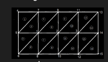
Formulación débil

$$\int_\Omega \nabla u \cdot \nabla w \, dV = 4\pi \int_\Omega \text{Sin}(\pi x)\,\text{Sin}(\pi y)\, w \, dV \quad (\Delta)$$

4. $-\nabla^2 u = 4\pi \,\text{Sin}(\pi x)\,\text{Sin}(\pi y)$

$\Omega = (0,1)\times(0,1)$

$\partial\Omega \equiv \Gamma$ y $u|_\Gamma = 0$



$V_i \to$ vértice i, $i=1,\dots,15$.

Sea V_i^j el i-ésimo vértice del triángulo j.

Nótese que $\forall i,j,\ \exists k$ t.q. $V_i^j = V_k$

Sea $I \in \mathbb{N}$ un conjunto de índices, sea h una biyección entre los índices y los elementos t.q.

i) El elemento T t.q. $h(2K) = T$, $K \in \mathbb{N}$ no existe $K^* \in \mathbb{N}$ t.q. $h(2K^*)$ comparta aristas con T.

ii) El elemento T es t.q. $h(2K+1) = T$, $K \in \mathbb{N}$ no existe $K^* \in \mathbb{N}$ t.q. $h(2K^*+1)$ comparta aristas con T.

Nota: $T_j = h(j)$, $j \in I$

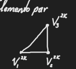
$I_i = \{ j \in I : \exists k,\ V_k^j = V_i \}$

Para el triángulo entre $(0,0), (1/4,0), (1/4,1/2)$

Se define $\varphi_i(x,y) = a_i + b_i x + c_i y$, con $i = 1,2,3$.
t.q.

$$\varphi_i(x_i,y_i) = 1 \text{ y } \varphi_i(x_j,y_j) = 0, \text{ con } j \neq i.$$

Elemento par



Para $V_1^4 = (0,0)$: $\varphi_1(0,0) = 1 = a_1$

$\varphi_1(1/4,0) = 0 = a_1 + \frac{b_1}{4} \to b_1 = -4$ $\quad \varphi_1(x,y) = -4x+1$

$\varphi_1(1/4,1/2) = 0 = a_1 + \frac{b_1}{4} + \frac{c_1}{2} \to c_1 = 0$ $\quad \nabla\varphi_1 = (-4,1)$

$V_2^4 = (1/4,0)$: $\varphi_2(0,0) = 0 = a_2$

$\varphi_2(1/4,0) = 1 = \frac{b_2}{4} \to b_2 = 4$ $\quad \varphi_2(x,y) = 4x - 2y$

$\nabla\varphi_2 = (4,-2)$

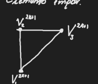
$\varphi_2(1/4,1/2) = 0 = 1 + \frac{c_2}{2} \to c_2 = -2$

$V_3^4 = (1/4,1/2)$: $\varphi_3(0,0) = 0 = a_3$

$\varphi_3(1/4,0) = 0 = \frac{b_3}{4} \to b_3 = 0$ $\quad \varphi_3(x,y) = 2y$

$\varphi_3(1/4,1/2) = 1 = \frac{c_3}{2} \to c_3 = 2$. $\quad \nabla\varphi_3 = (0,2)$

Elemento impar.



Para el triángulo entre $(0,0), (0,1/2), (1/4,1/2)$

Se define la función $\psi_i(x,y) = \alpha_i + \beta_i x + \gamma_i y$ para cada vértice $i = 1,2,3$

que satisface $\psi_i(x_i,y_i) = 1$

$\psi_i(x_j,y_j) = 0$, $i \neq j$.

Luego, para $V_1^3 = (0,0)$

$\psi_1(0,0) = 1 = \alpha_1$

$\psi_1(0,1/2) = 0 = 1 + \frac{\gamma_1}{2} \to \gamma_1 = -2$ $\quad \psi_1(x,y) = 1 - 2y$

$\psi_1(1/4,1/2) = 0 = 1 + \frac{\beta_1}{4} - 1 \to \beta_1 = 0$ $\quad \nabla\psi_1 = (0,-2)$

para $V_2^3 = (0,1/2)$

$\psi_2(0,0) = 0 = \alpha_2$ $\quad \psi_2(x,y) = -4x + 2y$

$\psi_2(0,1/2) = 1 = \frac{\gamma_2}{2} \to \gamma_2 = 2$ $\quad \nabla\psi_2 = (-4,2)$

$\psi_2(1/4,1/2) = 0 = \frac{\beta_2}{4} + 1 \to \beta_2 = -4$

para $V_3^3 = (1/4,1/2)$

$\psi_3(0,0) = 0 = \alpha_3$ $\quad \psi_3(x,y) = 4y$

$\psi_3(0,1/2) = 0 = \frac{\gamma_3}{2} \to \gamma_3 = 0$ $\quad \nabla\psi_3 = (0,4)$

$\psi_3(1/4,1/2) = 1 = \frac{\beta_3}{4} \to \beta_3 = 4$

Sea $\xi_i|_{T_j} = \begin{cases} \varphi_k, & \text{si } V_i = V_k^j \text{ y } j \text{ par.} \\ \psi_k, & \text{si } V_i = V_k^j \text{ y } j \text{ impar} \end{cases}$

$i \to$ Vértice
$j \to$ Triángulo

Las entradas de la matriz de rigidez se construyen con:

$$k_{ij} = \sum_{r \in I_i \cap I_j} \int_{T_r} \nabla\xi_i|_{T_r} \cdot \nabla\xi_j|_{T_r} \, dA_{T_r}$$

Obsérvese que $I_1 = \{2,3,4,5,6,7\}$

$I_2 = \{6,7,8,9,10,11\}$

$I_3 = \{10,11,12,13,14,15\}$

$$K_{11} = \sum_{r \in I_1} \int_{T_r} \|\nabla\xi_1|_{T_r}\|^2 dA_{T_r}$$

$$= \int_{T_2} \|\nabla\varphi_2\|^2 dA_{T_2} + \int_{T_3} \|\nabla\psi_3\|^2 dA_{T_3} + \int_{T_4} \|\nabla\varphi_3\|^2 dA_{T_4} + \int_{T_5} \|\nabla\psi_1\|^2 dA_{T_5} + \int_{T_6} \|\nabla\varphi_1\|^2 dA_{T_6} + \int_{T_7} \|\nabla\psi_2\|^2 dA_{T_7}$$

$$= \|(4,-2)\|^2 A_{T_2} + \|(4,0)\|^2 A_{T_3} + \|(0,2)\|^2 A_{T_4} + \|(0,-2)\|^2 A_{T_5} + \|(-4,0)\|^2 A_{T_6} + \|(-4,2)\|^2 A_{T_7}$$

$$= 80(1/16) = 5$$

$K_{12} = -2$
$K_{13} = 0 \quad (I_1 \cap I_3 = \phi)$

$K_{22} = 5$
$K_{23} = -2$

$K_{33} = 5$

Teniendo en cuenta que K es simétrica. $\Rightarrow K = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Aproximando el término derecho de (Δ):

$$l_i = \sum_{j \in I_i} \int_{T_j} \xi_i|_{T_j} f \, dA_{T_j}$$

por la regla de cuadratura:

$$\int_{T_j} \xi_i|_{T_j} f \, dA_{T_j} \approx \frac{A_{T_j}}{3}\left[\xi_i|_{T_j}(V_1^j) f(V_1^j) + \xi_i|_{T_j}(V_2^j) f(V_2^j) + \xi_i|_{T_j}(V_3^j) f(V_3^j) \right]$$

Como $j \in I_i$, $\exists k$ t.q. $V_k^j = V_i \Rightarrow \approx \frac{A_{T_j}}{3} \xi_i|_{T_j}(V_i) f(V_i)$

$$\approx \frac{A_{T_j}}{3} f(V_i)$$

$$l_i \approx \sum_{j \in I_i} \frac{A_{T_j}}{3} f(V_i) = |I_i| \frac{A_T}{3} f(V_i)$$

$$l_1 = \frac{6(1/16)}{3} f\left(\frac{1}{4}, \frac{1}{2}\right) = \left(\frac{1}{8}\right)(4\pi)\,\text{Sin}\left(\frac{\pi}{4}\right)\text{Sin}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{\pi\sqrt{2}}{4}$$

$$l_2 = \frac{6(1/16)}{3} f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{8}\right)(4\pi)\,\text{Sin}\left(\frac{\pi}{2}\right)\text{Sin}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot 1 \cdot 1 = \frac{\pi}{2}$$

$$l_3 = \frac{6(1/16)}{3} f\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{8}(4\pi)\,\text{Sin}\left(\frac{3\pi}{4}\right)\text{Sin}\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{\pi\sqrt{2}}{2}$$

$$\begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \pi\sqrt{2}/2 \\ \pi/2 \\ \pi\sqrt{2}/2 \end{bmatrix}$$

$$u_1 = \left(\frac{2+5\sqrt{2}}{34}\right)\pi, \quad u_2 = \left(\frac{5+4\sqrt{2}}{34}\right)\pi, \quad u_3 = \left(\frac{2+5\sqrt{2}}{34}\right)\pi$$