L8 - 24/11/23 - Optimization - Lecture of 17/11 pdf file

Augmented objective function: x^* minimizes f in $\mathbb{R}^n$
x^* minimizes f in X $\downarrow$ feasible set

- Discontinuity of f on the boundary of X
- The infinite values outside X
  - ↳ Sequence of unconstrained minimization problems
    - Penalty parameter is adjusted
    - Sequence of unconstrained minima converges

* Approximately minimize $f(x)$ by running an unconstrained algorithm on the penalized objective function
↳ reasonably close ↳ done using unimodal optimization method

Penalty function: $\varphi(x,t)$ is continuous
$\varphi(x,t) \geq 0$ for all g and t
$\varphi(x,t) = 0$ for $t \leq 0$ and φ is strictly increasing for both $x \geq 0$ and $t > 0$

* desirable t has at least one continuous derivative in t
↳ preferably two

Typical example: $\varphi(x,t) = \begin{cases} 0, & t < 0 \\ x t^n, & t \geq 0 \end{cases}$ where $n \geq 1$

$$\tilde{f}(x) = f(x) + \sum_{i=1}^{p} \varphi(\alpha_i, g_i(x)) + \sum_{j=1}^{m} [\varphi(\beta_j, h_j(x)) + \varphi(\beta_j, -h_j(x))]$$

- If any inequality is violated ($g_i(x) > 0$), a large penalty is involved
- The second summation penalizes equality constraints

* Minimize $\tilde{f}(x)$ with no constraints
↳ no finite choice for the penalty parameter keeps the solution to the feasible set

→

* increase the a_i and b_i and use the result of the last iterate as the initial guess for a new minimization

Increasing the penalty parameters: improve the accuracy
slows down the convergence
↳ causes a very large gradient

* As the penalty parameters are increased without bound, any convergent subsequence of solutions to the unconstrained penalized problems must converge to a solution of the constrained problem

Pros: ◦ "hands-off" method for converting constrained
↳ unconstrained problems of any type

◦ Don't have to worry about finding an initial feasible point

◦ Constraints are "soft"

Cons: ◦ Solution will not be exact

◦ Some cases can't be defined when the objective function is actually defined outside the feasible set as we increase the penalty parameters, the unconstrained optimization problem becomes ill-conditioned

Barrier function methods:

- Strictly feasible methods
- Use a barrier term that approaches the infinite penalty function $t(x)$
- The feasible set has a non-empty interior
- Possible to reach any boundary point of x by approaching it from the interior.

Logarithmic function: $\varphi(x) = -\sum_{i=1}^{p} \log(g_i(x))$

Inverse function: $\phi(x) = \sum_{i=1}^{p} \frac{1}{g_i(x)}$

* Solve a sequence of unconstrained minimization problems of the form min $f_\mu(x)$, $x = 0, 1, 2, ..$ for a sequence $\{\mu_k\}$ of positive barrier parameters that decrease monotonically to 0

1. Positive μ and feasible point x_0
2. Minimize with unconstrained algorithm using first-order necessary condition for optimality.
3. Decrease the value of μ and reoptimize
4. Iterate

* Issue of finding the initial feasible point
  ↳ find an initial point using penalty functions, but on constraints

- $\lambda(\mu)$ estimate of the langrange multiplier λ^*
- if x^* is a regular point of the contraints, then as $x(\mu)$ converges to x^*, $\lambda(\mu)$ converges to λ^*
- Increasingly difficult to solve as the barrier parameter decreases
- Condition number of the Hessian matrix becomes increasingly large
  ↳ rules out the method whose convergence rate is dependent Newton methods are easily sensitive