1. a) Observemos si es un producto interno. Sean $f, g, h \in V$:

i) $\langle f+h, g\rangle = \left(\int_0^1 (f(t)+h(t))\,dt\right)\left(\int_0^1 g(t)\,dt\right)$

$= \left(\int_0^1 f(t)\,dt + \int_0^1 h(t)\,dt\right)\left(\int_0^1 g(t)\,dt\right)$

$= \left(\int_0^1 f(t)\,dt\right)\left(\int_0^1 g(t)\,dt\right) + \left(\int_0^1 h(t)\,dt\right)\left(\int_0^1 g(t)\,dt\right)$

$= \langle f, g\rangle + \langle h, g\rangle$

ii) Sea, $a \in \mathbb{R}$:

$\langle af, g\rangle = \left(\int_0^1 af(t)\cdot dt\right)\left(\int_0^1 g(t)\,dt\right)$

$= a\left(\int_0^1 f(t)\,dt\right)\left(\int_0^1 g(t)\,dt\right) = a\langle f, g\rangle$

iii) $\langle f, g\rangle = \left(\int_0^1 f(t)\,dt\right)\left(\int_0^1 g(t)\,dt\right)$

$= \left(\int_0^1 g(t)\,dt\right)\left(\int_0^1 f(t)\,dt\right) = \langle g, f\rangle$

iv)

$\langle f, f\rangle = \left(\int_0^1 f(t)\,dt\right)^2$, Claramente lo anterior es positivo.

a) Supóngase que $f(t)=0$, luego:

$\langle f, f\rangle = \left(\int_0^1 f(t)\,dt\right)^2 = \left(\int_0^1 0\,dt\right)^2 = 0$

Supongamos $\langle f, f\rangle = 0$, luego:

$\langle f, f\rangle = \left(\int_0^1 f(t)\,dt\right)^2 = 0 \rightarrow \int_0^1 f(t)\,dt = 0$