Lecture 9: Statistics of stationary time series

Josep Vives

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1 Introduction

Assume we have a time series $y_1, \ldots, y_n$, supposed to be stationary. We want to propose a second order linear causal model to describe it. That is, a model

$$Y_j = \mu + \sum_{i=0}^{\infty} \psi_i Z_{j-i}$$

where Z is a centered white noise with variance σ^2 and $\sum_{i=0}^{\infty} \psi_i^2 < \infty$. Recall that, in particular, ARMA models are a distinguished class of this type of models. Recall also, as we saw in Lecture 2, that a second order stationary time series is completely determined by its mean and its auto-covariance function, or alternatively, by its mean, its variance and its auto-correlation function.

How to do it? What is the best model to describe our stationary time series data? How to estimate its parameters?

2 Estimating the mean

The standard estimator of the mean μ is the empirical mean

$$\bar{Y}_n := \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

For this estimator, we can obtain similar results as the results obtained by the elementary Mathematical Statistics when the random variables $Y_1, \ldots, Y_n$ are independent and identically distributed. Recall that here, this is not the case. But we will be able, under suitable hypotheses, to prove that our estimator $\bar{Y}_n$ is also consistent in mean square sense and asymptotically normal.

Estimator $\bar{Y}_n$ is obviously centered. In relation with its variance we have

$$\begin{aligned}
\mathbb{V}(\bar{Y}_n) &= \mathbb{C}\left(\frac{1}{n} \sum_{i=1}^{n} Y_i,\ \frac{1}{n} \sum_{j=1}^{n} Y_j \right) \\
&= \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{C}(Y_i, Y_j) \\
&= \frac{1}{n^2} \{ n\gamma(0) + 2(n-1)\gamma(1) + \cdots + 2\gamma(n-1) \} \\
&= \frac{\gamma(0)}{n} + \frac{2}{n} \sum_{l=1}^{n-1} (1 - \frac{l}{n}) \gamma(l).
\end{aligned}$$

Note that in particular we have

$$\mathbb{V}(\bar{Y}_n) \leq \frac{\gamma(0)}{n} + \frac{2}{n}\sum_{l=1}^{n-1}(1-\frac{l}{n})|\gamma(l)| \leq \frac{2}{n}\sum_{l=0}^{\infty}|\gamma(l)|.$$

Proposition 2.1 *If $\sum_{i=0}^{\infty}|\psi_i| < \infty$, $\bar{Y}_n$ is consistent in mean square sense, and so, it is also consistent in probability sense.*

Proof: Note, first of all, that for a second order linear causal model

$$Y_j = \mu + \sum_{i=0}^{\infty} \psi_i\, Z_{j-i}$$

we have, as we saw in Lecture 6,

$$\gamma(l) = \mathbb{C}(Y_j, Y_{j+l}) = \sum_{r=0}^{\infty} \psi_r\, \psi_{l+r}\, \sigma^2.$$

Then,

$$\begin{aligned}
\sum_{l=0}^{\infty}|\gamma(l)| &\leq \sigma^2 \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}|\psi_r|\,|\psi_{l+r}| \\
&= \sigma^2 \sum_{r=0}^{\infty}\sum_{s=r}^{\infty}|\psi_r|\,|\psi_s| \\
&\leq \sigma^2 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|\psi_r|\,|\psi_s| \\
&= \sigma^2 (\sum_{r=0}^{\infty}|\psi_r|)^2
\end{aligned}$$

This means that, under the assumption of the Proposition, $\sum_{l=0}^{\infty}\gamma(l)$ is a constant.
To prove the consistence of $\bar{Y}_n$ in the quadratic mean sense means to prove

$$\mathbb{V}(\bar{Y}_n) \stackrel{n\uparrow\infty}{\longrightarrow} 0.$$

But note that

$$n\,\mathbb{V}(\bar{Y}_n) = \gamma(0) + 2\sum_{l=1}^{n-1}(1-\frac{l}{n})\gamma(l) \stackrel{n\uparrow\infty}{\longrightarrow} \gamma(0) + 2\sum_{l=1}^{\infty}\gamma(l).$$

So, the result follows.
Let's check the last convergence. For any fixed l we have

$$(1-\frac{1}{n})\gamma(l) \stackrel{n\uparrow\infty}{\longrightarrow} \gamma(l),$$

and on other hand,

$$\sum_{l=1}^{n-1}\left|1-\frac{l}{n}\right||\gamma(l)| \leq \sum_{l=1}^{\infty}|\gamma(l)| < \infty.$$

So, dominated convergence guarantees the result. ■

Remark 2.2

1. *Note that not all second order linear causal models satisfy* $\sum_{i=0}^{\infty}|\psi_i| < \infty$ *because* $\sum_{i=0}^{\infty}\psi_i^2 < \infty$ *does not imply* $\sum_{i=0}^{\infty}|\psi_i| < \infty$*. For example, the series* $\sum_{n=1}^{\infty}\frac{1}{n^2}$ *is convergent whereas the series* $\sum_{n=1}^{\infty}\frac{1}{n}$ *is divergent.*

2. *Recall that the reverse is true. Concretely, if* $\sum_{i=0}^{\infty}|\psi_i| < \infty$*, the sequence of coefficients* $|\psi_i|$ *converge to* 0 *and so, after some* n_0*, all of them are less than* 1*. So, we can write*

$$\sum_{i=0}^{\infty}\psi_i^2 = \sum_{i=0}^{n_0}\psi_i^2 + \sum_{i>n_0}^{\infty}\psi_i^2 \leq \sum_{i=0}^{n_0}\psi_i^2 + \sum_{i>n_0}^{\infty}|\psi_i| < \infty.$$

The second key result that we have for $\bar{Y}_n$ is the asymptotic normality.

Proposition 2.3 *Let Z a centered IID noise with variance σ^2. We have*

$$\bar{Y}_n \sim AN(\mu, \sqrt{\frac{V}{n}})$$

where

$$V := \gamma(0) + 2\sum_{l=1}^{\infty}\gamma(l)$$

Proof: Using the representation

$$Y_j = \mu + \sum_{i=0}^{\infty}\psi_i\, Z_{j-i}$$

we have

$$\bar{Y}_n = \mu + \frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{\infty}\psi_i\, Z_{j-i} = \mu + \sum_{i=0}^{\infty}\psi_i(\frac{1}{n}\sum_{j=1}^{n}Z_{j-i}).$$

Using the Central Limit Theorem, and so, using the hypothesis that Z is an IID noise, the variables

$$U_i^{(n)} := \frac{1}{n}\sum_{j=1}^{n}Z_{j-i}$$

have approximately a $N(0, \frac{\sigma^2}{n})$ law and so, $\bar{Y}_n$ will converge to a Gaussian random variable.

Finally, it is enough to check

$$\mathbb{V}(\frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sqrt{V}}) \longrightarrow 1,$$

or equivalently,

$$n\mathbb{V}(\bar{Y}_n - \mu) \longrightarrow V.$$

We have

$$n \cdot \mathbb{V}(\bar{Y}_n - \mu) = n \cdot \mathbb{V}(\bar{Y}_n) = \gamma(0) + 2\sum_{l=1}^{n-1}\left(1 - \frac{l}{n}\right)\gamma(l)$$

and this converges to $V = \gamma(0) + 2\sum_{l=1}^{\infty}\gamma(l)$ as we have seen before. ■

Remark 2.4 *In case of independence of variables Y_j we have $V = \gamma(0)$ and we recuperate the classic result*

$$\bar{Y}_n \sim AN\left(\mu, \sqrt{\frac{\gamma(0)}{n}}\right).$$

This result allows to built confidence intervals for μ. In effect,

$$I_\gamma(\mu) = \left[\bar{Y}_n \pm z(\gamma)\sqrt{\frac{\hat{V}}{n}}\right]$$

is approximately an interval of $100\gamma\%$ confidence, using

$$\hat{V} = \widehat{\gamma}(0) + 2\sum_{k=1}^{\sqrt{n}}(1 - k/n)\,\widehat{\gamma}(k).$$

Recall that $z(\gamma) := \Phi^{-1}(\frac{\gamma+1}{2})$, where Φ is the cumulative probability function of the standard normal distribution.

3 Estimating the autocovariance and autocorrelation functions

The standard estimator for the auto-covariance function is

$$\widehat{\gamma}(l) := \frac{1}{n}\sum_{j=1}^{n-l}(Y_j - \bar{Y}_n)(Y_{j+l} - \bar{Y}_n)$$

for $l \le \frac{n}{3}$ or better, $l \le \log(n)$. The estimator of the variance is the particular case of $\gamma(0)$.

Obviously, a natural estimator for the auto-correlation is

$$\widehat{\rho}(l) := \frac{\widehat{\gamma}(l)}{\widehat{\gamma}(0)}.$$

The most important result in relation with $\hat{\rho}$ is the following asymptotic result

Proposition 3.1 *If Y is a linear causal model with a white noise Z such that $\mathbb{E}(Z_j^4) < \infty$ for any j, then*

$$(\widehat{\rho}(l) - \rho(l)) \sim AN(0, \frac{\omega_l}{n})$$

where

$$\omega_l \cong 1 + 2\sum_{j=1}^{l-1}\rho(j)^2$$

can be estimated by

$$\widehat{\omega}_l := 1 + 2\sum_{j=1}^{l-1} \widehat{\rho}^2(j).$$

This result, of course, allows us to built confidence intervals of type

$$I_\gamma(\rho_l) = \left[\widehat{\rho}(l) \pm z\,(\gamma)\sqrt{\frac{\widehat{\omega}_l}{n}}\right].$$

Remark 3.2 *If Y is a white noise we have*

$$\widehat{\omega}_l = 1$$

and so

$$\sqrt{n}\,\widehat{\rho}(l) \sim N(0,1), \quad l \geq 1,$$

and this is the result used to test the white noise hypothesis to a time series. See Lecture 3.

In relation with the partial auto-correlation function we can estimate $\alpha(l)$ with the last component of the vector x that satisfies

$$\hat{R}_m x = (\hat{\rho}_1, \ldots, \hat{\rho}_l)^t$$

Under AR(p) model we have

$$\hat{\alpha}_m \sim AN(0, \frac{1}{\sqrt{n}}),\ \forall m > p.$$

and this allows to test the hypothesis $\alpha_m = 0$ and so, to test if the model is an AR(p) or not.