$$E\left[e^{i\theta X}\right] = e^{i\mu\theta - \frac{\theta^2\sigma^2}{2}} \quad E\left[e^{\theta X}\right] = e^{\mu\theta + \frac{\theta^2\sigma^2}{2}}$$

UNIVERSIDAD EAFIT

DEPARTAMENTO DE CIENCIAS MATEMÁTICAS
INGENIERÍA MATEMÁTICA

PROCESOS ESTOCÁSTICOS II

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20 1) (20%) Suponga que $\{B_t\}_{t\geq 0}$ es un Movimiento Browniano Estándar y $\beta > 0$. Considere

MBEU

$$W_t = \alpha B_{\frac{t}{\beta}}$$

.

15 (a) (15%) Encuentre $\alpha > 0$ tal que el proceso estocástico W_t sea un Movimiento Browniano Estándar. Demuestre todas la condiciones.

5 (b) (5%) ¿Es W_t una martingala respecto $\mathcal{F}_s$? Si no lo es, ¿bajo que condiciones W_t constituye una martingala respecto de $\mathcal{F}_s$?

$\boxed{s<t}$ Bt es MBEU.

a) i) $h(t) = t/\beta \to$ Continua y positiva $\forall t \geq 0$.

$f(B_t) = \alpha B_{t/\beta} \to$ Continua $\forall t \geq 0$.

ii) Veamos que $W_t - W_s \sim N(0, t-s)$. $E[W_t - W_s] = \alpha E[B_{t/\beta} - B_{s/\beta}] = \alpha E[B_{\frac{t-s}{\beta}}] = 0$

$V[W_t - W_s] = E[(W_t - W_s)^2] = \alpha^2 E[(B_{t/\beta} - B_{s/\beta})^2] = \underline{\frac{\alpha^2(t-s)}{\beta}} \Rightarrow \boxed{\alpha = \sqrt{\beta}}$

$\phi_{W_t - W_s}(\theta) = E\left[e^{i\theta(W_t - W_s)}\right] = E\left[e^{\sqrt{\beta}i\theta(B_{t/\beta} - B_{s/\beta})}\right] = E\left[e^{\sqrt{\beta}i\theta(B_{\frac{t-s}{\beta}} - B_0)}\right] = E\left[e^{\sqrt{\beta}i\theta\left(\frac{t-s}{\beta}\right)^{1/2} B_1}\right]$

$= e^{-\frac{\theta^2(t-s)}{2}} \Rightarrow W_t - W_s \sim N(0, t-s)$.

iii) Cuando $s=0 \Rightarrow W_t \sim N(0,t)$.

iv) $E[W_s(W_t - W_s)] = E[\sqrt{\beta}B_{s/\beta}(\sqrt{\beta}B_{t/\beta} - \sqrt{\beta}B_{s/\beta})] = \beta E[B_{s/\beta}(B_{t/\beta} - B_{s/\beta})]$

$= \beta\{E[B_{s/\beta}B_{t/\beta}] - E[B_{s/\beta}^2]\} = \beta\{\min\{s/\beta, t/\beta\} - s/\beta\} = 0$.

Luego W_t tiene incrementos independientes $\Rightarrow W_t$ es un MBEU.

b) $E[W_t | \mathcal{F}_s] = E[\sqrt{\beta}B_{t/\beta} | \mathcal{F}_s] = \sqrt{\beta}(E[B_{t/\beta} - B_{s/\beta} | \mathcal{F}_s] + E[B_{s/\beta} | \mathcal{F}_s])$

$= \sqrt{\beta}(E[B_{t/\beta} - B_{s/\beta}] + B_{s/\beta}) = \sqrt{\beta}B_{s/\beta}$. Luego $E[W_t | \mathcal{F}_s] = W_s \Rightarrow$ Es Martingala.

0 → porque Bt es MBEU

25

$-6tB_t^2+3t$

2) (25%) Sea $\{B_t\}_{t\geq 0}$, un Movimiento Browniano Estándar y sea A_t un proceso estocástico tal que

$$W_t = B_t^4 + A_t$$

es una Martingala con respecto a $\mathcal{F}_s = \sigma(B_s, s \leq t)$ que se conoce como filtración natural.

20

(a) (20%) Determine el proceso A_t.

5

(b) (5%) ¿Es $W_t - 3t^2$ un Movimiento Browniano Estándar?

a) $E[W_t|\mathcal{F}_s] = E[B_t^4 + A_t|\mathcal{F}_s] = E[((B_t-B_s)+B_s)^4 + A_t|\mathcal{F}_s]$

$= E[(B_t-B_s)^4 + 4(B_t-B_s)^3B_s + 6(B_t-B_s)^2B_s^2 + 4(B_t-B_s)B_s^3 + B_s^4 + A_t|\mathcal{F}_s]$

$= E[(B_t-B_s)^4|\mathcal{F}_s] + 4E[(B_t-B_s)^3|\mathcal{F}_s]E[B_s|\mathcal{F}_s] + 6E[(B_t-B_s)^2|\mathcal{F}_s]E[B_s^2|\mathcal{F}_s] + 4E[(B_t-B_s)|\mathcal{F}_s]E[B_s^3|\mathcal{F}_s]$
$+ E[B_s^4|\mathcal{F}_s] + E[A_t|\mathcal{F}_s]$

$= E[(B_t-B_s)^4] + 4\underbrace{E[(B_t-B_s)^3]}_{0}B_s + 6E[(B_t-B_s)^2]B_s^2 + 4\underbrace{E[(B_t-B_s)]}_{0}B_s^3 + B_s^4 + E[A_t|\mathcal{F}_s]$

$= 3(t-s)^2 + 6(t-s)B_s^2 + B_s^4 + E[A_t|\mathcal{F}_s] \quad (1)$

Sea, $A_t = \psi(t)B_t^2 + \varphi(t) \Longrightarrow E[A_t|\mathcal{F}_s] = E[\psi(t)B_t^2 + \varphi(t)|\mathcal{F}_s] = \psi(t)E[B_t^2|\mathcal{F}_s] + \varphi(t)$

$= \psi(t)\{E[(B_t-B_s)^2|\mathcal{F}_s] + 2E[(B_t-B_s)B_s|\mathcal{F}_s] + E[B_s^2|\mathcal{F}_s]\} + \varphi(t)$.

$= \psi(t)\{E[(B_t-B_s)^2] + 2\underbrace{E[(B_t-B_s)]}_{0}E[B_s|\mathcal{F}_s] + B_s^2\} + \varphi(t)$

$= \psi(t)\{(t-s) + B_s^2\} + \varphi(t) \longleftarrow$ Reemplaza en (1).

$= 3t^2 - 6st + 3s^2 + 6tB_s^2 - 6sB_s^2 + B_s^4 + \psi(t)t - \psi(t)s + \psi(t)B_s^2 + \varphi(t)$

Sea $\psi(t) = -6t$:

$= 3t^2 - 6st + 3s^2 + 6tB_s^2 - 6sB_s^2 + B_s^4 - 6t^2 + 6st - 6tB_s^2 + \varphi(t)$

$= 3s^2 - 3t^2 - 6sB_s^2 + B_s^4 + \varphi(t)$

Sea $\varphi(t) = 3t^2$:

$= \underbrace{3s^2 - 6sB_s^2 + B_s^4}_{A(s)} \Longrightarrow E[W_t|\mathcal{F}_s] = W_s$ cuando $\boxed{A_t = -6tB_t^2 + 3t^2}$

$\Longrightarrow W_t$ es Martingala.

b) $Z_t = W_t - 3t^2 = B_t^4 - 6tB_t^2 + 3t^2 - 3t^2 = B_t^4 - 6tB_t^2 \longrightarrow h(t) = t \to$ continua y positiva

$f(B_t) = B_t^4 - 6tB_t^2 \to$ continua.

i) $Z_t - Z_s \sim N(0, t-s), \forall t$.

$E[Z_t - Z_s] = E[B_t^4 - 6tB_t^2 - B_s^4 + 6sB_s^2] = E[B_t^4] - 6tE[B_t^2] - E[B_s^4] + 6sE[B_s^2]$

Continua en hoja [1] de Anexos

↓

28 26

$Z_t = B_t - tB_1 \to Z_{t/(t+1)} = B_{t/(t+1)} - \frac{t}{(t+1)} B_1$

3) (25%) Sea $\{Z_t\}_{t\geq 0}$ un proceso Browniano Bridge. Considere $\to (t+1)Z_{t/(t+1)} = (t+1)B_{t/(t+1)} - tB_1$

$$W_t = (1+t)\, Z_{\frac{t}{t+1}}$$

y (la propia condición??

a) (20%) Muestre que W_t es un Movimiento Browniano Estándar.

b) (5%) Si $\{Z_t\}_{t\geq 0}$ es un proceso Drift. ¿Es W_t una martingala respecto $\mathcal{F}_s$? Si no lo es, ¿bajo que condiciones W_t constituye una martingala respecto de $\mathcal{F}_s$?

2.5

a) Ver NOTA: $W_t = (1+t)Z_{\frac{t}{t+1}} = (t+1)B_{t/(t+1)} - tB_1$

i) $h(t) = \frac{t}{t+1} \geq 0,\ \forall t \geq 0$ y continua $\forall t \geq 0$. $f(B_t) = (t+1)B_{t/(t+1)} - tB_1$ es continua.

ii) Veamos que $W_t - W_s \sim N(0, t-s),\ \forall s<t$

$\phi_{W_t-W_s}(\theta) = E\left[e^{i\theta(W_t - W_s)}\right]$

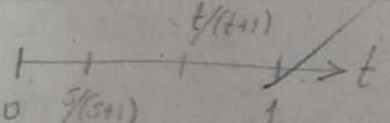
$= E\left[e^{i\theta[(t+1)B_{t/(t+1)} - tB_1 - (s+1)B_{s/(s+1)} + sB_1]}\right]$

$= E\left[e^{i\theta[tB_{t/(t+1)} + B_{t/(t+1)} - tB_1 - sB_{s/(s+1)} - B_{s/(s+1)} + sB_1 + (sB_{t/(t+1)} - sB_{t/(t+1)})]}\right]$

$= E\left[e^{i\theta[(B_{t/(t+1)} - B_{s/(s+1)}) - (t-s)B_1 + (t-s)B_{t/(t+1)} + s(B_{t/(t+1)} - B_{s/(s+1)})]}\right]$

$= E\left[e^{i\theta[(s+1)(B_{t/(t+1)} - B_{s/(s+1)}) - (t-s)(B_1 - B_{t/(t+1)})]}\right]$

$= E\left[e^{i\theta(s+1)(B_{t/(t+1)} - B_{s/(s+1)})}\, e^{i\theta(s-t)(B_1 - B_{t/(t+1)})}\right]$



Como $s<t \to s+st < t+st \to s(t+1) < (s+1)t$

$\to 0 < \frac{s}{s+1} < \frac{t}{t+1} < 1$

Luego $B_{t/(t+1)} - B_{s/(s+1)}$ y $B_1 - B_{t/(t+1)}$ son incrementos disjuntos y por lo tanto independientes (dado que B_t es MBEU).

Así, $= E\left[e^{i\theta(s+1)(B_{t/(t+1)} - B_{s/(s+1)})}\right] E\left[e^{i\theta(s-t)(B_1 - B_{t/(t+1)})}\right]$

$= E\left[e^{i\theta(s+1)(B_{t/(t+1) - s/(s+1)} - B_0)}\right] E\left[e^{i\theta(s-t)(B_{1-t/(t+1)} - B_0)}\right]$

$= e^{-\frac{\theta^2(s+1)^2}{2}\left[\frac{t}{t+1} - \frac{s}{s+1}\right] - \frac{\theta^2(s-t)^2}{2}\left[1 - \frac{t}{t+1}\right]} = e^{-\frac{\theta^2}{2}(s+1)^2\left[\frac{t(s+1) - s(t+1)}{(t+1)(s+1)}\right] - \frac{\theta^2}{2}(s-t)^2\left[\frac{1}{t+1}\right]}$

$= e^{-\frac{\theta^2}{2}\left[(s+1)^2\left(\frac{t-s}{(s+1)(t+1)}\right) + (s-t)^2\left(\frac{1}{t+1}\right)\right]} = e^{-\frac{\theta^2}{2}\left[\frac{(s+1)(t-s)}{(t+1)} + \frac{(t-s)^2}{(t+1)}\right]}$

↓ Sigue en hoja [1] Anexos

18.5

$W_t = e^{\alpha t + \lambda B_t}$

1) (30%) Considere el Movimiento Browniano Geométrico (W_t).

5 (a) (5%) ¿Es W_t un Movimiento Browniano Estándar?

7.5 (b) (15%) Determine la función covarianza dada por $Cov[W_t, W_s] = e^{(\alpha+\frac{1}{2}\lambda^2)(t+s)}\left[e^{\lambda^2 \min\{s,t\}} - 1\right]$.

6 (c) (10%) ¿Es W_t una martingala respecto de $\mathcal{F}_s$? Si no lo es, ¿bajo que condiciones W_t constituye una martingala respecto de $\mathcal{F}_s$?

a) $E[W_t] = E[e^{\alpha t+\lambda B_t}] = e^{\alpha t}E[e^{\lambda B_t}] = e^{\alpha t}E[e^{\lambda t^{1/2}B_1}] = e^{\alpha t + \frac{\lambda^2 t}{2}} \neq 0$

$\Rightarrow$ No es MBEU, pues $W_t \not\sim N(0,t)$. Sea $s<t$.

b) $Cov[W_t, W_s] = E[(W_t - E[W_t])(W_s - E[W_s])] = E[W_tW_s - W_tE[W_s] - E[W_t]W_s + E[W_t]E[W_s]]$

$= \underbrace{E[W_tW_s]}{} - E[W_s]E[W_t] \quad (1)$

? porque supones que W_t y W_s son independientes??

$E[W_tW_s] = E[e^{\alpha t+\lambda B_t+\alpha s+\lambda B_s}] = e^{\alpha(t+s)}E[e^{\lambda(B_t-B_s)}\cdot e^{2\lambda B_s}] = e^{\alpha(t+s)}E[e^{\lambda(B_t-B_s)}]E[e^{2\lambda B_s}]$

$= e^{\alpha(t+s)}E[e^{\lambda B_{t-s}}]E[e^{2\lambda B_s}] = e^{\alpha(t+s)}E[e^{\lambda(t-s)^{1/2}B_1}]E[e^{2\lambda s^{1/2}B_1}] = e^{\alpha(t+s)+\frac{\lambda^2(t-s)}{2}+2\lambda^2 s}$

supuesto de independencia??

Reemplazando en 1:

$= e^{\alpha(t+s)+\frac{\lambda^2(t-s)}{2}+2\lambda^2 s} - (e^{\alpha t+\frac{\lambda^2 t}{2}})(e^{\alpha s+\frac{\lambda^2 s}{2}}) = e^{\alpha(t+s)+\frac{\lambda^2 t}{2}+\frac{3}{2}\lambda^2 s} - e^{\alpha(t+s)+\frac{\lambda^2(t+s)}{2}}$

$= e^{\alpha(t+s)}\left(e^{\frac{\lambda^2 t}{2}+\frac{3}{2}\lambda^2 s} - e^{\frac{\lambda^2(t+s)}{2}}\right) = e^{\alpha(t+s)}e^{\frac{\lambda^2(t+s)}{2}}(e^{\lambda^2 s}-1) = e^{(\alpha+\frac{\lambda^2}{2})(t+s)}(e^{\lambda^2 s}-1)$

Como supusimos $s<t$, $\min\{s,t\}=s$. En general

$Cov[W_t, W_s] = e^{(\alpha+\frac{\lambda^2}{2})(t+s)}\left[e^{\lambda^2\min\{s,t\}} - 1\right]$

c) $E[W_t|\mathcal{F}_s] = E[e^{\alpha t+\lambda B_t}|\mathcal{F}_s] = e^{\alpha t}E[e^{\lambda B_t}|\mathcal{F}_s] = e^{\alpha t}\left[E[e^{\lambda(B_t-B_s)}|\mathcal{F}_s]E[e^{\lambda B_s}|\mathcal{F}_s]\right]$

$= e^{\alpha t}\left[E[e^{\lambda B_{t-s}}]e^{\lambda B_s}\right] = e^{\alpha t}\left[E[e^{\lambda(t-s)^{1/2}B_1}]e^{\lambda B_s}\right] = e^{\alpha t}e^{\frac{\lambda(t-s)}{2}+\lambda B_s}$

$= e^{\alpha t+\frac{\lambda(t-s)}{2}+\lambda B_s} = e^{(\alpha+\frac{\lambda}{2})t}e^{-\frac{\lambda s}{2}+\lambda B_s} \longrightarrow$ No es Martingala.

Si $\underbrace{\alpha = -\frac{\lambda}{2}}{} \Rightarrow E[W_t|\mathcal{F}_s] = e^{\alpha s+\lambda B_s} = W_s$

Para que sea Martingala, $\alpha = -\frac{\lambda}{2}$

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Anexos

$E[Z_t - Z_s] = E[B_t^4] - 6tE[B_t^2] - E[B_s^4] + 6sE[B_s^2]$ Punto 2.b)

Como B_t es MBEU $\rightarrow E[B_t^{2n}] = \frac{(2n)!}{2^n n!} t^n$, luego

$\Rightarrow = 3t^2 - 6t^2 - 3s^2 + 6s^2 = -3t^2 + 3s^2$

Como $E[Z_t - Z_s] \neq 0 \Rightarrow Z_t - Z_s \not\sim N(0, t-s)$. luego no es MBEU.

Punto 3.a)

$$e^{-\frac{\theta^2}{2}\left[\frac{(s+1)(t-s)}{t+1} + \frac{(t-s)^2}{t+1}\right]} = e^{-\frac{\theta^2}{2}(t-s)\left[\frac{s+1+t-s}{t+1}\right]} = e^{-\frac{\theta^2(t-s)}{2}} \qquad t - t^2$$

Como $\phi_{W_t - W_s}(\theta) = \phi_X(\theta)$, con $X \sim N(0, t-s)$, entonces $W_t - W_s \sim N(0, t-s)$.

iii) Cuando $s = 0 \rightarrow W_t \sim N(0, t)$.

iv) $E[W_s(W_t - W_s)] = E\left[(s+1)Z_{s/(s+1)}\left((t+1)Z_{t/(t+1)} - (s+1)Z_{s/(s+1)}\right)\right]$??

$= (s+1)E\left[(t+1)Z_{t/(t+1)}Z_{s/(s+1)} - (s+1)Z^2_{s/(s+1)}\right]$

$= (s+1)(t+1)E\left[Z_{t/(t+1)}Z_{s/(s+1)}\right] - (s+1)^2E\left[Z^2_{s/(s+1)}\right]$ $\quad W_t = (1+t)B_{\frac{t}{t+1}} - tB_1$

Como Z_t es Bridge, $Cov(Z_t, Z_s) = E[Z_t Z_s] = \min\{s,t\} - st$.

$= (s+1)(t+1)\left[\min\left\{\frac{t}{t+1}, \frac{s}{s+1}\right\} - \frac{st}{(s+1)(t+1)}\right] - (s+1)^2\left[\frac{s}{s+1} - \frac{s^2}{(s+1)^2}\right]$

$= s(t+1) - st - s(s+1) + s^2 = st + s - st - s^2 - s + s^2 = 0$

Luego W_t tiene incrementos independientes

Por lo tanto W_t es MBEU.

3.b) Drift: $Z_t = \mu t + \sigma B_t$; $W_t = (1+t)Z_{t/(t+1)}$

$= (1+t)\left[\frac{\mu t}{t+1} + \sigma B_{t/(t+1)}\right]$

$= \mu t + \sigma(1+t)B_{t/(t+1)}$

$E[W_t | F_s] = E[\mu t + \sigma(1+t)B_{t/(t+1)} | F_s] = \mu t + \sigma(1+t)E[B_{t/(t+1)} | F_s]$

$= \mu t + \sigma(1+t)\{E[(B_{t/(t+1)} - B_{s/(s+1)}) | F_s] + E[B_{s/(s+1)} | F_s]\} = \mu t + \sigma(1+t)B_{s/(s+1)}$

$\Longrightarrow$ No es Martingala.

y como sería martingala.