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Parcial 2: Economía Matemática

1. $z = f(x,y)$ Usando $\mathcal{L}(\lambda, x, y) = f(x,y) - \lambda G(x,y)$
s.t. $G(x,y) = 0$

La matriz Hessiana con borde se puede obtener del Jacobiano de $\nabla\mathcal{L}(\lambda,x,y)=0$

$$\nabla\mathcal{L}(\lambda,x,y) = \left(-G(x,y), \frac{\partial f(x,y)}{\partial x} - \lambda\frac{\partial G(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y} - \lambda\frac{\partial G(x,y)}{\partial y}\right)^T = 0_3$$

$$\left.\begin{aligned} F_1(\lambda,x,y) &= -G(x,y) = 0 \\ F_2(\lambda,x,y) &= \frac{\partial f(x,y)}{\partial x} - \lambda\frac{\partial G(x,y)}{\partial x} = 0 \\ F_3(\lambda,x,y) &= \frac{\partial f(x,y)}{\partial y} - \lambda\frac{\partial G(x,y)}{\partial y} = 0 \end{aligned}\right\} \text{Jacobiano} \to J(\lambda,x,y) = \tilde{H}(\lambda,x,y)$$

$$J(\lambda,x,y) = \tilde{H}(\lambda,x,y) = \begin{bmatrix} 0 & -\frac{\partial G(x,y)}{\partial x} & -\frac{\partial G(x,y)}{\partial y} \\ -\frac{\partial G(x,y)}{\partial x} & \frac{\partial^2 f(x,y)}{\partial x^2} - \lambda\frac{\partial^2 G(x,y)}{\partial x} & \frac{\partial^2 f(x,y)}{\partial y\partial x} - \lambda\frac{\partial^2 G(x,y)}{\partial y\partial x} \\ -\frac{\partial G(x,y)}{\partial y} & \frac{\partial^2 f(x,y)}{\partial x\partial y} - \lambda\frac{\partial^2 G(x,y)}{\partial x\partial y} & \frac{\partial^2 f(x,y)}{\partial y^2} - \lambda\frac{\partial^2 G(x,y)}{\partial y^2} \end{bmatrix}$$

2. $U(x_1,x_2) = x_1 x_2$
s.t. $x_1 + \frac{x_2}{1+r} = B$ $\quad \mathcal{L}(\lambda,x_1,x_2) = x_1x_2 + \lambda\left[B - x_1 - \frac{x_2}{1+r}\right]$

Condiciones de primer Orden: $\nabla\mathcal{L}(\lambda,x_1,x_2) \overset{!}{=} 0$

$$\underset{(1)}{\frac{\partial\mathcal{L}}{\partial\lambda} = B - x_1 - \frac{x_2}{1+r} = 0}, \quad \underset{(2)}{\frac{\partial\mathcal{L}}{\partial x_1} = x_2 + \lambda[-1] = 0}, \quad \underset{(3)}{\frac{\partial\mathcal{L}}{\partial x_2} = x_1 + \lambda\left[-\frac{1}{1+r}\right]}$$

de (2): $x_2 = \lambda$. de (3): $x_1 = \frac{\lambda}{1+r} \to \lambda = x_1(1+r) \Rightarrow x_2 = (1+r)x_1$. En (1): $B - x_1 - \frac{(1+r)x_1}{1+r} = 0$

$x_1^* = \frac{B}{2}. \Rightarrow x_2^* = \frac{B(1+r)}{2} \Rightarrow \lambda^* = \frac{B(1+r)}{2}$.

Condiciones de 2^{do} Orden: Matriz Hessiana con Borde

$$\tilde{H}(\lambda,x_1,x_2) = \begin{bmatrix} 0 & -1 & -\frac{1}{1+r} \\ -1 & 0 & 1 \\ -\frac{1}{1+r} & 1 & 0 \end{bmatrix} \to \det(\tilde{H}(\lambda,x_1,x_2)) = \frac{2}{1+r} > 0 \quad (|r|<1).$$

Como $\det(\tilde{H}(\lambda,x_1,x_2)) > 0 \Rightarrow$ los puntos $x_1^* = \frac{B}{2}$, $x_2^* = \frac{B(1+r)}{2}$, $\lambda^* = \frac{B(1+r)}{2}$ generan un máximo global para el problema

3. $\min f(x,y)=(x-4)^2+(y-4)^2$

s.t. $2x+3y \geq 6$

$-3x-2y \geq -12$

$x,y \geq 0.$

$\min f(x,y)=(x-4)^2+(y-4)^2$

s.t. $-2x-3y \leq -6$

$3x+2y \leq 12$

$x,y \geq 0$

Problema equivalente.

$\mathcal{L}(\lambda_1,\lambda_2,x,y)=(x-4)^2+(y-4)^2+\lambda_1[-6+2x+3y]+\lambda_2[12-3x-2y]$

$\boxed{KKT}$: $2(x-4)+2\lambda_1-3\lambda_2 \geq 0;\ x\geq 0;\ x[2(x-4)+2\lambda_1-3\lambda_2]=0$ (1)

$2(y-4)+3\lambda_1-2\lambda_2 \geq 0;\ y\geq 0;\ y[2(y-4)+3\lambda_1-2\lambda_2]=0$ (2)

$-6+2x+3y \geq 0;\ \lambda_1 \leq 0;\ \lambda_1[-6+2x+3y]=0$ (3)

$12-3x-2y \geq 0;\ \lambda_2 \leq 0;\ \lambda_2[12-3x-2y]=0.$ (4)

• Si $\lambda_1=\lambda_2=0$ y $x,y\neq 0 \Rightarrow$ De $(1.3)\Rightarrow x=4$. De $(2.3)\Rightarrow y=4$.

En (4.1): $12-3(4)-2(4)=-8\geq 0$ $(\rightarrow\leftarrow)$.

• Si $\lambda_1=0$ y $x,y,\lambda_2\neq 0$: De (4.3): $12-3x-2y=0$ $(*)$

De (1.3): $2(x-4)-3\lambda_2=0 \rightarrow \lambda_2=\frac{2}{3}(x-4)$

De (2.3): $2(y-4)-2\lambda_2=0 \rightarrow \lambda_2=(y-4)$ $(**)$

Luego, $\frac{2}{3}(x-4)=y-4 \rightarrow 2x-3y=-4$

$(*)\ -3x-2y=-12$

$6x-9y=-12$

$-6x-4y=-24$

$13y=-36 \rightarrow \boxed{y^*=\frac{36}{13}} \geq 0$

En $(*) \Rightarrow -3x-2\left(\frac{36}{13}\right)=-12 \rightarrow -3x=-12+\frac{72}{13} \rightarrow \boxed{x^*=\frac{28}{13}} \geq 0$

En $(**) \Rightarrow \boxed{\lambda_2^*=\frac{36}{13}-4=\frac{-16}{13}} \leq 0$

En (1.1): $2(x^*-4)-3\lambda_2^* \geq 0 \rightarrow 0\geq 0.$ ✓

En (2.1): $2(y^*-4)-2\lambda_2^* \geq 0 \rightarrow 0\geq 0$ ✓

En (4.1): $12-3x^*-2y^* \geq 0 \rightarrow 0\geq 0$ ✓

En (3.1): $-6+2x^*+3y^* \geq 0 \rightarrow \frac{86}{13}\geq 0$ ✓.

Luego, $x^*=\frac{28}{13}$, $y^*=\frac{36}{13}$, $\lambda_1^*=0$, $\lambda_2^*=\frac{-16}{13}$ son un mínimo para el problema.

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4. $Q_d = \alpha - \beta P$, $\alpha, \beta > 0$
$Q_s = -\gamma + \delta P$, $\gamma, \delta > 0$.

Equilibrio $Q_d = Q_s$
$\alpha - \beta P^* = -\gamma + \delta P^*$
$$\boxed{P^* = \frac{\alpha+\gamma}{\beta+\delta}}$$

Como $\dot{P}(t) \propto (Q_d - Q_s) \Rightarrow \exists K \in \mathbb{R}$ t.q. $\dot{P}(t) = K(Q_d - Q_s)$

$\dot{P}(t) = K[(\alpha - \beta P) - (-\gamma + \delta P)] = K(\alpha + \gamma - (\beta+\delta)P)$

a. $\begin{cases} \dot{P}(t) + K(\beta+\delta)P = K(\alpha+\gamma) \\ P(0) = P_0. \end{cases}$ $\Rightarrow$ Problema de Valor Inicial.

b. $\dot{P}(t) + K(\beta+\delta)P = \frac{K(\alpha+\gamma)}{(\beta+\delta)} \cdot (\beta+\delta)$

$\dot{P}(t) + K(\beta+\delta)P = K(\beta+\delta)P^* \rightarrow \dot{P}(t) + K(\beta+\delta)(P-P^*) = 0$ Sea $\omega = K(\beta+\delta)$

$\dot{P}(t) + \omega(P - P^*) = 0$

c. $\dot{P}_h(t) + \omega P_h(t) = 0$ $\leftarrow$ Solución Homogénea

$\dot{P}_h(t) = -\omega P_h(t)$ Como $P_h(t)$ es diferenciable $\rightarrow dP_h(t) = \dot{P}_h(t)\,dt$

$\rightarrow dP_h(t) = -\omega P_h(t)\,dt \rightarrow \frac{dP_h(t)}{P_h(t)} = -\omega dt \rightarrow \ln\left(\frac{P_h(t)}{P_h(0)}\right) = -\omega t$

$P_h(t) = P_h(0)e^{-\omega t}$

$P_h(t) = ce^{-\omega t}$, $c \in \mathbb{R}$.

Solución Particular:

$\dot{P}_p(t) + \omega P_p(t) = \omega P^*$ $(*)$

Claramente, $P_p(t) = P^*$ es solución particular, puesto que $\dot{P}_p(t) = 0$ en $(*) \rightarrow 0 + \omega P^* = \omega P^*$.

Solución General: $P(t) = P^* + ce^{-\omega t}$

$P_0 = P^* + c \rightarrow c = P_0 - P^*$

Precio: $\boxed{P(t) = P^* + (P_0 - P^*)e^{-\omega t}}$ $\rightarrow \lim_{t\to\infty} P(t) = \lim_{t\to\infty}\left(P^* + (P_0 + P^*)e^{-\omega t}\right)$

$\left[\lim_{t\to\infty} P(t) = P^*\right] \Leftrightarrow \omega > 0$. El precio es dinámicamente estable si $\omega > 0$.