WI4205 - 2022/23
Applied Finite Elements
Assignment 1.3
Deadline - 23:59, May 12, 2023

Instructions and assessment criteria to keep in mind:

- A submission for Assignment 1.3 is **required for passing** WI4205.
- This is an **individual/group assignment**. You can work by yourself, but if you wish you can also work in groups of maximum two and submit a joint report.
- The reports need to be **typed in LaTeX or Word**.
- The **deadline** for uploading your solutions to Brightspace is 23:59, May 12, 2023.
- Grading of late submissions will be done as per the **grading protocol posted on Brightspace**; make sure you are familiar with it.
- Provide **clear and motivated answers** to the questions. No/reduced points will be awarded if your solutions are unaccompanied by explanations.

Verification of the 1D Finite Element Method (5 points)

In this section, you will use your finite element implementation from the previous assignment (or the provided script) and run it multiple times to check that certain verification tests (based on the method of manufactured solutions) are passed.

Method of manufactured solutions
Choose $\Omega = (0, 1)$ and consider the following continuous and discrete weak problems:

$$
\begin{aligned}
(\mathrm{W}) \ &: \quad \text{find } u \in \mathcal{S} \text{ s.t. } \forall w \in \mathcal{W}, \quad B(u, w) = L(w) \ , \\
(\mathrm{W}_h) \ &: \quad \text{find } u_h \in \mathcal{S}_h \text{ s.t. } \forall w_h \in \mathcal{W}_h, \quad B(u_h, w_h) = L(w_h) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
B(u, w) := \int_\Omega w_{,x} u_{,x} \, dx \ , \quad L(w) &:= \int_\Omega f w \, dx \ , \\
\mathcal{S} &:= \{ w \in H^1(\Omega) \ : \ w(0) = g_0, \ w(1) = g_1 \} \ , \\
\mathcal{W} &:= \{ w \in H^1(\Omega) \ : \ w(0) = w(1) = 0 \} \ . \\
\mathcal{S}_h &:= \{ w \in \mathcal{F}(p, k; \mathcal{T}_h) \ : \ w(0) = g_0, \ w(1) = g_1 \} \ , \\
\mathcal{W}_h &:= \{ w \in \mathcal{F}(p, k; \mathcal{T}_h) \ : \ w(0) = w(1) = 0 \} \ .
\end{aligned}
$$

A. (2 points) Assume that you know the exact solution u_e to the weak problem. Provide pseudocode for how you can reuse the element-by-element assembly of integrals to compute the following error measures:

$$
\|u_h - u_e\|_0 \ , \quad \|u_h - u_e\|_1 \ .
$$

B. (1 point) Let $m = 4$, $p = 2$, $k = 1$, $n_q = 3$, $x_i = iL/m$. Choose a function u_e that is a non-constant polynomial contained in $\mathcal{F}(p, k; \mathcal{T}_h)$.
Note: Specify what this choice is explicitly, e.g., by saying that you pick $u_e = x^2 + 3x - 6$ (but do not use this specific example that I have given).

- (0.5 point) Show how f, g_0 and g_1 should be chosen such that u_e is an exact solution to the weak problem at hand.
- (0.5 point) Verify that the "approximate" solution your finite element code computes is exactly equal to u_e.
(Present plots + 1-2 line explanation in your report.)

C. (2 points) Choose a smooth (non-zero) non-polynomial function u_e defined on Ω.
Note: Specify what this choice is explicitly, e.g., by saying that you pick $u_e = \sin(x)$ (but do not use this specific example that I have given.)

- (0.5 point) Show how f, g_0 and g_1 should be chosen such that u_e is an exact solution to the weak problem at hand.
- (1.5 points) Verify that the approximate solution your finite element code computes converges to u_e with the right order for $p = 2$, $k = 1$, $n_q = 3$. This is called a convergence study and can be performed in the following steps:
  - First, a 0-level mesh $\mathcal{T}_h^{(0)}$ is chosen. For your computations, pick $m^{(0)} = 4$, $x_i^{(0)} = iL/m$.
  - Next, define the k-th level mesh $\mathcal{T}_h^{(k)}$, $k \geq 1$, as:

$$m^{(k)} = 2^k m^{(0)} , \quad x_i^{(k)} = iL/m^{(k)} .$$

  - Next, for $k = 0, 1, \ldots, 5$:
    * Compute the solution $u_h^{(k)}$ by choosing the finite element space $\mathcal{F}(p, k; \mathcal{T}_h^{(k)})$.
    * Compute the following errors:

$$e_0^{(k)} := \|u_h^{(k)} - u_e\|_0 , \quad e_1^{(k)} := \|u_h^{(k)} - u_e\|_1 .$$

  - Once done, verify that $e_0^{(k)}$ and $e_1^{(k)}$ converge with the expected power of the mesh size $h^{(k)} := 1/m^{(k)}$ as per the theory shown in class.
(Present appropriate plots for $h^{(k)}$ vs $e_j^{(k)}$, $j = 0, 1$, along with a brief explanation in your report saying what the expected rate of convergence is, and how did you verify that your solution shows this rate.)

A 1D Mixed Finite Element Method (10 points)

In this section, you will extend your finite element implementation from the previous assignment (or the provided script) to solved a mixed form of the Laplace problem. Specifically, we can decompose the Laplace problem:

$$\begin{aligned} -u_{,xx} &= f , \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

into the following first-order system:

$$\begin{aligned} \sigma + u_{,x} &= 0 , \\ \sigma_{,x} &= f , \\ u|_{\partial\Omega} &= 0 . \end{aligned}$$

Introducing a new unknown σ allows us to develop the following weak formulation where we can reduce the required regularity for u: find $(\sigma, u) \in \mathcal{S}_\sigma \times \mathcal{S}_u$ such that

$$\begin{aligned} (\tau, \sigma) - (\tau_{,x}, u) &= 0 \ , \qquad \tau \in \mathcal{W}_\sigma \\ (v, \sigma_{,x}) &= (v, f) \ , \qquad v \in \mathcal{W}_u \ , \end{aligned} \tag{1}$$

where

$$\begin{aligned} \mathcal{S}_u = \mathcal{W}_u &:= L^2(\Omega) \ , \\ \mathcal{S}_\sigma = \mathcal{W}_\sigma &:= H^1(\Omega) \ . \end{aligned}$$

The corresponding discrete problem is built in the usual way: find $(\sigma_h, u_h) \in \mathcal{S}_{\sigma,h} \times \mathcal{S}_{u,h}$ such that

$$\begin{aligned} (\tau_h, \sigma_h) - (\tau_{h,x}, u_h) &= 0 \ , \qquad \tau_h \in \mathcal{W}_{\sigma,h} \\ (v_h, \sigma_{h,x}) &= (v_h, f) \ , \qquad v_h \in \mathcal{W}_{u,h} \ , \end{aligned} \tag{2}$$

where

$$\begin{aligned} \mathcal{S}_{u,h} = \mathcal{W}_{u,h} &:= \mathcal{F}(p_u, k_u; \mathcal{T}_h) \ , \\ \mathcal{S}_{\sigma,h} = \mathcal{W}_{\sigma,h} &:= \mathcal{F}(p_\sigma, k_\sigma; \mathcal{T}_h) \ . \end{aligned}$$

That is, we look for discrete approximations to σ and u in finite element spaces that are built on the same mesh but with possibly different degrees and smoothness.

The problem in equation (1) is well-posed but, as you will see in this section, one can easily construct discretizations for it that are unstable.

D. (0.5 point) In equation (1), the trial spaces for u and σ do not seem to account for the boundary condition. Explain why this is OK.

E. (0.5 point) Derive the exact u and σ that solve the problem for the forcing function $f = \pi^2 \cos(\pi(x - 0.5))$.

F. (5 points) Provide pseudo-code where you show how you generalize your existing FEM implementation to tackle the mixed problem in equation (2).

G. (3 points) Use the following three finite element discretizations and plot your computed u and σ for each (with $m = 14$ for all three).
Hint: In the following, one of the choices will lead to an accurate solution (stable method), two will lead to highly inaccurate solutions (unstable methods). In fact, one of the following will even lead to a singular matrix.

$$\mathcal{T}_h : \quad 0 = x_0 < x_1 < \cdots < x_m = 1 \ , \qquad x_i = i/m$$

- Choice 1:

$$\begin{aligned} \mathcal{S}_{u,h} &: \ p_u = 0 \ , \ k_u = -1 \ , \\ \mathcal{S}_{\sigma,h} &: \ p_\sigma = 1 \ , \ k_\sigma = 0 \ . \end{aligned} \tag{3}$$

- Choice 2:

$$\begin{aligned} \mathcal{S}_{u,h} &: \ p_u = 1 \ , \ k_u = 0 \ , \\ \mathcal{S}_{\sigma,h} &: \ p_\sigma = 1 \ , \ k_\sigma = 0 \ . \end{aligned} \tag{4}$$

- Choice 3:

$$\begin{aligned} \mathcal{S}_{u,h} &: \ p_u = 0 \ , \ k_u = -1 \ , \\ \mathcal{S}_{\sigma,h} &: \ p_\sigma = 2 \ , \ k_\sigma = 0 \ . \end{aligned} \tag{5}$$

H. (1 point) Explain whether the choices of finite dimensional spaces above correspond to conforming finite element methods or a non-conforming finite element methods.