Lecture 12: ARFIMA models

Josep Vives

April 20, 2024

It can be shown, thanks to Wold decomposition theorem, see [1] (page 76), that any second order stationary time series X can be written as

$$X_j = \sum_{i=0}^{\infty} \psi_i Z_{j-i} + f(t)$$

where f is a deterministic function. So, we assume in all the chapter that our process X is a centered purely nondeterministic stationary time series that has the representation

$$X_j = \sum_{i=0}^{\infty} \psi_i Z_{j-i}$$

with $\sum_{i=0}^{\infty} \psi_i^2 < \infty$ and Z is a centered white noise with variance σ^2. Recall moreover, from Lecture 6, that these models include ARMA models and

$$\rho(l) = \frac{\sum_{r=0}^{\infty} \psi_r \psi_{r+l}}{\sum_{r=0}^{\infty} \psi_r^2}.$$

It is immediate to see that

$$|\rho(l)| \stackrel{l\uparrow\infty}{\longrightarrow} 0.$$

Note that we can bound

$$\sum_{r=0}^{\infty} |\psi_r||\psi_{r+l}| \leq (\sum_{r=0}^{\infty} |\psi_r|^2)^{\frac{1}{2}} (\sum_{r=0}^{\infty} |\psi_{r+l}|^2)^{\frac{1}{2}}$$

and the first term in the product is a finite constant and the second one can be re-written as

$$(\sum_{j=l}^{\infty} |\psi_j|^2)^{\frac{1}{2}}$$

that of course converge to 0 when $l \uparrow \infty$ because $\sum_{r=0}^{\infty} |\psi_r|^2$ is finite.

Definition 1 *It is said that a stationary time series that satisfies*

$$\sum_{l=0}^{\infty} |\rho(l)| < \infty \tag{1}$$

has short memory. On the contrary, if property (1) is false, that is,

$$\sum_{l=0}^{\infty} |\rho(l)| = \infty. \tag{2}$$

we say the series has long memory.

ARMA processes have short memory; its autocorrelation function eventually decreases exponentially. In fact it can be shown that

$$|\rho(l)| \leq \frac{c}{|\alpha|^l}$$

where $c \geq 0$ and α is the smaller root of the AR polynomial, that of course, it is out of the unit circle, that is $|\alpha| > 1$. But not all stationary processes are short memory processes.

In Lecture 9 we saw that for second order causal linear models we have

$$\sum_{l=0}^{\infty} |\gamma(l)| \leq \sigma^2 (\sum_{r=0}^{\infty} |\psi_r|)^2$$

So, $\sum_{r=0}^{\infty} |\psi_r| < \infty$ is a sufficient condition to guarantee short memory. But recall that

$$\sum_{k=0}^{\infty} |\psi_k|^2 < \infty \tag{3}$$

is weaker than

$$\sum_{k=0}^{\infty} |\psi_k| < \infty \tag{4}$$

Then, there is room for stationary processes with long memory. There exists stationary second order causal linear models such that $\sum_{l=0}^{\infty} |\gamma(l)|$ and $\sum_{r=0}^{\infty} |\psi_r|$ are infinite.

Long memory processes appears in many problems in Finance, Hydrology or Meteorology. Examples of long memory processes are ARIMA (or FARIMA) processes, that we define below. They are stationary second order causal linear models, and so, they satisfy (3) but they have long memory, and so, they not satisfy (4).

Let Z be a centered white noise with variance σ^2. Assume X satisfies

$$(Id - B)^d X = Z.$$

If $d = 0$, X is a white noise, or an ARIMA(0,0,0). If $d = 1$, X is a random walk, that is, an ARIMA(0,1,0). Can we consider a fractional $d \in (0, 1)$? The answer is given by the following theorem, whose proof can be found in [2].

Theorem 2 *Let Z be a centered white noise with variance σ^2. If $d \in (-\frac{1}{2}, \frac{1}{2})$, it exists a unique stationary solution of the equation*

$$(Id - B)^d X = Z$$

with

$$X_j = \sum_{i=0}^{\infty} \psi_i Z_{j-i},$$

and

$$\psi_i = \frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} = \prod_{k=1}^{i} \frac{k-1-d}{k}.$$

In particular,

$$\gamma(0) = \frac{\Gamma(1-2d)\sigma^2}{\Gamma^2(1-d)},$$

$$\rho(l) = \frac{\Gamma(1-d)\Gamma(i+d)}{\Gamma(1+i-d)\Gamma(d)} = \prod_{k=1}^{l} \frac{k-1+d}{k-d},\ l \geq 1$$

and

$$\alpha(l) = \frac{d}{l-d},\ l \geq 1.$$

Recall that Gamma function for $x > 0$ is defined as

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt.$$

According to Stirling formula, for x big enough, we have

$$\Gamma(x) \sim \sqrt{2\pi}\, e^{1-x} (x-1)^{x-\frac{1}{2}}.$$

Using this formula we can determine accurately the speed of convergence of the coefficients of the causal series and the autocorrelations. If j is big enough, the coefficients behave as

$$\psi_j \sim \frac{1}{j^{1-d}\Gamma(d)}$$

Indeed,

$$\begin{aligned}
\frac{\Gamma(i+d)}{\Gamma(i+1)\Gamma(d)} &= \frac{e^{1-d}(i+d-1)^{i+d-\frac{1}{2}}}{\Gamma(d) i^{i+\frac{1}{2}}} \\
&= \frac{1}{\Gamma(d)} e^{1-d} (1 + \frac{d-1}{i})^i \frac{(i+d-1)^{d-\frac{1}{2}}}{i^{\frac{1}{2}}} \\
&= \frac{i^{d-1}}{\Gamma(d)} e^{1-d} (1 + \frac{d-1}{i})^i \frac{(i+d-1)^{d-\frac{1}{2}}}{i^{d-\frac{1}{2}}} \\
&= \frac{i^{d-1}}{\Gamma(d)} e^{1-d} (1 + \frac{d-1}{i})^i (1 + \frac{d-1}{i})^{d-\frac{1}{2}} \\
&\sim \frac{i^{d-1}}{\Gamma(d)}.
\end{aligned}$$

Similar estimations give

$$\rho(j) \sim \frac{\Gamma(1-d)}{\Gamma(d)} \frac{1}{j^{1-2d}}.$$

Indeed, from

$$\rho(j) = \frac{\Gamma(1-d)}{\Gamma(d)} \frac{\Gamma(j+d)}{\Gamma(1+j-d)}$$

we need only to check

$$\frac{\Gamma(j+d)}{\Gamma(1+j-d)} \sim \frac{1}{j^{1-2d}}.$$

We have

$$\begin{aligned}
\frac{\Gamma(j+d)}{\Gamma(1+j-d)} &= \frac{e^{1-2d}(j+d-1)^{j+d-\frac{1}{2}}}{(j-d)^{j-d+\frac{1}{2}}} \\
&= \frac{e^{1-2d}(j-d+2d-1)^{j-d+\frac{1}{2}+2d-1}}{(j-d)^{j-d+\frac{1}{2}}} \\
&= \frac{(j-d+2d-1)^{j-d+\frac{1}{2}} e^{1-2d}(j+d-1)^{2d-1}}{(j-d)^{j-d+\frac{1}{2}}} \\
&= (1+\frac{2d-1}{j-d})^{j-d} e^{1-2d}(j+d-1)^{2d-1}(1+\frac{2d-1}{j-d})^{\frac{1}{2}} \\
&= (j+d-1)^{2d-1}(1+\frac{2d-1}{j-d})^{\frac{1}{2}} \\
&= \frac{(j+d-1)^{2d-\frac{1}{2}}}{(j-d)^{\frac{1}{2}}} \\
&= \frac{(j-d+2d-1)^{2d-\frac{1}{2}}}{(j-d)^{2d-\frac{1}{2}}(j-d)^{1-2d}} \\
&= \frac{1}{(j-d)^{1-2d}}(1+\frac{2d-1}{j-d})^{2d-\frac{1}{2}} \\
&= \frac{1}{j^{1-2d}}(1-\frac{d}{j})^{2d-1}(1+\frac{2d-1}{j-d})^{2d-\frac{1}{2}}.
\end{aligned}$$

The proof finishes noticing that $(1-\frac{d}{j})^{2d-1}$ and $(1+\frac{2d-1}{j-d})^{2d-\frac{1}{2}}$ converge to 1.

Note that the series of ψ_j is divergent in absolute value for any $d \in [0,1)$. If $d < 0$ the series is convergent and the model is of short memory.

In relation with coefficient $|\rho(l)|$, note that for $d \in (0, \frac{1}{2})$ the sequence $\rho(l)$ converge to 0 in absolute value but $1-2d \in (0,1]$ and so

$$\sum_{l=0}^{\infty} |\rho(l)| = \infty.$$

Note that when d approaches $\frac{1}{2}$ the long memory effect increases. On the contrary, for $d \geq 0$ the model is a short memory one. If $d \geq \frac{1}{2}$ it can be shown the model is non stationary.

Definition 3 *A process X is an ARFIMA (p,d,q) model with $-\frac{1}{2} < d < \frac{1}{2}$ if and only if $(Id-B)^d X$ is an ARMA(p,q). That is, if X is a stationary process that satisfies*

$$\Phi_p(B)(Id-B)^d X = \Theta_q(B)Z.$$

Note that we can write

$$\Phi_p(B)X = \Theta_q(B)(Id - B)^{-d}Z$$

so, we can interpret that an ARFIMA model is nothing more than an ARMA(p,q) model with a fractional noise given by $(Id - B)^{-d}Z$.

In relation with parameter estimation of ARFIMA models we have the following usual techniques

1. The mean is estimated by the empirical mean, but now this estimator is not asimptotically normal because the sum of the absolute values of the autocorrelations is not convergent.
2. The autocorrelation function is estimated by $\hat{\rho}$, but its asymptotic behaviour as estimator is not clear.
3. The d index and the parameters are estimated using least squares or maximum of likelihood.

References

[1] P. J. Brockwell and R. A. Davis (1996): *Introduction to Time Series and Forecasting.* Springer.

[2] W. Palma (2007): *Long memory time series, theory and methods.* Wiley.