1. $-u''(x) = \mathrm{Cos}\left(\frac{\pi x}{4}\right) \quad (*) \qquad \Omega = (0,2)$

$u(0) = u'(2) = 0, \quad V = \{v \in \mathcal{H}_0^1(\Omega) : v(0) = 0\}.$

a) Sea $v \in V$, multiplicando $(*)$ por v e integrando en Ω:

$$-\int_0^2 u''(x)v(x)\,dx = -u'(x)v(x)\Big|_0^2 + \int_0^2 u'(x)v'(x)\,dx$$

$$= -\cancelto{0}{u'(2)}v(2) + \cancelto{0}{u'(0)}v(0) + \int_0^2 u'(x)v'(x)\,dx$$

$$= \int_0^2 u'(x)v'(x)\,dx$$

Luego, $\displaystyle\int_0^2 u'(x)v'(x)\,dx = \int_0^2 \mathrm{Cos}\left(\frac{\pi x}{4}\right)v(x)\,dx$ ← Nótese que $f \in \mathbb{L}^2(\Omega)$

b). $V_2 = \mathrm{gen}\{x, x^2\}$

Supóngase $u_m(x) = \alpha_1 x + \alpha_2 x^2$

Reemplazando en la formulación débil:

$$(1.1) \quad \int_0^2 (\alpha_1 + 2\alpha_2 x)v'(x)\,dx = \int_0^2 \mathrm{Cos}\left(\frac{\pi x}{4}\right)v(x)\,dx$$

Como (1.1) es $\forall v \in V_2$, en particular para $v_1 = x$ y $v_2 = x^2$.

Para v_1: $\displaystyle\int_0^2 (\alpha_1 + 2\alpha_2 x)(1)\,dx = \int_0^2 x\,\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$

$$\alpha_1 x + \alpha_2 x^2\Big|_0^2 = \frac{4}{\pi}x\,\mathrm{Sin}\left(\frac{\pi x}{4}\right)\Big|_0^2 - \frac{4}{\pi}\int_0^2 \mathrm{Sin}\left(\frac{\pi x}{4}\right)dx$$

$$2\alpha_1 + 4\alpha_2 = \frac{8}{\pi} + \frac{16}{\pi^2}\mathrm{Cos}\left(\frac{\pi x}{4}\right)\Big|_0^2$$

$$2\alpha_1 + 4\alpha_2 = \frac{8}{\pi} - \frac{16}{\pi^2}$$

Para v_2: $\displaystyle\int_0^2 (\alpha_1 + 2\alpha_2 x)(2x)\,dx = \int_0^2 x^2\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$

$$\alpha_1 x^2 + \frac{4}{3}\alpha_2 x^3\Big|_0^2 = \frac{4}{\pi}x^2\mathrm{Sin}\left(\frac{\pi x}{4}\right)\Big|_0^2 - \frac{8}{\pi}\int_0^2 x\,\mathrm{Sin}\left(\frac{\pi x}{4}\right)dx$$

$$4\alpha_1 + \frac{32}{3}\alpha_2 = \frac{16}{\pi} + \cancelto{0}{\frac{32}{\pi^2}x\,\mathrm{Cos}\left(\frac{\pi x}{4}\right)\Big|_0^2} - \frac{32}{\pi^2}\int_0^2 \mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$$

$$4\alpha_1 + \frac{32}{3}\alpha_2 = \frac{16}{\pi} - \frac{128}{\pi^3}\mathrm{Sin}\left(\frac{\pi x}{4}\right)\Big|_0^2$$

$$4\alpha_1 + \frac{32}{3}\alpha_2 = \frac{16}{\pi} - \frac{128}{\pi^3}$$

$$\begin{cases} 2\alpha_1 + 4\alpha_2 = \frac{8}{\pi} - \frac{16}{\pi^2} & (-2) \\ 4\alpha_1 + \frac{32}{3}\alpha_2 = \frac{16}{\pi} - \frac{128}{\pi^3} \end{cases}$$

$$\frac{8}{3}\alpha_2 = \frac{32}{\pi^2} - \frac{128}{\pi^3}$$

$$\alpha_2 = \frac{12}{\pi^2} - \frac{48}{\pi^3}$$

$$2\alpha_1 + \frac{48}{\pi^2} - \frac{192}{\pi^3} = \frac{8}{\pi} - \frac{16}{\pi^2}$$

$$\alpha_1 = \frac{4}{\pi} - \frac{32}{\pi^2} + \frac{96}{\pi^3}$$

Luego $u_m(x) = \left(\frac{4}{\pi} - \frac{32}{\pi^2} + \frac{96}{\pi^3}\right)x + \left(\frac{12}{\pi^2} - \frac{48}{\pi^3}\right)x^2$

c) $V_3 = \mathrm{gen}\{x, x^2, x^3\}$. $u_m(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$

Para x: $\displaystyle\int_0^2 (\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2)(1)\,dx = \int_0^2 x\,\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$

$$\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3\Big|_0^2 = \frac{8}{\pi} - \frac{16}{\pi^2}$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = \frac{4}{\pi} - \frac{8}{\pi^2}$$

Para $v_2 = x^2$:

$$\int_0^2 (\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2)(2x)\,dx = \int_0^2 x^2\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$$

$$\alpha_1 x^2 + \frac{4}{3}\alpha_2 x^3 + \frac{6}{4}\alpha_3 x^4\Big|_0^2 = \frac{16}{\pi} - \frac{128}{\pi^3}$$

$$4\alpha_1 + \frac{32}{3}\alpha_2 + 24\alpha_3 = \frac{16}{\pi} - \frac{128}{\pi^3}$$

Para x^3:

$$\int_0^2 (\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2)(3x^2)\,dx = \int_0^2 x^3\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$$

$$\alpha_1 x^3 + \frac{6}{4}\alpha_2 x^4 + \frac{9}{5}\alpha_3 x^5\Big|_0^2 = \frac{4}{\pi}x^3\mathrm{Sin}\left(\frac{\pi x}{4}\right)\Big|_0^2 - \frac{12}{\pi}\int_0^2 x^2\mathrm{Sin}\left(\frac{\pi x}{4}\right)dx$$

$$8\alpha_1 + 24\alpha_2 + \frac{288}{5}\alpha_3 = \frac{32}{\pi} + \cancelto{0}{\frac{48}{\pi^2}x^2\mathrm{Cos}\left(\frac{\pi x}{4}\right)\Big|_0^2} - \frac{96}{\pi^2}\int_0^2 x\,\mathrm{Cos}\left(\frac{\pi x}{4}\right)dx$$

$$8\alpha_1 + 24\alpha_2 + \frac{288}{5}\alpha_3 = \frac{32}{\pi} - \frac{96}{\pi^2}\left[\frac{8}{\pi} - \frac{16}{\pi^2}\right]$$

$$\alpha_1 + 3\alpha_2 + \frac{36}{5}\alpha_3 = \frac{4}{\pi} - \frac{96}{\pi^3} + \frac{192}{\pi^4}$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = \frac{4}{\pi} - \frac{8}{\pi^2} \quad (i)$$

$$\alpha_1 + \frac{8}{3}\alpha_2 + 6\alpha_3 = \frac{4}{\pi} - \frac{32}{\pi^3} \quad (ii)$$

$$\alpha_1 + 3\alpha_2 + \frac{36}{5}\alpha_3 = \frac{4}{\pi} - \frac{96}{\pi^3} + \frac{192}{\pi^4} \quad (iii)$$

$(ii)-(i)$: $\frac{2}{3}\alpha_2 + 2\alpha_3 = \frac{8}{\pi^2} - \frac{32}{\pi^3} \quad (iv)$

$(iii)-(i)$: $\alpha_2 + \frac{16}{5}\alpha_3 = \frac{8}{\pi^2} - \frac{96}{\pi^3} + \frac{192}{\pi^4} \quad (v)$

$-\frac{2}{3}(v)$: $-\frac{2}{3}\alpha_2 - \frac{32}{15}\alpha_3 = -\frac{16}{3\pi^2} + \frac{192}{3\pi^3} - \frac{384}{3\pi^4}$

$(iv)-\frac{2}{3}(v)$: $-\frac{2\alpha_3}{15} = \frac{8}{3\pi^2} + \frac{96}{3\pi^3} - \frac{384}{3\pi^4}$

$$2\alpha_3 = -\frac{40}{\pi^2} - \frac{480}{\pi^3} + \frac{1920}{\pi^4}$$

$$\alpha_3 = -\frac{20}{\pi^2} - \frac{240}{\pi^3} + \frac{960}{\pi^4}$$

en (iv): $\frac{2}{3}\alpha_2 - \frac{40}{\pi^2} - \frac{480}{\pi^3} + \frac{1920}{\pi^4} = \frac{8}{\pi^2} - \frac{32}{\pi^3}$

$$\frac{2}{3}\alpha_2 = \frac{48}{\pi^2} + \frac{448}{\pi^3} - \frac{1920}{\pi^4}$$

$$\alpha_2 = \frac{72}{\pi^2} + \frac{672}{\pi^3} - \frac{2880}{\pi^4}$$

en (i):

$$\alpha_1 + \frac{144}{\pi^2} + \frac{1344}{\pi^3} - \frac{5720}{\pi^4} - \frac{80}{\pi^2} - \frac{960}{\pi^3} + \frac{3840}{\pi^4} = \frac{4}{\pi} - \frac{8}{\pi^2}$$

$$\alpha_1 = \frac{4}{\pi} - \frac{72}{\pi^2} - \frac{384}{\pi^3} + \frac{1880}{\pi^4}$$

$$u_m(x) = \left[\frac{4}{\pi} - \frac{72}{\pi^2} - \frac{384}{\pi^3} + \frac{1880}{\pi^4}\right]x + \left[\frac{72}{\pi^2} + \frac{672}{\pi^3} - \frac{2880}{\pi^4}\right]x^2 + \left[-\frac{20}{\pi^2} - \frac{240}{\pi^3} + \frac{960}{\pi^4}\right]x^3$$

d) $-\dfrac{d^2u}{dx^2} = \mathrm{Cos}\left(\frac{\pi x}{4}\right)$

$\lambda^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 0$

$u_h(x) = C_1 + C_2 x$

$u_p(x) = A\,\mathrm{Cos}\left(\frac{\pi x}{4}\right) + B\,\mathrm{Sin}\left(\frac{\pi x}{4}\right)$

$u_p'(x) = -\frac{A\pi}{4}\mathrm{Sin}\left(\frac{\pi x}{4}\right) + \frac{B\pi}{4}\mathrm{Cos}\left(\frac{\pi x}{4}\right)$

$u_p''(x) = -\frac{A\pi^2}{16}\mathrm{Cos}\left(\frac{\pi x}{4}\right) - \frac{B\pi^2}{16}\mathrm{Sin}\left(\frac{\pi x}{4}\right)$

$\frac{A\pi^2}{16}\mathrm{Cos}\left(\frac{\pi x}{4}\right) + \frac{B\pi^2}{16}\mathrm{Sin}\left(\frac{\pi x}{4}\right) = \mathrm{Cos}\left(\frac{\pi x}{4}\right)$

$B = 0$ y $A = \frac{16}{\pi^2}$ →

$u(x) = C_1 + C_2 x + \frac{16}{\pi^2}\mathrm{Cos}\left(\frac{\pi x}{4}\right)$

$u(0) = 0 = C_1 + \frac{16}{\pi^2} \rightarrow C_1 = -\frac{16}{\pi^2}$

$u'(2) = 0 = C_2 - \frac{4}{\pi}\mathrm{Sin}\left(\frac{\pi}{2}\right) \rightarrow C_2 = \frac{4}{\pi}$

$u(x) = \frac{16}{\pi^2}\left[\mathrm{Cos}\left(\frac{\pi x}{4}\right) - 1\right] + \frac{4}{\pi}x$