Taller AN2. $\mathbb{R}^n$

1\. a) $\|u\| = \frac{\|u\|_1}{3} + \frac{2}{3}\|u\|_\infty$

i) $\|u\| \geq 0$ dado que $\|u\|_1, \|u\|_\infty \geq 0$

ii) $\|\alpha u\| = \|\alpha u\|_1$

$\vdots$

$u = (x_1, \dots, x_n)$

b. $\|u\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

$\|u\|_p = \max_i |x_i|$

Sea $|x_j| = \max_i |x_i|$.

(1) Supongamos $|x_j| = 0$, luego $|x_i| = 0, \forall i = 1, \dots, n$.

$$\lim_{p\to\infty} \|u\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^{n} 0^p\right)^{1/p} = 0 = |x_j| = \|u\|_\infty$$

(2) Supongamos $|x_j| > 0$.

$$\|u\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \cdot \frac{|x_j|}{|x_j|}$$

$$= \left(\sum_{i=1}^{n} \left|\frac{x_i}{x_j}\right|^p\right)^{1/p} \cdot |x_j|$$

Sea $K = \{i : |x_i| < |x_j|\}$

$$= |x_j| \left(\sum_{i\in K} \left|\frac{x_i}{x_j}\right|^p + C(K^c)\right)^{1/p} \longrightarrow \text{Cardinalidad}$$

Como $|x_i| < |x_j|, \forall i \in K, \rightarrow \left|\frac{x_i}{x_j}\right| < 1 \rightarrow \left|\frac{x_i}{x_j}\right|^p < 1$

Lema 1: $\lim_{p\to\infty} (\phi^p) = 0$ con $\phi \in (0,1)$

Sea $\epsilon > 0$. Es claro que $\phi < 1$

$\phi^{p+1} < \phi^p \rightarrow \{\phi^p\}_{p\in\mathbb{N}}$ es decreciente.

$N = \lceil \log_\phi(\epsilon/2) \rceil$

Sabemos que $\log_\phi(\epsilon/2) \leq \lceil \log_\phi(\epsilon/2) \rceil$

$\phi^{\lceil \log_\phi(\epsilon/2) \rceil} \leq \phi^{\log_\phi(\epsilon/2)} = \epsilon/2 < \epsilon$

Como $\{\phi^p\}_{p\in\mathbb{N}}$ es decreciente, $\forall n > N$

$|\phi^n - 0| = \phi^n < \phi^N < \epsilon$

Luego $\phi^p \to 0$

Por el lema 1, $\left|\frac{x_i}{x_j}\right|^p \to 0$ cuando $p\to\infty$.

Luego, $\sum_{i\in K} \left|\frac{x_i}{x_j}\right|^p \to 0$ cuando $p\to\infty$

$$\lim_{p\to\infty} \|u\|_p = \lim_{p\to\infty} |x_j| \left[C(K^c)\right]^{1/p}$$

Claramente, $j \in K^c \rightarrow C(K^c) > 0$

Luego, $\lim_{p\to\infty} \|u\|_p = |x_j| \left[\cancel{C(K^c)}\right]^0$ 1

$\lim_{p\to\infty} \|u\|_p = |x_j| = \|u\|_\infty$

e) Sea $0 < p < 1$. ¿$\|u\|_p$ es norma?

Sea $u \in \mathbb{R}^n$

i) $\|u\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \geq 0$

i^2) $(\rightarrow)$

$\|u\|_p = 0$

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = 0 \longleftrightarrow \sum_{i=1}^{n} |x_i|^p = 0 \longleftrightarrow |x_i| = 0\ \forall i = 1, \dots, n \longleftrightarrow u = 0$$

pues $y^{1/p} = 0$ cuando $y = 0$, con $p > 0$

ii) $(5^2, 16) = u$

$(12^2, 9) = v$

$p = 0.5$

$$\|u+v\| = \left[(5^2+12^2)^{0.5} + (16+9)^{0.5}\right]^2 = (13+5)^2 = 18^2 = 324$$

$$\|u\| + \|v\| = \left[(5^2)^{0.5} + (16)^{0.5}\right]^2 + \left[(12^2)^{0.5} + (9)^{0.5}\right]^2 = 9^2 + 15^2 = 306$$

Sea $u = (u_1, \dots, u_n)$ y $v = (v_1, \dots, v_n)$

tal que $\sqrt{\sqrt{u_i} + \sqrt{v_i}} \in \mathbb{N}$, denotemos $u_i = a_i^2$, $v_i = b_i^2$, y $\exists i,j$ t.q. $u_i \neq u_j$

Sea $p = 0.5$.

$$\|u+v\|_{0.5} = \left(\sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2}\right)^2$$

$$\|u\|_{0.5} = \left(\sum_{i=1}^{n} a_i\right)^2, \quad \|v\|_{0.5} = \left(\sum_{i=1}^{n} b_i\right)^2$$

d) $\underbrace{|x_i| \leq \|x\|_2}\ \overbrace{\leq \sqrt{n}\,\|x\|_\infty}$

$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \geq |x_i|$$

$$\rightarrow \sqrt{n}\,\|x\|_\infty = \sqrt{n\|x\|_\infty^2}$$

$$= \sqrt{\sum_{i=1}^{n} \|x\|_\infty^2} \geq \sqrt{\sum_{j=1}^{n} |x_j|^2}$$

dado que $|x_j| \leq \max_i |x_i| = \|x\|_\infty$