Trabajo Teórico Unidad 3:

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1.º $dS_t = (1-\cos(2t))S_t dt + 2\sin^2(t) S_t dB_t$. Claramente, $1-\cos(2t) = 2\sin^2(t)$, luego
$dS_t = 2\sin^2(t)S_t dt + 2\sin^2(t)S_t dB_t$.

a) Se sabe que para $dX_t = f(t,S_t)dt + g(t,S_t)dB_t \rightarrow \phi(t) = -\dfrac{f(t,S_t)}{g(t,S_t)}$

Luego, $\phi(t) = -\dfrac{2\sin^2(t)S_t}{2\sin^2(t)S_t} = -1$.

Ahora, $h(t,\phi(t)) = -\dfrac{1}{2}\displaystyle\int_0^t [\phi(u)]^2 du + \int_0^t \phi(u) dB_u = -\dfrac{1}{2}\int_0^t (-1)^2 du + \int_0^t (-1) dB_u = -\dfrac{t}{2} - B_t$

$\xi_t^{\phi(t)} = e^{h(t,\phi(t))} = e^{-\frac{t}{2}-B_t}$. Verifiquemos que $E[\xi_t^{\phi(t)}] = e^{-\frac{t}{2}} E[e^{-B_t}] = e^{-\frac{t}{2}} e^{\frac{t}{2}} = 1$. ✓

b) $B_t^* = B_t - \displaystyle\int_0^t \phi(u) du = B_t - \int_0^t (-1) du = B_t + t$.

- $E^*[B_t^*] = E[\xi_t^{\phi(t)} B_t^*] = E[e^{-\frac{t}{2}-B_t}(B_t+t)] = e^{-\frac{t}{2}}\{E[B_t e^{-B_t}] + tE[e^{-B_t}]\}$ (1)

Calculemos primero $E[B_t e^{-\lambda B_t}]$: Sea $X \sim N(0,t)$

$E[B_t e^{-\lambda B_t}] \equiv E[X e^{-\lambda X}] = \displaystyle\int_{-\infty}^{\infty} x e^{-\lambda x} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx \rightarrow \left(\begin{array}{ll} u = e^{-\lambda x} & dv = \dfrac{x e^{-x^2/2t}}{\sqrt{2\pi t}} dx \\ du = -\lambda e^{-\lambda x} & v = -\dfrac{t e^{-x^2/2t}}{\sqrt{2\pi t}} \end{array} \right)$

$= \cancel{-t e^{-\lambda x} \dfrac{e^{-x^2/2t}}{\sqrt{2\pi t}}\Big|_{-\infty}^{\infty}}^{0} - \lambda t \displaystyle\int_{-\infty}^{\infty} e^{-\lambda x} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} dx = -\lambda t E[e^{-\lambda X}] = -\lambda t e^{\frac{\lambda^2 t}{2}}$ (*)

Ahora, (*) en (1); con $\lambda = 1$:

$E^*[B_t^*] = e^{-\frac{t}{2}}\{-te^{\frac{t}{2}} + te^{\frac{t}{2}}\} = 0$

- - $V^*[B_t^*] = \underbrace{E^*[(B_t^*)^2]}_{\square} - E^{2*}[B_t^*]$. Trabajemos primero en $\square$.

$E^*[(B_t^*)^2] = E[\xi_t^{\phi(t)}(B_t^*)^2] = E[e^{-\frac{t}{2}-B_t}(B_t+t)^2] = e^{-\frac{t}{2}} E[B_t^2 e^{-B_t} + 2tB_t e^{-B_t} + t^2 e^{-B_t}]$

$= e^{-\frac{t}{2}}\{\underbrace{E[B_t^2 e^{-B_t}]}_{\nabla} + 2tE[B_t e^{-B_t}] + t^2 E[e^{-B_t}]\}$. (2)

Trabajemos en ∇: ↓

$\nabla = E[B_t^2 e^{-B_t}]$. Encontremos $E[X^2 e^{-\lambda X}]$, con $\lambda \in \mathbb{R}$ y $X \sim N(0,t)$

$$= \int_{-\infty}^{\infty} x^2 e^{-\lambda x} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx \to \begin{pmatrix} u = xe^{-\lambda x} & dv = x\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx \\ du = (e^{-\lambda x} - \lambda x e^{-\lambda x})\,dx & v = -t\frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \end{pmatrix}$$

$$= \cancelto{0}{-t\frac{xe^{-\lambda x} e^{-x^2/2t}}{\sqrt{2\pi t}}\Big|_{-\infty}^{\infty}} + t\int_{-\infty}^{\infty} (e^{-\lambda x} - \lambda x e^{-\lambda x}) \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\,dx$$

$$= t\left[E[e^{-\lambda X}] - \lambda E[Xe^{-\lambda X}]\right] = t\left(e^{\frac{\lambda^2 t}{2}} + \lambda^2 t e^{\frac{\lambda^2 t}{2}}\right) = e^{\frac{\lambda^2 t}{2}} t(1+\lambda^2 t) \quad (**)$$

Reemplazando $(*)$ y $(**)$ en (2); con $\lambda = 1$.

$$E^*[(B_t^*)^2] = e^{-\frac{t}{2}}\left\{e^{\frac{t}{2}} t(1+t) + 2t(-te^{t/2}) + t^2 e^{t/2}\right\} = t$$

Como $E^*[B_t^*] = 0 \Rightarrow E^{*2}[B_t^*] = 0$. Luego, $V^*[B_t^*] = E^*[(B_t^*)^2] = t$ □.

c) $E^*[e^{i\theta(B_t^* - B_s^*)} | \mathcal{F}_s] = \frac{1}{\xi_s^{\phi(s)}} E[\xi_t^{\phi(t)} e^{i\theta(B_t^* - B_s^*)} | \mathcal{F}_s]$

$$= e^{s/2 + B_s} E[e^{-t/2 - B_t} e^{i\theta(B_t^* - B_s^*)} | \mathcal{F}_s] = e^{s/2 + B_s} E[e^{-\frac{t}{2} - B_t} e^{i\theta(B_t + t - B_s - s)} | \mathcal{F}_s]$$

$$= e^{-\frac{(t-s)}{2} + B_s} E[e^{-B_t} e^{i\theta(B_t - B_s) + i\theta(t-s)} | \mathcal{F}_s] = e^{-\frac{(t-s)}{2} + B_s + i\theta(t-s)} E[e^{-(B_t - B_s) - B_s} e^{i\theta(B_t - B_s)} | \mathcal{F}_s]$$

$$= e^{-\frac{(t-s)}{2} + B_s + i\theta(t-s)} E[e^{-B_s} e^{(i\theta - 1)(B_t - B_s)} | \mathcal{F}_s] = e^{-\frac{(t-s)}{2} + \cancel{B_s} + i\theta(t-s)} \cancel{E[e^{-B_s} | \mathcal{F}_s]} E[e^{(\theta i - 1)(B_t - B_s)} | \mathcal{F}_s]$$

$$= e^{-\frac{(t-s)}{2} + i\theta(t-s)} E[e^{(i\theta - 1)(B_t - B_s)}] = e^{-\frac{(t-s)}{2} + i\theta(t-s)} e^{-\frac{(i\theta - 1)^2 (t-s)}{2}}$$

$$= e^{\cancel{-\frac{(t-s)}{2}} + \cancel{i\theta(t-s)}} e^{-\frac{(-\theta^2 \cancel{- 2i\theta + 1})(t-s)}{2}} = e^{-\frac{\theta^2 (t-s)}{2}}$$

□

Luego, $B_t^* - B_s^* \sim N(0, t-s)$ y B_t^* tiene incrementos indep.

$\Rightarrow \{B_t^*\}_{t \geq 0}$ es MBEU en $(\Omega, \mathcal{F}, \mathbb{P}^*)$.

d). $dS_t = 2Sin^2(t)S_t dt + 2Sin^2(t)S_t dB_t \quad (3)$

$\Rightarrow B_t^* = B_t + t \Rightarrow dB_t = dB_t^* - dt$. Reemplazando en (3):

$dS_t = 2Sin^2(t)S_t dt + 2Sin^2(t)S_t[dB_t^* - dt]$

$dS_t = 2Sin^2(t)S_t dB_t^* \xrightarrow[\text{integral}]{\text{en forma}} S_t = 2\int_0^t Sin^2(u)S_u dB_u^* + S_0$

Veamos que S_t es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$ con $\mathcal{F}_s$, $s<t$.

$$E^*[S_t/\mathcal{F}_s] = S_0 + 2E^*\left[\int_0^t Sin^2(u)S_u dB_u^* \middle/ \mathcal{F}_s\right]$$

$$= S_0 + 2\left(E^*\left[\int_0^s Sin^2(u)S_u dB_u^* \middle/ \mathcal{F}_s\right] + \cancelto{0}{E^*\left[\int_s^t Sin^2(u)S_u dB_u^* \middle/ \mathcal{F}_s\right]}\right)$$

$$= S_0 + 2\int_0^s Sin^2(u)S_u dB_u^* = S_s \Longrightarrow \text{Martingala.}$$

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oo $dS_t = \left(\alpha Tan\left(\frac{t}{2}\right)\right)^2 S_t dt + \frac{(1-Cos(t))^2}{\sigma(1-Cos^2(t))} S_t dB_t$

Se sabe que $Tan\left(\frac{t}{2}\right) = \sqrt{\frac{1-Cos(t)}{1+Cos(t)}}$, luego

$$dS_t = \alpha^2\left(\frac{1-Cos(t)}{1+Cos(t)}\right)S_t dt + \frac{(1-Cos(t))^{\cancel{2}}}{\cancel{(1-Cos(t))}(1+Cos(t))}\frac{S_t}{\sigma}dB_t$$

$$\Rightarrow dS_t = \alpha^2\left(\frac{1-Cos(t)}{1+Cos(t)}\right)S_t dt + \left(\frac{1-Cos(t)}{1+Cos(t)}\right)\frac{S_t}{\sigma}dB_t.$$

a). $\bullet\ \phi(t) = -\frac{f(t,S_t)}{g(t,S_t)} = -\frac{\alpha^2\left(\frac{1-Cos(t)}{1+Cos(t)}\right)S_t}{\left(\frac{1-Cos(t)}{1+Cos(t)}\right)\frac{S_t}{\sigma}} = -\alpha^2\sigma$

$\bullet\ h(t,\phi(t)) = -\frac{1}{2}\int_0^t(-\alpha^2\sigma)^2 du + \int_0^t(-\alpha^2\sigma)dB_u = -\frac{\alpha^4\sigma^2 t}{2} - \alpha^2\sigma B_t$

$\bullet\ \xi_t^{\phi(t)} = e^{h(t,\phi(t))} = e^{-\frac{\alpha^4\sigma^2 t}{2} - \alpha^2\sigma B_t}$. Verifiquemos que $E[\xi_t^{\phi(t)}] = 1$.

$E[\xi_t^{\phi(t)}] = e^{-\frac{\alpha^4\sigma^2 t}{2}} E[e^{-\alpha^2\sigma B_t}] = e^{-\frac{\alpha^4\sigma^2 t}{2} + \frac{\alpha^4\sigma^2 t}{2}} = 1$ ✓

$\bullet\ B_t^* = B_t - \int_0^t(-\alpha^2\sigma)du = B_t + \alpha^2\sigma t$

b) • $E^*[B_t^*] = E[\xi_t^{\phi(t)}(B_t+\alpha^2\sigma t)] = E[e^{-\frac{\alpha^4\sigma^2 t}{2}-\alpha^2\sigma B_t}(B_t+\alpha^2\sigma t)]$

$= e^{-\frac{\alpha^4\sigma^2 t}{2}}\left[E[B_t e^{-\alpha^2\sigma B_t}] + \alpha^2\sigma t E[e^{-\alpha^2\sigma B_t}]\right]$. Por (*) con $\lambda=\alpha^2\sigma$, se tiene

$= e^{-\frac{\alpha^4\sigma^2 t}{2}}\left(-\alpha^2\sigma t e^{\frac{\alpha^4\sigma^2 t}{2}} + \alpha^2\sigma t e^{\frac{\alpha^4\sigma^2 t}{2}}\right) = 0.$

• $V^*[B_t^*] = E^*[(B_t^*)^2] - \cancel{E^{*2}[B_t^*]}^{\,0}$

$= E[\xi_t^{\phi(t)}(B_t^*)^2] = E\left[e^{-\frac{\alpha^4\sigma^2 t}{2}-\alpha^2\sigma B_t}(B_t+\alpha^2\sigma t)^2\right]$

$= e^{-\frac{\alpha^4\sigma^2 t}{2}}\left(E\left[B_t^2 e^{-\alpha^2\sigma B_t} + 2\alpha^2\sigma t B_t e^{-\alpha^2\sigma B_t} + \alpha^4\sigma^2 t^2 e^{-\alpha^2\sigma B_t}\right]\right)$

$= e^{-\frac{\alpha^4\sigma^2 t}{2}}\left(E[B_t^2 e^{-\alpha^2\sigma B_t}] + 2\alpha^2\sigma t E[B_t e^{-\alpha^2\sigma B_t}] + \alpha^4\sigma^2 t^2 E[e^{-\alpha^2\sigma B_t}]\right)$

De (*) y (**) con $\lambda=\alpha^2\sigma$, se tiene:

$= e^{-\frac{\alpha^4\sigma^2 t}{2}}\left(e^{\frac{\alpha^4\sigma^2 t}{2}} t(1+\alpha^4\sigma^2 t) - 2\alpha^2\sigma t^2(\alpha^2\sigma) e^{\frac{\alpha^4\sigma^2 t}{2}} + \alpha^4\sigma^2 t^2 e^{\frac{\alpha^4\sigma^2 t}{2}}\right) = t.$

c) $E^*[e^{i\theta(B_t^*-B_s^*)}|\mathcal{F}_s] = \frac{1}{\xi_s^{\phi(s)}} E\left[\xi_t^{\phi(t)} e^{i\theta(B_t+\alpha^2\sigma t-B_s-\alpha^2\sigma s)}|\mathcal{F}_s\right]$

$= e^{\frac{\alpha^4\sigma^2 s}{2}+\alpha^2\sigma B_s+i\theta\alpha^2\sigma(t-s)-\frac{\alpha^4\sigma^2 t}{2}} E\left[e^{-\alpha^2\sigma B_t} e^{i\theta(B_t-B_s)}|\mathcal{F}_s\right]$

$= e^{-\frac{\alpha^4\sigma^2(t-s)}{2}+\alpha^2\sigma B_s+i\theta(t-s)\alpha^2\sigma} E\left[e^{(i\theta-\alpha^2\sigma)(B_t-B_s)-\alpha^2\sigma B_s}|\mathcal{F}_s\right] = e^{-\frac{\alpha^4\sigma^2(t-s)}{2}+\cancel{\alpha^2\sigma B_s}+i\theta(t-s)} \cancel{E[e^{-\alpha^2\sigma B_s}|\mathcal{F}_s]}\, E\left[e^{(i\theta-\alpha^2\sigma)(B_t-B_s)}|\mathcal{F}_s\right]$

$= e^{-\frac{\alpha^4\sigma^2(t-s)}{2}+i\theta(t-s)\alpha^2\sigma} e^{\frac{(i\theta-\alpha^2\sigma)^2(t-s)}{2}} = e^{-\frac{\alpha^4\sigma(t-s)}{2}+i\theta(t-s)\alpha^2\sigma} e^{\left(\frac{-\theta^2-2i\theta\alpha^2\sigma+\alpha^4\sigma}{2}\right)(t-s)} = e^{-\frac{\theta^2(t-s)}{2}}$

$\Longrightarrow \{B_t^*\}_{t\geq 0}$ es MBEU en $(\Omega, \mathcal{F}, \mathbb{P}^*)$.

d) $dS_t = \alpha^2\left(\frac{1-\cos(t)}{1+\cos(t)}\right)S_t + \left(\frac{1-\cos(t)}{1+\cos(t)}\right)\frac{S_t}{\sigma}(dB_t^* - \alpha^2\sigma t)$

$dS_t = \left(\frac{1-\cos(t)}{1+\cos(t)}\right)\frac{S_t}{\sigma}dB_t^*$

Veamos que es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$ ↓

$$\delta_t = \delta_0 + \int_0^t \left(\frac{1-\cos(u)}{1+\cos(u)}\right)\frac{\delta_u}{\sigma}\,dB_u^*$$

$$E[\delta_t / \mathcal{F}_s] = E\left[\int_0^t \left(\frac{1-\cos(u)}{1+\cos(u)}\right)\frac{\delta_u}{\sigma}\,dB_u^*\right] + \delta_0$$

$$= E\left[\int_0^s \left(\frac{1-\cos(u)}{1+\cos(u)}\right)\frac{\delta_u}{\sigma}\,dB_u^* / \mathcal{F}_s\right] + \underbrace{E\left[\int_s^t \left(\frac{1-\cos(u)}{1+\cos(u)}\right)\frac{\delta_u}{\sigma}\,dB_u^*\right]}_{0} + \delta_0$$

$$= \delta_0 + \int_0^s \left(\frac{1-\cos(u)}{1+\cos(u)}\right)\frac{\delta_u}{\sigma}\,dB_u^* = \delta_s \implies \text{Es Martingala}$$

ooo $\delta_t = \frac{1}{2}X_t^2$, con $dX_t = \mu X_t dt + \sigma X_t dB_t$, $X_t^2 = 2\delta_t$

Aplicando Itô unidim. parte II: $d\delta_t = X_t dX_t + \frac{1}{2}(dX_t)^2$

$$d\delta_t = X_t(\mu X_t dt + \sigma X_t dB_t) + \frac{1}{2}\sigma^2 X_t^2 dt = (\mu + \tfrac{1}{2}\sigma^2)X_t^2 dt + \sigma X_t^2 dB_t$$

$$d\delta_t = (2\mu + \sigma^2)\delta_t dt + 2\sigma\delta_t dB_t$$

a). $\phi(t) = -\frac{f(t,\delta_t)}{g(t,\delta_t)} = -\frac{2\mu+\sigma^2}{2\sigma}\frac{\delta_t}{\delta_t} = \frac{-(2\mu+\sigma^2)}{2\sigma}$. Sea $\boxed{K = \frac{-(2\mu+\sigma^2)}{2\sigma}}$

- $h(t,\phi(t)) = -\frac{1}{2}\int_0^t K^2 du + \int_0^t K\,dB_u = -\frac{K^2}{2}t + KB_t$
- $\xi_t^{\phi(t)} = e^{h(t,\phi(t))} = e^{-\frac{K^2 t}{2} + KB_t}$ Verifiquemos $E[\xi_t^{\phi(t)}] = e^{-\frac{K^2 t}{2}}E[e^{KB_t}] = e^{-\frac{K^2 t}{2} + \frac{K^2 t}{2}} = 1$ ✓.

b) $B_t^* = B_t - \int_0^t K\,du = B_t - Kt$

- $E^*[B_t^*] = E[\xi_t^{\phi(t)} B_t^*] = E[e^{-\frac{K^2 t}{2} + KB_t}(B_t - Kt)] = e^{-\frac{K^2 t}{2}}\left(E[B_t e^{KB_t}] - Kt\,E[e^{KB_t}]\right)$

Reemplazando en (*) con $\lambda = -K$:

$$= e^{-\frac{K^2 t}{2}}\left(Kte^{\frac{K^2 t}{2}} - Kte^{\frac{K^2 t}{2}}\right) = 0.$$

- $V^*[B_t^*] = E^*[(B_t^*)^2] = E[\xi_t^{\phi(t)}(B_t - Kt)^2] = e^{-\frac{K^2 t}{2}}\left(E[B_t^2 e^{KB_t}] - 2Kt\,E[B_t e^{KB_t}] + K^2t^2 E[e^{KB_t}]\right)$

Reemplazando (*) y (**), con $\lambda = -K$, se tiene:

$$= e^{-\frac{K^2 t}{2}}\left(e^{\frac{K^2 t}{2}}t(1+K^2 t) - 2K^2t^2 e^{\frac{K^2 t}{2}} + K^2t^2 e^{\frac{K^2 t}{2}}\right) = t.$$

c) $E^*\left[e^{i\theta(B_t^*-B_s^*)}\,|\,\mathcal{F}_s\right] = \frac{1}{\xi_s^{\phi(s)}} E\left[\xi_t^{\phi(t)} e^{i\theta(B_t-Kt-B_s+Ks)}\,|\,\mathcal{F}_s\right]$

$= e^{\frac{K^2 s}{2}-KB_s-\frac{K^2 t}{2}-i\theta K(t-s)} E\left[e^{KB_t} e^{i\theta(B_t-B_s)}\,|\,\mathcal{F}_s\right] = e^{-\frac{K^2(t-s)}{2}-KB_s-i\theta K(t-s)} E\left[e^{(i\theta+K)(B_t-B_s)+KB_s}\,|\,\mathcal{F}_s\right]$

$= e^{-\frac{K^2(t-s)}{2}-\cancel{KB_s}-i\theta(t-s)K} E\left[e^{(i\theta+K)(B_t-B_s)}\,|\,\mathcal{F}_s\right] \cancel{E\left[e^{KB_s}\,|\,\mathcal{F}_s\right]}$

$= e^{-\frac{K^2(t-s)}{2}-i\theta K(t-s)} E\left[e^{(i\theta+K)(B_t-B_s)}\right] = e^{-\frac{K^2(t-s)}{2}-i\theta K(t-s)} e^{\frac{(i\theta+K)^2(t-s)}{2}}$

$= e^{-\frac{K^2(t-s)}{2}-i\theta K(t-s)} e^{\frac{(-\theta^2+2i\theta K+K^2)(t-s)}{2}} = e^{-\frac{\theta^2(t-s)}{2}}$

$\Longrightarrow \{B_t^*\}_{t\geq 0}$ es MBEU en $(\Omega, \mathcal{F}, \mathbb{P}^*)$.

d) $dS_t = (2\mu+\sigma^2)S_t dt + 2\sigma S_t(dB_t^* + K dt) = 2\sigma S_t dB_t^*$ ← Al reemplazar $K = -\frac{(2\mu+\sigma^2)}{2\sigma}$

En forma integral: $S_t = S_0 + \int_0^t 2\sigma S_t dB_t^*$.

Veamos que S_t es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$ con $\mathcal{F}_s$, $s<t$:

$E^*[S_t|\mathcal{F}_s] = S_0 + E^*\left[\int_0^s 2\sigma S_u dB_u^* + \cancel{\int_s^t 2\sigma S_u dB_u^*}\,|\,\mathcal{F}_s\right] = S_0 + \int_0^s 2\sigma S_u dB_u^* = S_s$

$\Longrightarrow$ Es Martingala.

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2. $\circ\, dS_t = (\alpha+\beta S_t)dt + \sigma dB_t$. Se define $S_t^* = e^{-rt}S_t$

a) Aplicando Itô unidim. II: $dS_t^* = -re^{-rt}S_t dt + e^{-rt}dS_t$

$dS_t^* = -re^{-rt}S_t dt + e^{-rt}\left[(\alpha+\beta S_t)dt + \sigma dB_t\right]$

$dS_t^* = \left[(\beta-r)S_t e^{-rt} + \alpha e^{-rt}\right]dt + \sigma e^{-rt}dB_t$, pero $e^{-rt}S_t = S_t^*$

$dS_t^* = \left[(\beta-r)S_t^* + \alpha e^{-rt}\right]dt + \sigma e^{-rt}dB_t$ (2.1.1.)

b) $\phi(t) = -\frac{(\beta-r)S_t^*+\alpha e^{-rt}}{\sigma e^{-rt}} = -\frac{(\beta-r)S_t^* e^{rt}+\alpha}{\sigma} = -\frac{(\beta-r)S_t+\alpha}{\sigma}$

$B_t^* = B_t + \int_0^t \frac{(\beta-r)S_u+\alpha}{\sigma}du = B_t + \frac{\alpha}{\sigma}t + \int_0^t \frac{(\beta-r)S_u du}{\sigma}$

$B_t = B_t^* - \frac{\alpha}{\sigma}t + \int_0^t \frac{(r-\beta)}{\sigma}S_u du$

$\Longrightarrow dB_t = dB_t^* + \left[\frac{(r-\beta)}{\sigma}S_u - \frac{\alpha}{\sigma}\right]dt$ (2.1.2)

(2.1.2) en (2.1.1):

$$dS_t^* = [\cancel{(\beta-r)S_t^* + \alpha e^{-rt}}]dt + \sigma e^{-rt}[dB_t^* + \cancel{\left(\frac{(r-\beta)}{\sigma}S_u - \frac{\alpha}{\sigma}\right)dt}]$$

$$dS_t^* = \sigma e^{-rt} dB_t^* \xrightarrow[\text{integral}]{\text{en forma}} S_t^* = S_0^* + \int_0^t \sigma e^{-ru} dB_u^*$$

Veamos que S_t^* es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$

$$E^*[S_t^* | \mathcal{F}_s] = E^*\left[\int_0^t \sigma e^{-ru} dB_u^* | \mathcal{F}_s\right] + S_0 = E^*\left[\int_0^s \sigma e^{-ru} dB_u^* + \cancel{\int_s^t \sigma e^{-ru} dB_u^*} \,\Big|\, \mathcal{F}_s\right]$$

$$= S_0 + \int_0^s \sigma e^{-ru} dB_u^* = S_s^*$$

c) $dS_t = (\alpha + \beta S_t)dt + \sigma dB_t$

$$dS_t = (\alpha + \beta S_t)dt + \sigma\left(\left(\frac{r-\beta}{\sigma}\right)S_t - \frac{\alpha}{\sigma}\right)dt$$

$$\boxed{dS_t = rS_t dt + \sigma dB_t^*} \quad \text{SDE Neutralizada}$$

d) Se hará el proceso para una SDE en general:

$dS_t = h(t, S_t)dt + g(t, S_t)dB_t$. Sea $f \equiv f(t, S_t)$ el precio de la opción en el mundo de riesgo neutral.

Aplicando Itô unidimensional II:

$$df(t, S_t) = \frac{\partial f(t, S_t)}{\partial t}dt + \frac{\partial f(t, S_t)}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial S_t^2}(dS_t)^2$$

$$df(t, S_t) = \frac{\partial f(t, S_t)}{\partial t}dt + \frac{\partial f(t, S_t)}{\partial S_t}[h(t, S_t)dt + g(t, S_t)dB_t] + \frac{1}{2}\frac{\partial^2 f(t, S_t)}{\partial S_t^2}[g^2(t, S_t)dt]$$

$$df(t, S_t) = \left[\frac{\partial f(t, S_t)}{\partial t} + h(t, S_t)\frac{\partial f(t, S_t)}{\partial S_t} + \frac{1}{2}g^2(t, S_t)\frac{\partial^2 f(t, S_t)}{\partial S_t^2}\right]dt + g(t, S_t)\frac{\partial f(t, S_t)}{\partial S_t}dB_t$$

Consideremos la discretización:

$$\Delta f(t, S_t) = \left[\frac{\partial f(t, S_t)}{\partial t} + h(t, S_t)\frac{\partial f(t, S_t)}{\partial S_t} + \frac{1}{2}g^2(t, S_t)\frac{\partial^2 f(t, S_t)}{\partial S_t^2}\right]\Delta t + g(t, S_t)\frac{\partial f(t, S_t)}{\partial S_t}\Delta B_t \quad (2.1.3)$$

y para la SDE:

$$\Delta S_t = h(t, S_t)\Delta t + g(t, S_t)\Delta B_t \quad (2.1.4)$$

Consideremos el portafolio $\pi = -f(t, S_t) + \frac{\partial f(t, S_t)}{\partial S_t}S_t \Rightarrow \Delta\pi = -\Delta f(t, S_t) + \frac{\partial f(t, S_t)}{\partial S_t}\Delta S_t$ (2.1.5)

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(2.1.3) y (2.1.4) en (2.1.5):

$$\Delta\pi = -\left\{\left[\frac{\partial f(t,S_t)}{\partial t} + \cancel{h(t,S_t)\frac{\partial f(t,S_t)}{\partial S_t}} + \frac{1}{2}g^2(t,S_t)\frac{\partial^2 f(t,S_t)}{\partial S_t^2}\right]\Delta t + \cancel{g(t,S_t)\frac{\partial f(t,S_t)}{\partial S_t}\Delta B_t}\right\}$$

$$+ \frac{\partial f(t,S_t)}{\partial S_t}\left[\cancel{h(t,S_t)\Delta t} + \cancel{g(t,S_t)\Delta B_t}\right]$$

$$\Delta\pi = -\left[\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2}g^2(t,S_t)\frac{\partial^2 f(t,S_t)}{\partial S_t^2}\right]\Delta t$$

Como es libre de riesgo $\Delta\pi = \pi r\Delta t$, $r \rightarrow$ tasa de interés libre de riesgo.

$$\pi r\cancel{\Delta t} = -\left[\frac{\partial f(t,S_t)}{\partial t} + \frac{1}{2}g^2(t,S_t)\frac{\partial^2 f(t,S_t)}{\partial S_t^2}\right]\cancel{\Delta t}$$

$$\left(-f(t,S_t) + S_t\frac{\partial f(t,S_t)}{\partial S_t}\right)r = -\frac{\partial f(t,S_t)}{\partial t} - \frac{1}{2}g^2(t,S_t)\frac{\partial^2 f(t,S_t)}{\partial S_t^2}$$

$$\boxed{rf(t,S_t) = \frac{\partial f(t,S_t)}{\partial t} + rS_t\frac{\partial f(t,S_t)}{\partial S_t} + \frac{1}{2}g^2(t,S_t)\frac{\partial^2 f(t,S_t)}{\partial S_t^2}} \quad (***)$$

En este caso, $h(t,S_t) = \alpha + \beta S_t$, $g(t,S_t) = \sigma$

$$rf(t,S_t) = \frac{\partial f(t,S_t)}{\partial t} + rS_t\frac{\partial f(t,S_t)}{\partial S_t} + \frac{\sigma^2}{2}\frac{\partial^2 f(t,S_t)}{\partial S_t^2}$$

---

a) $dS_t = (\alpha+\beta S_t)dt + \sigma S_t dB_t$. $S_t^* = e^{-rt}S_t \Rightarrow$ Itô:

$$dS_t^* = -re^{-rt}S_t dt + e^{-rt}dS_t = -re^{-rt}S_t dt + e^{-rt}\left[(\alpha+\beta S_t)dt + \sigma S_t dB_t\right]$$

$$dS_t^* = \left[(\beta-r)S_t e^{-rt} + \alpha e^{-rt}\right]dt + \sigma e^{-rt}S_t dB_t$$

$$dS_t^* = \left[(\beta-r)S_t^* + \alpha e^{-rt}\right]dt + \sigma S_t^* dB_t$$

b) $\phi(t) = -\frac{(\beta-r)S_t^* + \alpha e^{-rt}}{\sigma S_t^*}$. $B_t^* = B_t + \int_0^t \frac{(\beta-r)S_u^* + \alpha e^{-ru}}{\sigma S_u^*}du$.

$$B_t = B_t^* + \int_0^t \frac{(r-\beta)S_u^* - \alpha e^{-ru}}{\sigma S_u^*}du \Rightarrow dB_t = dB_t^* + \left[\frac{(r-\beta)S_t^* - \alpha e^{-rt}}{\sigma S_t^*}\right]dt$$

$$dS_t^* = \left[\cancel{(\beta-r)S_t^* + \alpha e^{-rt}}\right]dt + \sigma S_t^*\left[dB_t^* + \left(\cancel{\frac{(r-\beta)S_t^* - \alpha e^{-rt}}{\sigma S_t^*}}\right)dt\right]$$

$$dS_t^* = \sigma S_t^* dB_t^* \xrightarrow{\text{int.}} S_t^* = S_0^* + \int_0^t \sigma S_u^* dB_u^*$$

Veamos que S_t^* es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$

$$E^*[S_t^* | \mathcal{F}_s] = E^*\left[S_0 + \int_0^t \sigma S_u^* dB_u^* \Big| \mathcal{F}_s\right] = S_0 + E^*\left[\int_0^s \sigma S_u^* dB_u^* \Big| \mathcal{F}_s\right] + \cancel{E^*\left[\int_s^t \sigma S_u^* dB_u^* \Big| \mathcal{F}_s\right]}$$

$$= S_0 + \int_0^s \sigma S_u^* dB_u^* = S_s \Longrightarrow \text{Es Martingala}$$

c) $dS_t = (\alpha + \beta S_t)dt + \sigma S_t dB_t$, $\quad S_t = e^{rt} S_t^*$

$$dS_t = (\alpha+\beta S_t)dt + \sigma S_t\left[dB_t^* + \frac{(r-\beta)S_t^* - \alpha e^{-rt}}{\sigma S_t^*} dt\right] = (\alpha+\beta S_t)dt + \sigma S_t\left[\left(\frac{(r-\beta)}{\sigma} - \frac{\alpha}{\sigma S_t^* e^{rt}}\right)dt + dB_t^*\right]$$

$$dS_t = (\alpha+\beta S_t)dt + [(r-\beta)S_t - \alpha]dt + \sigma S_t dB_t^*$$

$\boxed{dS_t = rS_t dt + \sigma S_t dB_t^*}$ SDE Neutralizada.

d) En este caso $h(t,S_t) = (\alpha+\beta S_t)$, $g(t,S_t) = \sigma S_t$, reemplazando en (***):

$$rf(t,S_t) = \frac{\partial f(t,S_t)}{\partial t} + rS_t \frac{\partial f(t,S_t)}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f(t,S_t)}{\partial S_t^2}$$

ooo $dS_t = (\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t$. $S_t^* = e^{-rt} S_t$

a) $dS_t^* = -re^{-rt} S_t dt + e^{-rt} dS_t = -re^{-rt} S_t dt + e^{-rt}[(\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t]$

$dS_t^* = [(\beta - r)S_t e^{-rt} + \alpha e^{-rt}]dt + \sigma e^{-rt} S_t^\gamma dB_t$. Como $S_t^* = e^{-rt} S_t \Rightarrow (S_t^*)^\gamma = (e^{-rt}S_t)^\gamma$

$(S_t^*)^\gamma = e^{-\gamma rt} S_t^\gamma = e^{-(\gamma-1)rt} e^{-rt} S_t^\gamma$. Luego, $\boxed{dS_t^* = [(\beta-r)S_t^* + \alpha e^{-rt}]dt + \sigma e^{(\gamma-1)rt} (S_t^*)^\gamma dB_t}$

b) $\phi(t) = -\dfrac{(\beta-r)S_t^* + \alpha e^{-rt}}{\sigma e^{(\gamma-1)rt}(S_t^*)^\gamma}$ $\quad B_t^* = B_t + \displaystyle\int_0^t \frac{(\beta-r)S_u^* + \alpha e^{-ru}}{\sigma e^{(\gamma-1)ru}(S_t^*)^\gamma} du$

$$B_t = B_t^* + \int_0^t \frac{(r-\beta)S_u^* - \alpha e^{-rt}}{(S_t^*)^\gamma \sigma e^{(\gamma-1)ru}} du \rightarrow dB_t = dB_t^* + \frac{(r-\beta)S_t^* - \alpha e^{-rt}}{(S_t^*)^\gamma \sigma e^{(\gamma-1)rt}} dt$$

$$dS_t^* = [(\beta-r)S_t^* + \alpha e^{-rt}]dt + \sigma e^{(\gamma-1)rt}(S_t^*)^\gamma \left[dB_t^* + \frac{(r-\beta)S_t^* - \alpha e^{-rt}}{(S_t^*)^\gamma \sigma e^{(\gamma-1)rt}} dt\right]$$

$$dS_t^* = [\cancel{(\beta-r)S_t^*} + \cancel{\alpha e^{-rt}}]dt + \sigma e^{(\gamma-1)rt}(S_t^*)^\gamma dB_t^* + [\cancel{(r-\beta)S_t^*} - \cancel{\alpha e^{-rt}}]dt$$

$$dS_t^* = \sigma e^{(\gamma-1)rt}(S_t^*)^\gamma dB_t^* \xrightarrow{int} S_t^* = S_0^* + \int_0^t \sigma e^{(\gamma-1)ru}(S_u^*)^\gamma dB_u^*$$

Veamos que S_t^* es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$:

$$E^*[S_t^* | \mathcal{F}_s] = S_0 + E\left[\int_0^s \sigma e^{(\gamma-1)ru} (S_u^*)^\gamma dB_u^* \Big| \mathcal{F}_s\right] + \cancelto{0}{E\left[\int_s^t \sigma e^{(\gamma-1)ru} (S_u^*)^\gamma dB_u^* \Big| \mathcal{F}_s\right]}$$

$$= S_0 + \int_0^s \sigma e^{(\gamma-1)ru} (S_u^*)^\gamma dB_u^* = S_s \Rightarrow \text{Es Martingala.}$$

c) $$dS_t = (\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t = (\alpha+\beta S_t)dt + \sigma S_t^\gamma \left[dB_t^* + \frac{(r-\beta)S_t^* - \alpha e^{-rt}}{\sigma (S_t^*)^\gamma e^{(\gamma-1)rt}} dt\right]$$

$$= (\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t^* + \left[\frac{(r-\beta)S_t^* S_t^\gamma}{(S_t^*)^\gamma e^{(\gamma-1)rt}} - \frac{\alpha e^{-rt} S_t^\gamma}{(S_t^*)^\gamma e^{(\gamma-1)rt}}\right]dt$$

Como $(S_t^*)^\gamma = e^{-\gamma rt} S_t^\gamma \Rightarrow S_t^\gamma = e^{\gamma rt}(S_t^*)^\gamma$.

$$= (\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t^* + \left[\frac{(r-\beta)S_t^\gamma}{(S_t^*)^{\gamma-1} e^{(\gamma-1)rt}} - \frac{\alpha S_t^\gamma}{(S_t^*)^\gamma e^{\gamma rt}}\right]dt$$

$$= (\alpha+\beta S_t)dt + \sigma S_t^\gamma dB_t^* + [(r-\beta)S_t - \alpha]dt = rS_t dt + \sigma S_t^\gamma dB_t^*$$

$\boxed{dS_t = rS_t dt + \sigma S_t^\gamma dB_t^*}$ SDE Neutralizada.

d) En este caso $h(t,S_t) = (\alpha+\beta S_t)$, $g(t,S_t) = \sigma S_t^\gamma$. Reemplazando en (***)

$$r f(t,S_t) = \frac{\partial f(t,S_t)}{\partial t} + rS_t \frac{\partial f(t,S_t)}{\partial S_t} + \frac{\sigma^2 S_t^{2\gamma}}{2} \frac{\partial^2 f(t,S_t)}{\partial S_t^2}$$

3. Sea $\{W_t\}_{t\geq 0}$ un MBEU en $(\Omega, \mathcal{F}, \mathbb{P})$. Se define $W_t^* = \lambda t + W_t$, entonces

$$\frac{d\mathbb{P}^*(\omega^*)}{d\mathbb{P}(\omega)} = \xi_t^\lambda = e^{-\frac{\lambda^2 t}{2} - \lambda W_t}$$

En este caso $\mathbb{P}^*$ es una medida equivalente a $\mathbb{P}$. Sea $\hat{W}_t = -\lambda t + W_t^*$.
Veamos que $\hat{\mathbb{P}}$ es una medida equivalente a $\mathbb{P}^*$

Se quiere encontrar una función ψ_t^λ que satisfaga

① $E[(W_t^* - \lambda t)\psi_t^\lambda] = 0$ y ② $E[(W_t^* - \lambda t)^2 \psi_t^\lambda] = t$

③ ψ_t^λ es Martingala en $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.

Se sabe que $\{W_t^*\}_{t\geq 0}$ es un MBEU en $\{W_t^*\}_{t\geq 0}$. Se quiere

$$\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\omega^* - \lambda t)\psi_t^\lambda e^{-\frac{(\omega^*)^2}{2t}} d\omega^* = 0$$

Como $W_t^* \sim N(0,t)$ ← Cambio de variable.

$$I_t = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\omega^* - \lambda t) e^{-\frac{(\omega^* - \lambda t)^2}{2t}} d\omega^* = 0$$

$$= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\omega^* - \lambda t) e^{\frac{(\omega^*)^2}{2t}} \underbrace{e^{\lambda\omega^* - \frac{\lambda^2 t}{2}}}_{\psi_t^\lambda} d\omega^*$$

Verifiquemos la varianza:

$$I_t^2 = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\omega^* - \lambda t)^2 e^{-\frac{(\omega^* - \lambda t)^2}{2t}} d\omega^* = t$$

$$= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} (\omega^* - \lambda t)^2 e^{-\frac{(\omega^*)^2}{2t}} \underbrace{e^{\lambda\omega^* - \frac{\lambda^2 t}{2}}}_{\psi_t^\lambda} d\omega^* = t \checkmark$$

Luego, $\psi_t^\lambda = e^{\lambda W_t^* - \frac{\lambda^2 t}{2}}$. Veamos que ψ_t^λ es Martingala en $(\Omega, \mathcal{F}, \mathbb{P}^*)$.

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$$E^*[\psi_t^\lambda | \mathcal{F}_s] = E^*\left[e^{-\frac{\lambda^2 t}{2} + \lambda W_t^*} \middle| \mathcal{F}_s\right] = e^{-\frac{\lambda^2 t}{2}} E^*\left[e^{\lambda(W_t^* - W_s^*) + \lambda W_s^*} \middle| \mathcal{F}_s\right]$$

$$= e^{-\frac{\lambda^2 t}{2}} E^*\left[e^{\lambda(W_t^* - W_s^*)} \middle| \mathcal{F}_s\right] E^*\left[e^{\lambda W_s^*} \middle| \mathcal{F}_s\right] = e^{-\frac{\lambda^2 t}{2}} e^{\lambda W_s} E\left[e^{\lambda(W_t^* - W_s^*)}\right]$$

$$= e^{-\frac{\lambda^2 t}{2} - \lambda W_s + \frac{\lambda^2 (t-s)}{2}} = e^{\lambda W_s + \frac{\lambda^2 s}{2}} = \psi_s^\lambda \Longrightarrow \text{Es Martingala.}$$

Sea $g(W_t^* - \lambda t)$, con $\{W_t^*\}_{t \geq 0}$ MBEU en $(\Omega, \mathcal{F}, \mathbb{P}^*)$.

$$E[g(W_t^* - \lambda t)\psi_t^\lambda] = \int_{-\infty}^{\infty} g(\omega^* - \lambda t)\psi_t^\lambda \, d\mathbb{P}^*(\omega^*) \quad (3.1)$$

donde $d\mathbb{P}^*(\omega^*) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\omega^*)^2}{2t}} d\omega^*$. Si $\Omega = \mathbb{R}$ y $\mathcal{F} = \mathcal{B}$ - Borel, y suponiendo que $\hat{W}_t \sim N(0,t)$ en $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$, entonces

$$E[g(\hat{W}_t)] = \int_{-\infty}^{\infty} g(\hat{\omega}) d\hat{\mathbb{P}}(\hat{\omega}), \quad \hat{\mathbb{P}}(\hat{\omega}) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{\hat{\omega}^2}{2t}} d\hat{\omega}$$

$$= \int_{-\infty}^{\infty} g(\hat{\omega}) \frac{d\hat{\mathbb{P}}(\hat{\omega})}{d\mathbb{P}^*(\omega^*)} d\mathbb{P}^*(\omega^*) \quad (3.2)$$

De (3.1) y (3.2) $\Rightarrow \boxed{\frac{d\hat{\mathbb{P}}(\hat{\omega})}{d\mathbb{P}^*(\omega^*)} = \psi_t^\lambda}$ Derivada de Radon Nikodym

$\Longrightarrow$ La medida $\hat{\mathbb{P}}$ es equivalente a $\mathbb{P}^*$.

Veamos que $\hat{W}_t \sim N(0,t)$ en $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.

$$\hat{E}[e^{i\theta \hat{W}_t}] = E^*[e^{i\theta \hat{W}_t} \psi_t^\lambda] = E^*\left[e^{i\theta(W_t^* - \lambda t)} e^{\lambda W_t - \frac{\lambda^2 t}{2}}\right] = e^{-\frac{\lambda^2 t}{2} - i\theta t\lambda} E^*\left[e^{(i\theta + \lambda)W_t^*}\right]$$

$$= e^{-\frac{\lambda^2 t}{2} - i\theta t\lambda} e^{\frac{(i\theta+\lambda)^2 t}{2}} = e^{-\frac{\lambda^2 t}{2} - i\theta t\lambda} e^{\frac{(-\theta^2 t + 2i\theta t\lambda + \lambda^2 t)}{2}} = e^{-\frac{\theta^2 t}{2}}$$

$\Longrightarrow \hat{W}_t \sim N(0,t)$ en $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$.

4. Modelo de Black-Scholes:

Supuestos:

1. La dinámica de precios del activo está dada por un Movimiento Browniano geométrico. Sea S_t el precio del activo subyacente. Se debe verificar que $S_t \sim$ lognormal, $\forall t \in [t_0, T]$; o de forma equivalente, los rendimientos son normales: $\boxed{dS_t = \mu S_t dt + \sigma S_t dB_t}$
2. El redimiento medio esperado $\mu \in \mathbb{R}$ y la volatilidad $\sigma \in \mathbb{R}_+$, del precio del activo se mantienen constantes en $[t_0, T]$.
3. No existen costos de transacción.
4. Todos los activos son perfectamente divisibles.
5. No existen dividendos durante la vida de la opción.
6. No hay oportunidades de arbitraje.
7. La información es simétrica.
8. El activo puede negociarse ~~de~~ manera continua.
9. El retorno esperado de todos los activos es la tasa de interés libre de riesgo.
10. El valor de la opción es el pago esperado en un mundo de riesgo neutral descontado a la tasa de interés libre de riesgo.
11. El portafolio libre de riesgo está compuesto por una posición en la opción y otra en el activo subyacente.

La ecuación de Black-Scholes-Merton debe satisfacer que el precio f de cualquier instrumento derivado sobre un activo que no paga dividendos es

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \quad (4.1)$$

A continuación se describirá el proceso para llegar a (4.1).

$\boxed{1}$ El precio del activo subyacente satisface

(4.2) $dS_t = \mu S_t dt + \sigma S_t dB_t$, $t \in [t_0, T]$, $\mu \in \mathbb{R}_+$ y $\sigma \in \mathbb{R}_+$.

$\{B_t\}_{t \geq 0}$ es un MBEU.

$\boxed{2}$ Suponer que el precio de una opción call f sobre S_t es una función de S_t y del tiempo, i.e. $f \equiv f(t, S_t)$.

$\boxed{3}$ Aplicar fórmula de Itô unidimensional parte II a f y reemplazar (4.2).

$\boxed{4}$. Considerar una discretización de f y de (4.2), en intervalos de longitud Δt.

$\boxed{5}$ Se construye un portafolio π tal que el tenedor asuma una posición corta en el derivado y larga en una cantidad $\frac{\partial f}{\partial S_t}$ del activo S_t:

(4.4) $\pi = -f + S_t \frac{\partial f}{\partial S_t} \Rightarrow \Delta\pi = -\Delta f + \frac{\partial f}{\partial S_t} \Delta S_t$ (4.3)

$\boxed{6}$ Reemplazar las discretizaciones consideradas en $\boxed{4}$ en (4.3)

$\boxed{7}$ El $\Delta\pi$ quedará libre de riesgo, debido a que se elimina el diferencial Browniano. Por lo tanto se asume que los cambios en el portafolio son proporcionales al mismo portafolio y a la taza de interés libre de riesgo r en un intervalo Δt. Luego, $\Delta\pi = \pi r \Delta t$. (4.5)

$\boxed{8}$ Al reemplazar (4.4) y (4.5) en el resultado de $\boxed{6}$, se llega al resultado esperado.

Este es un caso específico de (***), con $h(t, S_t) = \mu S_t$ y $g(t, S_t) = \sigma S_t$.

La solución a la ecuación en derivadas parciales otorga el precio de una opción en el tiempo, sujeta a un activo S_t. Para la solución es necesario determinar condiciones de frontera apropiadas, que se definen de acuerdo a la clase de opción (put o call) y si es Europea, Americana o Mixta (tipo).