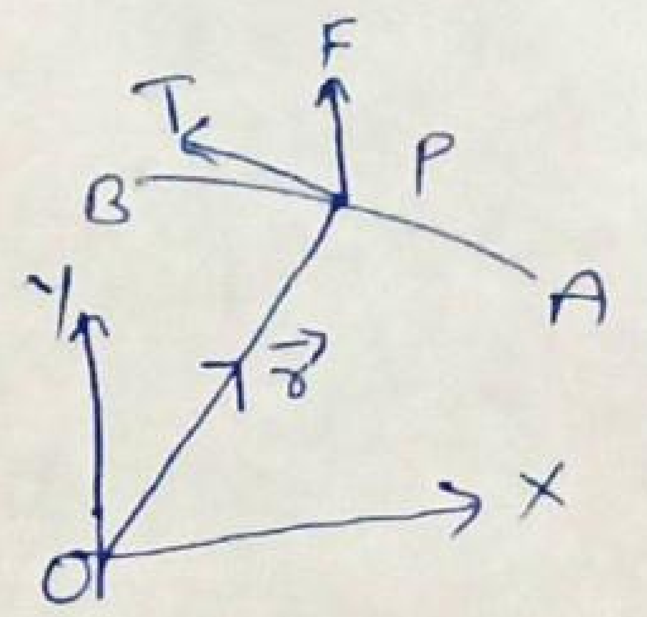
* Line Integral :-

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB

$$\left[\text{Line integral} = \int_C \vec{F} \cdot d\vec{r}\right]$$



*** Work :-** If $\vec{F}$ represent the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F} \cdot d\vec{r}$

*** Circulation :-** If $\vec{v}$ represent the velocity of a liquid then $\oint_C \vec{v} \cdot d\vec{r}$ is called the circulation of v round the closed curve c.

Q :- If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.

Solution : Work done $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_C \left(2x^2y\,\hat{i} + 3xy\,\hat{j}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= \int_C \left(2x^2y\,dx + 3xy\,dy\right)$$

Putting $y = 4x^2$

$dy = 8x\,dx$

and limit $x = 0$ to $x = 1$, we get

$$= \int_0^1 2x^2 4x^2\,dx + 3x\,4x^2\,8x\,dx$$

$$= \int_0^1 8x^4\,dx + 96x^4\,dx$$

$$= \int_0^1 104\,x^4\,dx$$

$$= 104\left[\frac{x^5}{5}\right]_0^1 = \frac{104}{5} \quad \text{Ans}$$

Q → A vector field is given by $\vec{F} = (2y+3)\,\hat{i} + xz\,\hat{j} + (yz - x)\,\hat{k}$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the path C is $x = 2t$, $y = t$, $z = t^3$ from $t = 0$ to $t = 1$.

Solution: $\int_C \vec{F}\cdot d\vec{r} = \int_C [(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}]\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \int_C (2y+3)\,dx + xz\,dy + (yz-x)\,dz$$

since $x = 2t \Rightarrow dx = 2dt$
$y = t \Rightarrow dy = dt$
$z = t^3 \Rightarrow dz = 3t^2\,dt$

and limit $t=0$ to $t=1$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (2t+3)2\,dt + 2t\cdot t^3\,dt + (t\cdot t^3 - 2t)3t^2\,dt$$

$$= \int_0^1 (4t + 6 + 2t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[4\frac{t^2}{2} + 6t + \frac{2}{5}t^5 + \frac{3}{7}t^7 - \frac{6}{4}t^4\right]_0^1$$

$$= \left(2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}\right) - 0$$

$$= 7.32857 \quad \text{Ans}$$

Q→ Suppose $\vec{F} = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from the $(1,2,3)$ to $(3,5,7)$.

Solution:- Work done $= \int_C \vec{F}\,d\vec{r} = \int_{(1,2,3)}^{(3,5,7)} (x^3\hat{i} + y\hat{j} + z\hat{k})(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \int_{(1,2,3)}^{(3,5,7)} (x^3\,dx + y\,dy + z\,dz)$$

$$= \int_1^3 x^3\,dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

$$= \left[\frac{x^4}{4}\right]_1^3 + \left[\frac{y^2}{2}\right]_2^5 + \left[\frac{z^2}{2}\right]_3^7$$

$$= \left(\frac{81}{4} - \frac{1}{4}\right) + \left(\frac{25}{2} - \frac{4}{2}\right) + \left(\frac{49}{2} - \frac{9}{2}\right)$$

$$= \frac{80}{4} + \frac{21}{2} + \frac{40}{2}$$

$$= \frac{202}{4} = 50.5 \text{ unit} \quad \text{A}$$

H.W
Q→ Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from $(0,0)$ to $(1,1)$ along $y^2 = x$. Ans $2/3$