## Master in Foundations of Data Science — 2023-2024

## NUMERICAL LINEAR ALGEBRA

Exercises from previous exams on linear equation solving, the least square problem and the singular value decomposition.

- **1.** Let  $A^{-1}$  be the inverse matrix of A.
  - (1) Show that  $A^{-1} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$  where *i*-th column  $c_i$  verifies

$$(0.1) Ac_i = e_i$$

with  $e_i$  the *i*-th vector in the standard basis of  $\mathbb{R}^n$ .

- (2) Given a PLU factorization of A, show that solving each equation in (0.1) costs  $4n^2/3$  flops.
- (3) Show that the time complexity of obtaining  $A^{-1}$  by computing a PLU factorization of A and then solving the equations (0.1) is  $2n^3 + O(n^2)$ .
- **2.** Let  $A \in \mathbb{R}^{m \times n}$  be a matrix of rank n and  $b \in \mathbb{R}^m$ .
  - (1) Explain how to use the QR factorization of A to solve the least square problem (LSP) that asks to find the vector  $x_{\min} \in \mathbb{R}^n$  minimizing the quantity  $||Ax b||_2$  for  $x \in \mathbb{R}^n$ , and give the expression for the residual error  $||Ax_{\min} b||_2$ .
  - (2) Find the affine function  $\ell(x) = \alpha x + \beta$  whose graph fits better the points (-1,1), (0,0) and (1,1), in the sense that the Euclidean norm of the vector

$$(\ell(-1) - 1, \ell(0) - 0, \ell(1) - 1) \in \mathbb{R}^3$$

is minimal among all possible choices of  $\alpha, \beta \in \mathbb{R}$ .

3. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$$

- (1) Compute its QR factorization using Householder reflexions.
- (2) Compute the same factorization, but this time using Givens rotations instead of reflexions.
- **5.** Consider the singular value decomposition (SVD)

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \cdot \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \cdot \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- (1) Compute the condition number of the matrix A with respect to the 2-norm.
- (2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations.
- **4.** Let  $A \in \mathbb{R}^{2 \times 2}$  such that the eigenvalues of  $A \cdot A^T$  are 9 and  $\frac{1}{4}$  with respective eigenvectors

$$\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix} \quad \text{ and } \quad \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix},$$

and the eigenvalues of  $A^T \cdot A$  are 9 and  $\frac{1}{4}$  with respective eigenvectors

$$\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \approx \begin{pmatrix} 0.50 \\ -0.87 \end{pmatrix} \quad \text{ and } \quad \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.50 \end{pmatrix}.$$

- (1) Compute a singular value decomposition (SVD) of A.
- (2) Using this SVD, compute condition number of A with respect to the 2-norm.
- (3) Determine the image of the unit disk of  $\mathbb{R}^2$  under the linear map defined by A.