4th Assignment - Optimization - Constrained opt, Lagrange multipliers

Exercise 4.1: Using necessary and sufficient conditions, solve the following optimization problem in terms of the parameter $\beta > 0$;

$f(x) = (x_1 - 1)^2 + x_2^2 \to \min$, $x \in \mathbb{R}^2$, subject to $h(x_1, x_2) = -x_1 + \beta x_2^2 \geq 0$

Interpret the solutions geometrically in terms of the level curves and the restrictions

Using the KKT conditions: the lagrangian for the problem is

$$L(x_1, x_2, \lambda) = (x_1 - 1)^2 + x_2^2 - \lambda(-x_1 + \beta x_2^2)$$

- $\frac{\partial L}{\partial x_1} = 2(x_1 - 1) + \lambda = 0 \Rightarrow \boxed{x_1 = 1 - \lambda/2}$ (1)
- $\frac{\partial L}{\partial x_2} = 2x_2 - 2\lambda\beta x_2 = 0 \Rightarrow \boxed{2x_2(1 - \lambda\beta) = 0}$ (2)
- $\boxed{\lambda(-x_1 + \beta x_2^2) = 0}$ (3)

If $\lambda = 0$: from eq.(2) $\boxed{x_2 = 0}$ and from eq.(1) $\boxed{x_1 = 1}$

If $\lambda \neq 0$: from eq.(3) $\boxed{x_1 = \beta x_2^2}$ (4)

$\overset{(1)}{\Rightarrow} 2\beta x_2^2 - 2 + \lambda = 0$ and $2x_2(1 - \lambda\beta) = 0$

If $x_2 = 0$: $\boxed{\lambda = 2}$ and from eq.(1) $\boxed{x_1 = 0}$

If $x_2 \neq 0$: $\boxed{\lambda = \frac{1}{\beta}}$ $\overset{eq.(1)}{\Rightarrow}$ $\boxed{x_1 = 1 - \frac{1}{2\beta}}$

↪ ≥ 0 since $\beta > 0$ $\overset{eq.(4)}{\Rightarrow}$ $\boxed{x_2 = \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}}$

Therefore the coordinates are:

$(1, 0, 0): h(1, 0) = -1 < 0$ not feasible

$$\left.\begin{array}{l} (0, 0, 2): h(0, 0) = 0 \\ \left(1 - \frac{1}{2\beta}, \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}, \frac{1}{\beta}\right) \; h\left(1 - \frac{1}{2\beta}, \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}\right) = 0 \end{array}\right\} \text{feasible}$$

→

Let's now find the local minima:

$$\nabla^2_{xx} L = \begin{pmatrix} 2 & 0 \\ 0 & 2(1-\lambda b) \end{pmatrix} \text{ and } \nabla_x h = \begin{pmatrix} -1 \\ 2bx_2 \end{pmatrix}$$

▶ $(0, 0, 2)$:

$$\nabla^2_{xx} L(0,0,2) = \begin{pmatrix} 2 & 0 \\ 0 & 2(1-2b) \end{pmatrix}$$

$$\nabla_x h(0,0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

- Let's say we have $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ such that $z^T \nabla_x h = 0$

$$\Rightarrow (z_1 \quad z_2) \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -z_1 = 0$$

↳ So the vectors that satisfy the above are in the following form: $\begin{pmatrix} 0 \\ z_2 \end{pmatrix}$

- Let's check these vector so $z^T \nabla^2_{xx} L \, z \geq 0$

$$(0 \quad z_2) \begin{pmatrix} 2 & 0 \\ 0 & 2(1-2b) \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \end{pmatrix} = (0 \quad 2z_2(1-2b)) \begin{pmatrix} 0 \\ z_2 \end{pmatrix}$$

$$= 2z_2^2(1-2b)$$

So, $2z_2^2(1-2b) \geq 0$ only when $\boxed{b \leq \frac{1}{2}}$

Because $\lambda = 2$, the sufficient conditions are satisfied when $b \leq \frac{1}{2} \Rightarrow \boxed{(0,0)}$ is a strict local minima

▶ $\left(1 - \frac{1}{2b}, \pm\sqrt{\frac{1}{b} - \frac{1}{2b^2}}, \frac{1}{b}\right)$ this exist only $b \geq \frac{1}{2}$

$$\nabla^2_{xx} L\left(1-\frac{1}{2b}, \pm\sqrt{\frac{1}{b}-\frac{1}{2b^2}}, 1/b\right) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\nabla_x h\left(1-\frac{1}{2b}, \pm\sqrt{\frac{1}{b}-\frac{1}{2b^2}}\right) = \begin{pmatrix} -1 \\ \pm 2b\sqrt{\frac{1}{b}-\frac{1}{2b^2}} \end{pmatrix}$$

(For $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x h = 0 \rightarrow (z_1 \quad z_2) \begin{pmatrix} -1 \\ \pm 2b\sqrt{\frac{1}{b}-\frac{1}{2b^2}} \end{pmatrix} = 0$

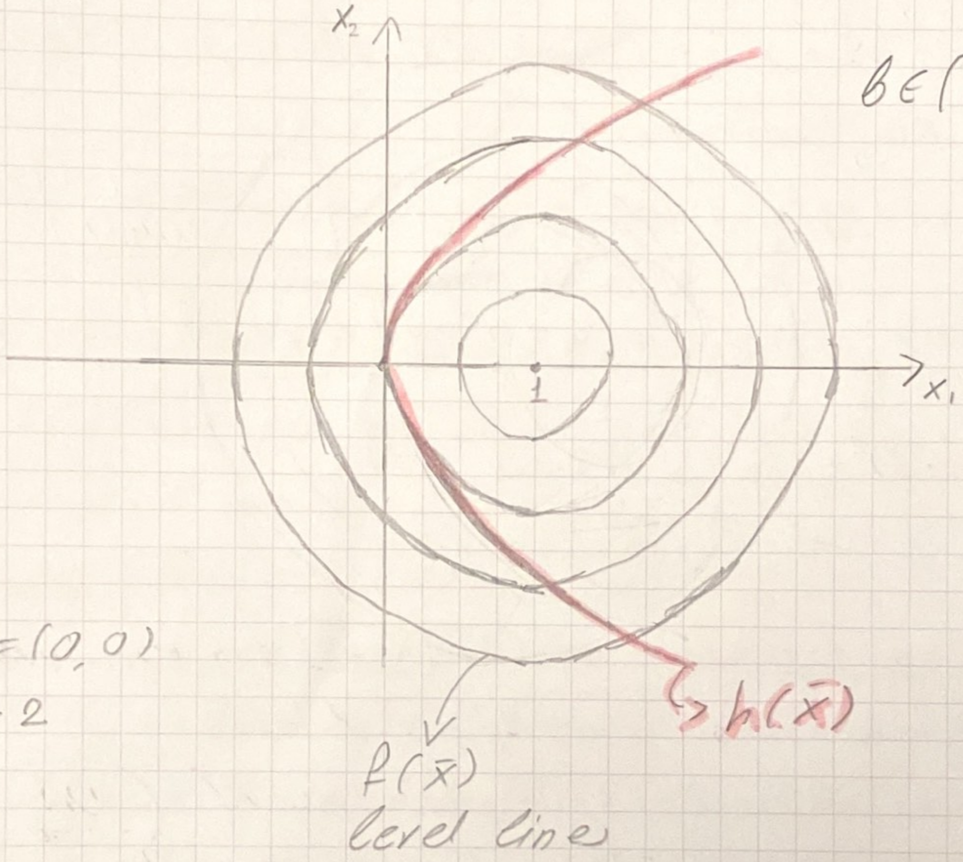
$\Rightarrow -z_1 \pm 2 z_2 b\sqrt{\frac{1}{b} - \frac{1}{2b^2}} = 0$)

→

Now, $(z_1 \ \ z_2)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (2z_1 \ \ 0)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2z_1^2 \geq 0 \ \forall \ z \in \mathbb{R}^n$

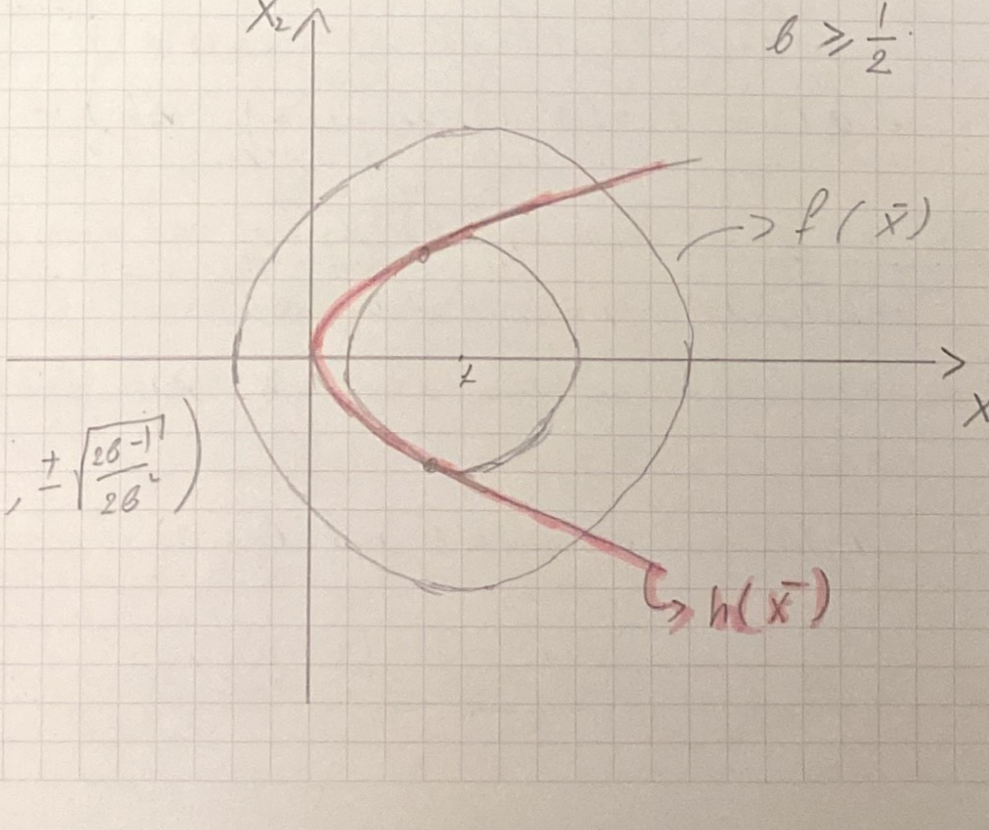
Also $\frac{1}{b} > 0$ because $b > 0$ therefore the points will be strict local minima.

Geometrically:



$x^* = (0, 0)$

$\lambda = 2$



$x^* = \left(1 - \frac{1}{2b}, \pm\sqrt{\frac{2b-1}{2b^2}}\right)$

$\lambda = \frac{1}{b}$

Exercise 4.2: Using necessary and sufficient conditions solve the following opt. problem: for $x=(x_1,x_2)$, $f(x)=x_1 \rightarrow min$ subject to mixed constraints

$g_1(x)=(x_1-3)^2+(x_2-2)^2-13=0$, $h_1(x)=16-(x_1-4)^2-x_2^2 \geq 0$

Lagrangian function: $L(x,\lambda,\mu)=f(x)-\lambda g_1(x)-\mu h_1(x)$

$L(x,\lambda,\mu)=x_1-\lambda[(x_1-3)^2+(x_2-2)^2-13]-\mu[16-(x_1-4)^2-x_2^2]$

The KKT conditions:

$$\frac{\partial L}{\partial x_1}=1-2\lambda(x_1-3)+2\mu(x_1-4)=0 \Rightarrow \boxed{x_1=\frac{8\mu-6\lambda-1}{2(\mu-\lambda)}} \ (1) \quad \mu\neq\lambda$$

$$\frac{\partial L}{\partial x_2}=-2\lambda(x_2-2)+2\mu x_2=0 \Rightarrow \boxed{x_2=-\frac{2\lambda}{\mu-\lambda}} \ (2) \quad \mu\neq\lambda$$

$$(x_1-3)^2+(x_2-2)^2-13=0 \quad (3)$$

$$\mu[16-(x_1-4)^2-x_2^2]=0 \quad (4)$$

If $\mu=0, \lambda>0$: from eq. (2) $\boxed{x_2=2}$ and from eq. (3) $\boxed{x_1=3\pm\sqrt{13}}$

Will check those values to see if $h_1(x)\geq 0$

$\Rightarrow h_1(3+\sqrt{13},2)=16-(3+\sqrt{13}-4)^2-4>0$ ✓

$\Rightarrow h_1(3-\sqrt{13},2)=16-(3-\sqrt{13}-4)^2-4<0$ ✗

$\Rightarrow g_1(3+\sqrt{13},2)=(3+\sqrt{13}-3)^2-13=0$ ✓

$\Rightarrow g_1(3-\sqrt{13},2)=(3-\sqrt{13}-3)^2-13=0$ ✓

Only $(3+\sqrt{13},2)$ is feasible

If $\lambda=0, \mu>0$: from eq. (2) $\boxed{x_2=0}$ and from eq. (3) $\boxed{x_1=0}$, $x_1=-6$

$\Rightarrow h(0,0)=16-16=0$ ✓ feasible

$\Rightarrow h(-6,0)=16-(-10)^2=-84<0$ ✗ not feasible

$\Rightarrow g(0,0)=9+4-13=0$ ✓

So, $(0,0)$ is acceptable and $(-6,0)$ is rejected

If $\mu \neq 0$ & $\lambda \neq 0$: from eq.(4) then $16-(X_1-4)^2-X_2^2=0$

$\Rightarrow X_2^2 = (4+X_1-4)(4-X_1+4) = X_1(8-X_1)$

$\Rightarrow \boxed{X_2 = \pm\sqrt{X_1(8-X_1)}}$ (5)

Substituting in (3)

$(X_1-3)^2+(\sqrt{X_1(8-X_1)}-2)^2-13=0$

$\Rightarrow X_1^2-6X_1\cancel{+9}+X_1(8-X_1)-4\sqrt{X_1(8-X_1)}\cancel{+4}\cancel{-13}=0$

$\Rightarrow \cancel{X_1^2}-6X_1+8X_1-\cancel{X_1^2}-4\sqrt{X_1(8-X_1)}=0$

$\Rightarrow 2X_1 = 4\sqrt{X_1(8-X_1)} \Rightarrow X_1^2 = 4X_1(8-X_1)$

$\Rightarrow X_1^2-32X_1+4X_1^2=0 \Rightarrow 5X_1^2-32X_1=0$

$\Rightarrow X_1(5X_1-32)=0 \Rightarrow X_1=0$ or $X_1 = 32/5$

The same if we use $X_2 = -\sqrt{X_1(8-X_1)}$

If $X_1=0$: from eq. (5) $\boxed{X_2=0}$

$\Rightarrow$ (0,0) is feasible the same as before

If $X_1 = 32/5$: from eq.(5) $\boxed{X_2 = \frac{16}{5}}$ and $X_2 = -\frac{16}{5}$

let's check these values:

$h(32/5, \frac{16}{5}) = 0$ ✓

$h(32/5, -\frac{16}{5}) = 0$ ✓

$g(32/5, 16/5) = 0$ ✓

$g(32/5, -16/5) = 128/5 \neq 0$ ✗

Only the $(32/5, 16/5)$ is feasible

Therefore the coordinates are:

- For $X_1 = 3+\sqrt{13}$, $X_2 = 2$, $\mu=0 \overset{(1)}{\Rightarrow} \lambda = \frac{1}{2\sqrt{13}}$: $(3+\sqrt{13}, 2, \frac{1}{2\sqrt{13}}, 0)$

- For $X_1=0$, $X_2=0$, $\lambda=0 \overset{(1)}{\Rightarrow} \mu = 1/8$: $(0, 0, 0, 1/8)$

- For $X_1 = \frac{32}{5}$, $X_2 = \frac{16}{5} \xrightarrow{(1),(2)} \mu = 3/40, \lambda = 1/5$: $(\frac{32}{5}, \frac{16}{5}, 1/5, 3/40)$

Let's find the minima:

$$\nabla^2_{xx} L(x_1, x_2, \lambda, \mu) = \begin{pmatrix} -2\lambda+2\mu & 0 \\ 0 & -2\lambda+2\mu \end{pmatrix}$$

$$\nabla_x h(x_1, x_2) = \begin{pmatrix} -2(x_1-4) \\ -2x_2 \end{pmatrix} \qquad \nabla_x g(x_1, x_2) = \begin{pmatrix} 2(x_1-3) \\ 2(x_2-2) \end{pmatrix}$$

▶ $(3+\sqrt{13}, 2, \frac{1}{2\sqrt{13}}, 0) = x_1^*$

$$\nabla^2_{xx} L(x_1^*) = \begin{pmatrix} -1/\sqrt{13} & 0 \\ 0 & -1/\sqrt{13} \end{pmatrix}$$

$$\nabla_x h(3+\sqrt{13}, 2) = \begin{pmatrix} 2-\sqrt{13} \\ -4 \end{pmatrix}$$

$$\nabla_x g(3+\sqrt{13}, 2) = \begin{pmatrix} 2\sqrt{13} \\ 0 \end{pmatrix}$$

Take $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x g(3+\sqrt{13}, 2) = 0$

(We don't compute the $z^T \nabla h(x_1^*)$ because when $\mu = 0$ is not active)

$$\Rightarrow (z_1 \quad z_2) \begin{pmatrix} 2\sqrt{13} \\ 0 \end{pmatrix} = 2\sqrt{13}\, z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow \begin{pmatrix} 0 \\ z_2 \end{pmatrix}$$

$(0\ z_2)$ is the vector that satisfy the above, let's now check if $(0, z_2) \Rightarrow z^T \nabla^2_{xx} L(x_1^*) z \geq 0$

$$(0, z_2) \begin{pmatrix} -1/\sqrt{13} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \end{pmatrix} = -z_2^2 \frac{1}{\sqrt{13}} < 0$$

Therefore $(3+\sqrt{13}, 2)$ <u>is not</u> a local minimum

~~▶ $\left(3-\sqrt{13}, 2, -\frac{1}{2\sqrt{13}}, 0\right) = x_2^*$~~

~~$\nabla^2_{xx} L(x_2^*) = \begin{pmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{pmatrix}$ $\nabla_x h(x_2^*) = \begin{pmatrix} 2+\sqrt{13} \\ -4 \end{pmatrix}$ $\nabla_x g(x_2^*) = \begin{pmatrix} -2\sqrt{13} \\ 0 \end{pmatrix}$~~

~~So $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$: $(z_1 \ z_2)\begin{pmatrix} -2\sqrt{13} \\ 0 \end{pmatrix} = -2z_1\sqrt{13} = 0 \Rightarrow z_1 = 0$~~

In similar way as x_1^*: $(3-\sqrt{13}, 2)$ local minima

▶ $(0, 0, 0, 1/8) = x_3^*$

$\nabla^2_{xx} L(x_3^*) = \begin{pmatrix} 2/8 & 0 \\ 0 & 2/8 \end{pmatrix}$ $\nabla_x h(x_3^*) = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ $\nabla_x g(x_3^*) = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$

Take $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x h(x_3^*) = 0$, $z^T \nabla_x g(x_3^*) = 0$

$$\left.\begin{array}{l} \Rightarrow (z_1 \ z_2)\begin{pmatrix} 8 \\ 0 \end{pmatrix} = 8z_1 = 0 \Rightarrow z_1 = 0 \\ \Rightarrow (z_1 \ z_2)\begin{pmatrix} -6 \\ -4 \end{pmatrix} = -6z_1 - 4z_2 = 0 \Rightarrow z_2 = 0 \end{array}\right\} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

the vector that satisfy

So $(0,0)$ is local minima

▶ $\left(\frac{32}{5}, \frac{16}{5}, \frac{1}{5}, \frac{3}{40}\right) = x_4^*$

$\nabla^2_{xx} L(x_4^*) = \begin{pmatrix} -1/4 & 0 \\ 0 & -1/4 \end{pmatrix}$, $\nabla_x h(x_4^*) = \begin{pmatrix} -24/5 \\ -32/5 \end{pmatrix}$, $\nabla_x g(x_4^*) = \begin{pmatrix} 34/5 \\ 12/5 \end{pmatrix}$

$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x h(x_4^*) = 0$ & $z^T \nabla_x g(x_4^*) = 0$

$$\left.\begin{array}{l} \Rightarrow (z_1 \ z_2)\begin{pmatrix} -24/5 \\ -32/5 \end{pmatrix} = -\frac{24}{5}z_1 - \frac{32}{5}z_2 = 0 \\ \Rightarrow (z_1 \ z_2)\begin{pmatrix} 34/5 \\ 12/5 \end{pmatrix} = \frac{34}{5}z_1 + \frac{12}{5}z_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} z_1 = 0 \\ z_2 = 0 \end{array}$$

and $\left(\frac{32}{5}, \frac{16}{5}\right)$ is a local minima