7. $-\nabla^2 u = xy \quad \Omega = (0,1)\times(0,1)$ (*)
$u = 0$ sobre $\partial\Omega$

$V = \mathcal{H}_0^1(\Omega)$

$V_4 = gen\{S\}$

$S =$

La formulación débil de (*) es:
Encontrar $u \in \mathcal{H}_0^1(\Omega)$ t.q.

$$\int_\Omega \nabla u \cdot \nabla v \, dA = \int_\Omega xy\, v(x,y)\, dA, \quad \forall v \in \mathcal{H}_0^1(\Omega) \quad (A)$$

Aproximando la solución en V_4 con u_4 donde

$$u_4(x,y) = a_1 \sin(\pi x)\sin(\pi y) + a_2 \sin(\pi x)\sin(2\pi y) + a_3 \sin(2\pi x)\sin(\pi y) + a_4 \sin(2\pi x)\sin(2\pi y)$$

Como (A) se cumple $\forall v \in V_4$, en particular para $s \in S$ (⊙)

Calculemos ∇u_4:

$$\frac{\partial u_4}{\partial x} = a_1\pi\cos(\pi x)\sin(\pi y) + a_2\pi\cos(\pi x)\sin(2\pi y) + 2a_3\pi\cos(2\pi x)\sin(\pi y) + 2a_4\pi\cos(2\pi x)\sin(2\pi y)$$

$$\frac{\partial u_4}{\partial y} = a_1\pi\sin(\pi x)\cos(\pi y) + 2a_2\pi\sin(\pi x)\cos(2\pi y) + a_3\pi\sin(2\pi x)\cos(\pi y) + 2a_4\pi\sin(2\pi x)\cos(2\pi y)$$

Por ⊙, en particular para $v_1(x,y) = \sin(\pi x)\sin(\pi y)$

$$\nabla v_1(x,y) = \left(\pi\cos(\pi x)\sin(\pi y), \pi\sin(\pi x)\cos(\pi y)\right)$$

$$\int_0^1\int_0^1 \nabla u_4 \cdot \nabla v_1 \, dx dy = \int_0^1\int_0^1 xy \sin(\pi x)\sin(\pi y)\, dx dy$$

$$\int_0^1\int_0^1 \Big(a_1\pi\cos(\pi x)\sin(\pi y) + a_2\pi\cos(\pi x)\sin(2\pi y) + 2a_3\pi\cos(2\pi x)\sin(\pi y) + 2a_4\pi\cos(2\pi x)\sin(2\pi y),\ a_1\pi\sin(\pi x)\cos(\pi y) + 2a_2\pi\sin(\pi x)\cos(2\pi y) + a_3\pi\sin(2\pi x)\cos(\pi y) + 2a_4\pi\sin(2\pi x)\cos(2\pi y)\Big)\cdot\left(\pi\cos(\pi x)\sin(\pi y), \pi\sin(\pi x)\cos(\pi y)\right) dx dy = \frac{1}{\pi^2}$$

$$\int_0^1\int_0^1 \Big[a_1\pi^2\cos^2(\pi x)\sin^2(\pi y) + a_2\pi^2\cos^2(\pi x)\sin(2\pi y)\sin(\pi y) + 2a_3\pi^2\cos(2\pi x)\cos(\pi x)\sin^2(\pi y) + 2a_4\pi^2\cos(2\pi x)\cos(\pi x)\sin(2\pi y)\sin(\pi y) + a_1\pi^2\sin^2(\pi x)\cos^2(\pi y) + 2a_2\pi^2\sin^2(\pi x)\cos(2\pi y)\cos(\pi y) + a_3\pi^2\sin(2\pi x)\sin(\pi x)\cos^2(\pi y) + 2a_4\pi^2\sin(2\pi x)\sin(\pi x)\cos(2\pi y)\cos(\pi y)\Big] dx dy = \frac{1}{\pi^2}$$

$$\frac{\pi^2 a_1}{2} + a_2\cdot 0 + a_3 \cdot 0 + a_4\cdot 0 = \frac{1}{\pi^2}$$

$$\boxed{a_1 = \frac{2}{\pi^4}}$$

Para $v_2(x,y) = \sin(\pi x)\sin(2\pi y)$

$$\nabla v_2 = \left(\pi\cos(\pi x)\sin(2\pi y), 2\pi\sin(\pi x)\cos(2\pi y)\right)$$

$$\int_0^1\int_0^1 \nabla u_4 \cdot \nabla v_2 \, dx dy = \int_0^1\int_0^1 xy \sin(\pi x)\sin(2\pi y)\, dx dy$$

$$\int_0^1\int_0^1 \Big(a_1\pi\cos(\pi x)\sin(\pi y) + a_2\pi\cos(\pi x)\sin(2\pi y) + 2a_3\pi\cos(2\pi x)\sin(\pi y) + 2a_4\pi\cos(2\pi x)\sin(2\pi y),\ a_1\pi\sin(\pi x)\cos(\pi y) + 2a_2\pi\sin(\pi x)\cos(2\pi y) + a_3\pi\sin(2\pi x)\cos(\pi y) + 2a_4\pi\sin(2\pi x)\cos(2\pi y)\Big)\cdot\left(\pi\cos(\pi x)\sin(2\pi y), 2\pi\sin(\pi x)\cos(2\pi y)\right) dx dy = -\frac{1}{2\pi^2}$$

$$\int_0^1\int_0^1 \Big[a_1\pi^2\cos^2(\pi x)\sin(\pi y)\sin(2\pi y) + a_2\pi^2\cos^2(\pi x)\sin^2(2\pi y) + 2a_3\pi^2\cos(2\pi x)\cos(\pi x)\sin(2\pi y)\sin(\pi y) + 2a_4\pi^2\cos(2\pi x)\cos(\pi x)\sin^2(2\pi y) + 2a_1\pi^2\sin^2(\pi x)\cos(\pi y)\cos(2\pi y) + 4a_2\pi^2\sin^2(\pi x)\cos^2(2\pi y) + 2a_3\pi^2\sin(\pi x)\sin(2\pi x)\cos(\pi y)\cos(2\pi y) + 4a_4\pi^2\sin(\pi x)\sin(2\pi x)\cos^2(2\pi y)\Big] dx dy = -\frac{1}{2\pi^2}$$

$$a_1\cdot 0 + \frac{5a_2\pi^2}{4} + a_3\cdot 0 + a_4 \cdot 0 = -\frac{1}{2\pi^2}$$

$$\boxed{a_2 = -\frac{2}{5\pi^4}}$$

Para $v_3(x,y) = \sin(2\pi x)\sin(\pi y)$

$$\nabla v_3 = \left(2\pi\cos(2\pi x)\sin(\pi x), \pi\sin(2\pi x)\cos(\pi y)\right)$$

$$\int_0^1\int_0^1 \Big(a_1\pi\cos(\pi x)\sin(\pi y) + a_2\pi\cos(\pi x)\sin(2\pi y) + 2a_3\pi\cos(2\pi x)\sin(\pi y) + 2a_4\pi\cos(2\pi x)\sin(2\pi y),\ a_1\pi\sin(\pi x)\cos(\pi y) + 2a_2\pi\sin(\pi x)\cos(2\pi y) + a_3\pi\sin(2\pi x)\cos(\pi y) + 2a_4\pi\sin(2\pi x)\cos(2\pi y)\Big)\cdot\left(2\pi\cos(2\pi x)\sin(\pi x), \pi\sin(2\pi x)\cos(\pi y)\right) dx dy = \int_0^1\int_0^1 xy\sin(2\pi x)\sin(\pi y)\, dx dy$$

$$\int_0^1\int_0^1 a_3\pi^2\left[4\cos^2(2\pi x)\sin^2(\pi x) + \sin^2(2\pi x)\cos^2(\pi y)\right] dx dy = -\frac{1}{2\pi^2}$$

$$\frac{5\pi^2}{4}a_3 = -\frac{1}{2\pi^2}$$

$$a_3 = \frac{-2}{5\pi^4}$$

Para $v_4(x,y) = \sin(2\pi x)\sin(2\pi y)$

$$\nabla v_4 = \left(2\pi\cos(2\pi x)\sin(2\pi y), 2\pi\sin(2\pi x)\cos(2\pi y)\right)$$

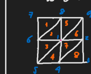
$$\int_0^1\int_0^1 \Big(a_1\pi\cos(\pi x)\sin(\pi y) + a_2\pi\cos(\pi x)\sin(2\pi y) + 2a_3\pi\cos(2\pi x)\sin(\pi y) + 2a_4\pi\cos(2\pi x)\sin(2\pi y),\ a_1\pi\sin(\pi x)\cos(\pi y) + 2a_2\pi\sin(\pi x)\cos(2\pi y) + a_3\pi\sin(2\pi x)\cos(\pi y) + 2a_4\pi\sin(2\pi x)\cos(2\pi y)\Big)\cdot\left(2\pi\cos(2\pi x)\sin(2\pi y), 2\pi\sin(2\pi x)\cos(2\pi y)\right) dx dy = \int_0^1\int_0^1 xy\sin(2\pi x)\sin(2\pi y)\, dx$$

$$\int_0^1\int_0^1 4a_4\pi^2\left[\cos^2(2\pi x)\sin^2(2\pi y) + \sin^2(2\pi x)\cos^2(2\pi y)\right] dx dy = \frac{1}{4\pi^2}$$

$$2a_4\pi^2 = \frac{1}{4\pi^2} \rightarrow a_4 = \frac{1}{8\pi^4}$$

$$u_4(x,y) = \frac{2}{\pi^4}\sin(\pi x)\sin(\pi y) - \frac{2}{5\pi^4}\sin(\pi x)\sin(2\pi y) - \frac{2}{5\pi^4}\sin(2\pi x)\sin(\pi x) + \frac{1}{8\pi^4}\sin(2\pi x)\sin(2\pi y)$$

$-\nabla^2 u = xy \quad \Omega = (0,1)\times(0,1)$
$u = 0$ sobre $\partial\Omega$



$I_1 = \{2,3,4,5,6,7\}$

Para triángulo par: $(0,0), (1/2,0), (1/2,1/2)$ $\quad \varphi_i(x,y) = a_i + b_i x + c_i y$

Para $v_1^4 = (0,0)$:
$\varphi_1(0,0) = 1 = a_1$
$\varphi_1(1/2,0) = 0 = 1 + \frac{b_1}{2} \rightarrow b_1 = -2.$
$\varphi_1(1/2,1/2) = 0 = 1 - 1 + \frac{c_1}{2} \rightarrow c_1 = 0$

$\varphi_1(x,y) = 1 - 2x$
$\nabla\varphi_1 = (-2, 0)$

Para $v_2^4 = (1/2, 0)$:
$\varphi_2(0,0) = 0 = a_2$
$\varphi_2(1/2,0) = 1 = \frac{b_2}{2} \rightarrow b_2 = 2$
$\varphi_2(1/2,1/2) = 0 = 1 + \frac{c_2}{2} \rightarrow c_2 = -2$

$\varphi_2(x,y) = 2x - 2y$
$\nabla\varphi_2 = (2, -2)$

Para $v_3^4 = (1/2, 1/2)$:
$\varphi_3(0,0) = 0 = a_3$
$\varphi_3(1/2,0) = 0 = \frac{b_3}{2} \rightarrow b_3 = 0$
$\varphi_3(1/2,1/2) = 1 = \frac{c_3}{2} \rightarrow c_3 = 2$

$\varphi_3(x,y) = 2y$
$\nabla\varphi_3 = (0, 2)$

Para triángulos impares: $(0,0), (0,1/2), (1/2,1/2)$

$\psi_i(x,y) = \alpha_i + \beta_i x + \gamma_i y$

Para $v_1^3 = (0,0)$:
$\psi_1(0,0) = 1 = \alpha_1$
$\psi_1(0,1/2) = 0 = 1 + \frac{\gamma_1}{2} \rightarrow \gamma_1 = -2$
$\psi_1(1/2,1/2) = 0 = 1 - 1 + \frac{\beta_2}{2} \rightarrow \beta_2 = 0$

$\psi_1(x,y) = 1 - 2y$
$\nabla\psi_1 = (0, -2)$

Para $v_2^3 (0,0)$:
$\psi_2(0,0) = 0 = \alpha_2$
$\psi_2(0,1/2) = 1 = \frac{\gamma_2}{2} \rightarrow \gamma_2 = 2$
$\psi_2(1/2,1/2) = 0 = 1 + \frac{\beta_2}{2} \rightarrow \beta_2 = -2$

$\psi_2(x,y) = -2x + 2y$
$\nabla\psi_2 = (-2, 2)$

Para $v_3^3 = (1/2, 1/2)$

$\psi_3(0,0) = 0 = \alpha_3$
$\psi_3(0,1/2) = 0 = \gamma_3$
$\psi_3(1/2,1/2) = 1 = \frac{\beta_3}{2} \rightarrow \beta_3 = 2$

$\psi_3(x,y) = 2x$
$\nabla\psi_3 = (2, 0)$

$$K_{11} = \sum_{r\in I_1}\int_{T_r} \|\nabla\xi_1|_{T_r}\|^2 dA_{T_r}$$

$$= \int_{T_2}\|\nabla\varphi_2\|^2 dA_{T_2} + \int_{T_3}\|\nabla\psi_3\|^2 dA_{T_3} + \int_{T_4}\|\nabla\varphi_3\|^2 dA_{T_4} + \int_{T_5}\|\nabla\psi_1\|^2 dA_{T_5} + \int_{T_6}\|\nabla\varphi_1\|^2 dA_{T_6} + \int_{T_7}\|\nabla\psi_2\|^2 dA_{T_7}$$

$$= \frac{1}{8}\left[8 + 4 + 4 + 4 + 4 + 8\right] = 4$$

Ahora, por cuadratura:

$$l_1 \approx \frac{|I_1| A_T}{3} f(v_1)$$

$$= \frac{(6)(1/8)(1/4)}{3} = \frac{1}{16}$$

Luego, $K_{11}u_1 = l_1 \rightarrow 4u_1 = \frac{1}{16} \rightarrow \boxed{u_1 = \frac{1}{64}}$