CHAPTER 4

Iterative methods

Iterative methods for solving the linear equation $A\,x = b$ are used when the direct methods, like GEPP, require either too much time or too much space. They are usually based on matrix-vector multiplications, and so they are particularly convenient when the cost of this operation is low, like it happens for sparse matrices.

These methods do not produce an exact answer after a finite number of steps, but rather decrease the error by some amount after each step. The final error depends on how many iterations are done, and on the properties of the considered method and the matrix to which it is applied.

4.1. Splittings and iterative methods

Let A be a nonsingular $n \times n$ matrix and b an n-vector. Given an initial n-vector x_0, an iterative method generate a sequence of n-vectors

$$(x_l)_{l \geq 0}$$

hopefully converging to the solution $x = A^{-1}\,b$, and where each vector is easy to compute from the previous one.

A *splitting* of A is a decomposition

$$A = M - K$$

with M nonsingular. Such a decomposition produces an iterative method as above in the following way: the equation $A\,x = b$ is equivalent to $M\,x = K\,x + b$ or still to

$$x = M^{-1}K\,x + M^{-1}b.$$

Decoupling both sides of this equality we obtain the iteration

$$x_l = R\,x_{l-1} + c \quad \text{for } l \geq 1 \tag{4.1}$$

with $R = M^{-1}K$ and $c = M^{-1}b$. When this iteration converges, its limit x_∞ satisfies the equation $x_\infty = R\,x_\infty + c$ or equivalently

$$A\,x_\infty = b,$$

and so this limit *is* the solution of the linear equation.

The convergence of this method depends on the *spectral radius* of the iteration matrix, denoted by $\rho(R)$ and defined as the maximum absolute value of its eigenvalues. Indeed, the iteration (4.1) converges for every choice of the initial vector x_0 if and only if

$$\rho(R) < 1. \tag{4.2}$$

To see this, let x be the solution of the linear equation $A\,x = b$. If $\rho(R) \geq 1$ then choose x_0 such that $x - x_0$ is an eigenvector for an eigenvalue λ of R of absolute value ≥ 1. Hence

$$x - x_l = R^l\,(x - x_0) = \lambda^l\,(x - x_0),$$

and so x_l does not converge to x in this case.

For the converse, suppose for simplicity that R is diagonalizable, so that we can factor this matrix as

$$R = S\,\Lambda\,S^{-1}$$

with S nonsingular and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ diagonal. Then for each $l \geq 0$ we have that

$$\begin{array}{rc} & x = R\,x + c \\ - & x_l = R\,x_{l-1} + c \\ \hline & x - x_l = R\,(x - x_{l-1}) \end{array}$$

which readily implies that

$$x - x_l = R^l(x - x_0) = S\,\Lambda^l\,S^{-1}(x - x_0).$$

Taking ∞-norms we obtain that

$$\begin{aligned} \|x - x_l\|_\infty &= \|S\,\Lambda^l\,S^{-1}(x - x_0)\|_\infty \\ &\leq \|S\|_\infty \|\Lambda\|_\infty^l \|S^{-1}\|_\infty \|x - x_0\|_\infty = \kappa(S)\,\rho(R)^l \|x - x_0\|_\infty \end{aligned} \tag{4.3}$$

because $\|\Lambda\|_\infty = \max_j |\lambda_j| = \rho(R)$. Hence if the condition (4.2) holds, then the upper bound in (4.3) tends to 0 when $l \to +\infty$ and to the iteration converges.

Moreover, if $x \neq 0$ or equivalently $b \neq 0$, then we deduce from (4.3) the upper bound for the relative error

$$\frac{\|x - x_l\|_\infty}{\|x\|_\infty} \leq \kappa(S)\,\rho(R)^l\,\frac{\|x - x_0\|_\infty}{\|x\|_\infty}.$$

Setting $\gamma(R) = -\log_\beta \rho(R) > 0$ with β the base of our floating point system, this upper bound can be rewritten as

$$-\log_\beta\Big(\frac{\|x - x_l\|_\infty}{\|x\|_\infty}\Big) \geq \gamma(R)\,l - \log_\beta \kappa(S) - \log_\beta\Big(\frac{\|x - x_0\|_\infty}{\|x\|_\infty}\Big)$$

which, following the rule of thumb (2.8), shows that when the condition (4.2) holds, the precision of the approximations increases *linearly* with rate $\gamma(R)$: the smaller the spectral radius is, the higher is the rate of convergence.

To design an efficient iterative scheme for a given problem, we need to find a splitting $A = M - K$ verifying the conditions:

(1) $R = M^{-1}K$ and $c = M^{-1}b$ are easy to compute,
(2) $\rho(R)$ is small.

4.2. The Richardson iteration

This is a simple iterative method that will mainly serve us as an example where the relevant aspects can be fully understood.

Given a nonsingular $n \times n$ matrix A and an n-vector b, the *Richardson iteration* aims at improving an approximate solution $\widehat{x}$ of the equation $A\,x = b$ by adding to it multiple of the residual $b - A\,\widehat{x}$. Precisely, for a parameter $\omega > 0$ this iteration starts from an initial n-vector x_0 and constructs the sequence $(x_l)_{l \geq 0}$ by setting

$$x_l = x_{l-1} + \omega\,(b - A\,x_{l-1}) \quad \text{for } l \geq 1. \tag{4.4}$$

In the context of the previous section, it corresponds to the splitting $A = M_\omega - K_\omega$ with $M_\omega = \omega^{-1}\,\mathbb{1}_n$ and $K_\omega = \omega^{-1}\,\mathbb{1}_n - A$, and so we can rewrite it as

$$x_l = R_\omega\,x_{l-1} + c_\omega$$

with $R_\omega = M_\omega^{-1} K_\omega = \mathbb{1}_n - \omega A$ and $c_\omega = M^{-1} b = \omega b$.

The eigenvalues of this iteration matrix are of the form $1 - \omega\lambda$ with $\lambda \in \mathbb{C}$ an eigenvalue of A. Hence if we denote by $\Lambda(A)$ the *spectrum* of A, that is, the set of its eigenvalues, then

$$\rho(R_\omega) = \max_{\lambda \in \Lambda(A)} |1 - \omega\lambda|.$$

Hence by the criterion in (4.2), the Richardson iteration (4.4) converges for every choice of initial vector x_0 if and only if

$$|1 - \omega\lambda| < 1 \quad \text{for all } \lambda \in \Lambda(A) \tag{4.5}$$

or equivalently, if the spectrum of A lies in the open disk centered at the point ω^{-1} and of radius ω^{-1}:

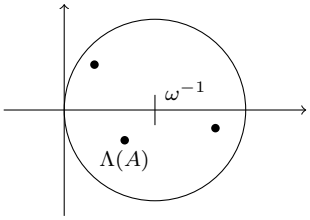

FIGURE 4.2.1. Convergence of the Richardson iteration

To make the full convergence analysis, we assume for simplicity that the eigenvalues of A are real. We respectively denote by $\lambda_{\min}$ and $\lambda_{\max}$ the minimal and the maximal ones, so that $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$ for all $\lambda \in \Lambda(A)$.

The condition for convergence in (4.5) then translates into $-1 < 1 - \omega\lambda_{\max}$ and $1 - \omega\lambda_{\min} < 1$, which are equivalent to

$$0 < \lambda_{\min} \quad \text{and} \quad \lambda_{\max} < \frac{2}{\omega}.$$

Hence in this situation, the iteration converges if and only if all the eigenvalues are positive and the parameter satisfies

$$\omega < \frac{2}{\lambda_{\max}}.$$

When this is the case, the spectral radius is given by the piecewise affine function

$$\rho(R_\omega) = \max(|1 - \omega\lambda_{\min}|, |1 - \omega\lambda_{\max}|),$$

whose graph is shown in Figure 4.2.2.

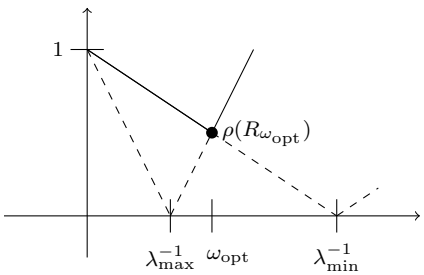

FIGURE 4.2.2. The spectral radius of R_ω

As shown in this figure, the best value ω_{opt}, that is the value that minimizes the spectral radius, is reached at the point where the graph of $\omega \mapsto |1 - \omega\,\lambda_{\max}|$ with positive slope crosses the graph of $\omega \mapsto |1 - \omega\,\lambda_{\min}|$ with negative slope, that is when $-1 + \lambda_1\,\omega_{\text{opt}} = 1 - \lambda_n \omega_{\text{opt}}$. This gives the value

$$\omega_{\text{opt}} = \frac{2}{\lambda_{\max} + \lambda_{\min}},$$

whose corresponding spectral radius is

$$\rho(\omega_{\text{opt}}) = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}.$$

If A has very large and very small eigenvalues, then the Richardson iteration will be extremely slow, even when using the optimal parameter ω_{opt}. Moreover, the determination of this optimal parameter requires knowledge of the maximal and minimal eigenvalue, which is not easily accessible in realistic problems.

4.3. Basic iterative methods

For the methods to be discussed in the section, we assume that all the diagonal entries of A are nonzero and we decompose it as

$$A = D - \widetilde{L} - \widetilde{U} = D\,(\mathbb{1}_n - L - U)$$

with D diagonal, $\widetilde{L}$ and L strictly lower triangular, and $\widetilde{U}$ and U strictly upper triangular.

Jabobi's method can be interpreted as going successively through each scalar linear equation, so that at the l-th iteration the updated value of j-th component of the approximate solution satisfies the j-th scalar linear equation when evaluated at the previous values of the other components. Precisely, for each $l \geq 1$ we want to have

$$a_{j,1}\,x_{l-1,1} + \cdots + a_{j,j-1}\,x_{l-1,j-1} + a_{j,j}\,x_{l,j} + a_{l,j+1}\,x_{l-1,j+1} + \cdots + a_{l,n}\,x_{l-1,n} = b_j$$

for $j = 1, \ldots, n$. This leads to an iterative method starting with a given initial vector x_0 and where the l-th iteration is defined by the rule:

Algorithm 4.3.1 (Jacobi's method)

for $j = 1, \ldots, n$
$\quad x_{l,j} \leftarrow \frac{1}{a_{j,j}} \left(b_j - \sum_{k \neq j} a_{j,k}\,x_{l-1,k}\right)$

It can be expressed in matrix notation as $x_l = R_{\text{J}}\,x_{l-1} + c_{\text{J}}$ with

$$R_{\text{J}} = D^{-1}(\widetilde{L} + \widetilde{U}) = L + U \quad \text{and} \quad c_{\text{J}} = D^{-1}b. \tag{4.6}$$

The *Gauss-Seidel method* is motivated by the observation that at the j-th step of Jacobi's method we have already computed improved values for the first $j-1$ components of the approximate solution, and so it is reasonable to take advantage of them when updating the j-th component: for each $l \geq 0$ we aim at

$$a_{j,1}\,x_{l,1} + \cdots + a_{j,j-1}\,x_{l,j-1} + a_{j,j}\,x_{l,j} + a_{l+1,j+1}\,x_{l-1,j+1} + \cdots + a_{l,n}\,x_{l-1,n} = b_j$$

for $j = 1, \ldots, n$. The corresponding iterative method is defined by the rule:

Algorithm 4.3.2 (Gauss-Seidel method)

for $j = 1, \dots, n$

$$x_{l,j} \leftarrow \tfrac{1}{a_{j,j}} \Big(b_j - \underbrace{\sum_{k<j} a_{j,k}\, x_{l,k}}_{\text{updated } x\text{'s}} - \underbrace{\sum_{k>j} a_{j,k}\, x_{l-1,k}}_{\text{older } x\text{'s}} \Big)$$

In matrix notation, we have that $x_{l+1} = R_{\mathrm{GS}}\, x_l + c_{\mathrm{GS}}$ with

$$R_{\mathrm{GS}} = (D - \widetilde{L})^{-1}\widetilde{U} = (\mathbb{1}_n - L)^{-1}U, \tag{4.7}$$
$$c_{\mathrm{GS}} = (D - \widetilde{L})^{-1}b = (\mathbb{1}_n - L)^{-1}D^{-1}b.$$

Successive overrelaxation for a parameter $\omega \in \mathbb{R}$ is a weighted average of the vectors x_{l+1} and x_l from the Gauss-Seidel method: it is defined by

$$x_l^{\mathrm{SOR}(\omega)} = (1-\omega)\, x_{l-1}^{\mathrm{GS}} + \omega x_l^{\mathrm{GS}} \qquad \text{for } l \geq 1.$$

This yields the following iterative scheme:

Algorithm 4.3.3 (SOR(ω))

for $j = 1, \dots, n$

$$x_{l,j} \leftarrow (1-\omega)x_{l-1,j} + \tfrac{\omega}{a_{j,j}} \big(b_j - \textstyle\sum_{k<j} a_{j,k}\, x_{l,k} - \sum_{k>j} a_{j,k}\, x_{l-1,k} \big)$$

In matrix notation, we have that $x_l = R_{\mathrm{SOR}(\omega)}\, x_{l-1} + c_{\mathrm{SOR}(\omega)}\, b$ with

$$R_{\mathrm{SOR}(\omega)} = (D - \omega\, \widetilde{L})^{-1}((1-\omega)\, D + \omega\, \widetilde{U}) = (\mathbb{1}_n - \omega\, L)^{-1}((1-\omega)\, \mathbb{1}_n + \omega\, U), \tag{4.8}$$
$$c_{\mathrm{SOR}(\omega)} = \omega\, (D - \omega\, \widetilde{L})\, b = \omega\, (\mathbb{1}_n - \omega\, L)\, D^{-1}b.$$

We distinguish three cases, depending on the value of the relaxation parameter ω: $\omega = 1$ is equivalent to the Gauss-Seidel method, $\omega < 1$ it is called *underrelaxation*, and $\omega > 1$ is called *overrelaxation*. A somewhat superficial motivation for overrelaxation is that if going from x_{l-1} to x_l is a good direction towards the solution, then moving $\omega > 1$ times as far should be even better.

Example 4.3.1. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad b = (1, 0).$$

The solution of the linear equation $A\, x = b$ is the 2-vector $x = (\frac{3}{2}, \frac{-1}{3})$.

The decompositions $A = D - \widetilde{L} - \widetilde{U} = D\,(\mathbb{1}_2 - L - U)$ are respectively given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{-1}{2} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} \right)$$

Hence

$$R_{\mathrm{J}} = L + U = \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix} \quad \text{and} \quad c_{\mathrm{J}} = D^{-1}b = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

and so the corresponding Jacobi iteration writes down as

$$\begin{bmatrix} x_{l,1} \\ x_{l,2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_{l-1,1} \\ x_{l-1,2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-x_{l-1,2}}{2} + \frac{1}{2} \\ \frac{x_{l-1,1}}{2} \end{bmatrix}.$$

The corresponding characteristic polynomial is

$$\chi_{R_{\mathrm{J}}} = \det \begin{bmatrix} -t & \frac{-1}{2} \\ \frac{-1}{2} & -t \end{bmatrix} = t^2 - \frac{1}{4}.$$

Its spectrum is $\Lambda(R_{\mathrm{J}}) = \{t \mid \chi_{R_{\mathrm{J}}}(t) = 0\} = \left\{ \pm \frac{1}{2} \right\}$ and so its spectral radius is

$$\rho(R_{\mathrm{J}}) = \frac{1}{2},$$

showing that the method converges to the solution x for every choice of initial vector.

We also have that

$$R_{\mathrm{GS}} = (\mathbb{1}_2 - L)^{-1} U = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & \frac{1}{4} \end{bmatrix},$$

$$c_{\mathrm{GS}} = (\mathbb{1}_2 - L)^{-1} D^{-1} b = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{4} \end{bmatrix}$$

and so the corresponding Gauss-Seidel iteration writes down as

$$\begin{bmatrix} x_{l,1} \\ x_{l,2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_{l-1,1} \\ x_{l-1,2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{4} \end{bmatrix} = \begin{bmatrix} \frac{-x_{l,2}}{2} + \frac{1}{2} \\ \frac{x_{l,2}}{4} - \frac{1}{4} \end{bmatrix}.$$

We have that

$$\chi_{R_{\mathrm{GS}}} = \det \begin{bmatrix} -t & \frac{1}{2} \\ 0 & -t + \frac{1}{4} \end{bmatrix} = t^2 - \frac{1}{4}\, t.$$

Hence $\Lambda(R_{\mathrm{GS}}) = \left\{0, \frac{1}{4}\right\}$ and so $\rho(R_{\mathrm{GS}}) = \frac{1}{4}$. We have that

$$\rho(R_{\mathrm{GS}}) = \rho(R_{\mathrm{J}})^2$$

and so in this example, the Gauss-Seidel method converges with the *double of the speed* of the Jacobi method.

Now for $\omega \in \mathbb{R}$ we have that

$$\begin{aligned} R_{\mathrm{SOR}(\omega)} &= (\mathbb{1}_2 - \omega\, L)^{-1}((1-\omega)\, \mathbb{1}_2 + \omega\, U) \\ &= \begin{bmatrix} 1 & 0 \\ \omega/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-\omega & -\omega/2 \\ 0 & 1-\omega \end{bmatrix} = \begin{bmatrix} 1-\omega & -\frac{\omega}{2} \\ \frac{\omega^2}{2} - \frac{\omega}{2} & \frac{\omega^2}{4} - \omega + 1 \end{bmatrix} \end{aligned}$$

and

$$c_{\mathrm{SOR}(\omega)} = \omega\, (\mathbb{1}_2 - \omega\, L)^{-1} D^{-1} b = \omega \begin{bmatrix} 1 & 0 \\ \frac{\omega}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\omega}{2} \\ \frac{-\omega^2}{4} \end{bmatrix},$$

and so the corresponding SOR(ω) iteration writes down as

$$\begin{aligned} \begin{bmatrix} x_{l,1} \\ x_{l,2} \end{bmatrix} &= \begin{bmatrix} 1-\omega & -\frac{\omega}{2} \\ \frac{\omega^2}{2} - \frac{\omega}{2} & \frac{\omega^2}{4} - \omega + 1 \end{bmatrix} \begin{bmatrix} x_{l-1,1} \\ x_{l-1,2} \end{bmatrix} + \begin{bmatrix} \frac{\omega}{2} \\ \frac{-\omega^2}{4} \end{bmatrix} \\ &= \begin{bmatrix} (1-\omega)\, x_{l-1,1} - \frac{\omega}{2}\, x_{l-1,2} + \frac{\omega}{2} \\ (\frac{\omega^2}{2} - \frac{\omega}{2})\, x_{l-1,1} + (\frac{\omega^2}{4} - \omega + 1)\, x_{l-1,2} - \frac{\omega^2}{4} \end{bmatrix}. \end{aligned}$$

We have that

$$\chi_{R_{\mathrm{SOR}(\omega)}} = \det \begin{bmatrix} 1-\omega-t & -\frac{\omega}{2} \\ \frac{\omega^2}{2} - \frac{\omega}{2} & \frac{\omega^2}{4} - \omega + 1 - t \end{bmatrix} = t^2 + \left(\frac{-\omega^2}{4} + 2\,\omega - 2 \right) t + (\omega^2 - 2\,\omega + 1).$$

The spectral radius $\rho(R_{\mathrm{SOR}(\omega)})$ is the maximum of the absolute value of the zeros of this polynomial, and its graph is shown for $\omega \in [0, 2]$ is shown in Figure 4.3.1.

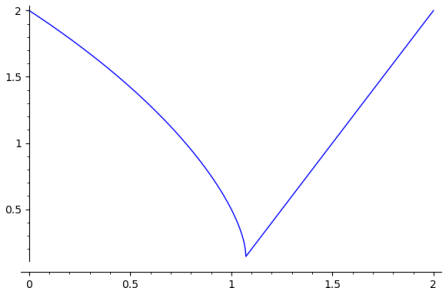

FIGURE 4.3.1. The spectral radius of SOR(ω)

The optimal value of the relaxation parameter is $\omega_{\text{opt}} = 1.0717$ up to 4 decimal digits, and the corresponding spectral radius is $\rho(R_{\text{SOR}(1.0717)}) = 0.1535$.

4.4. Convergence of the basic iterative schemes

We will give some criteria that, in some specific situation, guarantee the convergence of the basic iterative schemes. Since the spectral radius is difficult to compute, these convergence criteria can be useful in practice.

As in §4.3, we assume that all the diagonal entries of A are nonzero and we consider the decomposition

$$A = D - \widetilde{L} - \widetilde{U}$$

with D diagonal, $\widetilde{L}$ strictly lower triangular and $\widetilde{U}$ strictly upper triangular.

The matrix A is *(column) diagonally dominant* if the absolute value of each diagonal entry is greater the sum of the absolute values of the rest of the entries in its column, that is

$$|a_{j,j}| > \sum_{i \neq j} |a_{i,j}|, \quad j = 1, \dots, n.$$

If A is diagonally dominant then both the Jacobi and the Gauss-Seidel iterations will converge for every choice of initial vector x_0. To see this, let

$$R_{\text{J}} = D^{-1}(\widetilde{L} + \widetilde{U}) \quad \text{and} \quad R_{\text{GS}} = (D - \widetilde{L})^{-1}\widetilde{U}$$

be the corresponding iteration matrices as in (4.6) and (4.7). For each $\lambda \in \Lambda(R_{\text{J}})$ let x be a corresponding eigenvector for it, and k the index of its largest component. Up to a normalization, we can suppose that

$$x_k = 1 \quad \text{and} \quad |x_j| \leq 1 \text{ for all } j.$$

Then the equality $R_{\text{J}}\, x = \lambda\, x$ implies that $D^{-1}(+\widetilde{U})\, x = \lambda\, x$. The k-th entry of this later equality is

$$-\sum_{j \neq k} \frac{a_{k,j}}{a_{k,k}}\, x_j = \lambda,$$

which implies that

$$|\lambda| \leq \sum_{j \neq k} \frac{|a_{k,j}|}{|a_{k,k}|}\, |x_j| < 1,$$

because $|x_j| \leq 1$ for all j and A is diagonally dominant. Since this inequality holds for every eigenvalue of R_{J}, we deduce that $\rho(R_{\text{J}}) < 1$, which gives the result for the Jacobi method.

Similarly, for each $\lambda \in \Lambda(R_{\mathrm{GS}})$ let x a corresponding eigenvector, and k the index of the largest component of x. Again, we assume that $x_k = 1$ and $|x_j| \leq 1$ for all j. From the equality $R_{\mathrm{GS}}\, x = \lambda\, x$ we deduce that $\widetilde{U}\, x = \lambda\, (D - \widetilde{L})\, x$ and so

$$\sum_{j<k} a_{k,j}\, x_j = \lambda \left(a_{k,k} + \sum_{j>k} a_{k,j}\, x_j \right).$$

This implies that

$$|\lambda| \leq \frac{\sum_{j<k} |a_{k,j}|\, |x_j|}{|a_{k,k}| - \sum_{j>k} |a_{k,j}|\, |x_j|} \leq \frac{\sum_{j<k} |a_{k,j}|}{|a_{k,k}| - \sum_{j>k} |a_{,j}|} < 1,$$

again because $|x_j| \leq 1$ for all j and A is diagonally dominant, which implies that $|a_{k,k}| - \sum_{j>k} |a_{k,j}| > \sum_{j<k} |a_{k,j}|$. Since this inequality holds for every eigenvalue of R_{GS}, we deduce the convergence of the Gauss-Seidel method in this case.

For the successive overrelaxation method, the condition on the relaxation parameter

$$0 < \omega < 2 \tag{4.9}$$

is necessary for the convergence of the method. Indeed, by (4.8) the corresponding iteration matrix is

$$R_{\mathrm{SOR}(\omega)} = (\mathbb{1}_n - \omega\, L)^{-1}((1 - \omega)\, \mathbb{1}_n + \omega\, U)$$

and so its determinant can be computed as

$$\begin{aligned} \det(R_{\mathrm{SOR}(\omega)}) &= \det((\mathbb{1}_n - \omega\, L)^{-1}((1 - \omega)\, \mathbb{1}_n + \omega\, U)) \\ &= \det(\mathbb{1}_n - \omega\, L)^{-1} \det((1 - \omega)\, \mathbb{1}_n + \omega\, U) = (1 - \omega)^n. \end{aligned} \tag{4.10}$$

The determinant of a matrix coincides with the product of its eigenvalues, repeated with the corresponding multiplicity, and so

$$|\det(R_{\mathrm{SOR}(\omega)})| \leq \rho(R_{\mathrm{SOR}(\omega)})^n. \tag{4.11}$$

The inequalities (4.11) and (4.10) readily imply the lower bound

$$\rho(R_{\mathrm{SOR}(\omega)}) \geq |1 - \omega|,$$

and so the condition for the spectral radius in (4.2) can only be satisfied whenever the condition (4.9) holds.

On the other hand, if A is a symmetric and positive definite matrix with real coefficients then SOR(ω) converges for every $0 < \omega < 2$ [**Dem97**, Theorem 6.4]. In particular, the Gauss-Seidel method ($\omega = 1$) always converges in this situation.