2-3 BASIC THEOREMS AND PROPERTIES OF BOOLEAN ALGEBRA

Duality

The Huntington postulates have been listed in pairs and designated by part (a) and part (b). One part may be obtained from the other if the binary operators and the identity elements are interchanged. This important property of Boolean algebra is called the *duality principle*. It states that every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged. In a two-valued Boolean algebra, the identity elements and the elements of the set B are the same: 1 and 0. The duality principle has many applications. If the *dual* of an algebraic expression is desired, we simply interchange OR and AND operators and replace 1's by 0's and 0's by 1's.

Basic Theorems

Table 2-1 lists six theorems of Boolean algebra and four of its postulates. The notation is simplified by omitting the $\cdot$ whenever this does not lead to confusion. The theorems and postulates listed are the most basic relationships in Boolean algebra. The reader is advised to become familiar with them as soon as possible. The theorems, like the postulates, are listed in pairs; each relation is the dual of the one paired with it. The postulates are basic axioms of the algebraic structure and need no proof. The theorems must be proven from the postulates. The proofs of the theorems with one variable are presented below. At the right is listed the number of the postulate which justifies each step of the proof.

TABLE 2-1 Postulates and theorems of Boolean algebra

Postulate 2	(a) $x + 0 = x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x + x' = 1$	(b) $x \cdot x' = 0$
Theorem 1	(a) $x + x = x$	(b) $x \cdot x = x$
Theorem 2	(a) $x + 1 = 1$	(b) $x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Postulate 3, commutative	(a) $x + y = y + x$	(b) $xy = yx$
Theorem 4, associative	(a) $x + (y + z) = (x + y) + z$	(b) $x(yz) = (xy)z$
Postulate 4, distributive	(a) $x(y + z) = xy + xz$	(b) $x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a) $(x + y)' = x'y'$	(b) $(xy)' = x' + y'$
Theorem 6, absorption	(a) $x + xy = x$	(b) $x(x + y) = x$

THEOREM 1(a): $x + x = x$.

$$\begin{aligned}
x + x &= (x + x) \cdot 1 && \text{by postulate: 2(b)}\\
&= (x + x)(x + x') && \text{5(a)}\\
&= x + xx' && \text{4(b)}\\
&= x + 0 && \text{5(b)}\\
&= x && \text{2(a)}
\end{aligned}$$

THEOREM 1(b): $x \cdot x = x$.

$$\begin{aligned}
x \cdot x &= xx + 0 && \text{by postulate: 2(a)}\\
&= xx + xx' && \text{5(b)}\\
&= x(x + x') && \text{4(a)}\\
&= x \cdot 1 && \text{5(a)}\\
&= x && \text{2(b)}
\end{aligned}$$

Note that theorem 1(b) is the dual of theorem 1(a) and that each step of the proof in part (b) is the dual of part (a). Any dual theorem can be similarly derived from the proof of its corresponding pair.

THEOREM 2(a): $x + 1 = 1$.

$$\begin{aligned}
x + 1 &= 1 \cdot (x + 1) && \text{by postulate: 2(b)}\\
&= (x + x')(x + 1) && \text{5(a)}\\
&= x + x' \cdot 1 && \text{4(b)}\\
&= x + x' && \text{2(b)}\\
&= 1 && \text{5(a)}
\end{aligned}$$

THEOREM 2(b): $x \cdot 0 = 0$ by duality.

THEOREM 3: $(x')' = x$. From postulate 5, we have $x + x' = 1$ and $x \cdot x' = 0$, which defines the complement of x. The complement of x' is x and is also $(x')'$. Therefore, since the complement is unique, we have that $(x')' = x$.

The theorems involving two or three variables may be proven algebraically from the postulates and the theorems which have already been proven. Take, for example, the absorption theorem.

THEOREM 6(a): $x + xy = x$.

$$\begin{aligned} x + xy &= x \cdot 1 + xy && \text{by postulate 2(b)} \\ &= x(1 + y) && \text{by postulate 4(a)} \\ &= x(y + 1) && \text{by postulate 3(a)} \\ &= x \cdot 1 && \text{by theorem 2(a)} \\ &= x && \text{by postulate 2(b)} \end{aligned}$$

THEOREM 6(b): $x(x + y) = x$ by duality.

The theorems of Boolean algebra can be shown to hold true by means of truth tables. In truth tables, both sides of the relation are checked to yield identical results for all possible combinations of variables involved. The following truth table verifies the first absorption theorem.

x	y	xy	$x + xy$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

The algebraic proofs of the associative law and De Morgan's theorem are long and will not be shown here. However, their validity is easily shown with truth tables. For example, the truth table for the first De Morgan's theorem $(x + y)' = x'y'$ is shown below.

x	y	$x + y$	$(x + y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Operator Precedence

The operator precedence for evaluating Boolean expressions is (1) parentheses, (2) NOT, (3) AND, and (4) OR. In other words, the expression inside the parentheses must be evaluated before all other operations. The next operation that holds precedence is the complement, then follows the AND, and finally the OR. As an example, consider the truth table for De Morgan's theorem. The left side of the

EXAMPLE 2-1: Simplify the following Boolean functions to a minimum number of literals.

1. $x + x'y = (x + x')(x + y) = 1 \cdot (x + y) = x + y$
2. $x(x' + y) = xx' + xy = 0 + xy = xy$
3. $x'y'z + x'yz + xy' = x'z(y' + y) + xy' = x'z + xy'$
4. $$\begin{aligned} xy + x'z + yz &= xy + x'z + yz(x + x') \\ &= xy + x'z + xyz + x'yz \\ &= xy(1 + z) + x'z(1 + y) \\ &= xy + x'z \end{aligned}$$
5. $(x + y)(x' + z)(y + z) = (x + y)(x' + z)$ by duality from function 4.

Functions 1 and 2 are the duals of each other and use dual expressions in corresponding steps. Function 3 shows the equality of the functions F_3 and F_4 discussed previously. The fourth illustrates the fact that an increase in the number of literals sometimes leads to a final simpler expression. Function 5 is not minimized directly but can be derived from the dual of the steps used to derive function 4.

Complement of a Function

The complement of a function F is F' and is obtained from an interchange of 0's for 1's and 1's for 0's in the value of F. The complement of a function may be derived algebraically through De Morgan's theorem. This pair of theorems is listed in Table 2-1 for two variables. De Morgan's theorems can be extended to three or more variables. The three-variable form of the first De Morgan's theorem is derived below. The postulates and theorems are those listed in Table 2-1.

$$\begin{aligned} (A + B + C)' &= (A + X)' && \text{let } B + C = X \\ &= A'X' && \text{by theorem 5(a) (De Morgan)} \\ &= A' \cdot (B + C)' && \text{substitute } B + C = X \\ &= A' \cdot (B'C') && \text{by theorem 5(a) (De Morgan)} \\ &= A'B'C' && \text{by theorem 4(b) (associative)} \end{aligned}$$

De Morgan's theorems for any number of variables resemble in form the two-variable case and can be derived by successive substitutions similar to the method used in the above derivation. These theorems can be generalized as follows:

$$(A + B + C + D + \cdots + F)' = A'B'C'D' \ldots F'$$

$$(ABCD \cdots F)' = A' + B' + C' + D' + \ldots + F'$$

The generalized form of De Morgan's theorem states that the complement of a function is obtained by interchanging AND and OR operators and complementing each literal.

EXAMPLE 2-2: Find the complement of the functions $F_1 = x'yz' + x'y'z$ and $F_2 = x(y'z' + yz)$. Applying De Morgan's theorem as many times as necessary, the complements are obtained as follows:

$$F_1' = (x'yz' + x'y'z)' = (x'yz')'(x'y'z)' = (x + y' + z)(x + y + z')$$

$$F_2' = [x(y'z' + yz)]' = x' + (y'z' + yz)' = x' + (y'z')' \cdot (yz)'$$

$$= x' + (y + z)(y' + z')$$

A simpler procedure for deriving the complement of a function is to take the dual of the function and complement each literal. This method follows from the generalized De Morgan's theorem. Remember that the dual of a function is obtained from the interchange of AND and OR operators and 1's and 0's.

EXAMPLE 2-3: Find the complement of the functions F_1 and F_2 of Example 2-2 by taking their duals and complementing each literal.

1. $F_1 = x'yz' + x'y'z$.
   The dual of F_1 is $(x' + y + z')(x' + y' + z)$.
   Complement each literal: $(x + y' + z)(x + y + z') = F_1'$.

2. $F_2 = x(y'z' + yz)$.
   The dual of F_2 is $x + (y' + z')(y + z)$.
   Complement each literal: $x' + (y + z)(y' + z') = F_2'$.