Stochastic Processes II: Final Report

Juan Sebastián Cárdenas-Rodríguez
David Plazas Escudero
Mathematical Engineering, Universidad EAFIT

June 11, 2020

The code for this workshop can be found in the source code.

Option Assessment

Question 1. Select a financial action that does not pay dividends, for a full operation year. Make a complete description if the company, specifying the type of operation, market, etc. Additionally, verify that the price of the selected action follows a lognormal distribution, based on statistical tests.

The selected company was Facebook. According to Facebook's company profile in Bloomberg, "*Facebook, Inc. operates a social networking website. The Company website allows people to communicate with their family, friends, and coworkers. Facebook develops technologies that facilitate the sharing of information, photographs, website links, and videos. Facebook users have the ability to share and restrict information based on their own specific criteria.*" [5] It started as a website in Harvard University in 2004, developed by Mark Zuckerberg [1] and it rapidly grew into a worldwide company. The main products of Facebook Inc. include Facebook, Instagram, Messenger, WhatsApp and Oculus [6].

The prices were obtained from the Yahoo Finance page for Facebook Inc.. The obtained time series is presented in Figure 1.

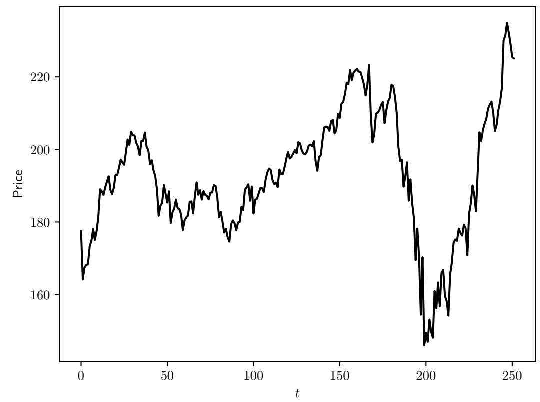


Figure 1: Time Series of Facebook Inc. prices.

In order to verify that the selected action follows a lognormal distribution, a standard Kolmogorov-Smirnov goodness-of-fit test and we obtained a p-value of 0.764, which implies that there is evidence to affirm that the prices follow a lognormal distribution (and a good fit, since the pvalue is high). Furthermore, the histogram and the fitted lognormal distribution are presented in Figure 2.

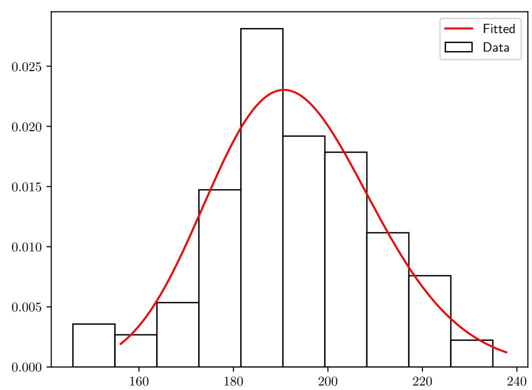


Figure 2: Histogram and lognormal distribution of prices.

Finally, the empirical and theoretical cumulative distribution functions, as well as the 95% confidence bands for the empirical distribution, are presented in Figure 3. The confidence bands where calculated using the Dvoretzky-Kiefer-Wolfowitz inequality (see e.g. [7]). The authors consider that the presented results give strong evidence that the prices follow a lognormal distribution.

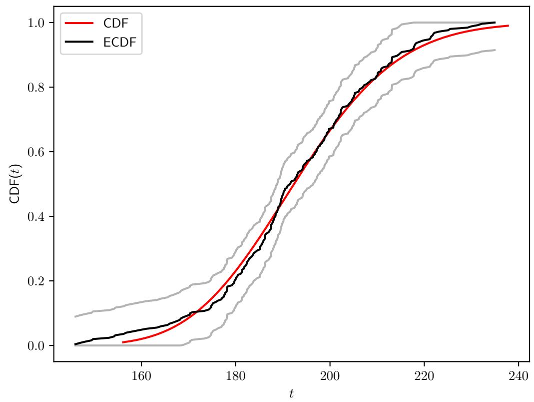


Figure 3: Cumulative distributions functions.

Question 2. Consult the risk-free interest rate for the financial market to which the selected action belongs. Additionally, estimate the volatility of the action returns using the maximum likelihood methodology studied in class.

The free risk rate of Facebook was taken from Guru Focus. The free risk rate was extracted from the website on the 2nd of June of 2020. The value for this rate is $r = 0.65\%$.

On the other hand, the volatility was estimated by using the Maximum Likelihood methodology. Let y_i for $i = 1, 2, \ldots, n$ be the prices of the stock. Then, in the first place, the returns ω_i are calculated by:

$$\omega_i = \frac{y_{i+1} - y_i}{y_i}$$

It is important to remark that the amount of returns is one less than the data. The time series for the returns can be seen in Figure 4. Finally, the volatility σ is estimated by the following equation:

$$\sigma = \frac{1}{(n-2)\Delta t} \sum_{i=1}^{n-1} \left(\omega_i - \frac{1}{n-1} \sum_{i=1}^{n-1} \omega_i \right)^2$$

The result for this estimation is $\sigma = 0.025$.

Question 3. Use the Black-Scholes formula to evaluate an **European call** option, where the underlying is the selected action (this will be named *Option 1*). Evaluate this option for an expiry of 1 year and three different strikes. The strikes must be selected such that the option could be executed ($S_0 > K$). Compare the results with the ones obtained using Montecarlo, finite differences and binomial trees. Analyze the results.

The stock pricing was realized with Black-Scholes, Monte Carlo simulation, binomial trees, and finite differences. The pricing obtained with Black-Scholes is considered to be the exact pricing for the Facebook stock.

Black and Scholes [2] created a methodology to correctly price a European call option. Nowadays, this methodology is known as the Black-Scholes method. Let σ be the volatility of the stock, T the time for that option. Then, two quantiles are calculated by:

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T \right), \qquad d_2 = d_1 - \sigma\sqrt{T}$$

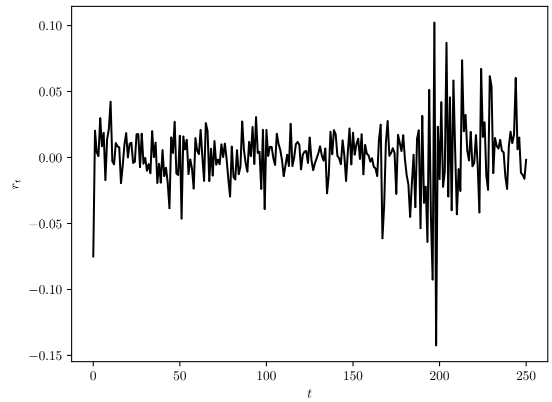


Figure 4: Time series for returns used for ML methology.

Lastly, using these quantiles the pricing for the option f is calculated as follows:

$$f = S_0 F(d_1) - K e^{-rT} F(d_2)$$

with $F(\cdot)$ the continuous distribution function for a standard normal distribution.

The second method used was a Monte Carlo simulation approach. This approach was created by Boyle [3]. The Monte Carlo simulation method first simulates the selected differential equation with M trajectories. In the case of the returns, the process is a Geometric Brownian Motion, which is discretized as:

$$S_{t+\Delta t} = S_t e^{\left(r-\frac{\sigma^2}{2}\right)+\sigma \Delta B_t}$$

Then, the payoff is calculated by:

$$p_i = \max(S_{i,T} - K, 0)$$

Consequently, then the f_{call} is calculated by the following equation:

$$f_{\text{call}} = e^{-rT}\mathbb{E}\left[p\right]$$

This process is repeated W times, saving the results of each simulation. Finally, the mean of all the f_{call} is calculated to estimate the pricing of the option.

The Finite Differences method consists in calculating the values for a grid of a given size. The grid has a size of $NT \times NS$, where NS is the number of S to be simulated and NT is the number of sample of times to be considered.

In the first place, three barrier conditions are calculated. Which are:

$$\begin{aligned} f_{NT,j} &= \max(j\Delta s - K, 0) \\ f_{j,NS} &= \max(S_{max} - K, 0) \\ f_{j,0} &= 0 \end{aligned}$$

where $S_{max} = 3S_0$ and $\Delta s = S_{max}/NS$. Finally, the rest of the grid is calculated based on the row forward, the equation is:

$$f_{i,j} = a_j f_{i+1,j-1} + b_j f_{i+1,j} + c_j f_{i+1,j+1}$$

where

$$
\begin{aligned}
a_j &= \frac{\Delta t}{r\Delta t + 1}\left(\frac{1}{2}\sigma^2 j^2 - \frac{1}{2}j\right) \\
b_j &= \frac{\Delta t}{r\Delta t + 1}\left(\frac{1}{\Delta t} - \sigma^2 j^2\right) \\
c_j &= \frac{\Delta t}{r\Delta t + 1}\left(\frac{1}{2}\sigma^2 j^2 + \frac{1}{2}rj\right)
\end{aligned}
$$

The Binomial Tree method was created by Cox, Ross, and Rubinstein [4]. This method consists of defining a barrier for a binomial tree and, then calculating the value for the root by regressing. An example of a binomial tree can be seen in Figure 5.

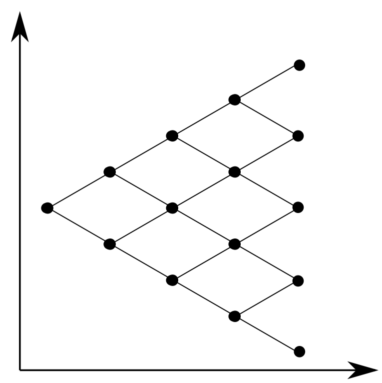


Figure 5: Example of binomial tree

Let N be the depth of the binomial tree, i.e. the number of nodes from the root to the last node. In the first place, the barrier is calculated by:

$$f_{N,j} = \max(S_0 ujd(N - j) - K, 0)$$

with:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p = \frac{e^{r\Delta t} - d}{u - d}$$

Lastly, the tree is traversed backwards calculating each node by:

$$f_{i,j} = e^{-r\Delta t}(pf_{i+1,j+1} + (1 - p)f_{i+1,j})$$

The pricing for the option is stored in the root of the tree, after completing all calculations.

Three strikes where selected such that for each K it happens that $S_0 > K$, with $S_0 \approx 225$. The strikes with the comparison of the result of each method is shown in Table 1. For Monte Carlo, the selected parameters were $M = 3000$ and $W = 500$. For finite differences, the selected parameter was $NS = 1000$. Finally, for the binomial tree the selected parameter was $N = 100$.

	Black Scholes	Monte Carlo	Finite Diffs.	Binomial Tree
$K = 200$	186.22	184.70	157.52	186.22
$K = 100$	205.65	204.29	191.91	205.65
$K = 50$	215.37	213.81	208.03	215.37

Table 1: Result for each method for two decimal places.

The method which was the closest one to the exact solution was the Binomial Tree as the numbers are equal to two decimals. Then, it is followed by the Monte Carlo simulation method, which was the second closest but the slowest algorithm. Finally, the Finite Difference was the worst at estimating the pricing. In this manner, by this small experiment, it can be stated that Binomial Tree is the most precise and efficient method.

It is important to notice, that the result of each method is the pricing for the respective derivative. Hence, it is important to see that for smaller values of K the price of the derivative becomes bigger. This happens because a smaller strike value would increase the price of the derivative.

Question 4. Suppose that you want to evaluate an **European call** option with the returns of the action selected (this will be named *Option 2*). Identify and justify the selection of the stochastic differential equation that models these returns. Evaluate the option considering three possible strikes, using a parameter for the proportionality of the market's risk, between 10% and 15%. Analyze the results.

For each method, a time series for the returns are calculated. Differently to the previous question, the returns are calculated by:

$$\omega_i = \log\left(\frac{y_{i+1}}{y_i}\right)$$

with y_i are the prices for the Facebook stock. Furthermore, it is important to notice that this returns also have one less data than the stock prices. This returns have to distribute normally. So, a fitting was made and tested by using Kolmorogov-Smirnov test, this test failed. The histogram for the fitting and the returns can be seen in Figure 6.

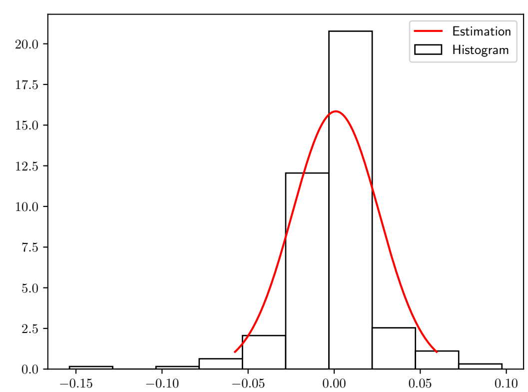


Figure 6: Histogram and fitting for returns.

Although the test failed, it is seen that the fitting works quite well. The test failed because the outliers that happen in the left tail but, for the rest of the data the distribution is well-fitted.

Nevertheless, the stochastic process to simulate the returns was an Ornstein-Uhlenbeck stochastic process. The parameters were estimated and the result of the fitted process is seen in Figure 7. The parameters estimated were $\alpha = 1.26$, $\mu = 0.0012$, $\sigma = 0.024$ and $\lambda = 0.125$.

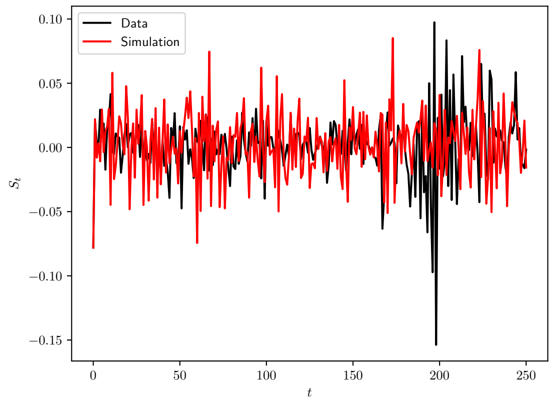


Figure 7: Time series for returns.

To price the derivative the Monte Carlo method was used, simulating by Euler-Maruyama's method. The result for the method can be seen in Table 2.

	Monte Carlo
$K = -0.1$	0.0192
$K = -1$	0.1941
$K = -3$	0.5829

Table 2: Result for each method for two decimal places.

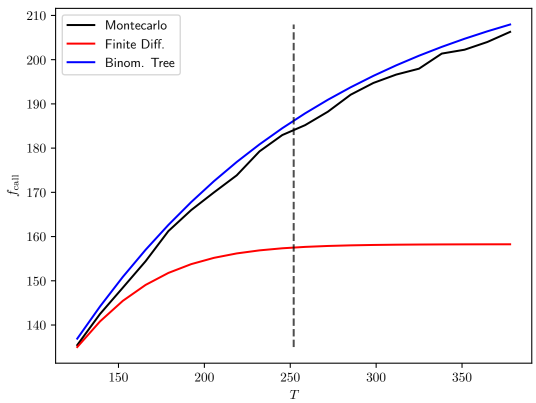
Sensitivity Analysis

Remark: all the sensitivity analysis were executed using the same parameters, except for the one being modified. The common parameters for the sensitivity analysis will be described: for the *Option 1*, the common parameters are $r = 0.0065$, $\sigma = 0.025$, $K = 200$, $S_0 = 225.09$ and $T = 252$. For the finite differences method, $N_s = 1000$ and N_t is calculated by the algorithm. For the binomial tree $N = 100$. Finally, for the Montecarlo, $M = 750$ and $W = 125$. For the *Option 2*, $\alpha = 1.258$, $\mu = 0.0012$, $\sigma = 0.024$, $\lambda = 0.125$, $K = -1$, $S_0 = -0.00164$, $T = 251$, with $M = 750$ and $W = 125$. Unless stated otherwise, the parameters were modified by a $p = 50\%$ factor and taking 20 samples in the range $[(1-p) \times \text{param}, (1+p) \times \text{param}]$.

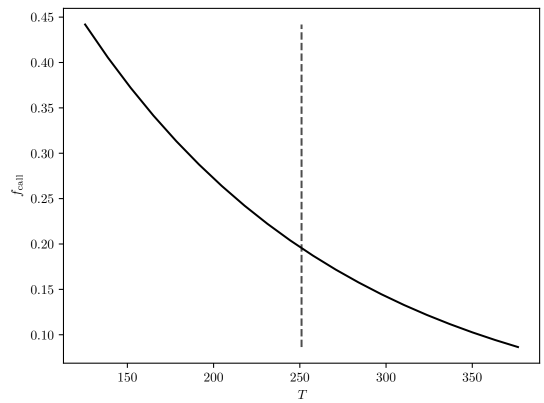
Preserving the same financial asset defined in the last section (with their parameters), analyze the effect on the option value of:

Question 5. An increment on the expiration date (T). This must be verified for all methods for *Option 1* and for Montecarlo in *Option 2*.

This first sensitivity was done with the default parameters. In this experiment, the objective is to find the influence that the final time of simulation T has on every method. The result for this



(a) Option 1



(b) Option 2

Figure 8: Results for first sensitivity for the first option.

sensitivity is found in Figure 8.

Furthermore, it is important to see that the finite difference method does not converge to the real solution therefore it would not be analyzed. In contrast, the other two methods show an increase in the pricing of the derivative. This has sense, as if the time expected to execute the derivative is larger it is possible that the stock will have a bigger change.

On the other hand, in Figure 8 the sensitivity realized for the second option can be seen.

The Monte Carlo simulation show a good behavior. It is seen that when the time of simulation increases the value for the derivative decreases. This has a similar behavior as the one shown in the previous section.

Question 6. An increment on the risk-free interest rate (r). This must be verified for all methods for *Option 1*.

This sensitivity was executed also with the default parameters. This experiment had the objective to test the influence that the free risk rate has in the pricing of the derivative. In Figure 9, the result for each method can be seen.

Similarly to the previous sensitivity, it is seen that for an increase on the risk free rate it increases the price of the derivative. That can be easily explained as, for a bigger rate the value of the stock should increase too. In this manner, bigger risk free rate generates an increase in the cost of the derivative.

Question 7. An increment on the volatility (σ). This must be verified for all methods for *Option 1* and for Montecarlo in *Option 2*.

The results for this sensitivity analysis are presented in Figure 10. Recall that the volatility is a parameter directly linked with the diffusion part in the stochastic differential equation (SDE), hence, as sigma increases, the trajectories of the SDE will be more spread for each point in time, i.e. the variance will increase. This can be evidenced in the lines for the Montecarlo methods, in both Options, since they present spikes and fluctuating behavior. Furthermore, the binomial tree method did not present any variation on the option price for changes in the volatility. Finally, the finite differences method showed little change in the option value, with a slight decrease as σ grew.

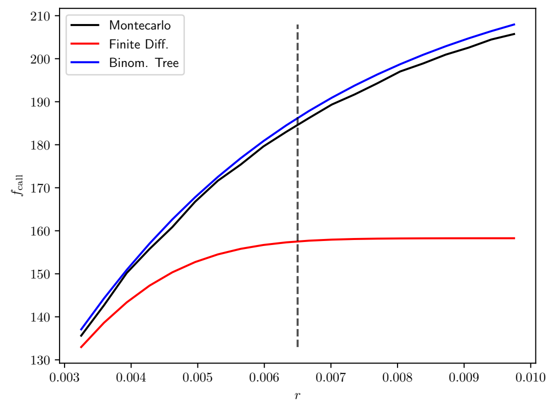
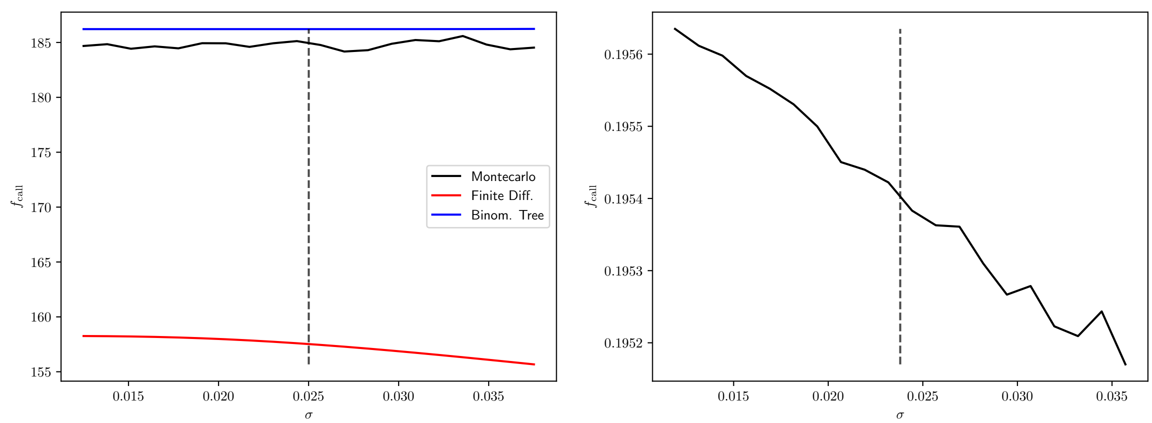


Figure 9: Results for second sensitivity for first option.

For the *Option 2*, the price decreases as well, this is due to the presence of $-\lambda\sigma$ in the trend of the SDE that models the underlying for *Option 2*: as σ increases, the trend term tends to get smaller, and therefore, the expected return too.



(a) *Option 1.*

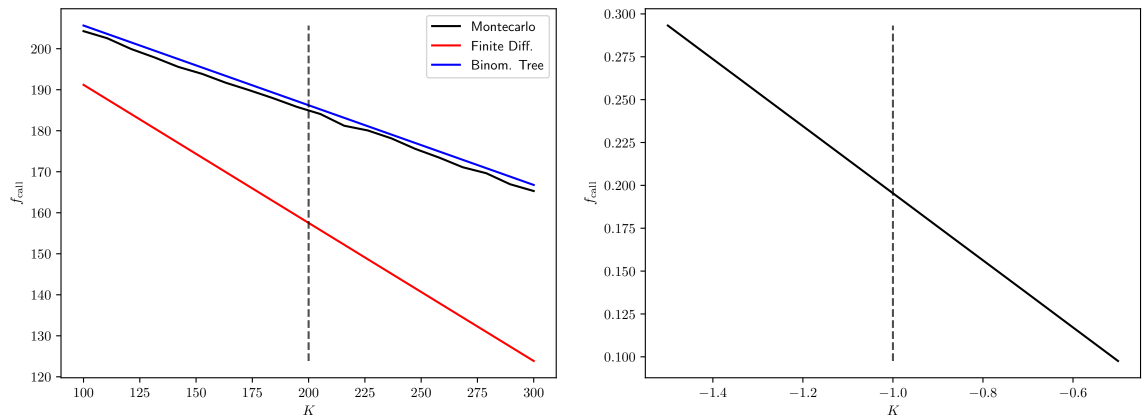
(b) *Option 2.*

Figure 10: Sensitivity on σ.

Question 8. An increment on the strike (K). This must be verified for all methods for *Option 1* and for Montecarlo in *Option 2*.

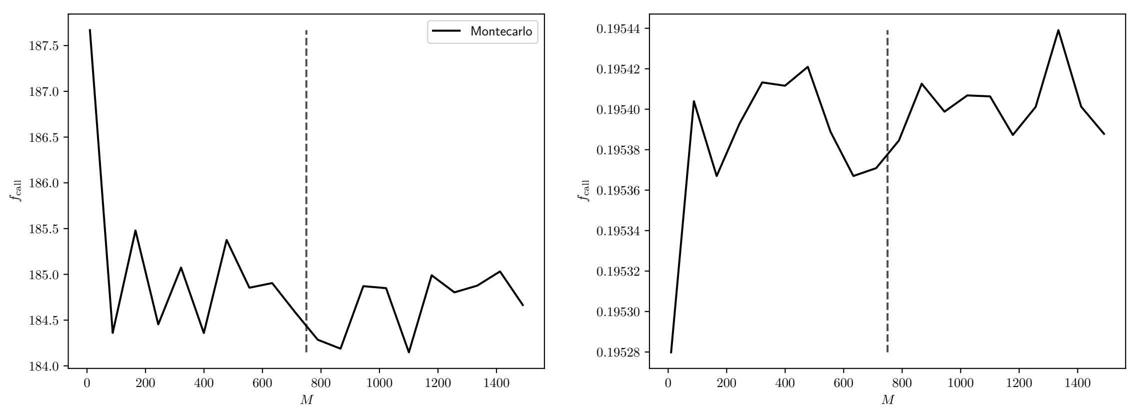
The results are presented in Figure 11. Note that for all methods, the same results (in terms of sensitivity output) were obtained: the option price decreases significantly as the strike K increases. Taking into consideration that these values represent the price of the contract today, as the strike increases, it makes sense that this price decreases since the buyer would be willing to pay more the underlying at the expiration date T.

Question 9. An increment on the number of trajectories (M), starting in 10. An increment on the number of repetitions (W), starting in 100. An increment on the number of subintervals of time (N), starting in 100. This must be verified for the Montecarlo method in both *Option 1* and *Option 2*.



(a) *Option 1.* (b) *Option 2.*

Figure 11: Sensitivity on the strike K.



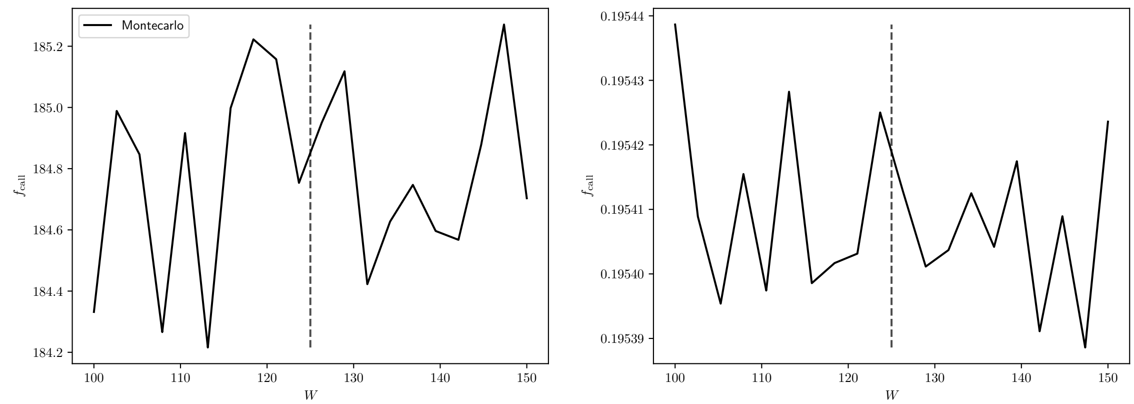
(a) Option 1 (b) Option 2

Figure 12: Results for first experiment.

In Figure 12, the result for the first experiment is seen. In this case, it is important to remark than an increase in the parameter M starts to change the Monte Carlo simulation to oscillate around a solution. This happens because when the number of trajectories increase, the method starts slowly converging to the real solution. In this manner, this experiment had the expected behavior. Furthermore, although not big enough M were tested due to computation time. it is seen that this parameter drastically improves the convergence of the algorithm.

In Figure 13, the results for the second experiment are seen. For increasing W, it is seen that the method tends to oscillate more around the same point instead of stabilizing to another point. Hence, with both experiments, one can conclude that the parameter M is the one who improves the accuracy of the method while W improves the speed of convergence.

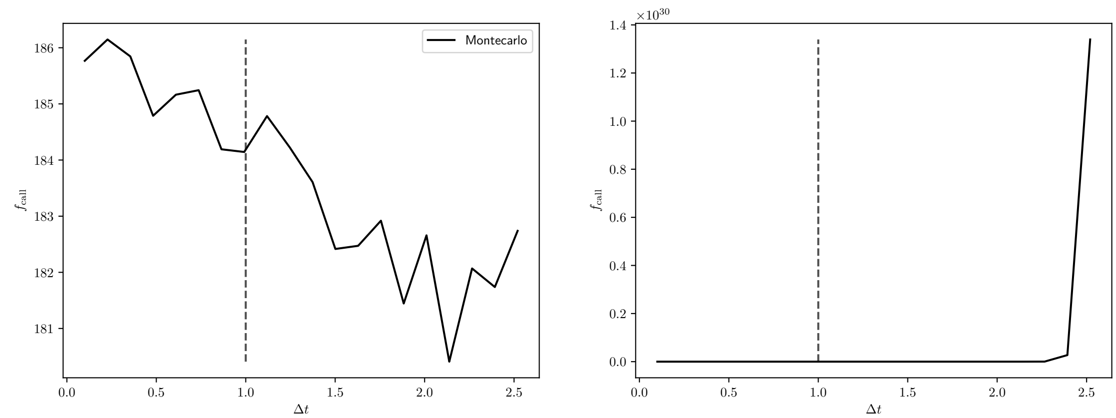
Finally, in Figure 14 the results for the last experiment is seen. It is important to remark that for both options a change in Δt has different impacts. In the *Option 1* it is seen that an increase on Δt does impact the convergence of the method but it does not affect the stability of it. On the other



(a) Option 1

(b) Option 2

Figure 13: Results for second experiment.



(a) Option 1

(b) Option 2

Figure 14: Results for second experiment.

hand, the *Option 2* does diverge as bigger Δt just are significantly bigger for the Euler-Maruyama method.

Question 10. An increment on the number of subintervals with respect to the underlying (N_s). This must be verified for the finite differences method for *Option 1*.

Note that scale of Figure 15 is relatively small, hence the price remains on similar values. It is clear that the price appears to be stabilizing between 157.4 and 157.6 since the amplitude of the fluctuations appear to be getting smaller; though this is pure speculation, since it would be needed to make more evaluations for more values for N_s and a bit more frequent to give a stronger conclusion.

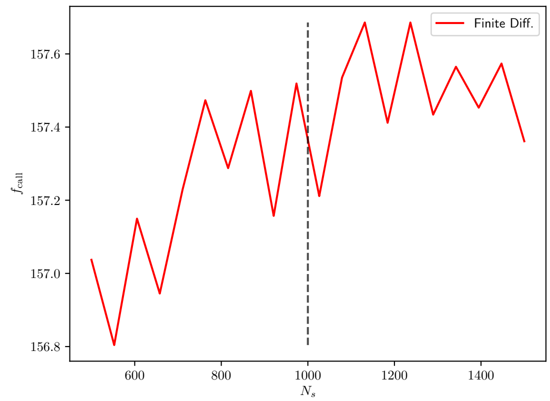


Figure 15: Sensitivity on N_s for *Option 1*.

Question 11. An increment in the number of subintervals in the tree with respect to time (N). this must be verified for the binomial tree method for *Option 1*.

Figure 16 shows the results for changing the number of subintervals in the binomial tree method for *Option 1*. It can be directly concluded that the change on the option price is insignificant (given the scale of the vertical axis).

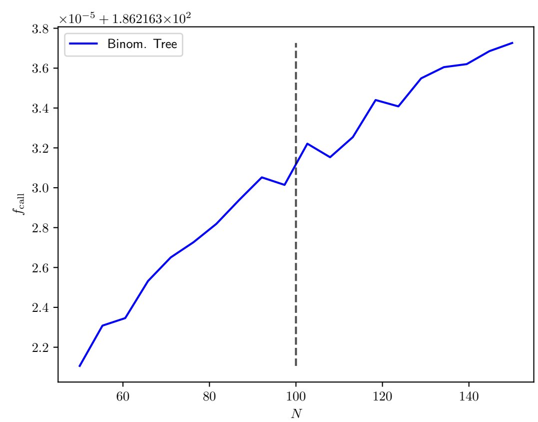


Figure 16: Sensitivity on N for the binomial tree of *Option 1*.

Question 12. An increment on the proportionality of the market's risk (λ). This must be verified for the Montecarlo method in *Option 2*.

The results for the sensitivity on the parameter λ can be found in Figure 17. It can be observed that the option price decreases as λ increases. The authors believe the this is due to the same reason as σ: this parameter is negatively linked to the trend of the SDE, therefore, the expected value decrements.

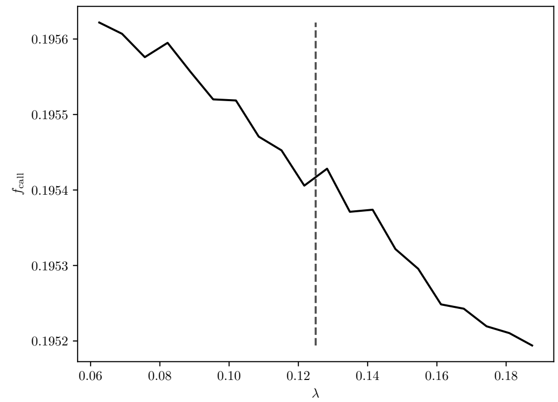


Figure 17: Sensitivity on λ for *Option 2*.

References

[1] "A brief history of Facebook". In: *The Guardian* (2007). URL: https://www.theguardian.com/technology/2007/jul/25/media.newmedia.

[2] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81.3 (1973), pp. 637–654.

[3] Phelim P Boyle. "Options: A Monte Carlo Approach". In: *Journal of Financial Economics* 4.3 (1977), pp. 323–338.

[4] John C Cox, Stephen A Ross, and Mark Rubinstein. "Option Pricing: A Simplified Approach". In: *Journal of Financial Economics* 7.3 (1979), pp. 229–263.

[5] "Facebook Inc - Company Profile". In: *Bloomberg* (2020). URL: https://www.bloomberg.com/profile/company/FB:US.

[6] "Facebook Inc Profile". In: *Reuters* (2020). URL: https://www.reuters.com/companies/FB.O.

[7] Larry Wasserman. *All of Nonparametric Statistics*. Springer Science & Business Media, 2006.