$$\ln|x| = \begin{cases} \ln(-x), & x<0 \\ \ln(x), & x>0 \end{cases}$$

$$\frac{d}{dx}(\ln|x|) = \begin{cases} -1/-x, & x<0 \\ 1/x, & x>0 \end{cases} = 1/x$$

13. Diaps. 44–46

14. Veamos que $\ln|x| \in \mathbb{L}^1_{loc}(\mathbb{R})$

Sea $K \subseteq \mathbb{R}$ compacto.

Luego, $\exists a>0 : K \subseteq [-a,a]$

$$\int_K |\ln|x||\,dx \le \int_{-a}^{a} |\ln|x||\,dx$$

Par

$$= 2\int_0^a |\ln|x||\,dx$$

$$= -2\int_0^1 \ln(x)\,dx + 2\int_1^a \ln(x)\,dx$$

$$= -2\lim_{\epsilon\to0^+}\int_\epsilon^1 \ln x + 2\int_1^a \ln x\,dx$$

$$= -2\lim_{\epsilon\to0^+}\left[x\ln x - x\right]_\epsilon^1 + 2\left[x\ln x - x\right]_1^a$$

$$= -2\lim_{\epsilon\to0^+}\left[-1-\epsilon\ln\epsilon+\epsilon\right] + 2\left[a\ln a - a + 1\right]$$

$$= 2 + 2\lim_{\epsilon\to0^+}\frac{\ln\epsilon}{1/\epsilon} + 2a(\ln a+1)+2$$

$$= 4 + 2\lim_{\epsilon\to0^+}\frac{1/\epsilon}{-1/\epsilon^2} + 2a(\ln a+1)$$

$$= 4 + 2\lim_{\epsilon\to0^+}\underbrace{\cancel{(-\epsilon)}}_{0} + 2a(\ln a+1)$$

$$= 4 + 2a(\ln a+1) < \infty$$

Luego, $\ln|x| \in \mathbb{L}^1_{loc}(\mathbb{R})$

$\therefore$ $\ln|x|$ define una distribución regular. (teorema (K)).

Derivada.

Sea $\varphi\in\mathcal{D}(\mathbb{R})$, $f(x)=\ln|x|$

$$\left\langle \frac{df}{dx}, \varphi\right\rangle = -\left\langle f, \frac{d\varphi}{dx}\right\rangle$$

$$= -\int_{\mathbb{R}} \ln|x|\varphi'(x)\,dx$$

$$= -\lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\ln|x|\varphi'(x)\,dx + \int_\epsilon^\infty \ln|x|\varphi'(x)\,dx\right]$$

$$= -\lim_{\epsilon\to0^+}\left[\ln|x|\varphi(x)\Big|_{-\infty}^{-\epsilon} - \int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx + \ln|x|\varphi(x)\Big|_\epsilon^\infty - \int_\epsilon^\infty \frac{\varphi(x)}{x}\,dx\right]$$

$$= -\lim_{\epsilon\to0^+}\left[\ln(\epsilon)\varphi(-\epsilon) - \int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx - \ln(\epsilon)\varphi(\epsilon) - \int_\epsilon^\infty \frac{\varphi(x)}{x}\,dx\right]$$

$$= \lim_{\epsilon\to0^+}\underbrace{\cancel{\ln(\epsilon)[\varphi(\epsilon)-\varphi(-\epsilon)]}}_{0} + \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx + \int_\epsilon^\infty \frac{\varphi(x)}{x}\,dx\right]$$

$$= \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x}\,dx + \int_\epsilon^\infty \frac{\varphi(x)}{x}\,dx\right]$$

$$= vp\int_{\mathbb{R}}\frac{\varphi(x)}{x}\,dx$$

Segunda derivada

16. b) Sea $\varphi\in\mathcal{D}(\mathbb{R})$, $f(x)=\ln|x|$

$$\left\langle \frac{d^2f}{dx^2}, \varphi\right\rangle = (-1)^2\left\langle f, \frac{d^2\varphi}{dx^2}\right\rangle$$

$$= \int_{\mathbb{R}}\ln|x|\varphi''(x)\,dx$$

$$= \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\ln|x|\varphi''(x)\,dx + \int_\epsilon^\infty \ln|x|\varphi''(x)\,dx\right]$$

$$= \lim_{\epsilon\to0^+}\left[\ln|x|\varphi'(x)\Big|_{-\infty}^{-\epsilon} - \int_{-\infty}^{-\epsilon}\frac{\varphi'(x)}{x}\,dx + \ln|x|\varphi'(x)\Big|_\epsilon^\infty - \int_\epsilon^\infty\frac{\varphi'(x)}{x}\,dx\right]$$

$$= \lim_{\epsilon\to0^+}\underbrace{\cancel{\ln(\epsilon)[\varphi'(-\epsilon)-\varphi'(\epsilon)]}}_{0} - \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi'(x)}{x}\,dx + \int_\epsilon^\infty\frac{\varphi'(x)}{x}\,dx\right]$$

$$= -\lim_{\epsilon\to0^+}\left[\frac{\varphi(x)}{x}\Big|_{-\infty}^{-\epsilon} + \int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x^2}\,dx + \frac{\varphi(x)}{x}\Big|_\epsilon^\infty + \int_\epsilon^\infty\frac{\varphi(x)}{x^2}\,dx\right]$$

$$= -\lim_{\epsilon\to0^+}\left[\frac{\varphi(x)}{x}\Big|_{-\infty}^{-\epsilon} + \frac{\varphi(x)}{x}\Big|_\epsilon^\infty\right] - \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x^2}\,dx + \int_\epsilon^\infty\frac{\varphi(x)}{x^2}\,dx\right]$$

Obsérvese que

$$\lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(0)}{x^2}\,dx + \int_\epsilon^\infty\frac{\varphi(0)}{x^2}\,dx\right] = -\lim_{\epsilon\to0^+}\left[\frac{\varphi(0)}{x}\Big|_{-\infty}^{-\epsilon} + \frac{\varphi(0)}{x}\Big|_\epsilon^\infty\right]$$

$$= -\lim_{\epsilon\to0^+}\frac{\varphi(0)}{-\epsilon} + \lim_{\epsilon\to-\infty}\frac{\varphi(0)}{\epsilon} - \lim_{\epsilon\to\infty}\frac{\varphi(0)}{\epsilon} + \lim_{\epsilon\to0^+}\frac{\varphi(0)}{\epsilon}$$

Como $\lim_{\epsilon\to\infty}\frac{\varphi(0)}{\epsilon} = 0 = \lim_{\epsilon\to\infty}\frac{\varphi(\epsilon)}{\epsilon}$

Como $\varphi\in\mathcal{D}(\mathbb{R})$, φ es continua luego $\lim_{\epsilon\to0^+}\frac{\varphi(\epsilon)}{\epsilon} = \lim_{\epsilon\to0^+}\frac{\varphi(0)}{\epsilon}$

Como $\lim_{\epsilon\to-\infty}\frac{\varphi(0)}{\epsilon} = 0 = \lim_{\epsilon\to-\infty}\frac{\varphi(\epsilon)}{\epsilon}$

Como $\varphi\in\mathcal{D}(\mathbb{R})$, φ es continua luego $\lim_{\epsilon\to0^+}\frac{\varphi(0)}{-\epsilon} = \lim_{\epsilon\to0^+}\frac{\varphi(-\epsilon)}{-\epsilon}$

$$= -\lim_{\epsilon\to0^+}\frac{\varphi(-\epsilon)}{-\epsilon} + \lim_{\epsilon\to-\infty}\frac{\varphi(\epsilon)}{\epsilon} - \lim_{\epsilon\to\infty}\frac{\varphi(\epsilon)}{\epsilon} + \lim_{\epsilon\to0^+}\frac{\varphi(\epsilon)}{\epsilon}$$

$$= -\lim_{\epsilon\to0^+}\left[\frac{\varphi(x)}{x}\Big|_{-\infty}^{-\epsilon} + \frac{\varphi(x)}{x}\Big|_\epsilon^\infty\right]$$

Luego,

$$= \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(0)}{x^2}\,dx + \int_\epsilon^\infty\frac{\varphi(0)}{x^2}\,dx\right] - \lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)}{x^2}\,dx + \int_\epsilon^\infty\frac{\varphi(x)}{x^2}\,dx\right]$$

$$= -\lim_{\epsilon\to0^+}\left[\int_{-\infty}^{-\epsilon}\frac{\varphi(x)-\varphi(0)}{x}\,dx + \int_\epsilon^\infty\frac{\varphi(x)-\varphi(0)}{x}\,dx\right]$$

$$= -vp\int_{\mathbb{R}}\frac{\varphi(x)-\varphi(0)}{x^2}\,dx$$