Lecture 13: Intervention analysis

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Let $\{x_1, \ldots, x_n, \ldots\}$ a time series. Assume we know that at a certain date $T > 0$, an external event has occurred and has influenced our series. We know there is an influence, but not its magnitude. For example, we can consider the effect of some presidential elections, some modifications of the official interest rate, some terrorist attacks or the lockdown caused by the covid-19 pandemic. Box and Tiao developed in 1975 the so called intervention analysis to treat this problem [2].

Let $\{X_j,\ j \in \mathbb{Z}\}$ be a SARIMA model. Intervention analysis consists in adding a deterministic series $\{M_j,\ j \in \mathbb{Z}\}$ to built a model like

$$Y_j = X_j + M_j, \quad j \in \mathbb{Z}.$$

The simplest example is to consider $M_j = a\, I_j^T$, where $I_j^T := \mathbb{1}_{\{j=T\}}$. In this case we have

$$Y_j = X_j + a\, I_j^T, \quad j \in \mathbb{Z}.$$

that is, $Y_j = X_j,\ \ \forall j \neq T$ and $\ Y_T = X_T + a$. We are simply adding to the model the parameter a. Note that we are assuming the effect of the external event is present only at T.

A second possibility is to consider

$$E_j^T := \mathbb{1}_{\{j \geq T\}}.$$

Note that this is a model for a permanent impact.

Then,

$$Y_j = \begin{cases} X_j & j < T \\ X_j + a & j \geq T \end{cases}$$

Note that

$$E_j^T - E_{j-1}^T = I_j^T = E_j^T - E_j^{T+1} = (\mathrm{Id} - B)\, E_j^T,$$

that is,

$$I^T = (\mathrm{Id} - B)\, E^T.$$

This idea allows us to generalize this type of models considering filters of type

$$\Lambda(B) := \frac{B^b\, U(B)\, u}{V(B)}$$

where $u \in \mathbb{R}$, $b \geq 0$ and U and V are polynomials of degrees c and g respectively, that is,

$$\begin{array}{rcl} U(B) & = & (\mathrm{Id}\, - u_1\, B - u_2\, B^2 - \cdots - u_c\, B^c) \\ V(B) & = & (\mathrm{Id}\, - v_1\, B - v_2\, B^2 - \cdots - v_g\, B^g). \end{array}$$

Then we can propose the model

$$Y_j = X_j + \Lambda(B)\, E_j^T.$$

Note that we are adding $g+c+1$ parameters, to be estimated, to the model

1 Example 1: Sudden start and permanent effect

Consider the operator

$$\Lambda(B) = u\, B.$$

Then,

$$Y_j = X_j + u \cdot B\, E_j^T = X_j + u\, E_{j-1}^T = X_j + u\, E_j^{T+1}$$

or equivalently,

$$Y_j = \begin{cases} X_j & j \leq T \\ X_j + u & j \geq T+1 \end{cases}$$

If we write $\Lambda(B) = u\; \mathrm{Id}$, the effect is permanent since date T.

2 Example 2: Gradual beginning and permanent effect

Use now

$$\Lambda(B) = \frac{u\, B}{\mathrm{Id}\, - v\, B}, \quad |v| < 1.$$

We have,

$$\Lambda(B) = \sum_{l=0}^{\infty} u \cdot B \cdot v^l \cdot B^l = \sum_{l=0}^{\infty} u \cdot v^l \cdot B^{l+1}$$

and so,

$$\Lambda(B)\,E_j^T = \sum_{l=0}^{\infty} u\,v^l\,E_j^{T+1+l} = u\,E_j^{T+1} + u\,v\,E_j^{T+2} + \cdots + u\,v^l\,E_j^{T+1+l} + \cdots,$$

that is, since $T+1$ we have

$$\begin{aligned}
Y_{T+1} &= X_{T+1} + u \\
Y_{T+2} &= X_{T+2} + u + uv \\
&\vdots \\
Y_{T+l} &= X_{T+l} + u + uv + uv^2 + \cdots + uv^l.
\end{aligned} \tag{1}$$

Note that $|u \cdot v^l|$ converges to 0 when l goes to infinity because $|v| < 1$.

3 Example 3: Sudden start and gradual decay

Now we use

$$\Lambda(B) = \frac{u\,B(\mathrm{Id}\,-B)}{\mathrm{Id}\,-v\,B}, \quad 0 < v < 1.$$

We have,

$$\begin{aligned}
\Lambda(B)\,E_j^T &= \sum_{l=0}^{\infty} u \cdot v^l \cdot B^{l+1} \cdot I_j^T = \sum_{l=0}^{\infty} u \cdot v^l \cdot I_j^{T+l+1} \\
&= u\,I_j^{T+1} + u\,v\,I_j^{T+2} + \cdots + u\,v^l\,I_j^{T+1+l} + \cdots
\end{aligned}$$

that is,

$$\begin{aligned}
Y_T &= X_T \\
Y_{T+1} &= X_{T+1} + u \\
Y_{T+2} &= X_{T+2} + u\,v \\
&\vdots \\
Y_{T+l} &= X_{T+l} + u\,v^{l-1}
\end{aligned}$$

therefore, there is a initial impact u at $T+1$ and afterwards, the effect decreases slowly because $0 < v < 1$.

4 Example 4: Gradual beginning and gradual decay

Now,

$$\Lambda(B) = \frac{u(\mathrm{Id}\,-B)}{\mathrm{Id}\,-v_1\,B - v_2\,B^2},$$

with $V(B)$ with the roots out of the unit circle.

We have

$$\Lambda(B)\, E_j^T = \sum_{l=0}^{\infty} \Psi_l\, B^l \cdot u \cdot I_j^T = \sum_{l=0}^{\infty} u \cdot \Psi_l \cdot I_j^{T+l}.$$

That is,

$$\begin{aligned} Y_T &= X_T + u\Psi_0 \\ Y_{T+1} &= X_{T+1} + u\Psi_1 \\ &\vdots \\ Y_{T+l} &= X_{T+l} + u\Psi_l \end{aligned}$$

To identify coefficients Ψ_k we impose

$$\begin{aligned} & (1 - v_1 x - v_2 x^2) \cdot \sum_{l=0}^{\infty} \Psi_l\, x^l = 1 \\ \iff \quad & \sum_{l=0}^{\infty} \Psi_l\, x^l - \sum_{l=0}^{\infty} \Psi_l\, v_1\, x^{l+1} - \sum_{l=0}^{\infty} \Psi_l\, v_2\, x^{l+2} = 1 \\ \iff \quad & \Psi_0 = 1,\ \Psi_1 - \Psi_0 v_1 = 0,\ \Psi_2 - \Psi_1 v_1 - \Psi_0\, v_2 = 0, \cdots \\ \iff \quad & \Psi_0 = 1,\ \Psi_1 = v_1,\ \Psi_2 = \Psi_1 v_1 + v_2, \cdots, \\ & \Psi_l = \Psi_{l-1} v_1 + \Psi_{l-2} v_2, \cdots \end{aligned}$$

Being $\sum_{k=0}^{\infty} \Psi_k^2 < \infty$, necessarily, $|\Psi_k| \xrightarrow{k \uparrow \infty} 0$.

5 Applied example

Consider the series D of car drivers killed or seriously injured quarterly in Great Britain in the years 1969-1984, see [1]. First differences to reduce seasonality are applied using the operator

$$(\mathrm{Id} - B^4)$$

and then, the series is centered, obtaining

$$Y_j = (\mathrm{Id} - B^4)\, D_j - 305.2.$$

In the graphics of page 334 in [1] we observe the external effect of the OPEP embargo in the fourth quarter of 1973. What was the magnitude of this effect?

We propose a model with sudden initial and permanent effect

$$\Lambda(B) = u\ \mathrm{Id}\,,$$

that is,

$$Y_j = \begin{cases} X_j & j < T \\ X_j + u & j \geq T. \end{cases}$$

To estimate u we minimize

$$\begin{aligned}
\sum_{j=1}^{n} y_j^2 &= \sum_{j=1}^{T-1} x_j^2 + \sum_{j=T}^{n} (x_j + u)^2 \\
&= S^2(1, T-1) + S^2(T, n) + (n - T + 1)u^2 + 2u \sum_{j=T}^{n} x_j \\
\Rightarrow \quad & 2(n + T + 1)u + 2 \sum_{j=T}^{n} x_j = 0 \\
\Longleftrightarrow \quad & \hat{u} = -\frac{\sum_{T}^{n} x_j}{n - T + 1} = -486.
\end{aligned}$$

Then, we define

$$U_j = Y_j - 486\, E_j^{16}$$

and we fit an ARMA model to the new data. We obtain a MA(4) model with

$$U_j = Z_j + 0.609\, Z_{j-1} + 0.477\, Z_{j-2} + 0.358\, Z_{j-3} - 0.369 Z_{j-4}$$

and $Z \sim WN(0, \sigma^2)$ with $\sigma^2 = 103560$.

Observing the autocorrelation function, the obtained noise is a white noise. So, finally,

$$(\mathrm{Id} - B^4)\, D_t - 305.2 - 486\, E_j^{16} = \Theta_4(B)\, Z_j$$

and therefore,

$$D_t = (\mathrm{Id} - B^4)^{-1}(\Theta_4(B)\, Z_j + 486\, E_j^{16} + 305.2).$$

References

[1] P. J. Brockwell and R. A Davis (1996): *Introduction to Time Series and Forecasting.* Springer.

[2] G. E. P. Box and G. C. Tiao (1975): *Intervention Analysis with Applications to Economic and Environmental Problems.* Journal of the American Statistical Association 70 (349): 70-79.