Final Work

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1 Option Pricing

1.1 Question 2

The free risk rate of Facebook was taken from Guru Focus. The free risk rate was extracted from the website on the 2nd of June of 2020. The value for this rate is $r = 0.65\%$.

On the other hand, the volatility was estimated by using the Maximum Likelihood methodology. Let y_i for $i = 1, 2, \ldots, n$ be the prices of the stock. Then, in the first place, the returns ω_i are calculated by:

$$\omega_i = \frac{y_{i+1} - y_i}{y_i}$$

It is important to remark that the amount of returns is one less than the data. The time series for the returns can be seen in Figure 1. Finally, the volatility σ is estimated by the following equation:

$$\sigma = \frac{1}{(n-2)\Delta t} \sum_{i=1}^{n-1} \left(\omega_i - \frac{1}{n-1} \sum_{i=1}^{n-1} \omega_i \right)^2$$

The result for this estimation is $\sigma = 0.025$.

1.2 Question 3

The stock pricing was realized with Black-Scholes, Monte Carlo simulation, binomial trees, and finite differences. The pricing obtained with Black-Scholes is considered to be the exact pricing for the Facebook stock.

For each method, a time series for the returns are calculated. Differently to the previous question, the returns are calculated by:

$$\omega_i = \log\left(\frac{y_{i+1}}{y_i}\right)$$

with y_i are the prices for the Facebook stock. Furthermore, it is important to notice that this returns also have one less data than the stock prices.

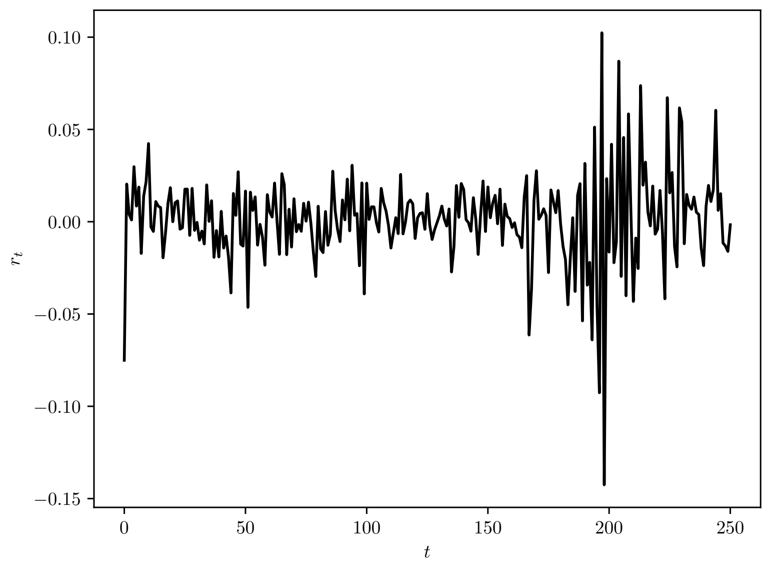

Figure 1: Time series for returns used for ML methology.

Black and Scholes [1] created a methodology to correctly price a European call option. Nowadays, this methodology is known as the Black-Scholes method. Let σ be the volatility of the stock, T the time for that option. Then, two quantiles are calculated by:

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right), \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Lastly, using these quantiles the pricing for the option f is calculated as follows:

$$f = S_0 F(d_1) - K e^{-rT} F(d_2)$$

with $F(\cdot)$ the continuous distribution function for a standard normal distribution.

The second method used was a Monte Carlo simulation approach. This approach was created by Boyle [2]. The Monte Carlo simulation method first simulates the selected differential equation with M trajectories. In the case of the returns, the process is a Geometric Brownian Motion, which is discretized as:

$$S_{t+\Delta t} = S_t e^{\left(r - \frac{\sigma^2}{2}\right) + \sigma \Delta B_t}$$

Then, the payoff is calculated by:

$$p_i = \max(S_{i,T} - K, 0)$$

Consequently, then the f_{call} is calculated by the following equation:

$$f_{\text{call}} = e^{-rT}\mathbb{E}\left[p\right]$$

This process is repeated W times, saving the results of each simulation. Finally, the mean of all the f_{call} is calculated to estimate the pricing of the option.

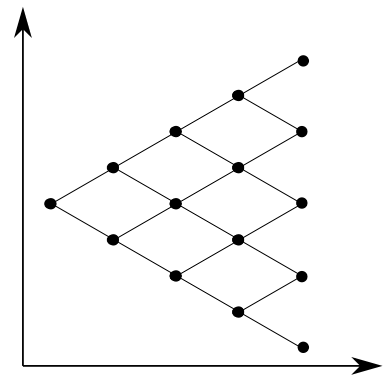

Figure 2: Example of binomial tree

The Binomial Tree method was created by Cox, Ross, and Rubinstein [3]. This method consists of defining a barrier for a binomial tree and, then calculating the value for the root by regressing. An example of a binomial tree can be seen in Figure 2.

Let N be the depth of the binomial tree, i.e. the number of nodes from the root to the last node. In the first place, the barrier is calculated by:

$$f_{N,j} = \max(S_0 ujd(N-j) - K, 0)$$

with:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p = \frac{e^{r\Delta t} - d}{u - d}$$

Lastly, the tree is traversed backwards calculating each node by:

$$f_{i,j} = e^{-r\Delta t}(pf_{i+1,j+1} + (1-p)f_{i+1,j})$$

The pricing for the option is stored in the root of the tree, after completing all calculations.

Three strikes where selected such that for each K it happens that $S_0 > K$, with $S_0 \approx 225$. The strikes with the comparison of the result of each method is shown in Table 1. For Monte Carlo, the selected parameters were $M = 3000$ and $W = 500$. For finite differences, the selected parameter was $NS = 1000$. Finally, for the binomial tree the selected parameter was $N = 100$.

	Black Scholes	Monte Carlo	Finite Diffs.	Binomial Tree
$K = 200$	186.22	184.70	157.52	186.22
$K = 100$	205.65	204.29	191.91	205.65
$K = 50$	215.37	213.81	208.03	215.37

Table 1: Result for each method for two decimal places.

The method which was the closest one to the exact solution was the Binomial Tree as the numbers are equal to two decimals. Then, it is followed by the Monte Carlo simulation method, which was the second closest but the slowest algorithm. Finally, the Finite Difference was the worst at estimating the pricing. In this manner, by this small experiment, it can be stated that Binomial Tree is the most precise and efficient method.

1.3 Question 4

2 Sensitivity Analysis

2.1 Question 1

2.2 Question 2

2.3 Question 5

References

[1] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81.3 (1973), pp. 637–654.

[2] Phelim P Boyle. "Options: A Monte Carlo Approach". In: *Journal of Financial Economics* 4.3 (1977), pp. 323–338.

[3] John C Cox, Stephen A Ross, and Mark Rubinstein. "Option Pricing: A Simplified Approach". In: *Journal of Financial Economics* 7.3 (1979), pp. 229–263.