Lecture 7: General Autoregressive Models

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1 Introduction

The purpose of this lecture is to present the general AR(p) model, after having seen the AR(1) case in the previous lecture.

In all the lecture $Z = \{Z_j,\ j \in \mathbb{Z}\}$ will be a centered white noise with variance $\sigma^2 > 0$, and $\{Y_j, j \in \mathbb{Z}\}$ will be a time series. Note that for simplicity in the presentation we consider time series on $\mathbb{Z}$. In practice we have $j \geq 0$ or $j \geq 1$. Recall the operator B introduced in Lecture 1 that acts as $BY_j := Y_{j-1}$.

We can write the AR(1) equation as

$$Y_j = \phi B Y_j + Z_j$$

or

$$(\mathrm{Id} - \phi B) Y_j = Z_j.$$

Therefore, if $(\mathrm{Id} - \phi B)^{-1}$ exists, we can write

$$Y_j = (\mathrm{Id} - \phi B)^{-1} Z_j.$$

But on other hand, using the linear causal expression of Y obtained in the previous lecture, we know that the AR(1) model can be written as

$$Y_j = \sum_{i=0}^{\infty} \phi^i B^i Z_j.$$

Therefore, we can define

$$(\mathrm{Id} - \phi B)^{-1} = \sum_{i=0}^{\infty} \phi^i B^i \tag{1.1}$$

provided $\sum_{i=0}^{\infty} \phi^i B^i Z_j$ is well defined in L^2. Note that as a consequence of Corollary 2.4 in the previous lecture this is equivalent to

$$\sum_{i=0}^{\infty} |\phi|^{2i} < \infty.$$

And it is well known that this convergence is equivalent to $|\phi| < 1$, that is the condition on ϕ that guarantees that the AR(1) model is a stationary model.

Notice that equation (1.1) is a formal version of the equality

$$\sum_{i=0}^{\infty} \phi^i x^i = \frac{1}{1 - \phi x}$$

that is true if and only if $|\phi x| < 1$. Equivalently, we can write

$$(1 - \phi x) \cdot \sum_{i=0}^{\infty} \phi^i x^i = 1, \quad \forall x \colon |x| < \frac{1}{|\phi|}.$$

2 The AR(p) model

Let $\Phi(x)$ be a polynomial of degree p. We can always write

$$\Phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p \tag{2.2}$$

up to a multiplicative constant, dividing the polynomial by the independent coefficient.

Note that this is an extension to any degree of the polynomial $1 - \phi x$ associated to the AR(1) model.

Definition 2.1 *We say that a polynomial $\Phi(x)$ is invertible if it exist a series $\sum_{i=0}^{\infty} \psi_i x^i$ such that*

$$\Phi(x) \sum_{i=0}^{\infty} \psi_i x^i = 1$$

with

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty.$$

Notice that if Φ is invertible we have

$$Y_j = \Phi(B)^{-1} Z_j = \sum_{i=0}^{\infty} \psi_i B^i Z_j$$

and this guarantees the causal and stationary character of Y.

So, the key question is, when Φ is invertible? We have seen that if $p = 1$, the answer is $|\phi| < 1$. But what is the answer when $p > 1$? The following theorem gives it.

Theorem 2.2 *A polynomial $\Phi_p(z)$, with $z \in \mathbb{C}$, is invertible if and only if all its roots are out of the unit circle, that is, if and only if $\{z \colon \Phi_p(z) = 0\} \subseteq \{z \colon |z| > 1\}$.*

Proof: See [1], pages 85-86. ■

Remark 2.3 *If $p = 1$, the previous theorem is clear. We have $\Phi(z) = 1 - \phi z$. In this case,*

$$\{z \colon 1 = \phi z\} = \{z \in \mathbb{C} \colon z = 1/\phi\},$$

and

$$\frac{1}{|\phi|} > 1 \Longleftrightarrow |\phi| < 1.$$

We define an AR(p) model as a stationary process $\{Y_j,\, j \in \mathbb{Z}\}$ that satisfies

$$\Phi(B)Y_j = Z_j, \quad j \in \mathbb{Z}. \tag{2.3}$$

where Φ is an invertible polynomial of degree p.

Note that given any polynomial of degree p we can divide it by the independent term and write it as $c\Phi(x)$ where c is a constant and Φ is of the form (2.2). In that case we can erase the constant c from the left hand side of (2.3) and change the variance of the white noise on the right hand side from σ^2 to $(\frac{\sigma}{c})^2$.

Since our AR(p) model is causal we have immediately that it is centered and its auto-covariance function is

$$\mathbb{C}(Y_j, Y_{j+l}) = \sigma^2 \sum_{r=0}^{\infty} \psi_r \psi_{r+l}.$$

But to identify the coefficients of the causal representation is not so easy.

One way is to do it recursively. Imposing

$$\Phi_p(x) \sum_{i=0}^{\infty} \psi_i x^i = 1$$

we have

$$\begin{aligned}
(1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p) \sum_{i=0}^{\infty} \psi_i x^i &= 1 \\
\Longleftrightarrow \sum_{i=0}^{\infty} \psi_i x^i - \phi_1 \sum_{i=0}^{\infty} \psi_i x^{i+1} - \cdots - \phi_p \sum_{i=0}^{\infty} \psi_i x^{i+p} &= 1 \\
\Longleftrightarrow \sum_{i=0}^{\infty} \psi_i x^i - \phi_1 \sum_{i=1}^{\infty} \psi_{i-1} x^i - \cdots - \phi_p \sum_{i=p}^{\infty} \psi_{i-p} x^i &= 1.
\end{aligned}$$

Equating the independent coefficient to 1 and the coefficients of the different powers of x to 0, we have

$$\begin{aligned}
&\psi_0 = 1 \\
&\psi_1 - \psi_0 \phi_1 = 0 \\
&\psi_2 - \psi_1 \phi_1 - \psi_2 \phi_2 = 0 \\
&\cdots\cdots,
\end{aligned}$$

that is, in general,

$$\psi_i = \phi_1 \psi_{i-1} + \cdots + \phi_p \psi_{i-p}, \quad i \geq p$$

So, recursively, $\psi_0 = 1$, $\psi_1 = \psi_0 \phi_1$, $\psi_2 = \psi_1 \phi_1 + \psi_0 \phi_2$ and so on.

Note that from the modeling point of view, given a centered second order stationary model, the only important object is the autocorrelation function. The variance is a constant, easy to estimate, and we cannot distinguish what is the weight of the variance of the white noise in the variance of the model.

To determine directly the autocorrelation function we have the following alternative method. Notice that

$$\begin{aligned}\mathbb{C}(Y_j, Y_{j+l}) &= \mathbb{C}(Y_j, \phi_1 Y_{j+l-1} + \cdots + \phi_p Y_{j+l-p} + Z_{j+l}) \\ &= \phi_1\gamma(l-1) + \phi_2\gamma(l-2) + \cdots + \phi_p\gamma(l-p), \quad l > 0,\end{aligned}$$

that is,

$$\gamma(l) = \phi_1\gamma(l-1) + \cdots + \phi_p\gamma(l-p), \quad l > 0.$$

Dividing by $\gamma(0)$ we obtain

$$\begin{aligned}\rho(l) &= \phi_1\rho(l-1) + \cdots + \phi_p\rho(l-p), \quad l > 0 \\ \rho(0) &= 1 \\ \rho(-l) &= \rho(l),\end{aligned}$$

that is a solvable system of equations of finite differences and the solution is the autocorrelation function of our AR(p) model. This system is called Yule-Walker equations.

3 Example: the AR(2) model

A AR(2) model is a process that satisfies the equation

$$Y_j - \phi_1 Y_{j-1} - \phi_2 Y_{j-2} = Z_j, \quad j \in \mathbb{Z}$$

with ϕ_1 and ϕ_2 real numbers such that the polynomial

$$1 - \phi_1 x - \phi_2 x^2$$

have all roots out of the unit circle.

As we have seen in the previous section, the autocorrelation function satisfies the following Yule-Walker equations:

$$\begin{aligned}\rho(l) &= \phi_1\,\rho(l-1) + \phi_2\,\rho(l-2), \quad l \in \mathbb{Z} \\ \rho(0) &= 1 \\ \rho(-l) &= \rho(l).\end{aligned}$$

The way to solve this equation, see for example [2], consists in studying the roots of the polynomial

$$1 - \phi_1 x - \phi_2 x^2,$$

or equivalently to distinguish the different possible values of the discriminant

$$\Delta := \phi_1^2 + 4\phi_2.$$

1. If $\Delta > 0$, the polynomial has two real roots η_1 and η_2. Then,

$$\rho(l) = c_1\left(\frac{1}{\eta_1}\right)^l + c_2\left(\frac{1}{\eta_2}\right)^l.$$

2. If $\Delta = 0$, $\phi_1^2 = -4\phi_2$, and we have a unique double root η. In this case,

$$\rho(l) = c_1 \left(\frac{1}{\eta}\right)^l + c_2\, l \left(\frac{1}{\eta}\right)^l .$$

3. Finally, if $\Delta < 0$, we have two conjugated complex roots η_1 and η_2. Then,

$$\rho(l) = c_1 \left(\frac{1}{\eta_1}\right)^l + c_2 \left(\frac{1}{\eta_2}\right)^l = \frac{1}{r^l}\{A_1 \sin(\alpha\, l) + A_2 \cos(\alpha\, l)\}$$

with $r = |\eta_1| = |\eta_2|$.

To determine the constants we can use the so called Yule-Walker equations. We can write the first two difference equations, for $l = 1$ and $l = 2$, and built the system

$$\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

and so,

$$\begin{aligned} \phi_1 + \rho_1\phi_2 &= \rho_1 \\ \rho_1\phi_1 + \phi_2 &= \rho_2 \end{aligned}$$

and isolating ρ_1 and ρ_2 we obtain

$$\begin{aligned} \rho_1 &= \frac{\phi_1}{1-\phi_2} \\ \rho_2 &= \phi_2 + \frac{\phi_1^2}{1-\phi_2}. \end{aligned}$$

Using $\rho(0) = 1$ and $\rho(1) = \frac{\phi_1}{1-\phi_2}$ we can determine c_1 and c_2 or A_1 and A_2. This technique can be generalized to AR(p) case writing the first p Yule-Walker equations.

As a particular and concrete example, consider the AR(2) model

$$X_j = 0.7X_{j-1} - 0.1X_{j-2} + Z_j,$$

with $Z \sim \mathrm{WN}(0, 1)$.
We have to solve equation

$$1 - 0.7x + 0.1x^2 = 0.$$

The roots are 2 and 5. Then, $\eta_1 = 2$ and $\eta_2 = 5$, that of course are out of the unit circle of $\mathbb{C}$. Then,

$$\rho(l) = c_1 \frac{1}{2^l} + c_2 \frac{1}{5^l}.$$

In particular $\rho(0) = 1$ and so $c_1 + c_2 = 1$. On other hand,

$$\rho(1) = \frac{\phi_1}{1-\phi_2} = \frac{0.7}{1+0.1} = \frac{7}{11}$$

Then, we have to solve the system

$$\begin{aligned}\frac{1}{2}\,c_1 + \frac{1}{5}\,c_2 &= \frac{7}{11}\\ \frac{1}{2}\,c_1 + \frac{1}{2}\,c_2 &= \frac{1}{2},\end{aligned}$$

obtaining

$$\begin{aligned}c_1 &= \frac{16}{11}\\ c_2 &= -\frac{5}{11}.\end{aligned}$$

Therefore,

$$\rho(l) = \frac{1}{11}\left\{16\,\frac{1}{2^l} - 5\,\frac{1}{5^l}\right\}, \quad l \geq 0.$$

Observe that

$$|\rho(l)| \leq \frac{c_1}{2^l} + \frac{c_2}{5^l} \leq \frac{k}{2^l} \overset{l\uparrow\infty}{\longrightarrow} 0.$$

In relation with the determination of ψ_i coefficients we have

$$\begin{aligned}&\psi_0 = 1\\ &\psi_1 = \phi_1 = 0.7\\ &\psi_2 = \phi_1^2 + \phi_2 = 0.7^2 - 0.1 = 0.39\\ &\psi_3 = \phi_1\psi_2 + \phi_2\psi_1 = 0.7 \times 0.39 - 0.1 \times 0.7 = 0.203\\ &\text{etc}\ldots\end{aligned}$$

Note that $|\psi_i| \overset{l\uparrow\infty}{\longrightarrow} 0$ because $\sum_i \psi_i^2 < \infty$.

References

[1] P. J. Brockwell and R. A. Davis (1991): *Time Series: Theory and Methods.* Springer.

[2] A. I. Markushévich (1981): *Sucesiones recurrentes.* MIR.