1. $g(t,X_t) = g_1(X_1(t)) + g_2(X_2(t)) + Z(t)$ $\qquad B_t = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$

$$dg(t,X_t) = g_1'(X_1(t))dX_1(t) + g_2'(X_2(t))dX_2(t) + Z'(t)dt + \frac{1}{2}\left[g_1''(X_1(t))(dX_1)^2 + g_2''(X_2(t))(dX_2)^2\right]$$

$$dg(t,X_t) = Z'(t)dt + g_1'(X_1(t))\left[Sin(t)dt + e^t dB_1(t)\right] + g_2'(X_2(t))\left[t^2 dB_2(t)\right]$$
$$+ \frac{1}{2}\left[g_1''(X_1(t))\left[Sin(t)dt + e^t dB_1(t)\right]^2 + g_2''(X_2(t))\left[t^2 dB_2(t)\right]\right]$$

$$dg(t,X_t) = \left[Z'(t) + Sin(t)g_1'(X_1(t)) + \frac{g_1''(X_1(t))}{2}e^{2t} + \frac{g_2''(X_2(t))}{2}t^4\right]dt$$
$$+ \left[e^t g_1'(X_1(t))dB_1(t) + t^2 g_2'(X_2(t))dB_2(t)\right] \longrightarrow \left[e^t g_1'(X_1(t)) \quad t^2 g_2'(X_2(t))\right]dB_t.$$

Como

$$dg(t,X_t) = \left[Cos(t)e^{Sin(t)} + 2Sin(t)Sin(X_1(t))Cos(X_1(t)) + e^{2t}(1 - 2Sin^2(X_1(t))) - \frac{1}{2}t^4 Sin(X_2(t))\right]dt$$
$$+ \left[2e^t Sin(X_1(t))\sqrt{1 - Sin^2(X_1(t))} \quad t^2\sqrt{1 - Sin^2(X_2(t))}\right]dB_t$$

Igualando términos en dB_t:

$$e^t g_1'(X_1(t)) = 2e^t Sin(X_1(t))\sqrt{1 - Sin^2(X_1(t))}$$
$$g_1'(X_1(t)) = 2Sin(X_1(t))Cos(X_1(t)) = 2Sin(2X_1(t)) \rightarrow \boxed{g_1(X_1(t)) = -Cos(2X_1(t))}$$
$$t^2\sqrt{1 - Sin^2(X_2(t))} = t^2 g_2'(X_2(t)) \rightarrow g_2'(X_2(t)) = Cos(X_2(t))$$
$$\rightarrow \boxed{g_2(X_2(t)) = Sin(X_2(t))}$$

Y Claramente, $Cos(t)e^{Sin(t)} = Z'(t) \longrightarrow \boxed{Z_t = e^{Sin(t)}}$

$$g(t,X_t) = -Cos(2X_1(t)) + Sin(X_2(t)) + e^{Sin(t)}$$

Aplicando Itô multidim. II:

$$dg(t,X_t) = \nabla g^T(t,X_t)\begin{bmatrix} dt \\ dX_t \end{bmatrix} + \frac{dX_t^T H_{X_t}(t,X_t)dX_t}{2}$$

$$dg(t,X_t) = \left[Cos(t)e^{Sin(t)} \quad 2Sin(2X_1(t)) \quad Cos(X_2(t))\right]\begin{bmatrix} dt \\ dX_1(t) \\ dX_2(t) \end{bmatrix}$$
$$+ \left[dX_1(t) \quad dX_2(t)\right]\begin{bmatrix} 4Cos(2X_1(t)) & 0 \\ 0 & -Sin(X_2(t)) \end{bmatrix}\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix}$$

$$dg(t,X_t)=\left[\cos(t)e^{\sin(t)}dt+2\sin(2X_1(t))dX_1(t)+\cos(X_2(t))dX_2(t)\right]$$
$$+\frac{1}{2}\left[4\cos(2X_1(t))(dX_1)^2+(-\sin(X_2(t)))(dX_2(t))^2\right]$$

$$dg(t,X_t)=\left[\cos(t)e^{\sin(t)}dt+2\sin(2X_1(t))\sin(t)dt+2\sin(2X_1(t))e^{t}dB_1(t)+t^2\cos(X_2(t))dB_2(t)\right]$$
$$+\frac{1}{2}\left[4\cos(2X_1(t))e^{2t}dt\ -\sin(X_2(t))t^4dt\right]$$

$$dg(t,X_t)=\left[\cos(t)e^{\sin(t)}+2\sin(2X_1(t))\sin(t)+2\cos(2X_1(t))e^{2t}-\frac{t^4}{2}\sin(X_2(t))\right]dt$$
$$+\begin{bmatrix}2\sin(2X_1(t))e^{t} & t^2\cos(X_2(t))\end{bmatrix}\begin{bmatrix}dB_1(t)\\ dB_2(t)\end{bmatrix}$$

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2. a) $dX_t = (F(t)X_t + f(t))dt + (G(t)X_t + g(t))dB_t$ (1)

Se propone una solución de la forma $X_t = U_t V_t$, donde $dU_t = F(t)U_t dt + G(t)U_t dB_t$ (1.1) y $dV_t = h(t)dt + l(t)dB_t$ (1.2), con $U_0 = 1$ y $V_0 = X_0$

Aplicando Itô multidimensional II: $dX_t = U_t dV_t + V_t dU_t + dU_t dV_t$ (2)

Reemplazando (1.2) y (1.1) en (2):

$$dX_t = U_t(h(t)dt + l(t)dB_t) + V_t(F(t)U_t dt + G(t)U_t dB_t) + (h(t)dt + l(t)dB_t)(F(t)U_t dt + G(t)U_t dB_t)$$

$$dX_t = (U_t h(t) + U_t V_t F(t) + G(t)l(t)U_t)dt + (U_t l(t) + V_t U_t G(t))dB_t \quad (3)$$

Igualando términos entre (1) y (3):

$$\begin{cases} (\cancel{F(t)X_t} + f(t)) = (U_t h(t) + \cancel{U_t V_t F(t)} + G(t)l(t)U_t) \longrightarrow h(t) = U_t^{-1}(f(t) - g(t)G(t)) \\ (\cancel{G(t)X_t} + g(t)) = (U_t l(t) + \cancel{U_t V_t G(t)}) \longrightarrow l(t) = U_t^{-1} g(t) \end{cases}$$

Luego, $\begin{cases} dU_t = F(t)U_t dt + G(t)U_t dB_t & (4.1) \\ dV_t = (f(t) - g(t)G(t))U_t^{-1} dt + g(t)U_t^{-1} dB_t & (4.2) \end{cases}$

(4.1): $\dfrac{dU_t}{U_t} = F(t)dt + G(t)dB_t \longrightarrow \displaystyle\int_0^t \frac{dU_s}{U_s} = \int_0^t F(s)ds + \int_0^t G(s)dB_s$ (5)

Sea $g(U_t) = \ln(U_t) \xrightarrow{Itô} dg(U_t) = \dfrac{dU_t}{U_t} - \dfrac{1}{2U_t^2}(\cancel{F(t)U_t dt} + G(t)U_t dB_t)^2$

$dg(U_t) = \dfrac{dU_t}{U_t} - \dfrac{1}{2}G^2(t)dt \xrightarrow{\int} \ln(U_t) = \displaystyle\int_0^t \frac{dU_s}{U_s} - \int_0^t \frac{G^2(s)}{2}ds$ (6)

(5) en (6): $\ln\left(\dfrac{U_t}{U_0}\right) = \displaystyle\int_0^t \left(F(s) - \frac{1}{2}G^2(s)\right)ds + \int_0^t G(s)dB_s$

$$\Longrightarrow U_t = U_0 \exp\left(\int_0^t \left(F(s) - \frac{G^2(s)}{2}\right)ds + \int_0^t G(s)dB_s\right) \quad (7)$$

(4.2): $dV_t = (f(t) - g(t)G(t))U_t^{-1}dt + g(t)U_t^{-1}dB_t$

$$\Longrightarrow V_t = V_0 + \int_0^t (f(s) - g(s)G(s))U_s^{-1}ds + \int_0^t g(s)U_s^{-1}dB_s$$

$$X_t = U_t\left(X_0 + \int_0^t (f(s) - g(s)G(s))U_s^{-1}ds + \int_0^t g(s)U_s^{-1}dB_s\right) \quad \text{con } U_t \text{ dado en (7)}$$

b) $F(t)=\lambda,\ f(t)=-\sigma\lambda,\ G(t)=0,\ g(t)=\rho$

$$X_t=U_t\left(x_0+\int_0^t(-\sigma\lambda-\rho\cdot 0^{0})U_s^{-1}ds+\int_0^t\rho\, U_s^{-1}dB_s\right)$$

Con $U_t=\exp\left(\int_0^t\left(\lambda-\frac{1}{2}(0)^{0}\right)ds+\cancel{\int_0^t 0\,dB_s}^{\,0}\right)=\exp\left(\int_0^t\lambda\, ds\right)=e^{\lambda t}$

Luego,

$$X_t=e^{\lambda t}\left(x_0+\int_0^t(-\sigma\lambda)e^{-\lambda s}ds+\int_0^t\rho e^{-\lambda s}dB_s\right)=e^{\lambda t}\left(x_0+\sigma e^{-\lambda s}\Big|_0^t+\int_0^t\rho e^{-\lambda s}dB_s\right)$$

$$X_t=e^{\lambda t}\left(x_0+\sigma(e^{-\lambda t}-1)+\int_0^t\rho e^{-\lambda s}dB_s\right)$$

• $E[X_t]=e^{\lambda t}\left(x_0+\sigma(e^{-\lambda t}-1)+\cancel{E\left[\int_0^t\rho e^{-\lambda s}dB_s\right]}\right)$ $\rightarrow 0 \rightarrow$ Propiedad Integral Itô.

$E[X_t]=e^{\lambda t}(x_0+\sigma e^{-\lambda t}-\sigma)=(x_0-\sigma)e^{\lambda t}+\sigma \rightarrow \lim_{t\to\infty}E[X_t]=\lim_{t\to\infty}\left[(x_0-\sigma)e^{\lambda t}+\sigma\right]=\infty.$

• $V[X_t]=E[X_t^2]-E^2[X_t]$. Hallemos $E[X_t^2]$

$$E[X_t^2]=e^{2\lambda t}E\left[\left(x_0+\sigma(e^{-\lambda t}-1)+\int_0^t\rho e^{-\lambda s}dB_s\right)^2\right]=e^{2\lambda t}E\Big[(x_0+\sigma(e^{-\lambda t}-1))^2+2(x_0+\sigma(e^{-\lambda t}-1))\int_0^t\rho e^{-\lambda s}dB_s+ \rightarrow$$

$$+\left(\int_0^t e^{-\lambda s}\rho\, dB_s\right)^2\Big]=e^{2\lambda t}\left[(x_0+\sigma(e^{-\lambda t}-1))^2+2(x_0+\sigma(e^{-\lambda t}-1))\cancel{E\left[\int_0^t\rho e^{-\lambda s}dB_s\right]}^{\,0}+E\left[\left(\int_0^t\rho e^{-\lambda s}dB_s\right)^2\right]\right]$$

$\leftarrow$ Isometría

$$=e^{2\lambda t}\left[(x_0+\sigma(e^{-\lambda t}-1))^2+\int_0^t E\left[(\rho e^{-\lambda s})^2\right]ds\right]=e^{2\lambda t}\left[(x_0+\sigma(e^{-\lambda t}-1))+\int_0^t\rho^2e^{-2\lambda s}ds\right]$$

$$=e^{2\lambda t}\left[(x_0+\sigma(e^{-\lambda t}-1))+\frac{\rho^2}{2\lambda}(1-e^{-2\lambda t})\right]=\left(x_0-\sigma+\frac{\rho^2}{2\lambda}\right)e^{2\lambda t}+\sigma e^{\lambda t}-\frac{\rho^2}{2\lambda}$$

En $V[X_t]=\left[\left(x_0-\sigma+\frac{\rho^2}{2\lambda}\right)e^{2\lambda t}+\sigma e^{\lambda t}-\frac{\rho^2}{2\lambda}\right]-\left[(x_0-\sigma)e^{\lambda t}+\sigma\right]^2$

$$V[X_t]=\left(x_0-\sigma+\frac{\rho^2}{2\lambda}\right)e^{2\lambda t}+\sigma e^{\lambda t}-\frac{\rho^2}{2\lambda}-(x_0-\sigma)^2e^{2\lambda t}-2\sigma(x_0-\sigma)e^{\lambda t}-\sigma^2$$

$$V[X_t]=\left(x_0-\sigma+\frac{\rho^2}{2\lambda}-(x_0-\sigma)^2\right)e^{2\lambda t}+(\sigma-2\sigma(x_0-\sigma))e^{\lambda t}-\sigma^2-\frac{\rho^2}{2\lambda}$$

c) Euler-Maruyama:

$$X_{t_i} = X_{t_{i-1}} + (\lambda X_{t_{i-1}} - \sigma\lambda)\Delta t_i + \rho \Delta B_{t_i}$$

Milstein:

$$X_{t_i} = X_{t_{i-1}} + (\lambda X_{t_{i-1}} - \sigma\lambda)\Delta t_i + \rho \Delta B_{t_i} + \frac{1}{2}(0)(0)((\Delta B_{t_i})^2 - \Delta t_i)$$

Son Iguales

d) Documento PDF

3. $X(t) = Z^2(t) + \int_0^t \sigma(s)\,dB_s$, $Z_t = E\left[\left(\int_0^t s\,dB_s\right)^2\right] \leftarrow$ Isometría.

$Z_t = \int_0^t E[s^2]\,ds = \int_0^t s^2\,ds = \frac{t^3}{3}$, luego $X(t) = \frac{t^6}{9} + \int_0^t \sigma(s)\,dB_s$.

$X(t) = \int_0^t \frac{6s^5}{9}\,ds + \int_0^t \sigma(s)\,dB_s \rightarrow$ Proceso de Itô.

$\Rightarrow dX(t) = \frac{6t^5}{9}dt + \sigma(t)\,dB_t$

a) $Y_t = e^{\mu^2 X(t)}$

Aplicando Itô unidimen. parte II:

$$dY_t = \mu^2 e^{\mu^2 X(t)}\,dX(t) + \frac{1}{2}\mu^4 e^{\mu^2 X(t)}\left(dX(t)\right)^2$$

$$dY_t = \mu^2 e^{\mu^2 X(t)}\left(\frac{6t^5}{9}dt + \sigma(t)\,dB_t\right) + \frac{\mu^4}{2} e^{\mu^2 X(t)}\left(\frac{6t^5}{9}dt + \sigma(t)\,dB_t\right)^2$$

$$dY_t = \left(\frac{6\mu^2 t^5}{9} e^{\mu^2 X(t)} + \frac{\mu^4 \sigma^2(t)}{2} e^{\mu^2 X(t)}\right)dt + \mu^2 \sigma(t) e^{\mu^2 X(t)}\,dB_t$$

Integrando:

$$Y_t - Y_0 = \int_0^t \left(\frac{6\mu^2 s^5}{9} + \frac{\mu^4 \sigma^2(s)}{2}\right) e^{\mu^2 X(s)}\,ds + \int_0^t \mu^2 \sigma(s) e^{\mu^2 X(s)}\,dB_s$$

Tomando Esperanza:

$$E[Y_t] = Y_0 + E\left[\int_0^t \left(\frac{6\mu^2 s^5}{9} + \frac{\mu^4 \sigma^2(s)}{2}\right) e^{\mu^2 X(s)}\,ds\right] + \overbrace{E\left[\int_0^t \mu^2 \sigma(s) e^{\mu^2 X(s)}\,dB_s\right]}^{0}$$

$$E[Y_t] = Y_0 + \int_0^t \left(\frac{6\mu^2 s^5}{9} + \frac{\mu^4 \sigma^2(s)}{2}\right) E\left[e^{\mu^2 X(s)}\right] ds.$$

Sea $\varphi(t) = E[Y_t] = E\left[e^{\mu^2 X(t)}\right]$

$$\varphi(t) = Y_0 + \int_0^t \left(\frac{6\mu^2 s^5}{9} + \frac{\mu^4 \sigma^2(s)}{2}\right)\varphi(s)\,ds \xrightarrow{\text{derivando}} \begin{cases} \dot{\varphi}(t) = \left(\frac{6\mu^2 t^5}{9} + \frac{\mu^4 \sigma^2(t)}{2}\right)\varphi(t) \\ \varphi(0) = 1 \end{cases} \quad \text{P.V.I.}$$

$$\frac{d\varphi(t)}{\varphi(t)} = \left(\frac{6\mu^2 t^5}{9} + \frac{\mu^4 \sigma^2(t)}{2}\right)dt \rightarrow \varphi(t) = \exp\left(\frac{t^6}{9}\mu^2 + \frac{\mu^4}{2}\int_0^t \sigma^2(s)\,ds\right)$$

Luego, $\boxed{E\left[e^{\mu X(t)}\right] = e^{\frac{\mu^2 t^6}{9} + \frac{\mu^4}{2}\int_0^t \sigma^2(s)\,ds}}$

b. $E[X(t)|\mathcal{F}_s] = E\left[\frac{t^6}{9} + \int_0^t \sigma(u)dB_u \Big| \mathcal{F}_s\right] = \frac{t^6}{9} + E\left[\int_0^t \sigma(u)dB_u \Big| \mathcal{F}_s\right]$

$= \frac{t^6}{9} + E\left[\int_0^s \sigma(u)dB_u + \cancel{\int_s^t \sigma(u)dB_u}^{\,0} \Big| \mathcal{F}_s\right] = \frac{t^6}{6} + \int_0^s \sigma(u)dB_u \rightarrow$ No es martingala.

$E[X(t)|\mathcal{F}_s] = E\left[Z^2(t) + \int_0^t \sigma(s)dB_u \Big| \mathcal{F}_s\right] = E[Z^2(t)|\mathcal{F}_s] + \int_0^s \sigma(u)dB_u$

Para que $X(t)$ verifique el teorema, se requiere que $Z^2(t)$ sea Martingala respecto a $\mathcal{F}_s$.

4.a) $dX_t = (Z_t - Tan(t)X_t)dt + X_t\sqrt{2Sec(t)}\,dB_t$, $Z_t = Sec(t)$

$dX_t = (Sec(t) - Tan(t)X_t)dt + X_t\sqrt{2Sec(t)}\,dB_t$ (EDE Lineal)

Parte Asociada con Difusión: $d\Phi_t = \Phi_t\sqrt{2Sec(t)}\,dB_t \leftarrow$ EDE Homogénea.

En el punto 1, en la ecuación (7) está la solución explícita de la EDE Homogénea. Luego, $\Phi_t = \Phi_0 \exp\left(\int_0^t \sqrt{2Sec(s)}\,dB_s - \int_0^t Sec(s)ds\right)$

$\Phi_t = \Phi_0\exp\left(\int_0^t\sqrt{2Sec(s)}\,dB_s - \ln(Sec(s)+Tan(s))\Big|_0^t\right) = \frac{\Phi_0}{Sec(t)+Tan(t)}\exp\left(\int_0^t\sqrt{2Sec(s)}\,dB_s\right)$

Sea $I_t^{-1} = \frac{1}{Sec(t)+Tan(t)}\exp\left(\int_0^t \sqrt{2Sec(s)}\,ds\right)$

Se propone una solución $X_t = I_t^{-1}Y_t \rightarrow Y_t = I_tX_t$. Aplicando Itô:

$dY_t = I_t dX_t + X_t dI_t + dX_t dI_t$. Hallemos dI_t: Se sabe que $\Phi_t = \Phi_0 I_t^{-1}$

$\rightarrow I_t = \Phi_0\Phi_t^{-1} \rightarrow dI_t = -\Phi_0\Phi_t^{-2}d\Phi_t + \Phi_0\Phi_t^{-3}(d\Phi_t)^2$. Reemplazando el $d\Phi_t$:

$dI_t = -\Phi_0\Phi_t^{-2}(\Phi_t\sqrt{2Sec(t)}\,dB_t) + \Phi_0\Phi_t^{-3}(\Phi_t\sqrt{2Sec(t)}\,dB_t)^2$

$dI_t = -I_t\sqrt{2Sec(t)}\,dB_t + I_t(2Sec(t))dt$. En dY_t:

Término que sobrevive.

$dY_t = I_t((Sec(t)-Tan(t)X_t)dt + \cancel{X_t\sqrt{2Sec(t)}\,dB_t}) + X_t(\cancel{-I_t\sqrt{2Sec(t)}\,dB_t} + \cancel{I_t(2Sec(t))dt}) + \cancel{(X_t\sqrt{2Sec(t)}\,dB_t)(-I_t\sqrt{2Sec(t)}\,dB_t)}$

$dY_t = (Sec(t)-Tan(t)X_t)I_t dt \rightarrow dY_t = (I_t Sec(t) - Tan(t)Y_t)dt$

$\rightarrow \dot{Y}_t + Tan(t)Y_t = I_t Sec(t) \rightarrow$ factor integrante: $\mu(t) = e^{\int Tan(t)dt} = \frac{1}{Cos(t)}$

$\rightarrow \frac{\dot{Y}_t}{Cos(t)} + \frac{Sin(t)Y_t}{Cos^2(t)} = I_t Sec^2(t) \rightarrow \frac{d}{dt}\left(Y_t/Cos(t)\right) = I_t Sec^2(t)$

$$\frac{Y_t}{Cos(t)} - \frac{Y_0}{Cos(0)} = \int_0^t I_s\, Sec^2(s)ds \rightarrow Y_t = Cos(t)\left(Y_0 + \int_0^t I_s Sec^2(s)ds\right)$$

Luego, $X_t = I_t^{-1} Cos(t)\left(X_0 + \int_0^t I_s Sec^2(s)ds\right)$

Con $I_t^{-1} = \dfrac{e^{\int_0^t \sqrt{2Sec(s)}\,dB_s}}{Sec(t)+Tan(t)}$.

b) $X(0)=0$

Solución Exacta Discretizada: $X_{t_{i+1}} = I_{t_{i+1}} Cos(t_{i+1})\left(X_{t_i} + \int_{t_i}^{t_{i+1}} I_s Sec^2(s)ds\right)$

$X_{t_{i+1}} = I_{t_i}\, Cos(t_i)\left(X_{t_i} + (I_{t_i} Sec^2(t_i))\Delta t_{i+1}\right)$, donde

$$I_{t_i} = \frac{1}{Sec(t_i)+Tan(t_i)} e^{\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}}}$$

Aproximando en I_{t_i} por Taylor:

$$e^{\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}}} = \sum_{K=0}^{\infty} \frac{\left(\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}}\right)^K}{K!}$$ Truncando en $K=2$:

$$e^{\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}}} \approx 1 + \sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}} + 2Sec(t_i)(\Delta B_{t_{i+1}})^2$$

Para Euler-Maruyama: $(\Delta B_{t_i})^2 = \Delta t_i$

$$I_{t_i} = \frac{1}{Sec(t_i)+Tan(t_i)}\left(1+\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}} + 2Sec(t_i)(\Delta t_i\right)$$

Para Milstein

$$I_{t_i} = \frac{1}{Sec(t_i)+Tan(t_i)}\left(1+\sqrt{2Sec(t_i)}\,\Delta B_{t_{i+1}} + 2Sec(t_i)(\Delta B_i\right)^2$$