Module - IV

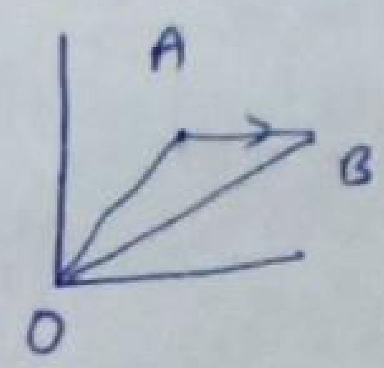
Vector: A vector is quantity having both magnitude and direction such as force, velocity, acceleration etc.

Unit vector: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Unit of $\vec{r} = \hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

* **Position vector of a point:**

$\vec{AB}$ = Position vector of B − position vector of A.



* **Product of two vectors:-**

Scalar or dot product:-

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, & $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Vector or Cross Product:-

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is unit vector $\perp$ to $\vec{a}$ & $\vec{b}$.

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$ and $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i}$, $\hat{j}\times\hat{k} = -\hat{k}\times\hat{j}$, $\hat{k}\times\hat{i} = -\hat{i}\times\hat{k}$

* if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

* **Vector function:** If vector r is a function of a scalar variable t then we write $\vec{r} = \vec{r}(t)$

Differentiation of vectors:

(i) $\dfrac{d}{dt}(\vec{F} + \vec{G}) = \dfrac{d\vec{F}}{dt} + \dfrac{d\vec{G}}{dt}$ (ii) $\dfrac{d(\vec{F}\phi)}{dt} = \dfrac{d\vec{F}}{dt}\cdot\phi + \vec{F}\dfrac{d\phi}{dt}$

(iii) $\dfrac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt}\cdot\vec{G}$ (iv) $\dfrac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\dfrac{d\vec{G}}{dt} + \dfrac{d\vec{F}}{dt}\times\vec{G}$

(v) $\dfrac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\dfrac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \dfrac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \dfrac{d\vec{c}}{dt}\right]$

(vi) $\dfrac{d}{dt}[\vec{a}\times(\vec{b}\times\vec{c})] = \dfrac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left(\dfrac{d\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\dfrac{d\vec{c}}{dt}\right)$

Point function : A variable quantity whose unit value at any point in a region of space depends upon the position of the point, is called a point function. There are two types of point fn^n
(i) Scalar point function (ii) Vector point function

* Vector Differential operator Del (∇) :-

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

* **Gradient of a Scalar function** :-

If $\phi(x,y,z)$ be a scalar function then

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

* **Normal** : If $\phi(x,y,z) = c$ represent a family of surfaces for different c.

Then $\nabla\phi$ is a vector normal to the surface $\phi(x,y,z) = c$.

* **Directional derivative** :-

The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi \cdot \hat{d}$ and it is called the directional derivative of ϕ in the direction $\vec{d}$.

Q→ 1 If $\phi = 3x^2y - y^3z^2$ find the grad ϕ at the point $(1,-2,-1)$.

Solution:

$$\text{grad } \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\text{grad } \phi \text{ at } (1,-2,-1) = \hat{i}(6\times1\times-2) + \hat{j}(3(1)^2 - 3(-2)^2(-1)^2) + \hat{k}(-2\times(-2)^3\times-1)$$

$$= -12\hat{i} + \hat{j}(3-12) + \hat{k}(-16)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$ **Ans**

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Q→ Find the directional derivative of $x^2y^2z^2$ at the point $(1,1,-1)$ in the direction of the tangent to the curve $x=e^t$, $y=\sin 2t+1$, $z=1-\cos t$ at $t=0$.

Solution: Let $\phi = x^2y^2z^2$

Directional derivative of $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$

$$\nabla\phi = 2xy^2z^2\,\hat{i} + 2x^2yz^2\,\hat{j} + 2x^2y^2z\,\hat{k}$$

Directional derivative at $(1,1,-1) = 2(1)(1)^2(-1)^2\hat{i} + 2(1)^2(1)(-1)^2\hat{j} + 2(1)^2(1)^2(-1)\hat{k}$

$$= 2\hat{i}+2\hat{j}-2\hat{k}$$

and given that $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{r} = e^t\,\hat{i} + (\sin 2t+1)\,\hat{j} + (1-\cos t)\,\hat{k}$$

Tangent vector $\frac{d\vec{r}}{dt} = e^t\,\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

Tangent (at $t=0$) $= e^0\,\hat{i} + 2\cos 0\,\hat{j} + \sin 0\,\hat{k} = \hat{i}+2\hat{j}$

⇒ Required direction derivative along tangent $= (2\hat{i}+2\hat{j}-2\hat{k})\cdot\frac{\hat{i}+2\hat{j}}{\sqrt{1^2+2^2}}$

$$= \frac{2\times1+2\times2+-2\times0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$ Ans

Q: Find the unit normal to the surface $xy^3z^2=4$ at $(-1,-1,2)$

Solution: Let $\phi(x,y,z) = xy^3z^2-4$

We know that $\nabla\phi$ is the vector normal to the surface $\phi(x,y,z)=0$

Normal vector $= \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}(y^3z^2)+\hat{j}(3xy^2z^2)+\hat{k}(2xy^3z)$$

⇒ Normal vector $= y^3z^2\,\hat{i} + 3xy^2z^2\,\hat{j} + 2xy^3z\,\hat{k}$

Normal vector at $(-1,-1,2) = -4\hat{i}-12\hat{j}+4\hat{k}$

Unit vector normal to the surface ϕ at $(-1,-1,2)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{i}-12\hat{j}+4\hat{k}}{\sqrt{16+144+16}} = \frac{1}{4\sqrt{11}}(-4\hat{i}-12\hat{j}+4\hat{k})$$

$$= \frac{-1}{\sqrt{11}}(\hat{i}+3\hat{j}-\hat{k})$$ Ans.

Q→

Find a unit vector normal to the surface $x^2+3y^2+2z^2=6$ at $P(2,0,1)$ Ans $\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$

Q- Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution: Normal on the surface $(x^2+y^2+z^2-9=0)$

$$=\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

Noral at point $(2,-1,2)$ $= 4\hat{i}-2\hat{j}+4\hat{k}$ — (1)

Normal on the surface $(x^2+y^2-z-3=0)$

$$=\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= 2x\hat{i}+2y\hat{j}-\hat{k}$$

Normal at point $(2,-1,2) = 4\hat{i}-2\hat{j}-\hat{k}$ — (2)

Let θ be angle between normals (1) and (2),

$$(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k}) = \sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$16+4-4 = 6\sqrt{21}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

Ans

Q:- Find the directional derivative of $\frac{1}{r}$ in the direction $\vec{r}$ where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Solution: Here $\phi(x,y,z) = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

Now $\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\hat{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\hat{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}\hat{k}$

$$=\left(-\frac{1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}\cdot 2x\right)\hat{i} + \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y\right)\hat{j} + \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right)\hat{k}$$

$$\nabla\phi = \nabla\left(\frac{1}{r}\right) = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}}$$

and $\hat{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

So, the required directional derivative

$$=\nabla\phi\cdot\hat{r} = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}}\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{1/2}}$$

$$= -\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{2}}$$

$$= -\frac{1}{x^2+y^2+z^2}$$

$$= -\frac{1}{r^2}$$

Ans

Q:- Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$.

Solution: Directional derivative $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

Directional derivative at the point $P(1,2,3) = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$$\vec{PQ} = \vec{Q} - \vec{P} = (5,0,4) - (1,2,3) = 4\hat{i} - 2\hat{j} + \hat{k}$$

Direction derivative along PQ $= (2\hat{i} - 4\hat{j} + 12\hat{k})\dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}} \text{ Ans}$$

Q:- If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ find the values of a, b and c.

Solution: Given $\phi = ax^2y + by^2z + cz^2x$

$$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(ax^2y + by^2z + cz^2x) = \hat{i}(2axy + cz^2) + \hat{j}(ax^2 + 2byz) + \hat{k}(by^2 + 2czx)$$

$$\nabla\phi \text{ at } (1,1,1) = (2a+c)\hat{i} + (a+2b)\hat{j} + (b+2c)\hat{k} \quad \text{——} (1)$$

We know that the maximum value of the directional derivative is in the direction of $\nabla\phi$.

i.e. $\nabla\phi \cdot \dfrac{\nabla\phi}{|\nabla\phi|} = 15$ or $|\nabla\phi| = 15$

$$\Rightarrow (2a+c)^2 + (a+2b)^2 + (b+2c)^2 = 15^2 \quad \text{——} (2)$$

But the directional derivative is given to be maximum parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ i.e parallel to vector $2\hat{i} - 2\hat{j} + \hat{k}$ —(3)

Comparing the coefficient of (1) and (3)

$$\Rightarrow \frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1} \Rightarrow \begin{cases} 2a+c = -(2b+a) \Rightarrow 3a+2b+c=0 \\ 2b+a = -2(2c+b) \Rightarrow a+4b+4c=0 \end{cases}$$

$$\Rightarrow \frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k \text{ (say)} \Rightarrow a = 4k,\ b = -11k \text{ and } c = 10k$$

Putting a, b, c in eq (2), we get $k = \pm\frac{5}{9}$ $\Rightarrow \left[a = \pm\frac{20}{9},\ b = \pm\frac{55}{9},\ c = \pm\frac{50}{9}\right]$

Q:- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

(i) $\text{grad } r = \frac{\vec{r}}{r}$ (ii) $\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

Solution: (i) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r$$

$$= \hat{i}\,\frac{x}{r} + \hat{j}\,\frac{y}{r} + \hat{k}\,\frac{z}{r}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$ Proved

(ii) $$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{r^3}$$

$$\left[\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}\right]$$ Proved

Q:- Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$.

Ans $\frac{28}{\sqrt{21}}$

Q:- For the function $\phi(x,y) = \frac{x}{x^2+y^2}$ find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at $(0,2)$.

Ans $\sqrt{3}/8$