2. $-\dfrac{d}{dx}\left(x\dfrac{du}{dx}\right) = x \quad \Omega = (1,2)$

$u(1) = u(2) = 0 \quad V = H_0^1(\Omega)$

$$-\frac{du}{dx} - x\frac{d^2u}{dx^2} = x$$

Multiplicando por x:

$$-x^2\frac{d^2u}{dx^2} - x\frac{du}{dx} = x^2 \quad (2.1)$$

Sea $x = e^z \longrightarrow z = \ln(x)$

$$\frac{du}{dx} = \frac{du}{dz}\frac{dz}{dx} = \frac{du}{dz}\frac{1}{x}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dz}\right)\frac{1}{x} - \frac{1}{x^2}\frac{du}{dz}$$

$$= \frac{d^2u}{dz^2}\cdot\frac{dz}{dx}\cdot\frac{1}{x} - \frac{1}{x^2}\left[x\frac{du}{dx}\right]$$

$$\frac{d^2u}{dz^2} = x^2\frac{d^2u}{dx^2} + x\frac{du}{dx}$$

en (2.1)

$$-\frac{d^2u}{dz^2} = e^{2z} \quad \tilde{\Omega} = (0, \ln(2))$$

$$\underbrace{u(0) = u(\ln(2)) = 0 \quad \tilde{V} = H_0^1(\tilde{\Omega})}_{\text{Problema de Dirichlet.}}$$

Formulación débil:

$$\int_0^{\ln(2)} u'(z)v'(z)\,dz = \int_0^{\ln(2)} e^{2z}v(z)\,dz$$

$$\tilde{V}_2 = \operatorname{gen}\{(e^z-1)(e^z-2), (e^z-1)(e^z-2)e^z\}$$

$$= \operatorname{gen}\{e^{2z}-3e^z+2,\ e^{3z}-3e^{2z}+2e^z\}$$

$$U_2(z) = \alpha_1(e^{2z}-3e^z+2) + \alpha_2(e^{3z}-3e^{2z}+2e^z)$$

$$= \alpha_2e^{3z} + (\alpha_1-3\alpha_2)e^{2z} + (2\alpha_2-3\alpha_1)e^z + 2\alpha_1$$

en la formulación débil:

$$\int_0^{\ln(2)} \left[3\alpha_2e^{3z} + 2(\alpha_1-3\alpha_2)e^{2z} + (2\alpha_2-3\alpha_1)e^z\right]v'(z)\,dz = \int_0^{\ln(2)} e^{2z}v(z)\,dz.$$

Para $v_1(z) = e^{2z}-3e^z+2$:

$$\int_0^{\ln(2)} \left[3\alpha_2e^{3z} + 2(\alpha_1-3\alpha_2)e^{2z} + (2\alpha_2-3\alpha_1)e^z\right]\left[2e^{2z}-3e^z\right]dz = \int_0^{\ln(2)} e^{2z}\left[e^{2z}-3e^z+2\right]dz$$

$$\frac{\alpha_1}{2} + \frac{47\alpha_2}{60} = \frac{-1}{4}$$

$$30\alpha_1 + 47\alpha_2 = -15$$

Para $v_2(z) = e^{3z}-3e^{2z}+2e^z$

$$\int_0^{\ln(2)} \left[3\alpha_2e^{3z} + 2(\alpha_1-3\alpha_2)e^{2z} + (2\alpha_2-3\alpha_1)e^z\right]\left[3e^{3z}-6e^{2z}+2e^z\right]dz = \int_0^{\ln(2)} e^{2z}\left[e^{3z}-3e^{2z}+2e^z\right]dz$$

$$\frac{1}{60}(47\alpha_1 + 78\alpha_2) = \left(\frac{e^{5z}}{5} - \frac{3}{4}e^{4z} + \frac{2}{3}e^{3z}\right)\Bigg|_0^{\ln(2)}$$

$$\frac{1}{60}(47\alpha_1 + 78\alpha_2) = \frac{32}{5} - 12 + \frac{16}{3} - \frac{1}{5} + \frac{3}{4} - \frac{2}{3}$$

$$47\alpha_1 + 78\alpha_2 = -23$$

De donde $\alpha_1 = \dfrac{-89}{131}$, $\alpha_2 = \dfrac{15}{131}$

$$U_m(z) = -\frac{89}{131}\left(e^{2z}-3e^z+2\right) + \frac{15}{131}\left(e^{3z}-3e^{2z}+2e^z\right)$$

$$U_m(x) = \frac{-89}{131}\left(x^2-3x+2\right) + \frac{15}{131}\left(x^3-3x^2+2x\right)$$

b) Sol exacta

$$-\frac{d^2u}{dz^2} = e^{2z}$$

$$U_h(z) = C_1 + C_2z$$

$$U_p(z) = Ae^{2z}$$

$$U_p'(z) = 2Ae^{2z}$$

$$U_p''(z) = 4Ae^{2z}$$

$$-4Ae^{2z} = e^{2z}$$

$$A = \frac{-1}{4}$$

$$u(z) = C_1 + C_2z - \frac{e^{2z}}{4}$$

$$u(0) = C_1 - \frac{1}{4} \Rightarrow C_1 = \frac{1}{4}$$

$$u(\ln(2)) = \frac{1}{4} + C_2\ln(2) - 1 = 0$$

$$C_2 = \frac{3}{4\ln(2)}$$

$$u(z) = \frac{1}{4} + \frac{3z}{4\ln(2)} - \frac{e^{2z}}{4}$$

$$u(x) = \frac{1}{4} + \frac{3}{4}\frac{\ln(x)}{\ln(2)} - \frac{x^2}{4}$$

$$u(x) = \frac{1}{4}\left[1 - x^2 + \log_2(x^3)\right]$$