Worked Out Exercises

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II. 5) $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $b=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

iterative methods for $Ax=b$ (Jacobi, Gauss-Seidel, SOR(ω))

$A=D-\tilde{L}-\tilde{U}=D(\mathbb{1}_2-U-L)$

Jacobi: splitting $A=D-(\tilde{L}+\tilde{U}) \rightarrow x_{k+1}=D^{-1}(\tilde{L}+\tilde{U})x_k+D^{-1}b$

$$A=\begin{bmatrix} 2 & \\ & 2 \end{bmatrix}-\begin{bmatrix} & \\ -1 & \end{bmatrix}-\begin{bmatrix} & -1 \\ & \end{bmatrix}=\begin{bmatrix} 2 & \\ & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}-\begin{bmatrix} & \\ -\frac{1}{2} & \end{bmatrix}-\begin{bmatrix} & -\frac{1}{2} \\ & \end{bmatrix}\right)$$

$R_J=\begin{bmatrix} & -\frac{1}{2} \\ -\frac{1}{2} & \end{bmatrix}$, $c_J=\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

iteration: $x_{k+1}=\begin{bmatrix} & -\frac{1}{2} \\ -\frac{1}{2} & \end{bmatrix}x_k+\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$$\begin{cases} x_{k+1,1}=-\frac{1}{2}x_{k_2}+\frac{1}{2} \\ x_{k+1,2}=-\frac{1}{2}x_{k_1} \end{cases}$$

Gauss-Seidel: $A=(D-\tilde{L})-\tilde{U}$

$R_{GS}=(D-\tilde{L})^{-1}\tilde{U}$, $c_{GS}=(D-\tilde{L})^{-1}b$

$$R_{GS}=\begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} & -1 \\ & \end{bmatrix}=\begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$c_{GS}=\begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

iteration: $x_{k+1}=\begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}x_k+\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$

SOR(ω): splitting - $A=M-K$

iteration - $x_{k+1}=M^{-1}Kx_k+M^{-1}b$

$$x_{k+1}^{SOR} = \omega x_{k+1}^{GS} + (1-\omega)x_k$$

$$R_{SOR}(\omega) = (\mathbb{1}_2 - \omega L)^{-1}((1-\omega)\mathbb{1}_2 + \omega U) = \begin{bmatrix} 1 & \\ \frac{\omega}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-\omega & -\frac{\omega}{2} \\ & 1-\omega \end{bmatrix} = \begin{bmatrix} 1-\omega & -\frac{\omega}{2} \\ \frac{\omega^2}{2} - \frac{\omega}{2} & \frac{\omega^2}{4} - \omega + 1 \end{bmatrix}$$

$$c_{SOR}(\omega) = \omega(\mathbb{1}_2 - \omega L)^{-1} D^{-1} b = \begin{bmatrix} \frac{\omega}{2} \\ -\frac{\omega^2}{4} \end{bmatrix}$$

convergence?

$\rho(R_J) = \frac{1}{2}$, the iterator converges: $x_k \xrightarrow{k \to \infty} 0$

$$\chi_J(t) = \det(R_J - t\mathbb{1}_2) = \det\begin{pmatrix} -t & -\frac{1}{2} \\ -\frac{1}{2} & -t \end{pmatrix} = t^2 - \frac{1}{4} = 0 \to t = \pm\frac{1}{2}$$

$$\chi_{GS}(t) = \det\begin{pmatrix} -t & -\frac{1}{2} \\ 0 & -t - \frac{1}{4} \end{pmatrix} = t\left(t + \frac{1}{4}\right) = 0 \to t = 0,\ t = -\frac{1}{4}$$

$\rho(R_{GS}) = \frac{1}{4}$

how many iterations for 30 correct decimal digits?

initial vector: $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

number of correct digits of $\tilde{x}$: $-\log_{10} \frac{\|x - \tilde{x}\|}{\|x\|}$

need a bound for $\|x - x_k\| \leq C \cdot \rho(R_J)^k = C \cdot 2^{-k}$

for 30 digits: need $\|x - x_k\| \leq 10^{-30}$

this is achieved when $10^{-30} \leq 2^{-k}$ (up to C)

$$-30 \log 10 \leq -k \log 2$$
$$\Rightarrow k \geq 30 \frac{\log(10)}{\log(2)} + c$$

II 1) $A \in \mathbb{C}^{n \times n}$

eigenvalues: $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$

which algorithm for (λ_i, v_i)? rate of convergence? (λ_i, v_i) eigenpair/unit eigenvector

power method

[illegible] $\left(|\lambda_2| / \ldots\right)^k$

power method

initial $x_0 \in \mathbb{C}^n$ unit. $\|v_1 - x_k\| \leq C \cdot \left(|\lambda_2| / |\lambda_1| \right)^k$

which algorithm for (λ_n, v_n)? rate of convergence?

$Av = \lambda v$

$A^{-1}(Av) = A^{-1}(\lambda v)$

$\Rightarrow A^{-1}v = \lambda^{-1}v$

apply the power method to A^{-1} to obtain (μ_k, v_k): approximate eigenpair of A^{-1} such that $\mu_k \to \lambda_n^{-1}$

$\rightsquigarrow$ (μ_k^{-1}, v_k) approximate eigenpair of A such that $\mu_k^{-1} \to \lambda_n$

$\|v_n - y_k\| \leq C \left(|\lambda_n| / |\lambda_{n-1}| \right)^k$

A singular? one eigenvector is zero, v_n generator of the $\ker(A)$

I 2) how to use the QR factorization to solve the least squares problem?

$A \in \mathbb{R}^{m \times n}$, $m \geq n$, $n = \operatorname{rank}(A)$. $b \in \mathbb{R}^m$, $\min_x \|Ax - b\|_2$

x_{min} solution of the "normal equations

$A^T A x = A^T b$

$A = QR$ and $A^T A x = (QR)^T QRx = R^T Q^T Q R x \Rightarrow R^T R x = R^T Q^T b$

orthogonal upper triangular with positive diagonal entries

$x = R^{-1} Q^T b$

$\ell x = \alpha x + \beta$ fitting $(-1,1), (0,0), (1,1)$

precisely: $(\ell(-1) - 1, \ell(0) - 0, \ell(1) - 1)$ minimum with respect to 2-norm

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ because } \begin{bmatrix} \ell(-1) \\ \ell(0) \\ \ell(1) \end{bmatrix} = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$a_1 \quad a_2$

Gram-Schmidt: $q_1 = \dfrac{a_1}{\|a_1\|_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} -0.7 \\ 0 \\ 0.7 \end{bmatrix}$

$$\tilde{a}_2 = a_2 - \langle a_2, q_1 \rangle q_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} -0.7 \\ 0 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{\tilde{a}_2}{\|\tilde{a}_2\|} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}$$

$$\begin{cases} a_1 = 1.4 q_1 \\ a_2 = 1.7 q_2 \end{cases} \iff A = (q_1 \ q_2) \begin{bmatrix} 1.4 & \\ & 1.7 \end{bmatrix} = \begin{bmatrix} -0.7 & 0.6 \\ 0 & 0.6 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 1.4 & \\ & 1.7 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = R^{-1} Q^T b = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}$$

$$\ell(x) = 0 \cdot x + 0.7$$

I. 4) $A \in \mathbb{R}^{2\times 2}$

AA^T: eigenvalues $9, \frac{1}{4}$ and eigenvectors $\overset{\tilde{u}_1}{\begin{bmatrix} 0.7 \\ -0.7 \end{bmatrix}}, \overset{\tilde{u}_2}{\begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}}$

A^TA: eigenvalues $9, \frac{1}{4}$ and eigenvectors $\overset{v_1}{\begin{bmatrix} 0.5 \\ -0.9 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix}}$

compute the SVD of A: $A = U \Sigma V^T$

$$A \cdot A^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$$

$$A^TA = V\Sigma^2 V^T$$

$$AA^T = \overset{(\tilde{u}_1, \tilde{u}_2)}{S} \cdot \underset{\begin{bmatrix} 9 & \\ & \frac{1}{4} \end{bmatrix}}{\Lambda} \cdot S^{-1}$$

$$(AA^T)S = S\Lambda$$

$\iff$ for each column, $(AA^T) u_i = \lambda_i u_i$

$$\Sigma = \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}, \quad \sigma_i^2 = 9, \tfrac{1}{4}$$

$$= \begin{bmatrix} 3 & \\ & \tfrac{1}{2} \end{bmatrix}$$

$Av_i = \sigma_i u_i, \ i = 1, 2$

$$V = \begin{bmatrix} 0.5 & 0.9 \\ -0.9 & 0.5 \end{bmatrix}$$

$u_i = \frac{1}{\sigma_i} Av_i$

$U = (\pm \tilde{u}_1, \pm \tilde{u}_2)$ (assume positive)

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} 3 & \\ & \tfrac{1}{2} \end{bmatrix} \begin{bmatrix} 0.5 & -0.9 \\ 0.9 & 0.5 \end{bmatrix}$$

$\|A\|_2$, operator true norm $= \sigma_1 = 3$

$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$A^{-1} = (U \Sigma V^T)^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1} = V \underbrace{\Sigma^{-1}}_{\begin{bmatrix} \frac{1}{3} & \\ & 2 \end{bmatrix}} U^T$ (U is orthogonal, $U^{-1} = U^T$)

$$A^{-1} = (v_2 \ \ v_1) \begin{bmatrix} 2 & \\ & \tfrac{1}{3} \end{bmatrix} (u_2, u_1)^T$$

$\kappa_2 = \sigma_1 \cdot \sigma_2^{-1}$

$= 3 \cdot 2$

$= 6$