Gauss to PLU fact.

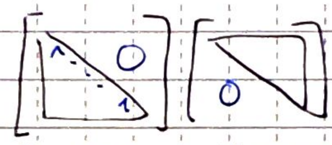
$A_{m \times n}$ $\quad b \in \mathbb{R}^m$

$Ax = b$ $\quad$ x sol. unique if $\ker(Ax) = \{0\}$

$\dim(Im(Ax)) = m \iff m = n$ and

$rank(A) = m$

$A \to$ nonsingular

$A = LU \quad [\quad]$ 

Lower unit $\quad$ Upper triang.

$A = PLU$

Ex

$$\begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 1 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad m_{ij} = \frac{a_{ij}}{a_{kk}} \text{ – pivot}$$

$\underline{J=1}$ $\quad a^1_{11} = 2$ pivot

No perm. $\frac{a_{21}}{a_{11}} = 1 = m^1_{21}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \overset{L}{\begin{bmatrix} 1 & & \\ -1 & 1 & \\ 1 & & 1 \end{bmatrix}} \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \frac{a_{31}}{a_{11}} = -1 = m^1_{31}$

$\underline{J=2}$ $\to$ pivoto $2^a \leftrightarrow 3^a$

$$m^2_{32} = \frac{a^2_{32} = 0}{a^2_{22} = 2} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

...cont!

$$L_{ij} = \begin{cases} 0 & j > i \\ -m_{ij}^k & i < j \\ 1 & i = j. \end{cases} \quad i > j$$

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(1) PLU by GEPP

$$A = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 7 & -1 \\ 0 & -1 & 0 \end{bmatrix} = PLU \qquad \begin{bmatrix} -4 & 7 & -1 \\ 2 & -4 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad -2 \quad 3'5 \quad -1/2$$

GEPP Algorithm (with overwriting)

for $j = 1, \dots, n-1$

choose $i_j \in \{1, \dots, n\}$ s.t. $|a_{i_j, j}| \to \max$ → mayor por columnas

swap row j and i_j

for $i = j+1, \dots, n$

$a_{ij} \leftarrow a_{ij} / a_{jj}$ → pivot

for $i, k = j+1, \dots, n$

$a_{ik} \leftarrow a_{ik} - \underbrace{a_{ij}}^{m_{ij}} a_{jk}$

Los elementos lower diag. pongo los multiplicadores (donde irían los 0). En el resto aplico las comb. lin.

$j = 1 \to |a_{21}| > |a_{11}| \to$ permute $P_1 \leftrightarrow P_2$ pivot $= -4$

$$\begin{bmatrix} -4 & 7 & -1 \\ -1/2 & -1/2 & -1/2 \\ 0 & -1 & 0 \end{bmatrix} \qquad m_{21} = \frac{2}{-4} = -1/2 \qquad m_{31} = 0$$

$j = 2 \to |a_{32}| > |a_{22}| \to P_2 \leftrightarrow P_3$ pivot $= -1$

1 SQUARE = ____________

$$\begin{bmatrix} -4 & 7 & -1 \\ 0 & -1 & 0 \\ -1/2 & -1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} -4 & 7 & -1 \\ 0 & -1 & 0 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$$

U

L

$$m_{32} = \frac{-1/2}{-1} = 1/2$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} -4 & 7 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$$

$$b = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

solve $Ax = b$ $\quad P\underbrace{L\overbrace{U\,x}^{y}}_{z} = b$ $\quad Pz = b$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} z_3 = 0 \\ z_1 = 2 \\ z_2 = -1 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$L \cdot y = z$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$y_3 = 2$

$y_2 = -1$

$-\frac{1}{2} y_1 + \frac{1}{2} \overset{-1}{y_2} + \overset{2}{y_3} = 0$

$-\frac{1}{2} y_1 = \frac{1}{2} - 2$

$-1 y_1 = -3$ $\quad y_1 = 3$

$$y = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

1 SQUARE = ____________

* Condition number → information about behaviour of x in front of small changes in (A, b) → data.
κ

* Ill conditioned → κ small/large compared to norm of A
↳ small changes in (A, b) → large changes in x.

$Ux = y$

$$\begin{bmatrix} -4 & 7 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$-4x_1 + 7\overset{1}{x_2} - \overset{-4}{x_3} = 3 \rightarrow x_1 = \frac{3-7-4}{(-4)} = \frac{-8}{-4} = 2$

$+x_2 = +1$

$-\frac{1}{2}x_3 = 2 \rightarrow x_3 = -4$

$$\boxed{x = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}}$$

Vector and matrix norms

$\|\cdot\| : F^n \rightarrow \mathbb{R}$

$\|x+y\| \leq \|x\| + \|y\|$ Triangle inequality

p-norm for $p \in 1, \infty$

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} & 1 \leq p \leq \infty \\ \max_i |x_i| & p = +\infty \end{cases}$$

Matrix norms: $\|AB\| \leq \|A\| \|B\|$ $A \in F^{m \times n}$, $B \in F^{n \times p}$

Frobenius $A^{m \times n}$ $\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$

(Euclidian)

orthogonal $Q^t Q = 1\!\!1_n$

$\|Q^t A Q\|_F = \|A\|_F$

(1) $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ maximum row sum

2) $\|A\|_1 = \max_j \sum_i |a_{ij}|$ " column sum

SQUARE = ... absolute value of eigenvalues

(3) $\|$

Symetric matrices $\boxed{LDL^T}$

$A = LU$ (without permutation)

If A symmetric $\rightarrow$ $U = DL^T$ $\quad D = \mathrm{diag}(u_{11}, \dots, u_{nn})$

$$A = LDL^T \qquad Ax = b \qquad \underbrace{LD\overbrace{L^T x}^{y}}_{z} = b$$

We assume A admits LU factorization $|a_{11}| \neq 0$

$$A = \left[\begin{array}{c|c} u_{11} & A_{12} \\ \hline A_{21} & A_{2,2} \end{array}\right] \qquad A^T = A \text{ (symetric)} \qquad A_{12} = A_{21}^T$$

$$= \begin{bmatrix} 1 & 0 \\ L_{2,1} & \mathbb{1}_{n-1} \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & L_{2,1}^T \\ 0 & \mathbb{1}_{n-1} \end{bmatrix}$$

$d_1 = a_{1,1}$

$L_{2,1} = a_{1,1}^{-1} \cdot A_{21}$

A_1: Schur complement

$\hookrightarrow D_1 = A_{22} - A_{2,1} L_{2,1}^T$

Algorithm

for $j = 1, \dots, n$

$\quad d_j \leftarrow a_{jj}^{j-1}$ $\quad$ j-th diagonal entry of D

$\quad$ for $i = j+1, \dots, n$

$\quad\quad l_{i,j} \leftarrow a_{ij}^{j-1} / a_{jj}^{j-1}$ $\quad$ jth column of L

$\quad$ for $i, k = j+1, \dots, n$

$\quad\quad a_{i,k}^{j} \leftarrow a_{i,k}^{j-1} - a_{i,j}^{j-1} l_{k,j}$ $\quad$ jth Schur complement

1 SQUARE =

$d_n \leftarrow a_{nn}^{n-1}$ $\quad$ nth diagonal entry of D

$\to$ J-th column of A

LDL^T with storage management:

for $J = 1, \dots, n$

 for $i = J+1, \dots, n$

  $\boxed{c_i \leftarrow a_{iJ}}$ *

  $\boxed{a_{iJ} \leftarrow a_{iJ}/a_{JJ}}$ *

 for $k = J+1, \dots, n$ $\quad i = k, \dots, n$

  $a_{ik} \leftarrow a_{ik} - c_i a_{kJ}$ *

elemento de su izquierda.

EX

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 17 \end{bmatrix}$$

c

La primera fila se queda igual

$J = 1$ $\quad c = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ *

$$A = \begin{bmatrix} 1 & & \\ -1/1 & 5-(-1)(-1) & \\ 2/1 & 2-(2)(-1) & 17-(2)(2) \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 4 & \\ 2 & \boxed{4} & 13 \end{bmatrix}$$

c

$J = 2$ $\quad c = \begin{bmatrix} * \\ 4 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & & \\ -1 & 4 & \\ 2 & 1 & 13 - 4 \cdot (1) \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 4 & \\ 2 & 1 & 9 \end{bmatrix}$$

4/4

D

L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}$$

$\to$ Ahora me fijo en el bloque reducido. De normal iría a buscar el elemento de arriba, como es simétrica el de lado

TSQUARE =

24 Jan 2021

1

(2) $A = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 7 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

LDL^T $J = 1$

$c = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & & \\ -2 & 7-(-4)(-2) & \\ 0 & -1-0\cdot 0 & 0-0\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & & \\ -2 & 1 & \\ 0 & -1 & 0 \end{bmatrix}$

$J = 2$

$c = \begin{bmatrix} * \\ -1 \end{bmatrix}$ $A = \begin{bmatrix} 2 & & \\ -2 & 1 & \\ 0 & -1 & -1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

1 SQUARE =

CHOLESKY

Symmetric positive definite SPD → eigenvalues > 0

$x^T A x > 0$

$$\boxed{\underset{n\times n}{A} = \underset{n\times n}{G} G^T}$$

G → lower triang with > 0 diag. entries

$A = LDL^T = GG^T$ ⟨ $L = G\Lambda^{-1}$, $D = \Lambda^2$ $\quad \Lambda = \text{diag}(G)$

$G = L \cdot \text{diag}(d_{11}^{1/2}, \ldots, d_{nn}^{1/2})$

Algorithm Cholesky

for $j = 1, \ldots, n$

$\quad a_{j,j} \leftarrow \left(a_{j,j} - \sum_{k=1}^{j-1} a_{jk}^2\right)^{1/2}$

$\quad$ for $i = j+1, \ldots, n$

$\quad\quad a_{ij} \leftarrow \left[a_{ij} - \sum_{k=1}^{j-1} a_{ik} a_{jk}\right] / a_{jj}$

(Daphne's way)

$$\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{matrix} k=1 \\ k=2 \\ k=3 \end{matrix}$$

$$l_{ki} = \left(a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}\right) / l_{ii} \quad k \neq i \text{ not diag}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2} \quad k = i \to \text{diag (!)}$$

Ex

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 17 \end{bmatrix}$$

$J=1 \quad K \neq 1, 0$

$a_{11} = (a_{11} - 0)^{1/2} = 1$

$a_{21} = a_{21} - 0 / a_{11} = -1/1$

$a_{31} = a_{31} - \sum_{k=1}^{0} X / a_{11} = 2/1$

$J=2, \; k=1$

$a_{22} = \left(5 - \sum_{k=1}^{1} \underset{1}{a_{21}^2}\right)^{1/2} = (5-1)^{1/2} = 2$

$a_{32} = \left[2 - \sum_{k=1}^{1} \underset{2}{a_{31}} \cdot \underset{-1}{a_{21}}\right] / \underset{2}{a_{22}} = +2$

$J=3$
$k=2$

$\overset{3}{a_{33}} = \Big(\overset{17}{a_{33}} - \sum_{1}^{2} \underset{4}{a_{31}^2} + \underset{4}{a_{32}^2}\Big)^{1/2}$

1 SQUARE =

Regla premotécnica Cholesky

Elems diag → ese elemento - suma de los cuadrados de los elementos que estan encima (o a la izquierda) de él.

$$\begin{bmatrix} \square & \bigcirc & \bullet \\ & \square & \bullet \\ & & \blacksquare \end{bmatrix}$$

$$\boxed{u_{JJ} = \left(a_{JJ} - \sum_{k=1}^{J-1} a_{Jk}^2\right)^{1/2}}$$

$$\blacksquare = \left(\blacksquare - \sum \bullet^2\right)^{1/2}$$

Elems no diag → ese elemento - Σ por columnas de:

$$\begin{bmatrix} x & & & \\ \Delta & x & & \\ \Delta & \Delta & x & \\ \Delta & \Delta & \Delta & x \end{bmatrix}$$

Π elementos columna anterior que no son de la diagonal.

$$\boxed{a_{iJ} = \left(u_{iJ} - \sum_{k=1}^{J-1} a_{ik} \cdot a_{Jk}\right) / a_{JJ}}$$

$$\Delta = \left(\Delta - \sum \Pi\Delta + \Pi\Delta\right) / x$$

SPLITING AND ITERATIVE METHODS

$A = M - K$ (Splitting) M non singular

$Ax = b \rightarrow Mx = Kx + b \rightarrow x = \underbrace{M^{-1}K}_{R} x + \underbrace{M^{-1}b}_{c}$

$\boxed{x_l = R\,x_{l-1} + c}$ $l \geq 1$

$A x_\infty = b$

$\boxed{\text{Convergence} \rightarrow \rho(R) < 1}$: spectral radius of the iteration matrix R.

maximum absolute value of its eigenvalues.

ITERATIVE

Assume all diag(A) $\neq 0$ $\qquad$ $L, \tilde{L} \to$ lower triang

$$A = D - \tilde{L} - \tilde{U} = D(\mathbb{1}_n - L - U) \qquad U, \tilde{U} \to \text{upper triang.}$$

Jacobi

$J = 1, \dots, n$

$$x_{lJ} \leftarrow \frac{1}{a_{JJ}} \left(b_J - \sum_{k \neq J} a_{J,k} \, x_{l-1,k} \right)$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

matrix notation

$$x_l = R_J \, x_{l-1} + C_J$$

$$R_J = D^{-1}(\tilde{L} + \tilde{U}) = L + U$$

$$C_J = D^{-1} b$$

at $l \to$ the j-th eq. is satisfied when taking

$$\underbrace{x_1^{l-1}, x_2^{l-1}, \dots}_{\text{previous } (l-1)}, \underbrace{x_J^{l}}_{\text{updated}}, \underbrace{x_{J+1}^{l-1}, \dots, x_n^{l-1}}_{\text{previous } (l-1)}$$

Gauss-Seidel $\to$ take advantage of the already computed $J-1$ components of the approx. solution.

for $J = 1, \dots, n$

$$x_{l,J} \leftarrow \omega \frac{1}{a_{JJ}} \left(b_J - \underbrace{\sum_{k<J} a_{Jk} \, x_{lk}}_{\text{updated } x\text{'s}} - \underbrace{\sum_{k>J} a_{Jk} \, x_{l-1,k}}_{\text{older } x\text{'s}} \right) + (1-\omega)\, x_{l-1,J}$$

SOR(ω) $\circledast$

Matrix not. GS

$$x_{l+1} = R_{GS} \, x_l + C_{GS}$$

$$R_{GS} = (\mathbb{1}_n - L)^{-1} U = (D - \tilde{L})^{-1} \tilde{U}$$

$$C_{GS} = (D - \tilde{L})^{-1} b = (\mathbb{1}_n - L)^{-1} D^{-1} b$$

Matrix not SOR(ω)

$$x_l = R_{SOR(\omega)} \, x_{l-1} + C_{SOR(\omega)} b$$

$$R_{SOR(\omega)} = (\mathbb{1}_n - \omega L)^{-1} \left((1-\omega) \mathbb{1}_n + \omega U \right) = (D - \omega\tilde{L})^{-1}\left((1-\omega) D + \omega \tilde{U} \right)$$

$$C_{SOR(\omega)} = \omega (\mathbb{1}_n - \omega L) D^{-1} b = \omega (D - \omega \tilde{L}) b$$

SOR(ω) $\to$ successive overrelaxation $\to$ weighted average x_{l+1} and x_l

$$x_l^{SOR(\omega)} = (1-\omega)\, x_{l-1}^{GS} + \omega \, x_l^{GS} \qquad l \geq 1 \quad \circledast$$

1 SQUARE =

$\boxed{\text{Jan 21 2019}}$ Ex $\boxed{1}$

SPD $A = L \cdot L^T$ $L \to$ lower triang.

$A = G G^T$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

$J=1$ $\quad J=2, k=1$ $\quad J=3, k=2$

$a_{11} = 1^{1/2} = 1$

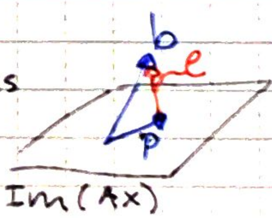
$a_{21} = -1/1 = -1$ $\quad a_{22} = (10 - (-1)^2)^{1/2} = 3$

$a_{31} = 0/1 = 1$ $\quad a_{32} = [3 - (-1 \cdot 0)]/3 = 1$ $\quad a_{33} = (5 - (1^2 + 1^2))^{1/2} = \sqrt{3}$

$\boxed{\text{LSP}}$ $\|Ax - b\|_2$

- Normal equations
- QR fact.
- SVD

Normal eq. $\quad A_{m \times n}$ $\quad p = A x_{LS}$



Full rank $m \geq n$, $Rank(A) = n$

$Ax \cdot e = 0$ $\quad (Ax)^T e = (Ax)^T (b - A x_{LS})$

$$\boxed{A^T A x_{LS} = A^T b}$$

$e \to$ residual $b-p$.
$\hookrightarrow$ minimum when $Ax \perp e$ ($p \to$ orthogonal projection of b in Im(Ax))

$A^T A \to n \times n$ SPD

1 SQUARE = ____________

Normal eq Algorithm $A^T A x_{LS} = A^T b$

$C \leftarrow A^T A$

$d \leftarrow A^T b$

Cholesky $C = G G^T$ $\qquad G \underbrace{G^T x_{LS}}_{y} = d$

$\boxed{\text{QR fact.}}$

Gram-Schmidt orthogonalization GSO

orthonormal set of vectors st

$\mathrm{span}(q_1, \dots, q_J) = \mathrm{span}(a_1, \dots, a_J)$ $\quad J \to$ colums of A

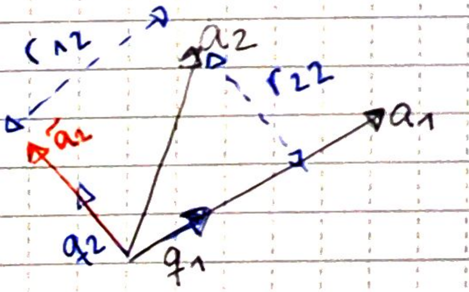
$\boxed{1}$ $r_{11} \leftarrow \|a_1\|_2$ $\quad \boxed{q_1 \leftarrow \frac{a_1}{r_{11}}}$

$\boxed{2}$ $r_{1,2} \leftarrow \langle q_1, a_2 \rangle$ $\quad \tilde{a}_2 \leftarrow a_2 - r_{12} q_1$ $\quad r_{22} \leftarrow \|\tilde{a}_2\|_2$ $\quad \boxed{q_2 \leftarrow \frac{\tilde{a}_2}{r_{22}}}$

a la dir. a_2 le quitamos la componente en la dir. u_1, para que sea ortog.

$\boxed{3}$ $r_{1,3} \leftarrow \langle q_1, a_3 \rangle$ $\quad r_{2,3} \leftarrow \langle q_2, a_3 \rangle$, $\quad \tilde{a}_3 \leftarrow a_3 - r_{13} q_1 - r_{23} q_2$

$r_{33} \leftarrow \|\tilde{a}_3\|_2$



$\boxed{q_3 \leftarrow \frac{\tilde{a}_3}{r_{33}}}$

1 SQUARE =

$$[a_1 \ \dots \ a_n] = [q_1 \ \dots \ q_n] \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$$

$r_{ij} = \langle q_i, a_j \rangle \quad 1 \le i \le j \le n$

$Q^T Q = \mathbb{1}_n \rightarrow$ orthogonal.

Thin QR

$A = QR$

full rank

$$\underset{m \times n}{A} = \underset{m \times n}{Q} \ \underset{n \times n}{R}$$

upper triang with positive diagonal entries

Full QR $\quad A = \tilde{Q}\tilde{R}$, $\tilde{Q}$: $m \times m$, $\tilde{R}$: $m \times n$

$$\tilde{R} = \begin{bmatrix} R \\ 0 \end{bmatrix} \begin{matrix} \}\, n \\ \}\, m-n \end{matrix}$$

$$\boxed{x_{LS} = (A^T A)^{-1} A^T b = R^{-1} Q^T b} \qquad A = QR$$

when A is close to rank defficient $\rightarrow$ not num. stable

Householder reflections $\quad A_{m \times n}$

orthogonal transformations : $P \quad P^T = P, \ P^T P = \mathbb{1}_m$ (reflections) Ex $m \times n = 4 \times 3$

1) Choose $P_1 \in \mathbb{R}^{m \times m}$ (4×4) $\qquad A^{(1)} = P_1 A \qquad \begin{bmatrix} x & x & \cdots \\ 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}$

2) $P_2' \in \mathbb{R}^{3 \times 3} \qquad A^{(2)} = P_2 \cdot A^{(1)} \qquad \begin{bmatrix} x & & \\ 0 & x & \\ 0 & 0 & \cdots \\ 0 & 0 & \end{bmatrix}$

3) $P_3' \in \mathbb{R}^{2 \times 2} \qquad A^{(3)} = P_3 A^{(2)} \qquad A^{(3)} = P_3 P_2 P_1 A$

$\tilde{Q}\tilde{R}$

1 SQUARE = ____________

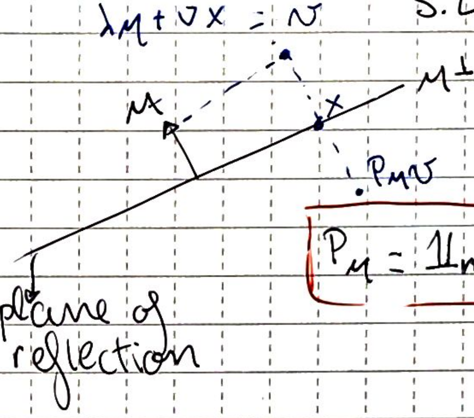
$\boxed{\tilde{Q} = P_1^T P_2^T P_3^T \Lambda}$ $\boxed{\tilde{R} = \Lambda A^{(3)}}$ $\Lambda = \mathrm{diag}(\pm 1, \ldots, \pm 1) \in \mathbb{R}^{4\times 4}$

s.t. $\mathrm{diag}\,\tilde{R} > 0$

Ex

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ -1 & -1 \end{bmatrix}$$

a_1



$\boxed{P_\mu = \mathbb{1}_m - 2\mu\mu^T}$

$P_\mu v = -\lambda\mu + \nu x$

Given point $a \in \mathbb{R}^m$ we want to find plane with vector $\mu \in \mathbb{R}^m$

s.t. $P_\mu a = c e_1$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m$ $\quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$

$\boxed{\tilde{\mu} = a + \mathrm{sign}(a_1)\|a\| e_1 \quad \mu = \frac{\tilde{\mu}}{\|\tilde{\mu}\|}}$ $\rightarrow$ Householder vector of a

$$\tilde{\mu} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \|a\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{2} \\ 0 \\ -1 \end{pmatrix} \quad , \mu = \frac{1}{2'61} \begin{pmatrix} 1+\sqrt{2} \\ 0 \\ -1 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0'925 \\ 0 \\ -0'383 \end{pmatrix} \begin{pmatrix} 0'925 & 0 & -0'383 \end{pmatrix}$$

$$= \begin{bmatrix} -0'711 & 0 & 0'7085 \\ 0 & 1 & 0 \\ 0'7085 & 0 & 0'7066 \end{bmatrix}$$

$$A^{(1)} = P_1 A = \begin{bmatrix} -1'415 & 1'4252 \\ 0 & 2 \\ 0 & -2'832 \end{bmatrix}$$

a_2

1 SQUARE = ____________

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1\!\!1_2 - 2\mu\mu^T \end{pmatrix}$$

$$\tilde{\mu}^{(2)} = \begin{pmatrix} 2 \\ -2'832 \end{pmatrix} + 3'46 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 + \sqrt{12'02} \\ -2'832 \end{pmatrix}$$

$$\mu = \frac{\tilde{\mu}}{\|\tilde{\mu}\|} = \frac{1}{6'16} \begin{pmatrix} 5'46 \\ -2'832 \end{pmatrix} = \begin{pmatrix} 0'887 \\ -0'46 \end{pmatrix}$$

$$\begin{pmatrix} -0'573 & 0'816 \\ 0'816 & 0'5768 \end{pmatrix}$$

$$A^{(2)} = \underset{2\times2}{P_2} \cdot A^{(1)} = \begin{bmatrix} -1'415 & 1'425 \\ 0 & -3'45 \\ 0 & 0 \end{bmatrix} \rightarrow \text{Full QR factor}$$

$$\tilde{Q} = P_1^T P_2^T \Lambda \quad \tilde{R} = \begin{pmatrix} 1'415 & -1'425 \\ 0 & 3'45 \\ 0 & 0 \end{pmatrix} \quad \Lambda = (-1, -1, 0)$$

Algorithm $A_{m\times n}$

QR by Householders

$i = 1$ to $\min(m-1, n)$

$\mu_i \leftarrow \text{House}(A(i:m, i))$ $[\,\square\,\square\,\square\,]$

$P_i' \leftarrow 1\!\!1_{m-i+1} - 2\mu_i\mu_i^T$

$A_i(i:m, i:n) \leftarrow P_i' A(i:m, i:n)$

Actually

$P_i' A(i:m, i:n) = A(i:m, i:n) - 2\mu_i(\mu_i^T A(i:m, i:n))$

$\hookrightarrow$ Don't need to explicitly compute P_i

1 SQUARE = ____________

Feb 3 2021 Ex 1

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \begin{array}{cc} a_1 & a_2 \end{array}$$

$A_{m\times n} = A_{3\times 2} = \underset{m\times n}{Q}\,\underset{n\times n}{R}$

QR fact. by Gram Schmidt orthogoralization

$$\underbrace{\begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}}_{R}$$

q_1, q_2

$\boxed{r_{11}} = \|a_1\| = 3$

$\boxed{q_1} = \frac{a_1}{r_1} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$

$\boxed{r_{12}} = \langle q_1, a_2 \rangle = (2/3 \;\; 1/3 \;\; -2/3) \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = -4/3 - 2/3 = \frac{-6}{3} = \boxed{-2}$

$\boxed{q_2} = \frac{\tilde{a}_2}{r_{22}} = \begin{pmatrix} -2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$

$\tilde{a}_2 = a_2 - r_{12} q_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - (-2) \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$

$\boxed{r_{22}} = \sqrt{2 \cdot \frac{4}{9} + \frac{1}{9}} = \boxed{1}$

b) LSP $b = (1, 0, 1)$

$\boxed{x_{LS}} = R^{-1} Q^T b = \underbrace{\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}^{-1}}_{A} \underbrace{\frac{1}{3} \begin{pmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{pmatrix}^T}_{B} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{C} = \boxed{\begin{pmatrix} -2 \\ -3 \end{pmatrix}}$

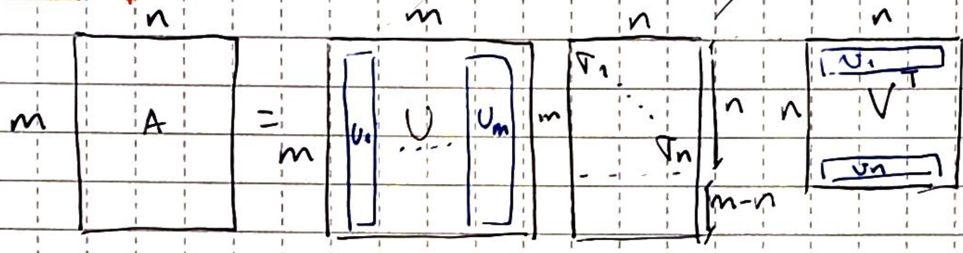
1 SQUARE = __________

Singular Value Decomp **SVD**

$A \in \mathbb{R}^{m \times n}$ $\quad m \geq n$ $\quad$ over reals.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

full SVD $\boxed{A = U \Sigma V^T}$ $\qquad AV = U\Sigma$



$\sigma_1, \dots \sigma_n \geq 0$

$U \to$ left singular vectors
orthogonal base $\mathbb{R}^m$
(eigenvectors of AA^t)

$V \to$ right singular vectors
orthogonal base $\mathbb{R}^n$
(eigenvectors of A^tA)

Thin SVD $\boxed{A = U_n \Sigma_n V^T}$

U_n: $m \times n$ $\quad \Sigma_n$: $n \times n$ diag $\quad V^T$: $n \times n$ as before

Reduced SVD $\boxed{A = U_r \Sigma_r V_r^T}$ $\qquad \operatorname{rank}(A) = r$

U_r: $m \times r$ $\quad \Sigma_r$: $r \times r$ $\quad V_r^T$: $r \times n$

$\to \sigma_1 \geq \dots \geq \sigma_r > 0$

$\sigma_{r+1} = \dots = \sigma_n = 0$

Nº of $\sigma_i \neq 0 \to \operatorname{rank}(A)$

$\to AA^t = U \Lambda_m U^T \in \mathbb{R}^{m \times m}$ $\qquad \Lambda_m = \operatorname{diag}(\sigma_1^2, \dots, \sigma_n^2, 0, \dots, 0) \in \mathbb{R}^{m \times m}$

$\to A^tA = V \Lambda_n V^T \in \mathbb{R}^{n \times n}$ $\qquad \Lambda_n = \operatorname{diag}(\sigma_1^2, \dots, \sigma_n^2) \in \mathbb{R}^{n \times n}$

Eigenvalues A^tA & $AA^t \Rightarrow \lambda_1, \dots, \lambda_n = \sigma_1^2, \dots, \sigma_n^2$

Singular values A^tA & $AA^t \Rightarrow \sigma_1, \dots, \sigma_n$
$\sqrt{\lambda_1}, \quad \sqrt{\lambda_n}$

1 SQUARE =

$(AB)^T = B^T A^T$

General case

1) $A^T A = \underline{Q \operatorname{diag}(\lambda_1, \dots, \lambda_n) Q^T}$ $\quad \lambda_1 \geq \dots \lambda_r \geq \lambda_{r+1} = \dots \lambda_n = 0$

$v_1, \dots v_n$ columns of Q $\quad \begin{bmatrix} v_1 & & \end{bmatrix}$

$v_1, \dots, v_r$ eigenvectors of $A^T A$ $\quad$ non-zero eigenv. $\lambda_1, \dots, \lambda_r$

2) $\boxed{\sigma_k = \lambda_k^{1/2}}$ $\quad \boxed{u_k = \frac{1}{\sigma_k} A v_k}$ $\quad k = 1, \dots, r$

we get the reduced SVD

To get the full SVD $\rightarrow$ $\left.\begin{matrix} v_{r+1}, \dots, v_n & \mathbb{R}^n \\ u_{r+1}, \dots, u_m & \mathbb{R}^m \end{matrix}\right\}$ complete to orthonormal bases

⊗ When A SPD $\rightarrow$ no cal fer $\underbrace{A^t A}_{SPD}$ ja és SPD

$A \in \mathbb{R}^{n \times n}$

$\lambda_1 \geq \dots \geq \lambda_n \geq 0$ eigenvalues of

$v_1, \dots, v_n$ eigenvectors of

$Q = [v_1, \dots, v_n]$ $\quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

$A = Q \Lambda Q^T$ $\quad Q = U = V$ $\quad \Sigma = \Lambda \rightarrow A = U \Sigma V^T$

Fun facts

$$\boxed{\begin{aligned} \|A\|_2 &= \|U \Sigma V^T\|_2 = \|\Sigma\|_2 = \sigma_1 \\ \|A\|_F &= \|\Sigma\|_F = \left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2} \\ \kappa_2(A) &= \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n \end{aligned}}$$

$\left(\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}\right)$

$\left(A^{-1} = V \Sigma^{-1} U^T\right)$

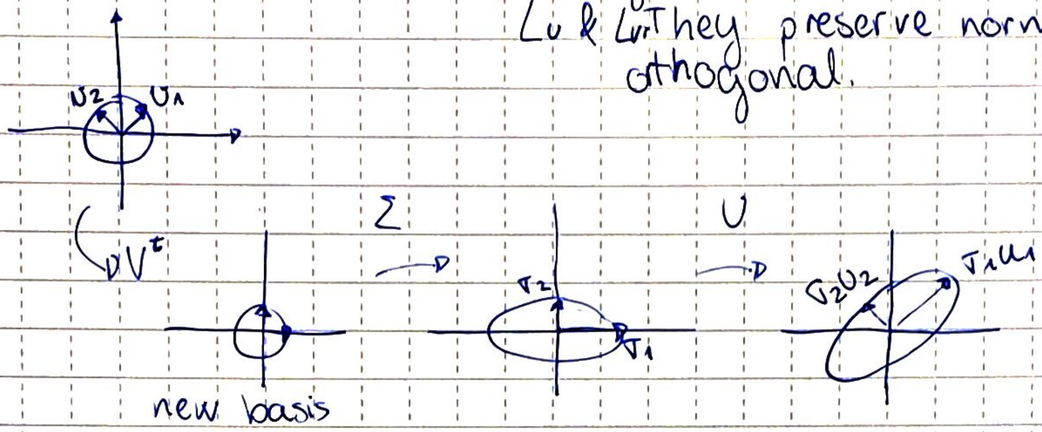
$A_{n \times n}$ non $\rightarrow$ $\kappa_2(A)$

$\rightarrow$ singular values $A^t A$

$\hat{=}$ A

1 SQUARE =

3) $L_A = L_U \circ L_\Sigma \circ L_{V^t}$ columns of U and V are orthogonal basis for $\mathbb{R}^m$ and $\mathbb{R}^n$ ($\mathbb{R}^2$ both in this case). L_U & L_{V^t} They preserve norm since they are orthogonal.



new basis $v_1, \dots, v_n$

If we take vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, their corresponding images by L_A would be:

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0'6369 \\ 1'11 \end{bmatrix}$ $\quad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0'29 \\ -0'763 \end{pmatrix}$

Jan 2019

3 1) $K_2(A) = \sigma_1 / \sigma_n = 1'73 / 0'71$

2) rank(1) $\rightsquigarrow$ $A_{\underset{1}{K}} = \sigma_1 u_1 v_1^T = 1'73 \begin{pmatrix} -0'82 \\ 0'41 \\ -0'41 \end{pmatrix} \begin{pmatrix} -0'71 & -0'71 & 0 \end{pmatrix}$

w.r.t. 2-norm

$\|A - A_1\|_2 = \sigma_{k+1} = \sigma_2 = 1$ $\quad = \begin{bmatrix} 1'0072 & 1'0072 & 0 \\ -0'503 & -0'503 & 0 \\ 0'5036 & 0'5036 & 0 \end{bmatrix}$

rank(2) $\rightsquigarrow$ $A_2 = \sum_{i=1}^{2} \sigma_i u_i v_i^T = U_K \Sigma_K V_K^T =$

$= \begin{pmatrix} -0'82 & 0 \\ 0'41 & -0'71 \\ -0'41 & -0'71 \end{pmatrix} \begin{pmatrix} 1'73 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -0'71 & -0'71 & 0 \\ 0'71 & -0'71 & 0 \end{pmatrix} =$

1 SQUARE = __________

$ax^2+bx+c=0$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

EIGEN PROBLEMS

$Av=\lambda v$ $\quad \boxed{\det(A-\lambda \mathbb{1}_n)=0}$

Schur decomposition Not unique!

$A=QTQ^*$ — unitary $Q^*=Q^{-1}$

T: upper triang.

colums of $Q=[q_1,\dots,q_n]$ Schur vectors of A

→ Eigenvalues of T ≡ eigenvalues A

$\forall(\lambda,y)$ s.t. $Ty=\lambda y \rightarrow AQy=QTy=\lambda Qy$

λ, y eigenpair of T

Qy: eigenvector of A with λ eigenvalue

From $(T-\lambda\mathbb{1}_n)y=0 \rightarrow \boxed{y=\begin{bmatrix}-(T_{1,1}-\lambda\mathbb{1}_{k-1})^{-1}T_{1,2}\\ 1\\ 0\end{bmatrix}}$

Computing eigenvectors/eigenvalues

QR iteration algorithm (real entries)

$H_0 \leftarrow Q_0^T A Q_0$ initialization

for $k=1,2,\dots$

$H_{k-1}=Q_kR_k$ (QR fact.)

$H_k \leftarrow R_kQ_k$

$H_k \xrightarrow[k\to\infty]{} H$ upper triangular (Schur form $A=QTQ^t$) T H?

Diagonalizable $A=S\Lambda S^{-1} \rightarrow \exists\, S$ nonsingular

$\exists\, \Lambda$ diagonal $\text{diag}(\lambda_1,\dots,\lambda_n)$

1 SQUARE = __________

Power method

$$\begin{bmatrix} x_0 \in \mathbb{C}^n \quad \text{with } \|x_0\|_2 = 1 \\ \text{for } k = 0, 1, 2, \dots \\ \quad y_{k+1} \leftarrow A x_k \\ \quad x_{k+1} \leftarrow \frac{y_{k+1}}{\|y_{k+1}\|_2} \end{bmatrix}$$

when stoping at $it = l$

$$(\tilde{x}, \tilde{\lambda}) \leftarrow (x_{l+1}, \; x_{l+1}^* A x_{l+1})$$

The residual:

$$\|\tilde{\lambda}_k - \lambda_1\|_2 \leq O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

Number of correct digits in base β of this aproximation:

$$-\log_\beta \|x_k - s_1\|_2 \; ; \; \boxed{-\log_\beta \|\tilde{\lambda}_k - \lambda_1\|_2 \geq k \log_\beta\left(\left|\frac{\lambda_1}{\lambda_2}\right|\right) + cte}$$

Recall $A = S \Lambda S^{-1}$ $\quad S = (s_1, \dots, s_n)$ $\quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$

eigenvector A $\qquad$ eigenv.

quan $\tilde{\lambda}_k - \lambda_1$ són molt propers, et dona

rollo $10^{-20} \to 20$ decimals correctes. $-\log\|\tilde{\lambda}_k - \lambda_1\| = 20 \geq k \log_\beta\left(\left|\frac{\lambda_1}{\lambda_2}\right|^k\right)$

$\to$ better with givens rotations than symmetries

Hessenberg reduction $\to$ variation of QR

Choose Q_0 carefully s.t. $H_0 = Q_0^T A Q_0$ upper Hessenberg mat

$\hookrightarrow h_{ij} = 0$ if $i > j+1$

lower secondary diag. and on are zero

1) $A_{5\times5}$

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1 \end{bmatrix} \quad \tilde{P}_1 \text{ Householder } 4\times4$$

$$P_1 A = \begin{bmatrix} x & & & \\ x & x & & \\ 0 & \cdot & \cdot & \\ 0 & & & \\ 0 & & & \cdot \end{bmatrix} \qquad A_1 = P_1 A P_1^T$$

2) $$P_2 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix} \qquad P_2 A_1 = \begin{bmatrix} x & x & & \\ x & x & & \\ 0 & x & \cdot & \\ 0 & 0 & & \\ 0 & 0 & & \cdot \end{bmatrix} \qquad A_2 = P_2 A_1 P_2^T \quad \dots \quad \boxed{Q_0 = P_n \dots P_1}$$

1 SQUARE = ____

QR fact by given Rotations

rotation $R(\theta): \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R(i,j,\theta) = \begin{bmatrix} \mathbb{1}_{i-1} & & & & \\ & \cos\theta & & -\sin\theta & \\ & & \mathbb{1}_{j-i-1} & & \\ & \sin\theta & & \cos(\theta) & \\ & & & & \mathbb{1}_{n-j-1} \end{bmatrix}$$

$1 \leq i \leq j \leq m$

Jan 2021

2 $A = UTU^T$

$$U = \begin{bmatrix} -0'61 & -0'48 & 0'63 \\ -0'25 & 0'87 & 0'42 \\ 0'75 & -0'09 & 0'65 \end{bmatrix} \qquad T = \begin{bmatrix} 0'34 & 0 & 0 \\ 0 & 9'3 & 0 \\ 0 & 0 & -0'64 \end{bmatrix}$$

Schur fact. $A = QTQ^T$

1) Eigenvalues of $A \equiv$ eigenvalues of T: $(0'34,\ 9'3,\ -0'64)$

Eigenvectors $T \rightsquigarrow (T - \lambda \mathbb{1}_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 8'96 & 0 \\ 0 & 0 & -0'96 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1 SQUARE = __________

Eigenvectors of A: $Q \cdot y \rightsquigarrow \begin{pmatrix} -0'61 \\ -0'25 \\ 0'75 \end{pmatrix} \begin{pmatrix} -0'48 \\ 0'87 \\ -0'09 \end{pmatrix} \begin{pmatrix} 0'63 \\ 0'42 \\ 0'65 \end{pmatrix}$

A SPD? $\to$ All eigenvalues should be > 0 $\rightsquigarrow$ No SPD

2) Rate convergence PM.

$$-\log \|\tilde{\lambda}_k - \lambda_1\|_2 \geq k \log_p \left|\frac{\lambda_1}{\lambda_2}\right|^k \qquad O\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)^k$$

$$|\lambda_1| = 9'3 \quad |\lambda_2| = 0'64 \rightsquigarrow k \log_p (14'53)^k$$

Ex) Iterative methods

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = D(\mathbb{1}_n - L - U) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1/2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1/2 \\ 0 & 0 \end{bmatrix} \right)$$

Jacobi

$$\begin{bmatrix} x_{l,1} \\ x_{l,2} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix}}_{R_J} \begin{bmatrix} x_{l-1,1} \\ x_{l-1,2} \end{bmatrix} + \underbrace{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}_{C_J} \qquad R_J = L + U = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix}$$

$$C_J = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$\rho(R_J) = 1/2 < 1 \to$ converges $\forall\ x_0$.

$\rho(R_{GS}) = \frac{1}{4} = \rho(R_J)^2 \to$ GS converges double of speed.

Convergence of Jacobi, G-S, SOR(w)

A Diagonally dominant → $|a_{jj}| > \sum_{i \neq j} |a_{i,j}|$ $j = 1, \dots, n$

diagonal entry

sum of the rest of the elements of the column.

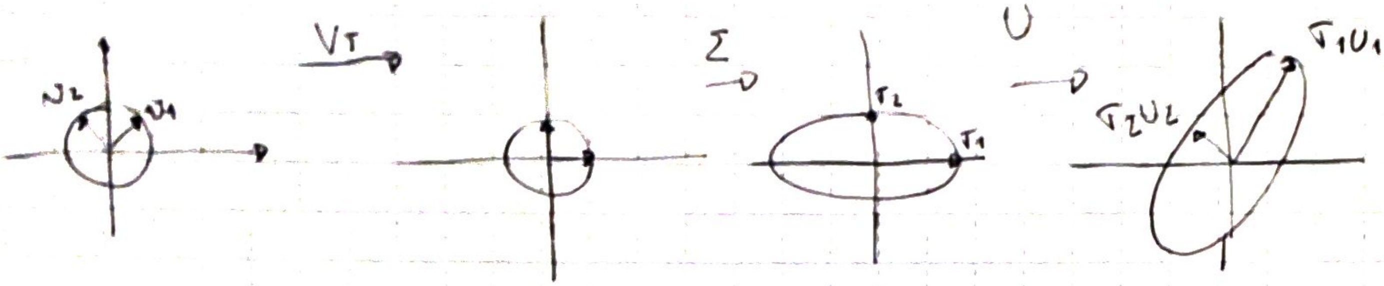
- A diagonally dominant ⇒ JACOBI & GS converge $\forall x_0$
- $0 < w < 2$ in SOR(w) ⇒ necessary (but not sufficient) condi. for convergence.
- $0 < w < 2$ and A SPD → SOR(w) converges $\forall x_0$.

1 SQUARE =

$$A = U \Sigma V^T$$

Geometry

$$L_A = L_U \circ L_\Sigma \circ L_{V^T}$$



Best low rank aprox

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad r = \mathrm{rank}(A)$$

Eckart-Young

$$\forall \; 1 \leq k \leq r \rightarrow \boxed{A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T = U_k \Sigma_k V_k^T} \rightarrow$$ best k rank approx for frob. and 2 norm

$$\|A - A_k\| = \|\Sigma - \Sigma_k\| \qquad \boxed{\begin{aligned} \|A - A_k\|_2 &= \sigma_{k+1} \\ \|A - A_k\|_F &= \left(\sum_{i=k+1}^{r} \sigma_i^2\right)^{1/2} \end{aligned}}$$

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[2] $A \quad \mathbb{R}^{2\times 2}$

AA^t eigenvalues $\begin{array}{cc} \lambda_1 & \lambda_2 \\ 2, & 1/3 \end{array}$ $\quad u_1 = \begin{pmatrix} 0'31 \\ 0'95 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0'95 \\ -0'31 \end{pmatrix}$

$A^t A$ eigenv. " " $\quad v_1 = \begin{pmatrix} 0'89 \\ -0'45 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0'45 \\ 0'89 \end{pmatrix}$

1) $$A = U \Sigma V^t = \underbrace{\begin{bmatrix} 0'31 & 0'95 \\ 0'95 & -0'31 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0'89 & -0'45 \\ 0'45 & 0'89 \end{bmatrix}}_{C} = \begin{bmatrix} 0'637 & 0'29 \\ 1'11 & -0'763 \end{bmatrix}$$

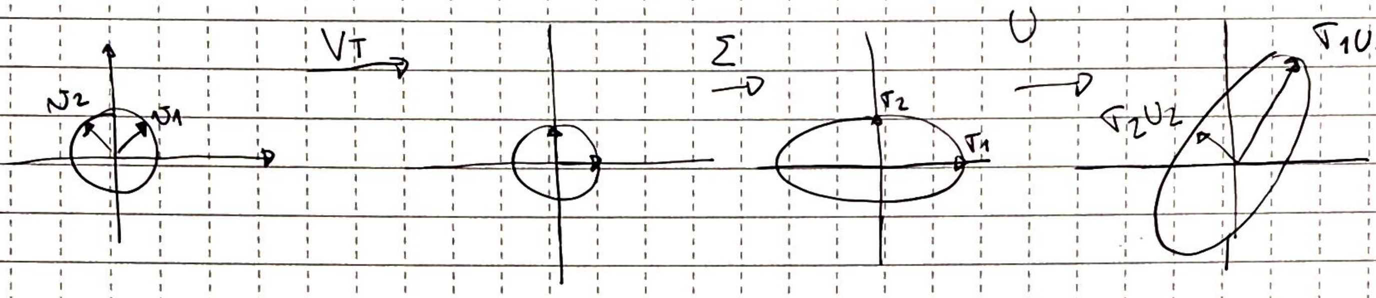
2) A non-singular

$$\boxed{k_2(A)} = \|A\|_2 \, \|A^{-1}\|_2 = \sigma_1 / \sigma_n = \sqrt{2} / \tfrac{1}{\sqrt{3}} = \boxed{\sqrt{6}}$$

1 SQUARE =

$$A = U \Sigma V^T$$

Geometry $L_A = L_U \circ L_\Sigma \circ L_{V^T}$



Best low rank aprox

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad r = \text{rank}(A)$$

Eckart-Young.

$$\forall \; 1 \leq k \leq r \rightarrow \boxed{A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T = U_k \Sigma_k V_k^T} \rightarrow$$ best k rank approx for frob. and 2-norm

$$\|A - A_k\| = \|\Sigma - \Sigma_k\| \quad \boxed{\rightarrow \|A - A_k\|_2 = \sigma_{k+1} \qquad \|A - A_k\|_F = \left(\sum_{i=k+1}^{r} \sigma_i^2\right)^{1/2}}$$

Feb. 3 2021

2 $A \; \mathbb{R}^{2\times 2}$

AA^t eigenvalues $\overset{\lambda_1}{2}, \overset{\lambda_2}{1/3}$ $u_1 = \begin{pmatrix} 0'31 \\ 0'95 \end{pmatrix}$ $u_2 = \begin{pmatrix} 0'95 \\ -0'31 \end{pmatrix}$

A^tA eigenv. " " $v_1 = \begin{pmatrix} 0'89 \\ -0'45 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0'45 \\ 0'89 \end{pmatrix}$

1) $$A = U \Sigma V^t = \underset{A}{\begin{bmatrix} 0'31 & 0'95 \\ 0'95 & -0'31 \end{bmatrix}} \underset{B}{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix}} \underset{C}{\begin{bmatrix} 0'89 & -0'45 \\ 0'45 & 0'89 \end{bmatrix}} = \begin{bmatrix} 0'637 & 0'29 \\ 1'11 & -0'763 \end{bmatrix}$$

2) A non-singular

$$\boxed{k_2(A)} = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n = \sqrt{2} / (1/\sqrt{3}) = \boxed{\sqrt{6}}$$

1 SQUARE = ____________