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Programme: MSc in Applied Mathematics

WI4205 Applied Finite Elements
Assignment 1.2

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1 The Finite Element Space

A. The procedures to create the mesh and the parametric map are presented in Algorithm 1 and 2 respectively.

Algorithm 1 `create_mesh`

Require: $\mathbf{x} = [x_0, \ldots, x_m] \in \mathbb{R}^{m+1}$ ▷ Grid point vector
1: $m \leftarrow \dim(\mathbf{x}) - 1$
2: $\texttt{elements} \leftarrow \mathbf{0} \in \mathbb{R}^{2\times m}$
3: $\texttt{elements}_{1,j} \leftarrow x_{j-1}$ for $j = 1, \ldots, m$
4: $\texttt{elements}_{2,j} \leftarrow x_j$ for $j = 1, \ldots, m$
Ensure: $\texttt{mesh} = (m, \texttt{elements})$

Algorithm 2 `create_param_map`

Require: $\texttt{mesh} = (m, \texttt{elements})$ ▷ $m \in \mathbb{N}$ (number of elements)
▷ $\texttt{elements} \in \mathbb{R}^{2\times m}$ (elements matrix)
1: Define $\texttt{map} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(\xi, x_1, x_2) \mapsto x_1 + \xi(x_2 - x_1)$ ▷ Map from reference element
2: $\texttt{map\_derivatives} \leftarrow \mathbf{0} \in \mathbb{R}^m$
$\texttt{map\_derivatives}_j \leftarrow \texttt{elements}_{2,j} - \texttt{elements}_{1,j}$ for $j = 1, \ldots, m$
3: $\texttt{imap\_derivatives} \leftarrow \mathbf{0} \in \mathbb{R}^m$
$\texttt{imap\_derivatives}_j \leftarrow \frac{1}{\texttt{map\_derivatives}_j}$ for $j = 1, \ldots, m$
Ensure: $\texttt{param\_map} = (\texttt{map}, \texttt{map\_derivatives}, \texttt{imap\_derivatives})$

F. The resulting u_h and its derivative are presented in Fig. 1.

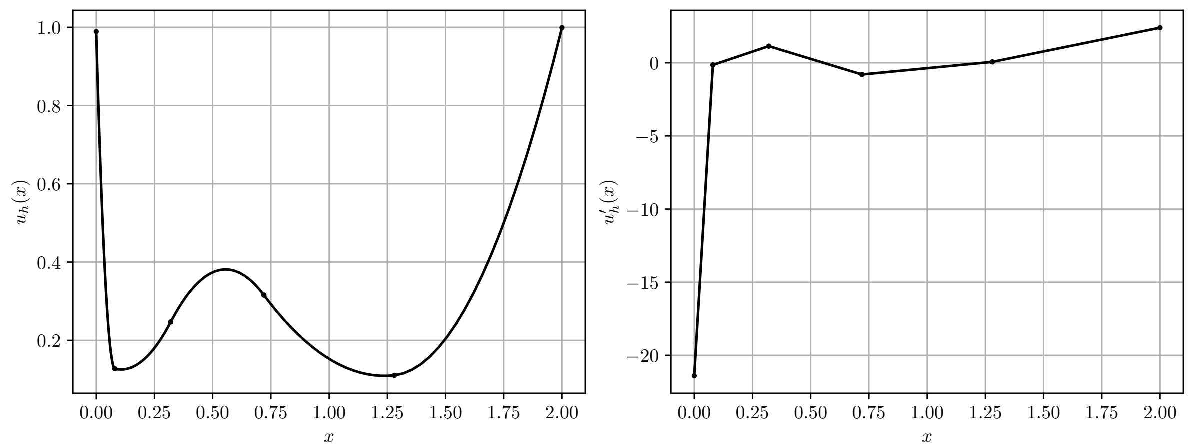

Figure 1: Resulting u_h and u'_h on Ω.

G. The finite element basis functions form a partition of unity if their sum is 1 in every element

$$\sum_{j=1}^{n} N_j(x) = 1 \iff \sum_{j=1}^{n} N_j(x)|_{\Omega_i} = 1, \qquad i = 1, 2, \ldots, m.$$

Taking into account that only some basis functions are non-zero in each element

$$\sum_{j=1}^{n} N_j(x)|_{\Omega_i} = \sum_{s=1}^{p+1} N_{j_s}(x)|_{\Omega_i}, \qquad i = 1, 2, .., m.$$

Using the extraction coefficients, the reference element basis functions, and reorganizing the summations we get that

$$\begin{aligned}
\sum_{s=1}^{p+1} N_{j_s}(x)|_{\Omega_i} &= \sum_{s=1}^{p+1}\sum_{k=1}^{p+1} e_{ski}\tilde{N}_k(\phi_i^{-1}(x)) \\
&= \sum_{k=1}^{p+1}\left(\tilde{N}_k(\phi_i^{-1}(x))\left(\sum_{s=1}^{p+1} e_{ski}\right)\right) \\
&= \sum_{k=1}^{p+1} \tilde{N}_k(\phi_i^{-1}(x)) = 1, \qquad i = 1, 2, .., m,
\end{aligned}$$

where we used the fact that $\sum_{s=1}^{p+1} e_{ski} = 1$ and that the basis functions on the reference element are a partition of unity.

2 The Finite Element Problem.

H. The procedure to assemble the FE problem (matrices A and b) is presented in Algorithm 3. In this section, the operator $\dim(\cdot)$ on a vector represents its length, e.g. if $v \in \mathbb{R}^k$ then $\dim(v) = k$.

Algorithm 3 `assemble_fe_problem`

Require: $\texttt{mesh} = (m, \texttt{elements})$
Require: $\texttt{space} = (n, \texttt{supported\_bases}, \texttt{extraction\_coefficients})$
Require: $\texttt{ref\_data} = (\texttt{deg}, \texttt{reference\_element}, \texttt{evaluation\_points}, \texttt{quadrature\_weights},$
$\texttt{reference\_basis}, \texttt{reference\_basis\_derivatives})$
Require: $\texttt{param\_map} = (\texttt{map}, \texttt{map\_derivatives}, \texttt{imap\_derivatives})$
Require: $\texttt{problem\_B}(\cdot,\cdot,\cdot,\cdot,\cdot)$ ▷ Bilinear weak form
Require: $\texttt{problem\_L}(\cdot,\cdot,\cdot)$ ▷ Linear form
Require: $[g_0, g_L]$ ▷ Boundary conditions

$\bar{A} \leftarrow \mathbf{0} \in \mathbb{R}^{n\times n}$
$\bar{b} \leftarrow \mathbf{0} \in \mathbb{R}^{n}$
for $\ell \in \{1,\dots,m\}$ **do** ▷ Iterate over all elements
  $x_1 \leftarrow \texttt{elements}_{1,\ell}$
  $x_2 \leftarrow \texttt{elements}_{2,\ell}$
  $q \leftarrow \dim(\texttt{evaluation\_points})$
  Construct $X \in \mathbb{R}^q$ where $X_i = \texttt{map}\left(\texttt{evaluation\_points}_i, x_1, x_2\right)$, for $i = 1,\dots,q$
  $i' \leftarrow 0$
  for $i \in \{\texttt{supported\_bases}_{rk} : r = \ell\}$ **do** ▷ Non-zero basis functions (l-th row)
    Construct $\mathbf{e} \in \mathbb{R}^{\texttt{deg}+1}$ where $\mathbf{e}_k = \texttt{extraction\_coefficients}_{\ell,i',k}$ for $k = 1,\dots,\texttt{deg}+1$
    $N_i \leftarrow \mathbf{e}^T\texttt{reference\_basis}$ ▷ Basis function on each evaluation point
    $dN_i \leftarrow \texttt{imap\_derivatives}_\ell\left(\mathbf{e}^T\texttt{reference\_basis\_derivatives}\right)$
    $L \leftarrow \texttt{problem\_L}(X, N_i, dN_i)$ ▷ Point-wise evaluation, $L \in \mathbb{R}^q$
    $\bar{b}_i \leftarrow \bar{b}_i + \texttt{map\_derivates}_\ell\left(L^T\texttt{quadrature\_weights}\right)$
    $j' \leftarrow 0$
    for $j \in \{\texttt{supported\_bases}_{rk} : r = \ell\}$ **do** ▷ l-th row
      Construct $\tilde{\mathbf{e}} \in \mathbb{R}^{\texttt{deg}+1}$ where $\tilde{\mathbf{e}}_k = \texttt{extraction\_coefficients}_{\ell,i',k}$ for $k = 1,\dots,\texttt{deg}+1$
      $N_j \leftarrow \tilde{\mathbf{e}}^T\texttt{reference\_basis}$
      $dN_j \leftarrow \texttt{imap\_derivatives}_\ell\left(\tilde{\mathbf{e}}^T\texttt{reference\_basis\_derivatives}\right)$
      $B \leftarrow \texttt{problem\_B}(X, N_i, dN_i, N_j, dN_j)$
      $\bar{A}_{i,j} \leftarrow \bar{A}_{i,j} + \texttt{map\_derivatives}_\ell\left(B^T\texttt{quadrature\_weights}\right)$
      $j' \leftarrow j' + 1$
    $i' \leftarrow i' + 1$
Construct $b \in \mathbb{R}^{n-2}$ where $b_i = \bar{b}_{i+1} - g_0\bar{a}_{i+1,1} - g_L\bar{a}_{i+1,n}$ for $i = 1,\dots,n-1$
Construct $A \in \mathbb{R}^{(n-2)\times(n-2)}$ where $a_{i,j} = \bar{a}_{i+1,j+1}$ for $i,j = 1,\dots,n-1$

Ensure: $A \in \mathbb{R}^{(n-2)\times(n-2)}$
Ensure: $b \in \mathbb{R}^{n-2}$

K. Saying that $\tilde{A}$ has zero rows or columns is equivalent, since, as we will prove in the next exercise, $B(\cdot,\cdot)$ is symmetric. We then have

$$\sum_{i=1}^{n} B(N_j, N_i) = 0, \qquad j = 1, 2, .., n.$$

Using the bilinearity of $B(\cdot,\cdot)$ and the fact that the basis functions form a partition of unity, we have

$$
\begin{aligned}
\sum_{i=1}^{n} B(N_j, N_i) =& B\left(N_j, \sum_{i=1}^{n} N_i\right) \\
=& \int_\Omega N_{j,x}\left(\sum_{i=1}^{n} N_{i,x}\right) dx \\
=& \int_\Omega N_{j,x}\left(\sum_{i=1}^{n} N_{i}\right)_{,x} dx \\
=& \int_\Omega N_{j,x}\,(1)_{,x}\, dx = 0, \qquad j = 1, 2, \ldots, n.
\end{aligned}
$$

L. First, note that $B(\cdot,\cdot)$ is symmetric

$$
B(N_i, N_j) = \int_\Omega N_{i,x} N_{j,x} dx = \int_\Omega N_{j,x} N_{i,x} = B(N_j, N_i),
$$

therefore,

$$
A_{ij} = B(N_{i+1}, N_{j+1}) = B(N_{j+1}, N_{i+1}) = A_{ji},
$$

from where we can conclude that $A = A^T$.

M. We know that $B(\cdot,\cdot)$ is coercive for functions in $H_0^1(\Omega)$, i.e.

$$
B(\mathbf{v},\mathbf{v}) \geq \alpha \|\mathbf{v}\|^2_{H^1(\Omega)} > 0, \quad \forall \mathbf{v} \in H_0^1(\Omega).
$$

Note also that our finite element test space is a subspace of $H_0^1(\Omega)$

$$
\mathcal{V}^h := \{\mathbf{v}_h \in \mathcal{F}(p, k : \Delta) :\ \mathbf{v}_h(0) = \mathbf{v}_h(L) = 0\} \subset H_0^1(\Omega).
$$

Therefore, for every $\mathbf{v}_h \in \mathcal{V}^h$, $B(\cdot,\cdot)$ is coercive. Every function of $\mathcal{V}^h$ can be expressed as $\mathbf{v}_h = \sum_{i=2}^{n-1} \phi_i N_i$ and similarly $\sum_{i=2}^{n-1} \phi_i N_i \in \mathcal{V}^h$, $\forall \boldsymbol{\phi} \in \mathbb{R}^{n-2}$. Using this, the bilinearity of $B(\cdot,\cdot)$ and the definition of A yields

$$
B(\mathbf{v}_h, \mathbf{v}_h) = \sum_{i=2}^{n-1}\sum_{j=2}^{n-1} B(\phi_i N_i, \phi_j N_j) = \sum_{i=2}^{n-1}\sum_{j=2}^{n-1} \phi_i B(N_i, N_j)\phi_j = \boldsymbol{\phi}^T A \boldsymbol{\phi} > 0, \quad \boldsymbol{\phi} \in \mathbb{R}^{n-2}\backslash\{0\}.
$$

This implies that A is a positive definite matrix, and only the zero vector is part of its null space.

N. The resulting u_h and its derivative are presented in Fig. 2.

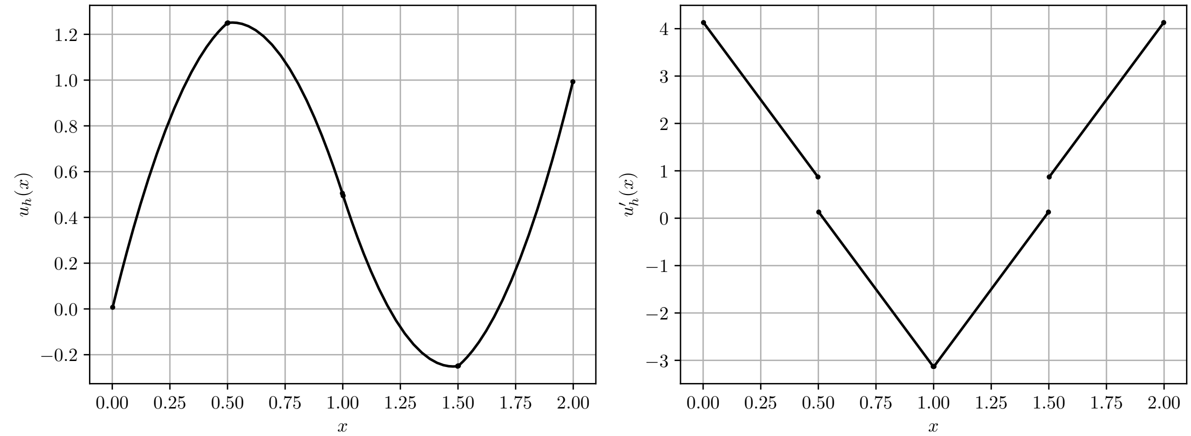

Figure 2: Resulting u_h and u_h' on Ω.

O. The resulting u_h and its derivative are presented in Fig. 3.

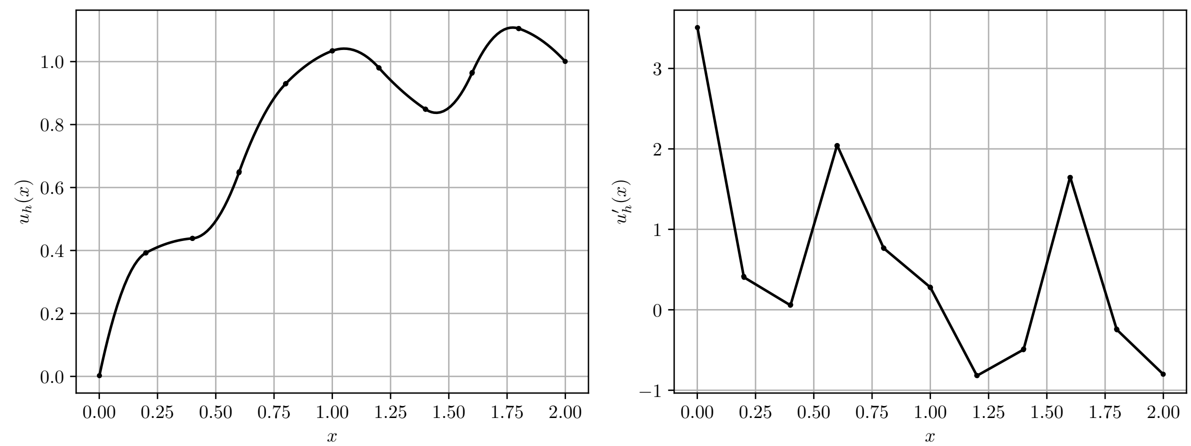

Figure 3: Resulting u_h and u_h' on Ω.