Master in Foundations of Data Science — 2016-2017

NUMERICAL LINEAR ALGEBRA

Reevaluation exam, January 26th, 2017, from 15:00h till 19:00h at room B1

1. Cholesky's algorithm computes, given a positive symmetric matrix $A \in \mathbb{R}^{n\times n}$, a factorization

$$A = L \cdot L^T$$

where L is a lower triangular matrix.

(1) Write down Cholesky's algorithm in pseudocode notation (indicating the types of the variables) and describe how it works.
(2) Is it necessary to use a pivoting strategy for this algorithm?
(3) Describe how you proceed to solve $Ax = b$, once computed the factorization $A = L \cdot L^T$.

3. Compute the QR factorization, using Householder reflexions, of the matrix

$$A = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$$

4. Consider the singular value decomposition (SVD)

$$A = \begin{pmatrix} 1 & -1 & 1/\sqrt{6} \\ 0 & 1 & 1/\sqrt{6} \\ 1 & 0 & -1/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Compute the best rank 1 approximation of the matrix A with respect to the operator norm $\| \cdot \|_2$ associated to the Euclidean metric.
(2) Compute the condition number of A with respect to the same norm.

5. The Schur form of an $n \times n$-matrix A is a factorization

$$A = QRQ^T$$

where Q is an orthogonal matrix and R is upper triangular.

(1) How can you compute the eigenvalues and the (right) eigenvectors of A using its Schur form?
(2) Given a matrix A, which algorithm would you apply to compute its Schur form?

6. Consider the matrix and the vector

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(1) For this matrix and vector, compute the corresponding iterative scheme given by (a) the Jacobi method, (b) the Gauss-Seidel method, and (c) the SOR(ω) method, for an arbitrary parameter $\omega \in \mathbb{R}$.
(2) Check if you can guarantee that the Jacobi and Gauss-Seidel schemes for this system converge, using the criterium based on the spectral radius.