CHAPTER

38

GAMMA, BETA FUNCTION

38.1 GAMMA FUNCTION

(*U.P. I Semester Dec. 2007*)

$$\int_0^\infty e^{-x} x^{n-1} dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$$\boxed{\int_0^\infty e^{-x}\, x^{n-1}\, dx = \Gamma n}$$

Example 1.*Prove that* $\Gamma 1 = 1$

Solution. We know that, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Put $n = 1$, $\Gamma 1 = \int_0^\infty e^{-x}\, x^{1-1}\, dx = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$ **Proved.**

Example 2. *Prove that*

(*i*) $\Gamma(n+1) = n\,\Gamma n$ (*ii*) $\Gamma(n+1) = n!$ **(*Reduction formula*)**

Solution.

(*i*) We know that, $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$ ...(1)

Integrating by parts, we have

$$\Gamma n = \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1} dx$$

$$= \lim_{x\to 0}\left\{\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^n}{n!} + \ldots + \infty\right)x^{n-1}\right\} + (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$$= 0 + (n-1)\int_0^\infty x^{n-2} e^{-x} dx$$

$$\therefore \qquad \Gamma n = (n-1)\Gamma(n-1) \qquad \text{...(2)}$$

$$\boxed{\Gamma(n+1) = n\,\Gamma n}$$ Replacing n by $(n + 1)$ **Proved.**

(*ii*) Replacing n by $n - 1$ in (2), we get

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

Putting the value $\boxed{n-1}$ in (2), we get

$$\Gamma(n)=(n-1)(n-2)\Gamma(n-2)$$

Similarly, $\Gamma(n) = (n-1)(n-2) \ldots\ldots 3.2.1\ \Gamma(1)$ ...(3)

Putting the value of $\Gamma(1)$ in (3), we have

$$\Gamma(n) = (n-1)(n-2)\ldots\ldots 3.2.1.1$$

$$\Gamma(n) = (n-1)\,!$$

Replacing n by $n+1$, we have $\Gamma(n+1) = n\,!$ **Proved.**

Example 3. *Evaluate* $\Gamma\left(-\frac{1}{2}\right)$.

Solution. $\Gamma(n+1) = n\,\Gamma(n)$

$$\Gamma\left(-\frac{1}{2}+1\right)=-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \Rightarrow \Gamma\left(\frac{1}{2}\right)=-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \Rightarrow \sqrt{\pi}=-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \Rightarrow \Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi} \quad \textbf{Ans.}$$

Example 4. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx$

Solution. Let $I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\,dx$ ...(1)

Putting $\sqrt{x} = t \Rightarrow x = t^2$ so that $dx = 2t\,dt$ in (1), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2t\,dt = 2\int_0^\infty t^{3/2} e^{-t} dt = 2\int_0^\infty t^{\frac{5}{2}-1} e^{-t}\,dt$$

$$= 2\,\Gamma\left(\frac{5}{2}\right) \qquad \text{[By definition]}$$

$$= 2\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{2}\sqrt{\pi} \qquad \textbf{Ans.}$$

Example 5. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\,dx.$

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\,dx$ ...(1)

Putting $\sqrt[3]{x} = t \Rightarrow x = t^3$ so that $dx = 3\,t^2\,dt$ in (1), we get

$$I = \int_0^\infty t^{3/2} e^{-t}\, 3t^2\,dt = 3\int_0^\infty t^{7/2} e^{-t} dt = 3\int_0^\infty t^{\frac{9}{2}-1} e^{-t}\,dt = 3\,\Gamma\left(\frac{9}{2}\right) = 3\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{315}{16}\sqrt{\pi}$$

Ans.

Example 6. *Evaluate* $\int_0^\infty x^{n-1}\, e^{-h^2x^2}\,dx.$

Solution. Let $I = \int_0^\infty x^{n-1}\, e^{-h^2x^2}\,dx$ ...(1)

Putting $t = h^2 x^2 \Rightarrow x = \frac{\sqrt{t}}{h}$ so that $dx = \frac{dt}{2h\sqrt{t}}$, we get

$$I = \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\,\frac{dt}{2h\sqrt{t}}$$

$$= \frac{1}{2h^n}\int_0^\infty t^{\frac{n-1}{2}} e^{-t}\frac{dt}{\sqrt{t}} = \frac{1}{2h^n}\int_0^\infty t^{\frac{n-2}{2}} e^{-t} dt = \frac{1}{2h^n}\Gamma\left(\frac{n}{2}\right) \qquad \textbf{Ans.}$$

Example 7. *Evaluate* $\int_0^\infty \frac{x^a}{a^x}dx$, *hence show that* $\int_0^\infty \frac{x^7}{7^x}dx = \frac{7!}{(\log 7)^8}$ *(a >1)*

Solution. Here, we have $\int_0^\infty \frac{x^a}{a^x}dx$ ...(1)

Putting $a^x = e^t \Rightarrow x \log a = t \Rightarrow x = \frac{t}{\log a}, \Rightarrow dx = \frac{dt}{\log a}$ in (1), we have

$$\int_0^\infty \frac{x^a}{a^x}dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}t^a dt = \frac{1}{(\log a)^{a+1}}\int_0^\infty t^{(a+1)-1}e^{-t}dt$$

$$= \frac{1}{(\log a)^{a+1}}\overline{|a+1}$$

On putting $a = 7$, we get $\int_0^\infty \frac{x^7}{7^x}dx = \frac{7!}{(\log 7)^8}$ **Ans.**

38.2 PROVE THAT

$$\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}}\overline{|n+1}$$

Proof : Put $\log x = -t$ so that $x = e^{-t} \Rightarrow dx = -e^{-t}dt$

$$\therefore \quad x^m = e^{-mt}$$

$$(\log x)^n = (-t)^n$$

Now, $\int_0^1 x^m(\log x)^n dx = \int_\infty^0 e^{-mt}(-t)^n(-e^{-t})\,dt = \int_0^\infty (-1)^n e^{-mt-t}t^n dt$

Putting $(m + 1)\, t = u$ so that $(m + 1)\, dt = du$, we get

$$\therefore \quad I = \int_0^\infty (-1)^n e^{-u}\cdot\frac{u^n}{(m+1)^n}\frac{du}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^n du = \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^{(n+1)-1}du = \frac{(-1)^n}{(m+1)^{n+1}}\overline{|n+1}$$ **Proved.**

Example 8. *Prove that* $\int_0^1 (x\log x)^4 dx = \frac{4!}{5^5}$ *(M.U. II Semester, 2009)*

Solution. We know that

$$\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}}\cdot\overline{|n+1} \quad \text{...(1) [From Art 38.2]}$$

Now, $\int_0^1 (x\log x)^4 dx = \int_0^1 x^4(\log x)^4 dx$

Putting $m = n = 4$ in (1), we get

$$\int_0^1 x^4(\log x)^4 dx = \frac{(-1)^4}{(4+1)^{4+1}}\overline{|4+1} = \frac{\overline{|5}}{5^5} = \frac{4!}{5^5}$$ **Proved.**

Example 9. *Evaluate* $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

Solution. Let $-\log x = y \Rightarrow \log x = -y \Rightarrow e^{-y} = x$ so that $dx = -e^{-y}\,dy$

$$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_0^1 \frac{-e^{-y}dy}{\sqrt{y}} = \int_0^\infty y^{-\frac{1}{2}}e^{-y}\,dy = \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$$ **Ans.**

Example 10. *Evaluate* $\int_0^1 x^{n-1} \cdot \left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} \cdot dx$

Solution: Put $\log_e \frac{1}{x} = t$ or $x = e^{-t}$ $\quad \therefore\ dx = -e^{-t}\,dt$

$$\int_0^1 x^{n-1} \cdot \left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} dx = \int_\infty^0 \left(e^{-t}\right)^{n-1} [t]^{m-1} \left(-e^{-t}dt\right) = \int_0^\infty e^{-nt} t^{m-1}\,dt$$

Putting $\quad n\,t = u \Rightarrow t = \frac{u}{n}$ so that $dt = \frac{du}{n}$

$$= \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1} \frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}\,u^{m-1}du = \frac{1}{n^m}\overline{|m} \qquad \textbf{Ans.}$$

38.3 TRANSFORMATION OF GAMMA FUNCTION

Prove that (i) $\int_0^\infty e^{-ky}\,y^{n-1}dy = \frac{\overline{|n}}{k^n}$ (*AMIETE, Dec. 2010*) $\qquad (ii)$ $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$

(iii) $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \overline{|n}$ $\qquad (iv)$ $\Gamma\, n = \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx$

Solution: We know that $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx$ ...(1)

(i) Replace x by $k\,y$, so that $dx = k\,dy$; then (1) becomes

$$\overline{|n} = \int_0^\infty (k\,y)^{n-1}\,e^{-ky}\,k\,dy.$$

$$\overline{|n} = k^n\int_0^\infty e^{-ky}y^{n-1}\,dy$$

$$\therefore \qquad \boxed{\int_0^\infty e^{-ky}\,y^{n-1}\,dy = \frac{\overline{|n}}{k^n}} \qquad \text{...(2)} \textbf{ Proved.}$$

(ii) Replace x^n by y, so that $n\,x^{n-1}\,dx = dy$ in (1), then

$$\overline{|n} = \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{nx^{n-1}} = \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{ny^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\,dy$$

When $n = \frac{1}{2}$, $\qquad \overline{\left|\frac{1}{2}\right.} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}\,dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$

$$\boxed{\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}} \qquad \textbf{Proved.}$$

(iii) Putting $e^{-x} = y$, so that $-e^{-x}\,dx = dy$ and $-x = \log y$, $x = \log\frac{1}{y}$, (1) becomes

$$\overline{|n} = -\int_1^0\left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{e^{-x}} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{y} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1} dy \qquad \textbf{Proved.}$$

(iv) We know that, $\qquad \Gamma(n) = \int_0^\infty e^{-x}.\,x^{n-1}\,dx$ ...(1)

Putting $\qquad x^n = y \Rightarrow x = y^{1/n}$ so that $dx = \frac{1}{n}y^{\frac{1}{n}-1}dy$ in (1), we get

$$\Gamma n = \int_0^\infty e^{-y^{1/n}}\, y^{\frac{n-1}{n}} \cdot \frac{1}{n} y^{\frac{1}{n}-1}\, dy = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}}\, dy$$

$$\Gamma n = \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}\, dx.$$ **Proved.**

EXERCISE 38.1

Evaluate:

1. (*i*) $\overline{\left| -\frac{3}{2} \right.}$ (*ii*) $\overline{\left| -\frac{15}{2} \right.}$ (*iii*) $\overline{\left| \frac{7}{2} \right.}$ (*iv*) $\overline{\left| 0 \right.}$

Ans. (*i*) $\frac{4}{3}\sqrt{\pi}$ (*ii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iii*) $\frac{15\sqrt{\pi}}{8}$ (*iv*) ∞

2. $\int_0^\infty \sqrt{x}\, e^{-x} dx$ **Ans.** $\overline{\left| \frac{3}{2} \right.}$

3. $\int_0^\infty x^4\, e^{-x^2} dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$

4. $\int_0^\infty e^{-h^2x^2} dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

5. $\int_0^\infty\int_0^\infty e^{-\left(ax^2+by^2\right)} x^{2m-1} y^{2n-1} dx\,dy, \quad a,b,m,n>0$ **Ans.** $\frac{\overline{|m}\;\overline{|n}}{4\,a^m b^n}$

6. $\int_0^1 (x\log x)^3\, dx$ **Ans.** $-\frac{3}{128}$

7. $\int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}}$ **Ans.** $\sqrt{2\pi}$

8. Prove that 1.3.5.... (2 *n* – 1) = $\frac{2^n\,\overline{\left| n+\frac{1}{2} \right.}}{\sqrt{\pi}}$

9. $\int_0^\infty e^{-y^{1/m}}\, dy = m\,\overline{|m}$

10. $\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy = \frac{\Gamma(m)\,\Gamma(n)}{4\,a^m b^n}$, where *a*, *b*, *m*, *n* are positive.

11. $\int_0^{\pi/2} \frac{d\theta}{(a\cos^4\theta + b\sin^4\theta)} = \frac{(\Gamma 1/4)^2}{4(ab)^{1/4}\sqrt{\pi}}$ [**Hint.** Put $\tan\theta = t$ then $bt^4 = az$]

38.4 BETA FUNCTION

(*U.P. I Semester Dec. 2007*)

$$\int_0^1 x^{l-1}(1-x)^{m-1} dx \qquad (l>0,\ m>0)$$

is called the Beta function of *l, m.* It is also written as

$$\beta\,(l,\,m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx$$

38.5 EVALUATION OF BETA FUNCTION

$$\beta\,(l,\,m) = \frac{\overline{|l}\;\overline{|m}}{\overline{|l+m}}$$

Solution. We have, $\beta\,(l,\,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx = \int_0^1 (1-x)^{m-1} x^{l-1} dx$

Integrating by parts, we have

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right) dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2}\, x^l dx$$

Again integrating by parts, we get

$$= \frac{(m-1)(m-2)}{l(l+1)}\int_0^1 (1-x)^{m-3}\, x^{l+1} dx = \frac{(m-1)(m-2)\ldots 2.1}{l\,(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2} dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l\,(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1 = \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)(l+m-1)}$$

$$= \frac{(m-1)!}{l(l+1)\ldots\ldots(l+m-2)(l+m-1)}\times\frac{(l-1)(l-2)\ldots\ldots 1}{(l-1)(l-2)\ldots\ldots 1}$$

$$= \frac{(m-1)!\;(l-1)!}{1.2\ldots.(l-2)(l-1).l(l+1)\ldots(l+m-2)(l+m-1)} = \frac{(l-1)!\,(m-1)!}{(l+m-1)!} = \frac{\Gamma l\;\Gamma m}{\Gamma l+m}$$

And if only l is positive integer and not m then

$$\boxed{\beta\,(l,\,m) = \frac{(l-1)!}{m(m+1)\ldots(m+l-1)}}$$ **Ans.**

38.6 A PROPERTY OF BETA FUNCTION

$$\beta\,(l,\,m) = \beta(m,\,l\,)$$

Solution. We have

$$\beta\,(l,\,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{l-1}\left[1-(1-x)\right]^{m-1} dx = \int_0^1 (1-x)^{l-1}\, x^{m-1}\, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{l-1}\, dx$$

l and m are interchanged.

$$\boxed{\beta\,(l,\,m) = \beta\,(m,\,l)}$$ **Proved.**

Example 11. *Evaluate* $\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx$

Solution. Let $\sqrt{x} = t \;\Rightarrow\; x = t^2$ so that $dx = 2\,t\,dt$

$$\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx = \int_0^1 (t^2)^4\,(1-t)^5(2\,t\,dt)$$

$$= 2\int_0^1 t^9(1-t)^5\,dt = 2\,\beta\,(10,6) = 2\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\,.\,\frac{9!\,5!}{(15)!}$$

$$= 2.\frac{5!}{10\times11\times12\times13\times14\times15} = \frac{2\times1\times2\times3\times4\times5}{10\times11\times12\times13\times14\times15} = \frac{1}{11\times13\times7\times15} = \frac{1}{15015}$$ **Ans.**

Example 12. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}} dx$

Solution. Let $x^3 = y \;\Rightarrow\; x = y^{1/3}$ so that $dx = \frac{1}{3}y^{-\frac{2}{3}}\,dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}} dx = \int_0^1 (1-y)^{-\frac{1}{2}}\left(\frac{1}{3}y^{-\frac{2}{3}}\,dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}}(1-y)^{-\frac{1}{2}}\,dy = \frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\frac{1}{3}\;\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

38.7 TRANSFORMATION OF BETA FUNCTION

We know that

$$\beta(l,\, m) \;=\; \int_0^1 x^{l-1}(1-x)^{m-1}\,dx,$$

Putting $x=\dfrac{1}{1+y}$ so that $d\,x=-\dfrac{1}{(1+y)^2}dy$ and $1-x=\dfrac{y}{1+y}$ in (1), we get

$$\beta(l,\, m)=\int_\infty^0\left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}\left[-\frac{1}{(1+y)^2}dy\right]=\int_0^\infty\frac{y^{m-1}}{(1+y)^{l+m}}dy$$

Since $l,\, m$ can be interchanged in $\beta\,(l,\, m)$,

$$\beta\,(l,\, m) \;=\; \int_0^\infty\frac{y^{l-1}}{(1+y)^{m+l}}dy \quad\Rightarrow\quad \beta\,(l,\, m)=\int_0^\infty\frac{x^{l-1}}{(1+x)^{m+l}}dx \qquad ...(1)$$

Example 13. *Evaluate* $\displaystyle\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

Solution. We know that

$$\beta\,(\,m,\; n) = \int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx \quad\Rightarrow\quad \int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx=\beta\,(m,n)$$

$$\Rightarrow\quad \int_0^1\frac{x^{m-1}}{(1+x)^{m+n}}dx+\int_1^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx \;=\; \beta\,(m,\, n) \qquad ...(1)$$

Consider $$\int_1^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_1^0\frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}dt\right)=\int_0^1\frac{\left(\frac{1}{t}\right)^{m-1}\frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n}(t+1)^{m+n}}dt \quad \left(\text{Put } x=\frac{1}{t}\right)$$

$$=\int_0^1\frac{t^{n-1}}{(1+t)^{m+n}}dt=\int_0^1\frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Putting the value of $\displaystyle\int_1^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$ in (1), we get

$$\int_0^1\frac{x^{m-1}}{(1+x)^{m+n}}dx+\int_0^1\frac{x^{n-1}}{(1+x)^{m+n}}dx \;=\; \beta\,(m,\, n)$$

$$\Rightarrow\qquad \int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx \;=\; \beta\,(m,\, n) \qquad \textbf{Ans.}$$

38.8 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that, $\displaystyle\Gamma l=\int_0^\infty e^{-x}\,x^{l-1}dx,$ [Put $zx = y$]

$$\frac{\Gamma l}{z^l}=\int_0^\infty e^{-zx}\,x^{l-1}dx$$

$$\Gamma l \;=\; \int_0^\infty z^l\,e^{-zx}x^{l-1}dx$$

Multiplying both sides by $e^{-z}\,z^{m-1}$, we have

$$\Gamma l\,.\,e^{-z}.\,z^{m-1} \;=\; \int_0^\infty e^{-z}.z^{m-1}.z^l.e^{-zx}.x^{l-1}dx$$

$$\Gamma l . e^{-z} . z^{m-1} = \int_0^\infty e^{-(1+x)z} . z^{l+m-1} . x^{l-1} dx$$

Integrating both sides w.r.t. 'z', we get

$$\int_0^\infty \Gamma l\, e^{-z} z^{m-1}\, dz = \int_0^\infty \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx\, dz$$

$$\Gamma l\, \Gamma m = \int_0^\infty x^{l-1}\, dx \int_0^\infty e^{-(1+x)z} z^{l+m-1} dz$$

$$= \int_0^\infty x^{l-1} dx \cdot \frac{\Gamma l+m}{(1+x)^{l+m}} \qquad \text{[From (1), Art 38.7]}$$

$$\Gamma l\, \Gamma m = \Gamma l+m \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx = \Gamma l+m \ . \ \beta\ (l,\ m)$$

$$\therefore \qquad \boxed{\beta\ (l,\ m) = \frac{\Gamma l\ \Gamma m}{\Gamma l+m}}$$

This is the required relation.

38.9. SHOW THAT

$$\boxed{\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$$

Solution. We know that

$$\beta\ (m,\ n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \qquad ...(1)$$

Putting $\qquad x = \sin^2 \theta,\ dx = 2 \sin \theta \cos \theta\ d\theta$

and $\qquad 1 - x = 1 - \sin^2 \theta = \cos^2 \theta$

Then (1) becomes

$$\beta\ (m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2} \theta \cos^{2n-2} \theta\ \ 2 \sin \theta \cos \theta\ d\theta$$

or $$\frac{\Gamma m\ \Gamma n}{\Gamma m+n} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta\ d\theta$$

Putting $\qquad 2m - 1 = p,\quad i.e.,\quad m = \dfrac{p+1}{2}$

and $\qquad 2n - 1 = q,\quad i.e.,\quad n = \dfrac{q+1}{2}$

$$\frac{\Gamma \frac{p+1}{2}\ \Gamma \frac{q+1}{2}}{\Gamma \frac{p+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^p \theta \cdot \cos^q \theta\ d\theta$$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta\ \cos^q\theta\ \ d\theta = \frac{\Gamma \frac{p+1}{2}\ \Gamma \frac{q+1}{2}}{2\,\Gamma \frac{p+q+2}{2}}} \qquad \textbf{Proved.}$$

Example 14. *Prove that* $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}} d\theta = \pi$ (*AMIETE, June 2009*)

Solution. L.H.S. $= \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}} d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^0\theta\, d\theta \times \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} \times \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{5}{4}\right)} \times \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{4\,\Gamma\left(\frac{5}{4}\right)} = \frac{(\sqrt{\pi})\,(\sqrt{\pi})\,\Gamma\left(\frac{1}{4}\right)}{4\,\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)} = \pi \ = \text{R.H.S.}$$ **Proved.**

Example 15. *Prove that* $\int_0^1 \frac{x^2}{\sqrt{(1-x^4)}} dx \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$ (*AMIETE, Dec. 2009*)

Solution. Here, we have $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$ ... (1)

Let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx$ Put $x^2 = \sin\theta \Rightarrow 2x\, dx = \cos\theta\, d\theta$

$x = \sqrt{\sin\theta} \Rightarrow dx = \frac{1}{2}(\sin\theta)^{-\frac{1}{2}} \cos\theta\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \frac{1}{2} (\sin\theta)^{-\frac{1}{2}} \cos\theta\, d\theta.$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{1-\frac{1}{2}}}{\cos\theta} .\cos\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{1}{2}} (\cos\theta)^0\, d\theta$$

$$= \frac{1}{2}\left(\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \qquad \text{... (2)}$$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$ $\left[\text{Put } x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta} \Rightarrow dx = \frac{1}{2}(\tan\theta)^{-\frac{1}{2}} \sec^2\theta\, d\theta\right]$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{(\tan\theta)^{-\frac{1}{2}} \sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\tan\theta)^{-\frac{1}{2}} \sec\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin\theta)^{-\frac{1}{2}} . (\cos\theta)^{-\frac{1}{2}}\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sqrt{\frac{2}{2\sin\theta\cos\theta}}\, d\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin 2\theta}} \qquad \text{Put } 2\theta = t \quad \Rightarrow \quad d\theta = \frac{dt}{2}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\sin t}} = \frac{1}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} (\sin t)^{-\frac{1}{2}} (\cos\theta)^{\circ}\, dt$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{\Gamma\left(\frac{1-\frac{1}{2}}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \right) = \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \ldots (3)$$

Putting the value in (1) from equation (1) and (2), we get

$$\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \times \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{16\sqrt{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{16\sqrt{2}} = \frac{\pi\,\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{\pi}{4\sqrt{2}} \quad \textbf{Proved.}$$

Example 16. *Prove that* $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ (*AMIETE, June 2010*)

Solution. Here, we have $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$, Put $x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta}$

$$\Rightarrow \quad dx = \frac{1}{2} (\sin\theta)^{-\frac{1}{2}} \cos\theta \,.\, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{-\frac{1}{2}} \cos\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{-\frac{1}{2}} \cos\theta}{\cos\theta}\, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{1}{2}} (\cos\theta)^0\, d\theta = \frac{1}{2} \left(\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \right) = \frac{1}{4} \frac{\Gamma\left(\frac{2}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad \textbf{Proved.}$$

Example 17. *Find the value of* $\Gamma\left(\frac{1}{2}\right)$.

Solution. We have already solved this problem in Art. 38.3 (ii) Transformation of the Gamma Function.

Now, by **Second method:** We know that,

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Putting $p = q = 0$, we get $\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma 1} \Rightarrow [\,\theta\,]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \Rightarrow \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2$

$$\Rightarrow \qquad \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi \qquad \Rightarrow \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \textbf{Ans.}$$

Example 18. *Show that*

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} \qquad \text{...(1)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$$

On applying formula (1), we have

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)} = \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \qquad \textbf{Proved.}$$

Example 19. *Using Beta and Gamma functions, evaluate*

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$$

Solution.
$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx \qquad \text{...(1)}$$

Putting $x^3 = \sin^2\theta$, so that $x = \sin^{\frac{2}{3}}\theta$, $dx = \frac{2}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta$ in (1), we get

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx = \int_0^{\pi/2} \left(\frac{\sin^2\theta}{1-\sin^2\theta}\right)^{\frac{1}{2}} \frac{2}{3}\sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right) \sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} \sin^{\frac{2}{3}}\theta\, d\theta$$

$$= \frac{2}{3}\,\frac{\Gamma\left(\frac{\frac{2}{3}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{2}{3}+1+1}{2}\right)} = \frac{2}{3}\,\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{3}\right)} = \frac{2}{3}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)} \qquad \textbf{Ans.}$$

Example 20. *Evaluate* $\int_{-1}^{+1} (1+x)^{p-1}(1-x)^{q-1}\, dx$.

Solution. Put $x = \cos 2\theta$, then $dx = -2\sin 2\theta\, d\theta$

$$\int_{-1}^{+1} (1+x)^{p-1}(1-x)^{q-1}\, dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{p-1}(1-\cos 2\theta)^{q-1}(-2\sin 2\theta\, d\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} \left(1+2\cos^2\theta-1\right)^{p-1}\left(1-1+2\sin^2\theta\right)^{q-1}(-4\sin\theta\cos\theta\, d\theta)$$

$$=4\int_0^{\frac{\pi}{2}} 2^{p-1}\cos^{2p-2}\theta \,.\, 2^{q-1}\sin^{2q-2}\theta \,.\, \sin\theta\cos\theta\, d\theta \;=\; 2^{p+q}\int_0^{\infty} \sin^{2q-1}\theta\cos^{2p-1}\theta\, d\theta$$

$$= 2^{p+q}\,\frac{\Gamma\left(\frac{2q}{2}\right)\Gamma\left(\frac{2p}{2}\right)}{2\,\Gamma\left(\frac{2p+2q}{2}\right)} = 2^{p+q-1}\,\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$ **Ans.**

Example 21. *Show that* $\Gamma(n)\,\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$ $\quad(0 < n < 1)$

Solution. We know that

$$\beta\,(m,\ n) = \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$ [From (1), Art 38.7]

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Putting $m + n = 1$ or $m = 1- n$, we get

$$\frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)} = \int_0^{\infty}\frac{x^{n-1}}{(1+x)^{1}}\,dx$$

$$\Gamma(1-n)\,\Gamma(n) = \int_0^{\infty}\frac{x^{n-1}}{1+x}\,dx \qquad \left[\int_0^{\infty}\frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}\right]$$

$$\Rightarrow \quad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$ **Proved.**

Example 22. *Assuming* $\Gamma(n)\,\Gamma(1-n) = \pi \operatorname{cosec} n\pi.\ 0 < n < 1,$ *show that*

$$\int_0^{\infty}\frac{x^{p-1}}{1+x}\,dx = \left(\frac{\pi}{\sin p\pi}\right).\quad 0 < p < 1$$ *(U.P., I Semester , Dec 2009)*

Solution: Here, we have π cosec $n\,\pi$ = $\Gamma(n)\,\Gamma(1-n)$

$$\Rightarrow \qquad \frac{\pi}{\sin n\pi} = \Gamma(n)\,\Gamma(1-n)$$

We know that $\displaystyle\int_0^{\infty}\frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$ ...(1)

Setting $m + n = 1$ so that $m = 1 - n$ in (1), we get

$$\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta\,(m,n)$$ **Proved.**

Example 23. *Prove that* $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Solution. Putting $n = \dfrac{1}{4}$ in result of example 22, we obtain

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Rightarrow \qquad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \quad\Rightarrow\quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$$ **Proved.**

Example 24. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $$dx = \frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$$

$$\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta\cos\theta}{(1-\sin^2\theta)^{1/n}}\,d\theta = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta\cos\theta}{(\cos^2\theta)^{1/n}}\,d\theta$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta\,\cos^{1-2/n}\theta\,d\theta$$

$$= \frac{2}{n}\,\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right)} = \frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{\Gamma(1)} \qquad \left(\because\ \Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right)$$

$$= \frac{\pi}{n\sin\frac{\pi}{n}}$$ **Ans.**

Example 25. *Show that* $\int_0^a \frac{dx}{\sqrt[n]{(a^n-x^n)}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$, *where $n > 1$.(M.U. II Semester 2009)*

Solution. Let $x^n = a^n\sin^2\theta \quad \Rightarrow \quad x = a\sin^{\frac{2}{n}}\theta$

So that $$dx = \frac{2a}{n}\sin^{\frac{2}{n}-1}\theta\cos\theta\,d\theta$$

$$\therefore \quad \int_0^a \frac{dx}{\sqrt[n]{a^n-x^n}} = \int_0^{\frac{\pi}{2}} \frac{a\times\frac{2}{n}\sin^{\frac{2}{n}-1}\theta\cos\theta}{(a^n-a^n\sin^2\theta)^{\frac{1}{n}}}\,d\theta = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{2}{n}-1}\theta\cos\theta}{\cos^{\frac{2}{n}}\theta}\,d\theta$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{\frac{2}{n}-1}\theta\cos^{1-\frac{2}{n}}\theta\,d\theta = \frac{2}{n}\,\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right)}$$

$$= \frac{1}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{\Gamma(1)} \qquad \left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right]$$

$$= \frac{\pi}{n\sin\frac{\pi}{n}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$ **Proved.**