Master in Foundations of Data Science 2017-2018

NUMERICAL LINEAR ALGEBRA

Final exam, January 12th, 2018, from 15:00h till 19:00h.

All answers should be suitably justified. Please write each problem in a separate sheet of paper.

1. Given a matrix $A \in \mathbb{R}^{n\times n}$, Gaussian elimination with partial pivoting (GEPP) computes a factorization

$$A = PLU$$

where P is a permutation matrix, L is lower triangular, and U is upper triangular.

(1) Write down the GEPP algorithm in pseudocode notation and describe how it works.

(2) Prove that the complexity of this algorithm in terms of floating point operations *(flops)* is bounded by $\frac{2}{3}n^3 + O(n^2)$.

(3) Describe how to solve $Ax = b$ from the PLU factorization of the matrix A.

(4) Give a concrete example of a matrix for which Gaussian elimination without pivoting is possible but numerical unstable.

2. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$$

(1) Compute its QR factorization using Householder reflexions.

(2) Compute the same factorization, but this time using Givens rotations instead of reflexions.

3. Consider the singular value decomposition (SVD)

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(1) Compute the best rank 1 approximation of the matrix A with respect to the operator norm $\|\cdot\|_2$ associated to the Euclidean metric.

(2) Compute the vector $x_{min} \in \mathbb{R}^2$ that solves the least squares problem

$$\|Ax_{min} - b\|_2 = \min_{x\in\mathbb{R}^2} \|Ax - b\|_2$$

for the vector $b = (1, 2, 0) \in \mathbb{R}^3$.

4. Let $A \in \mathbb{R}^{n\times n}$ a matrix whose eigenvalue ssatisfy the strict inequalities

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

(1) Which iterative algorithm would you apply to compute the eigenvalue λ_1 of largest absolute value? Give an estimate for its rate of convergence.

(2) How would you proceed to compute the eigenvalue λ_n of smallest absolute value? Similarly as before, give an estimate for its rate of convergence.

5. Consider the matrix and the vector

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(1) For this matrix and vector, compute the corresponding iterative scheme given by (a) the Jacobi method, (b) the Gauss-Seidel method, and (c) the SOR(ω) method, for an arbitrary parameter $\omega \in \mathbb{R}$.

(2) Check if you can guarantee that the Jacobi and Gauss-Seidel schemes for this system converge, using the criterium based on the spectral radius.