* Matrix *

An arrangements of m, n numbers in form of Rectangular array in m row & n column is said to be matrix of order 'm' by n'.

E.g — $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3}$ $[a_{ij}]_{m\times n}$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

Types of Matrix

① **Row Matrix** :- A matrix having only one row & n- column is said to be row matrix.

E.g → $[1 \ 2 \ 3]_{1\times 3}$ $[1 \ 0]_{1\times 2}$ $[1]_{1\times 1}$

② **Column Matrix** :- A matrix having only one column and n numbers of Rows is said to be Column Matrix.

E.g :- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$ $[1]_{1\times 1}$

③ **Square Matrix**

A Matrix having equal No. of rows & column is said to be Square Matrix $(m=n)$

Eg :- $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{3\times 3}$

④ **Diagonal Matrix**

A Square Matrix having all non-diagonal elements equals to zero is said to be diagonal matrix.

Eg :- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2}$

⑤ **Zero / Null Matrix**

A matrix having all elements are equal to zero is said to be Zero Matrix.

Eg :- $[0]_{1\times 1}$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

(6) Identity (Unit) Matrix

A matrix having its all non-diagonal elements are equal to zero and diagonal equal to 1. It is denoted by I.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) Scalar Matrix

A Square Matrix having all non-diagonal element are equal to zero and all diagonal elements are same is said to be Scalar matrix.

*** All Scalar matrix are not Identity Matrix but all Identity Matrix are Scalar.

Eg:- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$

Upper triangular Matrix :-

A Square matrix having all elements below the diagonal are zero is called Upper triangular Matrix.

Eg:- $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}_{2\times 2}$

Lower triangular Matrix

A Square matrix having all elements above the diagonal are zero is called lower triangular Matrix.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}_{3\times 3}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Transport Matrix

Transport of a Matrix is obtained by inter changing the rows and columns and denoted by A' and A^T.

Eg: $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix}_{3\times3}$ $\quad A^T = \begin{bmatrix} 1 & 4 & 1 \\ -2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}_{3\times3}$

$B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}_{3\times2}$ $\quad \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}_{2\times3}$

Symmetric Matrix

A Square Matrix is said to be symmetric if $A = A^T$.

Eg: $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$ $\quad A' = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 5 \\ 2 & 5 & 4 \end{bmatrix}$

$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

Skew Symmetric Matrix

A Square Matrix is said to be Skew matrix if

$A' = -A$ or $A = -A'$.

Eg $\Rightarrow \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & -4 \\ 1 & -4 & 0 \end{bmatrix}_{3\times3}$

$A' = \begin{bmatrix} 0 & 2 & \cancel{4}\,1 \\ -2 & 0 & -4 \\ 1 & -4 & 0 \end{bmatrix}_{3\times3}$, $\quad A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -4 \\ 1 & -4 & 0 \end{bmatrix}_{3\times3}$

Orthagonal Matrix

A matrix is said to be orthogonal matrix if $A \cdot A' = I$

Algebra of Matrix

(1) Addition of Matrix

$$[A]_{m \times n} + [B]_{m \times n} = [C]_{m \times n}$$

Eg:- $\begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}_{2 \times 3} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}_{2 \times 3} = \begin{bmatrix} 2+1 & 3+2 & -1+3 \\ 2+1 & 1+0 & 0+4 \end{bmatrix}_{2 \times 3}$

$$= \begin{bmatrix} 3 & 5 & 2 \\ 3 & 1 & 4 \end{bmatrix}_{2 \times 3}$$

(2) Scalar Multiplication

$$K \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} Ka & Kb & Kc \\ Kd & Ke & Kf \end{bmatrix}$$

(3) Matrix Multiplication

$$[A]_{m \times n} [B]_{n \times p} = [C]_{m \times p}$$

The no of column in A = no. of rows in B.

Eg:- $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}_{3 \times 2}$

$$\begin{bmatrix} 1 \times 1 + 2 \times 3 + 0 \times 2 & 1 \times 2 + 2 \times 1 + 0 \times 2 \\ 2 \times 1 + (-1) \times 3 + 1 \times 2 & 2 \times 2 - 1 \times 1 + 1 \times 2 \\ 0 \times 1 + 1 \times 3 + 2 \times 2 & 0 \times 2 + 1 \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 1 & 5 \\ 7 & 5 \end{bmatrix}_{3 \times 2}$$

(ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}_{2 \times 2} \quad \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}_{2 \times 2}$

$$\begin{bmatrix} 1 \times 1 + (-1) \times 4 & 1 \times 3 + (-1) \times 1 \\ 2 \times 1 + 3 \times 4 & 2 \times 3 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 14 & 9 \end{bmatrix}_{2 \times 2}$$

(iii) $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}_{3 \times 1} \quad \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}_{1 \times 3}$

$$\begin{bmatrix} 1 \times 1 & 1 \times 0 & 1 \times 2 \\ 3 \times 1 & 3 \times 0 & 3 \times 2 \\ 1 \times 1 & 1 \times 0 & 1 \times 2 \end{bmatrix}_{3 \times 3} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 6 \\ 1 & 0 & 2 \end{bmatrix}_{3 \times 3}$$

Equality of Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A \neq B$$
$$A = C$$

Conjugate of Matrix

Conjugate of a Matrix A is denoted by $\overline{A}$

$$A = \begin{bmatrix} a+ib & c+id \\ id & a-ie \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} a-ib & c-id \\ -id & a+ie \end{bmatrix}$$

$$A = \begin{bmatrix} 2+i & 3-i \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2-i & 3+i \\ -1 & 0 \end{bmatrix}$$

Transpose of Conjugate of Matrix

It is denoted A^{θ} and given by $\left[A^{\theta} = (\overline{A})'\right]$

Eg :- $A = \begin{bmatrix} 2+i & 3-i \\ -1 & 0 \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 2-i & 3+i \\ -1 & 0 \end{bmatrix} = A^{\theta} = (\overline{A})' = \begin{bmatrix} 2-i & -1 \\ 3+i & 0 \end{bmatrix}$$

Hermitian Matrix

A Square Matrix is said to be Hermitian matrix if $A = A^{\theta}$ or $A = (\overline{A})'$.

$$A = \begin{bmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & -i \\ 2+i & i & 3 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 1 & 1-i & 2+i \\ 1+i & 2 & i \\ 2-i & -i & 3 \end{bmatrix}$$

$$A^{\theta} = (\overline{A})' = \begin{bmatrix} 1 & 1+i & 2-i \\ 1-i & 2 & -i \\ 2+i & i & 3 \end{bmatrix}$$

Skew Hermitian Matrix

A square matrix is said to be skew Hermitian.
if $A = -A^\theta$ or $A^\theta = -A$

Eg:- $A = \begin{bmatrix} 0 & 2+i & 1-i \\ -2+i & 0 & 3+i \\ -1-i & -3+i & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 0 & 2-i & 1+i \\ -2-i & 0 & 3-i \\ -1+i & -3-i & 0 \end{bmatrix}$

$A^\theta = (\bar{A})' = \begin{bmatrix} 0 & -2-i & -1+i \\ 2-i & 0 & -3-i \\ 1+i & 3-i & 0 \end{bmatrix}$, $-A^\theta = \begin{bmatrix} 0 & 2+i & 1-i \\ -2+i & 0 & 3+i \\ -1-i & -3+i & 0 \end{bmatrix} = A$

Idempotent Matrix $A^2 = A$ Eg $= I^2 = I$

Nilpotent Matrix $A^k = 0$
where k is Index of Nilpotent Matrix

Eg:- $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. $A \cdot A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$A^2 = 0$
where 2 is index of Nilpotent Matrix.

Involuntary Matrix $\boxed{A^2 = I}$

Determinant of a Matrix

If A is a matrix then determinant of A is denoted by $|A|$.

① $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $|A| = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = 2 \times 4 - 1 \times 3 = 5$

Singular Matrix

A Square matrix is said to be Singular
if $|A| = 0$

Eg:- $A = \begin{bmatrix} 4 & 4 & 3 \\ 7 & 4 & 5 \\ 8 & 8 & 6 \end{bmatrix}$

$|A| = 0$

Non-Singular ~~x~~ Matrix

A Square matrix is said to be Non-Singular if $|A| \neq 0$

Sub Matrix

A matrix is obtained by deleting some rows or columns or both is called submatrix of given matrix.

Eg :- $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times 2} \quad \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}_{2\times 2}$

$\begin{bmatrix} 2 & 0 & 3 \end{bmatrix}_{1\times 3} \quad \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}$

Minor :- The determinant of any Square matrix of a matrix is called Minor.

E.g :- $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix}_{3\times 3} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}_{2\times 2} \quad |B| = 1\times 0 - 2\times 3 = -6$

$|B|$ is minor of A.

Rank of a Matrix :- The number 'r' is said to rank of the matrix A if

(i) There is at least one non-zero minor of matrix, A of order 'r'.

(ii) Every minor of order greater than 'r' is zero.

e.g :- (i) $A = \begin{bmatrix} 4 & 4 & 3 \\ 7 & 4 & 5 \\ 8 & 8 & 6 \end{bmatrix} \quad |A| = 0$

$|B| = \begin{vmatrix} 4 & 4 \\ 7 & 4 \end{vmatrix} = 16 - 28 = -12 \neq 0$

rank of $A = 2$.

Elementry row (column) operation →

There are mainly three elementry row (column) operation -

(i) $R_i \longleftrightarrow R_j \quad (C_i \longleftrightarrow C_j)$

(ii) $R_i \longrightarrow K R_i \quad (C_i \longrightarrow K C_i)$

(iii) $R_i \rightarrow R_i + K R_j \quad (C_i \rightarrow C_i + K C_j)$

⇒ **Echelon form of a matrix :-** A matrix A is said to in Echelon form.

(i) Every row of A which has all its entries '0' occurs below every row which has a non-zero entry.

(ii) The first non-zero entry in each non-zero row is equal to 1.

(iii) The no. of zero's before the first non-zero element in a row is less than the no. of such zero's in the next row.

Ex :- $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4}$, $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

All Identity matrix is a Echelon matrix.

The rank of matrix in Echelon form defn :- The number of non-zero rows in echelon form is said to be rank of matrix.

Q. Reduce the matrix in echelon form and find its rank

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Soln We have,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}, \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to -\frac{1}{2} R_3$$

This is the required echelon form & rank of $A = \rho(A) = 3$ Ans

Q. Reduce the matrix in echelon form and find the rank of matrix.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

$$A \sim \begin{bmatrix} -1 & 2 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \quad R_3 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 7 & 8 \\ 0 & 7 & 8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 8/7 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to -R_1 \\ R_2 \to \frac{1}{7} R_2 \end{matrix}$$

This is the in echelon form.

Rank of $A = 2$ Ans.

Q. $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$ Reduce the matrix in echelon form and find its rank.

Solⁿ $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 4 & -4 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ $R_2 \to \frac{1}{2} R_2$, $R_3 \to \frac{1}{4} R_3$

$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

This is in echelon form.

rank of $(A) = 2$

Q. Reduce the matrix in echelon form.

$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Solⁿ We have, $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + R_1$.

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 - R_2 \\ R_3 \to \frac{1}{4} R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - 5R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 4R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form
& Rank $(A) = 3$

Normal form of a matrix :- A matrix can be reduced by row and column elementary transformation into one of the following form.

(i) $\begin{bmatrix} I_n & O \\ O & O \end{bmatrix}$ (ii) $\begin{bmatrix} I_n \\ O \end{bmatrix}$ (iii) $\begin{bmatrix} I_n & O \end{bmatrix}$ (iv) $\begin{bmatrix} I_n \end{bmatrix}$

where I_n is identity matrix of order 'n'.
and O is zero matrix
This form is called normal form (Canonical form) and rank of matrix is n.

1. Reduce the matrix in normal form and find its rank

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

Solⁿ we have,

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1,\ R_3 \to \frac{1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2,\ C_4 \to C_3 + C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_1,\ C_4 \to C_4 - C_1$$

$$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

This is required normal form of A.

$\boxed{\text{Rank}(A) = 2}$

(ii) $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

We have,

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 4C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 - R_2 \\ R_3 \to \frac{1}{4} R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - 5R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 4R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_4 \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

rank(A) = 3

Types of system of linear equation :-

(i) consistent :- A system of linear eq^n is said to be consistent if it has one or more than one sol^n.

Ex :- $x - 2y = -3$
$x + y = 3$

(ii) $x + 3y = 4$
$2x + 6y = 8$

(ii) inconsistent :- A system of linear eq^n is said to Inconsistent if it has no sol^n.

ex :- $x + 4y = 6$
$x + 4y = 9$

Consistency of system of linear eq^n :-

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}\right] \text{— } (1)$$

The matrix form of (1) $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\Rightarrow$ Augmented matrix is $C = [A : B]$

Consistent eqn.: If rank A = Rank C

(a) Unique soln: If rank A = rank C = n (no. of unknown variable)

(b) Infinite soln: If rank A = rank C < n

(ii) Inconsistent Eqn.: If rank A ≠ rank C.

Q. Discuss the consistency of system of linear eqn:

$$2x+6y=-11$$
$$6x+20y-6z=-3$$
$$6y-18z=-1$$

Soln.

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 0 & -11/2 \\ 0 & 1 & -3 & 15 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_3 \to \frac{-1}{91}R_3 \end{array}$$

⇒ Rank A = 2 & rank C = 3

i.e rank A ≠ Rank C

So the given system of linear eqn is not consistent.

Q. Discuss the consistency of system of linear eq^n and find its sol^n.

$$2x + 3y + 4z = 11$$
$$x + 5y + 7z = 15$$
$$3x + 11y + 13z = 25$$

Sol^n

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$$

$$C = [A : B]$$

$$C = \begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{bmatrix} \quad R_2 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{bmatrix} \quad R_3 \to -\frac{1}{4} R_3$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 0 & 4 & 16 \end{bmatrix} \quad R_3 \to 7R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & 10/7 & 19/7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 \to -\frac{1}{7} R_2 \\ R_3 \to \frac{1}{4} R_3 \end{array}$$

$\Rightarrow$ rank $A = 3$ & rank $C = 3$

rank A = rank B = 3 = no. of unknown Variable

$\Rightarrow$ Given system of linear eqn is consistent & unique soln.

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 10/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19/7 \\ 4 \end{bmatrix}$$

$x + 5y + 7z = 15$ — (1)

$y + \frac{10}{7}z = \frac{19}{7}$ — (2)

$\boxed{z = 4}$ — (3)

$y + \frac{10 \times 4}{7} = \frac{19}{7}$

$y = \frac{19}{7} - \frac{40}{7} = -3$

$x + 5 \times -3 + 7 \times 4 = 15$

$x - 15 + 28 = 15$

$x = 15 + 15 - 28$

$\boxed{x = 2}$

Q. Investigate the value of λ and μ for the equations

$2x + 3y + 5z = 9$

$7x + 3y - 2z = 8$

$2x + 3y + \lambda z = \mu$

have (i) no soln (ii) a unique soln (iii) an infinite no. of soln.

Soln.

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 & -5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & \frac{-47}{2} \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \quad \begin{bmatrix} R_2 \to R_2 - \frac{7}{2}R_1 \\ R_3 \to R_3 - R_1 \end{bmatrix}$$

(i) for no solⁿ:

rank $A \neq$ rank C

for the $\lambda = 5$, rank $A = 2$

$\mu \neq 9$, rank $C = 3$

for no solⁿ $\lambda = 5$ and $\mu \neq 9$ Ans

(ii) for unique solⁿ:

rank A = rank $C = 3$

for $\lambda \neq 5$, rank $A = 3 =$ rank C

for $\lambda \neq 5$, it has unique solⁿ.

(iii) for infinite no of solⁿ:

rank (A) = rank $C < 3$

For $\lambda = 5$, $\mu = 9$

For $\lambda = 5$, $\mu = 9$ it has infinite number of solutions.

Q.

$$x + y + z = 6$$
$$x + 2y + 5z = 10$$
$$2x + 3y + \lambda z = \mu$$

(i) No solⁿ (ii) a unique solⁿ (iii) an infinite no. of solⁿ

Solⁿ

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 5 & 10 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 1 & \lambda-2 & \mu-12 \end{bmatrix} \qquad \begin{bmatrix} R_3 \to R_3 - 2R_1 \\ R_2 \to R_2 - R_1 \end{bmatrix}$$

(i) ~~For no sol^n~~

~~rank A~~ $\neq$ rank C

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & \lambda-6 & \mu-16 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

(i) for no sol^n

rank$(A) \neq$ rank(C), then rank$(A) = 2$ and rank$(C) = 3$

$\lambda - 6 = 0$ & $\mu - 16 \neq 0$

$\lambda = 6$ $\qquad \mu \neq 16$

(ii) for unique sol^n

rank(A) = rank$(C) = 3 =$ no. of unknowns

then $\lambda - 6 \neq 0$ ~~&~~ μ can have any value.

$\lambda \neq 6$

(iii) Infinite sol^n

rank(A) = rank$(C) = 2 < 3 =$ no. of unknowns

then $\lambda - 6 = 0$ & $\mu - 16 = 0$

$\lambda = 6$ $\qquad \mu = 16$

System of Linear Equations

Homogenous eq^n — $\boxed{AX = 0}$

Non-Homogenous sol^n. — $\boxed{AX = B}$, $B \neq 0$

Homogenous system of Linear equation : $(AX=0)$

A homogenous system of Linear equation $AX=0$ has

(i) $X=0$ is always a sol^n of $AX=0$ and it is called zero sol^n or trivial solution.

(ii) If rank(A) = number of unknown variable then it has only trivial sol^n.

(iii) If rank(A) < no. of unknown variable then it has infinite number non-trivial sol^n.

Q. Solve the homogenous sol' eq^n.

$$4x+3y-z=0$$
$$3x+4y+z=0$$
$$x-y-2z=0$$
$$\cancel{5x-4y-4z=0} \quad 5x+y-4z=0$$

Sol^n

$$\begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 1 & -1 & -2 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 3 & -1 & 0 \\ 3 & 4 & 1 & 0 \\ 1 & -1 & -2 & 0 \\ 5 & 1 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 3 & -1 & 0 \\ 5 & 1 & -4 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 6 & 6 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 5R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{1}{7} R_2 \\ R_3 \to \frac{1}{7} R_3 \\ R_4 \to \frac{1}{6} R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$\Rightarrow$ Rank $(A) = 2 < 3$

So, it has infinite number of non-trivial solution.

$\cancel{z - y - 2x = 0}$

$\cancel{y + x = 0}$

$\cancel{x = -y = k}$

$\cancel{\Rightarrow z - (-k) - 2k = 0}$

$\cancel{z + k - 2k}$

$\Rightarrow x - y - 2z = 0$

$y + z = 0$

$y = -z = k$

$\boxed{z = -k}$

$x - k - 2(-k) = 0$

$x - k + 2k = 0$

$\boxed{x = -k}$

$x = -k$

$y = k$

$z = -k$

Q. Find the value of k for which the system of linear equation

$$x + ky + 3z = 0$$
$$4x + 3y + kz = 0$$
$$2x + y + 2z = 0$$

has non-trivial solⁿ.

Solⁿ.

$$\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & k & 3 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 4 & 3 & k & 0 \\ 1 & k & 3 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & 2k-1 & 4 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & k-\frac{1}{2} & 2 & 0 \end{bmatrix} \quad R_3 \to \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & 1 & \frac{2}{k-1/2} & 0 \end{bmatrix} \quad R_3 \to \frac{1}{k-1/2} R_3$$

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & 0 & \frac{2}{k-1/2} - (k-4) & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

For rank $(A) = 2$

$$\frac{2}{k-1/2} - (k-4) = 0$$

$$\frac{2-(k-4)(k-1/2)}{k-1/2} = 0$$

$$2 - k^2 + \frac{1}{2}k + 4k - \frac{4}{2} = 0$$

$$2 - k^2 + \frac{9}{2}k - \frac{4}{2} = 0$$

$$-k^2 + \frac{9}{2}k = 0$$

$$k\left(\frac{9}{2} - k\right) = 0$$

$\boxed{k = 0}$, $\frac{9}{2} - k = 0$

$\boxed{k = 9/2}$

Q. For what value of λ following equations have a solⁿ and find the values of x, y and z for each λ.

$$x + y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 10z = \lambda^2$$

Solⁿ

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 1 & 3 & \frac{\lambda^2-1}{3} \end{bmatrix} \quad R_3 \to \frac{1}{3}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \left(\frac{\lambda^2-1}{3}\right)-(\lambda-1) \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\left(\frac{\lambda^2-1}{3}\right)-(\lambda-1) = 0$$

$$\frac{(\lambda^2-1)-3(\lambda-1)}{3} = 0$$

$$(\lambda-1)[(\lambda-1)-3] = 0 \qquad \lambda^2-1-3\lambda+3=0$$

$$\lambda^2-3\lambda+2=0$$

$$\lambda^2-2\lambda-\lambda+2=0$$

$$(\lambda-1)(\lambda-2)=0$$

$$\boxed{\lambda=1,2}$$

rank $(A) = 2$.

rank $(C) = 2$.

$\boxed{\lambda=1}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x+y+z=1$$
$$y+3z=0$$
$$y=-3z=k$$
$$\boxed{z=-k/3}$$

$$x+y+z=1$$
$$x+k-\frac{k}{3}=1$$
$$x-\frac{2k}{3}=1$$
$$\boxed{x=1+\frac{2k}{3}}$$

$$x=1+\frac{2k}{3},\ y=k,\ z=-k/3$$

For $\lambda=2$

$$x+y+z=1$$
$$y+3z=1$$
$$y=1-3z \quad (z=k)$$
$$y=1-3k$$
$$x+1-3k+k=1$$
$$x-2k=0$$
$$\boxed{x=2k}$$

$$x=2k,\ y=1-3k,\ z=k$$

Q. Solve the following Equations

$$x+y+z=8$$
$$x-y+2z=6$$
$$9x+5y-7z=14$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 9 & 5 & -7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 8 \\ 1 & -1 & 2 & 6 \\ 9 & 5 & -7 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & -4 & -16 & 58 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & +2 & +8 & 29 \end{bmatrix} \quad R_3 \to \frac{1}{-2} R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 9 & 27 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{1}{-2} R_2 \\ R_3 \to \frac{1}{9} R_3 \end{array}$$

72
14
58

-7-9

rank (A) = 3
rank (C) = 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}$$

$$x+y+z=8$$
$$y - \frac{z}{2} = 1$$
$$\boxed{z = 3}$$

$$y - \frac{3}{2} = 1$$
$$y = 1 + \frac{3}{2}$$
$$\boxed{y = 5/2}$$

$$x + \frac{5}{2} + 3 = 8$$
$$x = 8 - 3 - \frac{5}{2}$$
$$\boxed{x = \frac{10-5}{2} = 5/2}$$

Q. $3x + 2y + 4z = 7$

$2x + y + z = 4$

$x + 3y + 5z = 2$

Solⁿ.

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 4 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 2 & 1 & 1 & 4 \\ 3 & 2 & 4 & 7 \end{bmatrix} R_1 \leftrightarrow R_3 \quad \sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 2 & 1 & 1 & 4 \\ 3 & 2 & 4 & 7 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & -5 & -9 & 0 \\ 0 & -7 & -9 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & -5 & -9 & 0 \\ 0 & -7 & -9 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 9/5 & 0 \\ 0 & 1 & 11/7 & -1/7 \end{bmatrix} \begin{matrix} R_2 \to \frac{1}{-5} R_2 \\ R_3 \to \frac{1}{-7} R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 9/5 & 0 \\ 0 & 0 & -8/35 & -1/7 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 2 \\ 0 & 1 & 9/5 & 0 \\ 0 & 0 & 1 & 5/8 \end{bmatrix} R_3 \to \frac{-35}{8} R_3$$

rank(A) = 3, rank(C) = 3

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 9/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5/8 \end{bmatrix}$$

$x + 3y + 5z = 2$

$y + \frac{9}{5} z = 0$

$\boxed{z = 5/8}$

$y + \frac{9}{5} \times \frac{5}{8} = 0 \quad \boxed{y = -9/8}$

$x - 3 \times \frac{9}{8} + 5 \times \frac{5}{8} = 2$

$x - \frac{27}{8} + \frac{25}{8} = 2$

$x - \frac{2}{8} = 2$

$x = 2 + \frac{2}{8}$

$x = \frac{16+2}{8} = \frac{18}{8} = \frac{9}{4}$

$\boxed{x = \frac{9}{4}}$

Q. $x+y+z=3$
$x+2y+3z=4$
$x+4y+9z=6$

Sol^n

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 8/3 & 1 \end{bmatrix} \quad R_3 \to \frac{1}{3} R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2/3 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \to \frac{3}{2} R_3$$

Rank$(A) = 3$, Rank $(C) = 3$

$$\begin{bmatrix} 1 & 1 & 1 & \cancel{3} \\ 0 & 1 & 2 & \cancel{1} \\ 0 & 0 & 1 & \cancel{0} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$x+y+z=3$
$y+2z=1$
$z=0$

$y+2\times 0=1$
$y=1$

$x+1+0=3$
$x=3-1$

$\boxed{x=2}$

Linear dependence and Independence Vectors:

Vectors (Matrices) $X_1, X_2 \ldots\ldots X_n$ are said to linearly dependent if:

(i) all the vectors (matrices) are of the same order.

(ii) and $\exists$ → (there exist) $r_1, r_2 \ldots\ldots r_n$ s.t. (such that) where all $r_1 \neq 0$

$$r_1 x_1 + r_2 x_2 + \ldots\ldots r_n x_n = 0$$

otherwise they are linearly Independent.

By Rank Method: (1) If the rank of the matrix of given vectors is equal to the number of vectors, then the vectors are linearly Independent.

(2) If the rank of the matrix of given vectors is less then the number of vectors then the vectors are L.D (linearly dependent)

Q. Check the linearly dependency or Independency by matrix method of the following vectors.

$$X = [1, 2, -3, 4],\ Y = [3, -1, 2, 1],\ Z = [1, -5, 8, -7]$$

Sol^n. Let us form a matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & -7 & 11 & -11 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$\Rightarrow$ rank $(A) = \rho(A) = 2 < 3$ (no. of vectors)

Given vectors are L.D.

Q. Show that the vectors $X = [2, 5, 2, -3]$ $Y = [3, 6, 5, 2]$, $Z = [4, 5, 14, 14]$, $P = [5, 10, 8, 4]$ are linearly Independent

Sol^n

$$A = \begin{bmatrix} 2 & 5 & 2 & -3 \\ 3 & 6 & 5 & 2 \\ 4 & 5 & 14 & 14 \\ 5 & 10 & 8 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 3 & 6 & 5 & 2 \\ 1 & -1 & 9 & 12 \\ 2 & 4 & 3 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to -(R_1 - R_2) \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 3 & -4 & -13 \\ 0 & -2 & 6 & 7 \\ 0 & 2 & -3 & -8 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & -6 \\ 0 & -2 & 6 & 7 \\ 0 & 2 & -3 & -8 \end{bmatrix} \quad R_2 \to R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 10 & -5 \\ 0 & 0 & -7 & 4 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & -4/7 \end{bmatrix} \quad \begin{array}{l} R_3 \to \frac{1}{10} R_3 \\ R_4 \to -\frac{1}{7} R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -1/14 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$\Rightarrow$ rank $(A) = 4 =$ no. of vectors

So given vectors are linearly Independent.

Q. $X = (1, -1, 1)$, $Y = (2, 1, 1)$, $Z = (3, 0, 2)$

Solⁿ.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 3 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{1}{3} R_2$$

rank $(A) = 2 < 3$ (no. of vectors)

Given vectors are L.D (Linear ~~Independent~~ dependent).

Ans

* Characteristic Matrix :-

Let $A = [a_{ij}]_{n \times n}$ be any n-row square matrix and λ is an indeterminant then the matrix $A - \lambda I$ is called characteristic matrix where I is identity matrix of order n

e.g → $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 1 & 1-\lambda \end{bmatrix}$ is characteristic matrix of A.

* Characteristic Polynomial & Equations

The determinant of characteristic matrix i.e $|A - \lambda I|$ is called ch- polynomial

and $|A - \lambda I| = 0$ is called charactheristic equations of matrix A.

* Characteristic Values / Eigen Values / characteristic roots :-

The root of equation $|A - \lambda I| = 0$ is called eigen values / characteristic values.

* Characteristic Vector :- If λ be the eigen value of a $n \times n$ - matrix A then a non-zero vector X is called a eigen vector / characteristic vector if $AX = \lambda X$ corresponding to λ.

* Some Important Properties of Eigen Values :-

① Any square matrix A and A' have same Eigen Values.

② The sum of Eigen values of a square matrix is equal to the trace of matrix (sum of elements on principal diagonal)

③ Product of Eigen values of a square matrix is equal to the determinant of that matrix.

④ If $\lambda_1, \lambda_2, \lambda_3 \dots\dots \lambda_n$ are the ~~dif~~ eigen value of a square matrix $A = [a_{ij}]_{n \times n}$

then eigen values of the matrix ⓐ kA are $k\lambda_1, k\lambda_2 \dots\dots k\lambda_n$

ⓑ A^n are $\lambda_1^n, \lambda_2^n \dots\dots \lambda_n^n$

⑤ $\frac{1}{A}$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots\dots \frac{1}{\lambda_n}$

Q. Find the characteristic value / Eigen Value of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol^n. The characteristic matrix of A is $A - \lambda I = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$

& ch-matrix equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (6-\lambda)\{(3-\lambda)(3-\lambda) - (-1\times -1)\} - (-2)\{-2(3-\lambda) - (-1)\times(2)\} + 2\{-2\times -1 - (3-\lambda)(2)\}$

$\Rightarrow (6-\lambda)(9+\lambda^2-6\lambda-1) + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$

$\Rightarrow (6-\lambda)(\lambda^2-6\lambda+8) + 2(2\lambda-4) + 2(2\lambda-4) = 0$

$\Rightarrow 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$

$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 16\lambda - 32 = 0$

$\Rightarrow \lambda^2(\lambda-2) - 10\lambda(\lambda-2) + 16(\lambda-2) = 0$

$\Rightarrow (\lambda-2)\{\lambda^2 - 10\lambda + 16\} = 0$

$\Rightarrow (\lambda-2)\{\lambda^2 - 2\lambda - 8\lambda + 16\} = 0$

$\Rightarrow (\lambda-2)\{\lambda(\lambda-2) - 8(\lambda-2)\} = 0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\Rightarrow \lambda = 2, 2, 8$ Ans.

(2) Find the ch. value of the matrix.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

$\underline{Sol^n}$. $A - \lambda I = \begin{bmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{bmatrix}$

Ch-eq^n of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{bmatrix} = 0$$

$(2-\lambda)\{(1-\lambda)(-4-\lambda) - 3\times 2\} - (-3)\{3(-4-\lambda) - (3\times -5)\} + 1\{3\times 2 - (1-\lambda)(-5)\} = 0$

$(2-\lambda)\{-4-\lambda+4\lambda+\lambda^2-6\} + 3\{-12-3\lambda+15\} + 1\{6+5-5\lambda\} = 0$

$(2-\lambda)\{\lambda^2+3\lambda-10\} + 3\{-3\lambda+3\} + 1\{-5\lambda+11\} = 0$

$2\lambda^2 + 6\lambda - 20 - \lambda^3 - 3\lambda^2 + 10\lambda - 9\lambda + 9 - 5\lambda + 11 = 0$

$-\lambda^3 - \lambda^2 + 2\lambda = 0$

$-\lambda(\lambda^2+\lambda-2) = 0$

$\lambda(\lambda+2\lambda-\lambda-2) = 0$

$\lambda(\lambda+2)(\lambda-1) = 0$

$\lambda = 0, 1, -2$ Ans

(3) Find the Eigen value of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{Sol^n}$. $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

Ch-eq^n of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$(2-\lambda)\{(2-\lambda)(1-\lambda) - 1\times 0\} - 1\{1(1-\lambda) - 1\times 0\} + 1\{1\times 0 - (2-\lambda)0\} = 0$

$(2-\lambda)\{2 - 2\lambda - \lambda + \lambda^2\} - 1 + \lambda + 0 = 0$

$4 - 4\lambda - 2\lambda + 2\lambda^2 - 2\lambda + 2\lambda^2 + \lambda^2 - \lambda^3 - 1 + \lambda = 0$

$$3 - 9\lambda + 5\lambda^2 - \lambda^3 = 0$$
$$\lambda^3 - 5\lambda^2 + 9\lambda - 3 = 0$$
$$\lambda^2(\lambda - 5) \quad \lambda^3 - 5\lambda^2 + 9\lambda = 3$$
$$\lambda(\lambda^2 - 5\lambda + 9) = 3$$

$$(1-\lambda)(2-\lambda)^2 - 1 + \lambda = 0$$
$$(1-\lambda)(2-\lambda)^2 - (1-\lambda) = 0$$
$$(1-\lambda)\{(2-\lambda)^2 - 1\} = 0$$
$$(1-\lambda)\{4 + \lambda^2 - 4\lambda - 1\} = 0$$
$$(1-\lambda)\{\lambda^2 - 4\lambda + 3\} = 0$$
$$(1-\lambda)\{\lambda^2 - 3\lambda - \lambda + 3\} = 0$$
$$(1-\lambda)(\lambda - 3)(\lambda - 1) = 0$$
$$\lambda = 1, 1, 3$$

Ans

(2) $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigen values of $3A^3 + 5A^2 - 6A + 2I$

Solⁿ. The ch-matrix of A is $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = 0$

$$(1-\lambda)\{(3-\lambda)(-2-\lambda) - 2\times 0\} - 2\{0\times(-2-\lambda) - 2\times 0\} - 3\{0\times 0 - 3-\lambda \times 0\} = 0$$
$$(1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, 3, -2$$

Ans.

There are three eigen value of $3A^3 + 5A^2 - 6A + 2I$

1st eigen value, $\lambda = 1$

$$3(1)^3 + 5\times(1)^2 - 6\times 1 + 2 = 3 + 5 - 6 + 2 = 4$$

2^{nd} eigen value $\lambda = 3$

$$3\times(3)^3+5\times(3)^2-6\times3+2 = 81+45-18+2 = 110$$

3^{rd} eigen value $\lambda = -2$

$$3\times(-2)^3+5\times(-2)^2-6(-2)+2 = -24+20+12+2 = 10$$

$\Rightarrow$ **Cayley-Hamilton Theorem :-** Every square matrix is satisfying its own characteristic equation.

Q. Verify the Cayley Hamilton th^m for the matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$ and also find A^{-1}

Sol^n. The ch-eq^n of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow -1-\lambda+\lambda+\lambda^2-4 = 0$

$\Rightarrow \lambda^2 - 5 = 0$

We have, $A^2 - 5I$

$$A^2 = A\cdot A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2 & 1\times2+2\times-1 \\ 2\times1+-1\times2 & 2\times2+-1\times-1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\because A^2 - 5I = 0$.

$\therefore$ the Cayley Hamilton th^m is verified.

$$A^2 - 5I = 0$$

Multiplying by A^{-1} on both side in (1)

$$A - 5A^{-1} = 0$$

$$5A^{-1} = A$$

$$A^{-1} = \frac{1}{5} A$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$ Ans.

Q. Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$ and find A^{-1}

Solⁿ. We have,

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The ch- eq of matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(1-\lambda)(-1-\lambda) - 3\times 1\} - 2\{1(-1-\lambda) - 1\times 1\} + (-2)\{1\times 3 - 1(1-\lambda)\} = 0$

$\Rightarrow (1-\lambda)(\{-(1-\lambda)^2 - 3\} - 2\{-1-1-\lambda-1\} - 2\{3-1+\lambda\} = 0$

$\Rightarrow (1-\lambda)\{-1+\lambda^2 - 3\} - 2\{-2-\lambda\} - 2\{2+\lambda\} = 0$

$$\lambda^2 - 4 - \lambda^3 + 4\lambda = 0$$

We replace λ by A in LHS of above eqⁿ

$$A^3 - A^2 - 4A + 4I$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1 + 2\times 1 - 2\times 1 & 2+2-6 & -\cancel{2}+\cancel{2}+2 \\ 1+1+1 & 1\times 2+1\times 1+3 & -2 \cancel{+1} \cancel{-1} \\ 1+3-1 & 2+3-3 & -2+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$A^3 = A \cdot A \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1\cancel{-2}\cancel{+2} & \cancel{2}-\cancel{2}+6 & -2-2-2 \\ 3+6-2 & 6\cancel{+6}\cancel{-6} & \cancel{-6}\cancel{+6}+2 \\ 3+2+2 & 6+2+6 & -6\cancel{+2}\cancel{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$$

$$A^3 - A^2 - 4A + 4I$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 8 & -8 \\ 4 & 0 & 4 \\ 4 & 12 & -8 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -8 \\ 4 & 4 & 4 \\ 4 & 12 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \left[A^3 - A^2 - 4A + 4I = 0 \right]$ Hence, Cayley Hamilton th^m is verify.

Multiplying by A^{-1} in eq^n (*), we have

$$A^2 - A - 4AI + 4A^{-1} = 0$$

$$A^{-1} = \frac{A + 4I - A^2}{4}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{4}\begin{bmatrix} 1+4-1 & 2+2 & -2-2 \\ 1-3 & 1+4-6 & 1+2 \\ 1-3 & 3-2 & -1+4-2 \end{bmatrix}$$

$$\boxed{A^{-1} \Rightarrow \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}}$$

Ans.

Q. Find the A^{-1} using Cayley-Hamilton th^m.

Sol^n

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

The ch-eq^n of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$4-\lambda\left\{(1-\lambda)(1-\lambda)+4\right\} - 3\left\{2(1-\lambda)+2\right\} + 1\left\{4-(1-\lambda)\right\} = 0$$

$$4-\lambda\left\{(1-\lambda)^2+4\right\} - 6 + 6\lambda - 6 + 4 - 1 + \lambda = 0$$

$$(4-\lambda)(1-\lambda)^2 + 16 - 4\lambda - 12 + 3 + 7\lambda = 0$$

$$(4-\lambda)(1+\lambda^2-2\lambda) + 7 + 3\lambda = 0$$

$$4 + 4\lambda^2 - 8\lambda - \lambda - \lambda^3 + 2\lambda^2 + 7 + 3\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0$$

$$A^3 - 6A^2 + 6A - 11I = 0$$

$$A^2 - 6A + 6 - 11A^{-1} = 0$$

$$11A^{-1} = A^2 - 6A + 6I$$

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6+1 & 12+3+2 & 4-6+1 \\ 8\cancel{+2}\cancel{-2} & 6+1-4 & 2\cancel{+2}\cancel{-2} \\ 4+4+1 & 3+2+2 & 1-4+1 \end{bmatrix} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 23-24+6 & 17-18 & -1-6 \\ 8-12 & 3\cancel{-6}\cancel{+6} & -2+12 \\ 9-6 & 7-12 & -2\cancel{-6}\cancel{+6} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Ans.

Characteristic vector / Eigen vector :-

$$AX = \lambda X$$

$$\boxed{(A - \lambda I)X = 0}$$

Properties of Eigen Vectors :-

* The eigen vectors of a matrix A is not unique.
* If $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are distinct eigen value of $n \times n$ matrix then corresponding eigen vector $X_1, X_2 \ldots\ldots X_n$ are linearly independent.
* If two or more then two eigen values are equal it may or may not be possible to set linearly independent eigen vectors corresponding to equal to eigen values.

Orthogonal vectors :- Two vectors X_1 & X_2 are said to orthogonal if

$$X_1' X_2 = X_2' X_1 = 0$$

* Eigen vectors of a symmetric matrix corresponding to distinct eigen value are orthogonal.

Q. Find the eigen vector of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ corresponding to each eigen values and check whether they are orthogonal.

Sol^n. Given matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The ch. eq^n of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\{(2-\lambda)(3-\lambda) - 2\} - 0 + (-1)\{1 \times 2 - 2(2-\lambda)\} = 0$$

$$(1-\lambda)\{6 - 2\lambda - 3\lambda + \lambda^2 - 2\} - 1\{2 - 4 + 2\lambda\} = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 1, 2, 3$$

The ch-vector X of the matrix A corresponding to the eigen value λ is given by:

$$(A-\lambda I)X = 0$$

For $\lambda = 1$,

$$\begin{bmatrix} 1-1 & 0 & -1 \\ 1 & 2-1 & 1 \\ 2 & 2 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$-z = 0 \Rightarrow z = 0$

$x + y + z = 0$

$x + y = 0$

$x = -y = k$ (say)

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$ is the eigen vector corresponding to $\lambda = 1$.

For $\lambda = 2$,

$$\Rightarrow \begin{bmatrix} 1-2 & 0 & -1 \\ 1 & 2-2 & 1 \\ 2 & 2 & 3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$\Rightarrow x + z = 0$

$\Rightarrow 2x + 2y + z = 0$

$$\Rightarrow \frac{x}{0-2} = \frac{y}{2-1} = \frac{z}{2-0}$$

$$\Rightarrow \frac{x}{-2} = \frac{y}{1} = \frac{z}{2} = k$$

$\Rightarrow x = -2k,\ y = k,\ z = 2k$

So, $X_2 = \begin{bmatrix} -2k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

for $k = 1$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda = 3$.

$$\begin{bmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x - z = 0$

$x - y + z = 0$

$2x + 2y = 0$

$\frac{x}{0-2} = \frac{y}{2-0} = \frac{z}{2+2}$

$\frac{x}{-2} = \frac{y}{1} = \frac{z}{4}$

$\frac{x}{-1} = \frac{y}{1} = \frac{z}{2} = k$ (say)

$x = -k,\ y = k,\ z = 2k$

So. $X_3 = \begin{bmatrix} -k \\ k \\ 2k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

For $k = 1$

$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$X_1' X_2 = \begin{bmatrix} 1 & -1, & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = -2-1+0 = -3 \neq 0$$

$$X_2' X_3 = \begin{bmatrix} -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 2+1+4 = 7 \neq 0$$

$$X_3' X_1 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = (-1)+(-1)+0 = -2 \neq 0$$

So, X_1, X_2, X_3 are not orthogonal vectors

Q. Find the eigen vector of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Solⁿ. The ch-eqⁿ. of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$3-\lambda \{(2-\lambda)(5-\lambda) - 0\} - 1\{0\} + 4\{0\} = 0$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda = 3, 2, 5$$

The ch-vector X of the matrix A corresponding to the eigen value λ is given by $(A - \lambda I)X = 0$

For $\lambda = 3$

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \cancel{R_3 = \frac{1}{2} R_3}$$

$$y + \cancel{4z = 0}$$
$$-y + 6z = 0$$
$$z = 1$$

$$y + 4z = 0$$
$$-y + 6z = 0$$
$$2z = 0$$
$$z = 0$$

$$\boxed{y = 0}$$

$$\cancel{X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\frac{x}{6+4} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{10} = \frac{y}{0} = \frac{z}{0}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k \text{ (say)}$$

$$X_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Ans

Case II $\lambda = 2$

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

$\Rightarrow x + y + 4z = 0$

$6z = 0 \Rightarrow z = 0$

$\Rightarrow x + y + 4(0) = 0$

$x + y = 0$

$x = -y = k$ (say)

So, $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Case III $\lambda = 5$

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + y + 4z = 0$

$0x + -3y + 6z = 0$

$\frac{x}{6+12} = \frac{y}{12} = \frac{z}{6} = k$ (say)

$\frac{x}{18} = \frac{y}{12} = \frac{z}{6} = k \Rightarrow \frac{x}{3} = \frac{y}{2} = \frac{z}{1} = k$

$x = 3,\ y = 2,\ z = 1 = k$

$X_3 = \begin{bmatrix} 3k \\ 2k \\ 1k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Ans

Q. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solⁿ. The ch-eqⁿ of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow 2-\lambda \{(2-\lambda)(1-\lambda) - 0\} - 1\{\cancel{0-0}\ 1-\lambda-0\} + 1\{0\} = 0$

$\Rightarrow (2-\lambda)\{2 - 2\lambda - \lambda + \lambda^2\} - 1 + \lambda = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda + 2) - 1 + \lambda = 0$

$\Rightarrow 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda - 1 + \lambda = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda^2 + 4\lambda + 3\lambda - 3 = 0$

$\Rightarrow \lambda^2(\lambda-1) - 4\lambda(\lambda-1) + 3(\lambda-1) = 0$

$\Rightarrow (\lambda-1)(\lambda^2 - 4\lambda + 3) = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-3) = 0$

$\Rightarrow \lambda = 1, 1, 3$

The eigen vectors X of the matrix A corresponding to eigen value λ is given

$$(A - \lambda I)X = 0$$

when $\lambda = 1$

$$\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$\Rightarrow x + y + z = 0$

$x = k_1, \; y = k_2, \; z = -(k_1 + k_2)$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -(k_1 + k_2) \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda=3}$

$$\begin{bmatrix} 2-3 & 1 & 1 \\ 1 & 2-3 & 1 \\ 0 & 0 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$-x+y+z=0$

$2z=0 \Rightarrow z=0$

$-x+y+0=0$ or $x=y=k$ (say)

$$X_3 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ Ans.

Q. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$$

Sol^n. The ch-eq^n of matrix A is

$$\begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\{(4-\lambda)(-3-\lambda) - 2\times(-6)\} + 6(0) + (-4)(0) = 0$$

$$(1-\lambda)\{-12-4\lambda+3\lambda+\lambda^2+12\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-\lambda) = 0$$

$$(1-\lambda)\lambda(\lambda-1) = 0$$

$$\lambda = 0, 1, 1$$

The eigen vectors are given by $(A-\lambda I) \cdot X = 0$

Case I $\lambda=0$

$$\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x-6y-4z=0$$
$$+4y+2z=0$$
$$-6y-3z=0$$

$$\Rightarrow \frac{x}{-12+16}=\frac{y}{0-2}=\frac{z}{4-0}$$

$$\Rightarrow \frac{x}{4}=\frac{y}{-2}=\frac{z}{4}$$

$$\Rightarrow \frac{x}{2}=\frac{y}{-1}=\frac{z}{2}=k \text{ (say)}$$

$$x=2k,\ y=-k,\ z=2k$$

So $X_1=\begin{bmatrix}2k\\-k\\2k\end{bmatrix}=k\begin{bmatrix}2\\-1\\2\end{bmatrix}$ or $\begin{bmatrix}2\\-1\\2\end{bmatrix}$

Case II $\lambda=1$

$$\begin{bmatrix}1-1 & -6 & -4\\0 & 4-1 & 2\\0 & -6 & -3-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & -6 & -4\\0 & 3 & 2\\0 & -6 & -4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-6y-4z=0$$
$$3y+2z=0$$

Let $y=2,\ z=-3$, We take $x=k$

So, $X=\begin{bmatrix}k\\2\\-3\end{bmatrix}$

$X_2=\begin{bmatrix}0\\2\\-3\end{bmatrix}$, $X_3=\begin{bmatrix}1\\2\\-3\end{bmatrix}$

Ans

Diagonalisation of a matrix:- A square matrix A is said to be diagonalizable if it is similar to an $n \times n$ diagonal matrix D. Then the square matrix A is diagonalizable if there exists a non-singular matrix P such that.

$$D = P^{-1}AP$$ (here, P is said to be modal matrix)

Th^m: A matrix of $n \times n$ is diagonalizable if and only if it possesses n linearly independent eigen vectors.

Q. Find a matrix P which is diagonalizable to matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, and verify $P^{-1}AP = D$

Sol. The ch-eq of A is $\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)(3-\lambda) - 2 = 0$
$\Rightarrow 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$
$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$
$\Rightarrow \lambda^2 - 2\lambda - 5\lambda + 10 = 0$
$\Rightarrow (\lambda - 2)(\lambda - 5) = 0$
$\Rightarrow \lambda = 2, 5$

Eigen vectors are given by $(A - \lambda I).X = 0$

Case I $\lambda = 2$.

$$\Rightarrow \begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + y = 0$

let $x = 1$, $y = -2$

$$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Case II $\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$-x + y = 0$

$x = y = k$ (say)

For $k = 1$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, the modal matrix $(P) = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ Ans

$Adj(P) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$P^{-1} = \dfrac{Adj(P)}{|P|} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

Now, $P^{-1}AP = D$

$\Rightarrow \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow \dfrac{1}{3}\begin{bmatrix} 2 & -2 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow \dfrac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$ verified

Module II Differential Calculus

Successive differentiation:-

If $y = f(x)$

The first derivative of y is $\frac{dy}{dx}$.

Second " " is $\frac{d^2y}{dx^2}$

n^{th} " " " $= \frac{d^ny}{dx^n}$

n^{th} derivative of some standard function:-

* If $y = x^m$ then $y_n = m(m-1)(m-2)\ldots(m-n+1).\, x^{m-n}$.
* If $y = e^{ax}$ then $y_n = a^n e^{ax}$
* If $y = a^{mx}$ then $y_n = a^{mx}(\log a)^n$
* If $y = (ax+b)^{-1}$ then $y_n = (-1)^n n!\, a^n (ax+b)^{-n-1}$
* If $y = (ax+b)^m$ then $y_n = m(m-1)\ldots(m-n+1)\, a^n (ax+b)^{m-n}$
* If $y = \log(ax+b)$ then $y_n = (-1)^{n-1}(n-1)!\, a^n (ax+b)^{-n}$
* If $y = \sin(ax+b)$ then $y_n = a^n \sin(ax+b+n\frac{\pi}{2})$
* If $y = \cos(ax+b)$ then $y_n = a^n \cos(ax+b+n\frac{\pi}{2})$

Q. If $y = \frac{1}{1-5x+6x^2}$ then, find y_n.

Soln.
$$y = \frac{1}{1-5x+6x^2}$$

$$y = \frac{1}{6x^2-3x-2x+1}$$

$$= \frac{1}{3x(2x-1)-1(2x-1)} = \frac{1}{(2x-1)(3x-1)}$$

$$y = 2(2x-1)^{-1} - 3(3x-1)^{-1}$$

$$y_n = 2(-1)^n n!\, 2^n (2x-1)^{-n-1} - 3(-1)^n n!\, 3^n (3x-1)^{-n-1}$$

$$= (-1)^n n! \left[2^{n+1}(2x-1)^{-n-1} - 3^{n+1}(3x-1)^{-n-1}\right] \text{ Ans}$$

Leibnitz theorem:- If u and v are the factor of x function. then

$$\frac{d^n}{dx^n}(u\cdot v) = (u\cdot v)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + {}^nC_n u\cdot v_n$$

Proof:- This thm is proved by mathematical induction

for $n = 1$,

$$(uv)_1 = u_1 v + u\cdot v_1$$

differentiating, above eqn we get,

$$(uv)_2 = \frac{d}{dx}[u_1 v + u\cdot v_1] = u_2 v + u_1 v_1 + u_1 v_1 + u_0 u_2$$

$$= u_2 v + 2u_1 v_1 + u v_2$$

$$\left[(uv)_2 = {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2\right]$$

Since it is true for $n = 1, 2$

let us consider it is also time for $n = m$.

$$(uv)_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots + {}^mC_r u_{m-r} v_r + \dots + {}^mC_m u\cdot v_m$$

Differenting above eqn, we get,

$$\Rightarrow (uv)_{m+1} = \left({}^mC_0 u_{m+1} v + {}^mC_0 u_m v_1\right) + \left({}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2\right) + \left({}^mC_2 u_{m-1} v_2 + {}^mC_2 u_{m-2} v_3\right) + \dots + \left({}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r} v_{r+1}\right) + \dots + \left({}^mC_m u_1 v_m + {}^mC_m u v_{m+1}\right)$$

$$\Rightarrow {}^{m+1}C_0 u_{m+1} v + \left({}^mC_0 + {}^mC_1\right) u_m v_1 + \left({}^mC_1 + {}^mC_2\right) u_{m-1} v_2 + \dots + \left({}^mC_r + {}^mC_{r+1}\right) u_{m-r} v_{r+1} + \dots + {}^{m+1}C_{m+1} u v_{m+1}$$

$$\Rightarrow {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \dots + {}^{m+1}C_{r+1} u_{m-r} v_{r+1} + \dots + {}^{m+1}C_{m+1} u v_{m+1}$$

Hence, it is true for $n = m+1$.

So it is always true.

Q. Differentiate n-times the eq^n

$$x^2 y_2 + x y_1 + y = 0$$

Sol^n. Given eq^n is –

$$x^2 y_2 + x y_1 + y = 0$$

Using Leibnitz th^m.

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots\dots + {}^nC_n u v_n$$

$$\Rightarrow \left\{{}^nC_0 y_{n+2} \cdot x^2 + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n \cdot 2 + 0\right\} + \left\{{}^nC_0 y_{n+1} \cdot x + {}^nC_1 y_n (1) + 0\right\}$$
$$+ y_n = 0$$

$$\Rightarrow y_{n+2} x^2 + 2nx y_{n+1} + \frac{1}{\cancel{2}} n(n-1) y_n \cdot \cancel{2} + x y_{n+1} + n y_n + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1) x y_{n+1} + \{n^2 - n + n + 1\} y_n = 0$$

$$\boxed{{}^nC_2 = \frac{n!}{2!(n-2)!}}$$

$$\Rightarrow x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0 \quad \text{Ans.}$$

Q2 Differentiate 'n' times the eq^n.

$$(1-x^2) y_2 + x y_1 + a^2 y = 0$$

$\Rightarrow$ Using Leibnitz th^m

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + {}^nC_n u v_n$$

$$\Rightarrow \left\{{}^nC_0 y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) + 0\right\} + \left\{{}^nC_0 y_{n+1} x + \right.$$
$$\left. {}^nC_1 y_n (1) + 0\right\} + a^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} + \frac{n(n-1)}{2}(-2) y_n + x y_{n+1} + n y_n +$$
$$a^2 y_n = 0$$

$$\Rightarrow (1-x^2) \cdot y_{n+2} - x y_{n+1}(2n-1) + y_n(-n^2 + n + n + a^2) = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n-1) x y_{n+1} + (a^2 + 2n - n^2) y_n = 0$$

Q. Differentiate 'n' times the function

$$y = \log\sqrt{4x^2+8x+3}$$

Solⁿ. We have,

$$y = \log\sqrt{4x^2+8x+3}$$

$$\Rightarrow \log(4x^2+8x+3)^{1/2} = \frac{1}{2}\log(4x^2+8x+3)$$

$$\Rightarrow \frac{1}{2}\log(4x^2+2x+6x+3)$$

$$\Rightarrow \frac{1}{2}\log(2x(2x+1)+3(2x+1))$$

$$\Rightarrow \frac{1}{2}\log[(2x+1)(2x+3)]$$

$$y = \frac{1}{2}\log(2x+1)+\frac{1}{2}\log(2x+3)$$

Diff. 'n' times w.r to 'x' we get

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\,2^n(2x+1)^{-n}+\frac{1}{2}(-1)^{n-1}(n-1)!\,2^n(2x+3)^{-n}$$

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\,2^n[(2x+1)^{-n}+(2x+3)^{-n}]$$

$$y_n = (-1)^{n-1}(n-1)!\,2^{n-1}[(2x+1)^{-n}+(2x+3)^{-n}] \text{ Ans}$$

Q. If $y = \cos(m\sin^{-1}x)$ then show that

$$(1-x^2)y_{n+2} - (2n-1)x\,y_{n+1} + (m^2-n^2)y_n = 0$$

Solⁿ We have, $y = \cos(m\sin^{-1}x)$

diff. above w.r to 'x' we get

$$y_1 = -\sin(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = -m\sin(m\sin^{-1}x)$$

Squaring on both sides we get

$$(1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)y_1^2 = m^2[1-\cos^2(m\sin^{-1}x)]$$

$$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$$

Again diff we get

$$(1-x^2)2y_1y_2 - 2xy_1^2 = m^2(-2yy_1)$$

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2 y = 0$ On dividing $2y_1$

Using Leibnitz th^m.

$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_n u v_n$

$\Rightarrow \{ {}^nC_0 y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) + 0\} - \{ {}^nC_0 y_{n+1} x + {}^nC_1 y_n (1) + 0\} + m^2 y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1} n 2x + \frac{n(n-1)}{\cancel{2}}(-\cancel{2}) y_n - y_{n+1} x - n y_n + m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + \{m^2 - \cancel{n} - n^2 + \cancel{n}\} y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + \{m^2 - n^2\} y_n = 0$ Proved

Q. If $y = e^{\tan^{-1} x}$ then show that

$(1+x^2) y_{n+2} + [\{2(n+1)\} x - 1] y_{n+1} + n(n+1) y_n = 0$

Sol^n $\quad y = e^{\tan^{-1} x}$ — (1)

Differentiating (1) w.r.to 'x' we get

$y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$

$\Rightarrow (1+x^2) y_1 = e^{\tan^{-1} x}$

$\Rightarrow (1+x^2) y_1 = y$ — (2)

Again differentiating (2) w.r.to 'x' we get.

$(1+x^2) y_2 + 2x y_1 = y_1$

$\Rightarrow (1+x^2) y_2 + (2x-1) y_1 = 0$

Differentiating 'n' times by Leibnitz th^m,

$\Rightarrow (1+x^2) y_{n+2} + {}^nC_1 (2x) y_{n+1} + {}^nC_2 (2) y_n + (2x-1) y_{n+1} + {}^nC_1 (2) y_n = 0$

$\Rightarrow (1+x^2) y_{n+2} \{2nx + 2x - 1\} + y_n \left\{\frac{n(n-1)}{2} \cdot 2 + 2n\right\} = 0$

$\Rightarrow (1+x^2) y_{n+2} + \{2x(n+1) - 1\} y_{n+1} + y_n \{n^2 - n + 2n\} = 0$

$\Rightarrow (1+x^2)y_{n+2} + \{2x(n+1)-1\}y_{n+1} + n(n+1)y_n = 0$ Proved.

Q. If $y = a\cos(\log x) + b\sin(\log x)$ then show that $x^2 y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0$

Sol^n $y = a\cos(\log x) + b\sin(\log x)$

diff (1) w.r to x.

$y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\frac{1}{x}$.

$y_1 = \frac{1}{x}\left(b\cos(\log x) - a\sin(\log x)\right)$

$x\,y_1 = b\cos(\log x) - a\sin(\log x)$

~~differentiate (2) w.r to x.~~

~~$y_2 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x)\frac{1}{x}$~~

~~$x y_2 = -(b\sin(\log x) + a\cos(\log x))$~~

~~$x y_2 = -y$~~

~~$x y_2 + y = 0$~~

~~Using Leibnitz th^m.~~

~~$\Rightarrow \{{}^nC_0\, x\, y_{n+2} + {}^nC_1 \cdot 1\, y_{n+1}\} + y_n = 0$~~

~~$\Rightarrow x y_{n+2} + n y_{n+1}$~~

$x y_2 + y_1 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x)\frac{1}{x}$.

$x^2 y_2 + x y_1 = -(y)$

$x^2 y_2 + x y_1 + y = 0$

Using Leibnitz th^m.

$\Rightarrow \{{}^nC_0\, x^2 y_{n+2} + {}^nC_1\, 2x\cdot y_{n+1} + {}^nC_2\, 1\, y_n\} + \{{}^nC_0\, x\, y_{n+1} + {}^nC_1\, 1\, y_n\} + y_n = 0$

$\Rightarrow \{x^2 y_{n+2} + 2xn\cdot y_{n+1} + \frac{n(n-1)}{\cancel{2}}\cancel{2}\, y_n + x y_{n+1} + n y_n + y_n = 0$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + \left\{\frac{n(n-1)}{2} + n + 1\right\} y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + \left\{\frac{n^2 \cancel{- n + 2n} + 2}{2}\right\} y_n$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$$

Proved

Q. Find the n^{th} derivative of $x^{n-1}\log x$.

Sol^n.

$$y = x^{n-1}\log x$$

Differentiating w.r.t. to x. we have

$$y_1 = x^{n-1}\cdot\frac{1}{x} + (n-1)x^{n-2}\log x$$

$$y_1 = x^{n-2} + (n-1)\frac{x^{n-1}}{x}\log x$$

$$y_1 = x^{n-2} + (n-1)\frac{y}{x}$$

$$x y_1 = x^{n-1} + (n-1)y$$

Differentiating $(n-1)$ times by Leibnitz th^m, we get,

$$\Rightarrow {}^{n-1}C_0\, x y_n + {}^{n-1}C_1 (1) y_{n-1} = (n-1)! + (n-1) y_{n-1}$$

$$\Rightarrow x y_n + \cancel{(n-1) y_{n-1}} = (n-1)! + \cancel{(n-1) y_{n-1}}$$

$$\Rightarrow \boxed{y_n = \frac{(n-1)!}{x}}$$

Ans

Q. If $y^{1/m} + y^{-1/m} = 2x$ Prove that

$$(x^2-1)y_{n+2} + (2n+1)x y_{n+1} + (n^2-m^2)y_n = 0$$

Sol^n Let, $y^{1/m} = z$

$$y^{-1/m} = \frac{1}{z}$$

$\Rightarrow z + \frac{1}{z} = 2x$

$\Rightarrow \frac{z^2+1}{z} = 2x$

$\Rightarrow z^2 - 2xz + 1 = 0$

$\Rightarrow z = \frac{2x \pm \sqrt{4x^2-4}}{2}$

$\left[z = x \pm \sqrt{x^2-1} \right]$

$z = x + \sqrt{x^2-1}$

$y^{1/m} = x + \sqrt{x^2-1}$ —— (1) $\Rightarrow y = \left(x+\sqrt{x^2-1}\right)^m$

Differentiate (1) w.r to 'x' we get

$\cancel{\frac{1}{m} y^{1/m-1} \cdot y_1 = 1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x}$

$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1} \cdot \left(1 + \frac{1 \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right)$

$y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1} \left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)$

$y_1 = \frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}} = \frac{m\left(y^{1/m}\right)^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$

$y_1^2 = \frac{m^2y^2}{x^2-1}$

or $(x^2-1)y_1^2 = m^2y^2$

(2) differentiating w.r to 'x' we get.

$\Rightarrow (x^2-1)2y_1y_2 + 2xy_1^2 = 2m^2yy_1$

$\Rightarrow (x^2-1)y_2 + xy_1 - m^2y = 0$

Diff 'n' times by Leibnitz thm.

$\Rightarrow {}^nC_0(x^2-1)y_{n+2} + {}^nC_1 2x\,y_{n+1} + {}^nC_2(2)y_n + {}^nC_0\,x y_{n+1} + {}^nC_1(1)y_n - m^2y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 \cancel{-n} + \cancel{n} - m^2)y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$ **Proved.**

Q. Find the n^{th} derivative of.

$$\tan^{-1}\left[\frac{2x}{1-x^2}\right]$$

Solⁿ. Let, $y = \tan^{-1}\left[\frac{2x}{1+x^2}\right]$

or $y = 2\tan^{-1}x$

$$y_1 = \frac{2}{1+x^2}$$

or $y_1 = \frac{2}{(x-i)(x+i)}$

$$\Rightarrow \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

$$y_1 \Rightarrow \frac{1}{i}\left[(x-i)^{-1} - (x+i)^{-1}\right]$$

Diff $(n-1)$ times w.r to x', we get $\left[\begin{array}{l} y = (ax+b)^{-1} \\ y_n = (-1)^n n!\, a^n (ax+b)^{-n-1} \end{array}\right]$

$$y_n = \frac{1}{i}\left[(-1)^{n-1}(n-1)!\,(x-i)^{-n} - (-1)^{n-1}!\,(x+i)^{-n}\right]$$

$$\Rightarrow (-1)^{n-1}(n-1)!\left[(x-i)^{-n} - (x+i)^{-n}\right] \text{ Ans}$$

Q. Find n^{th} derivative of.

$$y = \tan^{-1}\left[\frac{1+x}{1-x}\right]$$

$$y = \tan^{-1}\left[\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\pi/4 \tan\theta}\right], \text{ Let } x = \tan\theta$$

$$\Rightarrow \tan^{-1}\left[\tan\left(\pi/4 + \theta\right)\right]$$

$$y = \frac{\pi}{4} + \theta$$

$$y = \frac{\pi}{4} + \tan^{-1}x$$

$$y_1 = 0 + \cancel{\tan}\frac{1}{1+x^2}$$

$$y_1 = \frac{1}{1+x^2}$$

or $y_1 = \frac{1}{(x+i)(x-i)}$

$= \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$

$\Rightarrow \frac{1}{2i}\left[(x-i)^{-1} - (x+i)^{-1}\right]$

Diff $(n-1)$ times w.r.t 'x', we get

$y_n = \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x-i)^{-n} - (-1)^{n-1}!\,(x+i)^{-n}\right]$ Ans