$\min\{s,t\} = E[B_t B_s] = Cov[B_t, B_s]$

sura

Taller 1

2. $B_t \to$ MBEU. Sea $t_0 \geq 0$ fijo. Pruebe que $W_t = B_{t_0+t} - B_{t_0}$ es MBEU.

a) $W_0 = B_{t_0+0} - B_{t_0} = 0$ ✓. $\to$ $h(t) = t_0 + t$ es continua y positiva para $t \geq 0$.

$f(t) = B_{t+t_0} - B_{t_0}$ es continua, porque B_t es continua $\forall t$. (MBEU).

b) Veamos si $W_t - W_s \sim N(0, t-s)$ $\forall t > s$. Nótese

$W_t - W_s = B_{t+t_0} - B_{t_0} - (B_{s+t_0} - B_{t_0}) = B_{t+t_0} - B_{s+t_0}$.

Como B_t es un movimiento browniano estándar unidimensional, tiene incrementos estacionarios. Luego, dado que t_0 es fijo, se cumple que $B_{t+t_0} - B_{s+t_0} \overset{d}{=} B_t - B_s$. Por definición del MBEU, $B_t - B_s \sim N(0, t-s)$. Luego $W_t - W_s \sim N(0, t-s)$. En particular $W_t \sim N(0,t)$, cuando $s=0$.

- $E[W_s(W_t - W_s)] = E[(B_{s+t_0} - B_{t_0})[B_{t+t_0} - B_{t_0} - B_{s+t_0} + B_{t_0}]]$

$= E[(B_{s+t_0} - B_{t_0})(B_{t+t_0} - B_{s+t_0})]$

$= E[B_{s+t_0} B_{t+t_0}] - E[B_{s+t_0}^2] - E[B_{t_0} B_{t+t_0}] + E[B_{t_0} B_{s+t_0}]$

Se sabe que $V[B_t] = t$ y $Cov[B_t, B_s] = \min\{s,t\}$. Luego,

$= \min\{s+t_0, t+t_0\} - s + t_0 - \min\{t_0, t+t_0\} + \min\{t_0, s+t_0\}$.

$= s + t_0 - s + t_0 - t_0 + t_0 = 0$

Luego, W_t tiene incrementos independientes $\Rightarrow W_t$ es MBEU.

3. Sea B_t un MBEU. Sea $t_0 \geq 0$ fijo. Sea el proceso $W_t = B_t + \lambda t_0$, $t > 0$. ¿Es W_t un MBEU?

a) $W_0 = B_0 + \lambda t_0 = \lambda t_0 \longrightarrow$ No necesariamente $= 0 \Rightarrow$ No es MBEU.

4. Sea B_t un MBEU. Pruebe que los siguientes proc. estoc. son MBEU.

a) $W_t = \begin{cases} t B_{1/t}, & t > 0 \\ 0, & t = 0 \end{cases}$

a) $W_0 = 0$ ✓ Por definición. $h(t) = 1/t$ es una función continua y positiva $\forall t > 0$. $f(t) = t B_{1/t}$ es continua $\forall t \geq 0$.

b) Veamos si $W_t - W_s \sim N(0, t-s)$ $\forall t > s$.

- $E[W_t - W_s] = E[t B_{1/t} - s B_{1/s}] = t E[B_{1/t}] - s E[B_{1/s}]$. Como B_t es un MBEU, $E[B_t] = 0$ $\forall t$. Luego $E[W_t - W_s] = 0$.
- $V[W_t - W_s] = V[t B_{1/t} - s B_{1/s}] = E[(t B_{1/t} - s B_{1/s})^2] - \cancel{E^2[t B_{1/t} - s B_{1/s}]}^{0}$

$$= E[t^2 B_{1/t}^2 - 2st B_{1/t} B_{1/s} + s^2 B_{1/s}^2]$$

$$= t^2 E[B_{1/t}^2] - 2st E[B_{1/t} B_{1/s}] + s^2 E[B_{1/s}^2]$$

$$= t^2 (1/t) - 2st \cdot \min\{1/t, 1/s\} + s^2 (1/s)$$

$$= t - (2st \cdot 1/t) + s = t - s.$$

$$\sqrt{\tfrac{t}{s}}\, B_{\frac{1}{t}} - \sqrt{\tfrac{s}{t}}\, B_{\frac{1}{s}} \overset{d}{=} B_{\frac{1}{s}} - B_{\frac{1}{t}}$$

- Veamos ahora que su distribución es normal:

$$E\left[e^{i\theta(W_t - W_s)}\right] = E\left[e^{i\theta(tB_{1/t} - sB_{1/s})}\right] = E\left[e^{i\theta\sqrt{ts}\left(\frac{t}{\sqrt{ts}}B_{1/t} - \frac{s}{\sqrt{ts}}B_{1/s}\right)}\right]$$

$$= E\left[e^{i\theta\sqrt{ts}\left(\sqrt{\frac{t}{s}}B_{1/t} - \sqrt{\frac{s}{t}}B_{1/s}\right)}\right]$$

Nota: Es claro que $\sqrt{t}B_{1/t} \overset{d}{=} B_1$ y $\sqrt{s}B_s \overset{d}{=} B_1$ luego $\sqrt{t}B_{1/t} \overset{d}{=} \sqrt{s}B_{1/s}$

Luego, $\sqrt{t/s}\,B_{1/t} \overset{d}{=} B_{1/s}$ y $\sqrt{s/t}\,B_{1/s} \overset{d}{=} B_{1/t}$. $\sqrt{\frac{t}{s}}B_{1/t} - \sqrt{\frac{s}{t}}B_{1/s} \overset{d}{=} B_{1/s} - B_{1/t}$

$$\Rightarrow E\left[e^{i\theta\sqrt{ts}(B_{1/s} - B_{1/t})}\right] = E\left[e^{i\theta\sqrt{ts}(B_{1/s - 1/t})}\right]$$

$$= E\left[e^{i\theta\sqrt{ts}\cdot(1/s - 1/t)^{1/2}B_1}\right] = E\left[e^{i\theta\frac{(t-s)^{1/2}}{\sqrt{ts}}\sqrt{ts}\,B_1}\right] = e^{-\frac{\theta^2(t-s)}{2}}$$

Cont. Taller 1.

4. b) $W_t = a^{-1/2} B_{at}$, $a > 0$.

i) $W_0 = a^{-1/2} B_0 = 0$

$h(t) = at$, $a > 0$ es función continua y es positiva en su dominio $t \geq 0$.

$f(B_t) = a^{-1/2} B_{at}$ es continua.

ii) $\forall s < t$, $W_t - W_s \sim N(0, t-s)$

$E[W_t - W_s] = E[a^{-1/2} B_{at} - a^{-1/2} B_{as}] = a^{-1/2}\{E[B_{at}] - E[B_{as}]\} = 0.$

$$V[W_t - W_s] = E[(W_t - W_s)^2] = E[(a^{-1/2} B_{at} - a^{-1/2} B_{as})^2]$$

$$= a^{-1} E[(B_{at} - B_{as})^2] = a^{-1} E[(B_{at-as} - \cancel{B_{as-as}})^2]$$

$$= a^{-1} E[B_{at-as}^2] = a^{-1}(at - as) = t - s.$$

Veamos que su distribución es normal.

$$E[e^{i\theta(W_t - W_s)}] = E[e^{i\theta a^{-1/2}(B_{at} - B_{as})}] = E[e^{i\theta a^{-1/2}(B_{a(t-s)} - \cancel{B_0})}]$$

Se sabe que $B_t \sim N(0, t)$, porque B_t es un MBEU. Luego $B_{a(t-s)} \sim N(0, a(t-s))$

$$\rightarrow E[e^{i(\theta a^{-1/2}) B_{a(t-s)}}] = e^{-\frac{(\theta a^{-1/2})^2 a(t-s)}{2}} = e^{-\frac{\theta^2 (t-s)}{2}}$$

Por lo tanto, $W_t - W_s \sim N(0, t-s)$

iii) Cuando $s = 0 \Rightarrow W_t \sim N(0, t)$

iv) $E[W_s(W_t - W_s)] \stackrel{!}{=} 0$

$$E[a^{-1/2} B_{as}(a^{-1/2} B_{at} - a^{-1/2} B_{as})] = a^{-1} E[B_{as}(B_{at} - B_{as})] = a^{-1} E[B_{as} B_{at} - B_{as}^2]$$

$$= a^{-1}\{E[B_{as} B_{at}] - E[B_{as}^2]\} = a^{-1}(\min\{as, at\} - as) = 0.$$

Luego W_t es un MBEU.

$Z_t = B_t - tB_1$ $\quad s<t \implies \frac{t}{t+1} > \frac{s}{s+1}$ $\quad 1-\frac{t}{t+1} = \frac{1}{t+1}$

$s<t$

5. Sea $\{Z_t\}_{t\geq 0}$ un proceso Browniano Bridge. Muestre que $W_t := (t+1)Z_{t/(t+1)}$ es un MBEU.

i) $W_0 = Z_0 = B_0 - 0\cdot B_1 = 0$ ✓

$W_t = (t+1)Z_{t/(t+1)} = (t+1)\left[B_{t/(t+1)} - \frac{t}{t+1}B_1\right] \to = B_{t/(t+1)} - tB_{1/(t+1)}$

$= (t+1)B_{t/(t+1)} - tB_1 = B_{t/(t+1)} - t(B_1 - B_{t/(t+1)})$

$h(t) = t/(t+1)$ es una función continua y positiva $\forall t \geq 0$.

$f(t) = (t+1)B_{t/(t+1)} - tB_1$ es continua $\forall t \geq 0$.

ii) $\forall s<t$, $W_t - W_s \sim N(0, t-s)$

$E[W_t - W_s] = E\left[(t+1)B_{t/(t+1)} - tB_1 - (s+1)B_{s/(s+1)} + sB_1\right]$

$= (t+1)E[B_{t/(t+1)}] - tE[B_1] - (s+1)E[B_{s/(s+1)}] + sE[B_1] = 0$ ✓

$V[W_t - W_s] = E[(W_t - W_s)^2] = E\left[\left((t+1)Z_{t/(t+1)} - (s+1)Z_{s/(s+1)}\right)^2\right]$

$= E\left[(t+1)^2 Z^2_{t/(t+1)} - 2(t+1)(s+1)Z_{t/(t+1)}Z_{s/(s+1)} + (s+1)^2 Z^2_{s/(s+1)}\right]$

$= (t+1)^2 E\left[Z^2_{t/(t+1)}\right] - 2(t+1)(s+1)E\left[Z_{t/(t+1)}Z_{s/(s+1)}\right] + (s+1)^2 E\left[Z^2_{s/(s+1)}\right]$

Como Z_t es Bridge, $E[Z_t] = 0$ y, luego $V[Z_t] = E[Z_t^2] = t - t^2$.

Además, $Cov[Z_t, Z_s] = \min\{s,t\} - st = E[Z_t Z_s]$

$= (t+1)^2\left(\frac{t}{t+1} - \frac{t^2}{(t+1)^2}\right) - 2(t+1)(s+1)\left(\frac{s}{s+1} - \frac{st}{(s+1)(t+1)}\right) + (s+1)^2\left(\frac{s}{s+1} - \frac{s^2}{(s+1)^2}\right)$

$= t(t+1) - t^2 - 2(t+1)s + 2st + s(s+1) - s^2$

$= \cancel{t^2} + t - \cancel{t^2} - \cancel{2st} - 2s + \cancel{2st} + \cancel{s^2} + s - \cancel{s^2}$

$= t - s$

$x(x-1)(x+2)$

x^3+x^2-2x

Normalidad:

$$E\left[e^{i\theta(W_t-W_s)}\right]=E\left[e^{i\theta(W_t-W_s)}\right]=E\left[e^{i\theta\left[(t+1)B_{t/(t+1)}-tB_1-(s+1)B_{s/(s+1)}+sB_1\right]}\right]$$

$\Longrightarrow$ hoja atrás.

$$=E[W_s(W_t-W_s)]=E\left[(s+1)Z_{s/(s+1)}\left((t+1)Z_{t/(t+1)}-(s+1)Z_{s/(s+1)}\right)\right]$$

$$=(s+1)E\left[(t+1)Z_{s/(s+1)}Z_{t/(t+1)}-(s+1)Z_{s/(s+1)}Z_{s/(s+1)}\right]$$

$$=(s+1)\left\{(t+1)E\left[Z_{s/(s+1)}Z_{t/(t+1)}\right]-(s+1)E\left[Z_{s/(s+1)}Z_{s/(s+1)}\right]\right\}$$

$$=(s+1)\left\{(t+1)Cov(Z_{s/(s+1)},Z_{t/(t+1)})-(s+1)Cov(Z_{s/(s+1)},Z_{s/(s+1)})\right\}$$

$$=(s+1)\left\{(t+1)\left[\min\left\{s/(s+1),t/(t+1)\right\}-\frac{st}{(t+1)(s+1)}\right]-(s+1)\left[\frac{s}{s+1}-\frac{s^2}{(s+1)^2}\right]\right\}$$

$$=(s+1)(t+1)\left[\frac{s}{s+1}-\frac{st}{(s+1)(t+1)}\right]-(s+1)^2\left[\frac{s}{s+1}-\frac{s^2}{(s+1)^2}\right]$$

$$=s(t+1)-st-s(s+1)+s^2=st+s-st-s^2-s+s^2=0.$$

$\Rightarrow W_t$ tiene incrementos independientes,

$\Rightarrow W_t$ es MBEU.

$$=E\left[e^{i\theta\left(tB_{t/(t+1)}+B_{t/(t+1)}-tB_1-sB_{s/(s+1)}-B_{s/(s+1)}+sB_1\right)}\right]$$

$$=E\left[e^{i\theta\left(tB_{t/(t+1)}+B_{t/(t+1)}-tB_1-sB_{s/(s+1)}-B_{s/(s+1)}+sB_1+sB_{t/(t+1)}-sB_{t/(t+1)}\right)}\right]$$

$$=E\left[e^{i\theta\left[(t-s)B_{t/(t+1)}+(s+1)B_{t/(t+1)}-(t-s)B_1-(s+1)B_{s/(s+1)}\right]}\right]$$

$$=E\left[e^{i\theta\left[(t-s)\left(B_{t/(t+1)}-B_1\right)+(s+1)\left(B_{t/(t+1)}-B_{s/(s+1)}\right)\right]}\right]$$

$$=E\left[e^{-i\theta(t-s)\left(1-\frac{t}{t+1}\right)^{1/2}B_1+(s+1)\left(\frac{t}{t+1}-\frac{s}{s+1}\right)^{1/2}B_1}\right]$$

$$=e^{-\frac{\theta^2}{2}(t-s)^2\left(1-\frac{t}{t+1}\right)-\frac{\theta^2}{2}(s+1)^2\left(\frac{t}{t+1}-\frac{s}{s+1}\right)}$$

$$=e^{-\frac{\theta^2}{2}\left[(t-s)^2\left(\frac{1}{t+1}\right)+(s+1)\left(\frac{t-s}{t+1}\right)\right]}=e^{-\frac{\theta^2}{2}(t-s)\left[\frac{t-s}{t+1}+\frac{s+1}{t+1}\right]}=e^{-\frac{\theta^2}{2}(t-s)}$$

$W_t-W_s\sim N(0,t-s)\rightarrow W_t\sim N(0,t)$ cuando $s=0$.

6. Sea X_t y Y_t MBEUs t.q. $X_t \perp Y_t$

a) ¿Es $Z_t = X_t - Y_t$ es un MBEU?

i) $Z_0 = X_0 - Y_0 = 0$ ✓

$h_1(t) = t = h_2(t)$ son continuas y positivas $\forall t \geq 0$.

$f_1(X_t)$ y $f_2(Y_t)$ son ambas continuas.

ii)

$V[Z_t - Z_s] = V[X_t - Y_t - X_s + Y_s] = V[X_t - X_s] + V[Y_t - Y_s]$

$= t - s - t + s = 0 \Rightarrow$ No es MBEU

7. $B_t \to$ MBEU. $\beta > 0$. Encuentre $\alpha > 0$ t.q $W_t = \alpha B_{t/\beta^2}$ es MBEU.

i) $W_0 = \alpha B_0 = 0$ ✓

$h(t) = t/\beta^2 \geq 0, \ \forall t \geq 0$ y es continua.

$f(B_t) \to$ continua

ii) $E[W_t - W_s] = \alpha E[B_{t/\beta^2} - B_{s/\beta^2}] = 0$.

$V[W_t - W_s] = V[\alpha B_{t/\beta^2} - \alpha B_{s/\beta^2}] = \alpha^2 V[B_{t/\beta^2} - B_{s/\beta^2}] = \alpha^2 V[B_{t/\beta^2 - s/\beta^2}]$

$= \frac{\alpha^2 (t-s)}{\beta^2}$

$\frac{\alpha^2 (t-s)}{\beta^2} \overset{!}{=} (t-s) \to \alpha^2 = \beta^2 \to \alpha = \beta$.

$E\left[e^{i\theta(W_t - W_s)}\right] = E\left[e^{i\theta(\beta B_{t/\beta^2} - \beta B_{s/\beta^2})}\right] = E\left[e^{i\theta\beta(B_{t/\beta^2 - s/\beta^2})}\right] = E\left[e^{i\theta \frac{\beta (t-s)^{1/2}}{\beta} B_1}\right]$

$= e^{-\frac{\theta^2 (t-s)}{2}}$

Luego $W_t - W_s \sim N(0, t-s)$

iii) Si $s=0 \to W_t \sim N(0,t)$.

iv) $E[W_s(W_t-W_s)] = E[\beta B_{s/\beta^2}(\beta B_{t/\beta^2} - \beta B_{s/\beta^2})] = \beta^2 E[B_{s/\beta^2}(B_{t/\beta^2} - B_{s/\beta^2})]$

$= \beta^2 E[B_{s/\beta^2} B_{t/\beta^2} - B_{s/\beta^2}^2] = \beta^2 \{Cov[B_{s/\beta^2}, B_{t/\beta^2}] - V[B_{s/\beta^2}]\}$

$= \beta^2(\min\{s/\beta^2, t/\beta^2\} - s/\beta^2) = 0.$

Luego, W_t es un MBEU.

8. Sea $\{W_t\}_{t\geq 0}$ un proceso estocástico tal que $W_0 = 0$, posee incrementos independientes y estacionarios, y tiene trayectorias continuas. Si $W_t \sim N(\mu t, \sigma^2 t)$, $\mu \in \mathbb{R}$, $\sigma > 0$ y además $E[W_s W_t] = \sigma^2 s + \mu^2 ts$.

¿Es $Z_t = (W_t - \mu t)/\sigma$ un MBEU?

i) $Z_0 = (W_0 - \mu(0))/\sigma = 0$ ✓

$h(t) = t \geq 0$ y continua $\forall t \geq 0$.

$f(W_t)$ es continua.

ii) $E[Z_t - Z_s] = E\left[\frac{W_t - \mu t}{\sigma} - \frac{W_s - \mu s}{\sigma}\right] = \frac{E[W_t] - \mu t - E[W_s] + \mu s}{\sigma} = 0.$

$V[Z_t - Z_s] = \frac{1}{\sigma^2} V[W_t - \mu t - W_s + \mu s] = \frac{1}{\sigma^2} E[(W_t - \mu t - W_s + \mu s)^2]$

$= \frac{1}{\sigma^2} E[(W_t - \mu t)^2 - 2(W_t - \mu t)(W_s - \mu s) + (W_s - \mu s)^2]$

$= \frac{1}{\sigma^2}(E[(W_t - \mu t)^2] - 2E[(W_t - \mu t)(W_s - \mu s)] + E[(W_s - \mu s)^2])$

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$$E[W_s^2] = V[W_s] + E^2[W_s] = \sigma^2 s + \mu^2 s^2 \quad (\Delta)$$

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$$= \frac{1}{\sigma^2}\left(V[W_t] - 2E[W_t W_s - \mu t W_s - \mu s W_t + \mu^2 s t] + V[W_s]\right)$$

$$= (1/\sigma^2)(\sigma^2 t - 2(E[W_t W_s] - \mu t \mu s - \mu s \mu t + \mu^2 s t) + \sigma^2 s)$$

$$= (1/\sigma^2)(\sigma^2 t - 2(\sigma^2 s + \mu^2 t s) + 2\mu^2 t s + \sigma^2 s) = t - s$$

$$E\left[e^{i\theta(Z_t - Z_s)}\right] = E\left[e^{\frac{i\theta}{\sigma}(W_t - \mu t - W_s + \mu s)}\right] = E\left[e^{-\frac{i\theta}{\sigma}(\mu t - \mu s)} e^{\frac{i\theta}{\sigma}(W_t - W_s)}\right].$$

Como W_t tiene incrementos estacionarios, $W_t - W_s \overset{d}{=} W_{t+h} - W_{s+h}$

$$= E\left[e^{\frac{i\theta}{\sigma}(\mu s - \mu t)} e^{\frac{i\theta}{\sigma}(W_{t-s})}\right] = e^{\frac{i\theta}{\sigma}(\mu s - \mu t)} e^{\frac{i\mu(t-s)\theta}{\sigma} - \frac{\theta^2 \sigma^2 (t-s)}{2\sigma^2}} = e^{-\frac{\theta^2 (t-s)}{2}}.$$

Luego $Z_t - Z_s \sim N(0, t-s)$

iii) Cuando $s = 0 \Rightarrow Z_t \sim N(0, t)$

iv) $E[Z_s(Z_t - Z_s)] = (1/\sigma^2) E[(W_s - \mu s)(W_t - \mu t - W_s + \mu s)]$

$$= (1/\sigma^2) E[W_s W_t - \mu t W_s - W_s^2 + \mu s W_s - \mu s W_t + \mu^2 s t + \mu s W_s - \mu^2 s^2]$$

$$= (1/\sigma^2)(E[W_s W_t] - \mu^2 s t - \Delta + \mu^2 s^2 - \mu^2 s t + \mu^2 s t + \mu^2 s^2 - \mu^2 s^2])$$

$$= (1/\sigma^2)(\sigma^2 s + \mu^2 s t - \mu^2 s t - \sigma s^2) = 0.$$

Luego, Z_t es un MBEU.

Si $Z_t = W_t - \mu t_0$,

$$E[Z_t] = \frac{E[W_t] - \mu t_0}{\sigma} = \frac{\mu(t - t_0)}{\sigma} \neq 0 \text{ si } t \neq t_0.$$

$\longrightarrow$ No podría ser MBEU.

9. Un proceso estocástico $\{X_t; t\geq 0\}$ se dice que es estacionario si $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ tiene la misma distribución conjunta que $X_{t_1+a}, X_{t_2+a}, \ldots, X_{t_n+a}$, $\forall n, a, t_1, \ldots, t_n$. Suponga que $\{X_t\}_{t\geq 0}$ es un browniano y sea $V_t = e^{-\alpha t/2} X_{\alpha e^{\alpha t}}$. Muestre que $\{V_t; t\geq 0\}$ es un proceso estacionario.

Porque X_t es MBEU.

i) $E[V_t] = E[e^{-\alpha t/2} X_{\alpha e^{\alpha t}}] = e^{-\alpha t/2} E[X_{\alpha e^{\alpha t}}] = 0$

ii) $Cov(V_t, V_{t+s}) = E[(V_t - E[V_t])(V_{t+s} - E[V_{t+s}])] = E[V_t V_{t+s}]$

$= E[e^{-\alpha t/2} X_{\alpha e^{\alpha t}} e^{-\frac{\alpha(t+s)}{2}} X_{\alpha e^{\alpha(t+s)}}]$

$= e^{-\frac{\alpha t}{2}} e^{-\frac{\alpha(t+s)}{2}} E[X_{\alpha e^{\alpha t}} X_{\alpha e^{\alpha(t+s)}}]$

$= e^{-\frac{\alpha t}{2}} e^{-\frac{\alpha(t+s)}{2}} \alpha e^{\alpha t} = \alpha e^{-\alpha s/2} \longrightarrow$ No depende de t.

$\Rightarrow$ Proceso estacionario de segundo orden.