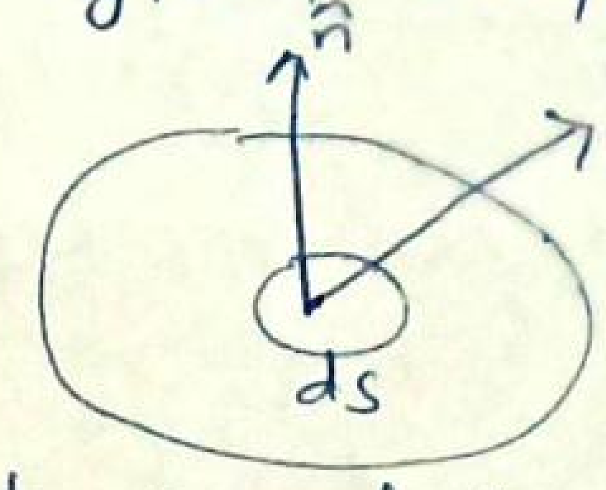
* Surface Integral :

Let $\vec{F}$ be a vector function and S be the given surface then surface integral of $\vec{F}$ over S



$$= \iint_S (\vec{F} \cdot \hat{n})\, ds$$

where $\hat{n}$ is unit normal to ds.

If $\iint_S (\vec{F} \cdot \hat{n})\, ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

* Volume Integral :

Let $\vec{F}$ be a vector point function and volume V enclosed surface. The volume integral $= \iiint_V \vec{F}\, dV$

* Green's theorem :

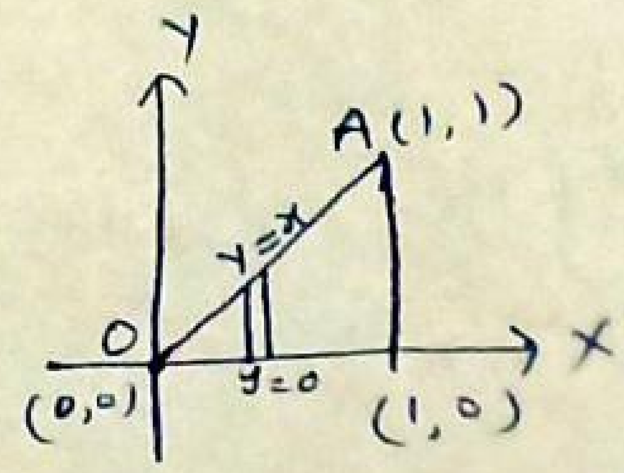
Statement :- If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in xy-plane, then

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

Probm - ① Using Green's theorem, evaluate $\int_C (x^2 y\, dx + x^2 dy)$, where c is the boundary described counter clockwise of the triangle with vertices (0,0), (1,0), (1,1).

Solution: By Green theorem, we have

$$\int_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$



$$\int_C (x^2 y\, dx + x^2 dy) = \iint_R (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, dx \int_0^x dy = \int_0^1 (2x - x^2)\, dx\, [y]_0^x$$

$$= \int_0^1 (2x - x^2)\, x\, dx = \int_0^1 (2x^2 - x^3)\, dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Ans

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Q> Use Green's theorem to evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by lines $y=\pm 1$, $x=\pm 1$.

Solution: $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$

By Green theorem, $\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$

here $\phi = x^2+xy \Rightarrow \frac{\partial \phi}{\partial y} = x$ & $\psi = x^2+y^2 \Rightarrow \frac{\partial \psi}{\partial x} = 2x$

then,

$$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{-1}^{+1}\int_{-1}^{+1} (2x - x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = \int_{-1}^{1} x\,dx \int_{-1}^{1} dy$$

$$= \int_{-1}^{1} x\,dx\,[y]_{-1}^{1} = \int_{-1}^{1} 2x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{-1}^{1} = (1)^2 - (-1)^2 = 0$$

Ans.

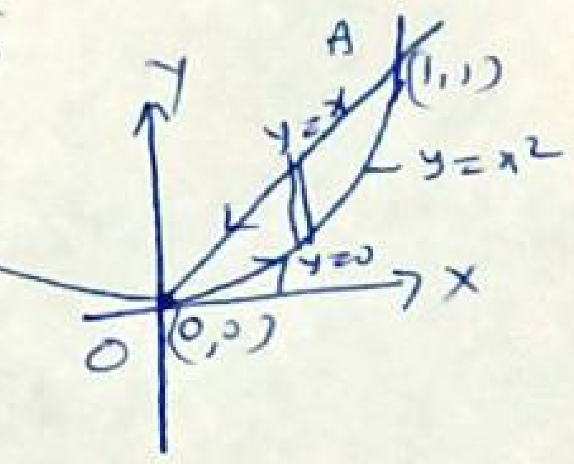
Prob^m - Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$, where C is bounded by $y=x$ and $y=x^2$.

Sol^n We have, Green's theorem

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\int_C (xy+y^2)\,dx + x^2\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy = \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1 dx \int_{x^2}^{x} (x - 2y)\,dy = \int_0^1 \left[xy - \cancel{x}\frac{y^2}{\cancel{x}}\right]_{x^2}^{x} dx$$



$$= \int_0^1 (x^2 - x^2) - (x\cdot x^2 - x^4)\,dx = \int_0^1 x^4 - x^3\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$$

Again

$$\int_C (xy+y^2)\,dx + x^2\,dy = \int_{\substack{\text{along } OA \\ y=x^2}} (xy+y^2)\,dx + x^2\,dy + \int_{\substack{\text{along } AO \\ y=x}} (xy+y^2)\,dx + x^2\,dy$$

$$= \int_0^1 (x\cdot x^2 + x^4)\,dx + x^2\cdot 2x\,dx + \int_1^0 (x^2+x^2)\,dx + x^2\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx + \int_1^0 3x^2\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx + \int_1^0 3x^2\,dx = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_1^0$$

$$= \frac{3}{4} + \frac{1}{5} - 1$$

$$= \frac{15+4-20}{20} = -\frac{1}{20}$$

Hence, Green theorem verify.