

b) Veamos que $f_n(t)$ es de Cauchy. // Sea $\varepsilon > 0$ y $k \in \mathbb{N}$ tal que $2\left(\frac{3}{4}\right)^{\frac{k+1}{k}} \leq \varepsilon/2$. Se ve que $\forall n, m > k$:

Observar que:

$$\|f_n(t)\| = \int_0^1 |f_n(t)| dt = \int_{1/4}^1 \left|t - \frac{1}{4}\right|^{1/n} dt = \int_{1/4}^1 (t - 1/4)^{1/n} dt$$

$$\begin{aligned} u &= t - \frac{1}{4} \\ du &= dt \\ &= \int_0^{3/4} u^{1/n} du = \frac{n}{n+1} u^{\frac{n+1}{n}} \Big|_0^{3/4} = \frac{n}{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{n}} \end{aligned}$$

$$\begin{aligned} \text{Luego: } \|u_n - u_m\| &\leq \|u_n\| + \|u_m\| \\ &\leq \frac{n}{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{n}} + \frac{m}{m+1} \left(\frac{3}{4}\right)^{\frac{m+1}{m}} \leq \left(\frac{3}{4}\right)^{\frac{n+1}{n}} + \left(\frac{3}{4}\right)^{\frac{m+1}{m}} \\ &\leq 2 \left(\frac{3}{4}\right)^{\frac{k+1}{k}} \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Luego, $f_n(t)$ es de Cauchy. Sin embargo, converge a una función discontinua. Luego \forall no es de Banach.