4^{th} Assignment - Optimization - Constrained opt, Lagrange multipliers

Exercise 4.1: Using necessary and sufficient conditions, solve the following optimization problem in terms of the parameter $b>0$;

$f(x)=(x_1-1)^2+x_2^2 \to \min$, $x\in\mathbb{R}^2$, subject to $h(x_1,x_2)=-x_1+b x_2^2 \geq 0$

Interpret the solutions geometrically in terms of the level curves and the restrictions

Using the KKT conditions: the lagrangian for the problem is

$$L(x_1,x_2,\lambda)=(x_1-1)^2+x_2^2-\lambda(-x_1+b x_2^2)$$

- $\frac{\partial L}{\partial x_1}=2(x_1-1)+\lambda=0 \Rightarrow \boxed{x_1=1-\lambda/2}$ (1)
- $\frac{\partial L}{\partial x_2}=2x_2-2\lambda b x_2=0 \Rightarrow \boxed{2x_2(1-\lambda b)=0}$ (2)
- $\boxed{\lambda(-x_1+b x_2^2)=0}$ (3)

If $\lambda=0$: from eq. (2) $\boxed{x_2=0}$ and from eq. (1) $\boxed{x_1=1}$

If $\lambda\neq 0$: from eq. (3) $\boxed{x_1=b x_2^2}$ (4)

$\overset{(1)}{\Rightarrow} 2b x_2^2-2+\lambda=0$ and $2x_2(1-\lambda b)=0$

If $x_2=0$: $\boxed{\lambda=2}$ and from eq. (1) $\boxed{x_1=0}$

If $x_2\neq 0$: $\boxed{\lambda=\frac{1}{b}}$ $\overset{eq.(1)}{\Rightarrow}$ $\boxed{x_1=1-\frac{1}{2b}}$

≥ 0 since $b>0$ $\overset{eq.(4)}{\Rightarrow}$ $\boxed{x_2=\pm\sqrt{\frac{1}{b}-\frac{1}{2b^2}}}$

Therefore the coordinates are:

$(1,0,0)$: $h(1,0)=-1<0$ not feasible

$(0,0,2)$: $h(0,0)=0$

$\left(1-\frac{1}{2b}, \pm\sqrt{\frac{1}{b}-\frac{1}{2b^2}}, \frac{1}{b}\right)$ $h\left(1-\frac{1}{2b}, \pm\sqrt{\frac{1}{b}-\frac{1}{2b^2}}\right)=0$ } feasible

→

Let's now find the local minima:

$$\nabla^2_{xx} L = \begin{pmatrix} 2 & 0 \\ 0 & 2(1-2\beta) \end{pmatrix} \text{ and } \nabla_x h = \begin{pmatrix} -1 \\ 2\beta X_2 \end{pmatrix}$$

▶ $(0,0,2)$:

$$\nabla^2_{xx} L(0,0,2) = \begin{pmatrix} 2 & 0 \\ 0 & 2(1-2\beta) \end{pmatrix}$$

$$\nabla_x h(0,0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

- Let's say we have $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ such that $z^T \nabla_x h = 0$

$$\Rightarrow (z_1 \; z_2)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -z_1 = 0$$

↳ So, the vectors that satisfy the above are in the following form: $\begin{pmatrix} 0 \\ z_2 \end{pmatrix}$

- Let's check these vector so $z^T \nabla^2_{xx} L \, z \geq 0$

$$(0 \; z_2)\begin{pmatrix} 2 & 0 \\ 0 & 2(1-2\beta) \end{pmatrix}\begin{pmatrix} 0 \\ z_2 \end{pmatrix} = (0 \;\; 2z_2(1-2\beta))\begin{pmatrix} 0 \\ z_2 \end{pmatrix}$$

$$= 2z_2^2(1-2\beta)$$

So, $2z_2^2(1-2\beta) \geq 0$ only when $\boxed{\beta \leq \frac{1}{2}}$

Because $\lambda = 2$, the sufficient conditions are satisfied when $\beta \leq 1/2 \Rightarrow \boxed{(0,0)}$ is a strict local minima

▶ $\left(1 - \frac{1}{2\beta}, \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}, \frac{1}{\beta}\right)$ this exist only $\beta \geq \frac{1}{2}$

$$\nabla^2_{xx} L\left(1-\frac{1}{2\beta}, \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}, 1/\beta\right) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\nabla_x h\left(1-\frac{1}{2\beta}, \pm\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}}\right) = \begin{pmatrix} -1 \\ \pm 2\beta\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}} \end{pmatrix}$$

$$\left(\text{For } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \text{ s.t. } z^T \nabla_x h = 0 \Rightarrow (z_1 \; z_2)\begin{pmatrix} -1 \\ \pm 2\beta\sqrt{\frac{1}{\beta} - \frac{1}{2\beta}} \end{pmatrix} = 0 \right.$$

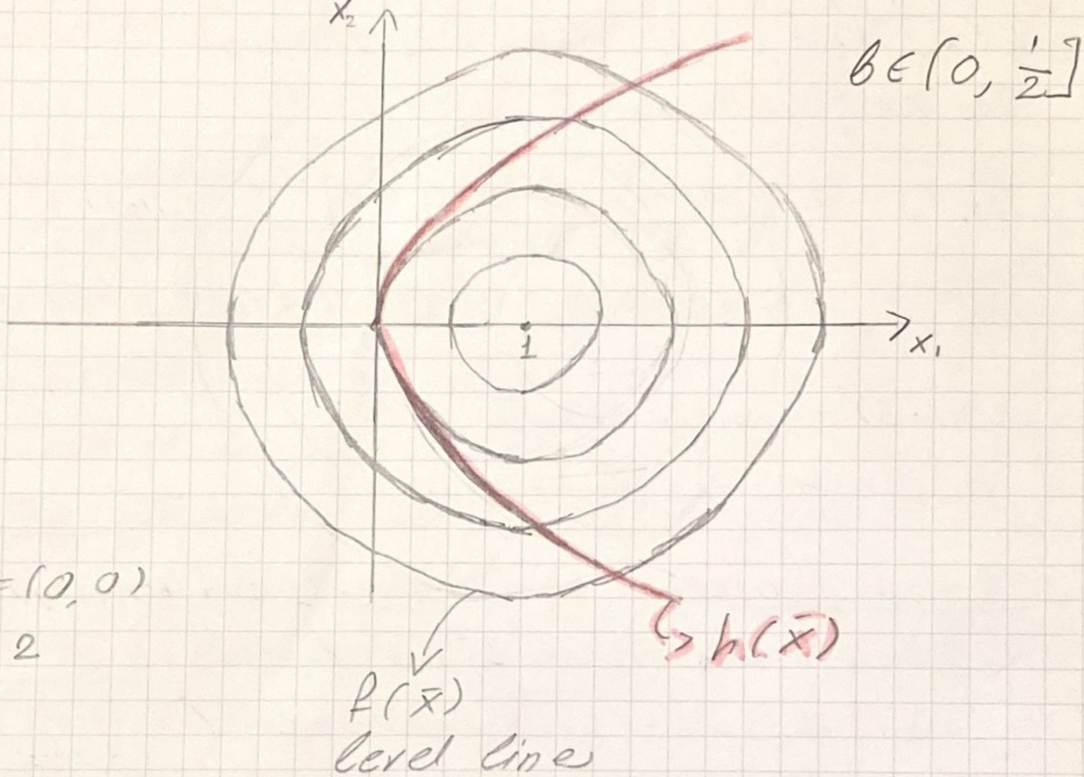
$$\left. \Rightarrow -z_1 \pm 2 z_2 \beta\sqrt{\frac{1}{\beta} - \frac{1}{2\beta^2}} = 0 \right)$$

→

Now, $(z_1 \ z_2)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (2z_1 \ 0)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2z_1^2 \geq 0 \ \forall \ z \in \mathbb{R}^n$

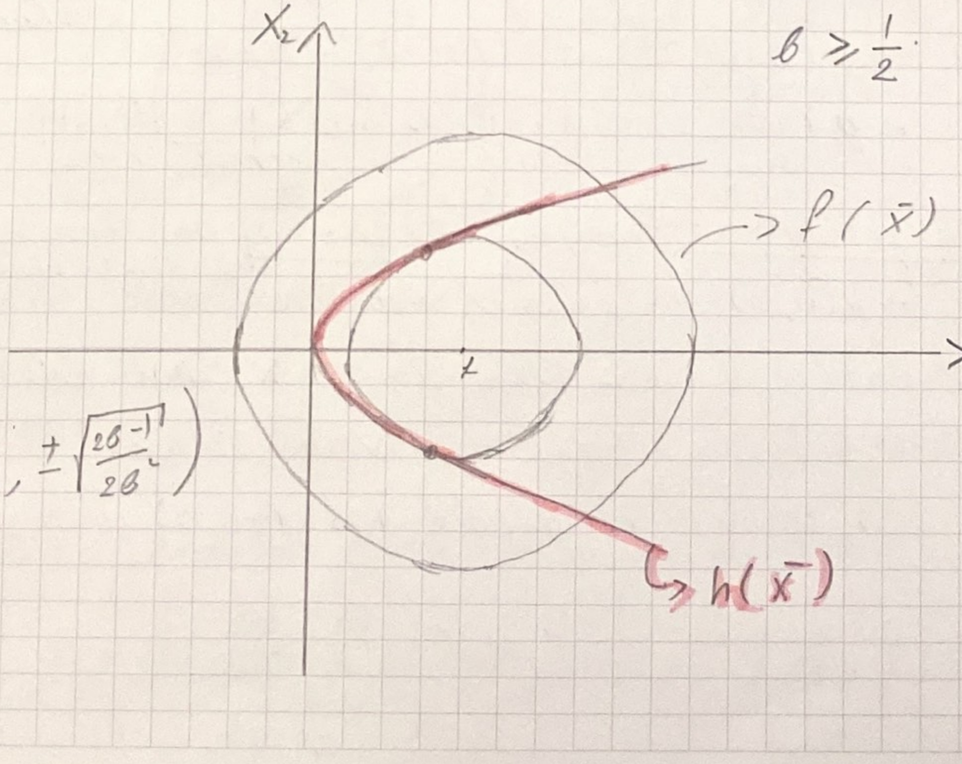
Also $\frac{1}{b} > 0$ because $b > 0$ therefore the points will be strict local minima.

Geometrically:



$x^* = (0, 0)$

$J = 2$



$x^* = \left(1 - \frac{1}{2b}, \pm\sqrt{\frac{2b-1}{2b^2}}\right)$

$J = \frac{1}{b}$

Exercise 4.2: Using necessary and sufficient conditions solve the following opt. problem: for $x=(x_1,x_2)$, $f(x)=x_1 \to min$ subject to mixed constraints

$g_1(x)=(x_1-3)^2+(x_2-2)^2-13=0$, $h_1(x)=16-(x_1-4)^2-x_2^2 \geq 0$

Lagrangian function: $L(x,\lambda,\mu)=f(x)-\lambda g_1(x)-\mu h_1(x)$

$L(x,\lambda,\mu) = x_1 - \lambda[(x_1-3)^2+(x_2-2)^2-13]-\mu[16-(x_1-4)^2-x_2^2]$

The KKT conditions:

$$\frac{\partial L}{\partial x_1} = 1-2\lambda(x_1-3)+2\mu(x_1-4)=0 \Rightarrow \boxed{x_1 = \frac{8\mu-6\lambda-1}{2(\mu-\lambda)}} \ (1) \quad \mu \neq \lambda$$

$$\frac{\partial L}{\partial x_2} = -2\lambda(x_2-2)+2\mu x_2 = 0 \Rightarrow \boxed{x_2 = -\frac{2\lambda}{\mu-\lambda}} \ (2) \quad \mu \neq \lambda$$

$(x_1-3)^2+(x_2-2)^2-13=0 \quad (3)$

$\mu[16-(x_1-4)^2-x_2^2]=0 \quad (4)$

If $\mu=0$, $\lambda>0$: from eq. (2) $\boxed{x_2=2}$ and from eq. (3) $\boxed{x_1=3\pm\sqrt{13}}$

Will check those values to see if $h_1(x) \geq 0$

$\Rightarrow h_1(3+\sqrt{13}, 2)=16-(3+\sqrt{13}-4)^2-4>0$

$\Rightarrow h_1(3-\sqrt{13}, 2)=16-(3-\sqrt{13}-4)^2-4>0$

$\Rightarrow g_1(3+\sqrt{13}, 2)=(3+\sqrt{13}-3)^2-13=0$

$\Rightarrow g_1(3-\sqrt{13}, 2)=(3-\sqrt{13}-3)^2-13=0$

$(3\pm\sqrt{13}, 2)$ feasible

If $\lambda=0$, $\mu>0$: from eq. (2) $\boxed{x_2=0}$ and from eq. (3) $\boxed{x_1=0}$, $x_1=-6$

$\Rightarrow h(0,0)=16-16=0$ ✓ feasible

$\Rightarrow h(-6,0)=16-(-10)^2=-84<0$ ✗ not feasible

$\Rightarrow g(0,0)=9+4-13=0$ ✓

So, $(0,0)$ is acceptable and $(-6,0)$ is rejected

If $\mu \neq 0$ & $\lambda \neq 0$: from eq.(4) then $16-(x_1-4)^2-x_2^2=0$

$$\Rightarrow x_2^2=(4+x_1-4)(4-x_1+4)=x_1(8-x_1)$$

$$\Rightarrow \boxed{x_2=\pm\sqrt{x_1(8-x_1)}} \quad (5)$$

Substituting in (3)

$$(x_1-3)^2+(\sqrt{x_1(8-x_1)}-2)^2-13=0$$

$$\Rightarrow x_1^2-6x_1+\cancel{9}+x_1(8-x_1)-4\sqrt{x_1(8-x_1)}\cancel{+4}\cancel{-13}=0$$

$$\Rightarrow \cancel{x_1^2}-6x_1+8x_1-\cancel{x_1^2}-4\sqrt{x_1(8-x_1)}=0$$

$$\Rightarrow 2x_1=4\sqrt{x_1(8-x_1)} \Rightarrow x_1^2=4x_1(8-x_1)$$

$$\Rightarrow x_1^2-32x_1+4x_1^2=0 \Rightarrow 5x_1^2-32x_1=0$$

$$\Rightarrow x_1(5x_1-32)=0 \Rightarrow x_1=0 \text{ or } x_1=32/5$$

The same if we use $x_2=-\sqrt{x_1(8-x_1)}$

If $x_1=0$: from eq. (5) $\boxed{x_2=0}$

$\Rightarrow (0,0)$ is feasible the same as before

If $x_1=32/5$: from eq.(5) $\boxed{x_2=\frac{16}{5}}$ and $x_2=-\frac{16}{5}$

let's check these values:

$$\left.\begin{aligned} &h(32/5, \tfrac{16}{5})=0 \checkmark \\ &h(32/5,-\tfrac{16}{5})=0 \checkmark \\ &g(32/5, 16/5)=0 \checkmark \\ &g(32/5,-16/5)=128/5 \neq 0 \times \end{aligned}\right\}$$ Only the $(32/5, 16/5)$ is feasible

Therefore the coordinates are:

For $x_1=3+\sqrt{13}$, $x_2=2$, $\mu=0 \overset{(1)}{\Rightarrow} \lambda=\frac{1}{2\sqrt{13}}$: $(3+\sqrt{13}, 2, \frac{1}{2\sqrt{13}}, 0)$

For $x_1=3-\sqrt{13}$, $x_2=2$, $\mu=0 \overset{(1)}{\Rightarrow} \lambda=-\frac{1}{2\sqrt{13}}$: $(3-\sqrt{13}, 2, -\frac{1}{2\sqrt{13}}, 0)$

For $x_1=0$, $x_2=0$, $\lambda=0 \overset{(1)}{\Rightarrow} \mu=1/8$: $(0,0,0,1/8)$

For $x_1=\frac{32}{5}$, $x_2=\frac{16}{5} \xrightarrow{(1),(2)} \mu=3/40$, $\lambda=1/5$: $(\frac{32}{5}, \frac{16}{5}, \frac{1}{5}, \frac{3}{40})$

Let's find the minima:

$$\nabla^2_{xx} L(x_1, x_2, \mu, \lambda) = \begin{pmatrix} -2\lambda + 2\mu & 0 \\ 0 & -2\lambda + 2\mu \end{pmatrix}$$

$$\nabla_x h(x_1, x_2) = \begin{pmatrix} -2(x_1 - 4) \\ -2x_2 \end{pmatrix} \qquad \nabla_x g(x_1, x_2) = \begin{pmatrix} 2(x_1 - 3) \\ 2(x_2 - 2) \end{pmatrix}$$

▶ $(3 + \sqrt{13}, 2, \frac{1}{2\sqrt{13}}, 0) = x_1^*$

$$\nabla^2_{xx} L(x_1^*) = \begin{pmatrix} -1/\sqrt{13} & 0 \\ 0 & -1/\sqrt{13} \end{pmatrix}$$

$$\nabla_x h(3 + \sqrt{13}, 2) = \begin{pmatrix} 2 - \sqrt{13} \\ -4 \end{pmatrix}$$

$$\nabla_x g(3 + \sqrt{13}, 2) = \begin{pmatrix} 2\sqrt{13} \\ 0 \end{pmatrix}$$

Take $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x g(3 + \sqrt{13}, 2) = 0$

(We don't compute the $z^T \nabla h(x_1^*)$ because when $\mu = 0$ is not active)

$$\Rightarrow (z_1 \quad z_2) \begin{pmatrix} 2\sqrt{13} \\ 0 \end{pmatrix} = 2\sqrt{13}\, z_1 = 0 \Rightarrow z_1 = 0 \Rightarrow \begin{pmatrix} 0 \\ z_2 \end{pmatrix}$$

$(0\ z_2)$ is the vector that satisfy the above, let's now check if $(0, z_2) \Rightarrow z^T \nabla^2_{xx} L(x_1^*) z \geq 0$

$$(0, z_2) \begin{pmatrix} -\frac{1}{\sqrt{13}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \end{pmatrix} = -z_2^2 \frac{1}{\sqrt{13}} \geq 0$$

Therefore $(3 + \sqrt{13}, 2)$ is a local minimum

▶ $(3-\sqrt{13}, 2, -\frac{1}{2\sqrt{13}}, 0) = x_2^*$

$\nabla_{xx}^2 L(x_2^*) = \begin{pmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{pmatrix}$ $\nabla_x h(x_2^*) = \begin{pmatrix} 2+\sqrt{13} \\ -4 \end{pmatrix}$ $\nabla_x g(x_2^*) = \begin{pmatrix} -2\sqrt{13} \\ 0 \end{pmatrix}$

So $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$: $(z_1 \ z_2)\begin{pmatrix} -2\sqrt{13} \\ 0 \end{pmatrix} = -2z_1\sqrt{13} = 0 \Rightarrow z_1 = 0$

In similar way as x_1^* : $(3-\sqrt{13}, 2)$ local minima

▶ $(0, 0, 0, 1/8) = x_3^*$

$\nabla_{xx}^2 L(x_3^*) = \begin{pmatrix} 2/8 & 0 \\ 0 & 2/8 \end{pmatrix}$ $\nabla_x h(x_3^*) = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ $\nabla_x g(x_3^*) = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$

Take $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x h(x_3^*) = 0$, $z^T \nabla_x g(x_3^*) = 0$

$\Rightarrow (z_1 \ z_2)\begin{pmatrix} 8 \\ 0 \end{pmatrix} = 8z_1 = 0 \Rightarrow z_1 = 0$

$\Rightarrow (z_1 \ z_2)\begin{pmatrix} -6 \\ -4 \end{pmatrix} = -6z_1 - 4z_2 = 0 \Rightarrow z_2 = 0$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the vector that satisfy

So $(0,0)$ local minima

▶ $(\frac{32}{5}, \frac{16}{5}, 1/5, 3/40) = x_4^*$

$\nabla_{xx}^2 L(x_4^*) = \begin{pmatrix} -1/4 & 0 \\ 0 & -1/4 \end{pmatrix}$, $\nabla_x h(x_4^*) = \begin{pmatrix} -24/5 \\ -32/5 \end{pmatrix}$, $\nabla_x g(x_4^*) = \begin{pmatrix} 34/5 \\ 12/5 \end{pmatrix}$

$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ s.t. $z^T \nabla_x h(x_4^*) = 0$ & $z^T \nabla_x g(x_4^*) = 0$

$\Rightarrow (z_1 \ z_2)\begin{pmatrix} -24/5 \\ -32/5 \end{pmatrix} = -\frac{24}{5}z_1 - \frac{32}{5}z_2 = 0$

$\Rightarrow (z_1 \ z_2)\begin{pmatrix} 34/5 \\ 12/5 \end{pmatrix} = \frac{34}{5}z_1 + \frac{12}{5}z_2 = 0$

$\Rightarrow z_1 = 0$, $z_2 = 0$

So we have that $(0,0)$ is a global minimum