19/09/23 **Optimization: Ex0 - Dafni Tziakouri**

Exercise 1.1.

$$f(x) = \sum_{i=1}^{m} w_i \| x^* - y_i \|$$

(i) Firstly $f(x) > 0$, because $w_1, \dots, w_n$ are positive
(ii) real numbers and from the properties of norms. Also, f is continuous and continuous functions always has a minimum within a closed and bounded set.
We may assume now that f has at least one global minimum

To show that the function has a unique minimum, We will first prove that norms are convex functions:

* Need to show that $\| t x + (1-t) y \| \leq t \|x\| + (1-t) \|y\|$ for any $x, y \in V$, which V is a vector space in which norm belongs and $t \in [0,1]$.

Proof: $\| t x + (1-t) y \| \leq \| t x \| + \| (1-t) y \|$
$= |t| \|x\| + |1-t| \|y\|$
$= t \|x\| + (1-t) \|y\|$ ∎

Now, we need to show that the weighted sum of convex functions is still convex.

* Let g be a convex function and $a > 0$, $t \in [0,1]$

$\Rightarrow (ag)(t x + (1-t) y) = a(g(t x + (1-t) y))$
$\leq a(t g(x) + (1-t) g(y))$
$= t (ag)(x) + (1-t)(ag)(y)$

$\Rightarrow ag$ is a convex function ∎

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* If g, h convex ~~fun~~ functions, we will show that $g+h$ convex function.
Let $t \in [0,1]$, due to convexity of g, h:

$$\begin{aligned}(g+h)(tx+(1-t)y) &= g(tx+(1-t)y) + h(tx+(1-t)y) \\ &\le tg(x)+(1-t)g(y) + th(x)+(1-t)h(y) \\ &= t(g(x)+h(x)) + (1-t)(g(y)+h(y)) \\ &= t(g+h)(x) + (1-t)(g+h)(y) \quad \blacksquare\end{aligned}$$

From all of the above calculations, we see that f is convex.

- Let's proof that the minimum is unique:
Suppose that f has two global minimum x_1, x_2, then $x_1 < x_2$, $f(x_1) = f(x_2)$ and $f(x) > f(x_1) = f(x_2)$ for any other x in the domain of the function.
Define $d(x_1, x_2)$ the shortest path connecting both points. Taking $t \in d(x_1, x_2) \Rightarrow f(t) > f(x_1) = f(x_2)$ and because we are working in a convex set $\Rightarrow \exists\, s \in (0,1)$ s.t. $t = sx_1 + (1-s)x_2$.
From convexity of $f \Rightarrow$

$$\begin{aligned}f(t) = f(sx_1+(1-s)x_2) &\le sf(x_1)+(1-s)f(x_2) \\ &= sf(x_1)+(1-s)f(x_1) = f(x_1) \quad \blacksquare\end{aligned}$$

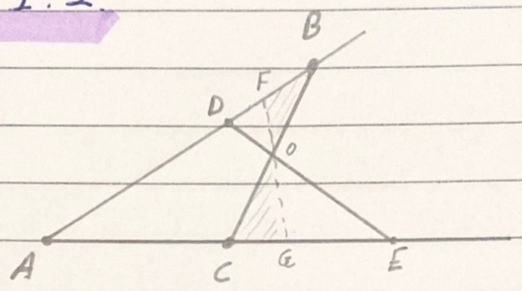
Therefore, the global minimum exist and it is unique

(iii) A physical interpretation of the solution is
(iv) the Fermat-Torricelli Problem in which the equilibrium point ~~in~~ is represented by the point where all ropes are knotted together. This point will move until it reaches the equilibrium
It is obvious that the higher weights will put the knot closer to the holes. This pull efect minimizes

$\rightarrow$

both the potential energy and the distances between the equilibrium point and knot.

Exercise 1.2



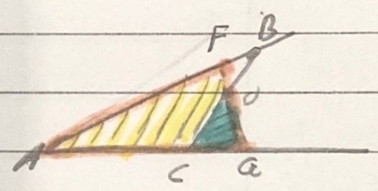
Suppose that $OB = CO$ and consider another line FG going through O.

Then we have: $\widehat{COG} = \widehat{FOB}$,

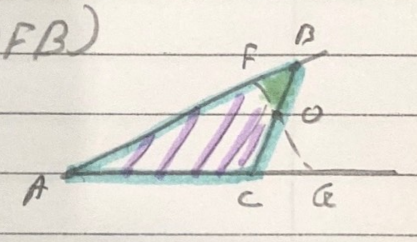
$\widehat{OCG} = \widehat{CAB} + \widehat{ABC} > \widehat{ABC}$

So, Area (OFB) < Area (OGC), because $\Delta OFB \subset \Delta OGC$

Area (AFG) = Area $(AFOC)$ + Area (OGC)



Area (ABC) = Area $(AFOC)$ + Area (OFB)



=> Area (ABC) < Area (AFG)

Similarly for line DE.