Master in Foundations of Data Science — 2020-2021

NUMERICAL LINEAR ALGEBRA

Reevaluation exam, February 3rd, 2021, from 15:00h till 18:00h. Exercises should be delivered in separated pages. All answers should be suitably justified.

1. Consider the matrix

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

(1) Compute the QR factorization of A.
(2) Solve the least square problem (LSP) for this matrix and the vector $b = (1, 0, 1)^T$ using the QR factorization of A.

2. Let $A \in \mathbb{R}^{2\times 2}$ such that the eigenvalues of $A \cdot A^T$ are 2 and $\frac{1}{3}$ with respective unit eigenvectors

$$\begin{pmatrix} 0.31 \\ 0.95 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.95 \\ -0.31 \end{pmatrix},$$

and the eigenvalues of $A^T \cdot A$ are also 2 and $\frac{1}{3}$ with respective unit eigenvectors

$$\begin{pmatrix} 0.89 \\ -0.45 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.45 \\ 0.89 \end{pmatrix}.$$

(1) Compute a singular value decomposition (SVD) of A.
(2) Using this SVD, compute the condition number of A with respect to the 2-norm.
(3) Determine the image of the unit disk of $\mathbb{R}^2$ with respect to the linear map defined by A.

3. Let

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(1) For this matrix and vector, write down the iterative scheme given by the Jacobi method, the Gauss-Seidel method, and the SOR(ω) method for a parameter $\omega \in \mathbb{R}$.
(2) Using the criterium based on the spectral radius, check if you can guarantee if the Jacobi and the Gauss-Seidel methods for A and b converge for any choice of initial vector $x_0 \in \mathbb{R}^2$.
(3) Choosing the starting vector $x_0 = (0, 0)^T$, how many iterations of the Jacobi and the Gauss-Seidel methods for A and b are necessary to compute 20 decimal digits of the solution? And how many if we want to compute 70 decimal digits?

4. Let $A \in \mathbb{C}^{n\times n}$ be a matrix whose eigenvalues satisfy the strict inequalities

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

(1) Explain a procedure to approximate the eigenvalue λ_1 of largest absolute value in a computationally efficient way, and give an estimate for its rate of convergence.
(2) How would you proceed to approximate the eigenvalue λ_n of smallest absolute value? Similarly as before, give an estimate for the rate of convergence of this procedure.