Lecture 14: ARCH and GARCH models

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1 Introduction

A centered white noise $Z = \{Z_j, j \in \mathbb{Z}\}$ is the simplest example of second order weak stationary process. Recall that it is a collection of uncorrelated centered random variables with constant variance σ^2. The theory of ARMA processes and its extensions consists in proposing a functional structure to a time series such that residuals become a white noise.

But uncorrelation is weaker than independence. There is still a structure underlying a white noise. Given a white noise we don't know anything about the law of Z_j nor about the law of any vector $(Z_{j_1}, \ldots, Z_{j_m})$ for any m. Note that to assume Z is IID noise, that is, to assume independence and identical distribution between the random variables Z_j, is much more informative, because in that case, we know that the law of the vector is the product of the laws of variables Z_j. And the only pending problem is to estimate this law that can be done from the histogram of the observed data because in that case, observed data constitute a true sample, that is, a collection of independent and identically distributed data. But this is not the case for a white but not IID noise.

On other hand, there is a very important case where to try to describe deeply the structure of a white noise is crucial: prices. According to the Osborne-Samuelson model, that claims that a price S_t has a lognormal law, if X denotes a process of logarithmic prices, its increments have to be a sample of a normal law. That is, if $\{S_n,\ n \geq 0\}$ is a price process and $\{X_n,\ n \geq 1\}$ is the corresponding log-price process $X_n := \log S_n$, then, process $\{Y_n\, n \geq 1\}$, given by random variables

$$Y_n := X_n - X_{n-1},$$

is a $\mathrm{GWN}(\alpha, \sigma^2)$. In other words, X is a $\mathrm{ARIMA}(0,1,0)$ with mean α and variance associated to the corresponding white noise σ^2.

If $Y \sim \mathrm{GWN}(\alpha, \sigma^2)$, we can define

$$Z_n := \frac{Y_n - \alpha}{\sigma}$$

and then, $Z \sim \{Z_j, j \in \mathbb{Z}\}$ is, necessarily, a $\mathrm{GWN}(0,1)$.

The question is: is the Osborne-Samuelson hypothesis true? We can check it doing on Y a test of normality. Usually, the answer is no. Several types of discrepancies frequently appear. The most important are:

1. Nonlinear correlation:

   Certainly, the autocorrelation function of $\{Z_j,\ j \geq 1\}$ is usually null for any lag greater than one, that is Z is usually a white noise. But $\{Z_j^2,\ j \geq 1\}$ or $\{|Z_j|, j \geq 1\}$ show frequently an autocorrelogram not compatible with a white noise. Therefore, Z is a white noise but not an IID noise. In other words, there is no correlation between random variables Z_j but there is correlation between random variables Z_j^2 or $|Z_j|$ and so, nor independence between them.

2. Intermittency:

   Periods of big movements are followed by periods of low movements and viceversa.

3. Empirical kurtosis:

   The empirical kurtosis, that is, the fourth order empirical moment, is greater than the standard normal Gaussian kurtosis, that is equal to 3.

These and other features of price processes are called in the literature *stylized facts*. The question is: can we construct a model for Z able to explain them?

Robert Engle solved the challenge developing during the first 80's the so called autoregressive conditional heteroskedastic (ARCH) models and received the Nobel price for this in 2003; see [2] and [3]. The family of ARCH models was extended to GARCH models (generalized ARCH) later by Tim Bollerslev, see [1]. See [4] for a presentation of the theory of GARCH models.

Therefore, the main application of GARCH models is to describe the probabilistic structure of time series that show a white noise but not an IID noise structure. In Finance, increments of log prices show frequently this fact. On other hand, after adjusting a SARIMA or a FARIMA model to a time series, sometimes the residual series becomes a white noise but not an IID noise, so, to adjust a GARCH model to this residual series can be the last step of the analysis.

Currently, theory of GARCH models is very used in modeling in areas as Finance, Insurance, Teletraffic and Climatology.

2 ARCH (p)

A process $\{Z_j,\, j \in \mathbb{Z}\}$ is an ARCH (p) process if it is a second order stationary and causal process such that

$$Z_j = \sigma_j\, \varepsilon_j, \quad j \in \mathbb{Z},$$

where $\varepsilon \sim \mathrm{GWN}(0,1)$ and σ is a process that satisfies

$$\sigma_j^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i\, Z_{j-i}^2, \quad p \geq 1.$$

Parameters $\alpha_0, \alpha_1, \cdots, \alpha_p$ are the parameters of the model and they are assumed all positive.

Why are they called autoregressive conditional heteroskedastic?

Define $\mathcal{F}_j := \sigma\{Z_l,\, l \leq j\}$, the $\sigma-$algebra generated by the history of process Z. Then,

$$\mathbb{E}[Z_j^2 \,|\, \mathcal{F}_{j-1}] = \mathbb{E}[\sigma_j^2\ \varepsilon_j^2 \,|\, \mathcal{F}_{j-1}] = \sigma_j^2\, \mathbb{E}[\varepsilon_j^2 \,|\, \mathcal{F}_{j-1}] = \sigma_j^2 \mathbb{E}[\varepsilon_j^2] = \sigma_j^2. \tag{2.1}$$

because σ_j^2 depends on Z until time $j-1$ and therefore it is $\mathcal{F}_{j-1}-$measurable, and ε is a standard Gaussian white noise and so, ε_j is independent of $\mathcal{F}_{j-1}$.

On other hand, note that $\mathbb{E}[Z_j \,|\, \mathcal{F}_{j-1}] = \sigma_j \cdot \mathbb{E}[\varepsilon_j] = 0$ and so $\sigma_j^2 = \mathbb{E}[Z_j^2 \,|\, \mathcal{F}_{j-1}]$ is actually the conditional variance of Z_j known $\mathcal{F}_{j-1}$, and is not a constant. This is the reason why we call these models heteroskedastic.

Moreover,

$$\sigma_j^2 = \mathbb{E}[Z_j^2 \,|\, \mathcal{F}_{j-1}] = \mathbb{V}(Z_j | \mathcal{F}_{j-1}) = \alpha_0 + \alpha_1\, Z_{j-1}^2 + \cdots + \alpha_p\, Z_{j-p}^2,$$

and being

$$\mathbb{E}(Z_{j-l}^2 | \mathcal{F}_{j-1}) = \mathbb{V}(Z_{j-l} | \mathcal{F}_{j-1}) = Z_{j-l}^2$$

for any $l \geq 1$ we can write

$$\mathbb{V}(Z_j|\mathcal{F}_{j-1}) = \alpha_0 + \sum_{l=1}^{p} \alpha_l \mathbb{V}(Z_{j-l}|\mathcal{F}_{j-1}),$$

and we can say that the conditional variance is autoregressive.

To analyze deeply this kind of models avoiding technicalities we will concentrate in the ARCH(1) case.

3 ARCH (1)

The ARCH(1) model is a second order stationary and causal process that satisfies the following equations:

$$\begin{aligned} Z_j &= \sigma_j \, \varepsilon_j \quad j \in \mathbb{Z}, \\ \varepsilon &\sim \mathrm{GWN}(0,1) \\ \sigma_j^2 &= \alpha_0 + \alpha_1 \, Z_{j-1}^2, \end{aligned}$$

with α_0 and α_1, positive constants.

First question is to show that, under suitable hypotheses on coefficients α_0 and α_1, this process exists as a second order causal process. We have the following result.

Theorem 3.1 *If $\alpha_0 \geq 0$ and $\alpha_1 \in [0,1)$, the process*

$$Z_j = \sqrt{\sum_{l=0}^{\infty} \alpha_0 \, \alpha_1^l \, \varepsilon_j^2 \cdots \varepsilon_{j-l}^2} \tag{3.2}$$

is a strictly stationary process that solves ARCH(1) equations.

Proof: First of all, from (3.2) it is clear that Z_j is strictly stationary because the law of $\varepsilon_j^2 \cdots \varepsilon_{j-l}^2$ depends on l but is the same for all j. This implies in particular that the expectation and the variance of Z_j are constant on j.

We have

$$\mathbb{E}(Z_j^2) = \frac{\alpha_0}{1-\alpha_1}. \tag{3.3}$$

Indeed, applying the double expectation theorem to equation (2.1) we have $\mathbb{E}(Z_j^2) = \mathbb{E}(\sigma_j^2)$. Therefore,

$$\mathbb{E}(Z_j^2) = \mathbb{E}(\sigma_j^2) = \mathbb{E}(\alpha_0 + \alpha_1 Z_{j-1}^2) = \alpha_0 + \alpha_1 \mathbb{E}(Z_{j-1}^2)$$

and using $\mathbb{E}(Z_j^2)$ is a constant for any j we prove (3.3). Note that (3.3) can be proved also directly from (3.2).

To verify that (3.2) satisfy the ARCH(1) equations note that

$$
\begin{aligned}
\sigma_j^2\,\varepsilon_j^2 &= (\alpha_0 + \alpha_1\,Z_{j-1}^2)\varepsilon_j^2 \\
&= \alpha_0\,\varepsilon_j^2 + \alpha_1\,Z_{j-1}^2\,\varepsilon_j^2 \\
&= \alpha_0\,\varepsilon_j^2 + \alpha_1\,\sigma_{j-1}^2\,\varepsilon_{j-1}^2\varepsilon_j^2 \\
&= \alpha_0\,\varepsilon_j^2 + \alpha_0\,\alpha_1\,\varepsilon_{j-1}^2\,\varepsilon_j^2 + \alpha_1^2\,\varepsilon_{j-1}^2\,\varepsilon_j^2\,Z_{j-2}^2 \\
&= \cdots \\
&= \alpha_0\sum_{l=0}^{n}\alpha_1^l\,\varepsilon_j^2\cdots\varepsilon_{j-l}^2 + \alpha_1^{n+1}\,\varepsilon_j^2\cdots\varepsilon_{j-n}^2\,Z_{j-n-1}^2.
\end{aligned}
$$

Moreover,

$$\mathbb{E}[\alpha_1^{n+1}\varepsilon_j^2\cdots\varepsilon_{j-n}^2\,Z_{j-n-1}^2] = \alpha_1^{n+1}\mathbb{E}(Z_{j-n-1}^2) = \alpha_1^{n+1}\frac{\alpha_0}{1-\alpha_1}$$

and this converges to 0 when $n\uparrow\infty$ if and only if $\alpha_1 < 1$. This guarantees the convergence of the series

$$\sum_{l=0}^{\infty}\alpha_0\,\alpha_1^l\,\varepsilon_j^2\cdots\varepsilon_{j-l}^2.$$

Then

$$\sigma^2\varepsilon_j^2 = Z_j^2.$$

■

Next propositions show that an ARCH(1) model is a centered white noise and its square is an AR(1) model.

Proposition 3.2 *Process Z defined in (3.2) is a centered white noise.*

Proof: Process Z is centered. We have already seen $E[Z_j|\mathcal{F}_{j-1}] = 0$ and taking expectations another time this shows $\mathbb{E}(Z_j) = 0$. On other hand we have seen that Z is a strictly stationary time series with second order moment (and variance since Z is centered) given by $\frac{\alpha_0}{1-\alpha_1}$. Finally, its autocovariance function is

$$
\begin{aligned}
\mathbb{C}(Z_j, Z_{j+l}) &= \mathbb{E}[Z_j\,Z_{j+l}] \\
&= \mathbb{E}[E[Z_j\,Z_{j+l}\,|\,\mathcal{F}_{j+l-1}]] \\
&= \mathbb{E}[Z_j\,E[Z_{j+l}\,|\,\mathcal{F}_{j+l-1}]] \\
&= \mathbb{E}[Z_j\,\sigma_{j+l}E[\varepsilon_{j+l}\,|\,\mathcal{F}_{j+l-1}]] = 0
\end{aligned}
$$

because ε_{j+l} is independent of $\mathcal{F}_{j+l-1}$ and centered.

Therefore, Z is a centered white noise. ■

Proposition 3.3 *Process $Z^2 = \{Z_j^2, j\in\mathbb{Z}\}$ is an AR(1) model*

Proof: Observe that

$$
\begin{aligned}
Z_j^2 &= \sigma_j^2 + Z_j^2 - \sigma_j^2 \\
&= \alpha_0 + \alpha_1\,Z_{j-1}^2 + \sigma_j^2(\varepsilon_j^2 - 1) \\
&= \alpha_0 + \alpha_1\,Z_{j-1}^2 + v_j
\end{aligned}
$$

with $v_j = \sigma_j^2(\varepsilon_j^2 - 1)$, and also observe that

$$\mathbb{E}(v_j) = \mathbb{E}(\sigma_j^2)\,\mathbb{E}\,(\varepsilon_j^2 - 1) = 0,$$

$$\begin{aligned}
\mathbb{C}(v_j,\, v_{j+l}) &= \mathbb{C}(\sigma^2(\varepsilon_j^2 - 1),\, \sigma_{j+l}^2(\varepsilon_{j+l}^2 - 1)) \\
&= \mathbb{E}[\sigma_j^2\sigma_{j+l}^2(\varepsilon_j^2 - 1)(\varepsilon_{j+l}^2 - 1)] \\
&= \mathbb{E}[\sigma_j^2(\varepsilon_j^2 - 1)E[\sigma_{j+l}^2(\varepsilon_{j+l}^2 - 1) \,|\, \mathcal{F}_{j+l-1}]] \\
&= \mathbb{E}[\sigma_j^2(\varepsilon_j^2 - 1)\sigma_{j+l}^2\mathbb{E}[(\varepsilon_{j+l}^2 - 1)]] = 0
\end{aligned}$$

for any $l \geq 1$ and

$$\begin{aligned}
\mathbb{V}(v_j) &= \mathbb{E}[(\sigma_j^4\,(\varepsilon_j^2 - 1)^2] \\
&= \mathbb{E}[E[\sigma_j^4(\varepsilon_j^2 - 1)^2 \,|\, \mathcal{F}_{j-1}]] \\
&= \mathbb{E}[\sigma_j^4 E[(\varepsilon_j^2 - 1)^2 \,|\, \mathcal{F}_{j-1}]] \\
&= \mathbb{E}[\sigma_j^4]\mathbb{V}[\varepsilon_j^2] \\
&= \mathbb{E}(\sigma_j^4)(\mathbb{E}(\varepsilon_j^4) - (\mathbb{E}(\varepsilon_j^2))^2) \\
&= \mathbb{E}(\sigma_j^4)(3 - 1) \\
&= 2\mathbb{E}(\sigma_j^4).
\end{aligned}$$

Therefore, Z^2 is an AR(1) model with associated centered white noise v with variance $2\mathbb{E}(\sigma_j^4)$ and this quantity is constant because

$$\mathbb{E}[Z_j^4] = \mathbb{E}[\sigma_j^4\,\varepsilon_j^4] = \mathbb{E}[\sigma_j^4] \cdot 3. \tag{3.4}$$

and $\mathbb{E}[Z_j^4]$ is constant because Z is strictly stationary.

Of course, being Z^2 an AR(1) we have

$$\mathbb{V}(Z_j^2) = \frac{1}{1 - \alpha_1^2}\, 2\mathbb{E}(\sigma_j^4).$$

■

Let's investigate a little more what is the four moment of Z. We have

$$\mathbb{E}[\sigma_j^4] = \mathbb{E}\left[(\alpha_0 + \alpha_1\, Z_{j-1}^2)^2\right] = \alpha_0^2 + \alpha_1^2\,\mathbb{E}(Z_{j-1}^4) + 2\,\alpha_0\,\alpha_1\,\mathbb{E}(Z_{j-1}^2)$$

Now, if $c := \mathbb{E}(Z_j^2)$ and $k := \mathbb{E}(Z_j^4)$ for any j, we have from (3.4),

$$k = 3\alpha_0^2 + 3\alpha_1^2 k + 6\alpha_0\alpha_1 c.$$

and using

$$c = \mathbb{E}[Z_j^2] = \frac{\alpha_0}{1 - \alpha_1}.$$

we obtain

$$\mathbb{E}(Z_j^4) = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)}$$

that is a well defined constant provided

$$\alpha_1 < \frac{1}{\sqrt{3}}.$$

Note that if we impose $\frac{\alpha_0}{1-\alpha_1^2} = 1$, Z becomes a standard white noise. Note that in this case

$$\mathbb{E}[Z_j^4] = 3 \cdot \frac{1-\alpha_1^2}{1-3\,\alpha_1^2} \geq 3$$

because $1-\alpha_1^2 \geq 1-3\,\alpha_1^2$.

As a summary we can say that Z is a centered standard white noise not IID but strictly stationary and with kurtosis greater than 3. Note that if $\alpha_1 = 0$ the model reduces to a Gaussian white noise. Note also that the correlation between Z_j^2 and Z_{j+l}^2 is α_1^l, that is, α_1 is the parameter that captures the autocorrelation of process $\{Z_j^2,\, j \in Z\}$. Finally,

$$\sigma_j^2 = \alpha_0 + \alpha_1\, Z_{j-1}^2$$

captures partially the idea that big square values of Z are followed by big changes of process σ.

4 GARCH model

The GARCH model is a generalization of ARCH model defined by the equations

$$\begin{aligned}
&Z_j = \sigma_j\, \varepsilon_j \\
&\varepsilon \sim \mathrm{GWN}(0,1) \\
&\sigma_j^2 = \alpha_0 + \sum_{l=1}^{q} \beta_l\, \sigma_{j-l}^2 + \sum_{r=1}^{p} \alpha_r\, Z_{j-r}^2.
\end{aligned}$$

Here all the parameters β_l and α_r are assumed positive. This model is called GARCH(p,q).

Under suitable conditions, $\{Z_j^2,\, j \in \mathbb{Z}\}$ is an ARMA process with parameters m and p and its noise is

$$v_j = \sigma_j^2\, (\varepsilon_j^2 - 1)$$

with $m = p \vee q$.

For the GARCH(1,1) model it can be shown that the condition that guarantees stationarity is $\alpha_1 + \beta_1 < 1$.

5 Statistics of GARCH process

GARCH models are conditionally Gaussian. This allows to use the maximum likelihood method to estimate the parameters. To fit a GARCH model we fit an ARMA model to $\{Z_j^2,\, j \in \mathbb{Z}\}$.

References

[1] T. Bollerslev (1986): *Generalized Autoregressive Conditional Heteroskedasticity*. Journal of Econometrics 31: 307-327.

[2] R. Engle (1982): *Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK inflation*. Econometrica 50: 987-1008.

[3] R. Engle (1995): *ARCH, selected readings*. Oxford.

[4] C. Gouriéroux (1997): *ARCH Models and Financial Applications*. Springer.