Jacobians:

If u and v are functions of the two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Similarly, Jacobian of u, v, w with respect to x, y and z is ~~written~~ ~~as~~

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Q:- If $x = r\cos\theta$, $y = r\sin\theta$ evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Solution:

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\frac{\partial y}{\partial \theta} = r\cos\theta$

then $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r(\cos^2\theta + \sin^2\theta) = r$ $\underline{A}$

Now, $r^2 = x^2 + y^2$, $\theta = \tan^{-1} y/x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{-y}{r^2} \\ \frac{y}{r} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{1}{r^3}(x^2+y^2)$$

$= \frac{1}{r^3}\cdot r^2 = \frac{1}{r}$ $\underline{\underline{A}}$

Q→ Find the Jacobian $J\left(\frac{u,v}{x,y}\right)$ for $u = e^x \sin y$ and $v = x\log\sin y$

Ans $e^x\cos y\,[x - \log\sin y]$

Q→ If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$

Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

Q→ If $u = \frac{x}{y-z}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Properties of Jacobians :-

① **First Property** :- If u and v are the functions of x & y

then $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Proof: let

$$u = f(x,y) \quad \text{—} ①$$
$$v = \phi(x,y) \quad \text{—} ②$$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix} \quad ③$$

On diff. (1) and (2) w.r. to u and v, we get

$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} \end{aligned}\right\} - ④$$

from ③ and ④, we get $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Ans

$\frac{\partial(x,y)}{\partial(u,v)}$

Q:- If $x = u.v$ and $y = \frac{u+v}{u-v}$, find $\frac{\partial(u,v)}{\partial(x,y)}$ Ans $= \frac{4uv}{(u-v)^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv} \text{ A}$$

Q→ If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

find $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$

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Example 11. *If* $x = uv,\ y = \dfrac{u+v}{u-v}$, *find* $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution. Here it is easy to find $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$. But to find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ is comparatively difficult. So we first find $\dfrac{\partial(x,y)}{\partial(u,v)}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \dfrac{-2v}{(u-v)^2} & \dfrac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{uv}{(u-v)^2}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = \frac{uv}{(u-v)^2}(2+2) = \frac{4uv}{(u-v)^2}$$

But $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{4uv}{(u-v)^2} = 1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}$$ Ans

Example 12. *If* $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$, *find* $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution. Since u, v, w are explicitly given, so first we evaluate

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$$J' = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$J' = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= yz(2y - 2z) - zx(2x - 2z) + xy(2x - 2y) = 2[yz(y - z) - zx(x - z) + xy(x -$

$= 2[x^2y - x^2z - xy^2 + xz^2 + y^2z - yz^2] = 2[x^2(y - z) - x(y^2 - z^2) + yz(y - z)]$

$= 2(y - z)[x^2 - x(y + z) + yz] = 2(y - z)[y(z - x) - x(z - x)]$

$= 2(y - z)(z - x)(y - x) = -2(x - y)(y - z)(z - x)$

Hence, by $JJ' = 1$, we have

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$$ Ans

(2) Second Property of Jacobian :-

If u, v are the function of r, s where r, s are functions of x, y then $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$

Proof: $$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \quad \underline{\text{Ans}}$$

Similarly, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$$

Q→ Find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

Solution: here, $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Now, by Property of Jacobian

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 4r^2 \cdot r = 4r^3 \quad \underline{\text{Ans}}$$

Q→ If $u = x+y+z$, $uv = y+z$, $uvw = z$. Evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Soln: let $x+y+z = r$, $y+z = s$, $z = t$

$\Rightarrow u = r$, $uv = s$, $uvw = t$

Now.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial(x,y,z)}{\partial(r,s,t)} \cdot \frac{\partial(r,s,t)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(r,s,t)}{\partial(x,y,z)}} \times \frac{\partial(r,s,t)}{\partial(u,v,w)} \quad (1)$$

$$\frac{\partial(r,s,t)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \text{ and } \frac{\partial(r,s,t)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = u^2v$$

Putting these value in (1),

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{1} \times u^2v = u^2v \quad \underline{\text{Ans}}$$

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Q:- If $u = x+y+z$, $u^2 v = y+z$, $u^3 w = z$,

show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}$

③ Third Property of Jacobian –

If functions u, v, w of three independent variables x, y, z are not independent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Q:- If $u = xy+yz+zx$, $v = x^2+y^2+z^2$ and $w = x+y+z$ show that they are functionally related and find the relation between them.

Solution: $u = xy + yz + zx$, $v = x^2+y^2+z^2$, $w = x+y+z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1+R_2$$

$(R_1 = R_3)$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence u, v, w are functionally related.

Now, $w^2 = (x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx)$

$\Rightarrow \boxed{w^2 = v+2u}$ which is required relationship

Q:- If $u = 3x+2y-z$, $v = x-2y+z$, $w = x(x+2y-z)$

Show that they are functionally related and find relation between them.

Solution Here we have

$u = 3x+2y-z$, $\frac{\partial u}{\partial x} = 3$, $\frac{\partial u}{\partial y} = 2$, $\frac{\partial u}{\partial z} = -1$

$v = x-2y+z$, $\frac{\partial v}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -2$, $\frac{\partial v}{\partial z} = 1$

$w = x^2+2xy-xz$, $\frac{\partial w}{\partial x} = 2x+2y-z$, $\frac{\partial w}{\partial y} = 2x$, $\frac{\partial w}{\partial z} = -x$

Now

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix} = 3(2x-2x) - 2(-x-2x-2y+z) - 1(2x+4x+4y-2z) = 0$$

Hence the functional relationship exists between u, v and w them.

Now, we find the relationship between them

$u+v = 4x$ and $u-v = 2x+4y-2z$

$(u+v)(u-v) = 4x(2x+4y-2z) = 8x(x+2y-z)$

$\Rightarrow \boxed{u^2-v^2 = 8w}$ → which is required relationship.

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Q:- Verify whether the following functions are functionally dependent, and if so find the relation between them. $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Hence, u and v are functionally related.

Now,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\Rightarrow v = \tan^{-1}u$$

$$\Rightarrow \boxed{u = \tan v}$$ Ans

* Jacobian of Implicit functions:-

The variables x, y, u, v are connected by implicit functions

$$f_1(x,y,u,v) = 0 \quad \text{---} (1)$$

$$f_2(x,y,u,v) = 0 \quad \text{---} (2)$$

where u, v are implicit functions of x, y.

then

$$\left[\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}\right]$$

Ex- If $x^2+y^2+u^2-v^2=0$ and $uv+xy=0$, prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2-y^2}{u^2+v^2}.$$

Solution:- $f_1 = x^2+y^2+u^2-v^2$, $f_2 = uv+xy$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial x} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2(x^2-y^2)$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2(u^2+v^2)$$

But $\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)} = \frac{2(x^2-y^2)}{2(u^2+v^2)} = \frac{x^2-y^2}{u^2+v^2}$ Proved

Q:- If $x+y+z = u$, $y+z = uv$, $z = uvw$ show that

$$\frac{\partial(x,y,z)}{\partial(u,v,z)} = u^2 v$$

Solution: Let

$$f_1 = x+y+z-u$$
$$f_2 = y+z-uv$$
$$f_3 = z-uvw$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

and

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = -u^2 v$$

But

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)/\partial(u,v,w)}{\partial(f_1,f_2,f_3)/\partial(x,y,z)} = (-1)^3 \frac{-u^2 v}{1} = u^2 v$$

Proved.

Q:- If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$ and $u+v+w = x^2+y^2+z^2$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Proof. Let

$$f_1 = u^3+v^3+w^3-x-y-z$$
$$f_2 = u^2+v^2+w^2-x^3-y^3-z^3$$
$$f_3 = u+v+w-x^2-y^2-z^2$$

Now,

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} = (-1)(-2)(-3)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

$$= -6\begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x-y & y-z & z \end{vmatrix} = -6\{0-0+1[(x^2-y^2)(y-z)-(x-y)(y^2-z^2)]\}$$

$$= -6(x-y)(y-z)(x+\cancel{y}-\cancel{y}-z)$$

$$= -6(x-y)(y-z)(x-z) = 6(x-y)(y-z)(z-x)$$

and

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix} = 6\begin{vmatrix} u^2-v^2 & v^2-w^2 & w^2 \\ u-v & v-w & w \\ 0 & 0 & 1 \end{vmatrix} = 6(u-v)(v-w)(u-w)$$
$$= -6(u-v)(v-w)(w-u)$$

But

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)/\partial(x,y,z)}{\partial(f_1,f_2,f_3)/\partial(u,v,w)} = (-1)^3 \frac{6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-u)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Proved