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Gaussian Estimation of Single-Factor Continuous Time Models of The Term Structure of Interest Rates

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ABSTRACT

This article presents the first application in finance of recently developed methods for the Gaussian estimation of continuous time dynamic models. A range of one factor continuous time models of the short-term interest rate are estimated using a discrete time model and compared to a recent discrete approximation used by Chan, Karolyi, Longstaff, and Sanders (1992a, hereafter CKLS). Whereas the volatility of short-term rates is highly sensitive to the level of rates in the United States, it is not in the United Kingdom.

THE ECONOMETRIC ESTIMATION of continuous time models of the short-term interest rate has been a relatively recent development in empirical finance (see, for example, Brown and Dybvig (1986), Melino and Turnbull (1986), Barone, Cuoco, and Zautzik (1991), Babbs (1992), Abken (1993), Chen and Scott (1993), Das (1993), Gibbons and Ramaswamy (1993), Pearson and Sun (1994), Lund (1994), Pfann, Schotman, and Tschernig (1995), Aït-Sahalia (1995), and Broze, Scaillet, and Zakoian (1995)).[1] In an important study Chan, Karolyi, Longstaff, and Sanders (1992a, hereafter CKLS) develop a general framework to estimate and compare a range of different single-factor term structure models for the short-term interest rate. Their major conclusion is that term structure models with volatilities more highly sensitive to the level of interest rates than more generally used models have a closer empirical fit to the data. In addition, the functional form of the drift, with or without mean reversion, is of secondary importance to the correct specification of the conditional heteroskedasticity.

In CKLS the continuous time models of the term structure are estimated using a discrete approximation. We present in this article an alternative

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[1] Melino (1994) presents an interesting discussion of the estimation of continuous time models in finance.

discrete time model that also nests the CKLS approximation but has the advantage of reducing some of the temporal aggregation bias. We estimate the models using recently developed Gaussian estimation methods for continuous time models by Bergstrom (1983, 1985, 1986, 1990). The empirical analysis provides an important result. Using one month British sterling rate data we find that the volatility of the short-term interest rate is not highly sensitive to the level of the interest rate. This differs sharply from the results reported by CKLS. We also find using the U.S. T-bill data used in CKLS that the volatility of interest rate changes are highly sensitive to the level of the riskless rate.

Section I below reviews the one-factor continuous time models used in CKLS. Section II discusses the econometric methodology used in estimating the model parameters. Section III describes the data used in the study. Section IV presents the empirical results. Section V contains a summary and concluding remarks.

I. The Continuous Time Interest Rate Models

The general stochastic differential equation used by CKLS to specify the dynamic adjustment of the interest rate is represented by equation (1) below. This general equation has the advantage that it nests a wide range of different term structure models (see also Marsh and Rosenfeld (1983) and Melino and Turnbull (1986)). The equation allows the conditional mean and variance to depend on the level r.

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)dZ \quad (t \geq 0) \tag{1}$$

where $\{r(t), t > 0\}$ is a real continuous time random process, α, β, γ, and σ are unknown structural parameters. Following Bergstrom (1983, 1984 Theorem 2) we make the following assumption with regard to dZ:[2]

ASSUMPTION 1: *Z is a random measure defined on all subsets of the half line $0 < t < \infty$ with finite Lebesgue measure, such that:* $\mathrm{E}[dZ] = 0$ *and* $\mathrm{E}[dZ^2] = dt$ *and* $\mathrm{E}[Z(\Delta_1)Z(\Delta_2)] = 0$ *for any disjoint sets Δ_1 and Δ_2 on the half line $0 < t < \infty$*. (See Bergstrom (1984, p. 1157) for a discussion of random measures and their application to continuous time stochastic models.)

This is much weaker than the assumption that the innovations are generated by Brownian motion. The assumptions about the innovation process include the case in which the innovations are a mixture of Brownian motion and Poisson processes and allow for more general innovation processes in which the increments are not independent but merely orthogonal.

Following CKLS we can obtain the various term structure models tested in CKLS as special cases by imposing the necessary restrictions on the structural parameters α, β, γ, and σ. The resulting term structure specifications are given below and summarized in Table I. We also follow CKLS in ordering the models

[2] Alternatively we call dZ $Z(dt)$. Compare Bergstrom (1983, Assumption 1).

Table I

Parameter Restrictions Imposed by Alternative Models of Short-Term Interest Rate

The alternative term structure models for r are obtained from imposing the appropriate parameter restrictions on the unrestricted model

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)dZ.$$

	Model	α	β	σ^2	γ
Merton	$dr(t) = \alpha dt + \sigma dZ$		0		0
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$				0
CIR SR	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$				½
Dothan	$dr(t) = \sigma r(t)dZ$	0	0		1
GBM	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$	0			1
Brennan-Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$				1
CIR VR	$dr(t) = \sigma r^{3/2}(t)dZ$	0	0		3⁄2
CEV	$dr(t) = \beta r(t)dt + \sigma r^{\gamma}(t)dZ$	0			

starting with the Merton and Vasicek models which assume that conditional volatility of changes in the interest rate are constant through to models with higher levels of dependence of the volatility of the short-rate process to the level of the rate itself.

1. Merton (1973) $dr(t) = \alpha dt + \sigma dZ$
2. Vasicek (1977) $dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
3. Cox, Ingersoll, and Ross (1985) $dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
4. Dothan (1978) $dr(t) = \sigma r(t)dZ$
5. Geometric Brownian motion $dr(t) = \beta r(t)dt + \sigma r(t)dZ$
6. Brennan and Schwartz (1980) $dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
7. Cox, Ingersoll and Ross (1980) $dr(t) = \sigma r^{3/2}(t)dZ$
8. Constant Elasticity of Variance $dr(t) = \beta r(t)dt + \sigma r^{\gamma}(t)dZ$

These models represent a number of well known models in the literature, and their use was summarized in CKLS.

The stochastic differential equations considered by Bergstrom (1983, 1984, 1985, 1986) have not (as pointed out by Melino (1994)) been used in the empirical finance literature to date. The reason is that Bergstrom assumes that the conditional second moment is constant which is seldom satisfied for financial data. We now assume as an approximation to the true underlying model given by equation (1) that over the interval [0, T], $r(t)$ satisfies the stochastic differential equation

$$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma\{r(t' - 1)\}^{\gamma}dZ \tag{2}$$

where $t' - 1$ is the largest integer less than t (i.e., t' is the smallest integer greater than or equal to t). We assume in equation (2) that the volatility of the interest rate changes at the beginning of the unit observation period and then remains constant. This assumption allows us to use the exact discrete model of

Bergstrom (1984, Theorem 2) in the next section to obtain the Gaussian estimates modified for heteroskedasticity.

We shall interpret equation (2) as meaning that $r(t)$ satisfies the stochastic integral equation

$$r(t) - r(t' - 1) = \int_{t'-1}^{t} [\alpha + \beta r(s)]\, ds + \sigma\{r(t' - 1)\}^{\gamma} \int_{t'-1}^{t} dZ(s) \quad (3)$$

for all t in $[t' - 1, t']$ where

$$t' - 1 < t \leq t' \qquad \text{and} \qquad \int_{t'-1}^{t} dZ(s) = Z[t' - 1, t].$$

II. Gaussian Estimation

In this section the econometric methods used to estimate the underlying structural parameters of the interest rate models are discussed. The approach is based on the Gaussian estimation methods developed by Bergstrom (1983, 1985, 1986) (see also, Bergstrom (1990) and Nowman (1991)) for estimating the parameters of open continuous time systems from discrete stock and flow data using an exact discrete model which takes account of the exact restrictions on the distribution of the discrete data implied by the continuous time model. Bergstrom (1983, 1985, 1986) provides a detailed analysis of using exact discrete models as a basis for the Gaussian estimation of the structural parameters of the continuous time model. We now proceed to discuss the discrete model used for the estimation of the particular one-factor models.

It follows from Bergstrom (1984, Theorem 2) that the discrete model corresponding to equation (3) used for estimation is given by

$$r(t) = e^{\beta} r(t - 1) + \frac{\alpha}{\beta}(e^{\beta} - 1) + \eta_t \qquad (t = 1,2, \ldots ,T) \quad (4)$$

where η_t $(t = 1, 2, \ldots, T)$ satisfies the conditions

$$E(\eta_s \eta_t) = 0 \qquad (s \neq t) \quad (5)$$

$$E(\eta_t^2) = \int_{t-1}^{t} e^{2(t-\tau)\beta} \sigma^2 \{r(t - 1)\}^{2\gamma}\, d\tau = \frac{\sigma^2}{2\beta}(e^{2\beta} - 1)\{r(t - 1)\}^{2\gamma} = m_{tt}^2. \quad (6)$$

Let the complete vector of parameters be defined as $\theta = [\alpha, \beta, \gamma, \sigma^2]$. Following Bergstrom (1985, 1986) we define L (θ) as minus twice the logarithm of the Gaussian likelihood function

$$L(\theta) = \sum_{t=1}^{T} \left[2 \log m_{tt} + \frac{\{r(t) - e^{\beta} r(t - 1) - (\alpha/\beta)(e^{\beta} - 1)\}^2}{m_{tt}^2} \right]. \quad (7)$$

We then have

$$L(\theta) = \sum_{t=1}^{T}[2 \log m_{tt} + \varepsilon_t^2] \tag{8}$$

where $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_T]$ is a vector whose elements can be computed from

$$m_{tt}\varepsilon_t = \eta_t. \tag{9}$$

We also estimate the interest rate models with the discrete approximation used in CKLS given as

$$r_{t+1} - r_t = \alpha + \beta r_t + \eta_{t+1} \tag{10}$$

$$E(\eta_{t+1}) = 0, \qquad E(\eta_{t+1}^2) = \sigma^2 r_t^{2\gamma}. \tag{11}$$

It should be noted that the CKLS approximation can be obtained from the discrete model (4) by expanding e^{β} and neglecting higher-order terms. Although both approaches use discrete approximations, one of the advantages of using the specification in equation (3) and the resulting discrete model given by equation (4) is that it allows us to use an exact maximum likelihood estimator. This should help to reduce some of the temporal aggregation bias (see, for example, Grossman, Melino, and Shiller (1987)). It should be noted that, in the case of the Merton and Vasicek models, Bergstrom's (1984, Theorem 2) holds.

III. The Data

The British short-rate used in this study is the one-month sterling interbank rate, middle rate, obtained from *Datastream.* The data are monthly covering the period from March 1975 to March 1995 giving a total of 241 observations. We also use the U.S. Treasury bill one-month yield data used in the CKLS study (see CKLS for details) that are monthly covering the period from June 1964 to December 1989 giving a total of 307 observations. These data were

Table II

Summary Statistics

Means, standard deviations, and autocorrelations of monthly sterling one-month interbank rate and first differences are computed from March 1975 to March 1995. The variable $r(t)$ denotes the sterling one-month interbank rate and $\Delta r(t)$ is the monthly change. ρ_j denotes the autocorrelation coefficient of order j. T represents the number of observations used. ADF denotes the Augmented Dickey-Fuller unit root statistic with a 5 percent critical value of −3.43.

Variable	T	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ADF
$r(t)$	241	0.1097	0.0316	0.97	0.92	0.87	0.83	0.78	0.74	−2.5273
$\Delta r(t)$	240	−0.0002	0.0083	0.12	0.11	−0.09	−0.04	0.04	−0.01	−5.4099

Table III

Gaussian Estimates of Continuous Time Models of the Short-Term Interest Rate

Gaussian estimates of alternative one-factor models of the short-term interest rate $r(t)$ (sterling one-month interbank rate) from March 1975 to March 1995 (241 observations). The models are

Unrestricted	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)dZ$
Merton	$dr(t) = \alpha dt + \sigma dZ$
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
Cox, Ingersoll, and Ross	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
Dothan	$dr(t) = \sigma r(t)dZ$
Geometric Brownian motion	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$
Brennan and Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
Cox, Ingersoll, and Ross	$dr(t) = \sigma r^{3/2}(t)dZ$
Constant Elasticity of Variance	$dr(t) = \beta r(t)dt + \sigma r^{\gamma}(t)dZ.$

Gaussian estimates with t-statistics in parentheses are presented for each model. Likelihood ratio tests evaluate restrictions imposed by different models against the unrestricted model. The χ^2 test statistics are reported with p-values in parentheses and associated degrees of freedom (df). S test statistics denote the Box–Pierce type portmanteau test statistic proposed by Bergstrom for a check on dynamic specification of the model with a critical value of 21.03 at 5% significance for the null hypothesis of white noise residuals. The Gaussian estimates are obtained from the following system of equations

$$r(t) = e^{\beta}r(t-1) + \frac{\alpha}{\beta}(e^{\beta} - 1) + \eta_t$$

$$E(\eta_s\eta_t) = 0 \quad (s \neq t), \qquad E(\eta_t^2) = \int_{t-1}^{t} e^{2(t-\tau)\beta}\sigma^2\{r(t-1)\}^{2\gamma}\,d\tau = \frac{\sigma^2}{2\beta}(e^{2\beta} - 1)\{r(t-1)\}^{2\gamma}.$$

The CKLS Gaussian estimates are obtained from the following system

$$r_{t+1} - r_t = \alpha + \beta r_t + \eta_{t+1}, \qquad E(\eta_{t+1}) = 0, \qquad E(\eta_{t+1}^2) = \sigma^2 r_t^{2\gamma}$$

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df	S Test
Unrestricted	0.0030	−0.0291	0.0003	0.2898	1037.9510			19.72
	(1.6065)	(−1.7112)	(1.3395)	(1.7620)				
CKLS	0.0029	−0.0287	0.0003	0.2898	1037.9510			
	(1.6342)	(−1.7440)	(1.3456)	(1.7720)				

Table III—*Continued*

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df	S Test
Merton	−0.0002	0.0	0.0001	0.0	1034.7780	6.3460	2	20.03
	(−0.4180)		(10.9760)			(0.0419)		
CKLS	−0.0002	0.0	0.0001	0.0	1034.7780			
	(−0.4180)		(10.9760)					
Vasicek	0.0032	−0.0311	0.0001	0.0	1036.4100	3.0820	1	20.86
	(1.5941)	(−1.7709)	(10.7828)			(0.0792)		
CKLS	0.0031	−0.0306	0.0001	0.0	1036.4100			
	(1.6371)	(−1.8240)	(10.9787)					
CIR SR	0.0029	−0.0279	0.0007	0.5	1037.0813	1.7394	1	19.28
	(1.6513)	(−1.6958)	(10.8072)			(0.1872)		
CKLS	0.0028	−0.0276	0.0006	0.5	1037.0813			
	(1.6653)	(−1.7121)	(10.9781)					
Dothan	0.0	0.0	0.0067	1.0	1025.6063	24.6894	3	18.56
			(10.9731)			(<.0001)		
CKLS	0.0	0.0	0.0067	1.0	1025.6063			
			(10.9731)					
GBM	0.0	0.0006	0.0067	1.0	1025.6128	24.6764	2	18.57
	(0.1135)	(10.9585)			(<.0001)			
CKLS	0.0	0.0006	0.0067	1.0	1025.6128			
		(0.1135)	(10.9765)					
Brennan-Schwartz	0.0028	−0.0269	0.0068	1.0	1027.2555	21.3910	1	19.41
	(1.7913)	(−1.6534)	(10.8084)			(<.0001)		
CKLS	0.0027	−0.0266	0.0066	1.0	1027.2555			
	(1.8297)	(−1.6900)	(10.9775)					
CIR VR	0.0	0.0	0.0776	1.5	1002.0835	71.7350	3	19.53
		(10.9753)			(<.0001)			
CKLS	0.0	0.0	0.0776	1.5	1002.0835			
		(10.9753)						
CEV	0.0	−0.0030	0.0003	0.2863	1036.6151	2.6718	1	18.92
		(−0.6133)	(1.3503)	(1.7552)		(0.1021)		
CKLS	0.0	−0.0030	0.0003	0.2863	1036.6151			
		(−0.6181)	(1.3414)	(1.7451)				

obtained from the Center for Research in Security Prices (CRSP). (See Duffee (1995) for an interesting recent discussion on the use of Treasury bill yields). Table II reports the descriptive statistics for the British data. The table displays the means, standard deviations, and first six autocorrelations of the one-month rate and monthly changes in the one-month rate. We also report the augmented Dickey–Fuller (ADF) statistic of Said and Dickey (1984) for the presence of a unit root. The average level of the British one-month rate is 10.97 percent with a standard deviation of 3.16 percent. The autocorrelations for the level fall off slowly and those of the first differences are small and not systematically positive or negative. See CKLS for summary details on the U.S. data. The ADF statistics do not reject the null hypothesis of a unit root at the 5 percent level.

Table IV

Gaussian Estimates of Continuous Time Models of the Short-Term Interest Rate

Gaussian estimates of alternative one-factor models of the short-term interest rate $r(t)$ (annualized one-month U.S. Treasury bill yield) from June 1964 to December 1989 (306 observations). The models are

Unrestricted	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{\gamma}(t)dZ$
Merton	$dr(t) = \alpha dt + \sigma dZ$
Vasicek	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma dZ$
Cox, Ingersoll and Ross	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r^{1/2}(t)dZ$
Dothan	$dr(t) = \sigma r(t)dZ$
Geometric Brownian motion	$dr(t) = \beta r(t)dt + \sigma r(t)dZ$
Brennan and Schwartz	$dr(t) = \{\alpha + \beta r(t)\}dt + \sigma r(t)dZ$
Cox, Ingersoll, and Ross	$dr(t) = \sigma r^{3/2}(t)dZ$
Constant Elasticity of Variance	$dr(t) = \beta r(t)dt + \sigma r^{\gamma}(t)dZ.$

Gaussian estimates with t-statistics in parentheses are presented for each model. Likelihood ratio tests evaluate restrictions imposed by different models against the unrestricted model. The χ^2 test statistics are reported with p-values in parentheses and associated degrees of freedom (df). S test statistics denotes the Box–Pierce type portmanteau test statistic proposed by Bergstrom for a check on dynamic specification of the model with a critical value of 21.03 at 5% significance for the null hypothesis of white noise residuals. The Gaussian estimates are obtained from the following system of equations

$$r(t) = e^{\beta} r(t-1) + \frac{\alpha}{\beta}(e^{\beta} - 1) + \eta_t$$

$$E(\eta_s \eta_t) = 0 \quad (s \neq t), \qquad E(\eta_t^2) = \int_{t-1}^{t} e^{2(t-\tau)\beta} \sigma^2 \{r(t-1)\}^{2\gamma}\, d\tau = \frac{\sigma^2}{2\beta}(e^{2\beta} - 1)\{r(t-1)\}^{2\gamma}.$$

The CKLS Gaussian estimates are obtained from the following system

$$r_{t+1} - r_t = \alpha + \beta r_t + \eta_{t+1}, \qquad E(\eta_{t+1}) = 0, \qquad E(\eta_{t+1}^2) = \sigma^2 r_t^{2\gamma}.$$

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df	S Test
Unrestricted	0.0020	−0.0273	0.0701	1.3610	1412.0632			16.97
	(2.2203)	(−1.5400)	(1.7443)	(13.2834)				
CKLS	0.0020	−0.0269	0.0682	1.3610	1412.0632			
	(2.2610)	(−1.5597)	(1.7259)	(13.1523)				

Table IV—*Continued*

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df	S Test
Merton	0.0001	0.0	0.0001	0.0	1316.9808	190.1648	2	21.30
	(0.1872)		(12.3693)			(<0.0001)		
CKLS	0.0001	0.0	0.0001	0.0	1316.9808			
	(0.1872)		(12.3693)					
Vasicek	0.0035	−0.0506	0.0001	0.0	1321.0003	182.1258	1	18.93
	(2.6632)	(−2.7852)	(12.0758)			(<0.0001)		
CKLS	0.0034	−0.0493	0.0001	0.0	1321.0003			
	(2.7191)	(−2.8511)	(12.3793)					
CIR SR	0.0026	−0.0373	0.0008	0.5	1375.3886	73.3492	1	17.80
	(2.3858)	(−2.1537)	(12.0983)			(<0.0001)		
CKLS	0.0025	−0.0366	0.0007	0.5	1375.3886			
	(2.4190)	(−2.1806)	(12.3682)					
Dothan	0.0	0.0	0.0098	1.0	1402.4291	19.2682	3	17.75
			(12.3642)			(0.0002)		
CKLS	0.0	0.0	0.0098	1.0	1402.4291			
			(12.3642)					
GBM	0.0	0.0068	0.0097	1.0	1403.1628	17.8008	2	17.96
		(1.2252)	(12.3403)			(0.0001)		
CKLS	0.0	0.0069	0.0098	1.0	1403.1628			
		(1.2202)	(12.3727)					
Brennan-Schwartz	0.0021	−0.0297	0.0099	1.0	1405.7277	12.6710	1	17.32
	(2.2464)	(−1.7230)	(12.1002)			(0.0004)		
CKLS	0.0021	−0.0293	0.0096	1.0	1405.7277			
	(2.2629)	(−1.7277)	(12.3688)					
CIR VR	0.0	0.0	0.1530	1.5	1406.3987	11.3290	3	17.46
			(12.3660)			(0.0101)		
CKLS	0.0	0.0	0.1530	1.5	1406.3987			
			(12.3660)					
CEV	0.0	0.0101	0.0683	1.3600	1409.5431	5.0402	1	17.55
		(1.8179)	(1.7193)	(13.0702)		(0.0248)		
CKLS	0.0	0.0102	0.0689	1.3600	1409.5431			
		(1.8539)	(1.7422)	(13.2678)				

IV. Results

In this section we present the Gaussian estimation results from estimating the unrestricted model and the eight different term structure models obtained after imposing the appropriate restrictions on the general model. Following Chan, Karolyi, Longstaff, and Sanders (1992b), the explanatory power of each model is compared to the unrestricted model using the maximized Gaussian log likelihood function value. We also report a formal check on the dynamic specification of the different models using a recently proposed Box-Pierce type portmanteau test by Bergstrom (1990).

A. U.K. Gaussian Estimation Results

In Table III we present the Gaussian coefficient estimates, asymptotic t-statistics, maximized log likelihoods for the unrestricted and eight nested models, the likelihood ratio tests comparing the nested models with the unrestricted model, and the portmanteau test of Bergstrom (1990). A comparison of the Gaussian estimates indicates that the bias resulting from using the CKLS approximation is very small compared to the discrete approximation proposed in this article. We now concentrate on the Gaussian estimation results using the discrete model proposed in this article. Based on maximized Gaussian log likelihood values compared to the unrestricted model, the CIR SR model performs the best followed by the CEV and Vasicek models. Although the CEV model allows the conditional volatility to depend on the level of interest rates, the difference in the log likelihoods compared to the Vasicek model are very small. The unrestricted model estimates imply that $\gamma = 0.2898$ but is insignificant. Also, there appears only to be weak evidence of mean reversion in the short-term rate. Our results with regard to the dependence of conditional volatility on the level of the interest rate are in contrast to the results of CKLS (see also, Chan, Karolyi, Longstaff, and Sanders (1992b)) who find a greater dependence with $\gamma = 1.499$. Based on the χ^2 likelihood ratio test under the null hypothesis that the nested model restrictions are valid, the results imply that we can reject the Merton, Dothan, GBM, Brennan-Schwartz, and CIR VR models.

As a more formal check on the dynamic specification of each model we compute the Box–Pierce type portmanteau test statistic proposed by Bergstrom (1990). This uses the vector of transformed residuals which, if the model is correct, are independent and have variance 1. The statistic is given by

$$S = \frac{1}{n(T-l)} \sum_{i=1}^{l} \left(\sum_{t=l+1}^{T} \hat{\varepsilon}_t' \hat{\varepsilon}_t \hat{\varepsilon}_{t-1} \right)^2 \tag{12}$$

and is asymptotically distributed as chi-squared with l degrees of freedom ($n = 1$). The S-statistic is calculated for $l = 12$. The null hypothesis of white noise residuals is not rejected for any of the models at the 5 percent level.

B. U.S. Gaussian Estimation Results

In Table IV we present the empirical results using the CKLS data set. A comparison of the Gaussian estimates indicates that the bias resulting from the use of the CKLS approximation is again very small. We now concentrate on the Gaussian estimation results using the discrete model proposed in this article. Based on maximized Gaussian log likelihood values compared to the unrestricted model, the CEV model performs the best followed by the CIR VR and Brennan-Schwartz models. The unrestricted model estimate of γ is 1.3610 and highly significant with a t-statistic of 13.28. This compares to the CKLS reported estimated value of γ of 1.4999. Also, there appears to be only weak evidence of mean reversion in the short-term rate; the parameter β is insignificant in the unrestricted model. The discrepancy in the results could be attributed to the differences in the method of estimation since CKLS use

Generalized Method of Moments methods. But the results still imply that the volatility of the interest rate process is highly sensitive to the level of the riskless rate as found by CKLS. Based on the χ^2 likelihood ratio test, the results imply that all the models are rejected. On the basis of the Bergstrom (1990) S-statistic, the null hypothesis of white noise residuals is not rejected for any of the models at the 5 percent level.

V. Conclusion

In this article we present the first application in finance of recently developed methods for the Gaussian estimation of continuous time dynamic models. A range of one-factor continuous time models of the short-term interest rate are estimated using data on the sterling one month interbank rate and the U.S. Treasury bill yield data used in CKLS. We adopt the approach of CKLS, which allows the nesting of eight well known models within a general stochastic differential equation. We obtain estimates from using a discrete model and compare them with the discrete approximation used in CKLS. We find that the asymptotic bias from using the CKLS discrete approximation is very small. Based on the Gaussian estimates using our proposed discrete model, we find for the British data that the volatility of the short-term interest rate is not highly sensitive to the level of the interest rate. This result differs sharply from recent results for U.S. data.

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