Stoke's theorem :- (Relation between Line Integral and Surface Integral)

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

Mathematically,
$$\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds}$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal to any surface ds.

Prob^m → (1) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by stoke's Theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is boundary of triangle with vertices at (0,0,0), (1,0,0) and (1,1,0).

Solution :- We have, curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$

$$\text{curl}\,\vec{F} = 0\cdot\hat{i} + \hat{j} + 2(x-y)\hat{k}$$

We observe that z-co-ordinate of each vertex of the triangle is zero. so the triangle is in xy-plane

$$\therefore \quad \underline{\hat{n} = \hat{k}}$$

$$\therefore \quad \text{curl}\,\vec{F}\cdot\hat{n} = \hat{j} + 2(x-y)\hat{k}\cdot\hat{k} = 2(x-y)$$

Now,
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}\cdot\hat{n})\, ds$$
$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\, dx\, dy$$
$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$$
$$= 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right) dx = 2\int_0^1 \frac{x^2}{2}\, dx$$
$$= \frac{\cancel{2}}{\cancel{2}}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \quad \underline{\text{Ans}}$$

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Q→ Apply stoke's theorem to find the value of $\int_C (y\,dx + z\,dy + x\,dz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.

Solution: By stoke's theorem

$$\int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_C (y\hat{i} + z\hat{j} + x\hat{k}) \cdot d\vec{r}$$

$$= \iint_S \text{curl}\,(y\hat{i} + z\hat{j} + x\hat{k}) \cdot \hat{n}\,ds \quad \text{—— (1)}$$

where S is the circle formed by intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+z-a)}{|\nabla\phi|}$$

$$\hat{n} = \frac{\hat{i}+\hat{k}}{\sqrt{1+1}} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}} \quad \text{—— (2)}$$

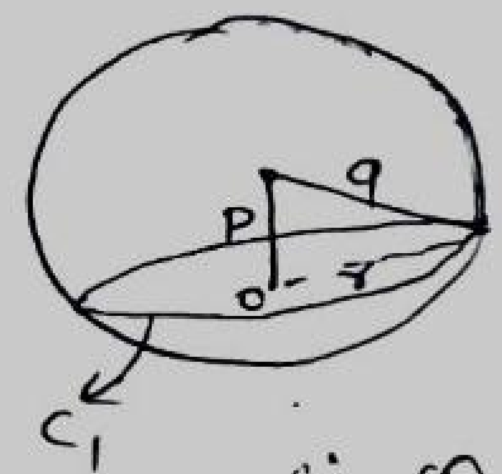

fig (1)

and $$\text{curl}\,(y\hat{i} + z\hat{j} + x\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k} \quad \text{—— (3)}$$

From (1), (2) and (3), we have,

$$\int_C y\,dx + z\,dy + x\,dz = \iint_S -(\hat{i}+\hat{j}+\hat{k}) \cdot \left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}\right) ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) ds$$

$$= \frac{-2}{\sqrt{2}} \iint_S ds$$

$$= -\frac{2}{\sqrt{2}} (\text{area of circle})$$

$$= -\frac{2}{\sqrt{2}} \left(\pi \frac{a^2}{2}\right)$$

$$= -\frac{\pi a^2}{\sqrt{2}} \quad \text{Ans}$$

From fig

$$\left\{ \because r^2 = R^2 - p^2 = a^2 - \frac{a^2}{2}, \quad r^2 = a^2/2 \right\}$$

* Gauss's theorem of Divergence :-

The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically,
$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

Q.1 use Divergence theorem to evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$, and $z=3$.

Solution: By Gauss's Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\,dV$$

$$= \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_0^3 (4 - 4y + 2z)\,dz$$

$$= \iint dx\,dy \left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3$$

$$= \iint (12 - 12y + 9)\,dx\,dy = \iint (21 - 12y)\,dx\,dy$$

let put $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 (21 - 12r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^2 (21r - 12r^2\sin\theta)\,dr = \int_0 d\theta \left[\frac{21r^2}{2} - \frac{12r^3\sin\theta}{3}\right]_0^2$$

$$= \int_0^{2\pi} (42 - 32\sin\theta)\,d\theta = \left[42\theta + 32\cos\theta\right]_0^{2\pi}$$

$$= (84\pi + 32\cos 2\pi) - (0 + 32) = \underline{84\pi} \quad \text{A}$$

Q-1) Evaluate by Gauss's divergence theorem $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

Solution: By Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_V (\nabla\cdot\vec{F})\, dV$$

$$= \iint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}\right) dV$$

$$= \iiint_V \left[\frac{\partial (4xz)}{\partial x} + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - 2y + y)\, dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z - \cancel{y})\, dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(4\frac{z^2}{2} - yz\right)_0^1 dx\,dy$$

$$= \int_0^1\int_0^1 (2-y)\, dx\,dy = \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}[x]_0^1$$

$$= 3/2$$

Ans