Taller 1 David Plazas Escudero 201710005101

1\. a) $\left.\begin{array}{l} f(x,y) = xy+2x \\ \text{s.t} \quad 4x+2y=60 \end{array}\right\}$ $\mathcal{L}(\lambda,x,y) = xy+2x+\lambda[60-4x-2y]$

$\nabla\mathcal{L}(\lambda,x,y) \overset{!}{=} 0 \rightarrow \begin{bmatrix} 60-4x-2y \\ y+2-4\lambda \\ x-2\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{array}{l} (1) \\ (2) \rightarrow \lambda = (y+2)/4 \\ (3) \rightarrow \lambda = \dfrac{x}{2} \end{array}$ $\left.\begin{array}{l} \dfrac{y+2}{4} = \dfrac{x}{2}. \\ -4x+2y = -4 \ (4) \end{array}\right.$

$(1)+(4)$: $\begin{array}{r} -4x+2y=-4 \\ 4x+2y=60 \\ \hline 4y=56 \end{array} \rightarrow \boxed{y=14} \rightarrow$ en (4): $x = \dfrac{2y+4}{4} \rightarrow \boxed{x=8}$

en (3): $\boxed{\lambda=4}$

Hessiano con borde: $\tilde{H}(\lambda,x,y) = \begin{bmatrix} 0 & -4 & -2 \\ -4 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \rightarrow \det(\tilde{H}(\lambda,x,y)) = 4(2)+(-2)(-4) = 16 > 0.$

Luego, el punto crítico es máximo.

b) $\left.\begin{array}{l} f(x,y) = xy \\ \text{s.t} \quad x+y=6 \end{array}\right\}$ $\mathcal{L}(\lambda,x,y) = xy+\lambda[6-x-y]$

$\nabla\mathcal{L}(\lambda,x,y) \overset{!}{=} 0 \rightarrow \begin{bmatrix} 6-x-y \\ y-\lambda \\ x-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$ De (2) y (3): $x=\lambda=y$.
En (1) $6-2x=0$
$x=3=\lambda=y$.

Hessiano con borde:
$\tilde{H}(\lambda,x,y) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ $\det(\tilde{H}(\lambda,x,y)) = 1(1)-1(-1) = 2 > 0$

Luego, $x^*=y^*=\lambda^*=3$ es máximo.

c) $\left.\begin{array}{l} f(x,y) = x-3y-xy \\ \text{s.t} \quad x+y=6 \end{array}\right\}$ $\mathcal{L}(\lambda,x,y) = x-3y-xy+\lambda[6-x-y]$

$\nabla\mathcal{L}(\lambda,x,y) \overset{!}{=} 0 \rightarrow \begin{bmatrix} 6-x-y \\ 1-y-\lambda \\ 3-x-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$ $\left.\begin{array}{l} (2): \lambda = 1-y \\ (3): \lambda = 3-x \end{array}\right\}$ $\begin{array}{l} 1-y = 3-x \\ x-y = 2 \ (4) \end{array}$

$(4)+(1)$: $\begin{array}{r} x-y=2 \\ x+y=6 \\ \hline 2x=8 \end{array} \rightarrow \boxed{x=4}$ En (1): $4+y=6 \rightarrow \boxed{y=2}$ En (2) $\boxed{\lambda=-1}$

Hessiano con borde.
$\tilde{H}(\lambda,x,y) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ $\det(\tilde{H}(\lambda,x,y)) = 1(-1)-(1)(1) = -2 < 0.$

Luego $\begin{array}{l} x^*=4 \\ y^*=2 \\ \lambda=-1 \end{array}$ son un mínimo del problema.

d) $\left.\begin{array}{l} f(x,y) = 7-y+x^2 \\ \text{s.t} \quad x+y=0 \end{array}\right\}$ $\mathcal{L}(\lambda,x,y) = 7-y+x^2+\lambda[-x-y]$

$\nabla\mathcal{L}(\lambda,x,y) \overset{!}{=} 0 \rightarrow \begin{array}{l} -x-y=0 \rightarrow x=-y \rightarrow \boxed{y=1/2} \\ 2x-\lambda=0 \rightarrow 2x=-1 \rightarrow \boxed{x=-1/2} \\ -1-\lambda=0 \rightarrow \boxed{\lambda=-1} \end{array}$

Hessiano con borde.
$\tilde{H}(\lambda,x,y) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \rightarrow \det(\tilde{H}(\lambda,x,y)) = (-1)(2) = -2 < 0$

Luego, $x^*=-1/2$, $y^*=1/2$, $\lambda^*=-1$ es el mínimo del problema.

2. $U(x,y) = x_1 x_2$
s.t. $x_1 + \frac{x_2}{1+r} = B$

$$\mathcal{L}(\lambda, x_1, x_2) = x_1 x_2 + \lambda\left[B - x_1 - \frac{x_2}{1+r}\right]$$

$$\nabla \mathcal{L}(\lambda, x, y) \overset{!}{=} 0 \quad \begin{bmatrix} B - x_1 - \frac{x_2}{1+r} \\ x_2 - \lambda \\ x_1 - \frac{\lambda}{1+r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1) \quad \to \begin{array}{l} x_2 = \lambda \\ x_1 = \frac{x_2}{1+r} \quad (4) \end{array}$$

(4) en (1):

$$B - \frac{x_2}{1+r} - \frac{x_2}{1+r} = 0 \to \boxed{x_2 = \frac{B(1+r)}{2}}$$

$$\boxed{x_1 = \frac{B}{2}} \quad y \quad \boxed{\lambda = \frac{B(1+r)}{2}}$$

Hessiano con borde $\tilde{H}(\lambda, x_1, x_2) = \begin{bmatrix} 0 & -1 & -1/(1+r) \\ -1 & 0 & 1 \\ -1/(1+r) & 1 & 0 \end{bmatrix}$ $\det(\tilde{H}(\lambda, x_1, x_2)) = \frac{2}{1+r} > 0$

Luego, se genera una utilidad máxima en $x_1^* = \frac{B}{2}$, $x_2^* = \frac{B(1+r)}{2}$, $\lambda^* = \frac{B(1+r)}{2}$.

3. a) $z = f(x,y)$ con $\mathcal{L}(\lambda, x, y) = f(x,y) + \lambda[g(x,y) - C]$
s.t. $g(x,y) = C$

$$\nabla \mathcal{L}(\lambda, x, y) = \begin{bmatrix} g(x,y) - C \\ \frac{\partial f(x,y)}{\partial x} + \lambda \frac{\partial g(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} + \lambda \frac{\partial g(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \left\} \begin{array}{l} F_1(\lambda, x, y) = g(x,y) - C = 0 \\ F_2(\lambda, x, y) = f_x(x,y) + \lambda g_x(x,y) = 0 \\ F_3(\lambda, x, y) = f_y(x,y) + \lambda g_y(x,y) = 0 \end{array} \right] (\Delta)$$

El Hessiano con borde se puede calcular como el Jacobiano de (Δ).

$$\tilde{H}(\lambda, x, y) = \begin{bmatrix} 0 & g_x(x,y) & g_y(x,y) \\ g_x(x,y) & f_{xx}(x,y) + \lambda g_{xx}(x,y) & f_{xy}(x,y) + \lambda g_{xy}(x,y) \\ g_y(x,y) & f_{yx}(x,y) + \lambda g_{yx}(x,y) & f_{yy}(x,y) + \lambda g_{yy}(x,y) \end{bmatrix}$$

b) $z = f(x,y)$ con $\mathcal{L}(\lambda, x, y) = f(x,y) - \lambda G(x,y)$
s.t. $G(x,y) = 0$

$$\nabla \mathcal{L}(\lambda, x, y) = \begin{bmatrix} G(x,y) \\ f_x(x,y) - \lambda G_x(x,y) \\ f_y(x,y) - \lambda G_y(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_1(\lambda, x, y) \\ F_2(\lambda, x, y) \\ F_3(\lambda, x, y) \end{bmatrix}$$

Hessiano con borde: Jacobiano de

$$\tilde{H}(\lambda, x, y) = \begin{bmatrix} 0 & -G_x(x,y) & -G_y(x,y) \\ -G_x(x,y) & f_{xx}(x,y) - \lambda G_{xx}(x,y) & f_{xy}(x,y) - \lambda G_{xy}(x,y) \\ -G_y(x,y) & f_{yx}(x,y) - \lambda G_{yx}(x,y) & f_{yy}(x,y) - \lambda G_{yy}(x,y) \end{bmatrix}$$

4. a) $\max f(x,y) = x$ $\quad \mathcal{L}(\lambda, x, y) = x + \lambda[1 - x^2 - y^2]$

s.t. $x^2 + y^2 \leq 1$

$x, y \geq 0$

KKT: $1 - 2\lambda x \leq 0;\ x \geq 0;\ x[1 - 2\lambda x] = 0 \quad (1)$

$-2\lambda y \leq 0;\ y \geq 0;\ y[-2\lambda y] = 0 \quad (2)$

$1 - x^2 - y^2 \geq 0;\ \lambda \geq 0;\ \lambda[1 - x^2 - y^2] = 0 \quad (3).$

· De (2): $\lambda = 0 \vee y = 0$

· Supongamos $\lambda = 0$ y $x, y \neq 0$: De (1): $x = 0$ $(\rightarrow\leftarrow)$.

· Supongamos $\boxed{y = 0}$ y $\lambda, x \neq 0$: De (3): $1 - x^2 = 0 \rightarrow x = \pm 1$, pero $x \geq 0$ de (1), luego $\boxed{x = 1}$. En (1): $1 - 2\lambda = 0 \rightarrow \boxed{\lambda = 1/2}$.

Hay máximo global en $x^* = 1$, $y^* = 0$ y $\lambda^* = 1/2$

b) $\min f(x,y) = x$ $\quad \min f(x,y) = x$ $\quad \mathcal{L}(\lambda, x, y) = x + \lambda[-1 - y + x^2]$

s.t. $x^2 - y \geq 1$ $\quad$ s.t. $y - x^2 \leq -1$

$x, y \geq 0.$ $\quad$ $x, y \geq 0$

KKT

$1 + \lambda(2x) \geq 0;\ x \geq 0;\ x[1 + 2x\lambda] = 0 \quad (1)$

$-\lambda \geq 0;\ y \geq 0;\ y[-\lambda] = 0 \quad (2)$

$-1 - y + x^2 \geq 0;\ \lambda \leq 0;\ \lambda[-1 - y + x^2] = 0 \quad (3)$

· De (2): $y = 0 \vee \lambda = 0.$

· Supongamos $\lambda = 0$ y $x, y \neq 0$. De (1) $x = 0$. $(\rightarrow\leftarrow)$.

· Supongamos $\boxed{y = 0}$ y $x, \lambda \neq 0$. De (3) $-1 + x^2 = 0 \rightarrow x = \pm 1$, pero $x \geq 0$ de (1). Luego $\boxed{x = 1}$. En (1). $1 + 2\lambda = 0 \rightarrow \boxed{\lambda = -1/2}$

Hay mínimo global del problema en $x^* = 1$, $y^* = 0$ y $\lambda^* = -1/2$.

5. $Q(K, L) = A[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}$, $A > 0$, $0 < \delta < 1$ y $\rho > -1$, $\rho \neq 0$.

a) $Q(tK, tL) = A[\delta(tK)^{-\rho} + (1-\delta)(tL)^{-\rho}]^{-1/\rho}$

$= A[\delta t^{-\rho} K^{-\rho} + (1-\delta) t^{-\rho} L^{-\rho}]^{-1/\rho}$

$= A[t^{-\rho}]^{-1/\rho}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}$

$= tQ(K, L) \Rightarrow$ Luego es una función homogénea de grado 1.

b) $\dfrac{\partial Q}{\partial L} = \dfrac{-A}{\cancel{\rho}}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{\frac{-(\rho+1)}{\rho}}(-\cancel{\rho} L^{-\rho-1}(1-\delta))$

$= -A(1-\delta)[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{\frac{-(\rho+1)}{\rho}} L^{-(\rho+1)}$

$= \dfrac{A^{\rho+1}}{A^{\rho}}(1-\delta)\left\{\dfrac{[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}}{L}\right\}^{\rho+1}$

$= \dfrac{1-\delta}{A^{\rho}}\left(\dfrac{Q(K,L)}{L}\right)^{\rho+1} > 0$

$\dfrac{\partial Q}{\partial K} = \dfrac{-A}{\cancel{\rho}}[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{\frac{-(\rho+1)}{\rho}}(-\cancel{\rho} K^{-\rho-1}\delta)$

$= \dfrac{A^{\rho+1}}{A^{\rho}}\delta\left\{\dfrac{[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho}}{K}\right\}^{\rho+1}$

$= \dfrac{\delta}{A^{\rho}}\left(\dfrac{Q(K,L)}{K}\right)^{\rho+1} > 0$

Como $\dfrac{\partial Q}{\partial L}$ y $\dfrac{\partial Q}{\partial K}$ son ambas positivas, la producción es directamente proporcional al capital y a la labor.

c) $Q(K,L) = Q_0 \longrightarrow F(K,L;Q_0) = Q(K,L) - Q_0$

$$= A[\delta K^{-\rho} + (1-\delta)L^{-\rho}]^{-1/\rho} - Q_0$$

Por derivación implícita: $\dfrac{dK}{dL} = -\dfrac{\partial F/\partial L}{\partial F/\partial K} = -\dfrac{\partial Q/\partial L}{\partial Q/\partial K}$

Luego,

$$\frac{dK}{dL} = \frac{\frac{(1-\delta)}{A^\rho}\left(\frac{Q}{L}\right)^{\rho+1}}{\frac{\delta}{A^\rho}\left(\frac{Q}{K}\right)^{\rho+1}} = -\frac{1-\delta}{\delta}\left(\frac{K}{L}\right)^{\rho+1}$$

Ahora $\dfrac{d^2K}{dL^2} = -\dfrac{1-\delta}{\delta}\dfrac{d}{dL}\left(\left(\dfrac{K}{L}\right)^{\rho+1}\right) = -\left(\dfrac{1-\delta}{\delta}\right)K^{\rho+1}\dfrac{d}{dL}\left(L^{-\rho-1}\right) = -\dfrac{(1-\delta)K^{\rho+1}}{\delta}(-\rho-1)L^{-\rho-2}$

$$= \frac{(1-\delta)(\rho+1)}{\delta L}\left(\frac{K}{L}\right)^{\rho+1} > 0.$$

d) Teorema de Euler para funciones homogéneas de grado 1:

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = Q$$

Derivando respecto a K: $\dfrac{\partial Q}{\partial K} + K\dfrac{\partial^2 Q}{\partial K^2} + L\dfrac{\partial^2 Q}{\partial K \partial L} = \dfrac{\partial Q}{\partial K} \rightarrow \dfrac{\partial^2 Q}{\partial K^2} = -\dfrac{L}{K}\dfrac{\partial^2 Q}{\partial K \partial L}$ (1)

Derivando respecto a L: $K\dfrac{\partial^2 Q}{\partial K \partial L} + \dfrac{\partial Q}{\partial L} + L\dfrac{\partial^2 Q}{\partial L^2} = \dfrac{\partial Q}{\partial L} \rightarrow \dfrac{\partial^2 Q}{\partial L^2} = -\dfrac{K}{L}\dfrac{\partial^2 Q}{\partial L \partial K}$ (2)

Ahora, se sabe que

$\dfrac{\partial Q}{\partial K} = \dfrac{\delta}{A^\rho}\left(\dfrac{Q}{K}\right)^{\rho+1} \leftarrow$ derivemos respecto a L y hallemos la cruzada.

$\dfrac{\partial^2 Q}{\partial L \partial K} = \dfrac{\delta(\rho+1)}{A^\rho K^{\rho+1}}Q^\rho \cdot \dfrac{\partial Q}{\partial L} \rightarrow$ Como $\dfrac{\partial Q}{\partial L} > 0 \Rightarrow \dfrac{\partial^2 Q}{\partial L \partial K} = \dfrac{\partial^2 Q}{\partial K \partial L} > 0$

Luego, en (1) y (2): $\dfrac{\partial^2 Q}{\partial K^2} = -\dfrac{L}{K}\dfrac{\partial^2 Q}{\partial K \partial L} < 0$

$$\frac{\partial^2 Q}{\partial L^2} = -\frac{K}{L}\frac{\partial^2 Q}{\partial L \partial K} < 0$$

e) Matriz Hessiana.

$$H(K,L) = \begin{bmatrix} \frac{\partial^2 Q}{\partial K^2} & \frac{\partial^2 Q}{\partial K \partial L} \\ \frac{\partial^2 Q}{\partial L \partial K} & \frac{\partial^2 Q}{\partial L^2} \end{bmatrix}$$

Menores principales: $D_1 = \left[\dfrac{\partial^2 Q}{\partial K^2}\right]$

Luego, $\det(D_1) = \dfrac{\partial^2 Q}{\partial K^2} < 0$

$D_2 = H(K,L)$

Luego, $\det(D_2) = \det(H(K,L))$

$$= \left(\frac{\partial^2 Q}{\partial K^2}\right)\left(\frac{\partial^2 Q}{\partial L^2}\right) - \left(\frac{\partial^2 Q}{\partial K \partial L}\right)^2 \quad (3)$$

Reemplazando (1) y (2) en (3)

$$\det(D_2) = \left(\frac{-L}{K}\frac{\partial^2 Q}{\partial K \partial L}\right)\left(\frac{-K}{L}\frac{\partial^2 Q}{\partial L \partial K}\right) - \left(\frac{\partial^2 Q}{\partial K \partial L}\right)^2$$

$$= 0.$$

Luego, se cumple que los menores principales distintos de 0 tienen el mismo signo de $(-1)^k$

$\rightarrow (-1)^k \det(D_k) \geq 0.$

Luego, $Q(K,L)$ es cóncava.