3. $-\dfrac{d^2u}{dx^2} = 20$ $\quad \phi_i^1(x) = \dfrac{x_i - x}{x_i - x_{i-1}}$ $\quad i = 1, \dots, 4$

$u(0) = 40$ $\quad \phi_i^2(x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$ $\quad$ 4 elementos

$u(16) = 36$

$$\overset{0}{x_0} \;\overset{1}{-}\; x_1 \;\overset{2}{-}\; x_2 \;\overset{3}{-}\; x_3 \;\overset{4}{-}\; \overset{16}{x_4}$$

La distribución de temperatura para un sólo elemento es approx. por

$$\bar{u}_i = \phi_i^1 u_{i-1} + \phi_i^2 u_i \quad (14)$$

Donde $u_i \in \mathbb{R} \;\forall i = 0, \dots, 4$

Al sustituir la solución aprox. (14) en $\dfrac{d^2u}{dx^2} + 20 = 0$

$$\Downarrow$$

$$\frac{d^2\bar{u}_i}{dx^2} + 20 = R_i$$

Se quiere que $\displaystyle\int_{x_{i-1}}^{x_i} R_i \phi_i^j \, dx = 0, \; j = 1, 2, \; i = 1, \dots, 4$

Luego, $\displaystyle\int_{x_{i-1}}^{x_i} \left[\frac{d^2\bar{u}_i}{dx^2} + 20\right] \phi_i^j(x)\, dx = 0$

$$\int_{x_{i-1}}^{x_i} \frac{d^2\bar{u}_i}{dx^2} \phi_i^j(x)\, dx = -\int_{x_{i-1}}^{x_i} 20\, \phi_i^j(x)\, dx$$

Integrando por partes

$$\phi_i^j(x) \frac{d\bar{u}_i}{dx}\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} \frac{d\bar{u}_i}{dx} \frac{d\phi_i^j}{dx} dx = -20 \int_{x_{i-1}}^{x_i} \phi_i^j(x)\, dx$$

$\phi_i^1(x) = \dfrac{x_i - x}{x_i - x_{i-1}}$

$\phi_i^1(x_{i-1}) = 1$

$\phi_i^1(x_i) = 0$

$\dfrac{d\phi_i^1}{dx} = -\dfrac{1}{x_i - x_{i-1}}$

$$\int_{x_{i-1}}^{x_i} \phi_i^1(x) dx = \int_{x_{i-1}}^{x_i} \left(\frac{x_i - x}{x_i - x_{i-1}}\right) dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{x_i}{x_i - x_{i-1}}\right) dx - \int_{x_{i-1}}^{x_i} \left(\frac{x}{x_i - x_{i-1}}\right) dx$$

$$= x_i - \frac{1}{2} \frac{x^2}{x_i - x_{i-1}} \Big|_{x_{i-1}}^{x_i}$$

$$= x_i - \frac{1}{2} \frac{x_i^2 - x_{i-1}^2}{x_i - x_{i-1}} = x_i - \frac{(x_i + x_{i-1})}{2}$$

$$= \frac{x_i - x_{i-1}}{2}$$

$\phi_i^2(x) = \dfrac{x - x_{i-1}}{x_i - x_{i-1}}$

$\phi_i^2(x_{i-1}) = 0$

$\phi_i^2(x_i) = 1$

$\dfrac{d\phi_i^2}{dx} = \dfrac{1}{x_i - x_{i-1}}$

$$\int_{x_{i-1}}^{x_i} \phi_i^2(x) dx = \frac{x_i - x_{i-1}}{2}$$

Para $j=1$: $\displaystyle -\frac{d\bar{u}_i}{dx}\Big|_{x=x_{i-1}} + \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \frac{d\bar{u}_i}{dx} dx = -10(x_i - x_{i-1})$

$$-\frac{d\bar{u}_i}{dx}\Big|_{x=x_{i-1}} + \int_{x_{i-1}}^{x_i} \frac{u_i - u_{i-1}}{(x_i - x_{i-1})^2} dx = -10(x_i - x_{i-1})$$

$$-\frac{d\bar{u}_i}{dx}\Big|_{x=x_{i-1}} + \frac{u_i - u_{i-1}}{x_i - x_{i-1}} = -10(x_i - x_{i-1})$$

Para $j=2$: $\displaystyle \frac{d\bar{u}_i}{dx}\Big|_{x=x_i} - \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \left(\frac{u_i - u_{i-1}}{x_i - x_{i-1}}\right) dx = -10(x_i - x_{i-1})$

$$\frac{d\bar{u}_i}{dx}\Big|_{x=x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}} = -10(x_i - x_{i-1})$$

$$\frac{1}{x_i - x_{i-1}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_{i-1} \end{bmatrix} = \begin{bmatrix} \frac{d\bar{u}_i}{dx}\Big|_{x=x_{i-1}} - 10(x_i - x_{i-1}) \\ -\frac{d\bar{u}_i}{dx}\Big|_{x=x_i} - 10(x_i - x_{i-1}) \end{bmatrix}$$

Para cada elemento $i = 1, \dots, 4$.

$$\frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{i-1} \\ u_i \end{bmatrix} = \begin{bmatrix} -\bar{u}_i'(x_{i-1}) + 40 \\ \bar{u}_i'(x_i) + 40 \end{bmatrix}$$

Ensamblando:

$$\begin{bmatrix} 1/4 & -1/4 & 0 & 0 & 0 \\ -1/4 & 1/2 & -1/4 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 0 & -1/4 & 1/2 & -1/4 \\ 0 & 0 & 0 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -\bar{u}_1'(x_0) + 40 \\ 80 \\ 80 \\ 80 \\ \bar{u}_4'(x_4) + 40 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1/4 & -1/4 & 0 & 0 & 0 \\ -1/4 & 1/2 & -1/4 & 0 & 0 \\ 0 & -1/4 & 1/2 & -1/4 & 0 \\ 0 & 0 & -1/4 & 1/2 & -1/4 \\ 0 & 0 & 0 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 40 \\ u_1 \\ u_2 \\ u_3 \\ 36 \end{bmatrix} = \begin{bmatrix} -\bar{u}_1'(x_0) + 40 \\ 80 \\ 80 \\ 80 \\ \bar{u}_4'(x_4) + 40 \end{bmatrix}$$

$$\left.\begin{aligned} 10 - \frac{u_1}{4} &= -\bar{u}_1'(x_0) + 40 \\ -10 + \frac{u_1}{2} - \frac{u_2}{4} &= 80 \\ -\frac{u_1}{4} + \frac{u_2}{2} - \frac{u_3}{4} &= 80 \\ -\frac{u_2}{4} + \frac{u_3}{2} - 9 &= 80 \\ -\frac{u_3}{4} + 9 &= \bar{u}_4'(x_4) + 40 \end{aligned}\right\} \quad \begin{aligned} u_1 &= 519 \\ u_2 &= 678 \\ u_3 &= 517 \\ \bar{u}_1'(0) &= 639/4 \\ \bar{u}_4'(16) &= -\frac{641}{4} \end{aligned}$$

$$-\frac{d^2u}{dx^2} = 20 \rightarrow -\frac{du}{dx} = 20x + c_1$$

$$\downarrow$$

$u(x) = -10x^2 + c_1 x + c_2$

$u(0) = 40 = c_2$

$u(16) = 36 = -10(256) + 16c_1 + 40$

$c_1 = \dfrac{2556}{16} = \dfrac{639}{4}$

$u(x) = -10x^2 + \dfrac{639}{4}x + 40$