* Curl

The curl of a vector point function F is defined as below

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

Curl $\vec{F}$ is a vector quantity.

If curl $\vec{F} = 0$, then the vector $\vec{F}$ is irrotational.

Q: Find the divergence and curl of $\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at point $(2, -1, 1)$.

Sol'n We have,

$$\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

$$\text{Div } \vec{v} = \nabla \cdot \vec{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$$\text{div } \vec{v} \text{ at } (2,-1,1) = -1\times1 + 3(2)^2 + 2\times2\times1 - (-1)^2 = -1 + 12 + 4 - 1$$

$$= 14 \quad \text{Ans}$$

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= (-2yz)\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k}$$

curl at $(2, -1, 1)$

$$= -2(-1)(1)\hat{i} - (1^2 - 2\times-1)\hat{j} + (6\times2\times-1 - (-1)(1))\hat{k}$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \quad \text{Ans}$$

Q A vector $\vec{r}$ is defind by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. If $|\vec{r}| = r$ then show that vector $r^n\vec{r}$ is irrotational.

Solution

$$\text{Curl } \vec{F} = \text{Curl } r^n\vec{r} = \nabla \times r^n\vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times r^n(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \left(\frac{\partial r^n z}{\partial y} - \frac{\partial r^n y}{\partial z}\right)\hat{i} - \left(\frac{\partial}{\partial x} r^n z - \frac{\partial}{\partial z} r^n x\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x} r^n y - \frac{\partial}{\partial y} r^n x\right)\hat{k}$$

$$= \left(n r^{n-1} \frac{\partial r}{\partial y} \cdot z - n r^{n-1} \frac{\partial r}{\partial z} \cdot y\right)\hat{i}$$

$$+ \left(n r^{n-1} \frac{\partial r}{\partial x} \cdot z - n r^{n-1} \frac{\partial r}{\partial z} \cdot x\right)\hat{j}$$

$$+ \left(n r^{n-1} \frac{\partial r}{\partial x} \cdot y - n r^{n-1} \frac{\partial r}{\partial y} \cdot x\right)\hat{k}$$

$$= n r^{n-1}\left(\frac{y}{r} \cdot z - \frac{z}{r} \cdot y\right)\hat{i} + n r^{n-1}\left(\frac{x}{r} \cdot z - \frac{z}{r} \cdot x\right)\hat{j}$$

$$+ n r^{n-1}\left(\frac{x}{r} \cdot y - \frac{y}{r} \cdot x\right)\hat{k}$$

$\Rightarrow$ $\underline{\text{curl } r^n \vec{r} = 0}$ so $r^n \vec{r}$ is irrotational.

Q:- Find the divergence and curl of the vector field $V = (x^2 - y^2)\hat{i} + 2xy\,\hat{j} + (y^2 - xy)\hat{k}$.

Ans $\underline{\text{Div } \vec{V} = 4x}$. $\underline{\text{Curl } \vec{V} = (2y - x)\hat{i} + y\hat{j} + 4y\hat{k}}$

Q→ Prove that for every vector field $\vec{V}$, $\underline{\text{div curl } \vec{V} = 0}$

Sol^n Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{div curl } \vec{V} = \nabla \cdot (\nabla \times \vec{V}) = \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_1}{\partial y \partial z} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

$$= 0$$

Proved