TALLER DE MARTINGALAS

1. Sea B_t un MBEU. Compruebe si los siguientes procesos son Martingalas en $\mathcal{F}_s$, $\forall t \geq s$.

a) Done in class.

b) $W_t = -B_t$

$$E[W_t / \mathcal{F}_s] = E[-B_t / \mathcal{F}_s] = -E[(B_t - B_s) / \mathcal{F}_s] - E[B_s / \mathcal{F}_s]$$
$$= -E[B_t - B_s] - E[B_s / \mathcal{F}_s] = -B_s \Rightarrow \text{Es Martingala.}$$

c) $W_t = tB_t$

$$E[W_t / \mathcal{F}_s] = E[tB_t / \mathcal{F}_s] = E[t(B_t - B_s) + tB_s / \mathcal{F}_s] -$$
$$= tE[B_t - B_s / \mathcal{F}_s] + tE[B_s / \mathcal{F}_s] = tE[\cancel{B_t - B_s}] + tB_s = tB_s$$

d) $W_t = 2B_t + t$

$$E[W_t / \mathcal{F}_s] = E[2B_t + t / \mathcal{F}_s] = 2E[B_t - B_s / \mathcal{F}_s] + 2E[B_s / \mathcal{F}_s] + E[t / \mathcal{F}_s]$$
$$= 2E[B_t - B_s] + 2B_s + t = 2B_s + t$$

e) $W_t = B_t + 4t$

$$E[W_t|\mathcal{F}_s] = E[B_t + 4t|\mathcal{F}_s] = E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] + 4t$$

$$= E[B_t - B_s] + B_s + 4t = B_s + 4t \rightarrow \text{No MG.}$$

f) $W_t = B_{t+1}$

$$E[W_t|\mathcal{F}_s] = E[B_{t+1}|\mathcal{F}_s] = E[B_{t+1} - B_{s+1}|\mathcal{F}_s] + E[B_{s+1}|\mathcal{F}_s]$$

$$= \cancel{E[B_{t+1} - B_{s+1}]}^{0} + E[B_{s+1} - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s]$$

$$= \cancel{E[B_{s+1} - B_s]}^{0} + B_s = B_s. \rightarrow \text{No MG.}$$

g) $W_t = B_t - B_{t_0}$, $0 < t_0 < s$ fijo

$$E[W_t|\mathcal{F}_s] = E[B_t - B_{t_0}|\mathcal{F}_s] = E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] - E[B_{t_0}|\mathcal{F}_s]$$

Como $t_0 < s \Rightarrow E[B_{t_0}|\mathcal{F}_s] = B_{t_0}$

$$= E[B_t - B_s] + B_s - B_{t_0} = B_s - B_{t_0}. \Rightarrow \text{Es MG.}$$

h) $W_t = B_T + B_t$, $T > 0$ t.q $s < t < T$

$$E[W_t|\mathcal{F}_s] = E[B_T + B_t|\mathcal{F}_s] = E[B_T - B_s|\mathcal{F}_s] + E[B_t - B_s|\mathcal{F}_s] + 2E[B_s|\mathcal{F}_s]$$

$$= \cancel{E[B_T - B_s]}^{0} + \cancel{E[B_t - B_s]}^{0} + 2B_s = 2B_s. \rightarrow \text{No MG.}$$

i) $W_t = B_t^2 \rightarrow$ Already done in class.

If X & Y are independent r.v. $\Rightarrow f(X)$ & $g(Y)$ are independent.

$$E[(B_t-B_s)^{2n}] = \frac{(2n)!}{2^n n!}(t-s)^n$$

j) $W_t = B_t^3 - tB_t$

$W_t = (B_t - B_s + B_s)^3 - tB_t$: $E[W_t|\mathcal{F}_s] = E[((B_t-B_s)+B_s)^3|\mathcal{F}_s] - tE[B_t|\mathcal{F}_s]$

$= E[(B_t-B_s)^3|\mathcal{F}_s] + 3E[B_s(B_t-B_s)^2|\mathcal{F}_s] + 3E[B_s^2(B_t-B_s)|\mathcal{F}_s] + E[B_s^3|\mathcal{F}_s] - t\underbrace{E[B_t|\mathcal{F}_s]}$

$= E[(B_t-B_s)^3] + 3E[B_s|\mathcal{F}_s]E[(B_t-B_s)^2] + 3E[B_s^2|\mathcal{F}_s]E[(B_t-B_s)] + B_s^3 - tB_s$ ← Since B_t is MG

$= {}^0\, 3B_s V[B_t-B_s] + B_s^3 - tB_s = 3B_s(t-s) + B_s^3 - tB_s = B_s^3 - 3sB_s + 2tB_s$

→ No es MG. Si fuera $W_t = B_t^3 - 3tB_t$ sí sería MG.

k) $W_t = B_t^4$.

$E[W_t|\mathcal{F}_s] = E[B_t^4|\mathcal{F}_s] = E[((B_t-B_s)+B_s)^4|\mathcal{F}_s]$

$= E[(B_t-B_s)^4|\mathcal{F}_s] + 4E[B_s(B_t-B_s)^3|\mathcal{F}_s] + 6E[B_s^2(B_t-B_s)^2|\mathcal{F}_s] + 4E[B_s^3(B_t-B_s)|\mathcal{F}_s]$

$+ E[B_s^4|\mathcal{F}_s] = E[(B_t-B_s)^4] + 4E[B_s|\mathcal{F}_s]\cancel{E[(B_t-B_s)^3|\mathcal{F}_s]} + 6E[B_s^2|\mathcal{F}_s]E[(B_t-B_s)^2|\mathcal{F}_s]$

$+ 4E[B_s^3|\mathcal{F}_s]\cancel{E[(B_t-B_s)|\mathcal{F}_s]}^{0} + E[B_s^4|\mathcal{F}_s] = E[(B_t-B_s)^4] + 6B_s^2 E[(B_t-B_s)^2]$

$+ B_s^4 = [(4!)/(2^2\cdot 2!)](t-s)^2 + 6B_s^2(t-s) + B_s^4 = 3(t-s)^2 + 6B_s^2(t-s) + B_s^4$

→ No es MG.

l) $W_t = B_t^4 - 4t$. $E[W_t|\mathcal{F}_s] = E[B_t^4 - 4t|\mathcal{F}_s] = E[B_t^4|\mathcal{F}_s] - 4E[t|\mathcal{F}_s]$

$= 3(t-s)^2 + 6B_s^2(t-s) + B_s^4 - 4t = 3t^2 - 12st + 3s^2 + 6tB_s^2 - 6sB_s^2 + B_s^4 - 4t$

⇒ No es MG

$E[e^{i\theta X}] = e^{i\mu\theta - \frac{\sigma^2\theta^2}{2}}$, $E[e^{\theta X}] = e^{\mu\theta + \frac{\sigma^2\theta^2}{2}}$, $X \sim N(\mu, \sigma^2)$

m) $W_t = e^{t/2}\sin(B_t)$

$E[W_t|F_s] = E[e^{t/2}\sin B_t|F_s] = e^{t/2}E[\sin(B_t - B_s + B_s)|F_s]$

$= e^{t/2}\{E[\sin(B_t-B_s)\cos(B_s) + \sin(B_s)\cos(B_t-B_s)|F_s]\}$

$= e^{t/2}\{E[\sin(B_t-B_s)|F_s]E[\cos(B_s)|F_s] + E[\sin(B_s)|F_s]E[\cos(B_t-B_s)|F_s]\}$.

$= e^{t/2}\{E[\sin(B_t-B_s)]\cos(B_s) + E[\cos(B_t-B_s)]\sin(B_s)\}$.

$= e^{t/2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n E[(B_t-B_s)^{2n+1}]}{(2n+1)!}\cos(B_s) + \sum_{n=0}^{\infty}\frac{(-1)^n E[(B_t-B_s)^{2n}]}{(2n)!}\sin(B_s)\right\}$

$= e^{t/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{(2n)!}{2^n n!}(t-s)^n \sin(B_s) = e^{t/2}e^{(s-t)/2}\sin(B_s) = e^{s/2}\sin(B_s)$.

$\Rightarrow$ Es MG.

n) $W_t = e^{\sigma B_t - \frac{1}{2}\sigma^2 t}$, $\sigma > 0$.

$E[W_t|F_s] = E[e^{\sigma B_t - \frac{\sigma^2 t}{2}}|F_s] = e^{-\frac{\sigma^2 t}{2}}E[e^{\sigma(B_t-B_s)+\sigma B_s}|F_s]$

$= e^{-\frac{\sigma^2 t}{2}}E[e^{\sigma(B_s-B_t)}|F_s]E[e^{\sigma B_s}|F_s] = e^{-\frac{\sigma^2 t}{2}}E[e^{\sigma(B_s-B_t)}]e^{\sigma B_s}$

$= e^{-\frac{\sigma^2 t}{2}}e^{\frac{\sigma^2(t-s)}{2}}e^{\sigma B_s} = e^{\sigma B_s - \frac{\sigma^2 s}{2}} \longrightarrow$ Es MG.

2. $W_t = e^{B_t - \frac{t^2}{2}}$. $E[W_t|F_s] = E[e^{B_t - \frac{t^2}{2}}|F_s] = e^{-t^2/2}E[e^{B_t-B_s}|F_s]E[e^{B_s}|F_s]$

$= e^{-t^2/2}E[e^{B_{t-s}}]e^{B_s} = e^{-t^2/2}e^{\frac{t-s}{2}}e^{B_s} = e^{B_s - \frac{s}{2} + \frac{t-t^2}{2}}$,

3. $W_t = (B_t + t)e^{-B_t - \frac{t}{2}}$

$$E[W_t|\mathcal{F}_s] = E[(B_t+t)e^{-B_t-\frac{t}{2}}|\mathcal{F}_s] = e^{-t/2}\{E[B_t e^{-B_t}|\mathcal{F}_s] + E[te^{-B_t}|\mathcal{F}_s]\}$$

$$= e^{-t/2}[E[B_t e^{-B_t}|\mathcal{F}_s] + E[te^{-B_t}|\mathcal{F}_s]]$$

$$= e^{-t/2}[E[B_t e^{-B_t}|\mathcal{F}_s] + tE[e^{-B_t}|\mathcal{F}_s]]$$

$$= e^{-t/2}\{E[(B_t - B_s + B_s)e^{-(B_t-B_s)-B_s}|\mathcal{F}_s] + tE[e^{-B_t}|\mathcal{F}_s]$$

$$= e^{-t/2}\{E[(B_t-B_s)e^{-(B_t-B_s)}|\mathcal{F}_s]e^{-B_s} + E[B_s e^{-(B_t-B_s)-B_s}|\mathcal{F}_s] + tE[e^{-B_t}|\mathcal{F}_s]\}$$

$$= e^{-t/2}\{E[B_{t-s}e^{-B_{t-s}}]e^{-B_s} + E[B_s e^{-B_s}e^{-(B_t-B_s)}|\mathcal{F}_s] + tE[e^{-(B_t-B_s)-B_s}|\mathcal{F}_s]\}$$

$$= e^{-t/2}\{-(t-s)e^{\frac{(t-s)}{2}}e^{-B_s} + E[B_s e^{-B_s}|\mathcal{F}_s]E[e^{-(B_t-B_s)}|\mathcal{F}_s] + tE[e^{-(B_t-B_s)}|\mathcal{F}_s]e^{-B_s}\}$$

$$= e^{-t/2}\{-(t-s)e^{\frac{(t-s)}{2}}e^{-B_s} + B_s e^{-B_s}E[e^{-B_{t-s}}] + tE[e^{-B_{t-s}}]e^{-B_s}\}$$

$$= e^{-t/2}\{-(t-s)e^{(t-s)/2}e^{-B_s} + B_s e^{-B_s}e^{(t-s)/2} + te^{(t-s)/2}e^{-B_s}\}$$

$$= -(t-s)e^{-s/2}e^{-B_s} + B_s e^{-s/2-B_s} + te^{-s/2}e^{-B_s}$$

$$= se^{-s/2-B_s} + B_s e^{-s/2-B_s} = (B_s+s)e^{-B_s-s/2}$$ es Martingala.

$A_t = k_t B_t + l_t$

sura

4. $W_t = B_1(t) B_2(t)$

$E[W_t | \mathcal{F}_s] = E[B_1(t) B_2(t) | \mathcal{F}_s] = E[B_1(t) | \mathcal{F}_s] E[B_2(t) | \mathcal{F}_s]$

$= [E[B_1(t) - B_1(s) | \mathcal{F}_s] + E[B_1(s) | \mathcal{F}_s]][E[B_2(t) - B_2(s) | \mathcal{F}_s] + E[B_2(s) | \mathcal{F}_s]]$

$= [\cancel{E[B_1(t) - B_1(s)]} + B_1(s)][\cancel{E[B_1(t) - B_1(s)]} + B_2(s)] = B_1(s) B_2(s) \Rightarrow E\ MG$

$W_s = B_s^4 + A_t$

5. Determine A_t tal que $W_t = B_t^4 + A_t$ es MG.

$E[W_t | \mathcal{F}_s] = E[B_t^4 + A_t | \mathcal{F}_s] = E[B_t^4 | \mathcal{F}_s] + E[A_t | \mathcal{F}_s]$

$= 3(t-s)^2 + 6B_s^2(t-s) + B_s^4 + E[A_t | \mathcal{F}_s]$

$= E[(B_t - B_s)^4 | \mathcal{F}_s] + 6E[B_s^2(B_t - B_s)^2 | \mathcal{F}_s] + E[B_s^4 | \mathcal{F}_s] + E[A_t | \mathcal{F}_s]$

Sea $A_t = K(t) B_t^2 = -3(t-s)^2 + 6B_s^2(t-s) + B_s^4 + K(t)[t - s + B_s^2]$

$= -3t^2 - 6ts + 3s^2 + 6B_s^2 t - 6B_s^2 s + B_s^4 + K(t)t - K(t)s + K(t)B_s^2$

Sea $K(t) = -6t$

$= 3t^2 - 6st + 3s^2 + \cancel{6tB_s^2} - 6sB_s^2 + B_s^4 - 6t^2 + \cancel{6st} - \cancel{6tB_s^2}$

$= -3t^2 + 3s^2 - 6sB_s^2 + B_s^4$ Se quiere que sea Martingala.

$-3t^2 + 3s^2 - 6sB_s^2 + B_s^4 = B_t^4 - 6tB_t^2$

$\Rightarrow B_s^4 - 6sB_s^2 + 3s^2 = B_t^4 - 6tB_t^2 + 3t^2$

$\Rightarrow$ Luego $A_t = -6tB_t^2 + 3t^2$ hace que W_t sea Martingala.

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6. $\{W_n\}_{n\in\mathbb{N}}$ t.q. $E[W_n | \mathcal{F}_{n-1}] = \alpha W_{n-1} + \beta W_{n-2}$, $\alpha+\beta = 1$. Determine A tal que $Y_n = AW_n + W_{n-1}$ sea Martingala.

$E[Y_n | \mathcal{F}_{n-1}] = E[AW_n + W_{n-1} | \mathcal{F}_{n-1}] = AE[W_n | \mathcal{F}_{n-1}] + E[W_{n-1} | \mathcal{F}_{n-1}]$

$= A[\alpha W_{n-1} + \beta W_{n-2}] + W_{n-1} = A\alpha W_{n-1} + A\beta W_{n-2} + W_{n-1} = (A\alpha+1) W_{n-1} + A\beta W_{n-2}$

Para que sea Martingala,

$(A\alpha+1) W_{n-1} + A\beta W_{n-2} = AW_{n-1} + W_{n-2}$

$\Rightarrow A\alpha+1 = A \rightarrow A(1-\alpha) = 1 \rightarrow A = \dfrac{1}{1-\alpha}$ | $A\beta = 1 \Rightarrow A = \dfrac{1}{\beta} = \dfrac{1}{1-\alpha}$

$Y_n = \frac{1}{\beta} W_n + W_{n-1} \Rightarrow E[Y_n | \mathcal{F}_{n-1}] = \frac{1}{\beta} E[W_n | \mathcal{F}_{n-1}] + E[W_{n-1} | \mathcal{F}_n]$

$= (1/\beta)(\alpha W_{n-1} + \beta W_{n-2}) + W_{n-1} = (\alpha/\beta + 1) W_{n-1} + W_{n-2} = (1/\beta) W_{n-1} + W_{n-2}$

$\Rightarrow MG.\ A = 1/\beta.$

7. $W_n = \begin{cases} \alpha + \beta W_{n-1}, & \text{con probabilidad } W_{n-1} \\ \beta W_{n-1}, & \text{con probabilidad } 1-W_{n-1} \end{cases}$, Si $\alpha+\beta = 1 \Rightarrow W_n$ es Martingala.

$E[W_n | \mathcal{F}_{n-1}] = (\alpha + \beta W_{n-1}) W_{n-1} + \beta W_{n-1}(1-W_{n-1})$

$= (\alpha+\beta) W_{n-1} = W_{n-1}$