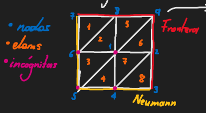
5. $-\nabla^2 u = 1 \quad \Omega = (0,1)\times(0,1)$

$u_x(0,y) = u_y(x,0) = 0$

$u(1,y) = u(x,1) = 0$



$\longrightarrow A_{T_j} = 1/8$

$I_1 = \{2,3,4,5,6,7\}$
$I_4 = \{4,7,8\}$
$I_5 = \{3,4\}$
$I_6 = \{1,2,3$

Para triangulo par: $(0,0),(1/2,0),(1/2,1/2)$

Para $V_1^A = (0,0)$: $\varphi_1(0,0) = 1 = a_1$

$\varphi_1(1/2,0) = 0 = 1 + \frac{b_1}{2} \Rightarrow b_1 = -2 \qquad \varphi_1(x,y) = 1-2x$

$\varphi_1(1/2,1/2) = 0 = 1-1+\frac{c_1}{2} \Rightarrow c_1 = 0 \qquad \nabla\varphi_1 = (-2,0)$

Para $V_2^A = (1/2,0)$: $\varphi_2(0,0) = 0 = a_2$

$\varphi_2(1/2,0) = 1 = \frac{b_2}{2} \to b_2 = 2 \qquad \varphi_2(x,y) = 2x-2y$

$\varphi_2(1/2,1/2) = 0 = 1 + \frac{c_2}{2} \Rightarrow c_2 = -2 \qquad \nabla\varphi_2 = (2,-2)$

Para $V_3^A = (1/2,1/2)$: $\varphi_3(0,0) = 0 = a_3$

$\varphi_3(1/2,0) = 0 = \frac{b_3}{2} \to b_3 = 0 \qquad \varphi_3(x,y) = 2y$

$\varphi_3(1/2,1/2) = 1 = \frac{c_3}{2} \to c_3 = 2 \qquad \nabla\varphi_3 = (0,2)$

Para triángulo impar: $(0,0),(0,1/2),(1/2,1/2)$

Para $V_1^3 = (0,0)$

$\psi_1(0,0) = 1 = \alpha_1 \qquad \psi_1(x,y) = 1-2y$

$\psi_1(0,1/2) = 1 + \frac{\gamma_1}{2} \to \gamma_1 = -2$ ($=0$) $\qquad \nabla\psi_1 = (0,-2)$

$\psi_1(1/2,1/2) = 1-1+\frac{\beta_1}{2} = 0 \to \beta_1 = 0$

Para $V_2^3 = (0,1/2)$:

$\psi_2(0,0) = 0 = \alpha_2 \qquad \psi_2(x,y) = -2x+2y$

$\psi_2(0,1/2) = 1 = \frac{\gamma_2}{2} \to \gamma_2 = 2 \qquad \nabla\psi_2 = (-2,2)$

$\psi_2(1/2,1/2) = 1 + \frac{\beta_2}{2} = 0 \to \beta_2 = -2$

Para $V_3^3 = (1/2,1/2)$

$\psi_3(0,0) = 0 = \alpha_3$

$\psi_3(0,1/2) = 0 = \frac{\gamma_3}{2} \to \gamma_3 = 0 \qquad \psi_3(x,y) = 2x$

$\psi_3(1/2,1/2) = 1 = \frac{\beta_3}{2} \to \beta_3 = 2 \qquad \nabla\psi_3(x,y) = (2,0)$

$$K_{11} = \sum_{r\in I_1} \int_{T_r} \nabla\xi_1|_{T_r} \cdot \nabla\xi_1|_{T_r}\, dA_{T_r}$$

$$= \int_{T_2} \|\nabla\varphi_2\|^2 dA_{T_2} + \int_{T_3} \|\nabla\psi_3\|^2 dA_{T_3} + \int_{T_4} \|\nabla\varphi_3\|^2 dA_{T_4} + \int_{T_5} \|\nabla\psi_1\|^2 dA_{T_4} + \int_{T_6} \|\nabla\varphi_1\|^2 dA_{T_6} + \int_{T_7} \|\nabla\psi_2\|^2 dA_{T_7}$$

$$= (1/8)[8+4+4+4+4+8] = 4$$

$$K_{14} = \sum_{r\in I_1\cap I_4} \int_{T_r} \nabla\xi_1|_{T_r} \cdot \nabla\xi_4|_{T_r}\, dA_{T_r}$$

$$= \int_{T_4} \nabla\varphi_3 \cdot \nabla\varphi_2\, dA_{T_4} + \int_{T_7} \nabla\psi_2 \cdot \nabla\psi_1\, dA_{T_7}$$

$$= \frac{1}{8}[-4-4] = -1$$

$$K_{15} = \int_{T_3} \nabla\psi_3 \cdot \nabla\psi_1\, dA_{T_3} + \int_{T_4} \nabla\varphi_3 \cdot \nabla\varphi_1 dA_{T_4}$$

$$= \frac{1}{8}[0+0] = 0$$

$$K_{16} = \int_{T_2} \nabla\varphi_2 \cdot \nabla\varphi_1\, dA_{T_2} + \int_{T_3} \nabla\psi_3 \cdot \nabla\psi_2\, dA_{T_3}$$

$$= \frac{1}{8}[-4-4] = -1$$

$$K_{44} = \int_{T_4} \|\nabla\varphi_2\|^2 dA_{T_4} + \int_{T_7} \|\nabla\psi_1\|^2 dA_{T_7} + \int_{T_8} \|\nabla\varphi_1\|^2 dA_{T_8}$$

$$= \frac{1}{8}[8+4+4] = 2$$

$$K_{45} = \int_{T_4} \nabla\varphi_2 \cdot \nabla\varphi_1\, dA_{T_4} = \frac{-1}{2}$$

$K_{46} = 0$

$$K_{55} = \int_{T_3} \|\nabla\psi_1\|^2 dA_{T_3} + \int_{T_4} \|\nabla\varphi_1\|^2 dA_{T_4}$$

$$= \frac{1}{8}[4+4] = 1$$

$$K_{56} = \int_{T_3} \nabla\psi_1 \cdot \nabla\psi_2\, dA_{T_3} = \frac{1}{8}[-4] = \frac{-1}{2}$$

$$K_{66} = \int_{T_1} \|\nabla\psi_1\|^2 dA_{T_1} + \int_{T_2} \|\nabla\varphi_1\|^2 dA_{T_2} + \int_{T_3} \|\nabla\psi_2\|^2 dA_{T_3}$$

$$= \frac{1}{8}[4+4+8] = 2$$

Luego

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ -1 & 0 & -1/2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/8 \\ 1/12 \\ 1/8 \end{bmatrix}$$

$$u_1 = \frac{17}{96},\ u_4 = \frac{11}{48},\ u_5 = \frac{5}{16},\ u_6 = \frac{11}{48}$$

$$l_i \approx \frac{|I_i| A_T}{3} f(x_i)$$

$l_1 \approx \frac{6(1/8)}{3} = \frac{1}{4}$

$l_4 \approx \frac{3(1/8)}{3} = \frac{1}{8}$

$l_5 \approx \frac{2(1/8)}{3} = \frac{1}{12}$

$l_6 \approx \frac{3(1/8)}{3} = \frac{1}{8}$

$$\begin{array}{c|cccc} & 1 & 4 & 5 & 6 \\ \hline 1 & 4 & -1 & 0 & -1 \\ 4 & -1 & 2 & -1/2 & 0 \\ 5 & 0 & -1/2 & 1 & -1/2 \\ 6 & -1 & 0 & -1/2 & 2 \end{array}$$

Formulación débil:

$-\nabla^2 u = 1 \quad \Omega = (0,1)\times(0,1)$

$u_x(0,y) = u_y(x,0) = 0$

$u(1,y) = u(x,1) = 0$

Sea v una función de prueba.

Luego $-\int_\Omega v\nabla^2 u\, dx = \int_\Omega v\, dx$.

Por la primera identidad de Green:

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Gamma u \frac{\partial v}{\partial \nu} dS = \int_\Omega v\, dx$$

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_{\partial\Omega} u \cancel{\frac{\partial v}{\partial \nu}} dS + \int_\Omega v\, dx$$

0

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega v\, dx$$