Q:- For Beta function show that

$$\beta(m,n) = \beta(m+1, n) + \beta(m, n+1)$$

Solution:- We have,

$$\beta(m+1, m) + \beta(m, n+1)$$

$$= \int_0^1 x^m (1-x)^{n-1}\, dx + \int_0^1 x^{m-1} (1-x)^{n}\, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \, \{ x + (1-x) \}$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \cdot 1$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$= \beta(m,n)$$

Proved

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Duplication formula :-

$$\Gamma m \, \Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m \quad , \ m \text{ is } +\text{ve}.$$

Hence show that $\beta(m, m) = 2^{1-2m} \beta\left(m, \frac{1}{2}\right)$.

Proof: We know that

$$\frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2 \, \Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

Putting $p = q$, we get

$$\frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{p+1}{2}}{2 \, \Gamma p+1} = \int_0^{\pi/2} \sin^p\theta \, \cos^p\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta \cos\theta)^p \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2^p} (2\sin\theta\cos\theta)^p \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2^p} (\sin 2\theta)^p \, d\theta$$

Putting $2\theta = t$

$\Rightarrow d\theta = \frac{dt}{2}$

$$= \frac{1}{2^p} \int_0^{\pi} \sin^p t \, \frac{dt}{2}$$

$$= \frac{1}{2^p} \cdot \frac{2}{2} \int_0^{\pi/2} \sin^p t \cdot \cos^0 t \, dt$$

$$\frac{\cancel{\Gamma\frac{p+1}{2}} \, \Gamma\frac{p+1}{2}}{2 \, \Gamma p+1} = \frac{1}{2^p} \, \frac{\cancel{\Gamma\frac{p+1}{2}} \, \Gamma\frac{0+1}{2}}{2 \, \Gamma\frac{p+0+2}{2}}$$

$$\frac{\Gamma\frac{p+1}{2}}{\cancel{2} \, \Gamma p+1} = \frac{1}{2^p} \, \frac{\Gamma 1/2}{\cancel{2} \, \Gamma\frac{p+2}{2}} = \frac{\sqrt{\pi}}{2^p \, \Gamma\frac{p+2}{2}}$$

Take, $\frac{p+1}{2} = m \Rightarrow p = 2m - 1$

$$\Rightarrow \frac{\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}} \, \frac{\sqrt{\pi}}{\Gamma\frac{2m+1}{2}}$$

$$\Rightarrow \boxed{\Gamma m \, \Gamma m+\tfrac{1}{2} = 2^{1-2m} \sqrt{\pi} \, \Gamma 2m}$$

Proved

multibly by Γm

$$\Rightarrow \frac{\Gamma m \, \Gamma m}{\Gamma 2m} = 2^{1-2m} \, \frac{\Gamma\frac{1}{2} \, \Gamma m}{\Gamma m+\frac{1}{2}}$$

Proved

$$\Rightarrow \left\{ \beta(m, m) = 2^{1-2m} \beta\left(m, \tfrac{1}{2}\right) \right\}$$