# Lecture 8: MA and ARMA models

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## 1 Moving average models (MA)

Given a centered white noise $Z$ with variance $\sigma^2$ and a natural number $q \geq 0$, we define the MA(q) model as the process $\{Y_j,\ j \in \mathbb{Z}\}$ defined as

$$Y_j = Z_j + \theta_1 Z_{j-1} + \cdots + \theta_q Z_{j-q}.$$

where $\theta_1, \ldots, \theta_q$ are real numbers.

The immediate properties of this process are the following:

1. It is a second order process because $Z$ it is.
2. It is a centered process because $Z$ is centered and

   $$E(Y_j) = \sum_{l=0}^{q} \theta_l E(Z_{j-l}) = 0.$$

   Note that by convention $\theta_0 = 1$.
3. The auto-covariance function is given by

   $$\mathbb{C}(Y_j, Y_{j+k}) = \sigma^2 \sum_{i=0}^{q-|k|} \theta_i \theta_{i+|k|}, \quad |k| \leq q$$

   and 0 in all the other cases.

   Indeed, we can choose $k \geq 0$ without loosing generality, and then,

   $$\begin{aligned} \mathbb{C}(Y_j, Y_{j+k}) &= \mathbb{C}\left(\sum_{r=0}^{q} \theta_r Z_{j-r}, \sum_{s=0}^{q} \theta_s Z_{j+k-s}\right) \\ &= \sum_{r=0}^{q} \sum_{s=0}^{q} \theta_r \theta_s \mathbb{C}(Z_{j-r}, Z_{j+k-s}) \\ &= \sum_{r=0}^{q} \sum_{s=0}^{q} \theta_r \theta_s \sigma^2 \mathbb{1}_{\{j-r=j+k-s\}} \\ &= \sum_{r=0}^{q} \sum_{s=0}^{q} \theta_r \theta_s \sigma^2 \mathbb{1}_{\{s=r+k\}} \\ &= \sum_{r=0}^{q-k} \theta_r \theta_{r+k} \sigma^2. \end{aligned}$$

Therefore, it is a stationary process because the auto-covariance between $Y_j$ and $Y_{j+k}$ does not depend on $j$.

4. The variance is

$$\mathbb{V}(Y_j) = \sigma^2 \sum_{r=0}^{q} \theta_r^2, \quad \forall j \in \mathbb{Z}$$

5. The auto-correlation function is given by

$$\rho(k) = \frac{\sum_{r=0}^{q-k} \theta_r \theta_{r+k}}{\sum_{r=0}^{q} \theta_r^2}, \quad 0 \leq k \leq q.$$

and so, it is not null only for $k = 0, 1, \ldots, q$. Notice that this is completely different from the AR case.

In terms of lag operator $B$ we have

$$Y_j = Z_j + \theta_1 B Z_j + \theta_2 B^2 Z_j + \cdots + \theta_q B^q Z_j$$

and putting $B^0 = \mathrm{Id}$ we can write

$$Y_j = \Theta_q(B) Z_j$$

where

$$\Theta_q(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q$$

is a polynomial of order $q \geq 0$.

**Example 1.1 (MA(1) model)** *MA(1) model is given by*

$$Y_j = Z_j + \theta Z_{j-1}, \quad j \in \mathbb{Z}, \ \theta \in \mathbb{R}.$$

*Then, the variance is*

$$\mathbb{V}(Y_j) = \sigma^2(1 + \theta^2)$$

*and the auto-covariance function is null except for lags* $0$ *and* $1$*. In these cases it takes values*

$$\gamma(0) = \sigma^2(1 + \theta^2)$$

*and*

$$\gamma(1) = \sigma^2 \theta.$$

*Equivalently,* $\rho(0) = 1$,

$$\rho(1) = \frac{\theta}{1 + \theta^2}$$

*and* $\rho(l) = 0$ *for any other lag.*

**Remark 1.2** *Consider the process*

$$X_j = V_j + \frac{1}{\theta} V_{j-1}$$

*with* $V \sim WN(0, \theta^2\sigma^2)$. *This is a centered MA(1) process with variance*

$$\left(1 + \frac{1}{\theta^2}\right)\theta^2\sigma^2 = \sigma^2(1+\theta^2)$$

*and auto-correlation of order* 1 *given by*

$$\rho(1) = \frac{1/\theta}{1+1/\theta^2} = \frac{\theta}{1+\theta^2}.$$

*Therefore, an MA(1) model with parameters* $\theta$ *and* $\sigma^2$ *and an MA(1) of parameters* $\frac{1}{\theta}$ *and* $\theta^2\sigma^2$ *have the same structure as second order stationary processes. So, we can assume without loosing generality that* $|\theta| \leq 1$.

# 2 ARMA models

Let $P_n(x) = 1 - \alpha_1 x - \alpha_2 x - \cdots - \alpha_n x^n$ be a polynomial of degree $n$ with all the roots out of the unit circle, that is, invertible. Then,

$$Y = P(B)Z$$

is a MA(n) model and

$$Y = P(B)^{-1}Z \Longleftrightarrow Z = P(B)Y$$

is an AR(n) model.

**Definition 2.1** *Let* $p$ *and* $q$ *two fixed natural numbers,* $Z \sim WN(0, \sigma^2)$ *and* $\Phi_p(\cdot)$ *and* $\Theta_q(\cdot)$ *two invertible polynomials of degrees* $p$ *and* $q$ *respectively without roots in common. Then, a model ARMA(p,q) is a model that satisfies the equation*

$$\Phi_p(B)Y_j = \Theta_q(B)Z_j, \quad j \in \mathbb{Z}.$$

**Remark 2.2** *Using the fact that* $\Phi_p(\cdot)$ *is invertible we can write*

$$Y_j = \Phi_p(B)^{-1}\Theta_q(B)Z_j = \Theta_q(B)\Phi_p(B)^{-1}Z_j.$$

*This fact proves that ARMA models are linear and causal models.*

**Remark 2.3** *Note that to put positive or negative signs in front of coefficients of polynomial* $\Phi$ *and* $\Theta$ *is irrelevant because coefficients are real numbers. From now on we will consider always polynomials with negative sign in front of all the parameters.*

**Example 2.4 (ARMA(1,1) model)** *ARMA(1,1) model satisfies*

$$Y_j - \phi\, Y_{j-1} = Z_j - \theta\, Z_{j-1}, \quad j \in \mathbb{Z},$$

*with* $|\phi| < 1$, $|\theta| < 1$ *and* $\phi \neq \theta$.

*Solving the equation*

$$(1-\theta\,x) = (1-\phi\,x)\sum_{i=0}^{\infty}\psi_i x^i$$

*we can obtain the causal expression. We have*

$$\begin{aligned}
1-\theta\,x &= \sum_{i=0}^{\infty}\psi_i\,x^i - \sum_{i=0}^{\infty}\phi\,\psi_i\,x^{i+1} \\
&= \sum_{i=0}^{\infty}\psi_i\,x^i - \sum_{j=1}^{\infty}\phi\,\psi_{j-1}\,x^j \\
&= \psi_0 + \sum_{j=1}^{\infty}(\psi_j - \phi\,\psi_{j-1})x^j
\end{aligned}$$

*and so,*

$$\psi_0 = 1$$
$$\psi_1 - \phi\,\psi_0 = -\theta$$
$$\psi_j = \phi\,\psi_{j-1}, \quad \forall j \geq 2.$$

*Finally*

$$\psi_0 = 1$$
$$\psi_1 = \phi - \theta$$
$$\psi_i = \phi\,\psi_{i-1} = \cdots = \phi^{i-1}\,\psi_1 = \phi^{i-1}(\phi-\theta), \quad i \geq 1.$$

*We know that the auto-covariance function is given by*

$$\gamma(l) = \sigma^2\sum_{i=0}^{\infty}\psi_i\,\psi_{i+l}.$$

*Then, for $l = 0$ we have*

$$\begin{aligned}
\gamma(0) &= \sigma^2\sum_{i=0}^{\infty}\psi_i^2 \\
&= \sigma^2\left\{1+(\phi-\theta)^2+\sum_{i=2}^{\infty}\phi^{2(i-1)}(\phi-\theta)^2\right\} \\
&= \sigma^2\left\{1+(\phi-\theta)^2\sum_{i=1}^{\infty}\phi^{2(i-1)}\right\} \\
&= \sigma^2\left(1+(\phi-\theta)^2\frac{1}{1-\phi^2}\right),
\end{aligned}$$

*and for $l \geq 1$ we have*

$$
\begin{aligned}
\gamma(l) &= \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+l} \\
&= \sigma^2 \{\psi_l + \sum_{i=1}^{\infty} \phi^{i-1}(\phi-\theta)^2 \phi^{i+l-1}\} \\
&= \sigma^2 \left\{ \phi^{l-1}(\phi-\theta) + (\phi-\theta)^2 \sum_{i=1}^{\infty} \phi^{2i+l-2} \right\} \\
&= \sigma^2 \left\{ \phi^{l-1}(\phi-\theta) + \phi^l(\phi-\theta)^2 \frac{1}{1-\phi^2} \right\} \\
&= \sigma^2 \phi^l \left\{ \frac{\phi-\theta}{\phi} + \frac{(\phi-\theta)^2}{1-\phi^2} \right\} \\
&= \sigma^2 (\phi-\theta) \phi^l \left\{ \frac{1}{\phi} + \frac{\phi-\theta}{1-\phi^2} \right\} \\
&= \sigma^2 \phi^l \frac{(\phi-\theta)(1-\phi\theta)}{\phi(1-\phi^2)}.
\end{aligned}
$$

*That is,*

$$
\begin{aligned}
\gamma(l) &= \sigma^2\, \phi^l\, k(\phi, \theta), \quad l \geq 1 \\
\gamma(0) &= \sigma^2\, k^*(\phi, \theta)
\end{aligned}
$$

*for certain functions $k$ and $k^*$.*

*Therefore,*

$$\rho(l) = c\,(\phi, \theta)\, \phi^l$$

*for a certain function $c(\phi, \theta)$. Note that this means that the autocorrelation function eventually decreases like an AR model.*

# 3 Partial autocorrelation in ARMA models

Recall the definition of partial autocorrelation $\alpha$ introduced in Lecture 5.

If $X$ is an AR(1) model we have $\rho_2 = \rho_1^2$ and therefore $\alpha(2) = 0$. In general, for an AR(p) model we have

$$\alpha(k) = 0, \quad \forall k \geq p+1.$$

But if $X$ is an MA(1) we have $\rho_2 = 0$. Then,

$$\alpha(2) = -\frac{\rho_1^2}{1-\rho_1^2}$$

and using

$$\rho_1 = \alpha_1 = -\frac{\theta}{1+\theta^2}$$

we obtain

$$\alpha(2) = \frac{-\theta^2/(1+\theta^2)^2}{1-\theta^2/(1+\theta^2)^2} = \frac{-\theta^2}{(1+\theta^2)^2-\theta^2} = \frac{-\theta^2}{1+\theta^4+\theta^2}.$$

For an AR(p) model, the partial autocorrelation is null for any lag greater than $p$. On the contrary, for an $MA$ model, the partial autocorrelation decreases exponentially.

# 4 Non centered ARMA models

It is said that a process $\{Y_j,\ j \in \mathbb{Z}\}$ is an ARMA process with mean $\mu$ if

$$X_j := Y_j - \mu, \quad j \in \mathbb{Z}, \tag{4.1}$$

is an ARMA process.

For example in the case $p = q = 1$ we have

$$X_j - \phi\, X_{j-1} = Z_j - \theta\, Z_{j-1}$$

implies

$$(Y_j - \mu) - \phi(Y_{j-1} - \mu) = Z_j - \theta\, Z_{j-1}$$

or equivalently

$$Y_j - \phi\, Y_{j-1} = (\mu - \phi\mu) + Z_j - \theta\, Z_{j-1} = \delta + Z_j - \theta\, Z_{j-1},$$

with

$$\delta = \mu(1-\phi).$$

Notice also that from (4.1) we have immediately

$$Y_j = \mu + \sum_{i=0}^{\infty} \psi_i\, Z_{j-i}.$$

# 5 List of exercises

1. Determine what of the following series are causal or invertible. Recall that $Z$ is a white noise.

   (a) $Y_j + 0.2Y_{j-1} - 0.48Y_{j-2} = Z_j$.

   (b) $Y_j + 1.9Y_{j-1} - 0.88Y_{j-2} = Z_j + 0.2Z_{j-1} + 0.7Z_{j-2}$.

   (c) $Y_j + 0.6Y_{j-2} = Z_j + 1.2Z_{j-1}$

   (d) $Y_j + 1.8Y_{j-1} + 0.81Y_{j-2} = Z_j$

   (e) $Y_j + 1.6Y_{j-1} = Z_j - 0.4Z_{j-1} - 0.04Z_{j-2}$.

2. Consider the time series

$$Y_j = 0.4Y_{j-1} + 0.45Y_{j-2} + Z_j + Z_{j-1} + 0.25Z_{j-2}$$

where $Z \sim WN(0, \sigma^2)$.

(a) Write the equation in terms of the lag operator $B$ and show that the models is causal.

(b) Can this model be simplified? In this case, what are the values $(p, q)$ of the model?

(c) Find the general form of the coefficients of the causal expression of this series.

3. Consider process $\{X_j, j \in \mathbb{Z}\}$ such that $X_j = \phi X_{j-2} + Z_j$ where $|\phi| < 1$ and $Z$ is a standard white noise. Find the auto-covariance function, the auto-correlation function and the partial auto-correlation function and identify the model.

4. Let $\{X_j, j \in \mathbb{Z}\}$ be a process such that $X_j = Z_j + \theta Z_{j-2}$ where $Z$ is a standard white noise.

(a) Find auto-covariance, auto-correlation and partial auto-correlation functions for $k = 1, 2, 3, 4$.

(b) Identify the model.

(c) Compute the variance of $\frac{1}{4}\sum_{j=1}^{4} X_j$ for $\theta = 0.8, 0$ and $-0.8$. Comment the results.

5. Let $Y_j = X_j + W_j$ be a time series where $W$ a white noise with variance $\tau^2$ and $X$ is a centered AR(1) model with parameters $\phi$ and $\sigma^2$. Assume the white noise associated to $X$ is uncorrelated with $W$.

(a) Show the series $Y$ is stationary and find its auto-covariance function.

(b) Show the series $U_j = Y_j - \phi Y_{j-1}$ is an MA(1) process.

(c) Show that $Y$ is an ARMA(1,1) process.