3. Slide 15

4. a) Slide 39.

b) Sea $V = C[a,b]$ $\left(\frac{x-a}{b}\right)^n$

Sean las funciones $f_n(t) = \left(\frac{x-a}{b}\right)^n$

Se puede ver que $f_n(\cdot)$ son funciones continuas en $\mathbb{R}$, en particular en $[a,b]$ (son Polinomios)

Veamos que $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ es sucesión de Cauchy con $\|f(\cdot)\|_1$

Sea $\epsilon > 0$, como $a \le t \le b$

$\left(\frac{t-a}{b-a}\right)^{n+1} \le \left(\frac{t-a}{b-a}\right)^n \Leftarrow 0 \le \frac{t-a}{b-a} \le 1$ $\Rightarrow$ Suc. decreciente.

Sea $K = \left\lceil \frac{4(b-a)}{\epsilon} \right\rceil - 1$

Sean $n, m > K$, luego

$$\|f_n(t) - f_m(t)\| \le \|f_n(t)\| + \|f_m(t)\|$$
$$< \|f_K(t)\| + \|f_K(t)\| \quad \text{(suc. decre.)}$$
$$= 2\|f_K(t)\|$$
$$= 2\int_a^b \left(\frac{t-a}{b-a}\right)^K dt = \frac{2(t-a)^{K+1}}{(b-a)^K(K+1)}\Big|_a^b = 2\left(\frac{b-a}{b-a}\right)^K \frac{(b-a)}{(K+1)} = \frac{2(b-a)}{\left\lceil\frac{4(b-a)}{\epsilon}\right\rceil - 1 + 1} \le \frac{2(b-a)}{\frac{4(b-a)}{\epsilon}} = \frac{\epsilon}{2} < \epsilon$$

Luego, $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ es suc. de Cauchy.

Por otra parte,

Si $t \in [a,b)$ se tiene que $\frac{t-a}{b-a} < 1$

Por el lema 1, $\lim_{n\to\infty} \left(\frac{t-a}{b-a}\right)^n = 0$

Mientras que si $t = b \to f_n(t) = \left(\frac{b-a}{b-a}\right)^n = 1$

De donde $f_n(t) \to f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t = 1 \end{cases}$

y claramente $f(\cdot) \notin C[a,b]$.

5. a) $\langle x,y\rangle = \sum_{k=1}^{n} x_k|y_k|$

i) $\langle x+y, z\rangle = \sum_{k=1}^{n} (x_k+y_k)|z_k|$
$= \sum_{k=1}^{n} x_k|z_k| + \sum_{k=1}^{n} y_k|z_k|$
$= \langle x,z\rangle + \langle y,z\rangle$

ii) $\langle ax,y\rangle = a\sum_{k=1}^{n} x_k|y_k| = a\langle x,y\rangle$

iii) No se cumple

iv) $\langle x,x\rangle \ge 0$ ← no siempre se cumple

Si $x=0$, $\langle x,x\rangle = \sum_{k=1}^{n} 0\cdot 0 = 0$

Pero $\langle x,x\rangle = 0 = \sum_{k=1}^{n} x_i|x_i|$ y $x_i|x_i|$ no siempre es positivo.

b) $\langle x,y\rangle = \left|\sum_{k=1}^{n} x_k y_k\right|$

i) $\langle x+y,z\rangle = \left|\sum_{k=1}^{n}(x_i+y_i)z_i\right| \le \left|\sum_{k=1}^{n} x_i z_i\right| + \left|\sum_{k=1}^{n} y_i z_i\right|$

No se cumple siempre

ii) $\langle ax,y\rangle = \left|\sum_{k=1}^{n} a x_i y_i\right| = |a|\langle x,y\rangle \ne a\langle x,y\rangle$

iii) $\langle x,y\rangle = \left|\sum_{k=1}^{n} x_i y_i\right| = \left|\sum_{k=1}^{n} y_i x_i\right| = \langle y,x\rangle$

iv) $\langle x,x\rangle = \left|\sum_{k=1}^{n} x_i^2\right| \ge 0$

($\rightarrow$) $x=0 \to \langle x,x\rangle = \left|\sum_{k=1}^{n} 0^2\right| = 0$

($\leftarrow$) $\langle x,x\rangle = 0 = \left|\sum_{k=1}^{n} x_i^2\right|$

$\sum_{k=1}^{n} x_i^2 = 0 \to x_i^2 = 0\ \forall i$
$\Rightarrow x = 0$

c) $\langle x,y\rangle = \left(\sum_{k=1}^{n} x_k^2 y_k^2\right)^{1/2}$

i) $\langle x+y,z\rangle = \left(\sum_{k=1}^{n}(x_k+y_k)^2 z_k^2\right)^{1/2} \ne \left(\sum_{k=1}^{n} x_k^2 z_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2 z_k^2\right)^{1/2}$

ii) $\langle ax,y\rangle = |a|\langle x,y\rangle$

iii) $\langle x,y\rangle = \langle y,x\rangle$ ✓

iv) $\langle x,x\rangle \ge 0$ ✓

($\rightarrow$) $x=0 \to \langle x,x\rangle = \left(\sum_{k=1}^{n} 0^4\right)^{1/2} = 0$

($\leftarrow$) $\langle x,x\rangle = 0 = \left(\sum_{k=1}^{n} x_k^4\right)^{1/2} \Rightarrow x_k^4 = 0$
$x_k = 0\ \forall k$
$\Rightarrow x = 0$

6. a) $\langle f,g\rangle = f(1)g(1)$

i) $\langle f+h,g\rangle = [f(1)+h(1)]g(1)$
$= \langle f,g\rangle + \langle h,g\rangle$

ii) $\langle af,g\rangle = a\langle f,g\rangle$

iii) $\langle f,g\rangle = \langle g,f\rangle$

iv) $\langle f,f\rangle \ge 0$ ✓

$\longrightarrow$ $f=0$, $\langle f,f\rangle = f(1)f(1)$ ($\ne 0$)

$\longleftarrow$ $\langle f,f\rangle = 0 \to f(1) = 0$ pero no dice nada del resto del dominio

b) $\langle f,g\rangle = \int_0^1 f'(t)g'(t)\,dt$

i) $\langle f+h,g\rangle = \int_0^1 [f(t)+h(t)]'g'(t)\,dt$
$= \int_0^1 f'(t)g'(t)\,dt + \int_0^1 h'(t)g'(t)\,dt$
$= \langle f,g\rangle + \langle h,g\rangle$

ii) $\langle af,g\rangle = \int_0^1 af'(t)g'(t)\,dt = a\langle f,g\rangle$

iii) $\langle f,g\rangle = \int_0^1 f'(t)g'(t)\,dt = \int_0^1 g'(t)f'(t)\,dt = \langle g,f\rangle$

iv) $\langle f,f\rangle = \int_0^1 [f'(t)]^2\,dt$

Claramente, $[f'(t)]^2 \ge 0$, $\forall t$

Luego $\int_0^1 [f'(t)]^2\,dt \ge 0$.

($\rightarrow$) $f=0 \to \langle f,f\rangle = \int_0^1 0\,dt = 0$

($\leftarrow$) No se cumple.

Ejemplo $f(t) = c \to f'(t) = 0$
$\to \langle f,f\rangle = 0$

Cuando $f(t) \ne 0$.

c) $\langle f,g\rangle = \left(\int_0^t f(t)\,dt\right)\left(\int_0^t g(t)\,dt\right)$

i) $\langle f+h,g\rangle = \left(\int_0^1 [f(t)+h(t)]\,dt\right)\left(\int_0^1 g(t)\,dt\right)$
$= \langle f,g\rangle + \langle h,g\rangle$

ii) $\langle af,g\rangle = a\langle f,g\rangle$

iii) $\langle f,g\rangle = \langle g,f\rangle$

iv) $\langle f,f\rangle \ge 0$ ✓

($\rightarrow$) $f=0 \to \langle f,f\rangle = 0$

($\leftarrow$) Sea $f(t) = \sin(2\pi t)$

$\langle f,f\rangle = \Big(\underbrace{\int_0^1 \sin(2\pi t)\,dt}_{\Delta}\Big)^2$

$\Delta = \frac{1}{2\pi}(-\cos(2\pi t))\Big|_0^1 = \frac{1}{\pi}[1-1] = 0$

No se cumple.

7. a) $f(t) = \sqrt{t}$

$\langle f,f\rangle = \int_1^e (\ln t)\,t\,dt$ $\quad u = \ln t$, $dv = t\,dt$; $du = \frac{dt}{t}$, $v = \frac{t^2}{2}$

$= \frac{t^2\ln t}{2}\Big|_1^e - \int_1^e \frac{t}{2}\,dt = \frac{e^2}{2} - \frac{1}{4}t^2\Big|_1^e$

$= \frac{e^2}{2} - \left[\frac{e^2}{4} - \frac{1}{4}\right]$

$= \frac{e^2+1}{4} \to \|f\| = \frac{\sqrt{e^2+1}}{2}$ □

b) $\langle f,g\rangle = \int_1^e (a+bt)\ln t\,dt$

$= \int_1^e a\ln t\,dt + b\underbrace{\int_1^e t\ln t\,dt}_{\frac{b(e^2+1)}{4}}$

$u = \ln t$, $dv = dt$; $du = \frac{dt}{t}$, $v = t$

$t\ln t - \int dt$

$a(t\ln t - t)\Big|_1^e$

$= a(e\ln e - e) - a(1\ln 1 - 1)$

$= a$

$\langle f,g\rangle = \frac{b(e^2+1)}{4} + a = 0$

$a = -\frac{b(e^2+1)}{4}$ → familia de funciones ortogonales a $f(t) = 1$

Luego,