Lecture 2: Stochastic processes. Stationarity. Autocorrelation.

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1 Stochastic Processes

To progress in the analysis of time series requires to introduce probabilistic models. The probabilistic objects that model random phenomena that evolve in time are the so called stochastic processes. So, from now on, a time series will be seen as a realization of a stochastic process.

In this course we will move between two poles, equivalent to the poles defined by Probability Theory and Statistics. On one hand we will propose models, that is, stochastic processes, like the familiy of SARIMA models introduced by Box and Jenkins during the seventies, and we will analyze them. On other hand, we will make an effort to adjust our models to concrete observed time series, that is, we will do Statistical Inference of Time Series.

What is a stochastic process? We can see them from different points of view.

A first way to see them is as a collection $\{X_t, t \in \mathbb{T}\}$ of random variables indexed by a set $\mathbb{T}$. The set $\mathbb{T}$ is the index set and represents, usually, time. For example, if we have $\mathbb{T} = \mathbb{N}$ we can write $\{X_k, k \geq 0\}$ and we talk about discrete time stochastic processes. On the contrary, if $\mathbb{T} = [0, \infty)$, we write $\{X_t, t \geq 0\}$ and we talk about continuous time stochastic processes. In general, $\mathbb{T}$ is $\mathbb{N}$ or $\mathbb{Z}$ in the discrete time case and $[0, \infty)$ or $\mathbb{R}$ in the continuous time case.

A second point of view is to see a stochastic process as the random selection according a certain probability law of a trajectory in a set of trajectories, that is,

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathrm{F}(\mathbb{T}; \mathbb{S})$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathrm{F}(\mathbb{T}; \mathbb{S})$ is a space of functions that represents the space of possible trajectories. The set $\mathbb{S}$ is the set where the variables of the process take values; it is called state space. Frequently it is $\mathbb{R}$, but it could be a much more general space. If $\mathbb{S}$ is finite or countable we call the stochastic process, a chain.

In this course we treat only discrete time stochastic processes. Therefore we always write

$$\{X_k,\ k \in \mathbb{Z} \text{ or } \mathbb{N}\}$$

where X_k are random variables taking values in $\mathbb{R}$.

Given a random variable X we denote its expectation by $\mathbb{E}(X)$, its variance by $\mathbb{V}(X)$ and given two random variables X and Y we denote its covariance by $\mathbb{C}(X, Y)$.

We assume in this course that $\mathbb{E}(X_k^2) < \infty\ \forall k \geq 0$. This fact guarantees that $\mathbb{E}(X_k)$ and $\mathbb{V}(X_k)$ are well defined and finite. Indeed, recall that by Jensen inequality we have

$$|\mathbb{E}X_k| \leq \mathbb{E}|X_k| < \infty,$$

and by Cauchy-Schwarz inequality we have

$$\mathbb{E}|X_k| \leq (\mathbb{E}(X_k^2))^{1/2} < \infty.$$

.

Then, we have $\mathbb{V}(X_k) := \mathbb{E}(X_k^2) - (\mathbb{E}(X_k))^2 < \infty$.

Stochastic processes that satisfy the previous condition are named second order processes. From now on we assume that all stochastic processes that appear in this course are second order stochastic processes.

Note that for any second order stochastic process the covariance of two random variables of the process is well defined. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\mathbb{C}(X_j, X_{j+l})| &= |\mathbb{E}[(X_j - \mathbb{E}(X_j))(X_{j+l} - \mathbb{E}(X_{j+l}))]| &(1.1)\\
&\leq \mathbb{E}(|X_j - \mathbb{E}(X_j)||X_{j+l} - \mathbb{E}(X_{j+l})|) &(1.2)\\
&\leq (\mathbb{E}(|X_j - \mathbb{E}(X_j)|^2)^{1/2}(\mathbb{E}|X_{j+l} - \mathbb{E}(X_{j+l})|^2)^{1/2} &(1.3)\\
&= \mathbb{V}(X_j)^{1/2}\mathbb{V}(X_{j+l})^{1/2} &(1.4)
\end{aligned}$$

2 Stationary processes

To be useful, a stochastic processes has to be the mathematical description of a certain type of regularity. The simplest case is the case the process is a collection of independent and identically distributed (iid) random variables.

We say that a process $\{X_k,\ k \in \mathbb{Z}\}$ is *strictly stationary* if for any $k_1, \ldots, k_n$ and l, the vectors

$$(X_{k_1}, \cdots, X_{k_n})$$

and

$$(X_{k_1+l}, \cdots, X_{k_n+l})$$

have the same law. Note that in particular, taking $n = 1$, this implies that all variables X_k have the same law.

We say that a process is *stationary*, or weakly stationary, if

- $\mathbb{E}[X_k] = \mu \in \mathbb{R}, \quad \forall k \in \mathbb{Z}$
- $\mathbb{C}(X_k, X_{k+l}) = \gamma(l), \quad \forall\, k, l, \in \mathbb{Z}$, where γ denotes a certain function defined on $\mathbb{N}$ because, by the symmetry of the covariance, $\gamma(-l) = \gamma(l)$.

Note that, for a stationary process, the covariance depends only on the distance between the variables. Observe that in the case $l = 0$ we have

$$\gamma(0) = \mathbb{V}(X_k), \quad \forall k \geq 1.$$

Function γ is the so called *autocovariance function* and function

$$\rho(k) := \frac{\gamma(k)}{\gamma(0)}, \quad k \in \mathbb{Z},$$

is the *auto-correlation function*. Note that $\rho(l) = \rho(-l)$ and so, it is also sufficient to define ρ on $\mathbb{N}$.

It is obvious that strict stationarity implies stationarity. But the reverse is in general, false. But if the process is Gaussian in the sense that all finite dimensional distributions are Gaussian, then weakly stationarity implies strict stationarity because in this case the mean vector and the covariance matrix determine the law. Recall that a process is Gaussian if $\forall\, k_1, \ldots, k_n$, the vector $(X_{k_1}, \cdots, X_{k_n})$ has a multidimensional normal law. Note also that from (1.1) we have

$$|\gamma(l)| \leq \gamma(0), \quad \forall l \geq 0$$

and so,

$$-1 \leq \rho(l) \leq 1, \quad \forall\, l \geq 0.$$

Values l are called *lags*. Then, $\rho(l)$ is the autocorrelation of lag l. Its graphical representation is given by

$$\{\rho(l),\ l \geq 0\}$$

and is called *correlogram*.

Finally, observe that $\gamma(0) \geq 0$. If $\gamma(0) = 0$ we have $X_j = \mu \in \mathbb{R}$, $\forall j \geq 1$. If $\gamma(0) \neq 0$ we have $\rho(0) = 1$.

As a summary, a process is stationary if its mean and variance functions are constant and if its auto-correlation function depends only on the lag. A second order stationary process is determined by the pair (μ, γ) or the triplet (μ, σ, ρ). So, to identify a second order model for a time series means simply to estimate μ and γ or μ, σ^2 and ρ.

3 Examples

A) IID Noise

We say that $\{X_k,\ k \geq 1\}$ is a i.i.d. noise if the random variables are i.i.d. with mean μ and standard deviation σ. Note that in this case, $\rho(k) = 1$ if $k = 0$ and $\rho(k) = 0$ if $k > 0$. This is a model for a purely random time series.

B) Random walk

Let $\{X_k,\ k \geq 1\}$ be an i.i.d. noise. The series $\{S_k,\ k \geq 1\}$ with

$$S_k := X_1 + \cdots + X_k, \quad , k \geq 1,$$

is the so called random walk.

Note that the random walk is not stationary but the series of its first differences is an IID noise. Observe that we can write

$$S_k = S_{k-1} + X_k.$$

Then,

$$\mathbb{E}[S_k] = \mathbb{E}[S_{k-1}] + \mathbb{E}[X_k]$$

and so, only if $\mathbb{E}[X_k] = 0$, the mean function is constant.

If we assume $\mathbb{E}[X_k] = 0$, noticing that X_k is independent of S_{k-1}, we have

$$\mathbb{V}(S_n) = \mathbb{V}(S_{n-1}) + \mathbb{V}(X_n)$$

and

$$\mathbb{V}(S_n) = \mathbb{V}(S_{n-1}) \Longleftrightarrow \mathbb{V}(X_n) = 0,$$

that is, if $X_n \equiv 0$.

Note that if $\mathbb{E}(X_n) = \mu$ and $\mathbb{V}(X_n) = \sigma^2$, we have $\mathbb{E}(S_n) = n\mu$ and $\mathbb{V}(S_n) = n\sigma^2$. On other hand, if $n < m = n + p$,

$$\begin{aligned}
\mathbb{C}(S_n, S_m) &= \mathbb{C}(S_n, S_{n+p}) \\
&= \mathbb{C}(S_n, S_n) + \mathbb{C}(S_n, X_{n+1} + \cdots + X_{n+p}) \\
&= \mathbb{V}(S_n) \\
&= n\sigma^2.
\end{aligned}$$

Therefore,

$$\mathbb{C}(S_n, S_m) = (n \wedge m)\,\sigma^2$$

and

$$\rho(S_n, S_m) = \sqrt{\frac{n \wedge m}{n \vee m}}.$$

C) White noise

We say that a process $\{X_k,\ k \geq 1\}$ is a white noise if all the variables have expectation μ, variance σ^2 and they are uncorrelated. Note that an IID noise is a particular case of white noise, because independence implies uncorrelation.

D) Gaussian white noise

We say that a process $\{X_k,\ k \geq 1\}$ is a Gaussian white noise if it is a white noise and any vector $(X_{k_1}, \ldots, X_{k_n})$ has a normal law. In this case, because in a normal vector no correlation implies independence, our process is also an IID noise. That is, a Gaussian white noise and a Gaussian IID noise are the same object. If random variables X_k have standard normal law we say that X is a standard Gaussian white noise. As a summary, note that a Gaussian White noise is a particular case of IID noise, and an IID noise is a particular case of a White noise.

E) Periodic process

Let $X_k = A\cos(k\theta) + B\sin(k\theta)$ with θ a real parameter and A and B i.i.d. centered random variables with finite variance σ^2. Obviously, $\mathbb{E}(X_k) = 0$, $\forall k \in \mathbb{Z}$. The covariance is given by

$$\begin{aligned}
\mathbb{C}(X_k, X_{k+l}) &= \mathbb{E}[A\cos(k\theta + l\theta)\cdot B\sin(k\theta)] \\
&+ \mathbb{E}[A\cos(k\theta + l\theta)\cdot A\cos(k\theta)] \\
&+ \mathbb{E}[B\sin(k\theta + l\theta)\cdot A\cos(k\theta)] \\
&+ \mathbb{E}[B\sin(k\theta + l\theta)\cdot B\sin(k\theta)] \\
&= \sigma^2\{\cos(k\theta + l\theta)\cos(k\theta) + \sin(k\theta + l\theta)\sin(k\theta)\} \\
&= \sigma^2\cos(l\theta).
\end{aligned}$$

and so, it is a stationary series. We have applied the well-known formula

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

4 Exercises

1. Consider a monthly series such that

$$X_j = a + bj + S_j + Z_j,\; j \in \mathbb{Z},$$

   where S is a seasonal component of period 12 and Z is a white noise.

   Show that the operator $Id - B^{12}$ converts this series in a stationary series.

2. Let Z be a standard Gaussian white noise and a, b and c real constants. Study which of the following series are stationary, and if they are, determine its auto-covariance function.

   a) $X_j = a + bZ_j + cZ_{j-2}$.
   b) $X_j = a + bZ_0$.
   c) $X_j = Z_0\cos(cj)$.
   d) $X_j = Z_jZ_{j-1}$.

3. Consider the time series $Y_j = Asin(\omega j) + Z_j$, where A is a centered random variable with variance 1, ω is fixed constant in $(0,\pi)$ and Z is a centered white noise with variance σ^2, uncorrelated with A. Prove that Y is not stationary.

4. Let $Y_j = (-1)^jX$ where X is a fixed random variable. Give necessary and sufficient conditions on X in order to make Y stationary.

5. Let $\{Z_j,\, j \in \mathbb{Z}\}$ a standard Gaussian white noise. Define

$$X_j = \begin{cases} Z_j, & \text{if } j \text{ is even,} \\ \big(Z_{j-1}^2 - 1\big)/\sqrt{2}, & \text{if } j \text{ is odd.} \end{cases}$$

   Prove that $\{X_j,\, j \in \mathbb{Z}\}$ is a white noise but not an IID noise.

6. Let $\{Z_j,\, j \in \mathbb{Z}\}$ be an IID noise with uniform law on the interval $(-1,1)$. We define the following time series:

   a) $X_j = Z_1\cos\big(\pi j/4\big) + Z_2\sin\big(\pi j/4\big)$.

b) $X_j = Z_j + 0.8\, Z_{j-1}$.

Prove that both are stationary and compute its auto-covariance function. Check that in series (a) $\lim_{k\to\infty} \gamma(k)$ doesn't exist.

7. In certain applications in Finance the so called threshold series is used. An example of this type of series is the following: from the stationary series $X_j = 0.5\, Z_{j-1} + Z_j$, where Z is a standard Gaussian white noise, the following threshold series is built:

$$Y_j = \begin{cases} 1, & \text{if } X_j \geq 0, \\ 0, & \text{if } X_j < 0. \end{cases}$$

In other words,

$$Y_j = \mathbf{1}_{\{X_j \geq 0\}}.$$

Prove that the threshold series is strictly stationary.

8. Consider the model $Y_j = Z_j - \theta Z_{j-1}$ with $Z \sim WN(0, \sigma^2)$. Compute its auto-correlation function. If $\rho(1) = 0.4$, what are the possible values of θ?

9. Consider the model $X_j = Z_j + c(Z_{j-1} + Z_{j-2} + \cdots)$, with $j \geq 0$, c constant and $Z \sim WN(0, \sigma^2)$. Show that X is not stationary but the series of they first differences Y, it is. Find the auto-correlation function of Y.