7. a) Lineal

Sean $u, v \in V$ y sean $a, b \in \mathbb{R}$

$$P(au+bv) = \sum_{j=1}^{n} \langle au+bv, v_j \rangle v_j$$

$$= \sum_{j=1}^{n} \left(\langle au, v_j \rangle v_j + \langle bv, v_j \rangle v_j \right)$$

$$= a \sum_{j=1}^{n} \langle u, v_j \rangle v_j + b \sum_{j=1}^{n} \langle v, v_j \rangle v_j$$

$$= aPu + bPv.$$

Continuo:

$$\|Pu\| = \left\| \sum_{j=1}^{n} \langle u, v_j \rangle v_j \right\|$$

$$\leq \sum_{j=1}^{n} |\langle u, v_j \rangle| \, \cancel{\|v_j\|}^{1}$$

$$= \sum_{j=1}^{n} \|u\| \cancel{\|v_j\|}^{1} |\cos\theta|$$

$$\leq n \|u\|$$

Luego T es acotado con $M = n$.

Luego T es continuo.

b) $$P(Pu) = P\left(\sum_{j=1}^{n} \langle u, v_j \rangle v_j \right)$$

$$= \sum_{j=1}^{n} \langle u, v_j \rangle P(v_j)$$

Nótese que $P(v_j) = \sum_{i=1}^{n} \langle v_j, v_i \rangle v_i$

$$= \langle v_j, \cancel{v_j} \rangle v_j$$

$$= v_j$$

Luego, $$P(Pu) = \sum_{j=1}^{n} \langle u, v_j \rangle v_j = Pu$$

c) Sea $x \in R(P)$

$\exists v \in V : Pv = x$.

Obsérvese que: Sea $u \in V$

$$\langle Pv, u \rangle = \left\langle \sum_{j=1}^{n} \langle v, v_j \rangle v_j, u \right\rangle$$

$$= \sum_{j=1}^{n} \langle v, v_j \rangle \langle v_j, u \rangle$$

$$= \left\langle v, \sum_{j=1}^{n} \langle v_j, u \rangle v_j \right\rangle$$

$$= \langle v, Pu \rangle$$

Ahora, sea $u \in N(P)$

Es claro que $u \in V$.

$$\langle x, u \rangle = \langle Pv, u \rangle = \langle v, Pu \rangle = \langle v, 0 \rangle = 0$$

Luego, $R(P) \perp N(T)$