# Maxima and Minima of functions :-

**Maximum Value :-** A function $f(x,y)$ is said to have a maximum value at $x=a$, $y=b$, if there exists a small neighbourhood of $(a,b)$ such that

$$f(a,b) > f(a+h, b+k)$$

**Minimum Value :-** A function $f(x,y)$ is said to have a minimum value for $x=a$, $y=b$, if there exist a small neighbourhood of $(a,b)$ such that

$$f(a,b) < f(a+h, b+k).$$

**Saddle Point :-** A point where the function is neither maximum nor minimum is called saddle point

## Working Rule to Find Extremum Values :-

(i) Differentiate $f(x,y)$ and find out

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$$

(ii) Put $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ and solve it let it is $(a,b)$

(iii) Evaluate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for these values $(a,b)$.

(iv) If $rt - s^2 > 0$ and

(a) $r < 0$ then $f(x,y)$ has a maximum value

(b) $r > 0$ then $f(x,y)$ has a minimum value

(v) If $rt - s^2 < 0$ then $f(x,y)$ has no extremum value at the point $(a,b)$

(vi) If $rt - s^2 = 0$, then the case is doubtful and needs further investigation.

Note :- The point $(a,b)$ is called stationary points

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Q. Find the absolute maximum and minimum values of
$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

Solution:- We have, $f(x,y) = 2 + 2x + 2y - x^2 - y^2$

$$\frac{\partial f}{\partial x} = 2 - 2x, \quad \frac{\partial f}{\partial y} = 2 - 2y, \quad \frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$
$$\frac{\partial^2 f}{\partial y^2} = -2$$

for maxima and minima, $\frac{\partial f}{\partial x} = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2 - 2y = 0 \Rightarrow y = 1$

At $(1,1)$ $\quad rt - s^2 = (-2)(-2) - 0 = 4 > 0$

and $\quad r = \frac{\partial^2 f}{\partial x^2} = -2 \; (-ve) < 0$

Hence $f(x,y)$ is maximum at $(1,1)$.

Maximum value of $f(x,y) = 2 + 2 + 2 - 1 - 1 = 4$

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Q.2 Examine the function $f(x,y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values

Solution. We have $f(x,y) = y^2 + 4xy + 3x^2 + x^3$

$$\Rightarrow \frac{\partial f}{\partial x} = 4y + 6x + 3x^2, \quad \frac{\partial f}{\partial y} = 2y + 4x$$
$$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 4, \quad t = \frac{\partial^2 f}{\partial y^2} = 2$$

for maxima and minima,

$\frac{\partial f}{\partial x} = 0 \Rightarrow 4y + 6x + 3x^2 = 0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + 4x = 0 \Rightarrow y = -2x$

so $\quad 4(-2x) + 6x + 3x^2 = 0$

or $\quad 3x^2 - 2x = 0 \Rightarrow x(3x - 2) = 0$

$\Rightarrow [x = 0, 2/3]$

when $x = 0$ then $y = 0$

when $x = 2/3$ then $y = -2\left(\frac{2}{3}\right) = -4/3$

so stationary points are $(0,0)$, $\left(\frac{2}{3}, \frac{-4}{3}\right)$

| | $(0,0)$ | $\left(\frac{2}{3}, -\frac{4}{3}\right)$ |
|---|---|---|
| $r = 6 + 6x$ | 6 | 10 |
| $s = 4$ | 4 | 4 |
| $t = 2$ | 2 | 2 |
| $rt - s^2$ | $-4$ | $+4$ |

at $(0,0)$ there is no extremum value since $rt - s^2 < 0$.

at $\left(\frac{2}{3}, -\frac{4}{3}\right)$, $rt - s^2 > 0$, and $r > 0$

so $\left(\frac{2}{3}, -\frac{4}{3}\right)$ is point of maximum value

and it is equal to $= \left(-\frac{4}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(-\frac{4}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$

$= -4/27$ Ans

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**Example 11.** *Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.*

**Solution.** Let $x, y, z$ be the number whose sum is 120.

$$i.e., \quad x + y + z = 120 \Rightarrow z = 120 - x - y \quad \ldots(1)$$

Let $\quad f = xy + yz + zx$

$\Rightarrow \quad f = xy + y(120 - x - y) + x(120 - x - y)$ [Using (1)]

$\Rightarrow \quad f = xy + 120y - xy - y^2 + 120x - x^2 - xy$

$\Rightarrow \quad f = 120x + 120y - xy - x^2 - y^2$

$$p = \frac{\partial f}{\partial x} = 120 - y - 2x$$

$$q = \frac{\partial f}{\partial y} = 120 - x - 2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

For maxima and minima

$$\frac{\partial f}{\partial x} = 0$$

$\Rightarrow \quad 120 - y - 2x = 0$

$\Rightarrow \quad y = 120 - 2x \quad \ldots(2)$

and $\quad \dfrac{\partial f}{\partial y} = 0$

$\Rightarrow \quad 120 - x - 2y = 0 \quad \ldots(3)$

Putting the value of $y$ from (2) in (3), we get

$120 - x - 2(120 - 2x) = 0$

$\Rightarrow \quad 120 - x - 240 + 4x = 0 \quad \Rightarrow \quad 3x = 120 \quad \Rightarrow \quad x = 40$

Putting the value of $x$ in (2), we get

$$y = 120 - 2(40) = 120 - 80 = 40$$

Thus, the stationary pair is (40, 40).

| | (40, 40) |
|---|---|
| $r = -2$ | $-2$ |
| $s = -1$ | $-1$ |
| $t = -2$ | $-2$ |
| $rt - s^2$ | $+3$ |

At (40, 40), $r = -$ ve and $rt - s^2 = +$ ve

Hence, $f$ is maximum at (40, 40).

Putting $x = 40$, $y = 40$ in (1), we get

$$40 + 40 + z = 120 \quad \Rightarrow \quad z = 40$$

Hence, $f$ is maximum at $x = 40$, $y = 40$ and $z = 40$. **Ans.**

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## * Lagrange Method of Undetermined Multipliers :-

Let $f(x,y,z)$ be fuction of three variables $x, y, z$ and $x, y, z$ are connected by relation $\phi(x,y,z)=0$

for finding stationary vale of $f(x,y,z)$.

we solve,
$$\phi(x,y,z)=0 \quad — ①$$
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad — ②$$
$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad — ③$$
$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad — ④$$

On solving ①, ②, ③, ④ we find the value of $x, y, z$ and $\lambda$ for which $f(x,y,z)$ has stationy point.

Draw Back in Lagranges method is that the nature of stationary point can not be determined.

Q→ Find the point upon the plane $ax+by+cz=p$ at which the fuction $f = x^2+y^2+z^2$ has a minimum value and find this minimum $f$.

Solution:- We have $f = x^2+y^2+z^2$ — ①

and $ax+by+cz=p \Rightarrow \phi = ax+by+cz-p$ — ②

By Lagranges method,
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda a = 0 \Rightarrow x = -\frac{\lambda a}{2}$$
$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda b = 0 \Rightarrow y = -\frac{\lambda b}{2}$$
$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda c = 0 \Rightarrow z = -\frac{\lambda c}{2}$$

Substituting the values of $x, y, z$ in (2), we get
$$a\left(-\frac{\lambda a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) + c\left(-\frac{\lambda c}{2}\right) = p$$
$$\Rightarrow \lambda(a^2+b^2+c^2) = -2p \Rightarrow \lambda = \frac{-2p}{a^2+b^2+c^2}$$

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$\Rightarrow$ Stationary point is

$$x = \frac{ap}{a^2+b^2+c^2}, \quad y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}$$

The minimum value of $f = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2} \quad \underline{A}$$

Q→ Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution: Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular solid.

Volume of solid $V = 8xyz$ — ①

and $x^2+y^2+z^2 = R^2$ —— ②

$\Rightarrow \phi(x,y,z) = x^2+y^2+z^2-R^2$ —— ②

By Lagranges method.

$$\frac{\partial V}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 8yz + \lambda(2x) = 0 \quad — ③$$

$$\frac{\partial V}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 8xz + \lambda(2y) = 0 \quad — ④$$

$$\frac{\partial V}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 8xy + \lambda(2z) = 0 \quad — ⑤$$

from (3) $\quad -8yz = +2\lambda x \quad$ or $\quad -8xyz = 2\lambda x^2$

from (4) $\quad -8xz = 2\lambda y \quad$ or $\quad -8xyz = 2\lambda y^2$

from (5) $\quad -8xy = 2\lambda z \quad$ or $\quad -8xyz = 2\lambda z^2$

$$\Rightarrow 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$$

$$\Rightarrow x^2 = y^2 = z^2$$

$$\Rightarrow x = y = z$$

Hence rectangular solid is cube. Proved

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Q-) Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$

Solution:- Let the co-ordinate of the given point be $(x, y, z)$, then its distance D from (3, 4, 12)

$$D = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

$$\Rightarrow F(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$$

and $\quad x^2+y^2+z^2 = 1 \quad \text{—— (1)}$

$$\Rightarrow \phi(x,y,z) = x^2+y^2+z^2-1$$

Now By Lagrange's method

$$\frac{\partial F}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2(x-3)+2\lambda x = 0 \quad \text{—— (2)}$$

$$\frac{\partial F}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2(y-4)+2\lambda y = 0 \quad \text{—— (3)}$$

$$\frac{\partial F}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2(z-12)+2\lambda z = 0 \quad \text{—— (4)}$$

Multiplying (2) by x, (3) by y and (4) by z and adding, we get

$$\Rightarrow (x^2+y^2+z^2) - 3x - 4y - 12z + \lambda(x^2+y^2+z^2) = 0$$

$$\Rightarrow 1 - 3x - 4y - 12z + \lambda = 0 \quad \text{—— (5)}$$

From (2) $\quad x = \frac{3}{1+\lambda} \quad$ (6)

From (3) $\quad y = \frac{4}{1+\lambda} \quad$ (7)

From (4) $\quad z = \frac{12}{1+\lambda} \quad$ (8)

Putting these values of x, y, z in (5), we get

$$1+\lambda - \frac{9}{1+\lambda} - \frac{16}{1+\lambda} - \frac{144}{1+\lambda} = 0$$

$$\Rightarrow (1+\lambda)^2 = 169 \quad \text{or} \quad 1+\lambda = \pm 13$$

$$\Rightarrow \lambda = 12, -14$$

Putting the value of $\lambda$ in (6), (7), (8), we have the points

$$\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) \text{ and } \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$$

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The minimum distance $= \sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}$

$= 12$ Ans

and The maximum distance $= \sqrt{\left(3+\frac{3}{12}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}$

$= 14$ Ans

Q-) Use the method of the Lagrange's multipliers to find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Here, we have, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$\Rightarrow \phi(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid. Then volume $V = 2x \cdot 2y \cdot 2z = 8xyz$

By Lagrange's equations are,

$\frac{\partial V}{\partial x}+\lambda\frac{\partial\phi}{\partial x}=0 \Rightarrow 8yz+\lambda\frac{2x}{a^2}=0$ (1)

$\frac{\partial V}{\partial y}+\lambda\frac{\partial\phi}{\partial y}=0 \Rightarrow 8xz+\lambda\frac{2y}{b^2}=0$ (2)

$\frac{\partial V}{\partial z}+\lambda\frac{\partial\phi}{\partial z}=0 \Rightarrow 8xy+\lambda\frac{2z}{c^2}=0$ (3)

multiplying (1), (2), (3) by $x, y, z$ resp. and adding we get

$24xyz+2\lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right]=0 \Rightarrow 24xyz+2\lambda=0$

$\Rightarrow \boxed{\lambda=-12xyz}$

Putting these value in (1), (2), (3), we get

$x=\frac{a}{\sqrt{3}},\quad y=\frac{b}{\sqrt{3}},\quad z=\frac{c}{\sqrt{3}}$

then volume of largest rectangular parallelopiped $= 8xyz = 8\times\frac{a}{\sqrt{3}}\times\frac{b}{\sqrt{3}}\times\frac{c}{\sqrt{3}}$

$= \frac{8abc}{3\sqrt{3}}$ Ans

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