15/03/2024

Lecture 6: The AR(1) model

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April 30, 2022

1 Introduction

The family of ARMA(p,q) models (autoregressive and moving average models) is the most famous family of stationary stochastic processes used to describe time series. The theory of ARMA models was developed by G. Box i G. Jenkins during the seventies, see [1] or [2]. A particular subfamily is the family of AR(p) models. In this lecture we present the most simple AR model, the AR(1) model.

Before to develop the theory we need to recall some ideas related to mean square convergence. This is what we do in the first section.

2 Some results related with mean square convergence

Let $\{T_k,\ k \geq 1\}$ be a sequence of square integrable random variables. Consider the following definition:

Definition 2.1 *It is said that $\{T_k,\ k \geq 1\}$ converges in L^2, or in quadratic mean, to a square integrable random variable T, if and only if*

$$\mathbb{E}[(T - T_k)^2] \stackrel{n\uparrow\infty}{\longrightarrow} 0.$$

A well-known criterion to determine if a sequence $\{T_k,\ k \geq 1\}$ is convergent in L^2 is the following Cauchy type criterion:

Lemma 2.2 *It exist a square integrable random variable T such that*

$$T_k \underset{k\uparrow\infty}{\stackrel{L^2}{\longrightarrow}} T$$

if and only if

$$\mathbb{E}[(T_n - T_m)^2] \stackrel{n,m\uparrow\infty}{\longrightarrow} 0.$$

Proof: The proof, based on the Cauchy criterion for real numbers, can be found in [3], pages 68-69. ■

Note that this lemma says that we can prove that $\{T_k,\ k \geq 1\}$ has a limit in L^2 simply proving that

$$\mathbb{E}[(T_n - T_m)^2] \underset{n,m\to\infty}{\longrightarrow} 0.$$

Definition 2.3 *It is said that a series of square integrable random variables*

$$\sum_{k=1}^{\infty} U_k$$

converges in L^2 if its partial sums

$$T_N := \sum_{k=1}^{N} U_k$$

converge in L^2 to a certain square integrable random variable that we can call T. In this case we write

$$T := \sum_{k=1}^{\infty} U_k$$

The following proposition characterizes the L^2 convergent series.

Corollary 2.4 *If $\{U_k, k \geq 1\}$ is a sequence of centered and pairwise uncorrelated square-integrable random variables we have*

$$\sum_{k\geq 1} U_k \text{ is } L^2\text{-convergent} \Leftrightarrow \sum_{k\geq 1} \mathbb{V}(U_k) < \infty$$

Proof:

$$\begin{aligned} \sum_{k\geq 1} U_k \text{ is } L^2\text{-convergent} \quad &\Leftrightarrow \quad \mathbb{E}[(T_N - T_M)^2] = \mathbb{E}\big\{\big(\sum_{k=N+1}^{M} U_k\big)^2\big\} \underset{N,M\uparrow\infty}{\longrightarrow} 0 \\ &\Leftrightarrow \quad \sum_{k=N+1}^{M} \mathbb{V}(U_k) \underset{N,M\uparrow\infty}{\longrightarrow} 0. \end{aligned}$$

And by the Cauchy criterion for series of positive numbers, the last convergence is equivalent to

$$\sum_{k\geq 1} \mathbb{V}(U_k) < \infty.$$

■

With the previous notions we can prove the following lemma that will allow us to compute expectations and variances of infinite sums of series of square integrable random variables.

Lemma 2.5 *If $T_k \xrightarrow{L^2} T$ and $S_k \xrightarrow{L^2} S$ we have*

1. $\mathbb{E}[T_k] \longrightarrow \mathbb{E}[T]$
2. $\mathbb{E}[T_k^2] \longrightarrow \mathbb{E}[T^2]$, *and in particular,* $\mathbb{V}[T_k] \longrightarrow \mathbb{V}[T]$.
3. $\mathbb{E}[T_k \cdot S_k] \longrightarrow \mathbb{E}[T \cdot S]$, *and in particular,* $\mathbb{C}(T_k, S_k) \longrightarrow \mathbb{C}(T, S)$.

Proof:

1. $|E[T]-E[T_k]|=|E[T-T_k]|\leq E|T-T_k|\leq(E|T-T_k|^2)^{1/2} \underset{k\uparrow\infty}{\longrightarrow} 0$ by hypothesis.

2. Notice that
$$(T-T_k)^2 = T^2+T_k^2-2T_kT = T_k^2-T^2+2T(T-T_k).$$
Then
$$T_k^2-T^2 = (T-T_k)^2-2T(T-T_k)$$
and so,
$$\begin{aligned} |E[T_k^2]-E[T^2]| &\leq E(T-T_k)^2+2E|T||T-T_k| \\ &\leq E(T-T_k)^2+2E(T^2)^{1/2}(E(T-T_k)^2)^{1/2} \underset{k\uparrow\infty}{\longrightarrow} 0 \end{aligned}$$
The convergence of variances is immediate from this result and item (1) because $\mathbb{V}(T) = \mathbb{E}(T^2)-(\mathbb{E}T)^2$.

3.
$$|E[T\cdot S]-E[T_k\cdot S_k]| \leq |E(T(S-S_k)]+E[(T-T_k)S_k]|$$
$$\leq (E|T|^2)^{1/2}(E|S-S_k|^2)^{1/2}+(E|T-T_k|^2)^{1/2}(E|S_k|^2)^{1/2}$$
and this converges to 0 because, from the previous item,
$$E(S_k^2)\longrightarrow E(S^2)<\infty.$$
The convergence of the covariances is immediate from this result, previous items and the fact that $\mathbb{C}(S,T) := \mathbb{E}(S\cdot T)-\mathbb{E}(S)\mathbb{E}(T)$.

■

3 The AR(1) model

Model AR(1) is a very simple and natural model for a time series, and in some sense, as we are going to see, is an extension of the idea of random walk.

Its basic building block is the white noise process, so, from now on, $Z=\{Z_j, j\in\mathbb{Z}\}$ will be a centered white noise with variance $\sigma^2>0$. We will write
$$Z\sim \mathrm{WN}(0,\sigma^2).$$

Assume that every variable Y_j of our model is a linear function of the previous one, Y_{j-1}, plus an independent innovation represented by a variable Z_j of a white noise. More concretely, we say that $Y=\{Y_j,\ j\in\mathbb{Z}\}$ is an AR(1) process if it a stationary process and satisfies the equation
$$Y_j=\phi Y_{j-1}+Z_j,\quad j\in\mathbb{Z},\quad \phi\in(-1,1). \tag{3.1}$$

Notice that iterating Equation (3.1) we have

$$
\begin{aligned}
Y_j &= \phi(\phi Y_{j-2} + Z_{j-1}) + Z_j \\
&= \phi^2 Y_{j-2} + \phi Z_{j-1} + Z_j \\
&= \cdots \\
&= Z_j + \phi Z_{j-1} + \phi^2 Z_{j-2} + \cdots + \phi^k Z_{j-k} + \phi^{k+1} Y_{j-k-1} \\
&= \sum_{i=0}^{k} \phi^i Z_{j-i} + \phi^{k+1} Y_{j-k-1}.
\end{aligned}
$$

Notice also that $\phi^i Z_{j-i}$ are centered and pairwise uncorrelated square integrable random variables and therefore, using Corollary 2.4 we have

$$
\sum_{i=0}^{\infty} \phi^i Z_{j-i} < \infty \quad (L^2) \Longleftrightarrow \sum_{i=0}^{\infty} \mathbb{V}(\phi^i Z_{j-i}) = \sum_{i=0}^{\infty} \phi^{2i} \sigma^2 = \frac{\sigma^2}{1-\phi^2} < \infty,
$$

and so, if $|\phi| < 1$ we have

$$
Y_j = \sum_{i=0}^{\infty} \phi^i Z_{j-i} \quad (L^2)
$$

because

$$
\mathbb{E}|Y_j - \sum_{i=0}^{k} \phi^i Z_{j-i}|^2 = \phi^{2k+2} \mathbb{E}(Y_{j-k-1}^2),
$$

using the stationarity, $\mathbb{E}(Y_{j-k-1}^2)$ is a constant, and $|\phi|^{2k+2}$ converges to 0 when $k \uparrow \infty$ because $|\phi| < 1$.

As a summary, an AR(1) model is a well defined second order stationary process. Its expression as a square integrable infinite series allows to determine immediately its properties. Concretely, observe that for any $j \in \mathbb{Z}$ we have

1. $\mathbb{E}[Y_j] = 0$

2. $\mathbb{V}(Y_j) = \frac{\sigma^2}{1-\phi^2} > 0$

3. $$
\begin{aligned}
\gamma(l) &= \mathbb{C}(Y_j, Y_{j+l}) \\
&= \mathbb{C}\left(\sum_{r=0}^{\infty} \phi^r Z_{j-r}, \ \sum_{s=0}^{\infty} \phi^s Z_{j+l-s} \right) \\
&= \sum_{r,s=0}^{\infty} \phi^r \phi^s \mathbb{C}(Z_{j-r}, \ Z_{j+l-s}) \\
&= \sum_{r,s \geq 0} \phi^{r+s} \sigma^2 \mathbb{1}_{\{-r=l-s\}} \\
&= \sum_{r=0}^{\infty} \phi^{l+2r} \sigma^2 = \frac{\sigma^2 \phi^l}{1-\phi^2}.
\end{aligned}
$$

4. The auto-correlation function is $\rho(l) = \phi^l, \quad l \geq 0$.

So, the AR(1) model is a centered second order stationary model because the expectation is null, the variance is constant and the autocorrelation depend only on the lag l

Remark 3.1 *What happens if we assume $|\phi| = 1$?*
If $\phi = 1$ or -1 the series is not stationary. Note that it satisfies the equation

$$Y_j = Y_{j-1} + Z_j, \quad j \in \mathbb{Z},$$

that is exactly the equation of a random walk as we saw in Lecture 2, or the equation

$$Y_j = -Y_{j-1} + Z_j, \quad j \in \mathbb{Z},$$

that is also a non stationary process.

Remark 3.2 *And what happens if $|\phi| > 1$? We have*

$$Y_j = \phi Y_{j-1} + Z_j$$

and so

$$Y_{j-1} = \frac{1}{\phi} Y_j - \frac{1}{\phi} Z_j.$$

Then,

$$\begin{aligned} Y_j &= \frac{1}{\phi} Y_{j+1} - \frac{1}{\phi} Z_{j+1} \\ &= \frac{1}{\phi}\left(\frac{1}{\phi} Y_{j+2} - \frac{1}{\phi} Z_{j+2}\right) - \frac{1}{\phi} Z_{j+1} \\ &= \cdots \\ &= \frac{1}{\phi^k} Y_{j+k} - \sum_{i=1}^{k} \frac{1}{\phi^i} Z_{j+i} \end{aligned}$$

If $|\phi| > 1$, $|1/\phi| < 1$ and therefore,

$$Y_j = -\sum_{i=1}^{\infty} \frac{1}{\phi^i} Z_{j+i} \,(L^2). \tag{3.2}$$

This model is nonsense because we are saying that Y_j depends on the future values of the white noise, but note that from formula (3.2) we can prove that its variance is

$$\frac{\sigma^2}{\phi^2 - 1}$$

and its auto-correlation function is

$$\rho(l) = \phi^{-l}, \quad l \geq 0.$$

So, an AR(1) with parameters ϕ and σ^2, with $|\phi| > 1$, is equivalent, as a second order stationary process, to an AR(1) with parameters $\frac{1}{\phi}$ and $\frac{\sigma^2}{\phi^2}$.

4 Second order causal linear series

The structure of AR(1) model we have found reinforces the interest of next definition.

Definition 4.1 *Given Z, we define a second order causal linear series as a series*

$$Y_j = \sum_{k=0}^{\infty} \psi_k Z_{j-k}, \quad j \in \mathbb{Z}$$

such that

$$\sum_{k=0}^{\infty} \psi_k^2 < \infty.$$

This series is a second order series because it is convergent in the L^2-sense, is causal because it depends only on previous values of the white noise and linear by obvious reasons.

From the previous expression we can obtain immediately its properties as a stationary process. It is obviously centered and its auto-covariance is

$$\mathbb{C}\left(\sum_{r=0}^{\infty} \psi_r Z_{j-r}, \sum_{s=0}^{\infty} \psi_s Z_{j+l-s}\right) = \sum_{r,s\geq 0} \psi_r \psi_s \sigma^2 \mathbb{1}_{\{-r=l-s\}} = \sum_{r=0}^{\infty} \psi_r \psi_{r+l} \sigma^2.$$

In particular, its variance is

$$\mathbb{V}(Y_j) = \sigma^2 \sum_{r=0}^{\infty} \psi_r^2,$$

ans its auto-correlation function is given by

$$\rho(l) = \frac{\sum_{r=0}^{\infty} \psi_r \psi_{r+l}}{\sum_{r=0}^{\infty} \psi_r^2}.$$

In the AR(1) case, $\psi_r = \phi^r$ and we recuperate immediately $\rho(l) = \phi^l$.

References

[1] G. Box and G. Jenkins (1970): *Time Series Analysis: Forecasting and Control.* Holden-Day.

[2] G. Box, G. Jenkins and G. Reinsel (2015): *Time Series Analysis: Forecasting and Control.* Wiley.

[3] P. J. Brockwell and R. A. Davis (1991): *Time Series: Theory and Methods.* Springer.