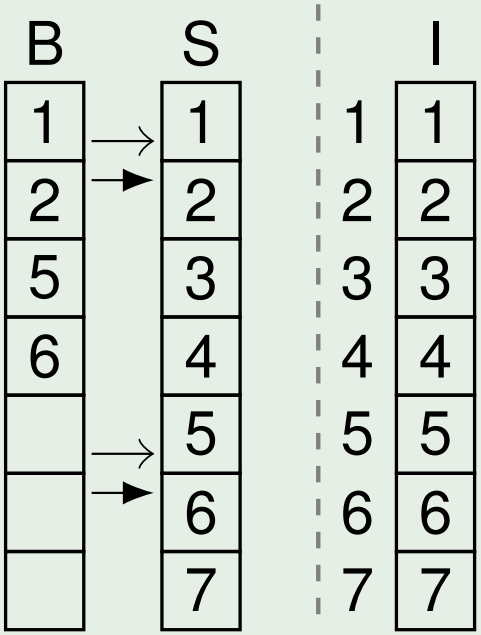


- Current procedure: DFS(7): w=5

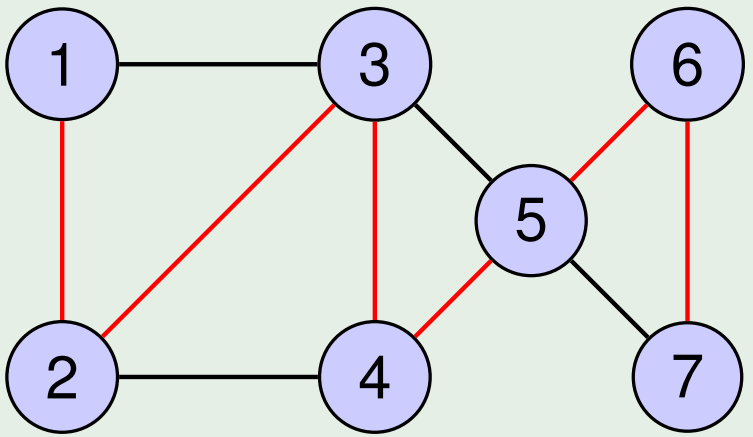
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

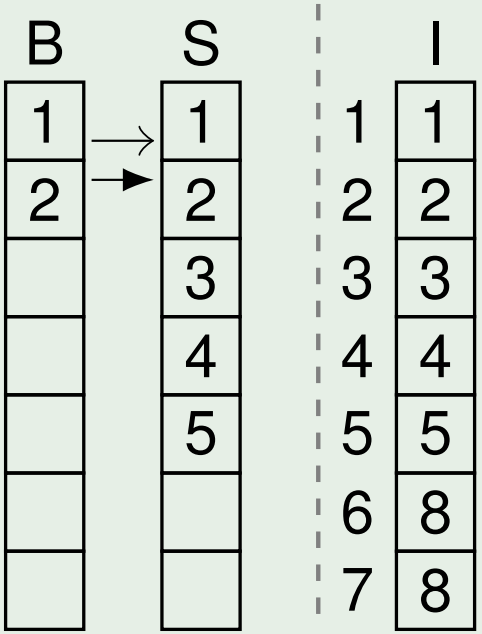


- Current procedure: DFS(7): End

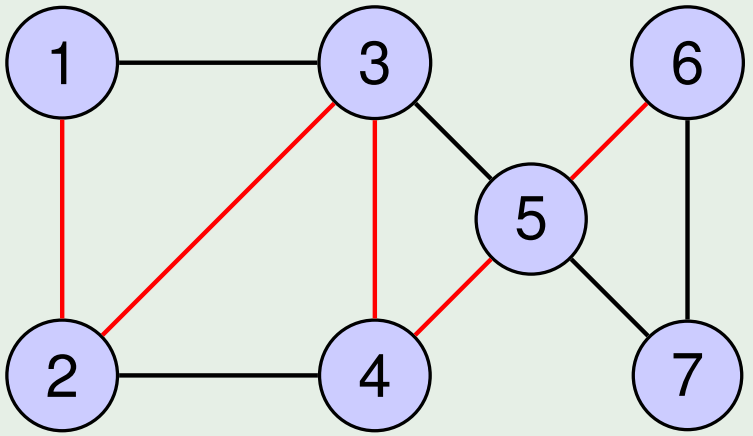
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

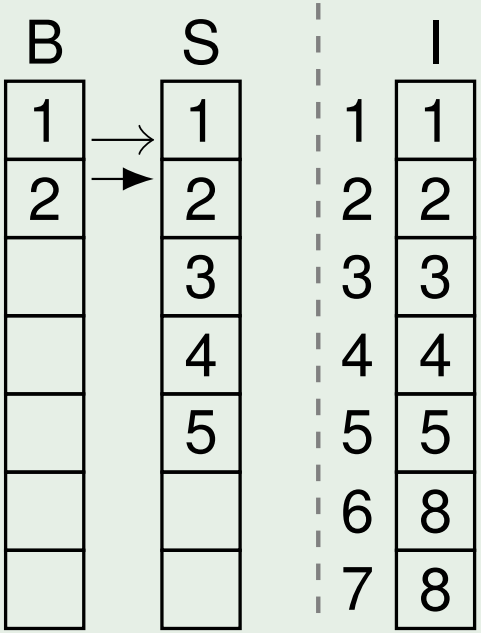


- Current procedure: DFS(6): End

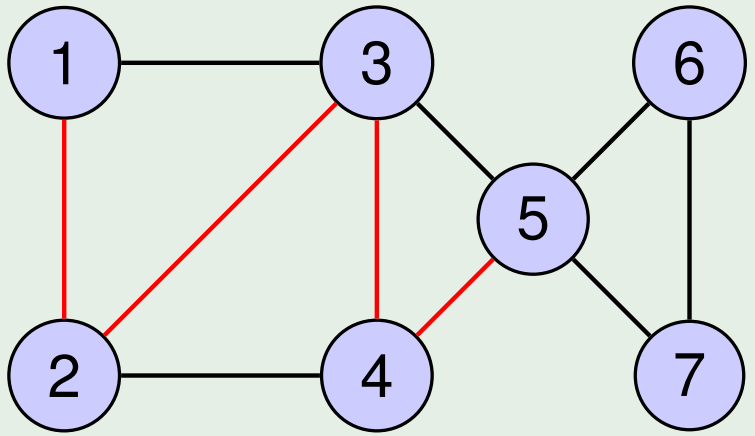
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



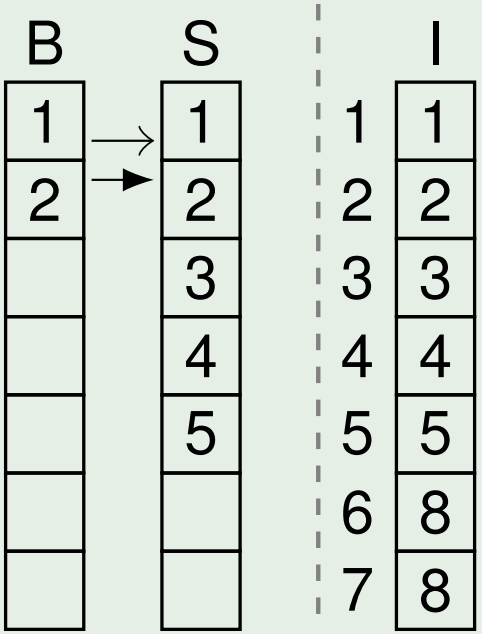
- Current procedure: when w=7)

DFS(5): End(No operation

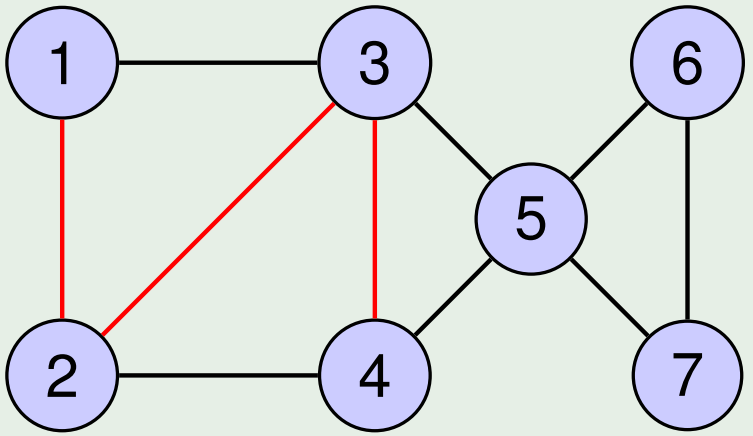
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph

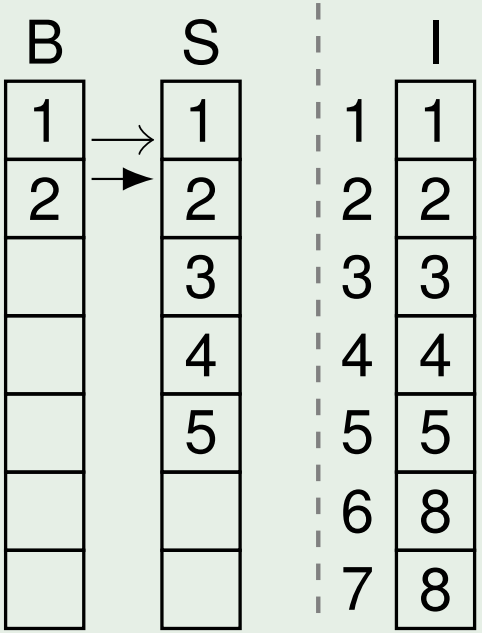


- Current procedure: DFS(4): End

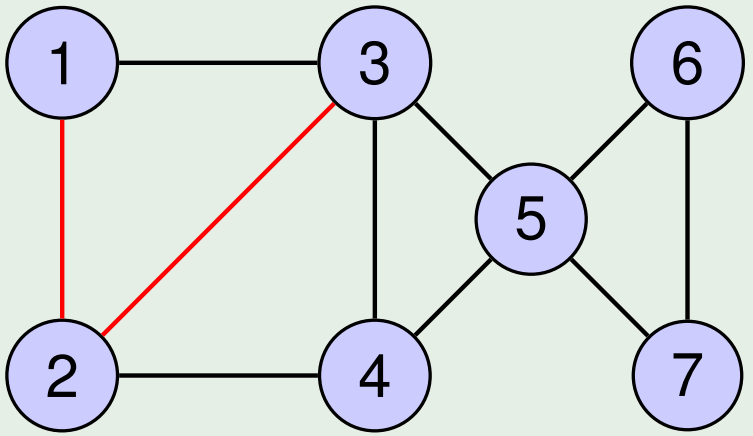
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.



Graph



- Current procedure: when w=5)

DFS(3): End(No operation

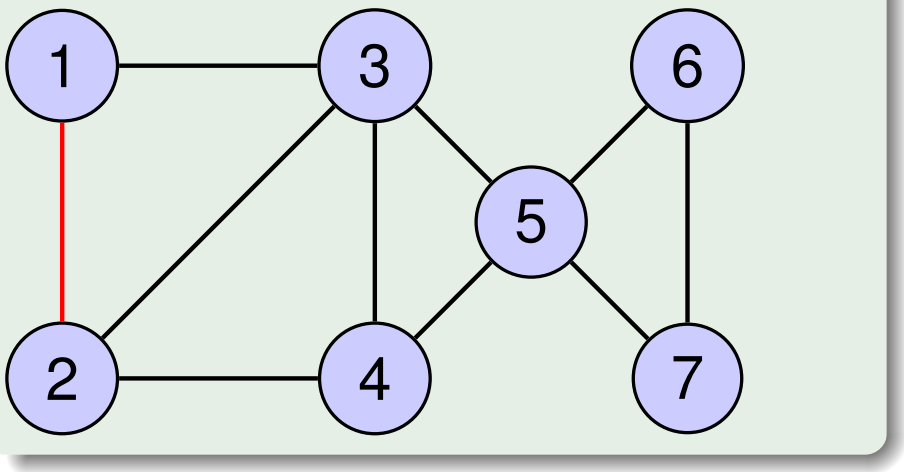
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
	1	1	1
		2	9
		3	9
		4	9
		5	9
		6	8
		7	8

Graph



- Current procedure: DFS(2): End

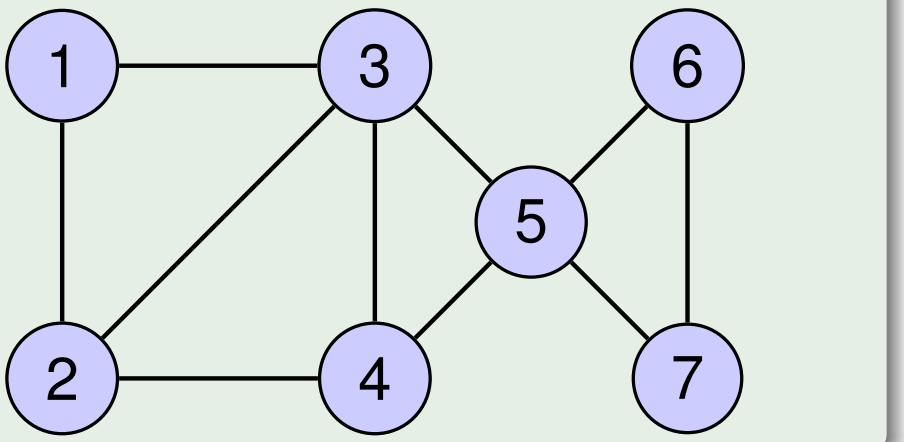
Demo: Gabow's biconnected components algorithm

Data in memory

B and S: stack. I: array.

B	S		I
		1	9
		2	9
		3	9
		4	9
		5	9
		6	8
		7	8

Graph



- Current procedure: DFS(1): End

Summary

- Gabow gave algorithms to find the strong components and biconnected components more effectively. They are one-pass algorithms while do not need a disjoint-set data structure.
- There is a close relationship between strong components and biconnected components, like two faces of a coin: one is directed graph, another is undirected graph.