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Programme: MSc in Applied Mathematics

**WI4205 Applied Finite Elements**
Assignment 1.4

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In this work, $\odot$ represents the element-wise (Hadamard) product of two matrices $A, B \in \mathbb{R}^{n\times m}$, defined as $A \odot B = C \in \mathbb{R}^{n\times m}$, with $c_{ij} = a_{ij}b_{ij}$.

# 1 The finite element space

A. The pseudocode for `create_geometric_map` is presented in Algorithm 1.

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**Algorithm 1** `create_geometric_map`

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**Require:** $\texttt{fe\_geometry} = (m, \texttt{map\_coefficients})$
**Require:** $\texttt{ref\_data} = (\mathbf{p}, \texttt{evaluation\_points}, \texttt{quadrature\_weights}, \texttt{reference\_basis}, \texttt{reference\_basis\_derivatives})$

$n_q \leftarrow (\dim(\texttt{evaluation\_points}))^2$
$p \leftarrow (\mathbf{p}_1 + 1)(\mathbf{p}_2 + 1)$
Initialize $\texttt{map} \leftarrow \mathbf{0} \in \mathbb{R}^{n_q\times 2\times m}$
Initialize $\texttt{map\_derivatives} \leftarrow \mathbf{0} \in \mathbb{R}^{n_q\times 4\times m}$
Initialize $\texttt{imap\_derivates} \leftarrow \mathbf{0} \in \mathbb{R}^{n_q\times 4\times m}$
**for** $i \in \{1, \dots, m\}$ **do**
  **for** $j \in \{1, 2\}$ **do**
    Construct $\mathbf{y} \in \mathbb{R}^{n_q}$ where $\mathbf{y}_\ell = \texttt{map\_coefficients}_{\ell,j,i}$ for $\ell = 1, \dots, n_q$
    $\texttt{map}_{\ell,j,i} \leftarrow (\texttt{reference\_basis} \cdot \mathbf{y})_\ell$ for $\ell = 1, \dots, n_q$
    **for** $k \in \{1, 2\}$ **do**
      Initialize $\mathbf{D} \leftarrow \mathbf{0} \in \mathbb{R}^{p\times n_q}$
      $\mathbf{D}_{\ell,\ell'} \leftarrow \texttt{reference\_basis\_derivatives}_{\ell,\ell',k}$ for $\ell = 1, \dots, p$ and $\ell' = 1, \dots, n_q$
      $\texttt{map\_derivatives}_{\ell,2k+j,i} \leftarrow (\mathbf{D} \cdot \mathbf{y})_\ell$ for $\ell = 1, \dots, n_q$
Construct the matrices $\mathbf{K}_i \leftarrow \texttt{map\_derivatives}_{\ell,i,\ell'}$ for $i = 1, \dots, 4$ with $\ell = 1, \dots, n_q$, $\ell' = 1, \dots, m$
$\Delta \leftarrow \mathbf{K}_1 \odot \mathbf{K}_4 - \mathbf{K}_2 \odot \mathbf{K}_3$
$v \leftarrow [4, 3, 2, 1]^T$
$w \leftarrow [1, -1, -1, 1]^T$
**for** $i \in \{1, 2, 3, 4\}$ **do**
  $\texttt{imap\_derivatives}_{\ell,i,\ell'} \leftarrow \frac{w_i}{\Delta_{\ell,\ell'}} \texttt{map\_derivatives}_{\ell,v_i,\ell'}$ for $\ell = 1, \dots, n_q$, $\ell' = 1, \dots, m$

**Ensure:** $\texttt{map} \in \mathbb{R}^{n_q\times 2\times m}$
**Ensure:** $\texttt{map} \in \mathbb{R}^{n_q\times 2\times m}$
**Ensure:** $\texttt{map\_derivatives} \in \mathbb{R}^{n_q\times 4\times m}$

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## 2 The finite element problem

**Algorithm 2** `assemble_fe_problem`

E. **Require:** $\texttt{fe\_space} = (n, \texttt{boundary\_bases}, \textbf{support\_and\_extraction})$,
where $\textbf{support\_and\_extraction}^\ell = (\texttt{supported\_bases}^\ell, \texttt{extraction\_coefficients}^\ell)$ for $\ell = 1, \ldots, m$

**Require:** $\texttt{ref\_data} = (\mathbf{p}, \texttt{evaluation\_points}, \texttt{quadrature\_weights}, \texttt{reference\_basis}, \texttt{reference\_basis\_derivatives})$

**Require:** $\texttt{geom\_map} = (\texttt{map}, \texttt{map\_derivatives}, \texttt{imap\_derivatives})$

**Require:** $\texttt{fe\_geometry} = (m, \texttt{map\_coefficients})$

**Require:** $\texttt{problem\_B}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$

**Require:** $\texttt{problem\_L}(\cdot,\cdot,\cdot,\cdot)$

Initialize $\bar{A} \leftarrow \mathbf{0} \in \mathbb{R}^{n \times n}$
Initialize $\bar{b} \leftarrow \mathbf{0} \in \mathbb{R}^{n}$
**for** $\ell \in \{1, \ldots, m\}$ **do**
  Construct the matrix $X \in \mathbb{R}^{n_q \times 2}$ as $X_{ij} \leftarrow \texttt{map}_{i,j,\ell}$, for $i = 1, \ldots, n_q$, $j = 1, 2$
  Construct the vectors $\mathbf{v}_i \leftarrow \texttt{map\_derivatives}_{j,i,\ell}$, for $i = 1, \ldots, 4$ with $j = 1, \ldots, n_q$
  $\Delta \leftarrow \mathbf{v}_1 \odot \mathbf{v}_4 - \mathbf{v}_2 \odot \mathbf{v}_3$
  $p \leftarrow (\mathbf{p}_1 + 1)(\mathbf{p}_2 + 1)$
  $i' \leftarrow 1$
  **for** $i \in \left\{\texttt{supported\_bases}^\ell_{rk} : r = \ell\right\}$ **do**
    Construct $\mathbf{e} \in \mathbb{R}^p$ where $\mathbf{e}_k = \texttt{extraction\_coefficients}^\ell_{i',k}$ for $k = 1, \ldots, p$
    $N_i \leftarrow \mathbf{e}^T \texttt{reference\_basis}$
    Construct the vectors $\mathbf{y}^t \in \mathbb{R}^{n_q}$ as
      $\mathbf{y}^t_r \leftarrow \texttt{imap\_derivatives}_{r,t,\ell}$ for $t = 1, 2, 3, 4$ with $r = 1, \ldots, n_q$
    Construct the matrices $\mathbf{D}^t \in \mathbb{R}^{p \times n_q}$ as
      $\mathbf{D}^t_{r,r'} \leftarrow \texttt{reference\_basis\_derivatives}_{r,r',t}$ for $t = 1, 2$
      with $r = 1, \ldots, p$, $r' = 1, \ldots, n_q$
    Construct the matrices $\mathbf{P}^t \in \mathbb{R}^{p \times n_q}$ as $\mathbf{P}^t_{r,r'} = \mathbf{D}^t_{r,r'} \mathbf{y}^t_{r'}$ for $t = 1, 2$
      with $r = 1, \ldots, p$, $r' = 1, \ldots, n_q$
    $dN^x_i \leftarrow \mathbf{e}^T \left(\mathbf{P}^1 + \mathbf{P}^2\right)$
    Construct the matrices $\mathbf{Q}^t \in \mathbb{R}^{p \times n_q}$ as $\mathbf{Q}^t_{r,r'} = \mathbf{D}^t_{r,r'} \mathbf{y}^{t+2}_{r'}$ for $t = 1, 2$
      with $r = 1, \ldots, p$, $r' = 1, \ldots, n_q$
    $dN^y_i \leftarrow \mathbf{e}^T \left(\mathbf{Q}^1 + \mathbf{Q}^2\right)$
    $L \leftarrow \texttt{problem\_L}(X, N_i, dN^x_i, dN^y_i) \in \mathbb{R}^{1 \times n_q}$
    $\bar{b}_i \leftarrow \bar{b}_i + (\Delta \odot L^T)^T \texttt{quadrature\_weights}$
    $j' \leftarrow 1$
    **for** $j \in \left\{\texttt{supported\_bases}^\ell_{rk} : r = \ell\right\}$ **do**
      Construct $\tilde{\mathbf{e}} \in \mathbb{R}^p$ where $\tilde{\mathbf{e}}_k = \texttt{extraction\_coefficients}^\ell_{j',k}$ for $k = 1, \ldots, p$
      $N_j \leftarrow \tilde{\mathbf{e}}^T \texttt{reference\_basis}$
      $dN^x_j \leftarrow \tilde{\mathbf{e}}^T \left(\mathbf{P}^1 + \mathbf{P}^2\right)$
      $dN^y_j \leftarrow \tilde{\mathbf{e}}^T \left(\mathbf{Q}^1 + \mathbf{Q}^2\right)$
      $B \leftarrow \texttt{problem\_B}(X, N_i, dN^x_i, dN^y_i, N_j, dN^x_j, dN^y_j) \in \mathbb{R}^{1 \times n_q}$
      $\bar{A}_{ij} \leftarrow \bar{A}_{ij} + \left(\Delta \odot B^T\right)^T \texttt{quadrature\_weights}$
      $j' \leftarrow j' + 1$
    $i' \leftarrow i' + 1$
$\mathcal{I} \leftarrow \{i : i = 1, \ldots, n\}$
$W \leftarrow \mathcal{I} - \texttt{boundary\_bases}$ ▷ Indices of non-boundary basis functions

**Ensure:** $A \in \mathbb{R}^{(n-n_D)\times(n-n_D)}$, where $A_{ij} = \bar{A}_{W_i, W_j}$, $\forall i, j = 1, \ldots, n - n_D$

**Ensure:** $b \in \mathbb{R}^{n-n_D}$, where $b_{ij} = \bar{b}_{W_i}$, $\forall i = 1, \ldots, n - n_D$

F. In general, Dirichlet boundary conditions need special treatment, as its values are not necessarily contained in $\mathcal{F}$, our finite element space, and, thus, the discretized solution $u_h \in \mathcal{F}$ is not guaranteed to satisfy said boundary conditions. Hence, we obtain the coefficients for the Dirichlet boundary by performing an $L^2$-projection of the boundary values, i.e. we find the best possible approximation of the boundary data in $\mathcal{F}$. Therefore, we have

$$\int_{\Gamma_D} \left( \sum_{\ell=1}^{n_D} u_{i_\ell} N_{i_\ell}|_{\Gamma_D} \right) N_j|_{\Gamma_D} d\Gamma = \int_{\Gamma_D} 0 \cdot N_j|_{\Gamma_D} d\Gamma = 0, \ \forall j = i_0, i_1, \ldots, i_{n_D}.$$

By doing a linear combination, we can obtain

$$\int_{\Gamma_D} \left( \sum_{\ell=1}^{n_D} u_{i_\ell} N_{i_\ell}|_{\Gamma_D} \right)^2 d\Gamma = 0 \implies \sum_{\ell=1}^{n_D} u_{i_\ell} N_{i_\ell}|_{\Gamma_D} = 0,$$

and since we have a basis that is linearly independent on the boundary, we obtain that $u_{i_\ell} = 0, \ \forall i_\ell$.

G. We have a system in the form $\mathbf{Au} = \mathbf{b}$. Now, we decompose $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where

$$\mathbf{u}_{1i} = \begin{cases} \mathbf{u}_i, & i \notin \{i_0, \ldots, i_{n_D}\} \\ 0, & i \in \{i_0, \ldots, i_{n_D}\} \end{cases}, \text{ and } \mathbf{u}_{2i} = \begin{cases} \mathbf{u}_i, & i \in \{i_0, \ldots, i_{n_D}\} \\ 0, & i \notin \{i_0, \ldots, i_{n_D}\} \end{cases}$$

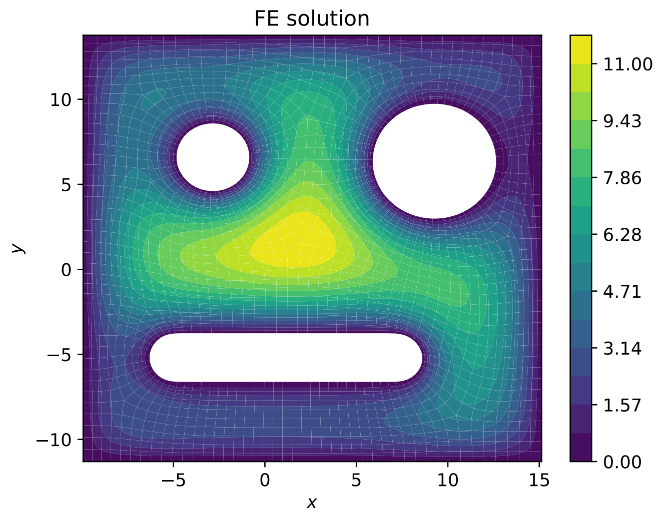
and therefore $\mathbf{u}_2$ is known from the boundary data. If we have then that

$$\mathbf{Au}_1 = b - \mathbf{Au}_2, \tag{1}$$

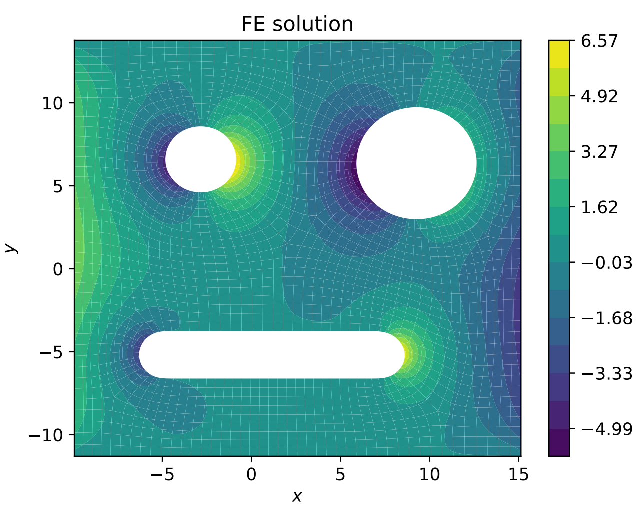
then note that each row of (1) corresponds with the equation obtained when multiplying by $N_j$, and since $N_j \notin \mathcal{W}_h$, for $j \in \{i_0, \ldots, i_{n_D}\}$, the rows $j \in \{i_0, \ldots, i_{n_D}\}$ can be omitted. Following a similar argument, the columns of $\mathbf{A}$ for which $\mathbf{u}_1$ is 0 can also be omitted.

H. The solution obtained through finite elements, and its partial derivatives, are presented in Fig. 1.

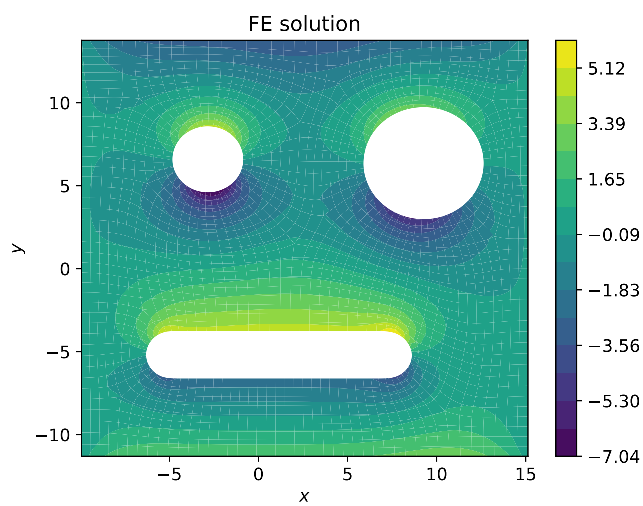
I. The solution obtained through finite elements, and its partial derivatives, are presented in Fig. 2.




(a) $u_h$

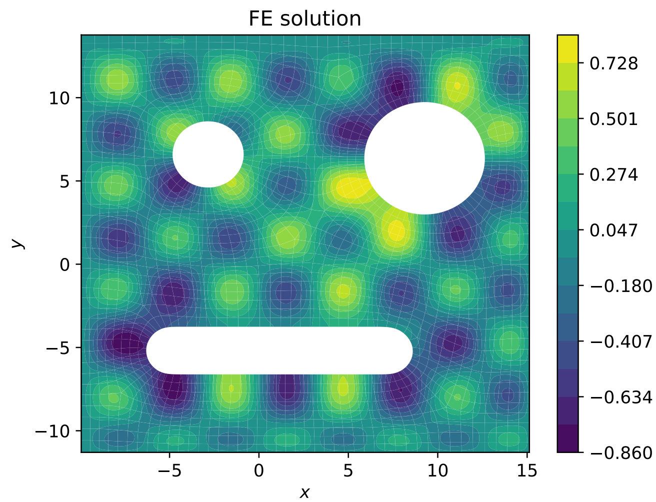



(b) $u_{h,x}$

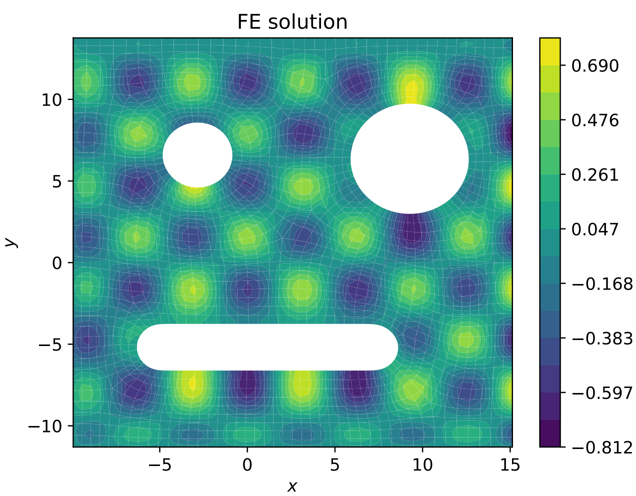



(c) $u_{h,y}$

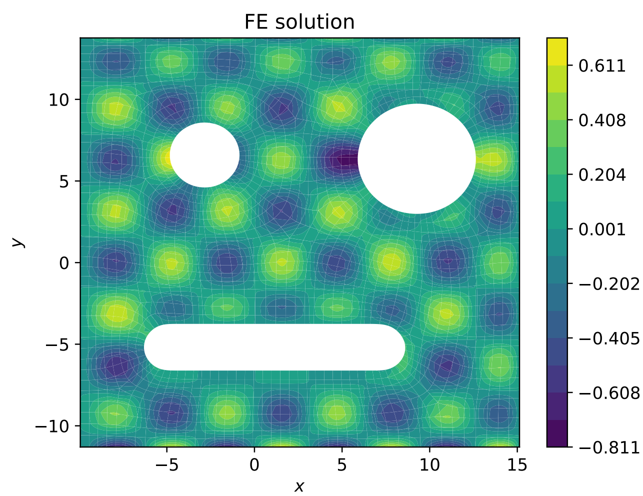
Figure 1: Finite element solutions.




(a) $u_h$




(b) $u_{h,x}$




(c) $u_{h,y}$

Figure 2: Finite element solutions.