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APPROXIMATIONS OF BOUNDARY CROSSING PROBABILITIES FOR A BROWNIAN MOTION

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Abstract

Using the Girsanov transformation we derive estimates for the accuracy of piecewise approximations for one-sided and two-sided boundary crossing probabilities. We demonstrate that piecewise linear approximations can be calculated using repeated numerical integration. As an illustrative example we consider the case of one-sided and two-sided square-root boundaries for which we also present analytical representations in a form of infinite power series.

Keywords: First passage times; Werner process; Girsanov transformation; numerical integration

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1. Introduction

The problem of the pricing of so-called time-dependent barrier options in mathematical finance can be reduced to the calculation of probabilities of the form

$$\mathrm{P}\{W_s < g(s), 0 \le s \le t;\, W_t > K\} := P_t(g, K)$$

and

$$\mathrm{P}\{f(s) < W_s < g(s), 0 \le s \le t;\, W_t > K\} := P_t(f, g, K),$$

where W_t is a standard Brownian motion (i.e. W_t is a continuous Gaussian process with $\mathrm{E}W_t = 0, \mathrm{E}W_t^2 = t$), $f(s)$ and $g(s)$ are some deterministic functions (see, for example, Frishling *et al.* (1998), Novikov *et al.* (1998), Roberts and Shortland (1997)).

This computational problem has attracted attention not only in finance but in many other fields of statistics and stochastic modelling: statistical sequential analysis (Siegmund (1985)), some biophysical models (Ricciardi (1977)), non-parametric statistics (Sen (1981)), etc.

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In this paper we consider approximations of $P_t(f, g, K)$ by $P_t(\hat{f}_n, \hat{g}_n, K)$ where, in particular, boundaries $\hat{f}_n$ and $\hat{g}_n$ are piecewise linear functions. The probability $P_t(\hat{f}_n, \hat{g}_n, K)$ can be calculated as an n-fold integral with a specified kernel (see Theorem 1 in Section 2). In the case of one-sided boundaries this approach was recently used by Wang and Pötzelberger (1997) who applied the Monte-Carlo method to calculate probabilities $P_t(\hat{g}_n, -\infty)$. In order to calculate probabilities $P_t(\hat{f}_n, \hat{g}_n, K)$ we use the method of repeated numerical integration which we found to be quite satisfactory.

Theorem 2 is one of the main results of this paper (see Section 3 below) which implies the following simple bound

$$|P_t(g, K) - P_t(\hat{g}_n, K)| \leq \mathrm{P}\{K < W_t < g(t)\}\left(\frac{1}{2\pi}\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}(\hat{g}_n(s) - g(s))\right)^2 \mathrm{d}s\right)^{1/2}.$$

The similar bound is found for $|P_t(f, g, K) - P_t(\hat{f}_n, \hat{g}_n, K)|$. As an application of these bounds we present an estimate for the rate of convergence of $P_t(\hat{f}_n, \hat{g}_n, K)$ to $P_t(f, g, K)$ as $n \to \infty$ where n is the number of equally spaced nodes (see Theorem 3, Section 4). This estimate appears to be of an order $O(\sqrt{(\log n)/n^3})$. To illustrate the accuracy of the approximations numerically we consider the case of square-root boundaries; for the last of these we present analytical representations of $P_t(g, K)$ and $P_t(-g, g, K)$ in the form of infinite power series (see Theorem 4, Section 5).

There are many other approaches to calculating $P_t(g, K)$ and $P_t(f, g, K)$ numerically: a series of multiple integrals (see Daniels (1996), Durbin (1992), Sacerdote and Tomasseti (1996)); a polynomial approximation (Ferebee (1983)); Girsanov's transformation technique (Novikov (1979), Salminen (1988)); Poisson approximation (Khmaladze and Shinjikashvili (1998)); see also other references in Durbin (1992) and Lerche (1986). One of the advantages of our approach is that (besides easily controlled accuracy) it can be used in the case of two-sided boundaries which may even be discontinuous (cf. the method in Sacerdote and Tomasseti (1996)).

2. Approximations in terms of conditional probabilities

Let $\hat{f}(s)$ and $\hat{g}(s)$ be boundaries on the interval $[0, t]$ which are considered as approximations for $f(s)$ and $g(s)$. For example, one may consider $\hat{f}(s)$ and $\hat{g}(s)$ as piecewise linear functions with nodes $t_i, t_0 = 0 < t_1 < \ldots t_n = t$ (but, generally, these functions may be nonlinear).

Denote

$$p(i, \hat{f}, \hat{g} \mid x, y) := \mathrm{P}\{\hat{f}(s) < W_s < \hat{g}(s), t_i \leq s \leq t_{i+1} \mid W_{t_i} = x, W_{t_{i+1}} = y\}. \tag{1}$$

In a particular case when $\hat{f} = -\infty$ and $\hat{g}$ is a linear boundary on the interval $[t_i, t_{i+1}]$ the last probability is equal to

$$\begin{aligned} &p(i, -\infty, \hat{g} \mid x_i, x_{i+1}) \\ &\quad = \mathbf{1}\{\hat{g}(t_i) > x_i, \hat{g}(t_{i+1}) > x_{i+1}\}\left[1 - \exp\left\{-\frac{2(\hat{g}(t_i) - x_i)(\hat{g}(t_{i+1}) - x_{i+1})}{t_{i+1} - t_i}\right\}\right], \end{aligned} \tag{2}$$

where $\mathbf{1}\{\}$ is an indicator function. This formula can be derived from the well-known formula for the first passage time of a Brownian motion through a linear boundary (see Remark 1).

It can also be considered as a particular case of the general formula for two-sided linear boundaries which is presented in Remark 2.

The next theorem which is the generalization of Theorem 1 from Wang and Pötzelberger (1997) gives the representation for $P_t(\hat{f}, \hat{g}, K)$ as an n-fold integral of $p(i, \hat{f}, \hat{g} \mid x, y)$ and transition probabilities of a Wiener process.

Theorem 1.

$$P_t(\hat{f}, \hat{g}, K) = \mathrm{E}\left[\mathbf{1}\{W_t > K\} \prod_{i=0}^{n-1} p(i, \hat{f}, \hat{g} \mid W_{t_i}, W_{t_{i+1}})\right]. \tag{3}$$

The proof of this formula can be easily carried out using the following Lemma 1.

Lemma 1. *For any partition* $\{0 = t_0 < \cdots < t_n = t\}$ *the random processes*

$$\eta_i(u) := W_{t_i+u} - W_{t_i}, \quad 0 \le u \le \Delta t_i = t_{i+1} - t_i, \quad i = 0, \ldots, n-1,$$

are conditionally independent with respect to the σ*-algebra*

$$F_n := \sigma\{W_{t_1}, \ldots, W_{t_n}\} = \sigma\{W_{t_1}, \Delta W_{t_1}, \ldots, \Delta W_{t_{n-1}}\}, \qquad (\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}).$$

Besides,

$$\mathrm{cov}(\eta_i(u), \eta_i(v) \mid F_n) = \min(u, v) - uv/\Delta t_i. \tag{4}$$

The result of this lemma is certainly well-known but we could not find any other reference besides Novikov (1980). For the sake of completeness of the exposition we present a short proof of it here.

Proof. As processes $\eta_i(u)$ are Gaussian, to check the independency of the processes $\eta_i(u)$ one only needs to verify that

$$\mathrm{cov}(\eta_i(u), \eta_j(v) \mid F_n) = 0 \qquad \text{for any } 0 \le u \le \Delta t_i, \quad 0 \le v \le \Delta t_j, \quad i \ne j.$$

Since the process W_t has independent increments we have, for any i and j,

$$\mathrm{cov}(\eta_i(u), \eta_j(v) \mid F_n) = \mathrm{cov}(\eta_i(u), \eta_j(v) \mid \Delta W_{t_i}, \Delta W_{t_j})$$

(all equations for conditional expectations hold, of course, almost surely with respect to the given probability measure). Note now that

$$\mathrm{cov}(\eta_i(u) - u\Delta W_{t_i}/\Delta t_i, \Delta W_{t_i}) = 0,$$

and so random variables $\eta_i(u) - u\Delta W_{t_i}/\Delta t_i$ and ΔW_{t_i} are independent. When $i \ne j$ we now have

$$\begin{aligned}
\mathrm{cov}(\eta_i(u), \eta_j(v) \mid F_n) &= \mathrm{cov}(\eta_i(u) - u\Delta W_{t_i}/\Delta t_i, \eta_j(v) - v\Delta W_{t_j}/\Delta t_j \mid \Delta W_{t_i}, \Delta W_{t_j}) \\
&= \mathrm{cov}(\eta_i(u) - u\Delta W_{t_i}/\Delta t_i, \eta_j(v) - v\Delta W_{t_j}/\Delta t_j) \\
&= 0.
\end{aligned}$$

When $i = j$ we have

$$\begin{aligned}\operatorname{cov}(\eta_i(u), \eta_i(v) \mid F_n) &= \operatorname{cov}(\eta_i(u) - u\Delta W_{t_i}/\Delta t_i, \eta_i(v) - v\Delta W_{t_i}/\Delta t_i) \\ &= \min(u, v) - uv/\Delta t_i,\end{aligned}$$

and so (4) holds.

The proof of Lemma 1 is completed.

Given Lemma 1 the proof of Theorem 1 becomes an easy exercise on properties of conditional expectations and it is omitted.

Remark 1. Let $p_\tau(t)$ be a density of the stopping time

$$\tau = \inf\{t \geq 0 : W_t \geq g(t)\},$$

where $g(t)$ is a continuous boundary, $g(0) > 0$ and

$$\varphi(x, \Delta) := \exp\left(-\frac{x^2}{2\Delta}\right) \Big/ \sqrt{2\pi\Delta}.$$

Then easy calculations (based on the continuity of trajectories of a Brownian motion) give for any y

$$\mathrm{P}\{\tau < t, W_t > y\} = \int_y^\infty \int_0^t p_\tau(u)\varphi(x - g(u), t - u)\, \mathrm{d}u\mathrm{d}x. \tag{5}$$

In the particular case of a linear function $g(s)$ this formula leads to the representation (2). There are other cases when density $p_\tau(t)$ can be found in an explicit form. Those cases are one-sided square-root boundaries (see Section 5 for details), quadratic boundaries (Salminen (1988)) and some other specific boundaries mentioned by Daniels (1996).

Remark 2. (The case of two-sided boundaries.) Anderson (1960) was the first to obtain the explicit representation for the distribution of the first passage time of a Brownian motion through linear boundaries f and g. The two-sided symmetric square-root boundary is the only other case for which we know the explicit representation of the first passage time density (see Section 5).

Recently Hall (1997) calculated the conditional distribution of crossing the upper linear boundary $g(s) = a_1 + b_1 s$ and the lower linear boundary $f(s) = a_2 + b_2 s$. His result gives the following formula for $p(i, f, g \mid x_i, x_{i+1})$ (in our notation)

$$p(i, f, g \mid x_i, x_{i+1}) = 1 - P_{\mathrm{U}} - P_{\mathrm{L}},$$

where

$$\begin{aligned}P_{\mathrm{U}} &:= \mathrm{P}(a_1, a_2, \hat{b}, x_i) \\ &= \sum_{j=1}^{\infty} \exp[2b(2j-1)(jc + a_2)] \exp\left[\frac{2(jc + a_2)}{\Delta t_i}(\Delta x_i - \hat{b}\Delta t_i - (jc + a_2))\right] \\ &\quad - \sum_{j=1}^{\infty} \exp[4bj(2j - \hat{a})] \exp\left[\frac{2}{\Delta t_i} jc(\Delta x_i - \hat{b}\Delta t_i - jc)\right],\end{aligned}$$

with

$$a_1 = g(t_{i+1}) - x_i, \quad a_2 = f(t_{i+1}) - x_i, \quad b_1 = \frac{g(t_{i+1}) - g(t_i)}{\Delta t_i},$$
$$b_2 = \frac{f(t_{i+1}) - f(t_i)}{\Delta t_i}, \quad c = a_1 - a_2, \quad b = (b_2 - b_1)/2, \quad \hat{b} = (b_2 + b_1)/2,$$
$$\hat{a} = (a_1 + a_2)/2,$$

and

$$P_{\mathrm{L}} = \mathrm{P}(-a_2, -a_1, -\hat{b}, -x_i).$$

Remark 3. (The recurrent algorithm for evaluation of (3).) Let

$$z_0(x) := p(0, \hat{f}_n, \hat{g}_n \mid 0, x)\varphi(x, t_1),$$
$$z_{k+1}(x) = \int_{\hat{f}_n(t_k)}^{\hat{g}_n(t_k)} z_k(u) p(k+1, \hat{f}_n, \hat{g}_n \mid u, x)\varphi(x-u, \Delta t_k)\, \mathrm{d}u, \qquad k = 0, \dots n-1.$$

Then

$$P_t(\hat{f}_n, \hat{g}_n, K) = \int_{\hat{f}_n(t_n)}^{\hat{g}_n(t_n)} \mathbf{1}\{u > K\} z_{n-1}(u)\, \mathrm{d}u. \tag{3'}$$

For small values of n (less then 5) this algorithm is reasonably fast even with interpreter packages like Mathematica or Maple. For moderate values of n (less than 100) this algorithm appeared to be reasonably fast when implemented in Fortran77 (it takes a few seconds to evaluate $\mathrm{P}_t(\hat{g}_{32}, -\infty)$ on a Pentium-II processor).

Generally speaking, Theorem 1 could be used for any spline approximation of boundaries f and g. The only problem is how to calculate the conditional probabilities $p(i, f, g \mid x_i, x_{i+1})$. In the case of one-sided boundaries and quadratic splines one can try to calculate these probabilities based on results of Salminen (1988) and formula (5).

3. Accuracy of approximations

The following theorem gives a simple bound for the difference of probabilities $P_t(f, g, K)$ and $P_t(\hat{f}, \hat{g}, K)$ in terms of the distance

$$\Delta_t(\hat{g}, g) := \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}(\hat{g}(s) - g(s))\right)^2 \mathrm{d}s.$$

We also assume

$$f(0) = \hat{f}(0), \qquad g(0) = \hat{g}(0),$$
$$f(t) = \hat{f}(t), \qquad g(t) = \hat{g}(t).$$

Write

$$g_{\max}(s) := \max(g(s), \hat{g}(s)), \quad g_{\min}(s) := \min(g(s), \hat{g}(s)).$$

Theorem 2.

$$|P_t(g, K) - P_t(\hat{g}, K)| \le \mathrm{P}\{K < W_t < g(t)\}\sqrt{\Delta_t(\hat{g}, g)/2\pi}; \tag{6}$$

$$\begin{aligned} |P_t(f, g, K) - P_t(\hat{f}, \hat{g}, K)| \le{}& \mathrm{P}\{K < W_t < g(t)\}\sqrt{\Delta_t(g_{\max}, g_{\min})/2\pi} \\ &+ \mathrm{P}\{\max(K, f(t)) < W_t\}\sqrt{\Delta_t(f_{\max}, f_{\min})/2\pi}. \end{aligned} \tag{7}$$

Proof. Write

$$\Xi_t(\hat{g}, g) = \exp\left\{\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\hat{g}(s) - g(s))\,\mathrm{d}W_s - \tfrac{1}{2}\Delta_t(\hat{g}, g)\right\},$$

assuming that $\Delta_t(\hat{g}, g) < \infty$ (otherwise, Theorem 2 is trivial). By the Girsanov theorem (see Liptser and Shiryaev (1977))

$$\mathrm{P}\{W_s + \hat{g}(s) - g(s) \in B_s, s \le t\} = \mathrm{E}[\mathbf{1}\{W_s \in B_s, s \le t\}\Xi_t(\hat{g}, g)],$$

where B_s are any Borel sets. From here, taking into account the equality $g(t) = \hat{g}(t)$ we get the following representation

$$P_t(g, K) = \mathrm{E}[\mathbf{1}\{W_s < \hat{g}(s), s \le t;\, W_t > K\}\,\Xi_t(\hat{g}, g)].$$

It implies

$$P_t(g, K) - P_t(\hat{g}, K) = \mathrm{E}[\mathbf{1}\{W_s < \hat{g}(s), s \le t;\, W_t > K\}(\Xi_t(\hat{g}, g) - 1)].$$

Therefore,

$$\begin{aligned} &|P_t(g, K) - P_t(\hat{g}, K)| \\ &\le \max(\mathrm{E}[\mathbf{1}\{K < W_t < g(t)\}(\Xi_t(\hat{g}, g) - 1)^+], \mathrm{E}[\mathbf{1}\{K < W_t < g(t)\}(1 - \Xi_t(\hat{g}, g))^+]), \end{aligned}$$

where we have used the notation $a^+ := \max(a, 0)$.

To estimate the last expectations one can use the fact that if

$$\Delta_t(\hat{g}, g) := \Delta < \infty \quad \text{and} \quad \Delta > 0,$$

the random variable $\log(\Xi_t(\hat{g}, g))$ has a Gaussian distribution with parameters

$$\mathrm{E}[\log(\Xi_t(\hat{g}, g))] = -\Delta/2, \qquad \mathrm{Var}[\log(\Xi_t(\hat{g}, g))] = \Delta.$$

Then

$$\begin{aligned} \mathrm{E}(\Xi_t(\hat{g}, g) - 1)^+ &= \int_0^\infty [\exp(x) - 1]\exp\left\{-\frac{(x + \Delta/2)^2}{2\Delta}\right\}\frac{\mathrm{d}x}{\sqrt{2\pi\Delta}} \\ &= \int_0^{\sqrt{\Delta/8}} \exp(-t^2)\frac{2\,\mathrm{d}t}{\sqrt{\pi}} \le \sqrt{\frac{\Delta}{2\pi}}. \end{aligned} \tag{8}$$

Since $\mathrm{E}[\Xi_t(g, \hat{g})] = 1$ we have

$$\mathrm{E}(1 - \Xi_t(\hat{g}, g))^+ = \mathrm{E}(\Xi_t(\hat{g}, g) - 1)^+.$$

To complete the proof of (6) one may note that random variables $\Xi_t(g, \hat{g})$ and W_t are independent. Indeed, the random variables $\log(\Xi_t(g, \hat{g}))$ and W_t are jointly Gaussian and by the imposed assumptions

$$\operatorname{cov}\left(W_t, \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\hat{g}(s) - g(s))\,\mathrm{d}W_s \right) = \hat{g}(t) - g(t) = 0.$$

This implies independency of $\Xi_t(\hat{g}, g)$ and W_t. Due to this fact and by (8)

$$\begin{aligned} \mathrm{E}[\mathbf{1}\{K < W_t < g(t)\}(\Xi_t(\hat{g}, g) - 1)^+] &= \mathrm{P}\{K < W_t < g(t)\}\mathrm{E}[(\Xi_t(\hat{g}, g) - 1)^+] \\ &\leq \mathrm{P}\{K < W_t < g(t)\}\sqrt{\Delta/2\pi}. \end{aligned}$$

This completes the proof of (6).

To prove (7) one may note, at first, that

$$|P_t(\hat{f}, \hat{g}, K) - P_t(f, g, K)| \leq |P_t(\hat{f}, \hat{g}, K) - P_t(\hat{f}, g, K)| + |P_t(\hat{f}, g, K) - P_t(f, g, K)|. \tag{9}$$

Consider now the term $P_t(\hat{f}, \hat{g}, K) - P_t(\hat{f}, g, K)$ and note

$$|P_t(\hat{f}, \hat{g}, K) - P_t(\hat{f}, g, K)| = |p_1 - p_2| \leq \max(p_1, p_2),$$

with

$$p_1 = P_t\{\hat{f}(s) < W_s < \hat{g}(s);\ \exists u \text{ such that } W_u \geq g(u), W_t > K\},$$

$$p_2 = P_t\{\hat{f}(s) < W_s < g(s);\ \exists u \text{ such that } W_u \geq \hat{g}(u), W_t > K\}.$$

It is clear that both of the last probabilities will only increase if we delete the lower bound $\hat{f}(s)$. Therefore,

$$\begin{aligned} p_1 &\leq P_t\{W_s < \hat{g}(s); \exists u \text{ such that } W_u \geq g(u), W_t > K\} \\ &\leq P_t\{W_s < g_{\max}(s); \exists u \text{ such that } W_u \geq g_{\min}(u), W_t > K\} \\ &= P_t(g_{\max}, K) - P_t(g_{\min}, K), \\ p_2 &\leq P_t\{W_s \leq g(s); \exists u \text{ such that } W_u > \hat{g}(u), W_t > K\} \\ &\leq P_t\{W_s < g_{\max}(s); \exists u \text{ such that } W_u \geq g_{\min}(u), W_t > K\} \\ &= P_t(g_{\max}, K) - P_t(g_{\min}, K). \end{aligned}$$

As $g_{\max}(0) = g_{\min}(0) = g(0)$ and $g_{\max}(t) = g_{\min}(t) = g(t)$ we can use the proved inequality (6) and then we have

$$|P_t(\hat{f}, \hat{g}, K) - P_t(\hat{f}, g, K)| \leq \mathrm{P}\{K < W_t < g(t)\}\sqrt{\Delta_t(g_{\max}, g_{\min})/2\pi}.$$

Taking into account the symmetry of the distribution of the Brownian motion and applying the same considerations as above we get

$$|P_t(\hat{f}, g, K) - P_t(f, g, K)| \leq \mathrm{P}\{\max(K, f(t)) < W_t\}\sqrt{\Delta_t(f_{\max}, f_{\min})/2\pi}.$$

To complete the proof one needs only to combine the last two inequalities with (9).

Remark 4. If $g(t_i) = \hat{g}(t_i)$ for some $t_i \le t, 1 \le i \le n$ then the same considerations as those concerning the independency of the random variables $\log(\Xi_t(g, \hat{g}))$ and W_{t_i} lead to the following inequality

$$|P_t(g, K) - P_t(\hat{g}, K)| \le \mathrm{P}\{K < W_t; W_{t_i} < g(t_i), 1 \le i \le n\} \sqrt{\Delta_t(\hat{g}, g)/2\pi}.$$

A similar improvement can be made for the case of two-sided boundaries. A further improvement can be achieved using a linear combination of upper and lower estimators.

Consider, for example, the case of one-sided boundaries and let $g_{\mathrm{l}}(s)$ and $g_{\mathrm{u}}(s)$ be lower and upper boundaries for $g(s)$, Δ_{l} and Δ_{u} be upper bounds for the accuracy of approximations accordingly:

$$0 \le P_t(g, K) - P_t(g_{\mathrm{l}}, K) \le \Delta_{\mathrm{l}},$$
$$0 \le P_t(g_{\mathrm{u}}, K) - P_t(g, K) \le \Delta_{\mathrm{u}}.$$

It is easy to see that for any $x \in [0, 1]$ by the inequality $|a^+ - b^+| \le \max(a^+, b^+)$

$$|P_t(g, K) - [x P_t(g_{\mathrm{l}}, K) + (1 - x) P_t(g_{\mathrm{u}}, K)]| \le \max\{x\Delta_{\mathrm{l}}, (1 - x)\Delta_{\mathrm{u}}\}.$$

The minimum of the last expression is achieved with $\hat{x} = \Delta_{\mathrm{u}}/(\Delta_{\mathrm{l}} + \Delta_{\mathrm{u}})$ and so we have

$$|P_t(g, K) - [\hat{x} P_t(g_{\mathrm{l}}, K) + (1 - \hat{x}) P_t(g_{\mathrm{u}}, K)]| \le \frac{\Delta_{\mathrm{l}}\Delta_{\mathrm{u}}}{\Delta_{\mathrm{l}} + \Delta_{\mathrm{u}}}. \tag{10}$$

Example 1. Consider the one-sided boundary $g(s) = \sqrt{s + 1}, 0 \le s \le t$, and take $t = 1$ and $K = -\infty$. The value of $P_1(g, -\infty) := \mathrm{P}(g)$ accurate to 5 decimal places is $\mathrm{P}(g) = 0.804\,003$ (see Section 5).

Consider the piecewise linear *lower* boundary $g_{\mathrm{l},n}(s)$ with values $g_{\mathrm{l},n}(t_i) = g(t_i)$, $g'_{\mathrm{l},n}(s) = (g(t_{i+1}) - g(t_i))/(t_{i+1} - t_i)$ for $s \in (t_i, t_{i+1})$, $i \le n$, $t_i = i/n$. For $n = 3$, the method of repeated numerical integration gives $\mathrm{P}(g_{\mathrm{l},3}) = 0.803\,42$ and

$$\mathrm{P}\big\{W_{i/3} < \sqrt{1 + i/3},\ i = 1, 2, 3\big\} = 0.892.$$

By Remark 4 the bound for the accuracy of approximation is

$$\Delta_{\mathrm{l}} = \mathrm{P}\big\{W_{i/3} < \sqrt{1 + i/3},\ i = 1, 2, 3\big\} \sqrt{\Delta_1(g_{\mathrm{l},3}, g)/2\pi} = 0.005\,04.$$

For the piecewise linear *upper* boundary $g_{\mathrm{u},3}$ with values $g_{\mathrm{u},3}(0) = 1$, $g_{\mathrm{u},3}(1/2) = g(1/2)$, $g_{\mathrm{u},3}(1) = g(1)$, $g'_{\mathrm{u},3}(s) = 1/2$ for $0 \le s \le 0.212 = t_1$, $g'_{\mathrm{u},3}(s) = g'(1/2)$ for $0.212 < s \le 0.723 = t_2$, $g'_{\mathrm{u},3}(s) = g'(1)$ for $0.723 < s \le 1$; we have $\mathrm{P}(g_{\mathrm{u},3}) = 0.804\,19$ and

$$\mathrm{P}\big\{W_{i/2} < \sqrt{1 + i/2},\ i = 1, 2\big\} = 0.9025.$$

By Remark 4 the bound for the accuracy of approximation is

$$\Delta_{\mathrm{u}} = \mathrm{P}\big\{W_{i/2} < \sqrt{1 + i/2},\ i = 1, 2\big\} \sqrt{\Delta_1(g_{\mathrm{u},3}, g)/2\pi} = 0.0073.$$

Now by (10) with $\hat{x} = 0.594$

$$|\mathrm{P}(g) - [\hat{x}\mathrm{P}(g_{\mathrm{l},3}) + (1 - \hat{x})\mathrm{P}(g_{\mathrm{u},3})]| \le 0.002\,97.$$

So, in the example under discussion the guaranteed accuracy of approximation by the linear combination of two 3-fold integrals is about 0.4%. One may note that the real accuracy of the estimator $\hat{x}\mathrm{P}(g_{\mathrm{l},3}) + (1 - \hat{x})\mathrm{P}(g_{\mathrm{u},3}) = 0.803\,73$ is about 0.03%. The next section provides some explanations as to why a real accuracy might be significantly better when compared with the bound from Theorem 2.

4. Rate of convergence of a piecewise linear approximation

Let $g(s)$ be a twice continuously differentiable function and $\hat{g}_n(s)$ be a piecewise linear approximation for $g(s)$ such that $\hat{g}_n(t_i) = g(t_i)$, $0 \le t_i \le t$. Then in the case of equally spaced nodes we have

$$\Delta_t(\hat{g}_n, g) \le \frac{\text{const}}{n^2}, \qquad n \ge 1, \tag{11}$$

and so by (6)

$$|P_t(g, K) - P_t(\hat{g}_n, K)| \le \frac{\text{const}}{n}.$$

The same bound holds in the case of two-sided probabilities. This rough estimate can be improved as follows.

Theorem 3. *Let $\hat{g}_n((i/n)t) = g((i/n)t)$, $\hat{f}_n((i/n)t) = f((i/n)t)$, $0 \le i \le n$, and (11) hold. Then*

$$|P_t(f, g, K) - P_t(\hat{f}_n, \hat{g}_n, K)| \le \text{const}\sqrt{\frac{\log(n)}{n^3}}. \tag{12}$$

Proof. By inequality (9) and the arguments for the two-sided case of Theorem 2, one needs to consider only the case of a one-sided boundary.

Denote

$$A_n = \{W_s < \hat{g}_n(s), 0 \le s \le t\}, \qquad B_n = \left\{W_{it/n} < g\left(\frac{i}{n}t\right), 0 \le i \le n\right\}.$$

By the Girsanov theorem we have

$$P_t(g, K) - P_t(\hat{g}_n, K) = \mathrm{E}[\mathbf{1}(A_n)\mathbf{1}(K < W_t)(\Xi_t(\hat{g}_n, g) - 1)].$$

As

$$\operatorname{cov}\left(W_{it/n}, \int_0^t (\hat{g}_n'(s) - g'(s))\,\mathrm{d}W_s\right) = \hat{g}_n(it/n) - g(it/n) = 0, \quad 1 \le i \le n,$$

one can conclude that the random variables $\mathbf{1}(B_n)$ and $\Xi_t(\hat{g}_n, g)$ are independent. Since $\mathrm{E}[\Xi_t(\hat{g}_n, g) - 1] = 0$ we have

$$P_t(g, K) - P_t(\hat{g}_n, K) = \mathrm{E}[(\mathbf{1}(A_n) - \mathbf{1}(B_n))\mathbf{1}(K < W_t < g(t))(\Xi_t(\hat{g}_n, g) - 1)].$$

It follows from here that for any $N > 0$

$$\begin{aligned} |P_t(g, K) - P_t(\hat{g}_n, K)| &\le \mathrm{E}[(\mathbf{1}(B_n) - \mathbf{1}(A_n))|\Xi_t(\hat{g}_n, g) - 1|] \\ &\le \mathrm{E}[\mathbf{1}\{|\Xi_t(\hat{g}_n, g) - 1| > N\sqrt{\log n}/n\}|\Xi_t(\hat{g}_n, g) - 1|] \\ &\quad + (\mathrm{P}(B_n) - \mathrm{P}(A_n))N\sqrt{\log n}/n. \end{aligned}$$

Note that

$$\mathrm{E}(\Xi_t(\hat{g}_n, g) - 1)^2 = \exp(\Delta_t(\hat{g}_n, g)) - 1,$$

and so by (11)

$$\mathrm{E}(\Xi_t(\hat{g}_n, g) - 1)^2 \le \frac{\text{const}}{n^2}.$$

Also, for any $0 < x < 1$

$$\begin{aligned}&\mathrm{P}\{|\Xi_t(\hat{g}_n, g) - 1| > x\}\\ &\qquad \le \mathrm{P}\{\log(\Xi_t(\hat{g}_n, g)) > \log(1+x)\} + \mathrm{P}\{-\log(\Xi_t(\hat{g}_n, g)) > -\log(1-x)\}.\end{aligned}$$

Now direct calculations based on the bound (11) and the normality of the random variable $\log(\Xi_t(\hat{g}_n, g))$ show that for sufficiently large N we have, for $n \to \infty$,

$$\mathrm{E}[\mathbf{1}\big\{|\Xi_t(\hat{g}_n, g) - 1| > N\sqrt{\log n}\,/n\big\}\,|\Xi_t(\hat{g}_n, g) - 1|] = O(n^{-3/2}).$$

To complete the proof one only needs the following lemma.

Lemma 2. *Under the conditions of Theorem 3*

$$\mathrm{P}\left\{W_{it/n} < g\left(\frac{i}{n}t\right), i \le n\right\} - \mathrm{P}\{W_s < \hat{g}_n(s), s \le t\} \le \text{const}/\sqrt{n}. \tag{13}$$

This result can actually be considered as a consequence of the general result of Nagaev (1970) (see also Sahanenko (1974) and Borovkov (1982)) concerning a rate of convergence of crossing probabilities of sums of independent normalized random variables with a finite third moment to the crossing probabilities of a Brownian motion.

5. Square-root boundaries

Denote

$$\tau_1 = \inf\{t \ge 0 : W_t \ge a + b\sqrt{t+c}\}, \qquad c \ge 0, \quad a + b\sqrt{c} > 0,$$

and

$$\tau_2 = \inf\{t \ge 0 : |W_t| \ge b\sqrt{t+c}\}, \qquad c > 0.$$

Stopping times like τ_1 and τ_2 were first studied by Breiman (1966), Shepp (1967), Novikov (1971) and later by Sato (1977) and others. In particular, Shepp (1967) derived the Mellin transformation of the density $p_2(t)$ of the stopping time τ_2. Using a different technique Novikov (1971) derived the Mellin transformation of the density $p_1(t)$ of the stopping time τ_1.

Let

$$\begin{aligned}HK(v, z) &:= \int_0^\infty \exp(zt - t^2/2)t^{-2v-1}\,\mathrm{d}t\\ &= D_{2v}(-z)\exp(z^2/4)/\Gamma(-2v), \qquad \mathrm{Re}(v) < 0\end{aligned}$$

where $D_v(z)$ is the parabolic cylinder function (see Gradshteyn and Ryzhik (1980), formula 9.241.2).

Denote

$$SH(v,z) := (HK(v,z) + HK(v,-z))/2,$$
$$v_n(z) = n\text{th positive root of } HK(v,z),$$
$$w_n(z) = n\text{th positive root of } SH(v,z).$$

Note that the functions $HK(v,z)$ and $SH(v,z)$ have no zeros when $\mathrm{Re}(v) < 0$ and z is real.

Theorem 4.

(i)

$$\mathrm{E}(\tau_1 + c)^v = \frac{HK(v, -a/\sqrt{c})c^v}{HK(v,b)}, \qquad \mathrm{Re}(v) < v_1(b) \tag{14}$$

and we take by continuity $HK(v, -a/\sqrt{c})c^v = a^{2v}\Gamma(-2v)$ *when* $c = 0$;

$$p_{\tau_1}(t) = \sum_{n=1}^{\infty} \frac{HK(v_n, -a\sqrt{c})c^{v_n}}{(\partial/\partial v HK(w_n, b))}(t+c)^{-v_n-1}; \tag{15}$$

(ii)

$$\mathrm{E}(\tau_2 + c)^v = \frac{SH(v,0)c^v}{SH(v,b)}, \qquad \mathrm{Re}(v) < w_1(b), \tag{16}$$

$$p_{\tau_2}(t) = \sum_{n=1}^{\infty} \frac{SH(w_n, 0)c^{w_n}}{(\partial/\partial v SH(w_n, b))}(t+c)^{-w_n-1}. \tag{17}$$

Formulas (14) and (16) were presented in Novikov (1971) and Shepp (1967) accordingly. Expansions (15) and (17) can be obtained by applying a standard analytical technique. Note that DeLong (1981) obtained the expansion for probability $\mathrm{P}\{\sup_{1<t<T}(|W_t|/\sqrt{t}) < c\}$ which is similar to (17).

The functions $HK(v,z)$ and $SH(v,z)$ can be expressed in terms of confluent hypergeometric functions. Numerical values of the roots $v_n(z)$ and $w_n(z)$ can be easily found with the Mathematica package which contains efficient algorithms for calculations of hypergeometric functions and their zeroes (see Wolfram (1996)).

Using (15) and (17) we get the following numerical values in the case $a = 0$, $b = 1$, $c = 1$:

$$\mathrm{P}\{\tau_1 > 1\} = 0.804\,003, \qquad \mathrm{P}\{\tau_2 > 1\} = 0.608\,560\,.$$

Numerical calculations with the algorithm ($3'$) using a classical Simpson method for $n = 64$ give the following approximations for these probabilities:

$$\mathrm{P}(\hat{g}_{64}) = 0.804\,002, \qquad \mathrm{P}(-\hat{g}_{64}, \hat{g}_{64}) = 0.608\,560.$$

The same numerical results (but essentially faster) we also obtained by the help of Gaussian quadratures.

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References

ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31**, 165–197.

BOROVKOV, K. A. (1982). Rate of convergence in a boundary problem. *Theory. Prob. Appl.* **27**, 148–149.

BREIMAN, L. (1966). First exit time from a square root boundary. *Proc. 5th Berkeley Symp. Math. Statist. Prob.* **2**, 9–16.

DANIELS, H. (1996). Approximating the first crossing-time density for a curved boundary. *Bernoulli* **2**, 133–143.

DELONG, D. (1981). Crossing probabilities for a square root boundary by a Bessel process. *Commun. Statist.-Theory Meth. A* **10**, 2197–2213.

DURBIN, J. (1992). The first-passage density of the Brownian motion process to a curved boundary. *J. Appl. Prob.* **29**, 291–304.

FEREBEE, B. (1983). An asymptotic expansion for one-sided Brownian exit densities. *Z. Wahreinscheinlichkeitsch.* **61**, 309–326.

FRISHLING, V., ANTIC, A., KUCHERA, A. AND RIDER, P. (1997). Pricing barrier options with time-dependent drift, volatility and barriers. Working paper, April 1997, Commonwealth Bank of Australia.

GRADSHTEYN, I. AND RYZHIK, I. (1980). *Table of Integrals, Series, and Products*. Academic Press, New York.

HALL, W. J. (1997). The distribution of Brownian motion on linear stopping boundaries. *Sequential Analysis* **4**, 345–352.

KHMALADZE, E. AND SHINJIKASHVILI, E. (1998). Calculations of non-crossing probabilities for Poisson processes and its corollaries. Preprint, The University of NSW, Australia.

LERCHE, H. R. (1986). *Boundary Crossing of Brownian Motion* (Lecture Notes in Statist. **40**). Springer, Berlin.

LIPTSER, R. S. AND SHIRYAEV, A. N. (1977). *Statistics of Random Processes*, Vol. 1. Springer, New York.

NAGAEV, S. V. (1970). The rate of convergence in a certain boundary value problem, part I. *Theory Prob. Appl.* **15**, 179–199; part II, **15**, 419–444.

NOVIKOV, A. A. (1971). On stopping times for a Wiener process. *Theory Prob. Appl.* **16**, 458–465.

NOVIKOV, A. A. (1979). On estimates and the asymptotic behavior of nonexit probabilities of a Wiener process to a moving boundary. *Mat. Sb.* **110**, 539–550. (English translation in *Math. USSR Sb.* **38**.)

NOVIKOV, A. A. (1980). Estimates and asymptotic behavior of the probability of not crossing moving boundaries by sums of independent random variables. *Izv. Akad. Nauk SSSR Ser. Mat.* **44**, 868–885.

NOVIKOV, A. A., KORDZAKHIA, N. AND WRIGHT, I. (1998). Time-dependent barrier options and boundary crossing probabilities. Working paper, Department of Statistics, the University of Newcastle, Australia.

RICCIARDI, L. M. (1977). *Diffusion Processes and Related Topics in Biology* (Lecture Notes in Biomath. **14**). Springer, Berlin.

ROBERTS, G. O. AND SHORTLAND, C. F. (1997). Pricing barriers options with time-dependent coefficients. *Math. Finance* **7**, 83–93.

SACERDOTE, L. AND TOMASSETI, F. (1996). On evaluations and asymptotics approximations of the first-passage-time probabilities. *Adv. Appl. Prob.* **28**, 270–284.

SAHANENKO, A. I. (1974). On the rate of convergence in a boundary value problem. *Theory. Prob. Appl.* **19**, 416–421.

SALMINEN, P. (1988). On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *J. Appl. Prob.* **20**, 411–426.

SATO, S. (1977). Evaluation of the first passage probability to a square root boundary for the Wiener process. *J. Appl. Prob.* **14**, 53–70.

SEN, P. K. (1981). *Sequential Nonparametrics: Invariance Principles and Statistical Inference*. Wiley, New York.

SHEPP, L. A. (1967). A first passage problem for the Wiener process. *Ann. Math. Statist.* **38**, 1912–1914.

SIEGMUND, D. (1985). *Sequential Analysis: Tests and Confidence Intervals*. Springer, New York.

WANG, L. AND PÖTZELBERGER, K. (1997). Boundary crossing probability for Brownian motion and general boundaries. *J. Appl. Prob.* **34**, 54–65.

WOLFRAM, ST. (1996). *The Mathematica Book*. Cambridge University Press.