

Numerical Analysis II: Homeworks

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Workshop 1

Exercise 14

Question 1. Prove that every normed and finite-dimensional vector space is a Banach space.

Proof. Let $(V, \|\cdot\|)$ be a normed and finite-dimensional vector space. As V is finite-dimensional, let $B = \{v_1, \dots, v_k\}$ be a basis for V , that is, $\forall v \in V$, $v = \sum_{j=1}^n \alpha_j v_j$. Define the norm $\|\cdot\|^* : V \rightarrow \mathbb{R}$ as

$$\|x\|^* = \sqrt{\sum_{j=1}^k (\alpha_j)^2}.$$

It is well known (see e.g. [1]) that all norms on a finite dimensional vector space are equivalent. Hence, $\forall x \in V$, $\exists M > 0$ such that

$$\frac{1}{M} \|x\|^* \leq \|x\| \leq M \|x\|^*.$$

Using this, let us prove that V is a Banach space. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in V . Now, it is clear that each x_n in this sequence can be written as

$$x_n = \sum_{j=1}^k \alpha_j^{(n)} v_j$$

for some constants $\alpha_j^{(n)} \in \mathbb{R}$. Now, if $n, m \geq 1$ then by the equivalence of norms we have

$$\begin{aligned} \|x_m - x_n\| &\geq \frac{1}{M} \|x_m - x_n\|^* \\ &\geq \frac{1}{M} \sqrt{\sum_{j=1}^k \left(\alpha_j^{(m)} - \alpha_j^{(n)}\right)^2} \\ &\geq \frac{1}{M} \left| \alpha_j^{(m)} - \alpha_j^{(n)} \right|, \quad \forall j = 1, \dots, k. \end{aligned}$$

Thus, $\{\alpha_1^{(n)}\}_{n \in \mathbb{R}}, \dots, \{\alpha_k^{(n)}\}_{n \in \mathbb{R}}$ are all Cauchy sequences in \mathbb{R} . As \mathbb{R} is complete, these sequences converge respectively to some constants $\alpha_1, \dots, \alpha_k$. Let $x = \sum_{j=1}^k \alpha_j v_j \in V$. Now, again for the equivalence of norms, $\forall n \in \mathbb{N}$ we have

$$\begin{aligned} \|x - x_n\| &\leq M \|x - x_n\|^* \\ &\leq M \sqrt{\sum_{j=1}^k (\alpha_j - \alpha_j^{(n)})^2} \end{aligned}$$

and taking the limit as $n \rightarrow \infty$, we obtain that $\{x_n\}_{n \in \mathbb{N}}$ converges to $x \in V$. Therefore, V is a Banach space. \square

Exercise 15

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ vectors in \mathbb{R}^n . Let $1 < p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Question 2. Prove Hölder's inequality for vectors

$$\sum_{j=1}^n |x_j y_j| \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}}$$

Proof. The case when $x = 0$ or $y = 0$ is trivial and the inequality holds. Assuming $x \neq 0$ and $y \neq 0$.

Recall Young's inequality (see e.g. [2]): let $a, b > 0$ real numbers and p, q selected as above, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (1)$$

the specific case for

$$a = \frac{|x_j|}{\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}}}, \quad b = \frac{|y_j|}{\left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}}}, \quad j = 1, \dots, n$$

yields

$$\frac{|x_j y_j|}{\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q} \frac{|y_j|^q}{\sum_{j=1}^n |y_j|^q}.$$

Now, taking the sum of the inequalities for $j = 1, \dots, n$, we obtain

$$\frac{\sum_{j=1}^n |x_j y_j|}{\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{\sum_{j=1}^n |x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q} \frac{\sum_{j=1}^n |y_j|^q}{\sum_{j=1}^n |y_j|^q} = \frac{1}{p} + \frac{1}{q} = 1$$

which directly implies

$$\sum_{j=1}^n |x_j y_j| \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |y_j|^q \right)^{\frac{1}{q}}$$

□

Question 3. Prove Minkowski's inequality for vectors

$$\left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}}$$

Proof. For $p > 1$, $q = \frac{p}{p-1}$ to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1}$$

and by the triangle inequality,

$$\begin{aligned} \sum_{j=1}^n |x_j + y_j|^p &\leq \sum_{j=1}^n (|x_j| + |y_j|) |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n |x_j| |x_j + y_j|^{p-1} + \sum_{j=1}^n |y_j| |x_j + y_j|^{p-1} \\ &= \sum_{j=1}^n |x_j (x_j + y_j)^{p-1}| + \sum_{j=1}^n |y_j (x_j + y_j)^{p-1}|. \end{aligned}$$

Now, by Hölder's inequality on both terms we have

$$\begin{aligned} &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1} \right)^{\frac{1}{q}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n |x_j + y_j|^{p-1} \right)^{\frac{1}{q}} \\ &= \left(\sum_{j=1}^n |x_j + y_j|^{p-1} \right)^{\frac{1}{q}} \left[\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \right] \\ &= \left(\sum_{j=1}^n |x_j + y_j|^{p-1} \right)^{\frac{p-1}{p}} \left[\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \right] \\ &= \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{p-1}{p}} \left[\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \right]. \end{aligned}$$

Thus,

$$\sum_{j=1}^n |x_j + y_j|^p \leq \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{p-1}{p}} \left[\left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \right],$$

and dividing both sides by the first term on the right side, we have

$$\begin{aligned} \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{1 - \frac{p-1}{p}} &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \\ \left(\sum_{j=1}^n |x_j + y_j|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p \right)^{\frac{1}{p}} \end{aligned}$$

□

Exercise 16

Let

$$\ell^p(\mathbb{R}) = \left\{ \{x_n\}_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

Question 4. Prove that $\ell^p(\mathbb{R})$ is a Banach space with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty$$

and the sum and scalar product defined by

$$x + y = \{x_n + y_n\}_{n \in \mathbb{N}}, \quad ax = \{ax_n\}_{n \in \mathbb{N}}, \quad \forall x, y \in \ell^p(\mathbb{R}), \quad \forall a \in \mathbb{R}$$

Proof. Proving that $\ell^p(\mathbb{R})$ is vector space with the operations defined above is a well-known result and will not be presented here. Let us prove that it is a Banach space (normed and complete).

Let us prove that $\|x\|_p$ is indeed a norm in $\ell^p(\mathbb{R})$: let $x, y \in \ell^p(\mathbb{R})$ and $a \in \mathbb{R}$. Let $p \in [1, \infty]$.

1. Since $|x_n| \geq 0, \forall n \in \mathbb{N}$, then

$$|x_n| \geq 0 \implies |x_n|^p \geq 0 \implies \sum_{n=1}^{\infty} |x_n|^p \geq 0 \implies \|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \geq 0.$$

2. “ \implies ” Let us suppose that $\|x\|_p = 0$, then

$$\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = 0 \implies |x_n|^p = 0$$

since $|x_n|^p \geq 0$, $\forall n \in \mathbb{N}$. Then $x_n = 0$, $\forall n \in \mathbb{N} \implies x = 0$ (with $0 = \{0\}_{n \in \mathbb{N}}$ in $\ell^p(\mathbb{R})$).

“ \Leftarrow ” Let us suppose that $x = 0$, then $x_n = 0$, $\forall n \in \mathbb{N}$ which directly implies that $\|x\|_p = 0$. Therefore, $\|x\|_p = 0 \iff x = 0$.

3.

$$\begin{aligned} \|ax\|_p &= \left(\sum_{n=1}^{\infty} |ax_n|^p \right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |a|^p |x_n|^p \right)^{\frac{1}{p}} = \left(|a|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \\ &= |a| \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} = |a| \|x\|_p. \end{aligned}$$

4.

$$\|x + y\|_p = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{p}}$$

by Minkowski's inequality, we have

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p.$$

Now, let us prove that $\ell^p(\mathbb{R})$ is complete. We will prove first the case for $p \in [1, \infty)$: let $1 \leq p < \infty$ and let $x_n = \{x_n\}_{k \in \mathbb{N}}$ be a Cauchy sequence in $\ell^p(\mathbb{R})$ for each $n \in \mathbb{N}$. Hence, for every $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that

$$|\{x_n\}_k - \{x_m\}_k| \leq \|x_n - x_m\|_p \leq \epsilon, \quad \forall m, n \geq N_\epsilon, \quad \forall k \in \mathbb{N}.$$

This implies that $\exists x_k = \lim_{n \rightarrow \infty} \{x_n\}_k$ in \mathbb{R} . Let $x = \{x_k\}_{k \in \mathbb{N}}$ and by Minkowski's inequality

$$\sum_{k=1}^K |\{x_n\}_k - \{x_m\}_k|^p \leq \|x_n - x_m\|_p^p \leq \epsilon^p, \quad \forall n, m \geq N_\epsilon, \quad K \in \mathbb{N}.$$

Taking $m \rightarrow \infty$, it follows

$$\sum_{k=1}^K |\{x_n\}_k - x_k|^p \leq \epsilon^p$$

and taking the supremum over $k \in \mathbb{N}$, we obtain

$$\sum_{k=1}^{\infty} |\{x_n\}_k - x_k|^p \leq \epsilon^p, \quad n \geq N_\epsilon.$$

Therefore, $x_n - x \in \ell^p(\mathbb{R})$ and converges to 0 in $\ell^p(\mathbb{R})$ as $n \rightarrow \infty$. Thus, $x \in \ell^p(\mathbb{R})$ and $x_n \rightarrow x$ in $\ell^p(\mathbb{R})$ and $\ell^p(\mathbb{R})$ is Banach space for $p \in [1, \infty)$.

Now, let $p = \infty$. The norm is given by

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Let $\{x^n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\ell^\infty(\mathbb{R})$, thus each $x^n = \{x_k^n\}_{k \in \mathbb{N}}$ is a bounded sequence in \mathbb{R} . Let $\epsilon > 0$ and since $\{x^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence $\exists N_\epsilon > 0$ such that

$$\|x^n - x^m\|_\infty < \epsilon, \quad \forall m, n > N_\epsilon.$$

Thus,

$$\sup_{k \in \mathbb{N}} |x_k^n - x_k^m| < \epsilon, \quad \forall m, n > N_\epsilon$$

which directly implies that

$$|x_k^n - x_k^m| < \epsilon, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_\epsilon.$$

This means that $\forall k \in \mathbb{N}$, the sequence $\{x_k^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} . It is well known that \mathbb{R} is a Banach space, hence this sequence is convergent. Let $x_k = \lim_{n \rightarrow \infty} x_k^n = x_k$ and let $x = \{x_k\}_{k \in \mathbb{N}}$.

Now, let us see that $\{x^n\}_{n \in \mathbb{N}}$ converges to x . Let $\gamma > 0$ and choose $\gamma/2$ as the ϵ in the previous discussion, then there exists $N_{\gamma/2}$ such that

$$|x_k^n - x_k^m| < \gamma/2, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_{\gamma/2}.$$

Taking the limit as $m \rightarrow \infty$, we have

$$|x_k^n - x_k| \leq \gamma/2, \quad \forall k \in \mathbb{N}, \quad \forall m, n > N_{\gamma/2},$$

and finally, taking the supremum in $k \in \mathbb{N}$, we obtain

$$\sup_{k \in \mathbb{N}} |x_k^n - x_k| = \|x^n - x\|_\infty \leq \gamma/2 < \gamma, \quad \forall n \in \mathbb{N}.$$

Therefore, x^n converges to x in $\ell^\infty(\mathbb{R})$. Thus, $\ell^\infty(\mathbb{R})$ is a Banach space.

We conclude that $\ell^p(\mathbb{R})$ is Banach space for $p \in [1, \infty]$. \square

Exercise 18

Question 5. Let V be a Hilbert space and suppose that $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal set in V . The following statements are equivalent:

1. $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for V .
2. The subspace $\text{span}\{v_j\}_{j \in \mathbb{N}}$ is dense in V .
3. Parseval's equality:

$$\|u\|^2 = \sum_{j \in \mathbb{N}} |\langle u, v_j \rangle|^2, \quad \forall u \in V.$$

4. $\forall u, v \in V$,

$$\langle u, v \rangle = \sum_{j \in \mathbb{N}} \langle u, v_j \rangle \langle v, v_j \rangle$$

5. $\forall u \in V$, if $\forall j \in \mathbb{N}$, $\langle u, v_j \rangle = 0$, then $u = 0$.

Proof. • “1. \implies 2.”: Suppose that $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for V and let $M = \overline{\text{span}\{v_j\}_{j \in \mathbb{N}}}$. It is clear that M is closed subspace. Furthermore, as $\{v_j\}_{j \in \mathbb{N}}$ is an orthonormal set in V , it is a maximal and hence $V^\perp = \{0\}$, then $M^\perp = \{0\}$ and finally $M = V$. Clearly, $\text{span}\{v_j\}_{j \in \mathbb{N}}$ is dense in V .

- “2. \implies 3.”: If $u = 0$, it is trivial. Let $u \in V$ such that $u \neq 0$ and let $\epsilon > 0$. Assume, without loss of generality, that $\|u\| > \epsilon$ and choose $y \in V$ such that $\|x - y\| < \epsilon$ and $y \in \text{span}\{v_j\}_{j=1}^k$, for some $k \in \mathbb{N}$. Moreover, let

$$z = \sum_{j=1}^k \langle u, v_j \rangle v_j.$$

Then, z minimizes $\|x - w\|$ over $w \in \text{span}\{v_j\}_{j=1}^k$ and hence $\|x - z\| \leq \|x - y\| < \epsilon$. Thus, $\|x\| < \|z\| + \epsilon$ and $(\|x\| - \epsilon)^2 < \|z\|^2$ and $\|z\|^2 = \sum_{j=1}^k |\langle u, v_j \rangle|^2 \leq \sum_{j \in \mathbb{N}} |\langle u, v_j \rangle|^2$. Taking $\epsilon \rightarrow 0$, $\|u\|^2 \leq \sum_{j \in \mathbb{N}} |\langle u, v_j \rangle|^2$. The other inequality is obtained from Bessel's inequality.

- “3 \implies 4”: Consequence of the polarization identity as only a countable number of terms in the sum are nonzero.
- “4 \implies 5”: Suppose that $\forall j \in \mathbb{N}$, $\langle u, v_j \rangle = 0$, then $\|u\|^2 = \langle u, u \rangle = 0$ and hence $u = 0$.
- “5 \implies 1”: If $\{v_j\}_{j \in \mathbb{N}}$ is not a base, it is not a maximal orthonormal set in V and hence, $\exists x \neq 0$ such that $\langle x, v_j \rangle = 0$, for all $j \in \mathbb{N}$, contradicting 5.

□

Workshop 2

Exercise 9 b)

Question 6. Let V be a Banach Space and λ a scalar. Suppose $T \in \mathcal{L}(V, V)$. Determine if the inverse operator of $\lambda I - T$ exists. If it does, find it.

Proof. Let

$$S = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Let's show that S is the inverse operator of $H = \lambda I - T$. First let's show that $H(S(x)) = x$:

$$\begin{aligned}
H(S(x)) &= \lambda S(x) - T(S(x)) \\
&= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} T\left(\frac{T^n(x)}{\lambda^{n+1}}\right) \\
&= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}(x)}{\lambda^{n+1}} \tag{2}
\end{aligned}$$

Changing the variable in the second summation to $n \rightarrow n - 1$ we obtain

$$\begin{aligned}
H(S(x)) &= \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=1}^{\infty} \frac{T^n(x)}{\lambda^n} \\
&= \frac{T^0(x)}{\lambda^0} \\
&= x
\end{aligned}$$

Lastly, let's show that $S(H(x)) = x$:

$$\begin{aligned}
S(H(x)) &= \sum_{n=0}^{\infty} \frac{T^n(\lambda x - T(x))}{\lambda^{n+1}} \\
&= \sum_{n=0}^{\infty} \underbrace{\frac{T^n(x)}{\lambda^n}}_{a_n} - \underbrace{\frac{T^{n+1}(x)}{\lambda^{n+1}}}_{b_n}
\end{aligned}$$

It is clear that both a_n and b_n are convergent as T is bounded (as it is continuous). In this manner, the sum can be separated as:

$$S(H(x)) = \sum_{n=0}^{\infty} \frac{T^n(x)}{\lambda^n} - \sum_{n=0}^{\infty} \frac{T^{n+1}(x)}{\lambda^{n+1}} \tag{3}$$

Note that Equation 3 is the same one as the Equation 2. Therefore, doing the same process we would obtain that:

$$S(H(x)) = x$$

Therefore, S is the inverse operator of H . □

Slide 23

Question 7. Let $\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear and bounded.}\}$. Show that $\mathcal{L}(V, W)$ is a normed vector space, with the norm for a $T \in \mathcal{L}(V, W)$ be

$$\|T\| = \sup_{\|u\|_V=1} \|Tu\|_W$$

Proof. Let's show that it is a vector space. In first place, it is seen that $\mathcal{L}(V, W) \subset \mathcal{C}(V, W)$, with $\mathcal{C}(V, W)$ being the space of all the continuous functions that have as domain V and as co-domain W . Furthermore, it is well known that $\mathcal{C}(V, W)$ is a vector space.

Hence, to show that $\mathcal{L}(V, W)$ is a vector space it is only needed to show that the sum and multiplication are closed under the same set.

- Let $T, H \in \mathcal{L}(V, W)$. Let's show that $T + H$ is linear.

$$\begin{aligned}(T + H)(\alpha x + \beta y) &= T(\alpha x + \beta y) + H(\alpha x + \beta y) \\ &= \alpha Tx + \beta Ty + \alpha Hx + \beta Hy \\ &= \alpha(Tx + Hx) + \beta(Ty + Hy) \\ &= \alpha(T + H)x + \beta(T + H)y\end{aligned}$$

Let's show that $T + H$ is bounded.

$$\begin{aligned}\|(T + H)x\|_W &= \|Tx + Hx\|_W \\ &\leq \|Tx\|_W + \|Hx\|_W \\ &\leq M_T \|x\|_V + M_H \|x\|_V \\ &= \underbrace{(M_T + M_H)}_M \|x\|_V \\ &= M \|x\|_V\end{aligned}$$

- Let $\lambda \in \mathbb{R}$. Let's show that λT is linear.

$$\begin{aligned}(\lambda T)(\alpha x + \beta y) &= \lambda T(\alpha x + \beta y) \\ &= \lambda \alpha Tx + \lambda \beta Ty \\ &= \alpha(\lambda T)x + \beta(\lambda T)y\end{aligned}$$

Let's show that λT is bounded.

$$\begin{aligned}\|(\lambda T)x\|_W &= \|\lambda Tx\|_W \\ &= |\lambda| \|Tx\|_W \\ &\leq |\lambda| \underbrace{M_T}_M \|x\|_V \\ &= M \|x\|_V\end{aligned}$$

Therefore, $\mathcal{L}(V, W)$ is a vector space. Now let's show that the defined norm satisfies the desired properties.

- Let's show that $\|T\| = 0 \iff T = 0$.

(\rightarrow) Let's suppose that $\|T\| = 0$. Therefore,

$$\begin{aligned}\|T\| &= 0 \\ \sup_{\|u\|_V=1} \|Tu\|_W &= 0\end{aligned}$$

Therefore, as the supremum is equals to 0 we obtain that for all $u \in V$ ($u \neq 0$, as every linear operator satisfies that $T0 = 0$)

$$\begin{aligned}
0 &\leq \|Tu\|_W \leq \sup \|Tu\|_W \\
0 &\leq \|Tu\|_W \leq \|u\|_V \sup \frac{\|Tu\|_W}{\|u\|_V} \\
0 &\leq \|Tu\|_W \leq \|u\|_V \sup \left\| T \left(\frac{u}{\|u\|_V} \right) \right\| \\
0 &\leq \|Tu\|_W \leq \|u\|_V \sup_{\|u\|_V=1} \|Tu\|_W \\
0 &\leq \|Tu\|_W \leq 0 \\
\|Tu\|_W &= 0 \\
Tu &= 0
\end{aligned}$$

Therefore, $\forall u \in V : Tu = 0$. Hence, $T = 0$.

(\leftarrow) Let's suppose that $T = 0$. Hence:

$$\begin{aligned}
\|T\| &= \sup_{\|u\|_V=1} \|Tu\|_W \\
&= \sup_{\|u\|_V=1} \|0\| \\
&= 0
\end{aligned}$$

Therefore, both implications are shown.

- Let's show that $\|\alpha T\| = |\alpha| \|T\|$.

$$\begin{aligned}
\|\alpha T\| &= \sup_{\|u\|_V=1} \|(\alpha T)u\| \\
\|\alpha T\| &= \sup_{\|u\|_V=1} \|\alpha Tu\| \\
&= \sup_{\|u\|_V=1} |\alpha| \|Tu\| \\
&= |\alpha| \sup_{\|u\|_V=1} \|Tu\| \\
&= |\alpha| \|T\|
\end{aligned}$$

- Let's show the triangular inequality. Let $T, H \in \mathcal{L}(V, W)$:

$$\begin{aligned}
\|T + H\| &= \sup_{\|u\|_V=1} \|(T + H)u\| &= \sup_{\|u\|_V=1} \|Tu + Hu\| \\
&\leq \sup_{\|u\|_V=1} (\|Tu\| + \|Hu\|) \\
&= \sup_{\|u\|_V=1} \|Tu\| + \sup_{\|u\|_V=1} \|Hu\| \\
&= \|T\| + \|H\|
\end{aligned}$$

□

Workshop 3

Exercise 5

Question 8. Solve the following mixed boundaries Poisson Equation using Triangular Elements:

$$\begin{aligned} -\nabla^2 u &= 1 \text{ in } \Omega = (0, 1) \times (0, 1) \\ u_x(0, y) &= u_y(x, 0) = 0 \\ u(1, y) &= u(x, 1) = 0 \end{aligned}$$

Proof. To solve this exercise the FEniCS project API for Python was used. The following code was used:

```
#!/usr/bin/env python3

import tools
from fenics import *

tol = DOLFIN_EPS

# Mesh
mesh = UnitSquareMesh(20, 20)
V = FunctionSpace(mesh, 'P', 1)

# Dirichlet boundary
def boundary(x, on_boundary):
    return on_boundary and (near(x[0], 1, tol) or (near(x[1], 1, tol)))

u0 = Constant(0.0)
bc = DirichletBC(V, u0, boundary)

# Variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("1", degree=0)
g = Expression("0", degree=0)

a = inner(grad(u), grad(v))*dx
L = f*v*dx + g*v*ds
```

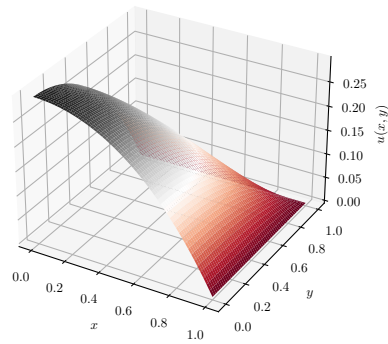
```

# Solution
u = Function(V)
solve(a == L, u, bc)

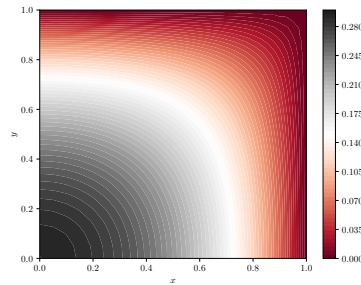
# Plotting
tools.plot3d(60, 60, u, "figs/exerc-5.pdf")
tools.plot(100, 100, u, "figs/exerc-5-2d.pdf", num_lines=60)

# Values
print(u(0.5, 0.5))

```



(a) 3D approximation.



(b) Contour lines in 2D.

Figure 1: Output of code.

□

Exercise 8

Question 9. Solve the following Poisson Equation using Triangular Elements:

$$-\nabla^2 u = xy \text{ in } \Omega = (0, 1) \times (0, 1)$$

$$u(x, 0) = 1 - e^{-x}$$

$$u(1, y) = 4 - e^{-y}$$

$$u = 0, \text{ otherwise.}$$

Proof. To solve this exercise the FEniCS project API for Python was used. The following code was used:

```

#!/usr/bin/env python3

import tools
from fenics import *

```

```

tol = DOLFIN_EPS

# Mesh
mesh = UnitSquareMesh(20, 20)
V = FunctionSpace(mesh, 'P', 1)

# Dirichlet boundary 1
def boundary(x, on_boundary):
    return on_boundary and (near(x[0], 0, tol) or (near(x[1], 1, tol)))

u0 = Constant(0.0)
bc1 = DirichletBC(V, u0, boundary)

# Dirichlet boundary 2
def boundary2(x, on_boundary):
    return on_boundary and near(x[1], 0, tol)

u0 = Expression("1 - exp(-x[0])", degree=2)
bc2 = DirichletBC(V, u0, boundary2)

# Dirichlet boundary 3
def boundary3(x, on_boundary):
    return on_boundary and near(x[0], 1, tol)

u0 = Expression("4 - exp(-x[1])", degree=2)
bc3 = DirichletBC(V, u0, boundary3)

# Variational problem
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("250 - pow(x[0], 2) - 6*pow(x[1], 2)", degree=2)

a = inner(grad(u), grad(v))*dx
L = f*v*dx

# Solution
u = Function(V)
solve(a == L, u, [bc1, bc2, bc3])

# Plotting
tools.plot3d(60, 60, u, "figs/exerc-8.pdf")
tools.plot(100, 100, u, "figs/exerc-8-2d.pdf", num_lines=60)

```

The result of this code can be seen in Figure 2.

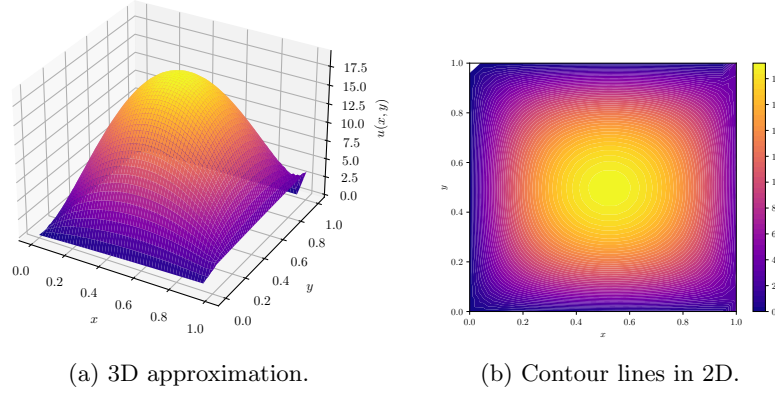


Figure 2: Output of code.

□

References

- [1] K. Conrad, “Equivalence of norms,” *Expository Paper, University Of Connecticut, Storrs*, heruntergeladen von <https://www-m10.ma.tum.de/foswiki/pub/Lehre/WS0910/TopologieKombinatorik/topskript.pdf>, vol. 17, p. 2018, 2018.
- [2] Z. Cvetkovski, “Hölder’s inequality, minkowski’s inequality and their variants,” in *Inequalities*. Springer, 2012, pp. 95–105.