Kalman Filter for Observer-ARMA Model with Parameter Estimation

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September 26, 2019

Preliminaries

Notation

- Observed data vector: $y_t^* := (y_0^T, y_1^T, \dots, y_t^T)^T$.
- $z_{t|t-1} := E(z_t | y_{t-1}^*)$ and $y_{t|t-1} := E(y_t | y_{t-1}^*)$.
- $\Delta y_t := y_t - y_{t|t-1}$ and $\Delta z_t := z_t - z_{t|t-1}$.
- $\Sigma_{t|t} := \mathrm{Cov}(\Delta z_t | y_t^*)$.

Preliminaries

Observer-ARMA Model

The general scheme for the observer-ARMA model is given by

$$
\begin{aligned}
z_{t+1} &= \sum_{s=0}^{l_1} a_s z_{t-s} + \sum_{s=0}^{l_2} b_s \xi_{t-s} \\
y_t &= h_t z_t + \zeta_t \\
z_0 &\in \mathbb{R},\ \ t = 0, 1, \dots
\end{aligned}
\tag{1}
$$

Preliminaries

Kalman Filter

The standard KF Procedure [Kalman 1960] has two main hypothesis:

H1

$$
\begin{aligned}
&E(\xi_t) = E(\zeta_t) = 0, \quad t = 0, 1, \dots \\
&E(\xi_t \xi_t^T) = Q_t, \quad E(\zeta_t \zeta_t^T) = R_t \\
&E(\xi_t \xi_s^T) = E(\zeta_t \zeta_s^T) = 0, \quad \forall s \neq t \\
&E(\xi_t \zeta_s^T) = 0, \quad \forall t, s = 0, 1, \dots \\
&E(z_t z_s^T) = 0, \quad \forall t \neq s
\end{aligned}
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&E(z_t z_s^T) = 0, \quad \forall t \neq s
\end{aligned}
$$

H2

$$\xi_t \sim \mathcal{N}(0, Q_t), \quad \zeta_t \sim \mathcal{N}(0, R_t), \quad \forall t = 0, 1, \dots$$

Preliminaries

Kalman Filter

The KF procedure is now presented:

1. **Initialization:**

$$z_{0|0} = z_0 \in \mathbb{R}^n,\ \Sigma_{0|0} = 0$$

2. **Prediction:**

$$\begin{aligned} z_{t|t-1} &= F_t z_{t-1|t-1} \\ \Sigma_{t|t-1} &= F_{t-1}\Sigma_{t-1|t-1}F_{t-1}^T + Q_{t-1} \end{aligned}$$

3. **Correction:**

$$\begin{aligned} z_{t|t} &= z_{t|t-1} + M_t^{\text{opt}}\Delta y_t \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - M_t^{\text{opt}} H_t \Sigma_{t|t-1} \end{aligned}$$

where $M_t^{\text{opt}} = \Sigma_{t|t-1}H_t^T \left(H_t\Sigma_{t|t-1}H_t^T + R_t\right)^{-1}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a given vector function.

$$\begin{cases} x_{t+1} = f(x_t) + \xi_t \\ y_t = H_t x_t + \zeta_t \\ x_0 \in \mathbb{R}^n, \ t = 0, 1, \dots \end{cases}$$

Preliminaries
Extended Kalman Filter

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a given vector function.

$$\begin{cases} x_{t+1} = f(x_t) + \xi_t \\ y_t = H_t x_t + \zeta_t \\ x_0 \in \mathbb{R}^n, \;\; t = 0, 1, \dots \end{cases} \implies \begin{cases} x_{t+1} = \tilde{F}_t x_t + \xi_t \\ y_t = H_t x_t + \zeta_t \\ x_0 \in \mathbb{R}^n, \;\; t = 0, 1, \dots \end{cases} \tag{2}$$

Where $\tilde{F}_t$ is:

$$\tilde{F}_t := \left.\frac{\partial f(x)}{\partial x}\right|_{x = x_t^{\mathrm{ref}}}$$

In this work we will use a simple ARMA model, with correlated state and one unknown stationary parameter a. The model in consideration is

$$\begin{aligned} z_{t+1} &= az_t + \xi_t + d\xi_{t-1} \\ y_t &= hz_t + \gamma_t \\ z_0 &\in \mathbb{R},\ t = 0, 1, \ldots \end{aligned} \tag{3}$$

Instrumental Extended Kalman Filter

Problem Definition

In this work we will use a simple ARMA model, with correlated state and one unknown stationary parameter a. The model in consideration is

$$
\begin{aligned}
&z_{t+1} = az_t + \xi_t + d\xi_{t-1} \\
&y_t = hz_t + \gamma_t \\
&z_0 \in \mathbb{R},\ \ t = 0, 1, \ldots
\end{aligned} \tag{3}
$$

Remark

The objective is find an estimator $\tilde{a}$ for the unknown parameter a using the data vector y_t^*.

Instrumental Extended Kalman Filter

Assumptions

It is assumed that the parameter $\tilde{a}$ is not constant, even if the original parameter is.

Instrumental Extended Kalman Filter

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It is assumed that the parameter $\tilde{a}$ is not constant, even if the original parameter is.

It is supposed that $\tilde{a}$ behaves as:

$$\tilde{a}_t = b\tilde{a}_{t-1} + \chi_{t-1} \tag{4}$$

with $0 < b < 1$ and $\chi_t \sim N(0, \sigma)$.

Instrumental Extended Kalman Filter

Final Model

Given the dynamics for $\tilde{a}_t$, we propose the complete dynamics for the ARMA model as

$$\begin{aligned} z_{t+1} &= \tilde{a}_t z_t + \xi_t + d\xi_{t-1} \\ \tilde{a}_{t+1} &= b\tilde{a}_t + \chi_t \\ z_0,\ \tilde{a}_0 &\in \mathbb{R},\ t = 0, 1, \ldots \end{aligned} \tag{5}$$

Instrumental Extended Kalman Filter

Final Model

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\tilde{a}_{t+1} &= b\tilde{a}_t + \chi_t \\
z_0,\ \tilde{a}_0 &\in \mathbb{R},\ t = 0, 1, \dots
\end{aligned}
\tag{5}
$$

Let $x_t := (z_t,\ \tilde{a}_t)^T$ and $x_t^{\mathrm{ref}} := (z_t^{\mathrm{ref}},\ \tilde{a}_t^{\mathrm{ref}})^T$. Extended Kalman Filter is applied.

$$
\tilde{F}_t = \begin{bmatrix} \tilde{a} & z \\ 0 & b \end{bmatrix}_{x = x_t^{\mathrm{ref}}}
$$

Instrumental Extended Kalman Filter

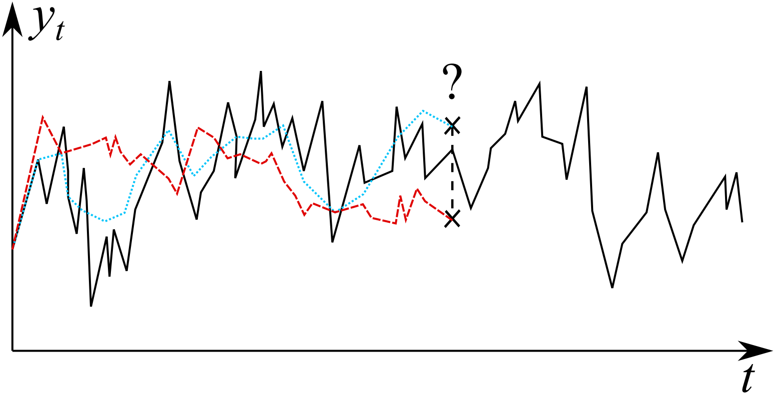
Final Model

Let $\tilde{x}_t := x_{2t}$, these transformation assures that the new state $\tilde{x}_t$ is not correlated with $\tilde{x}_{t-1}$.

$$\tilde{y}_t^* := \begin{cases} (y_0^T, y_2^T, \ldots, y_t^T)^T, & t \text{ even} \\ (y_0^T, y_2^T, \ldots, y_{t-1}^T)^T, & t \text{ odd} \end{cases}$$

Now, given the instrumental variables transformation and the linearization procedure for model, the standard KF procedure can be applied.

Discussion & Future Work



References I

Kalman, Rudolph Emil (1960). “A New Approach to Linear Filtering and Prediction Problems”. In: *Journal of basic Engineering* 82.1, pp. 35–45.

Thank you

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