10. $f(x) = \begin{cases} p(1/x)e^{-1/x}, & x>0 \\ 0, & x\le 0 \end{cases}$

$= \begin{cases} \sum_{i=0}^{n} a_i x^{-i} e^{-1/x}, & x>0 \\ 0, & x\le 0 \end{cases}$

Veamos que $f(x)$ es continua
Claramente $f(x)$ es continua para $x\neq 0$, pues la constante, el polinomio y la exponencial son continuas para $x\neq 0$.

En $x=0$: i) $f(0)=0$
ii) $\lim_{x\to 0} f(x) = 0$

$\lim_{x\to 0^-} 0 = 0$

$\lim_{x\to 0^+} \sum_{i=0}^{n} \frac{a_i e^{-1/x}}{x^i}$

Sea $y=\frac{1}{x} \rightarrow \lim_{x\to 0^+} \sum_{i=0}^{n} \frac{a_i e^{-1/x}}{x^i} = \lim_{y\to\infty} \sum_{i=0}^{n} \frac{a_i y^i}{e^y}$

$= \sum_{i=0}^{n} a_i \cancel{\lim_{y\to\infty} \frac{y^i}{e^y}}^{\,0}$

$=0$

$\therefore f(x)$ es continua: $f\in C[\mathbb{R}]$. $(\square)$

$p\left(\frac{1}{x}\right)e^{-1/x}$

$f'(x) = -p'\left(\frac{1}{x}\right)\frac{1}{x^2}e^{-1/x} + e^{-1/x}\frac{1}{x^2}\,p\left(\frac{1}{x}\right)$

$= -p'\left(\frac{1}{x}\right)\frac{e^{-1/x}}{x^2} + \frac{f(x)}{x^2}$

$f'(x) = q_1\left(\frac{1}{x}\right)e^{-1/x}$
$\therefore$ Continua por

$f''(x) = p''\left(\frac{1}{x}\right)\frac{e^{-1/x}}{x^4} - \frac{e^{-1/x}}{x^4}p'\left(\frac{1}{x}\right) + 2p'\left(\frac{1}{x}\right)\frac{e^{-1/x}}{x^3} + \frac{f'(x)}{x^2} - \frac{2f(x)}{x^3}$

$= -\frac{2}{x}\left[-p'\left(\frac{1}{x}\right)\frac{e^{-1/x}}{x^2} + \frac{f(x)}{x^2}\right] + \frac{e^{-1/x}}{x^4}\left[p''\left(\frac{1}{x}\right) - p'\left(\frac{1}{x}\right)\right] + \frac{f'(x)}{x^2}$

$\frac{e^{-1/x}}{x^4}\left[p''\left(\frac{1}{x}\right) - p'\left(\frac{1}{x}\right)\right] + f'(x)\left[\frac{1}{x^2} - \frac{2}{x}\right]$ Δ

2	
6	4
12	6
20	8

$f'''(x) = \frac{e^{-1/x}}{x^6}\left[p''\left(\frac{1}{x}\right) - p'\left(\frac{1}{x}\right)\right] - \frac{4e^{-1/x}}{x^5}\left[p''\left(\frac{1}{x}\right) - p'\left(\frac{1}{x}\right)\right] + \frac{e^{-1/x}}{x^6}\left[-p'''\left(\frac{1}{x}\right) + p''\left(\frac{1}{x}\right)\right] + f''(x)\left[\frac{1}{x^2} - \frac{2}{x}\right] - f'(x)\left[\frac{2}{x^3} - \frac{2}{x^2}\right]$

$= -\frac{4}{x}\left[\frac{e^{-1/x}}{x^4}\left(p''\left(\frac{1}{x}\right) - p'\left(\frac{1}{x}\right)\right) + f'(x)\left(\frac{1}{x^2} - \frac{2}{x}\right)\right] - \frac{e^{-1/x}}{x^6}\left[p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right] + f''(x)\left[\frac{1}{x^2} - \frac{2}{x}\right] + f'(x)\left[\frac{2}{x^3} - \frac{6}{x^2}\right]$

$= -\frac{4}{x}f''(x) - \frac{e^{-1/x}}{x^6}\left[p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right] + f''(x)\left[\frac{1}{x^2} - \frac{2}{x}\right] + 2f'(x)\left[\frac{1}{x^3} - \frac{3}{x^2}\right]$

$2(n-1)a_1^{n-1}(r) + a_1^{n}(r) = (n-r-1)a_1^{n-1}(r) - a_1^{n-1}(r-1)$

$= -\frac{e^{-1/x}}{x^6}\left[p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right] + f'(x)\left[\frac{2}{x^3} - \frac{6}{x^2}\right] + f''(x)\left[\frac{1}{x^2} - \frac{6}{x}\right]$

$f^{(4)}(x) = -\frac{e^{-1/x}}{x^8}\left[p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right] + \frac{6e^{-1/x}}{x^7}\left[p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right] + \frac{e^{-1/x}}{x^8}\left[p''\left(\frac{1}{x}\right) - 2p'''\left(\frac{1}{x}\right) + p^{(4)}\left(\frac{1}{x}\right)\right] + f'''(x)\left[\frac{1}{x^2} - \frac{6}{x}\right] - f''(x)\cancel{\left[\frac{2}{x^3} - \frac{6}{x^2}\right]} + f''(x)\cancel{\left[\frac{2}{x^3} - \frac{6}{x^2}\right]} - f'(x)\left[\frac{6}{x^4} - \frac{12}{x^3}\right]$

$= -\frac{6}{x}\left[\frac{e^{-1/x}}{x^6}\left(p'\left(\frac{1}{x}\right) - 2p''\left(\frac{1}{x}\right) + p'''\left(\frac{1}{x}\right)\right) + f''(x)\left[\frac{1}{x^2} - \frac{6}{x}\right] + f'(x)\left[\frac{2}{x^3} - \frac{6}{x^2}\right]\right] - \frac{e^{-1/x}}{x^8}\left[p'\left(\frac{1}{x}\right) - 3p''\left(\frac{1}{x}\right) + 3p'''\left(\frac{1}{x}\right) - p^{(4)}\left(\frac{1}{x}\right)\right] + f''(x)\left[\frac{6}{x^3} - \frac{36}{x^2}\right] + f'''(x)\left[\frac{1}{x^2} - \frac{6}{x}\right] + f'(x)\left[\frac{6}{x^4} - \frac{24}{x^3}\right]$

$= -\frac{6}{x}f'''(x) - \frac{e^{-1/x}}{x^8}\left[p'\left(\frac{1}{x}\right) - 3p''\left(\frac{1}{x}\right) + 3p'''\left(\frac{1}{x}\right) - p^{(4)}\left(\frac{1}{x}\right)\right] + f''(x)\left[\frac{6}{x^3} - \frac{36}{x^2}\right] + f'''(x)\left[\frac{1}{x^2} - \frac{6}{x}\right] + f'(x)\left[\frac{6}{x^4} - \frac{24}{x^3}\right]$

$= -\frac{e^{-1/x}}{x^8}\left[p'\left(\frac{1}{x}\right) - 3p''\left(\frac{1}{x}\right) + 3p'''\left(\frac{1}{x}\right) - p^{(4)}\left(\frac{1}{x}\right)\right] + f'(x)\left[\frac{6}{x^4} - \frac{24}{x^3}\right] + f''(x)\left[\frac{6}{x^3} - \frac{36}{x^2}\right] + f'''(x)\left[\frac{1}{x^2} - \frac{12}{x}\right]$

$f^{(5)}(x) = -\frac{e^{-1/x}}{x^{10}}\left[p'\left(\frac{1}{x}\right) - 4p''\left(\frac{1}{x}\right) + 6p'''\left(\frac{1}{x}\right) - 4p^{(4)}\left(\frac{1}{x}\right) + p^{(5)}\left(\frac{1}{x}\right)\right] + f'(x)\left[\frac{24}{x^5} - \frac{120}{x^4}\right] + f''(x)\left[\frac{36}{x^4} - \frac{240}{x^3}\right] + f'''(x)\left[\frac{12}{x^3} - \frac{120}{x^2}\right] + f^{(4)}(x)\left[\frac{1}{x^2} - \frac{20}{x}\right]$

	1	2	3	4
2	$\frac{1}{x^2} - \frac{2}{x}$			
3	$\frac{2}{x^3} - \frac{6}{x^2}$	$\frac{1}{x^2} - \frac{6}{x}$		
4	$\frac{6}{x^4} - \frac{24}{x^3}$	$\frac{6}{x^3} - \frac{36}{x^2}$	$\frac{1}{x^2} - \frac{12}{x}$	
5	$\frac{24}{x^5} - \frac{120}{x^4}$	$\frac{36}{x^4} - \frac{240}{x^3}$	$\frac{12}{x^3} - \frac{120}{x^2}$	$\frac{1}{x^2} - \frac{20}{x}$

$n(r+1)$			
1			
3	1		
12	6	1	
60	40	10	1

$P(n)$: $f^{(n)}(x) = \frac{e^{-1/x}}{x^{2n}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right]$

donde

$a^n(r) = \begin{cases} [0,0], & n\le r \\ [1, r(r+1)], & n=r+1 \\ [a_1^{n-1}(1), n!], & r=1, n>2 \\ [a_1^{n-1}(r), a_1^{n-1}(r-1) + (n+r-1)a_1^{n-1}(r)], & n>r+1 \end{cases}$

$a^1(1) = (1,2)$
$a^n(1) = (a_1^{n-1}(r), \quad n!)$

$a^n(r) = \begin{cases} (0,0), & n\le r \\ (1, r(r+1)), & n=r+1 \\ (a_1^{n-1}(r), -(n-r-1)a_1^{n-1}(r) + a_1^{n-1}(r-1) + 2(n-1)a_1^{n-1}(r)), & n>r+1 \end{cases}$

$n \backslash r$	1	2	3	4	5
2	(1,2)				
3	(2,6)	(1,6)			
4	(6,24)	(6,36)	(1,12)		
5	(24,120)	(36,240)	(12,120)	(1,20)	
6	(120,720)	(240,1560)	(120,1200)	(20,300)	(1,30)

$-1\cdot 20 + 120 + 2\cdot 5\cdot 20$

Paso base: $n=2$ se cumple en Δ.

Paso Inductivo. Supongamos que, $P(n)$ es verdad. Veamos que también lo es.

$P(n)$: $f^{(n)}(x) = \frac{e^{-1/x}}{x^{2n}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right]$

$f^{(n+1)}(x) = \frac{e^{-1/x}}{x^{2n+2}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) - 2n\frac{e^{-1/x}}{x^{2n+1}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) - \frac{e^{-1/x}}{x^{2n+2}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r+1)}\left(\frac{1}{x}\right) + \sum_{r=1}^{n-1} f^{(r+1)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right] + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{-(n-r+1)a_0^n(r)}{x^{n-r+2}} + \frac{(n-r)a_1^n(r)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2n+2}}\left[\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + \sum_{r=2}^{n+1}\binom{n-1}{r-2}(-1)^r p^{(r)}\left(\frac{1}{x}\right)\right] - \frac{2n}{x}\left[\frac{e^{-1/x}}{x^{2n}}\sum_{r=1}^{n}\binom{n-1}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right]\right]$
$+ \sum_{r=1}^{n-1} f^{(r+1)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right] + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{-(n-r+1)a_0^n(r)}{x^{n-r+2}} + \frac{(n-r)a_1^n(r)}{x^{n-r+1}}\right] + \frac{2n}{x}\sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{a_0^n(r)}{x^{n-r+1}} - \frac{a_1^n(r)}{x^{n-r}}\right]$

$= \frac{e^{-1/x}}{x^{2n+2}}\left[p'\left(\frac{1}{x}\right) + \sum_{r=2}^{n}\left\{\binom{n-1}{r-1} + \binom{n-1}{r-2}\right\}(-1)^r p^{(r)}\left(\frac{1}{x}\right)\right] - \frac{2n}{x}f^{(n)}(x) + \sum_{r=2}^{n} f^{(r)}(x)\left[\frac{a_0^n(r-1)}{x^{n-r+2}} - \frac{a_1^n(r-1)}{x^{n-r+1}}\right] + \sum_{r=1}^{n-1} f^{(r)}(x)\left[\frac{(2n-n+r-1)a_0^n(r)}{x^{n-r+2}} - \frac{(2n-n+r)a_1^n(r)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2(n+1)}}\sum_{r=1}^{n+1}\binom{n}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) - \frac{2n}{x}f^{(n)}(x) + f^{(n)}(x)\left[\frac{a_0^n(n-1)}{x^2} - \frac{a_1^n(n-1)}{x}\right] + f'(x)\left[\frac{a_0^n(1)n}{x^{n+1}} - \frac{(n+1)a_1^n(1)}{x^n}\right] + \sum_{r=2}^{n-1} f^{(r)}(x)\left[\frac{(n+r-1)a_0^n(r) + a_0^n(r-1)}{x^{n-r+2}} - \frac{(n+r)a_1^n(r) + a_1^n(r-1)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2(n+1)}}\sum_{r=1}^{n+1}\binom{n}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + f^{(n)}(x)\left[\frac{a_0^n(n-1)}{x^2} - \frac{2n + a_1^n(n-1)}{x}\right] + f'(x)\left[\frac{n!}{x^{n+1}} - \frac{(n+1)!}{x^n}\right] + \sum_{r=2}^{n-1} f^{(r)}(x)\left[\frac{(n+r-1)a_1^{n-1}(r) + a_1^{n-1}(r-1)}{x^{n-r+2}} - \frac{a_1^{n+1}(r)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2(n+1)}}\sum_{r=1}^{n+1}\binom{n}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + f^{(n)}(x)\left[\frac{1}{x^2} - \frac{n(n+1)}{x}\right] + f'(x)\left[\frac{a_0^{n+1}(1)}{x^{n+1}} - \frac{a_1^{n+1}(1)}{x^n}\right] + \sum_{r=2}^{n-1} f^{(r)}(x)\left[\frac{a_0^{n}(r)}{x^{n-r+2}} - \frac{a_1^{n+1}(r)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2(n+1)}}\sum_{r=1}^{n+1}\binom{n}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + f^{(n)}(x)\left[\frac{a_0^{n+1}(n)}{x^2} - \frac{a_1^{n+1}(n)}{x}\right] + f'(x)\left[\frac{a_0^{n+1}(1)}{x^{n+1}} - \frac{a_1^{n+1}(1)}{x^n}\right] + \sum_{r=2}^{n-1} f^{(r)}(x)\left[\frac{a_0^{n+1}(r)}{x^{n-r+2}} - \frac{a_1^{n+1}(r)}{x^{n-r+1}}\right]$

$= \frac{e^{-1/x}}{x^{2(n+1)}}\sum_{r=1}^{n+1}\binom{n}{r-1}(-1)^r p^{(r)}\left(\frac{1}{x}\right) + \sum_{r=1}^{n} f^{(r)}(x)\left[\frac{a_0^{n+1}(r)}{x^{n-r+2}} - \frac{a_1^{n+1}(r)}{x^{n-r+1}}\right]$

Puede ser reescrito como
$f^{(n)}(x) = q_n\left(\frac{1}{x}\right)e^{-1/x}$
$\therefore$ es continua por $(\square)$.

Luego, $P(n+1)$ es verdad
$\therefore P(n)$ es cierto $\forall n\ge 2$.