Taller Juan Pablo:

sura

(1) $g(t, W_t) = 3 + t + e^{W_t}$. Sea el proceso de Itô $X_t = e^{W_t}$

Luego, $\quad FIU1 \longrightarrow dX_t = e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt$

$g(t, X_t) = 3 + t + X_t \longrightarrow FIU2:$

$\longrightarrow dg(t, X_t) = dt + dX_t + \cancel{\frac{1}{2}(0)(dX_t)^2}^{\,0}$

$$= dt + \left(e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt\right) = \left(1 + \frac{e^{W_t}}{2}\right) dt + e^{W_t} dW_t$$

(2) $g(W_t) = W_t^3$, $dg(W_t) = 3W_t^2 dW_t + \frac{1}{2} 6 W_t dt$.

(3) $g(W_t) = W_t^4$, $dg(W_t) = 4W_t^3 dW_t + \frac{1}{2}(12 W_t^2 dt) = 4W_t^3 dW_t + 6W_t^2 dt$

$$g(W_t) = 4\int_0^t W_s^3 dW_s + 6\int_0^t W_s^2 ds, \quad E[g(W_t)] \overset{\text{Tomo } E[\cdot]}{=} 4E\left[\cancel{\int_0^t W_s^3 dW_s}^{\,0}\right] + 6E\left[\int_0^t W_s^2 ds\right]$$

$$= 6\int_0^t E[W_s^2] ds = 6\int_0^t s\, ds = 6 \frac{s^2}{2}\Big|_0^t = 3t^2.$$

4) a) $\int_0^t B_s^2\, dB_s = \frac{B_t^3}{3} - \int_0^t B_s\, ds$

Sea $g(B_t) = B_t^3 \rightarrow FIU1 \rightarrow dg(B_t) = 3B_t^2 \overset{dB_t}{} + \frac{1}{2}(6B_t) \overset{dt}{} = 3B_t^2 dB_t + 3B_t dt$

$g(B_t) = B_t^3 = \int_0^t 3B_s^2\, dB_s + 3\int_0^t B_s\, ds$

$\rightarrow \int_0^t B_s^2\, dB_s = \frac{B_t^3}{3} - \int_0^t B_s\, ds$

b) Sea $g(B_t) = B_t^5 \rightarrow FIU1 \rightarrow dg(B_t) = 5B_t^4 dB_t + \frac{1}{2}(20B_t^3)dt$

$B_t^5 = \int_0^t 5B_s^4\, dB_s + 10\int_0^t B_s^3\, ds \rightarrow \int_0^t B_s^4\, dB_s = \frac{B_t^5}{5} - 2\int_0^t B_s^3\, ds$

c) $\int_0^t B_s^n\, dB_s$. Sea $g(B_t) = B_t^{n+1}$, $dg(B_t) = (n+1)B_t^n dB_t + \frac{1}{2}(n+1)(n)B_t^{n-1}dt$

$\rightarrow B_t^{n+1} = \int_0^t (n+1)B_s^n\, dB_s + \frac{n(n+1)}{2}\int_0^t B_s^{n-1}\, ds$.

$\rightarrow \int_0^t B_s^n\, dB_s = \frac{B_t^{n+1}}{n+1} - \frac{n}{2}\int_0^t B_s^{n-1}\, ds$

5) a)

6) $X_t = e^{B_t^2} \rightarrow dX_t = 2B_t e^{B_t^2} dB_t + \frac{1}{2}\left[2\left(e^{B_t^2} + 2B_t^2 e^{B_t^2}\right)\right] dt.$

$$dX_t = 2B_t e^{B_t^2} dB_t + e^{B_t^2}(1 + 2B_t^2)\,dt.$$

b) $X_t = e^{\sigma B_t} \rightarrow dX_t = \sigma e^{\sigma B_t} dB_t + (1/2)\sigma^2 e^{\sigma B_t} dt.$

c) $X_t = e^{\sigma B_t - \frac{\sigma^2}{2} t}$

Sea $Y_t = -\frac{\sigma^2}{2} t + \sigma B_t = \int_0^t -\frac{\sigma^2}{2} dt + \int_0^t \sigma dB_t$ → Proceso de Itô

→ $x_0 = 0$

$f(t) = -\frac{\sigma^2}{2}$

$h(t) = \sigma^2$

$$dY_t = -\frac{\sigma^2}{2} dt + \sigma dB_t$$

Luego, $X_t = e^{Y_t} \rightarrow$ FIUZ. $dX_t = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} (dY_t)^2$

$$dX_t = e^{Y_t}\left(-\frac{\sigma^2}{2} dt + \sigma dB_t\right) + \frac{1}{2} e^{Y_t}\left(-\frac{\sigma^2}{2} dt + \sigma dB_t\right)^2$$

$$dX_t = \cancel{-\frac{\sigma^2 e^{Y_t}}{2} dt} + \sigma e^{Y_t} dB_t + \cancel{\frac{e^{Y_t}}{2}(\sigma^2 dt)} = \sigma e^{-\frac{\sigma^2}{2} t + \sigma B_t} dB_t$$

FIU1

7. a) $X_t = 2+t+e^{B_t}$ Sea $Y_t = e^{B_t} \rightarrow dY_t = e^{B_t} dB_t + \frac{e^{B_t}}{2} dt$

$X_t = 2+t+Y_t$

$dX_t = dt + dY_t + \frac{1}{2}(0)(dY_t)^2 = dt + e^{B_t} dB_t + \frac{e^{B_t}}{2} dt = \left(1+\frac{e^{B_t}}{2}\right)dt + e^{B_t} dB_t$

b) $X_t = B_1^2(t) + B_2^2(t)$, $B_t = (B_1(t), B_2(t))^T$

$dX_t = \frac{1}{2}\underbrace{[2+2]}_{f(t)}dt + \underbrace{[2B_1(t) \;\; 2B_2(t)]}_{h(t)}\begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix} = 2dt + 2B_1(t)dB_1(t) + 2B_2(t)dB_2(t)$

c) $X_t = (t_0+t, B_t)^T$, $B_t \rightarrow$ Unidimensional

FIU2: Sea $Y_t = B_t \rightarrow dY_t = dB_t$

$dX_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dY_t + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(dY_t)^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} dB_t$.

d) $X_t = \begin{bmatrix} B_1(t)+B_2(t)+B_3(t) \\ B_2^2(t) - B_1(t)B_3(t) \end{bmatrix}$, $B_t = (B_1(t), B_2(t), B_3(t))^T$

FIM1:

$dX_t = \frac{1}{2}\begin{bmatrix} 0 \\ 0+2+0 \end{bmatrix} dt + \begin{bmatrix} 1 & 1 & 1 \\ -B_3(t) & 2B_2(t) & -B_1(t) \end{bmatrix}\begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}$

8\. a) $Z(t)=e^{\alpha t}$, $dZ(t)=\alpha e^{\alpha t}dt$.

b) $Z(t)=\int_0^t g(s)dB_t$ $dZ(t)=g(t)dt$

c) $Z(t)=e^{\alpha W_t}\rightarrow dZ(t)=\alpha e^{\alpha W_t}dW_t+\frac{\alpha^2 e^{\alpha W_t}}{2}dt$.

d) $Z(t)=e^{\alpha X_t}$, donde $dX_t=\mu dt+\sigma dB_t$

$$dZ(t)=\alpha e^{\alpha X_t}(dX_t)+\frac{1}{2}(\alpha^2 e^{\alpha X_t})(dX_t)^2$$

$$=\alpha e^{\alpha X_t}(\mu dt+\sigma dB_t)+\frac{\alpha^2}{2}e^{\alpha X_t}(\mu dt+\sigma dB_t)^2$$

$$=\left(\alpha\mu e^{\alpha X_t}+\frac{\alpha^2\sigma^2}{2}e^{\alpha X_t}\right)dt+\sigma\alpha e^{\alpha X_t}dB_t.$$

e) $Z(t)=X_t^2$, donde $dX_t=\alpha X_t dt+\sigma X_t dB_t$.

$$dZ(t)=2X_t(dX_t)+\frac{1}{2}(2)(dX_t)^2$$

$$=2X_t(\alpha X_t dt+\sigma X_t dB_t)+(\alpha X_t dt+\sigma X_t dB_t)^2$$

$$=(2X_t^2+\sigma^2X_t^2)dt+\sigma X_t dB_t.$$

f) $Z(t)=X_t^{-1}$, donde $dX_t=\alpha X_t dt+\sigma X(t)dB_t$

$$dZ(t)=-X_t^{-2}dX_t+X_t^{-3}(dX_t)^2$$

$$=-X_t^{-2}(\alpha X_t dt+\sigma X_t dB_t)+X_t^{-3}(\alpha X_t dt+\sigma X_t dB_t)^2$$

$$=(-\alpha X_t^{-1}+\sigma^2X_t^{-2})dt-X_t^{-1}\sigma dB_t.$$

9. $B_t \in \mathbb{R}$, $B_0 = 0$. $\beta_k(t) := E[B_t^k]$, $k \in \mathbb{N}$, $t \geq 0$. Pruebe

$$\beta_k(t) = \frac{k(k-1)}{2} \int_0^t \beta_{k-2}(s)\,ds$$

Sea $g(B_t) = B_t^k$. Por Itô: $dg(B_t) = kB_t^{k-1}dB_t + \frac{k(k-1)}{2} B_t^{k-2}dt$

En forma integral

$$B_t^k = \int_0^t kB_s^{k-1}dB_s + \int_0^t \frac{k(k-1)}{2} B_s^{k-2}ds \quad \leftarrow \text{Tomando esperanza.}$$

$$E[B_t^k] = k\,E\left[\cancel{\int_0^t B_s^{k-1}dB_s}\right] + \frac{k(k-1)}{2} E\left[\int_0^t B_s^{k-2}ds\right] = \frac{k(k-1)}{2}\int_0^t E[B_s^{k-2}]ds$$

0

Luego,

$$\beta_k(t) = \frac{k(k-1)}{2}\int_0^t \beta_{k-2}(s)\,ds \quad \square$$

10. $W_t = (W_1(t), W_2(t))^T$. $g(W_t) = W_1^2(t) + W_2^2(t)$

$$dg(W_t) = \frac{1}{2}(2+2)dt + \begin{bmatrix} 2W_1(t) & 2W_2(t) \end{bmatrix}\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}$$

$$= 2dt + (2W_1(t)dW_1(t) + 2W_2(t)dW_2(t))$$

$$= \int_0^t 2ds + \int_0^t 2W_1(s)dW_1(s) + \int_0^t 2W_2(s)dW_2(s) \quad \square$$

$\{Z_t\}_{t\geq 0} \rightarrow$ Itô 2-dim.: $Z_t = (X_t, Y_t)^T$.

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11. X_t y Y_t, $t\geq 0$ dos procesos de Itô en $\mathbb{R}$. $g(t, Z_t) = X_t Y_t$

Apliquemos FIMR.

$$dg(t,Z_t) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial X_t}dX_t + \frac{\partial g}{\partial Y_t}dY_t + \frac{1}{2}\left(\cancelto{0}{\frac{\partial^2 g}{\partial X_t^2}}(dX_t)^2 + 2\frac{\partial^2 g}{\partial X_t Y_t}(dX_t dY_t) + \cancelto{0}{\frac{\partial^2 g}{\partial Y_t^2}}(dY_t)^2\right)$$

$$= \cancelto{0}{\frac{\partial(X_tY_t)}{\partial t}}dt + Y_t dX_t + X_t dY_t + \frac{\partial^2 g}{\partial X_t \partial Y_t}(dX_t dY_t)$$

$$= Y_t dX_t + X_t dY_t + dX_t dY_t$$

$$dg(t,Z_t) = Y_t dX_t + X_t dY_t + dX_t dY_t \leftarrow \text{Integrando}$$

$$\int_0^t dg(s,Z_s) = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_t dY_t$$

$$X_tY_t - X_0Y_0 = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_s dY_s$$

$$\Longrightarrow \int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s$$

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13) Sean $c, \alpha_1, \dots, \alpha_n \in \mathbb{R}$ y $B_t = (B_1(t), \dots, B_n(t))^T$

$X_t = \exp\left(ct + \sum_{j=1}^{n} \alpha_j B_j(t)\right)$. Sea $Y_t = ct + \sum_{j=1}^{n} \alpha_j B_j(t)$

Sea $Z_t = \sum_{j=1}^{n} \alpha_j B_j(t) \to dZ_t = \sum_{j=1}^{n} \alpha_j dB_j(t)$

$Y_t = ct + Z_t \to dY_t = c\,dt + dZ_t + \frac{1}{2}(0)(dZ_t)^2 = c\,dt + dZ_t = c\,dt + \sum_{j=1}^{n} \alpha_j dB_j(t)$

Luego

$X_t = e^{Y_t} \to dX_t = e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} (dY_t)^2$

$dX_t = e^{Y_t}\left(c\,dt + \sum_{j=1}^{n} \alpha_j dB_j(t)\right) + \frac{1}{2} e^{Y_t}\left(c\,dt + \sum_{j=1}^{n} \alpha_j dB_j(t)\right)^2$

$= c e^{Y_t} dt + e^{Y_t} \sum_{j=1}^{n} \alpha_j dB_j(t) + \frac{e^{Y_t}}{2} \sum_{j=1}^{n} \alpha_j^2 dt$

$= e^{Y_t}\left(c + \frac{1}{2} \sum_{j=1}^{n} \alpha_j\right) dt + e^{Y_t} \sum_{j=1}^{n} \alpha_j dB_j(t)$

$= X_t\left(c + \frac{1}{2} \sum_{j=1}^{n} \alpha_j\right) dt + X_t \sum_{j=1}^{n} \alpha_j dB_j(t)$

14. $X(t)=\int_0^t \sigma(s)dW(s)$, $u\in\mathbb{R}$.

Pruebe que: $E[e^{uX(t)}]=e^{\frac{u^2}{2}\int_0^t \sigma^2(s)ds}$.

Sea $g(t,X_t)=e^{uX(t)}$

$$dg(t,X_t)=ue^{uX(t)}dX(t)+\frac{u^2}{2}e^{uX(t)}(dX(t))^2.$$

$$dX(t)=\sigma(t)dW(t)$$

$$dg(t,X_t)=ue^{uX(t)}(\sigma(t)dW(t))+\frac{u^2}{2}e^{uX(t)}(\sigma(t)dW(t))^2$$

$$=ue^{uX(t)}\sigma(t)dW(t)+\frac{u^2}{2}e^{uX(t)}\sigma^2(t)dt.$$

$$e^{uX(t)}=\int_0^t ue^{uX(s)}\sigma(s)dW(s)+\int_0^t \frac{u^2e^{uX(t)}}{2}\sigma^2(s)ds \leftarrow \text{Tomando } E[\cdot].$$

$$E[e^{uX(t)}]=E\left[\cancel{\int_0^t ue^{uX(s)}\sigma(s)dW(s)}\right]+E\left[\int_0^t \frac{u^2e^{uX(t)}}{2}\sigma^2(s)ds\right]$$

0

$$E[e^{uX(t)}]=\frac{u^2}{2}\int_0^t E[e^{uX(s)}]\sigma^2(s)ds$$

Sea $\varphi(t)=E[e^{uX(t)}]$, diferenciando

$$\varphi'(t)=\frac{u^2}{2}\varphi(t)\sigma^2(t)\longrightarrow\frac{\varphi'(t)}{\varphi(t)}=\frac{u^2\sigma^2(t)}{2}\quad \varphi(t)=e^{\frac{u^2}{2}\int_0^t\sigma^2(s)ds}.$$

$\varphi(0)=1$

$2(S_t)^{1/2}$

(15). $dS_t = \left(\frac{1}{4}\sigma^2 - \alpha\sqrt{S_t}\right)dt + \sigma\sqrt{S_t}\,dW_t$; $X_t = 2\sqrt{S_t}$

$$dX_t = (S_t)^{-1/2} dS_t - \frac{1}{4}(S_t)^{-3/2}(dS_t)^2$$

$$= (S_t)^{-1/2}\left[\left(\frac{1}{4}\sigma^2 - \alpha\sqrt{S_t}\right)dt + \sigma\sqrt{S_t}\,dW_t\right] - \frac{1}{4}(S_t)^{-3/2}\left[\left(\frac{\sigma^2}{4} - \alpha\sqrt{S_t}\right)dt + \sigma\sqrt{S_t}\,dW_t\right]^2$$

$$= \cancel{\frac{\sigma^2 (S_t)^{-1/2}}{4}dt} - \alpha dt + \sigma dW_t - \cancel{\frac{(S_t)^{-3/2}}{4}(\sigma^2 S_t dt)}$$

$$= -\alpha dt + \sigma dW_t$$

(16). a) $X_t = e^{B_t}$ resuelve $dX_t = \frac{X_t}{2}dt + X_t dB_t$

$$dX_t = e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt = X_t dB_t + \frac{X_t}{2}dt. \quad \square$$

b) $X_t = (\alpha^{1/3} + B_t/3)^3$, $\alpha > 0$. resuelve $dX_t = \frac{X_t^{1/3}}{3}dt + X_t^{2/3}dB_t$.

$$dX_t = 3(\alpha^{1/3} + B_t/3)^2(1/3)dB_t + \frac{1}{2}\,6(\alpha^{1/3} + B_t/3)\left(\frac{1}{3}\right)^2 dt.$$

$$dX_t = (\alpha^{1/3} + B_t/3)^2 dB_t + \frac{1}{3}(\alpha^{1/3} + B_t/3)dt$$

$$dX_t = X_t^{2/3}dB_t + \frac{1}{3}X^{1/3}dt. \quad \square$$

c) $X_t = \frac{B_t}{t+1}$ resuelve $dX_t = -\frac{X_t}{t+1}dt + \frac{dB_t}{1+t}$, $X_0 = 0$

$dX_t = -\frac{B_t}{(t+1)^2}dt + \frac{dB_t}{t+1} + \cancel{\frac{1}{2}(0)(dB_t)^2} \Rightarrow dX_t = \frac{-B_t}{(t+1)} \cdot \frac{dt}{t+1} + \frac{dB_t}{t+1}$

$dX_t = -\frac{X_t\,dt}{t+1} + \frac{dB_t}{t+1}$ □

d) $X_t = \sin(B_t)$, $X_0 = a \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. $dX_t = -\frac{X_t\,dt}{2} + \sqrt{1-X_t^2}\,dB_t$

$dX_t = \cos(B_t)dB_t + \frac{1}{2}(-\sin(B_t))dt$, $\cos x = \sqrt{1-\sin^2 x}$ Luego

$dX_t = -\frac{X_t}{2}dt + \sqrt{1-X_t^2}\,dB_t$ □

e) $X(t) = (t, e^t B_t)^T$ resuelve $dX(t) = \begin{bmatrix} 1 \\ X_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1(t)} \end{bmatrix} dB_t$

$dX(t) = \begin{bmatrix} 1 \\ e^t B_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^t \end{bmatrix} dB_t + \cancel{\frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} dt}^{\,0}$

$= \begin{bmatrix} 1 \\ X_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ e^{X_1(t)} \end{bmatrix} dB_t$ □

f) $X(t) = [a\,Cos(B_t),\ b\,Sin(B_t)]^T$ resuelve $dX_t = -\frac{X(t)}{2}dt + M\,X(t)\,dB_t$

donde $M = \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix}$

$$dX(t) = \begin{bmatrix} -a\,Sin(B_t) \\ b\,Cos(B_t) \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} -a\,Cos(B_t) \\ -b\,Sin(B_t) \end{bmatrix} dt$$

$$= \begin{bmatrix} -(a/b)\,X_2(t) \\ b/a\,X_1(t) \end{bmatrix} dB_t - \frac{1}{2}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} 0 & -a/b \\ b/a & 0 \end{bmatrix}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dB_t - \frac{1}{2}X(t)\,dt = M\,X(t)\,dB_t - \frac{X(t)}{2}dt.$$

g) $X(t) = [Cosh(B_t)\ Sinh(B_t)]^T$ resuelven $dX(t) = \frac{1}{2}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \begin{bmatrix} X_2(t) \\ X_1(t) \end{bmatrix} dB_t$

$$dX(t) = \begin{bmatrix} Sinh(B_t) \\ Cosh(B_t) \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} Cosh(B_t) \\ Sinh(B_t) \end{bmatrix} dt$$

$$= \begin{bmatrix} X_2(t) \\ X_1(t) \end{bmatrix} dB_t + \frac{1}{2}\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt.$$

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