$v \in \mathbb{R}^3$

1. a) $T(v) = (v_1+v_2,\ v_1+3v_2,\ v_1+v_3)$

$T(av+bu) = (av_1+bu_1+av_2+bu_2,\ av_1+bu_1+3(av_2+bu_2),\ 2(av_1+bu_1)+3(av_3+bu_3))$

<u>Nulo</u>

$N(T) = \{v \in \mathbb{R}^3 : Tv = 0\}$

$N(T) = \{v \in \mathbb{R}^3 : v_1+v_2=0,\ v_1+3v_2=0,\ 2v_1+3v_3=0\}$

$$\begin{bmatrix}1&1&0\\1&3&0\\2&0&3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&0\\1&3&0&0\\2&0&3&0\end{array}\right] \xrightarrow[f_3-2f_1]{f_2-f_1} \left[\begin{array}{ccc|c}1&1&0&0\\0&2&0&0\\0&-2&3&0\end{array}\right]$$

$$\xrightarrow{f_3+f_2} \left[\begin{array}{ccc|c}1&1&0&0\\0&2&0&0\\0&0&3&0\end{array}\right]$$

$3v_3 = 0 \to v_3 = 0$

$\hookrightarrow 2v_2 = 0 \to v_2 = 0$

$v_1 + v_2 = 0 \to v_1 = 0$

$N(T) = \{(0,0,0)\}.$

<u>Recorrido</u>:

$R(T) = \{w \in \mathbb{R}^3 : w = Tv,$ para algún $v \in \mathbb{R}^3\}$

$R(T) = \{w \in \mathbb{R}^3 : w = (v_1+v_2,\ v_1+3v_2,\ 2v_1+3v_3),\ v_1,v_2,v_3 \in \mathbb{R}\}$

$= \{w \in \mathbb{R}^3 : w = (1,1,2)v_1 + (1,3,0)v_2 + (0,0,3)v_3,\ v_1,v_2,v_3 \in \mathbb{R}^3\}$

$= \text{gen}\{(1,1,2), (1,3,0), (0,0,3)\}.$

$= \mathbb{R}^3$

Matriz T

Base para $\mathbb{R}^3$: $\{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

$T(e_1) = (1,1,2)$

$T(e_2) = (1,3,0)$

$T(e_3) = (0,0,3)$

$$\longrightarrow \begin{bmatrix}1&1&2\\1&3&0\\0&0&3\end{bmatrix}$$

<u>Acotado</u>:

$\|Tv\|_2^2 = (v_1+v_2)^2 + (v_1+3v_2)^2 + (2v_1+3v_3)^2$

$= \cancel{v_1^2} + \cancel{2v_1v_2} + \cancel{v_2^2} + \cancel{v_1^2} + \cancel{6v_1v_2} + \cancel{9v_2^2} + \cancel{4v_1^2} + 12v_1v_3 + \cancel{9v_3^2}$

$= 6v_1^2 + 10v_2^2 + 9v_3^2 + 8v_1v_2 + 12v_1v_3$

$\leq \cancel{6v_1^2} + \cancel{10v_2^2} + 9v_3^2 + 8(\cancel{v_1^2} + \cancel{v_2^2}) + 12(\cancel{v_1^2} + v_3^2)$

$= 26v_1^2 + 18v_2^2 + 21v_3^2 \leq 26(v_1^2+v_2^2+v_3^2)$

$\|Tv\|_2^2 \leq 26\|v\|_2^2$

$\|Tv\|_2 \leq \sqrt{26}\,\|v\|_2$

b) $T: C^2[a,b] \to C[a,b]$

$Tu = u'' - 2u' - 3u$

$T(au+bv) = (au+bv)'' - 2(au+bv)' - 3(au+bv)$

$= au'' + bv'' - 2au' - 2bv' - 3au - 3bv$

$= a(u''-2u'-3u) + b(v''-2v'-3v)$

$= aTu + bTv$

<u>Nulo</u>

$N(T) = \{v \in C^2[a,b] : Tv = 0\}$

$Tv = 0 \to v'' - 2v - 3 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda+1)(\lambda-3) = 0$

$v(t) = c_1e^{-t} + c_2e^{3t}$

$N(T) = \{c_1e^{-t} + c_2e^{3t} : c_1, c_2 \in \mathbb{R}\}.$

$= \text{gen}\{e^{-t}, e^{3t}\}$

Recorrido

$R(T) = \{w \in C[a,b] : w = Tv$ para algún $v \in C^2[a,b]\}$

$R(T) = \{w \in C[a,b] : w = v'' - 2v' - 3v,$ para algún $v \in C^2[a,b]\}$

c) $TA = A - A^T$

$T(aA+bB) = aA + bB - (aA+bB)^T$

$= a(A-A^T) + b(B-B^T)$

$= aTA + bTB.$

<u>Nulo</u>:

$N(T) = \{A \in \mathbb{R}^{n\times n} \mid TA = 0\}$

$N(T) = \{A \in \mathbb{R}^{n\times n} \mid A = A^T\}$

↳ Matrices simétricas.

<u>Recorrido</u>

$R(T) = \{W \in \mathbb{R}^{n\times n} \mid W = TA$ para algún $A \in \mathbb{R}^{n\times n}\}$

$W = A - A^T,\ A \in \mathbb{R}^{n\times n}$

$W^T = A^T - A = -(A - A^T) = -W$

Luego, W es una matriz antisimétrica.

$R(T) =$ matrices antisimétricas

d) $T: \mathcal{P}^4 \to \mathcal{P}^3$

$T(p(t)) = p''(t)$

$T(ap(t)+bq(t)) = (ap(t)+bq(t))''$

$= ap''(t) + bq''(t)$

$= aT(p(t)) + bT(q(t))$

<u>Nulo</u>:

$N(T) = \mathcal{P}^1$

<u>Recorrido</u>:

$R(T) = \mathcal{P}^2$

$R(T) = \{q \in \mathcal{P}^3 \mid q = T(p(t)),\ p(t) \in \mathcal{P}^4\}$

$r' = r - 1$

$q(t) = \sum_{r=0}^{3} a_r t^r = \frac{d^2}{dt^2}\left[\sum_{r=0}^{4} b_r t^r\right] = \frac{d}{dt}\left[\sum_{r=1}^{4} r b_r t^{r-1}\right]$

$= \frac{d}{dt}\left[\sum_{r'=0}^{3} (r'+1) b_{r'+1} t^{r'}\right]$

$= \sum_{r'=1}^{3} r'(r'+1) b_{r'+1} t^{r'-1}$ $\quad r^* = r' - 1$

$= \sum_{r^*=0}^{2} (r^*+1)(r^*+2) b_{r^*+2} t^{r^*}$

Igualando coefs:

$a_3 = 0$ ← No hay grado 3.

$a_2 = 4\cdot 3\, b_4$

$a_1 = 3\cdot 2\, b_3$

$a_0 = 2\cdot 1\, b_2$

↳ Polinomios de grado 2 o menor.