2. a) $V = \{u \in C^1[0,1] \mid u(0) = 0\}$

$W = C^1[0,1]$

$Du = u'$

Veamos que $V \simeq W$.

Veamos que el operador D es invertible.

Definamos $D^{-1}w = \int_0^t w(s)ds$

$$D^{-1}(Du) = \int_0^t u'(s)ds = u(t) - \cancel{u(0)}^{0} = u(t)$$

$$D(D^{-1}w) = \frac{d}{dt}\left(\int_0^t w(s)ds\right) = w(t)$$

Como D tiene inversa, D es biyectivo.

Luego, $V \simeq W$

b) Sea $T: \mathbb{R}^3 \to P_2[0,1]$

$Tu = u_1 t(t-0.5) + u_2 t(t-1) + u_3(t-0.5)(t-1)$

Veamos que T es biyectivo

Sea $T^{-1}p(t) = (2p(1), -4p(0.5), 2p(0))$

$$T(T^{-1}p(t)) = T(2p(1), -4p(0.5), 2p(0))$$

$$= 2p(1)t(t-0.5) - 4p(0.5)t(t-1) + 2p(0)(t-0.5)(t-1)$$

$$= 2p(1)(t^2 - 0.5t) - 4p(0.5)(t^2 - t) + 2p(0)(t^2 - 1.5t + 0.5)$$

$$= (2p(1) - 4p(0.5) + 2p(0))t^2 - (p(1) - 4p(0.5) + 3p(0))t + p(0)$$

Obsérvese que

$$T(T^{-1}p(t))|_{t=0} = p(0)$$

$$T(T^{-1}p(t))|_{t=0.5} = (2p(1) - 4p(0.5) + 2p(0))0.25 - (p(1) - 4p(0.5) + 3p(0))0.5 + p(0)$$

$$= 0.5\cancel{p(1)} - p(0.5) + 0.5\cancel{p(0)} - 0.5\cancel{p(1)} + 2p(0.5) - 1.5\cancel{p(0)} + \cancel{p(0)}$$

$$= p(0.5)$$

$$T(T^{-1}p(t))|_{t=1} = 2p(1) - 4\cancel{p(0.5)} + 2\cancel{p(0)} - p(1) + 4\cancel{p(0.5)} - 3\cancel{p(0)} + \cancel{p(0)}$$

$$= p(1)$$

Luego $T(T^{-1}p(t)) = p(t)$ ← Sólo existe 1 polinomio de grado 2 que pase por 3 puntos difs.

$$T^{-1}(Tu) = T^{-1}(u_1 t(t-0.5) + u_2 t(t-1) + u_3(t-0.5)(t-1))$$

$$= (2Tu|_{t=1}, -4Tu|_{t=0.5}, 2Tu|_{t=0})$$

$$= \left(2(0.5u_1), -4\left(-\frac{u_2}{4}\right), 2(0.5u_3)\right)$$

$$= (u_1, u_2, u_3) = u$$

Luego $\mathbb{R}^3 \simeq P[0,1]$

b*) $Tu = u_1 + u_2 t + u_3 t^3$

$T^{-1}p(t) = (p(0), -p(1) + 4p(0.5) - 3p(0), 2p(1) - 4p(0.5) + 2p(0))$

$$T(T^{-1}p(t)) = T(p(0), -p(1) + 4p(0.5) - 3p(0), 2p(1) - 4p(0.5) + 2p(0))$$

$$= (2p(1) - 4p(0.5) + 2p(0))t^2 - (p(1) - 4p(0.5) + 3p(0))t + p(0)$$

Obsérvese que

$$T(T^{-1}p(t))|_{t=0} = p(0)$$

$$T(T^{-1}p(t))|_{t=0.5} = (2p(1) - 4p(0.5) + 2p(0))0.25 - (p(1) - 4p(0.5) + 3p(0))0.5 + p(0$$

$$= 0.5\cancel{p(1)} - p(0.5) + 0.5\cancel{p(0)} - 0.5\cancel{p(1)} + 2p(0.5) - 1.5\cancel{p(0)} + \cancel{p(0)}$$

$$= p(0.5)$$

$$T(T^{-1}p(t))|_{t=1} = 2p(1) - 4\cancel{p(0.5)} + 2\cancel{p(0)} - p(1) + 4\cancel{p(0.5)} - 3\cancel{p(0)} + \cancel{p(0)}$$

$$= p(1)$$

Luego, $T(T^{-1}p(t)) = p(t)$ ← Sólo existe 1 polinomio de grado 2 que pase por 3 puntos difs.

$$T^{-1}(Tu) = T^{-1}(u_1 + u_2 t + u_3 t^2)$$

$p(t) = u_1 + u_2 t + u_3 t^2$

$p(0) = u_1$, $p(0.5) = u_1 + 0.5u_2 + 0.25u_3$

$p(1) = u_1 + u_2 + u_3$.

$$T^{-1}p(t) = (p(0), -p(1) + 4p(0.5) - 3p(0), 2p(1) - 4p(0.5) + 2p(0))$$

$$= (u_1, -\cancel{u_1} - u_2 - \cancel{u_3} + 4\cancel{u_1} + 2u_2 + \cancel{u_3} - 3\cancel{u_1}, 2\cancel{u_1} + 2\cancel{u_2} + 2u_3 - 4\cancel{u_1} - 2\cancel{u_2} - u_3 + 2\cancel{u_1})$$

$$= (u_1, u_2, u_3)$$

Luego, $\mathbb{R}^3 \simeq P_2[0,1]$

~~Veamos que~~ $\mathbb{R}^3 \cong P_2[0,1]$

$$\|Tu\| = \|u_1 + u_2 t + u_3 t^2\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \|u\|$$