* Divergence of a Vector function :-

The divergence of a vector point function $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is denoted by $\operatorname{div}\vec{F}$ and defined as

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

($\operatorname{div}\vec{F}$ is a scalar function)

* Solenoidal Vector function :

A vector function $\vec{V}$ is called solenoidal vector function if $\operatorname{div}\vec{V} = 0$

Q:- If $u = x^2 + y^2 + z^2$, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\operatorname{div}(u\vec{r})$ in terms of u.

Solution:- $\operatorname{div}(u\vec{r}) = \nabla \times u\vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(x^2+y^2+z^2)(x\hat{i} + y\hat{j} + z\hat{k})\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x(x^2+y^2+z^2)\hat{i} + y(x^2+y^2+z^2)\hat{j} + z(x^2+y^2+z^2)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^3+xy^2+xz^2) + \frac{\partial}{\partial y}(x^2y+y^3+yz^2) + \frac{\partial}{\partial z}(x^2z+y^2z+z^3)$$

$$= (3x^2+y^2+z^2) + (x^2+3y^2+z^2) + (x^2+y^2+3z^2)$$

$$= 5(x^2+y^2+z^2)$$

$$= 5u \qquad \underline{\text{ans}}\ 5u$$

Q:- Find the value of n for which the vector $r^n\vec{r}$ is solenoidal, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution:- We know that

$$\operatorname{Div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot r^n\vec{r} = \nabla\cdot(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x^2+y^2+z^2)^{n/2}\cdot x\hat{i} + (x^2+y^2+z^2)^{n/2}y\hat{j} + (x^2+y^2+z^2)^{n/2}z\hat{k}\right]$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2x^2 + (x^2+y^2+z^2)^{n/2} + \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2y^2$$
$$+ (x^2+y^2+z^2)^{n/2} + \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\cdot 2z^2 + (x^2+y^2+z^2)^{n/2}$$

$$\text{div}\,\vec{F} = n(x^2+y^2+z^2)^{n/2-1}(x^2+y^2+z^2) + 3(x^2+y^2+z^2)^{n/2}$$
$$= n(x^2+y^2+z^2)^{n/2} + 3(x^2+y^2+z^2)^{n/2}$$
$$= (n+3)(x^2+y^2+z^2)^{n/2}$$

If $\vec{F} = r^n\vec{r}$ is solenoidal then $\text{div}\,\vec{F} = 0$

$$\Rightarrow (n+3)(x^2+y^2+z^2)^{n/2} = 0$$
$$\Rightarrow n+3 = 0 \quad (\because x^2+y^2+z^2 \neq 0)$$
$$\Rightarrow \boxed{n=-3}$$

Ans $n = -3$

Q→ Show that $\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$.

Solution:- We have,

$$\frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{n/2}}$$
$$= \frac{a_1x+a_2y+a_3z}{r^n}$$

let $\phi = \frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{r^n a_1 - (a_1x+a_2y+a_3z)\cdot nr^{n-1}\left(\frac{\partial r}{\partial x}\right)}{r^{2n}}$$

$\left\{ \because r^2 = x^2+y^2+z^2 \Rightarrow \frac{\partial r}{\partial x} = x/r \right.$

$$= \frac{a_1}{r^n} - \frac{n(a_1x+a_2y+a_3z)}{r^{2n}}\cdot r^{n-1}\left(\frac{x}{r}\right)$$
$$= \frac{a_1}{r^n} - \frac{n(a_1x+a_2y+a_3z)\cdot x}{r^{n+2}}$$

$$\therefore \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \frac{1}{r^n}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) - \frac{n}{r^{n+2}}\left[(a_1x+a_2y+a_3z)(x\hat{i}+y\hat{j}+z\hat{k})\right]$$

$$\nabla\phi = \frac{1}{r^n}\cdot\vec{a} - \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\cdot\vec{r}$$

$$\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \frac{\vec{a}}{r^n} - \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\vec{r}$$

Proved

Q:- Find the directional derivative of divergence of $x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$ at the point $(1,2,2)$ in the direction of the outer normal of the sphere $x^2+y^2+z^2=9$.

Solution: $\text{div}(\vec{u}) = \nabla\cdot\vec{u} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}\right)$

$= 4x^3 + 4y^3 + 4z^3$

outer normal of sphere $= \nabla(x^2+y^2+z^2-9)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

outer normal of sphere at $(1,2,2) = 2\hat{i} + 4\hat{j} + 4\hat{k}$

Directional derivative of $\text{div}(\vec{u}) = \nabla(\text{div}\,\vec{u})$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^3+4y^3+4z^3)$

$= 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$

Directional derivative at $(1,2,2) = 12\hat{i} + 48\hat{j} + 48\hat{k}$

Directional derivative along the outer normal

$= (12\hat{i} + 48\hat{j} + 48\hat{k})\cdot\dfrac{2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{4+16+16}}$

$= \dfrac{12\times2 + 48\times4 + 48\times4}{\sqrt{36}} = \dfrac{24+192+192}{6}$

$= \dfrac{408}{6} = 68$ Ans 68

Q:- If $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

show that (i) $\text{div}\left(\dfrac{\vec{r}}{r^2}\right) = \dfrac{1}{r^2}$ (ii) $\text{div}\left(\dfrac{\vec{r}}{|\vec{r}|^3}\right) = 0$

Solution: $r^2 = x^2+y^2+z^2$

$\Rightarrow \dfrac{\partial r}{\partial x} = x/r,\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r},\quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$\text{div}\left(\dfrac{\vec{r}}{r^2}\right) = \nabla\cdot\dfrac{\vec{r}}{r^2} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{r^2}\right) = \dfrac{\partial(x/r^2)}{\partial x} + \dfrac{\partial(y/r^2)}{\partial y} + \dfrac{\partial(z/r^2)}{\partial z}$

$= \dfrac{r^2\cdot1 - x\cdot2r\frac{\partial r}{\partial x}}{r^4} + \dfrac{r^2\cdot1 - y\cdot2r\frac{\partial r}{\partial y}}{r^4} + \dfrac{r^2\cdot1 - z\cdot2r\frac{\partial r}{\partial z}}{r^4}$

$= \dfrac{1}{r^4}\left[3r^2 - 2\left(x\cdot\cancel{r}\cdot\dfrac{x}{\cancel{r}} + y\cdot\cancel{r}\cdot\dfrac{y}{\cancel{r}} + z\cdot\cancel{r}\cdot\dfrac{z}{\cancel{r}}\right)\right]$

$= \dfrac{1}{r^4}\left[3r^2 - 2(x^2+y^2+z^2)\right] = \dfrac{1}{r^4}(3r^2-2r^2) = \dfrac{1}{r^2}$ A

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Q→ Show that $\text{div}(\text{grad}\, r^n) = n(n+1)\, r^{n-2}$,
where $r = \sqrt{x^2+y^2+z^2}$. Hence, show that $\nabla^2\left(\frac{1}{r}\right)=0$

Solution:

$$\text{grad}\,(r^n) = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}.\, n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}.\, n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}.\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$

Thus,

$$= n r^{n-1}\frac{\vec{r}}{r} = n r^{n-2}\vec{r} \quad \text{Proved}$$

$$\text{grad}\, r^n = n r^{n-2} x\hat{i} + n r^{n-2} y\hat{j} + n r^{n-2} z\hat{k}$$

$$\therefore \text{div grad}\, r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(n r^{n-2} x\hat{i} + n r^{n-2} y\hat{j} + n r^{n-2} z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(n r^{n-2} x) + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z)$$

$$= \left(n r^{n-2} + n(n-2) r^{n-3}\cdot x\frac{\partial r}{\partial x}\right) + \left(n r^{n-2} + n(n-2) r^{n-3}\cdot y\frac{\partial r}{\partial y}\right) + \left(n r^{n-2} + n(n-2) r^{n-3} z\frac{\partial r}{\partial z}\right)$$

$$= 3n r^{n-2} + n(n-2) r^{n-3}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-3}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-3}\left[\frac{x^2+y^2+z^2}{r}\right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-3}\cdot\frac{r^2}{r}$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= r^{n-2}\left[3n + n^2 - 2n\right]$$

$$\left[\text{div grad}\, r^n = n(n+1)\, r^{n-2}\right] \quad \text{Proved}$$

putting $n = -1$

$$\text{div grad}\, r^{-1} = -1(-1+1)\, r^{-1-2}$$

$$\text{div}\, \nabla\left(\frac{1}{r}\right) = 0$$

$$\nabla\cdot\nabla\left(\frac{1}{r}\right) = 0$$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = 0$$

Q→ Show that the vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.