# Numerical Analysis II: Homeworks

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## Workshop 1

### Exercise 14

*Question* 1. Prove that every normed and finite-dimensional vector space is a Banach space.

*Proof.* Let $(V, \|\cdot\|)$ be a normed and finite-dimensional vector space. As $V$ is finite-dimensional, let $B = \{v_1, \ldots, v_k\}$ be a basis for $V$, that is, $\forall v \in V$, $v = \sum_{j=1}^{n} \alpha_j v_j$. Define the norm $\|\cdot\|^* : V \to \mathbb{R}$ as

$$\|x\|^* = \sqrt{\sum_{j=1}^{k} (\alpha_j)^2}.$$

It is well known (see e.g. []) that all norms on a finite dimensional vector space are equivalent. Hence, $\forall x \in V$, $\exists M > 0$ such that

$$\frac{1}{M} \|x\|^* \leq \|x\| \leq M \|x\|^* .$$

Using this, let us prove that $V$ is a Banach space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $V$. Now, it is clear that each $x_n$ in this sequence can be written as

$$x_n = \sum_{j=1}^{k} \alpha_j^{(n)} v_j$$

for some constants $\alpha_j^{(n)} \in \mathbb{R}$. Now, if $n, m \geq 1$ then by the equivalence of norms we have

$$\begin{aligned}
\|x_m - x_n\| &\geq \frac{1}{M} \|x_m - x_n\|^* \\
&\geq \frac{1}{M} \sqrt{\sum_{j=1}^{k} \left(\alpha_j^{(m)} - \alpha_j^{(n)}\right)^2} \\
&\geq \frac{1}{M} \left|\alpha_j^{(m)} - \alpha_j^{(n)}\right|, \ \forall j = 1, \ldots, k.
\end{aligned}$$

Thus, $\{\alpha_1^{(n)}\}_{n\in\mathbb{R}}, \ldots, \{\alpha_k^{(n)}\}_{n\in\mathbb{R}}$ are all Cauchy sequences in $\mathbb{R}$. As $\mathbb{R}$ is complete, these sequences converge respectively to some constants $\alpha_1, \ldots, \alpha_k$. Let $x = \sum_{j=1}^k \alpha_j v_j \in V$. Now, again for the equivalence of norms, $\forall n \in \mathbb{N}$ we have

$$\begin{aligned}\|x - x_n\| &\le M \|x - x_n\|^* \\ &\le M \sqrt{\sum_{j=1}^k \left(\alpha_j - \alpha_j^{(n)}\right)^2}\end{aligned}$$

and taking the limit as $n \to \infty$, we obtain that $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in V$. Therefore, $V$ is a Banach space. □

## Exercise 15

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ vectors in $\mathbb{R}^n$. Let $1 < p < \infty$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

*Question* 2. Prove Hölder's inequality for vectors

$$\sum_{j=1}^n |x_j y_j| \le \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}}$$

*Proof.* The case when $x = 0$ or $y = 0$ it trivial and the inequality holds. Assuming $x \neq 0$ and $y \neq 0$.

Recall Young's inequality (see e.g. [1]): let $a, b > 0$ real numbers and $p, q$ selected as above, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{1}$$

the specific case for

$$a = \frac{|x_j|}{\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}},\ b = \frac{|y_j|}{\left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}}},\ j = 1, \ldots, n$$

yields

$$\frac{|x_j y_j|}{\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}}} \le \frac{1}{p}\frac{|x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q}\frac{|y_j|^q}{\sum_{j=1}^n |y_j|^q}.$$

Now, taking the sum of the inequalities for $j = 1, \ldots, n$, we obtain

$$\frac{\sum_{j=1}^n |x_j y_j|}{\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q\right)^{\frac{1}{q}}} \le \frac{1}{p}\frac{\sum_{j=1}^n |x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q}\frac{\sum_{j=1}^n |y_j|^q}{\sum_{j=1}^n |y_j|^q} = \frac{1}{p} + \frac{1}{q} = 1$$

which directly implies

$$\sum_{j=1}^{n} |x_j y_j| \leq \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} |y_j|^q \right)^{\frac{1}{q}}$$

$\square$

*Question* 3. Prove Minkowski's inequality for vectors

$$\left( \sum_{j=1}^{n} |x_j + y_j|^p \right)^{\frac{1}{p}} \leq \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}}$$

*Proof.* For $p > 1$, $q = \frac{p}{p-1}$ to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\sum_{j=1}^{n} |x_j + y_j|^p = \sum_{j=1}^{n} |x_j + y_j| \, |x_j + y_j|^{p-1}$$

and by the triangle inequality,

$$\begin{aligned}
\sum_{j=1}^{n} |x_j + y_j|^p &\leq \sum_{j=1}^{n} (|x_j| + |y_j|) \, |x_j + y_j|^{p-1} \\
&= \sum_{j=1}^{n} |x_j| \, |x_j + y_j|^{p-1} + \sum_{j=1}^{n} |y_j| \, |x_j + y_j|^{p-1} \\
&= \sum_{j=1}^{n} \left| x_j (x_j + y_j)^{p-1} \right| + \sum_{j=1}^{n} \left| y_j (x_j + y_j)^{p-1} \right|.
\end{aligned}$$

Now, by Hölder's inequality on both terms we have

$$\begin{aligned}
&\leq \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} \left| |x_j + y_j|^{p-1} \right|^q \right)^{\frac{1}{q}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} \left| |x_j + y_j|^{p-1} \right|^q \right)^{\frac{1}{q}} \\
&= \left( \sum_{j=1}^{n} \left| |x_j + y_j|^{p-1} \right|^q \right)^{\frac{1}{q}} \left[ \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}} \right] \\
&= \left( \sum_{j=1}^{n} \left| |x_j + y_j|^{p-1} \right|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left[ \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}} \right] \\
&= \left( \sum_{j=1}^{n} |x_j + y_j|^p \right)^{\frac{p-1}{p}} \left[ \left( \sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{n} |y_j|^p \right)^{\frac{1}{p}} \right].
\end{aligned}$$

Thus,

$$\sum_{j=1}^{n} |x_j + y_j|^p \leq \left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{\frac{p-1}{p}} \left[\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}\right],$$

and dividing both sides by the first term on the right side, we have

$$\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1-\frac{p-1}{p}} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

$$\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{\frac{1}{p}}$$

□

## Exercise 16

Let

$$\ell^p(\mathbb{R}) = \left\{ \{x_n\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

*Question* 4. Prove that $\ell^p(\mathbb{R})$ is a Banach space with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty$$

and the sum and scalar product defined by

$$x + y = \{x_n + y_n\}_{n\in\mathbb{N}}, \quad ax = \{ax_n\}_{n\in\mathbb{N}}, \quad \forall x, y \in \ell^p(\mathbb{R}),\ \forall a \in \mathbb{R}$$

*Proof.* Proving that $\ell^p(\mathbb{R})$ is vector space with the operations defined above is a well-known result and will not be presented here. Let us prove that it is a Banach space (normed and complete).

Let us prove that $\|x\|_p$ is indeed a norm in $\ell^p(\mathbb{R})$: let $x, y \in \ell^p(\mathbb{R})$ and $a \in \mathbb{R}$. Let $p \in [1, \infty]$.

1. Since $|x_n| \geq 0$, $\forall n \in \mathbb{N}$, then

$$|x_n| \geq 0 \implies |x_n|^p \geq 0 \implies \sum_{n=1}^{\infty} |x_n|^p \geq 0 \implies \|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \geq 0.$$

2. "$\implies$" Let us suppose that $\|x\|_p = 0$, then

$$\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = 0 \implies |x_n|^p = 0$$

since $|x_n|^p \geq 0, \ \forall n \in \mathbb{N}$. Then $x_n = 0, \ \forall n \in \mathbb{N} \implies x = 0$ (with $0 = \{0\}_{n\in\mathbb{N}}$ in $\ell^p(\mathbb{R})$).

" $\Longleftarrow$ " Let us suppose that $x = 0$, then $x_n = 0, \ \forall n \in \mathbb{N}$ which directly implies that $\|x\|_p = 0$. Therefore, $\|x\|_p = 0 \iff x = 0$.

3.

$$\|ax\|_p = \left(\sum_{n=1}^{\infty} |ax_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |a|^p |x_n|^p\right)^{\frac{1}{p}} = \left(|a|^p \sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$
$$= |a| \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = |a| \, \|x\|_p .$$

4.

$$\|x+y\|_p = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}}$$

by Minkowski's inequality, we have

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p .$$

Now, let us prove that $\ell^p(\mathbb{R})$ is complete. We will prove first the case for $p \in [1, \infty)$: let $1 \leq p < \infty$ and let $x_n = \{x_n\}_{k\in\mathbb{N}}$ by a Cauchy sequence in $\ell^p(\mathbb{R})$ for each $n \in \mathbb{N}$. Hence, for every $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that

$$|\{x_n\}_k - \{x_m\}_k| \leq \|x_n - x_m\|_p \leq \epsilon, \quad \forall m, n \geq N_\epsilon, \quad \forall k \in \mathbb{N}.$$

This implies that $\exists x_k = \lim_{n\to\infty} \{x_n\}_k$ in $\mathbb{R}$. Let $x = \{x_k\}_{k\in\mathbb{N}}$ and by Minkowski's inequality

$$\sum_{k=1}^{K} |\{x_n\}_k - \{x_m\}_k|^p \leq \|x_n - x_m\|_p^p \leq \epsilon^p, \quad \forall n, m \geq N_\epsilon, \quad K \in \mathbb{N}.$$

Taking $m \to \infty$, it follows

$$\sum_{k=1}^{K} |\{x_n\}_k - x_k|^p \leq \epsilon^p$$

and taking the supremum over $k \in \mathbb{N}$, we obtain

$$\sum_{k=1}^{\infty} |\{x_n\}_k - x_k|^p \leq \epsilon^p, \quad n \geq N_\epsilon.$$

Therefore, $x_n - x \in \ell^p\mathbb{R})$ and converges to 0 in $\ell^p(\mathbb{R})$ as $n \to \infty$. Thus, $x \in \ell^p(\mathbb{R})$ and $x_n \to x$ in $\ell^p(\mathbb{R})$ and $\ell^p(\mathbb{R})$ is Banach space for $p \in [1, \infty)$.

Now, let $p = \infty$. The norm is given by

$$\|x\|_{\infty} = \sup_{n\in\mathbb{N}} |x_n| .$$

Let $\{x^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^\infty(\mathbb{R})$, thus each $x^n = \{x^n_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Let $\epsilon > 0$ and since $\{x^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence $\exists N_\epsilon > 0$ such that

$$\|x^n - x^m\|_{\infty} < \epsilon, \quad \forall m, n > N_\epsilon.$$

Thus,

$$\sup_{k\in\mathbb{N}} |x^n_k - x^m_k| < \epsilon, \quad \forall m, n > N_\epsilon$$

which directly implies that

$$|x^n_k - x^m_k| < \epsilon, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_\epsilon.$$

This means that $\forall k \in \mathbb{N}$, the sequence $\{x^n_k\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. It is well known that $\mathbb{R}$ is a Banach space, hence this sequence is convergent. Let $x_k = \lim_{n\to\infty} x^n_k = x_k$ and let $x = \{x_k\}_{k\in\mathbb{N}}$.

Now, let us see that $\{x^n\}_{n\in\mathbb{N}}$ converges to $x$. Let $\gamma > 0$ and choose $\gamma/2$ as the $\epsilon$ in the previous discussion, then there exists $N_{\gamma/2}$ such that

$$|x^n_k - x^m_k| < \gamma/2, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_{\gamma/2}.$$

Taking the limit as $m \to \infty$, we have

$$|x^n_k - x_k| \le \gamma/2, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_{\gamma/2},$$

and finally, taking the supremum in $k \in \mathbb{N}$, we obtain

$$\sup_{k\in\mathbb{N}} |x^n_k - x_k| = \|x^n - x\|_{\infty} \le \gamma/2 < \gamma, \quad \forall n \in \mathbb{N}.$$

Therefore, $x^n$ converges to $x$ in $\ell^\infty(\mathbb{R})$. Thus, $\ell^\infty(\mathbb{R})$ is a Banach space.

We conclude that $\ell^p(\mathbb{R})$ is Banach space for $p \in [1, \infty]$. □

## Exercise 18

*Question* 5. Let $V$ be a Hilbert space and suppose that $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal set in $V$. The following statements are equivalent:

1. $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $V$.

2. The subspace $\operatorname{span}\{v_j\}_{j\in\mathbb{N}}$ is dense in $V$.

3. Parseval's equality:

$$\|u\|^2 = \sum_{j\in\mathbb{N}} |\langle u, v_j\rangle|^2 , \quad \forall u \in V.$$

4. $\forall u, v \in V$,
$$\langle u, v\rangle = \sum_{j\in\mathbb{N}} \langle u, v_j\rangle\langle v, v_j\rangle$$

5. $\forall u \in V$, if $\forall j \in \mathbb{N},\ \langle u, v_j\rangle = 0$, then $u = 0$.

*Proof.*

- "1. $\implies$ 2.": Suppose that $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $V$ and let $M = \overline{\text{span}\{v_j\}_{j\in\mathbb{N}}}$. It is clear that $M$ is closed subspace. Furthermore, as $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal set in $V$, it is a maximal and hence $V^\perp = \{0\}$, then $M^\perp = \{0\}$ and finally $M = V$. Clearly, $\text{span}\{v_j\}_{j\in\mathbb{N}}$ is dense in $V$.

- "2. $\implies$ 3.": If $u = 0$, it is trivial. Let $u \in V$ such that $u \neq 0$ and let $\epsilon > 0$. Assume, without loss of generality, that $\|u\| > \epsilon$ and choose $y \in V$ such that $\|x - y\| < \epsilon$ and $y \in \text{span}\{v_j\}_{j=1}^k$, for some $k \in \mathbb{N}$. Moreover, let
$$z = \sum_{j=1}^{k} \langle u, v_j\rangle v_j.$$
Then, $z$ minimizes $\|x - w\|$ over $w \in \text{span}\{v_j\}_{j=1}^k$ and hence $\|x - z\| \leq \|x - y\| < \epsilon$. Thus, $\|x\| < \|z\| + \epsilon$ and $(\|x\| - \epsilon)^2 < \|z\|^2$ and $\|z\|^2 = \sum_{j=1}^k |\langle u, v_j\rangle|^2 \leq \sum_{j\in\mathbb{N}} |\langle u, v_j\rangle|^2$. Taking $\epsilon \to 0$, $\|u\|^2 \leq \sum_{j\in\mathbb{N}} |\langle u, v_j\rangle|^2$. The other inequality is obtained from Bessel's inequality.

- "3 $\implies$ 4": Consequence of the polarization identity as only a countable number of terms in the sum are nonzero.

- "4 $\implies$ 5": Suppose that $\forall j \in \mathbb{N}$, $\langle u, v_j\rangle = 0$, then $\|u\|^2 = \langle u, u\rangle = 0$ and hence $u = 0$.

- "5 $\implies$ 1": If $\{v_j\}_{j\in\mathbb{N}}$ is not a base, it is not a maximal orthonormal set in $V$ and hence, $\exists x = 0$ such that $\langle x, v_j\rangle = 0$, for all $j \in \mathbb{N}$, contradicting 5.

$\square$

# Workshop 2

## Exercise 9 b)

*Question* 6. Let $V$ be a Banach Space and $\lambda$ a scalar. Suppose $T \in \mathcal{L}(V, V)$. Determine if the inverse operator of $\lambda I - T$ exists. If it does, find it.

*Proof.* Let
$$S = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Let's show that $S$ is the inverse operator of $H = \lambda I - T$. First let's show that $H(S(x)) = x$:

$$\begin{aligned}
H(S(x)) &= \lambda S(x) - T(S(x)) \\
&= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} T\left(\frac{T^n(x)}{\lambda^{n+1}}\right) \\
&= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}(x)}{\lambda^{n+1}} \qquad (2)
\end{aligned}$$

Changing the variable in the second summation to $n \to n-1$ we obtain

$$\begin{aligned}
H(S(x)) &= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=1}^{\infty} \frac{T^n(x)}{\lambda^n} \\
&= \frac{T^0(x)}{\lambda^0} \\
&= x
\end{aligned}$$

Lastly, let's show that $S(H(x)) = x$:

$$\begin{aligned}
S(H(x)) &= \sum_{n=0}^{\infty} \frac{T^n(\lambda x - T(x))}{\lambda^{n+1}} \\
&= \sum_{n=0}^{\infty} \underbrace{\frac{T^n(x)}{\lambda^n}}_{a_n} - \underbrace{\frac{T^{n+1}(x)}{\lambda^{n+1}}}_{b_n}
\end{aligned}$$

It is clear that both $a_n$ and $b_n$ are convergent as $T$ is bounded (as it is continuous). In this manner, the sum can be separated as:

$$S(H(x)) = \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}(x)}{\lambda^{n+1}} \qquad (3)$$

Note that Equation 3 is the same one as the Equation 2. Therefore, doing the same process we would obtain that:

$$S(H(x)) = x$$

Therefore, $S$ is the inverse operator of $H$. $\square$

## Slide 23

*Question* 7. Let $\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear and bounded.}\}$. Show that $\mathcal{L}(V, W)$ is a normed vector space, with the norm for a $T \in L(V, W)$ be

$$\|T\| = \sup_{\|u\|_V = 1} \|Tu\|_W$$

*Proof.* Let's show that it is a vector space. In first place, it is seen that $\mathcal{L}(V,W) \subset \mathcal{C}(V,W)$, with $\mathcal{C}(V,W)$ being the space of all the continuous functions that have as domain $V$ and as co-domain $W$. Furthermore, it is well known that $\mathcal{C}(V,W)$ is a vector space.

Hence, to show that $\mathcal{L}(V,W)$ is a vector space it is only needed to show that the sum and multiplication are closed under the same set.

- Let $T, H \in \mathcal{L}(V,W)$. Let's show that $T+H$ is linear.

$$\begin{aligned}(T+H)(\alpha x+\beta y) &= T(\alpha x+\beta y)+H(\alpha x+\beta y)\\ &= \alpha Tx+\beta Ty+\alpha Hx+\beta Hy\\ &= \alpha(Tx+Hx)+\beta(Ty+Hy)\\ &= \alpha(T+H)x+\beta(T+H)y\end{aligned}$$

  Let's show that $T+H$ is bounded.

$$\begin{aligned}\|(T+H)x\|_W &= \|Tx+Hx\|_W\\ &\leq \|Tx\|_W+\|Hx\|_W\\ &\leq M_T\|x\|_V+M_H\|x\|_V\\ &= \underbrace{(M_T+M_H)}_{M}\|x\|_V\\ &= M\|x\|_V\end{aligned}$$

- Let $\lambda \in \mathbb{R}$. Let's show that $\lambda T$ is linear.

$$\begin{aligned}(\lambda T)(\alpha x+\beta y) &= \lambda T(\alpha x+\beta y)\\ &= \lambda\alpha Tx+\lambda\beta Ty\\ &= \alpha(\lambda T)x+\beta(\lambda T)y\end{aligned}$$

  Let's show that $\lambda T$ is bounded.

$$\begin{aligned}\|(\lambda T)x\|_W &= \|\lambda Tx\|_W\\ &= |\lambda|\,\|Tx\|_W\\ &\leq \underbrace{|\lambda|\,M_T}_{M}\|x\|_V\\ &= M\|x\|_V\end{aligned}$$

Therefore, $\mathcal{L}(V,W)$ is a vector space. Now let's show that the defined norm satisfies the desired properties.

- Let's show that $\|T\|=0 \iff T=0$.

  $(\rightarrow)$ Let's suppose that $\|T\|=0$. Therefore,

$$\begin{aligned}\|T\| &= 0\\ \sup_{\|u\|_V=1}\|Tu\|_W &= 0\end{aligned}$$

Therefore, as the supremum is equals to 0 we obtain that for all $u \in V$ ($u \neq 0$, as every linear operator satisfies that $T0 = 0$)

$$
\begin{aligned}
0 \leq \|Tu\|_W &\leq \sup \|Tu\|_W \\
0 \leq \|Tu\|_W &\leq \|u\|_V \sup \frac{\|Tu\|_W}{\|u\|_V} \\
0 \leq \|Tu\|_W &\leq \|u\|_V \sup \left\| T\left(\frac{u}{\|u\|_V}\right) \right\| \\
0 \leq \|Tu\|_W &\leq \|u\|_V \sup_{\|u\|_V=1} \|Tu\|_W \\
0 \leq \|Tu\|_W &\leq 0 \\
\|Tu\|_W &= 0 \\
Tu &= 0
\end{aligned}
$$

Therefore, $\forall u \in V : Tu = 0$. Hence, $T = 0$.

($\leftarrow$) Let's suppose that $T = 0$. Hence:

$$
\begin{aligned}
\|T\| &= \sup_{\|u\|_V=1} \|Tu\|_W \\
&= \sup_{\|u\|_V=1} \|0\| \\
&= 0
\end{aligned}
$$

Therefore, both implications are shown.

- Let's show that $\|\alpha T\| = |\alpha| \, \|T\|$.

$$
\begin{aligned}
\|\alpha T\| &= \sup_{\|u\|_V=1} \|(\alpha T)u\| \\
\|\alpha T\| &= \sup_{\|u\|_V=1} \|\alpha Tu\| \\
&= \sup_{\|u\|_V=1} |\alpha| \, \|Tu\| \\
&= |\alpha| \sup_{\|u\|_V=1} \|Tu\| \\
&= |\alpha| \, \|T\|
\end{aligned}
$$

- Let's show the triangular inequality. Let $T, H \in \mathcal{L}(V, W)$:

$$
\begin{aligned}
\|T + H\| &= \sup_{\|u\|_V=1} \|(T + H)u\| & &= \sup_{\|u\|_V=1} \|Tu + Hu\| \\
&\leq \sup_{\|u\|_V=1} (\|Tu\| + \|Hu\|) \\
&= \sup_{\|u\|_v=1} \|Tu\| + \sup_{\|u\|_v=1} \|Hu\| \\
&= \|T\| + \|H\|
\end{aligned}
$$

□

# Workshop 3

## Exercise 5

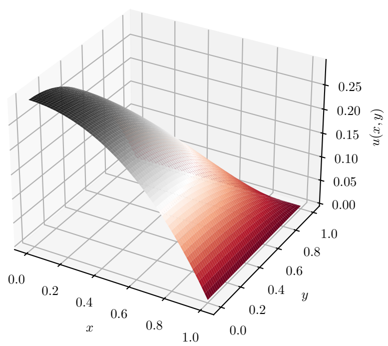
*Question* 8. Solve the following mixed boundaries Poisson Equation using Triangular Elements:

$$
\begin{aligned}
-\nabla^2 u &= 1 \text{ in } \Omega = (0,1) \times (0,1) \\
u_x(0,y) &= u_y(x,0) = 0 \\
u(1,y) &= u(x,1) = 0
\end{aligned}
$$

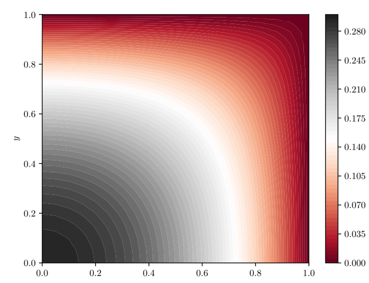
*Proof.* To solve this exercise the `FEniCS` project API for `Python` was used. The following code was used:

```
#!/usr/bin/env python3

import tools
from fenics import *

tol = DOLFIN_EPS

# Mesh
mesh = UnitSquareMesh(20, 20)
V = FunctionSpace(mesh, 'P', 1)

# Dirichlet boundary
def boundary(x, on_boundary):
    return on_boundary and (near(x[0], 1, tol) or (near(x[1], 1, tol)))

u0 = Constant(0.0)
bc = DirichletBC(V, u0, boundary)

# Variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("1", degree=0)
g = Expression("0", degree=0)

a = inner(grad(u), grad(v))*dx
L = f*v*dx + g*v*ds
```

```
# Solution
u = Function(V)
solve(a == L, u, bc)

# Plotting
tools.plot3d(60, 60, u, "figs/exerc-5.pdf")
tools.plot(100, 100, u, "figs/exerc-5-2d.pdf", num_lines=60)

# Values
print(u(0.5, 0.5))
```



(a) 3D approximation.



(b) Contour lines in 2D.

Figure 1: Output of code.

□

## Exercise 8

*Question* 9. Solve the following Poisson Equation using Triangular Elements:

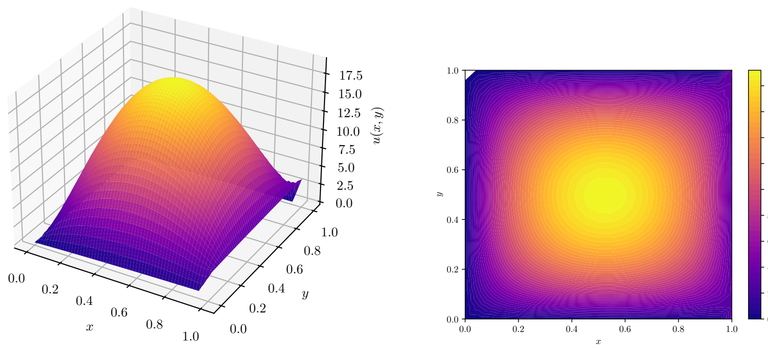
$$
\begin{aligned}
-\nabla^2 u &= xy \text{ in } \Omega = (0,1)\times(0,1) \\
u(x,0) &= 1 - e^{-x} \\
u(1,y) &= 4 - e^{-y} \\
u &= 0, \text{ otherwise.}
\end{aligned}
$$

*Proof.* To solve this exercise the `FEniCS` project API for `Python` was used. The following code was used:

```
#!/usr/bin/env python3

import tools
from fenics import *
```

```python
tol = DOLFIN_EPS

# Mesh
mesh = UnitSquareMesh(20, 20)
V = FunctionSpace(mesh, 'P', 1)

# Dirichlet boundary 1
def boundary(x, on_boundary):
    return on_boundary and (near(x[0], 0, tol) or (near(x[1], 1, tol)))

u0 = Constant(0.0)
bc1 = DirichletBC(V, u0, boundary)

# Dirichlet boundary 2
def boundary2(x, on_boundary):
    return on_boundary and near(x[1], 0, tol)

u0 = Expression("1 - exp(-x[0])", degree=2)
bc2 = DirichletBC(V, u0, boundary2)

# Dirichlet boundary 3
def boundary3(x, on_boundary):
    return on_boundary and near(x[0], 1, tol)

u0 = Expression("4 - exp(-x[1])", degree=2)
bc3 = DirichletBC(V, u0, boundary3)

# Variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("250 - pow(x[0], 2) - 6*pow(x[1], 2)", degree=2)

a = inner(grad(u), grad(v))*dx
L = f*v*dx

# Solution
u = Function(V)
solve(a == L, u, [bc1, bc2, bc3])

# Plotting
tools.plot3d(60, 60, u, "figs/exerc-8.pdf")
tools.plot(100, 100, u, "figs/exerc-8-2d.pdf", num_lines=60)
```

The result of this code can be seen in Figure 2.




(a) 3D approximation. (b) Contour lines in 2D.

Figure 2: Output of code.

□

# References

[1] Z. Cvetkovski, “Hölder’s inequality, minkowski’s inequality and their variants,” in *Inequalities*. Springer, 2012, pp. 95–105.