Theoretic Exercises

Andrés Felipe Tamayo
aftamayoa@eafit.edu.co

David Plazas Escudero
dplazas@eafit.edu.co

Juan Sebastián Cárdenas-Rodríguez
jscardenar@eafit.edu.co

Mathematical Engineering, Universidad EAFIT

September 4, 2020

Exercise 1

Exercise a)

Question. Prove that a decreasing sequence of sets (in the inclusion sense) is convergent. Give examples of a convergent and a divergent sequence of sets.

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of sets, that is, $\forall n \in \mathbb{N}$, $A_{n+1} \subset A_n$. Recall that a sequence of sets $\{B_n\}_{n\in\mathbb{N}}$ is said to be convergent if

$$\limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n$$

where

$$\limsup_{n\to\infty} B_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} B_m, \quad \text{and} \quad \liminf_{n\to\infty} B_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m=n}^{\infty} B_m.$$

Let us see that $\{A_n\}_{n\in\mathbb{N}}$ is convergent. Since $\{A_n\}_{n\in\mathbb{N}}$, it is clear that

$$\bigcup_{m=n}^{\infty} A_m = A_n$$

and that

$$\bigcap_{m=n}^{\infty} A_m = \bigcap_{m=1}^{\infty} A_m.$$

Hence,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m=n}^{\infty} A_m = \bigcap_{m=1}^{\infty} A_m = \bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} = \limsup_{m\to\infty} A_m$$

and by renaming m as n in the right-most expression we conclude that $\{A_n\}_{n\in\mathbb{N}}$ is convergent.

An example of convergent sequence in $\mathbb{R}$ is the sequence $\{A_n\}_{n\in\mathbb{N}}$, where $A_n = [-1, n]$. It is clear that this in an monotone increasing sequence of sets, since $n < n+1 \implies A_n \subset A_{n+1}$. Furthermore,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m=n}^{\infty} [-1, m] = \bigcup_{n\in\mathbb{N}} [-1, n] = [-1, \infty]$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m=n}^{\infty} [-1, m] = \bigcap_{n\in\mathbb{N}} [-1, \infty] = [-1, \infty]$$

On the other hand, an example of divergent sequence of sets is $\{B_n\}_{n\in\mathbb{N}}$, where $B_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$. In this case, it can be proved that

$$\bigcup_{m=n}^{\infty} B_m = \mathbb{Q} \cap [0, 1]$$

and therefore,

$$\limsup_{n\to\infty} B_n = \mathbb{Q} \cap [0, 1].$$

It can also be seen that

$$\bigcap_{m=n}^{\infty} B_m = \{0, 1\}$$

and thus,

$$\liminf_{n\to\infty} B_n = \{0, 1\}$$

Therefore, $\{B_n\}_{n\in\ltimes}$ is divergent. $\square$

Exercise b)

Question. Prove that any open ball is an open set.

Proof. Let (X, d) be a metric space. Let $x \in X$, the open ball with radius r_x is defined as the set

$$B(x, r_x) = \{y \in X \mid d(x, y) < r_x\}.$$

Let us see that $B(x, r_x)$ is an open set as well. For that we must prove that $\forall y \in B(x, r_x)$, $\exists r_y \in \mathbb{R}_+$ such that $B(y, r_y) \subseteq B(x, r_x)$.

Let $y \in B(x, r_x)$ and let $r_y \in \mathbb{R}_+$ such that

$$d(x, y) + r_y < r_x \tag{1}$$

Let $y' \in B(y, r_y)$, since $d(\cdot, \cdot)$ is a metric, it must satisfy the triangle inequality. Therefore, $d(x, y') \leq d(x, y) + d(y, y')$. Considering this with (1), we get $d(x, y') < r_x - r_y + r_y = r_x$. Hence, $y' \in B(x, r_x) \implies B(y, r_y) \subseteq B(x, r_x)$. Consequently, $B(x, r_x)$ is an open set. $\square$

Exercise c)

Question. The finite sum of metrics is a metric. Is the infinite sum of metrics a metric?

Proof. Let $d_i : V \times V \to \mathbb{R}^+$ be metrics, for $i = 1, \ldots, n$. Let's show that the sum of all the metrics is a metric, i.e.

$$d(x, y) = \sum_{i=1}^{n} d_i(x, y)$$

- Let's show that $d(x, y) = 0 \iff x = y$.

  ($\Rightarrow$) Let's suppose that $d(x, y) = 0$, for a $x, y \in V$.. In this manner, it is seen that:

  $$d(x, y) = \sum_{i=1}^{n} d_i(x, y) = 0$$

  As all d_i are metrics then it is clear that

  $$d_i(x, y) \geq 0$$

  Hence, for the sum to be 0 each of the components must be equal to 0. Therefore, for all i:

  $$d_i(x, y) = 0 \quad x = y$$

  As d_i is a metric.

  ($\Leftarrow$) Let's suppose that $x = y$. As d_i are metric, it occurs that if $x = y$, $d_i(x, y) = 0$. In this manner:

  $$\begin{aligned} d(x, y) &= \sum_{i=1}^{n} d_i(x, y) \\ &= \sum_{i=1}^{n} 0 \\ &= 0 \end{aligned}$$

  Hence, by the previous two proofs it is seen that $d_i(x, y) = 0 \iff x = y$.

- Let's show that $d(x, y) = d(y, x)$ for all $x, y \in V$. It is known that for all i it happens that $d_i(x, y) = d_i(y, x)$ as their are metrics. Then

  $$\begin{aligned} d(x, y) &= \sum_{i=1}^{n} d_i(x, y) \\ &= \sum_{i=1}^{n} d_i(y, x) \\ &= d(y, x) \end{aligned}$$

  Therefore, it is symmetric.

- Let's show that $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in V$. Similarly to previous proofs, it is known that for all i, $d_i(x,y) \leq d_i(x,z) + d_i(z,y)$ as their are metrics. Hence

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{n} d(x,y) \\ &\leq \sum_{i=1}^{n} (d_i(x,z) + d_i(z,y)) \\ &= \sum_{i=1}^{n} d_i(x,z) + \sum_{i=1}^{n} d_i(z,y) \\ &= d(x,z) + d(z,y) \end{aligned}$$

Therefore, the triangular inequality holds.

For all the three proofs done, it is concluded that the finite sum of metrics is also a metric. □

Exercise d)

Question. Show that a convex linear combination of metric is a metric.

Proof. Let $d_i : V \times V \to \mathbb{R}^+$ be metrics, for $i = 1, \ldots, n$. Let's show that for $\lambda_i \in [0,1]$ such that:

$$\sum_{i=1}^{n} \lambda_i = 1$$

the linear combination is also a metric, i.e.

$$d(x,y) = \sum_{i=1}^{n} \lambda_i d_i(x,y)$$

- Let's show that $d(x,y) = 0 \iff x = y$.

  ($\Rightarrow$) Let's suppose that $d(x,y) = 0$, for a $x, y \in V$.. In this manner, it is seen that:

$$d(x,y) = \sum_{i=1}^{n} \lambda_i d_i(x,y) = 0$$

  As all d_i are metrics and all λ_i are positives then it is clear that

$$\lambda_i d_i(x,y) \geq 0$$

  Hence, for the sum to be 0 each of the components must be equal to 0. Therefore, for all i:

$$\lambda_i d_i(x,y) = 0$$

  As the sum of all λ_i is equal to 1, there must be at least one λ_j such that $\lambda_j > 0$. In this manner, for $i = j$:

$$\lambda_j d_j(x,y) = 0$$

$$
\begin{aligned}
d_j(x,y) &= 0 \\
x &= y, \text{ As } d_j \text{ is a metric.}
\end{aligned}
$$

($\Leftarrow$) Let's suppose that $x = y$. As d_i are metric, it occurs that if $x = y$, $d_i(x,y) = 0$. In this manner:

$$
\begin{aligned}
d(x,y) &= \sum_{i=1}^{n} \lambda_i d_i(x,y) \\
&= \sum_{i=1}^{n} \lambda_i \cdot 0 \\
&= 0
\end{aligned}
$$

Hence, by the previous two proofs it is seen that $d_i(x,y) = 0 \iff x = y$.

- Let's show that $d(x,y) = d(y,x)$ for all $x, y \in V$. It is known that for all i it happens that $d_i(x,y) = d_i(y,x)$ as their are metrics. Then

$$
\begin{aligned}
d(x,y) &= \sum_{i=1}^{n} \lambda_i d_i(x,y) \\
&= \sum_{i=1}^{n} \lambda_i d_i(y,x) \\
&= d(y,x)
\end{aligned}
$$

  Therefore, it is symmetric.

- Let's show that $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in V$. Similarly to previous proofs, it is known that for all i, $d_i(x,y) \leq d_i(x,z) + d_i(z,y)$ as their are metrics. Hence

$$
\begin{aligned}
d(x,y) &= \sum_{i=1}^{n} \lambda_i d(x,y) \\
&\leq \sum_{i=1}^{n} \lambda_i \left(d_i(x,z) + d_i(z,y) \right) \\
&= \sum_{i=1}^{n} \lambda_i d_i(x,z) + \sum_{i=1}^{n} \lambda_i d_i(z,y) \\
&= d(x,z) + d(z,y)
\end{aligned}
$$

  Therefore, the triangular inequality holds.

For all the three proofs done, it is concluded that a convex linear combination of metrics is also a metric. □

Excercise e)

Question. Show that Mahalanobis distance is a metric.

Mahalanobis distance can be defined as a dissimilarity measure between two random vectors $\vec{x}$ and $\vec{y}$ of the same distribution with the covariance matrix S:

$$d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y})^T S^{-1} (\vec{x} - \vec{y})}$$

Proof. In order to prove that $d(\vec{x}, \vec{y}) \geq 0$ we have to show that $(\vec{x} - \vec{y})^T S^{-1} (\vec{x} - \vec{y}) \geq 0$. Note that this holds if we prove that S^{-1} is definite positive. Let's consider a sample of vector $x_i = (x_{i1}, \ldots, x_{ik})^T$, with $i = 1, \ldots, n$, the sample mean vector is

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

and the sample covariance matrix is

$$S = \frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})(x_i - \overline{x})^T$$

Then, for a nonzero vector $y \in \mathbb{R}^n$, we have

$$\begin{aligned} y^T S y &= y^T \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \overline{x})(x_i - \overline{x})^T \right) y \\ &= \frac{1}{n} \sum_{i=1}^{n} y^T (x_i - \overline{x})(x_i - \overline{x})^T y \\ &= \frac{1}{n} \sum_{i=1}^{n} ((x_i - \overline{x})^T y)^2 \geq 0 \quad * \end{aligned}$$

By this, S is always positive semi-definite. Now, we have to show that S is definite. Let's define $z_i = (x_i - \overline{x})$, for $i = 1, \ldots, n$. For any nonzero $y \in \mathbb{R}^n$, (*) is zero iff $z_i^T y = 0$, for each $i = 1, \ldots, n$. Let's suppose now that the set $\{z_1, \ldots, z_n\}$ spans over $\mathbb{R}^n$. Then there are real numbers $\alpha_1, \ldots, \alpha_n$ such hat $y = \alpha_1 z_1 + \ldots + \alpha_n z_n$. But then we have $y^T y = \alpha_1 z_1^T y + \ldots + \alpha_n z_n^T y = 0$, which yields that $y = 0$, a contradiction. Hence, if the z_i spans over $\mathbb{R}^n$, then S is positive definite.

We conclude that S is a definite positive matrix, hence

$$d(\vec{x}, \overline{x}) \geq 0$$

We have to prove now that $d(\vec{x}, \vec{y}) = 0$ iff $\vec{x} = \vec{y}$. This is obvious from the definition of the mahalanobis distance, because

$$d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y})^T S^{-1} (\vec{x} - \vec{y})} = \sqrt{(\vec{x} - \vec{x})^T S^{-1} (\vec{x} - \vec{x})}$$

since $\vec{x}$ and $\vec{y}$ have the same dimensions.

We continue with the proof that $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$. This holds because of S is a symmetric matrix, and hence we have finish.

At last, we have to prove the triangle inequality. Let S be a symmetric $n \times n$ matrix (This because the definition of covariance matrix). And let's rename the mahalanobis norm as

$$\|x\|_S = \sqrt{x^T S x}$$

We have shown that S is positive-definite. By spectral theorem for symmetrix matrices, there are a diagonal $n \times n$ matrix $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$ and an orthogonal $n \times n$ matrix Q (i.e. $Q^T Q = I$), such that $Q^T = Q^{-1}$ and:

$$S = Q^T \Lambda Q$$

Because of the matrix S is positive-definite we have that

$$\begin{aligned}
&\lambda_1 > 0 \\
&\lambda_2 > 0 \\
&\ldots \\
&\lambda_n > 0
\end{aligned}$$

Let the matrix

$$U = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})Q,$$

note that:

$$S = U^T U$$

set now $\overline{x} = Ux$ and $\overline{y} = Uy$. Let $\|v\|_E = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^n}$ the usual euclidean distance. Then is clearly that

$$\begin{aligned}
\|x\|_S &= \|\overline{x}\|_E \\
\|y\|_S &= \|\overline{y}\|_E \\
\|x + y\|_S &= \|\overline{x} + \overline{y}\|_E \quad \textbf{(**)}
\end{aligned}$$

By usual triangular inequality we have:

$$\|\overline{x} + \overline{y}\|_E \leq \|\overline{x}\|_E + \|\overline{y}\|_E$$

By the equality **(**)** we have

$$\|x + y\|_S \leq \|x\|_S + \|y\|_S$$

□

Exercise f)

Question. Prove that if $d : X \times X \to \mathbb{R}$ is a metric, then so is $\bar{d}(x, y) = \frac{d(x,y)}{1+d(x,y)}$.

Proof. Let (X, d) be a metric space. Let us prove that $\bar{d}(\cdot, \cdot)$ also satisfy the conditions to be metric. For the following steps, assume that $x, y, z \in X$.

1. Since $d(\cdot, \cdot)$ is a metric, it satisfies that $d(x, y) \geq 0$. Then, $d(x, y) + 1 \geq 1 > 0$ and finally $\bar{d}(x, y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$.

2. Since $d(\cdot,\cdot)$ is a metric, it satisfies that $d(x,y) = d(y,x)$. Then $\bar{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \bar{d}(y,x)$.

3. " $\implies$ " Suppose $\bar{d}(x,y) = 0 = \frac{d(x,y)}{1+d(x,y)}$, hence $d(x,y) = 0$ and as $d(\cdot,\cdot)$ is metric, $x = y$. " $\impliedby$ " Suppose that $x = y$, then $d(x,y) = 0$ since it is a metric; now, $\bar{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0$. Consequently, $\bar{d}(x,y) = 0 \iff x = y$.

4. Consider the function $f(t) = \frac{t}{1+t}$ on $[0,\infty)$. Note that $\bar{d}(x,y) = f(d(x,y))$. It is clear that $f'(t) = \frac{1}{(t+1)^2}$, and hence $f(t)$ is a positive increasing function on $[0,\infty)$. Now, as $d(x,y) = 0$ is a metric, it satisfies the triangle inequality, hence $d(x,y) \le d(x,z) + d(z,y)$. As $f(t)$ increases on $[0,\infty)$, the inequality is preserved when applied to this last expression: $f(d(x,y)) \le f(d(x,z) + d(z,y))$. This yields

$$\begin{aligned} f(d(x,z)+d(z,y)) &= \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &\le \bar{d}(x,z) + \bar{d}(z,y) \end{aligned}$$

and finally,
$$\bar{d}(x,y) = f(d(x,y)) \le f(d(x,z)+d(z,y)) \le \bar{d}(x,z) + \bar{d}(z,y).$$

□

Exercise i)

Question. Prove that the Frobenius norm satisfy the properties for a matrix norm.

Proof. Let $A \in \mathbb{R}^{m\times n}$. Recall that the Frobenius norm of A is given by

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

For the following proofs, assume $A, B \in \mathbb{R}^{m\times n}$ and $\alpha \in \mathbb{R}$.

1. $$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2} \ge 0, \ \forall a_{ij} \in \mathbb{R}.$$

2. $$\|\alpha A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(\alpha a_{ij})^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2} = |\alpha|\|A\|_F$$

3. " $\implies$ "
$$\|A\|_F = 0 \implies \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2} = 0$$

which directly implies that each $a_{ij} = 0$, since it is a positive sum.

"$\Longleftarrow$"

$$A = 0 \implies a_{ij} = 0,\ \forall i, j \implies \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij})^2} = 0 \implies \|A\|_F = 0.$$

Then, $\|A\|_F = 0 \iff A = 0$.

4. For the triange inequality, we may use the trace definition

$$\|A\|_F = \sqrt{\mathrm{tr}(A^T A)}$$

and use the fact that the Frobenius norm comes from the inner product defined as

$$\langle A, B \rangle = \mathrm{tr}(A^T B).$$

Let us work with the inner product as follows:

$$\begin{aligned} \langle A+B, A+B \rangle =& \langle A, A \rangle + 2\langle A, B \rangle + \langle B, B \rangle \\ \|A+B\|_F^2 =& \|A\|_F^2 + 2\langle A, B \rangle + \|B\|_F^2, \end{aligned}$$

using the Cauchy-Schwarz inequality $\langle A, B \rangle \leq \|A\|_F \|B\|_F$, we get

$$\begin{aligned} \|A+B\|_F^2 \leq& \|A\|_F^2 + 2\|A\|_F \|B\|_F + \|B\|_F^2 \\ \|A+B\|_F^2 \leq& (\|A\|_F + \|B\|_F)^2 \\ \|A+B\|_F \leq& \|A\|_F + \|B\|_F \end{aligned}$$

With the 4 points above proven, $\|A\|_F$ is a norm for matrices. □

Exercise g)

Question. If $d : X \times X \to \mathbb{R}$ is a metric, then $\overline{d}(x, y) = \min\{1, d(x, y)\}$ also is.

Proof. Let's show that $\overline{d}(x, y) = \min\{1, d(x, y)\}$ is a metric

- $\overline{d}(x, y) \geq 0$. We have three cases.
  - If $1 = d(x, y)$ then $\min\{1, d(x, y)\} = 1$, therefore $\overline{d}(x, y) \geq 0$.
  - If $1 < d(x, y)$ then $\min\{1, d(x, y)\} = 1$, therefore $\overline{d}(x, y) \geq 0$.
  - If $d(x, y) < 1$ then $\min\{1, d(x, y)\} = d(x, y)$. We have that $d(x, y)$ is a metric, $d(x, y) \geq 0$, therefore $\overline{d}(x, y) \geq 0$
- $\overline{d}(x, y) = 0$ iff $x = y$.
  - $\overline{d}(x, y) = 0 \Rightarrow x = y$.

    We have that $\overline{d}(x, y) = 0$, but this means $\min\{1, d(x, y)\} = 0$. Clearly $1 \neq 0$. Then $d(x, y) = 0$ iff $x = y$, but d is a metric. Therefore $x = y$.

- $x = y \Rightarrow \overline{d}(x,y) = 0$.

  Let's suppose that $x = y$, then $\overline{d}(x,y) = \min\{1, d(x,y)\} = 0$. This because d is a metric, and therefore $d(x,y) = 0$ if $x = y$ hence $\overline{d}(x,y) = 0$.

- $\overline{d}(x,y) = \overline{d}(y,x)$.

  $\overline{d}(x,y) = \min\{1, d(x,y)\}$, because d is a metric $d(x,y) = d(y,x)$, then $\overline{d}(x,y) = \min\{1, d(x,y)\} = \min\{1, d(y,x)\} = \overline{d}(y,x)$.

- $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$.

  $\overline{d}(x,z) = \min\{1, d(x,z)\}$. Because d is a metric we have $d(x,z) \leq d(x,y) + d(y,z)$. Therefore

$$\begin{aligned}\overline{d}(x,z) &\leq \min\{1, d(x,y) + d(y,z)\} \\ &\leq \min\{1, d(x,y)\} + \min\{1, d(y,z)\} \\ &= \overline{d}(x,y) + \overline{d}(y,z)\end{aligned}$$

□

Exercise h)

Question. If $A \subset B$, both subsets of $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$ and d a metric, we have that $d(x,A) \geq d(x,B)$.

Proof. By the definition of distance between sets we know that

$$d(x,B) = \inf\{d(x,b) : x \in \mathbb{R}^n, b \in B\}$$

Let's denote $\Gamma(x,A) = \{d(x,a) : x \in \mathbb{R}^n, a \in A\}$. Therefore, by definition $d(x,A) = \inf \Gamma(x,A)$. It is easily seen that for every other lower bound λ of $\Gamma(x,A)$ it happens that:

$$\lambda \leq d(x,A) \tag{2}$$

as the infimum is the largest lower bound of the set. On the other hand, it is clear that for all $b \in B$:

$$d(x,B) \leq d(x,b)$$

In particular, for all $b \in A$ as $A \subset B$. Therefore $d(x,B)$ is a lower bound for $\Gamma(x,A)$. In this manner, by Equation 2 it occurs that:

$$d(x,B) \leq d(x,A)$$

□

Exercise j)

Question. Show that the 2-norm of a real-valued matrix A of size $n \times n$ defined as

$$\|A\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

is the maximum eigen value of A.

Proof. Let λ_i be the eigen values of the matrix $B = A^T A$ for $i = 1, .., n$ and v_i be non-null vectors such that

$$(A^T A)v_i = \lambda_i v_i$$

It is seen that B is a Hermitian matrix, therefore it's eigen vectors are orthonormal ang generate a basis for the vector space. Therefore, for all vector x there exists a_i such that

$$x = \sum_{i=1}^{n} a_i v_i$$

Let $j \in \{1, ..., n\}$ such that $j = \max_i |\lambda_i|$.

($\leq$) Let's show that $\|A\|_2 \leq \sqrt{|\lambda_j|}$. Hence:

$$\begin{aligned}
\|Ax\|_2^2 &= \langle Ax, Ax \rangle > \\
&= \langle x, A^T Ax \rangle \\
&= \langle x, Bx \rangle \\
&= \langle \sum_{i=1}^{n} a_i v_i, B \sum_{i=1}^{n} a_i v_i \rangle \\
&= \langle \sum_{i=1}^{n} a_i V_i, \sum_{i=1}^{n} \lambda_i^2 a_i V_i \rangle \\
&= \sum_{i=1}^{n} a_i^2 \lambda_i \\
&\leq \lambda_j \sum_{i=1}^{n} a_i^2 \\
&= \lambda_j \|x\|_2^2
\end{aligned}$$

Therefore, $\|Ax\|_2 \leq \sqrt{\lambda_j} \|x\|_2$. Hence, $\|A\|_2 \leq \sqrt{\lambda_j}$.

($\geq$) Let's show that $\|A\|_2 \geq \sqrt{|\lambda_j|}$. Using the obtained previous result it can be seen that for v_j

$$\begin{aligned}
\|A\|_2^2 &\geq \frac{\langle v_j, Bv_j \rangle}{\|v_j\|_2} \\
&= \frac{\langle v_j, \lambda_j v_j \rangle}{\|v_j\|_2} \\
&= \lambda_j
\end{aligned}$$

Therefore, $\|A\|_2 \leq \sqrt{\lambda_j}$. Then, it must happen that $\|A\|_2 = \sqrt{\lambda_j}$. □

Exercise 2

Question. Define what it is a pseudometric and show a few examples of pseudometrics.

Definition 0.1. A pseudometrix space (E, d) is a set E together with a non-negative real-valued function $d : E \times E \to \mathbb{R}_{\geq 0}$ (called a **pseudometric**) such that for every $x, y, z \in E$,

1. $d(x, y) \geq 0$
2. $d(x, x) = 0$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Example 1. For a set E, define $d(x, y) = 0$ for all $x, y \in E$. We call d the trivial pseudometric on E: all distancs are 0.

Example 2. Every measure space $\Omega, \mathcal{A}, \mu)$ can be viewed as a pseudometric space by defining

$$d(A, B) := \mu(A \triangle B)$$

for all $A, B \in \mathcal{A}$, where the triangle denotes symmetric difference.

Example 3. For vector spaces V, a seminorm p induces a pseudometric on V, as

$$d(x, y) = p(x - y)$$