Module - IV Vector Calculus

* Vector:-

* Position of vector:-

$\vec{AB}$ = Position vector of B - Position vector of A

* unit vector:-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

Product:-

* (1) Dot / Scalar product:- ~~$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$~~

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$i\cdot i = j\cdot j = k\cdot k = 1$$

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

(2) Cross / vector product →

$$\vec{a}\times\vec{b} = \cancel{\vec{a}\otimes\vec{b}}\,|\vec{a}|\,|\vec{b}|\sin\theta$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}]$$

* Vector differentiation operation (∇) (del)

$$\nabla = \hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}$$

* Gradient of a scalar function :-

let $\phi(x,y,z)$ be a scalar function

than grad $\phi = \nabla\phi = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\phi$

Q> If $\phi = 3x^2y - y^3z^2$ find the grad ϕ at the point $(1,-2,-1)$

Sol we have $\phi = 3x^2y - y^3z^2$

~~observed~~

grad $\phi = \nabla\phi = \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)$

$(3x^2y - y^3z^2)$

$$\nabla\phi = i(6xy) + j(3x^2 - 3y^2 z^2) + \hat{k}(-2y^3z)$$

$\cancel{\nabla\phi =}$ grad ϕ at $(1,-2,-1)$

$$= 6(1)(-2)\hat{i} + \hat{j}(3(1)^2 - 3(-2)^2(-1)^2) + \hat{k}(-2(-2)^3(-1))$$

$$\Rightarrow -12\hat{i} - 9\hat{i} - 16\hat{k}$$ Ans

Normal:- If $\phi(x,y,z) = c$ represent a family of surfaces for difference c. then $\nabla\phi$ is vector normal to the surface $\phi(x,y,z) = c$.

the direction derivative of $\phi(x, y, z) = c$ in the direction of vector $\vec{d}$ is equal to $\nabla\phi \cdot \hat{d}$.

Q) Find the directional derivative of $x^2 y^2 z^2$ at the point $(1, 1, -1)$ in the direction of the tangent to the curve $x = e^t, y = \sin 2t - 1, z = 1 - \cos t$ at $t = 0$.

Sol we have $\phi = x^2 y^2 z^2$

grad $\phi = \nabla \phi$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x^2 y^2 z^2)$$

$$\cancel{\Rightarrow \hat{i}(2x) + \hat{j}(2y}$$

$$\Rightarrow \hat{i}(2xy^2z^2) + \hat{j}(2yx^2z^2) + \hat{k}(2zx^2y^2)$$

grad ϕ at point $(1, 1, -1)$

$$\text{grad}\,\phi = \hat{i}(2 \times 1 \times 1) + \hat{j}(2 \times 1 \times 1) + \hat{k}(-2 \times 1 \times 1)$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k}$$

At $t = 0$ curve is

$x = e^0 - 1$

$y = \sin 0 - 1 = -1$

$z = 0$

$\vec{r} = e^t\hat{i} + (\sin 2t - 1)\hat{j} + (1-\cos t)\hat{k}$

tangent vector $\left(\frac{d\vec{r}}{dt}\right) = e^t\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

tangent vector $\left(\frac{d\vec{r}}{dt}\right)_{t=0} = \hat{i} + 2\hat{j}$

$\vec{d} = \hat{i} + 2\hat{j}$

$\hat{d} = \frac{\hat{i}+2\hat{j}}{\sqrt{1+2^2}} = \frac{\hat{i}+2\hat{j}}{\sqrt{5}}$

required direction derivative of x^2y^2z along the tangent vector.

$= \nabla\phi \cdot \hat{d}$

$= (2\hat{i} + 3\hat{j} - 2\hat{k}) \cdot \frac{\hat{i}+2\hat{j}}{\sqrt{5}}$

$\Rightarrow \frac{2\times1 + 2\times2 + 2\times0}{\sqrt{5}}$

$= \frac{6}{\sqrt{5}}$ Ans

Q Find the unit normal vector to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

Sol ~~so~~ we have

$\phi = xy^3z^2 - 4$

normal vector of $\phi = \nabla\phi$

$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(xy^3z^2 - 4)$

$\Rightarrow \hat{i}(y^3z^2) + \hat{j}(3xy^2z^2) + \hat{k}(2xy^3z)$

normal vector of ϕ at $(-1,-1,2)$

$\Rightarrow -4\hat{i} - 12\hat{j} + 4\hat{k}$

unit normal vector of at $(-1,-1,2)$

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16+144+16}} = \frac{-4}{4\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$$

$$= \frac{-1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k}) \text{ Ans}$$

Q) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Sol Normal vector of surface

$$x^2+y^2+z^2-9=0$$

$$\nabla\phi = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x^2+y^2+z^2-9)$$

$$\Rightarrow 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal vector at $(2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$ —— (1)

and normal vector of surface

$$x^2+y^2-z-3=0$$

$$= \nabla\phi = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x^2+y^2-z-3)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

normal vector at point $(2,-1,2)$

$$= 4\hat{i} - 2\hat{j} - \hat{k}$$ —— (2)

Let θ be angle between ① & ②

$$\Rightarrow (4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k}) = \sqrt{16+4+16}\,\sqrt{16+4+1}\,\cos\theta$$

$$\Rightarrow 16+4-4 = 6\sqrt{21}\cos\theta$$

$$16 = 6\sqrt{21}\cos\theta$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

Q) find the directional derivative of $\frac{1}{r}$ in the direction $\vec{r}$.

Where $\vec{r} = xi + y\hat{j} + z\hat{k}$.

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

We have

$$\nabla\phi = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(\frac{1}{r}\right)$$

$$\Delta\phi = \hat{i}\left(-\frac{1}{r^2}\frac{\delta r}{\delta x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\delta r}{\delta y}\right) + k\left(-\frac{1}{r^2}\frac{\delta}{\delta z}\right)$$

$$\Delta\phi = -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$\Delta\phi = -\frac{1}{r^2}\frac{(xi+y\hat{j}+z\hat{k})}{r}$$

$$= \frac{-\vec{r}}{r^3}$$

So directional Derivative $= \frac{-\vec{r}}{r^3} \cdot \frac{\vec{r}}{|\vec{r}|}$

$\boxed{\vec{r} \cdot \vec{r} = r^2}$

$$\Rightarrow \frac{-r^2}{r^4} = \frac{-1}{r^2} \quad \underline{\text{Ans}}$$

~~Q) find the directional~~

Q) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that (i) $\text{grad}\, r = \frac{\vec{r}}{r}$

(ii) $\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

Sol (i) $\text{grad}\, r = \Delta r$

$$\Delta r = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(r)$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$\Rightarrow \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\Rightarrow \frac{\vec{r}}{r} \quad \underline{\text{Hence proved}}$$

(ii) $\text{grad}\left(\frac{1}{r}\right) = \Delta \frac{1}{r}$

$$\Delta\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(\frac{1}{r}\right)$$

$$\Rightarrow \hat{i}\left(-\frac{1}{r^2}\frac{\delta r}{\delta x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\delta r}{\delta y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\delta r}{\delta z}\right)$$

$$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$$

$$\Rightarrow -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\Rightarrow -\frac{\vec{r}}{r^3}$$ Hence proved

* **Divergence of a vector point function:-**

the divergence of a vector point function $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is defined as

$$\text{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\left[\text{div}\vec{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right]$$

(div $\vec{F}$ is a scalar function)

* **Solenoidal vector function:-**

A vector $\vec{V}$ is said to be solenoidal if div $\vec{V} = 0$.

Q) If $u = x^2 + y^2 + z^2$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then find the $\operatorname{div}(u\vec{r})$.

Sol we have

$$\operatorname{div}(u\vec{r}) = \nabla \cdot (u\vec{r})$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \cdot \left((x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})\right)$$

$$\Rightarrow \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \cdot \left(x(x^2+y^2+z^2)\hat{i} + y(x^2+y^2+z^2)\hat{j} + z(x^2+y^2+z^2)\hat{k}\right)$$

$$\Rightarrow \frac{\delta}{\delta x}(x^3 + xy^2 + xz^2) + \frac{\delta}{\delta y}(yx^2 + y^3 + yz^2) + \frac{\delta}{\delta z}(zx^2 + zy^2 + z^3)$$

$$\Rightarrow (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2)$$

$$\Rightarrow 5(x^2 + y^2 + z^2)$$

$$= \cancel{5u}$$

$$= 5u \quad Ans$$

Q) Find the value of n for which the vector $r^n\vec{r}$ is solenoidal where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol $\operatorname{div}\vec{V} = \operatorname{div}(r^n\vec{r})$

$$\nabla(r^n\vec{r}) = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k}\right)$$

$$\Rightarrow \frac{\delta}{\delta x}(r^n x) + \frac{\delta}{\delta y}(r^n y) + \frac{\delta}{\delta z}(r^n z)$$

$$\Rightarrow r^n + n r^{n-1}\frac{\delta r}{\delta x}\cdot x + r^n + n r^{n-1}\frac{\delta r}{\delta y}\cdot y$$

$$+ r^n + n r^{n-1}\frac{\delta r}{\delta z}\cdot z$$

$$\Rightarrow 3r^n + n r^{n-1}\left[\frac{x}{r}\cdot x + \frac{y}{r}\cdot y + \frac{z}{r}\cdot z\right]$$

$$= 3r^n + \frac{n r^{n-1}}{r}(x^2+y^2+z^2)$$

$$\text{div}\,\vec{v} = 3r^n + \frac{n r^{n-1}}{r} r^2$$

$$= 3r^n + n r^n = (3+n)r^n$$

for $\vec{v} = r^n\vec{r}$ is a solenoidal $\text{div}\,\vec{v} = 0$

$$= (3+n)r^n = 0$$

$$\Rightarrow 3 + n = 0$$

$$n = -3 \quad \text{Ans}$$

Q) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ $= \sqrt{x^2+y^2+z^2}$ then Show that

1. $\text{div}\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}$

2. $\text{div}\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0$

(3) $div(grad\, r^n) = n(n+1)\, r^{n-2}$ Hence Show that $\nabla^2\left(\frac{1}{r}\right) = 0$

Sol (1) $\underline{L.H.S}$ $div\left(\frac{\vec{r}}{r^2}\right)$

$$div(\vec{r}\cdot r) = \Delta(r\cdot\vec{r})$$

$$\Rightarrow \left(i\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(rx\hat{i} + ry\hat{j} + rz\hat{k}\right)$$

$$\Rightarrow r + \frac{\delta r}{\delta x}\cdot x + r + \frac{\delta r}{\delta y}\cdot y + r + \frac{\delta r}{\delta z}\cdot z$$

$$\Rightarrow 3r + \frac{x}{r}\cdot x + \frac{y}{r}\cdot y + \frac{z}{r}\cdot z$$

$$\Rightarrow \frac{3r+1}{r}\left(x^2+y^2+z^2\right)$$

$$\Rightarrow 3r + \frac{r^2}{r}$$

$$= 3r +$$

Q) Find the directional derivative of divergence of $\vec{V} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$ at the point (1, 2, 2) in the direction of order normal of the sphere $x^2+y^2+z^2=9$.

Sol $\text{Div}\,\vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)$

$(x^4\hat{i} + y^4\hat{j} + z^4\hat{k})$

$= \frac{\delta x^4}{\delta x} + \frac{\delta y^4}{\delta y} + \frac{\delta z^4}{\delta z}$

$\Rightarrow 4x^3 + 4y^3 + 4z^3$

Directional derivative of ~~div~~ $\vec{V}$ =

$\text{div}\,\vec{v} = \Delta(4x^3+4y^3+4z^3) = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)$

$(4x^3+4y^3+4z^3)$

$= 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$

Directional Derivative at (1, 2, 2)

$= 12\hat{i} + 48\hat{j} + 48\hat{k}$

* Curl :- Curl of a vector point function $\vec{V}$ is defined as

$$\text{Curl } \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

If Curl $\vec{V} = 0$ then $\vec{V}$ is irrational

Q) If

Q) find the divergences and curl of the $\vec{V} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ at $(2,-1,1)$.

Sol $\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)$

$\left(xyz\hat{i} + 3x^2y\hat{j} + \right.$

$\text{div}\vec{V} = -1 \times 1 + 3(2)^2 + 2 \times 2(1) - (-1)^2 = 14$

and curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$

$= \hat{i}\left[\frac{\delta}{\delta y}(xz^2 - y^2z) - \frac{\delta}{\delta z}(3x^2y)\right] - \hat{j}\left[\frac{\delta}{\delta x}(xz^2 - y^2z) - \frac{\delta}{\delta z}xyz\right]$

$\text{curl}\vec{V} = \hat{i}[-2yz] - \hat{j}[z^2 - xy] + \hat{k}[6xy - xz]$

at $(2,-1,1)$

$\text{curl}\vec{V} = \hat{i}[-2 \times -1 \times 1] - \hat{j}[1^2 - 2 \times -1] + \hat{k}[6 \times 2 \times -1 - 2 \times 1]$

$= 2\hat{i} - 3\hat{j} - 14\hat{k}$ Ans

Q) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r^2 = x^2 + y^2 + z^2$ then show that the vector $r^n\vec{r}$ is irrotational.

Sol: we have

$(curl(r^n\vec{r}) = \nabla \times r^n\vec{r}$

$$\Rightarrow \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times (r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$\Rightarrow \hat{i}\left[\frac{\delta}{\delta y} r^n z - \frac{\delta}{\delta z} r^n y\right] - \hat{j}\left[\frac{\delta}{\delta x} r^n z - \frac{\delta}{\delta z} r^n x\right]$$

$$+ \hat{k}\left[\frac{\delta}{\delta x} r^n y - \frac{\delta}{\delta y} r^n x\right]$$

$\Rightarrow \hat{i} \quad 0$

$r^n\vec{r}$ is irrotational

Q) find the divergent and curl of
$\vec{V} = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$

Q) Prove that for every vector field $\vec{v}$

$$\boxed{\operatorname{div}(\operatorname{curl} \vec{v}) = 0}$$

Sol We have

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\Rightarrow i\left[\frac{\delta}{\delta y} v_3 - \frac{\delta}{\delta z} v_2\right] + \hat{j}\left[\frac{\delta}{\delta x} v_3 - \frac{\delta}{\delta z} v_1\right]$$

$$+ \hat{k}\left[\frac{\delta}{\delta x} v_2 - \frac{\delta}{\delta y} v_1\right]$$

$\text{div}(\text{curl } \vec{v})$

$$= \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \cdot \left[i\left(\frac{\delta v_3}{\delta y} - \frac{\delta v_2}{\delta z}\right) - j\left(\frac{\delta v_3}{\delta x} - \frac{\delta v_1}{\delta z}\right) + \hat{k}\left(\frac{\delta v_2}{\delta x} - \frac{\delta v_1}{\delta y}\right) \right]$$

$$\Rightarrow \cancel{\frac{\delta^2 v_3}{\delta x \delta y}} - \cancel{\frac{\delta^2 v_2}{\delta x \delta z}} - \cancel{\frac{\delta^2 v_3}{\delta y \delta x}} + \cancel{\frac{\delta^2 v_1}{\delta y \delta z}} + \cancel{\frac{\delta^2 v_2}{\delta z \delta x}} - \cancel{\frac{\delta^2 v_1}{\delta z \delta y}}$$

$\Rightarrow 0$ proved

* Line Integral :- Let $\vec{F}(x, y, z)$ be a vector function and AB is given curve then line integral

$$= \int_c \vec{F} \cdot d\vec{r}$$

Work :- If $\vec{F}$ represent the variable force acting on a particle along arc AB then the total work done

$$= \int_A^B \vec{F} \cdot d\vec{r}$$

Q) If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle in xy-plane from $(0,0)$ to $(1,4)$ along a curve $y = 4x^2$. Find the work done.

Sol: work done $= \int_C \vec{F} \cdot \vec{dr}$

$$= \int_{(0,0)}^{(1,4)} (2x^2y\hat{i} + 3xy\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$

$$\Rightarrow \int_{(0,0)}^{(1,4)} 2x^2y\,dx + 3xy\,dy$$

$$= \int_0^1 2x^2 4x^2\,dx + 3x(4x^2)8x\,dx$$

$$= \int_0^1 (8x^4 + 96x^4)\,dx \qquad \left[\because \begin{array}{l} y = 4x^2 \\ dy = 8x\,dx \end{array}\right]$$

$$= \int_0^1 104x^4\,dx$$

$$\Rightarrow 104\left[\frac{x^5}{5}\right]_0^1$$

$$\Rightarrow 104 \times \frac{1}{5}$$

$$\Rightarrow \frac{104}{5} \text{ (Ans)}$$

Q) A vector field is given by

$$\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the path C is $x = 2t$, $y = t$, $z = t^3$ from $t=0$ to $t=1$.

Sol

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$\Rightarrow \int_C (2y+3)\,dx + xz\,dy + (yz-x)\,dz$$

We have $x = 2t \Rightarrow dx = 2dt$

$y = t \Rightarrow dy = dt$

$z = t^3 \Rightarrow dz = 3t^2 dt$ & limit of t, from $t=0$ to $t=1$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (2t+3)\,2\,dt + (2t\,t^3)\,dt + (t\,t^3 - 2t)\,3t^2\,dt$$

$$\rightarrow \int_0^1 [4t + 6 + 2t^4 + 3t^6 - 6t^3]\,dt$$

$$= \left[\frac{4t^2}{2} + 6t + \frac{2t^5}{5} + \frac{3t^7}{7} - \frac{6t^4}{4}\right]_0^1$$

$\Rightarrow \frac{4}{2} + 6 + \frac{2}{5} + \frac{3}{7} - \frac{6}{4}$

$\Rightarrow 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$

$\Rightarrow 8 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$

$\Rightarrow \frac{560 + 28 + 30 - 105}{70}$

$\Rightarrow \frac{513}{70}$ Ans

use green's theorm to evaluate $\oint_C$

$\int_C (x^2 + xy)\, dx + (x^2 + y^2)\, dy$ where C is

the square formed by line $x = \pm 1$, $y = \pm 1$

~~Sol~~ we ~~have~~ know that green's theorm

$$\int_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\delta \psi}{\delta x} - \frac{\delta \phi}{\delta y} \right) dx\, dy$$

$$\int_C (x^2 + xy)\, dx + (x^2 + y^2)\, dy = \iint_R (2x - x)\, dx\, dy$$

$$\Rightarrow \int_{-1}^{1} \int_{-1}^{1} x\, dx\, dy$$

$$\Rightarrow \int_{-1}^{1} x\, dx \int_{-1}^{1} dy = \int_{-1}^{1} x\, dx\, [y]_{-1}^{1}$$

$$= \int_{-1}^{1} 2x\, dx = 2 \left[\frac{x^2}{2} \right]_{-1}^{+1} = (1)^2 - (-1)^2 = 0$$

Ans

---

we have $\int_C x^2 y\, dx + x^2\, dy$

$$= \int_{\substack{\text{along } A \\ \text{i.e. } y = 0}} x^2 y\, dx + x^2\, dy + \int_{\substack{\text{on } AB \\ \text{i.e. } x = 1}} x^2 y\, dx + x^2\, dy$$

$$+ \int_{\substack{\text{dy o } B \\ y = x}} x^2 y\, dx + x^2\, dy$$

$$= 0 + \int_0^1 1^2 dy + \int_1^0 x^2 \cdot x\,dx + x^2\,dx$$

$$= \int_0^1 dy + \int_1^0 (x^3 + x^2)\,dx$$

$$= [y]_0^1 + \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_1^0$$

$$\Rightarrow 1 + \left[-\frac{1}{4} - \frac{1}{3}\right]$$

$$\Rightarrow \frac{5}{12} \text{ Ans}$$

Q verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2\,dy$ where C is bounded by $y = x$ & $y = x^2$

Sol $\int_C (xy + y^2)\,dx + x^2\,dy$

$$= \iint_R [2x - (x + 2y)]\,dx\,dy$$

$$= \iint (x - 2y)\,dx\,dy$$

$$= \int_0^1 dx \int_{x^2}^{x} (x - 2y)\,dy$$

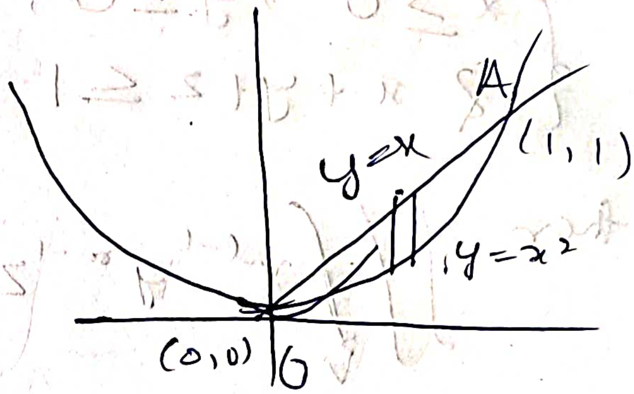
$$= \int_0^1 dx \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x}$$

$$= \int_0^1 (x^2 - x^2) - (x^3 - x^4)\, dx$$

$$= \int_0^1 x^4 - x^3\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \text{ Ans}$$



We have $\int_C (xy + y^2)\,dx + x^2\,dy$

$$= \int_{C_1} (xy + y^2)\,dx + x^2 dy + \int_{C_2} (xy + y^2)\,dx + x^2\,dy$$

$$= \int_0^1 (x x^2 + x^4)\,dx + x^2(2x\,dx) + \int (x^2 + x^2)\,dx + x^2\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx + \int_1^0 3x^2\,dx$$

$$\Rightarrow \left[\frac{5x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{3x^3}{3}\right]_1^0$$

$$\Rightarrow \left(\frac{3}{4} + \frac{1}{5}\right) + (0 - 1)$$

$$= \frac{15 + 4 - 20}{20} = -\frac{1}{20} \text{ Ans}$$

* Drichilet's theorem:-

$$\left\{ \begin{array}{l} x \geq 0, y \geq 0, z \geq 0 \\ \& \ x+y+z \leq 1 \end{array} \right\}$$

then $\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz$

$$\Rightarrow \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1}$$

Q) Find the mass of an octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the density at an point being $kxyz$.

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- $V = 2.68 \times 10^8$ m/s

Module - II

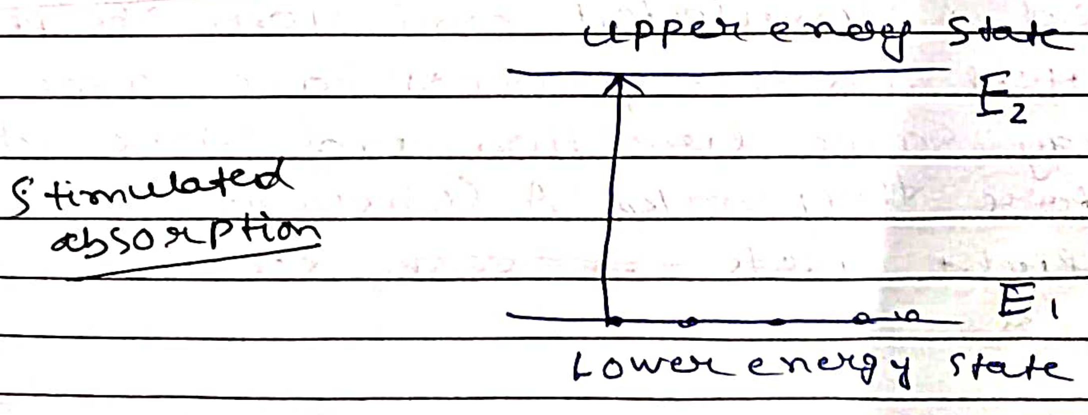
Laser

laser stands ~~for~~ light amplification by ostillmated emmison of readiation.

property of laser

1. Monochromatic
2. ~~Coherent~~ highly coherent beam.
3. Intensity
4. Unidirectional
5. Focussily

(1) Stimulated Absorption :- In normal condition most of the atoms exist in ground state. if we provide energy to them, atoms will get excited and jump to upper energy level this transition is called stimulated absorption.



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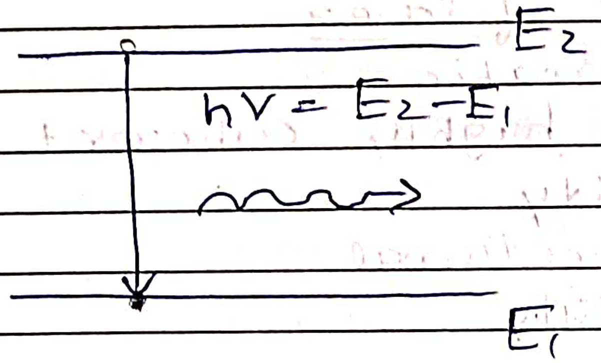
② Spontaneous emission.

the atoms can exist in excited state only for 10^{-8} seconds. after that they return to lower energy state by emitting a photon of energy $h\nu = E_2 - E_1$.
this Transition is called Spontaneous emission. the photons emitted by spontaneous emission process have random direction and random phase they make incoherent light beam

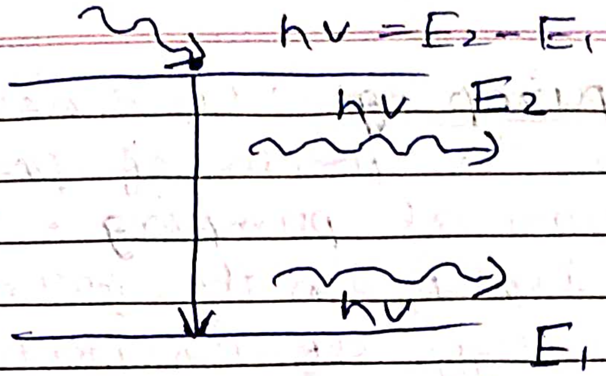
spontaneous emission



③ Stimulated emission

let an atom exist in excited state. and incoming Photon of energy ~~KE~~ = $h\nu = E_2 - E_1$ forces this atom to return lower energy state. two photons of energy $h\nu$ are emitted this is Called : Stimulated emission. the photons emitted in this process have same energy same direction and same phase therefore they make A Coherent beam of light ~~made as lase~~ Known as Laser.

Stimulated emission



Population Inversion

In ordinary condition number of atoms N_1 in lower energy state is greater than number of atom ~~and~~ N_2 those in upper state. therefore probability of stimulated emission is very low. If by some technique the number of atom in excited state is made much larger than lower state, probability of stimulated emission increases.

" the situation in which number of atom in upper state is greater than number of atom in lower state is called population inversion.

$$\boxed{N_2 > N_1}$$

* Pumping

the method use to increase numbers of atom in excited state is called pumping. several techniques are used to provide energy to the ground state atom. Commonly used methods are

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① **Optical pumping** → if energy is provided in the form of photon pumping is called optical pumping. the light source gives light in the form of pulses. the energy of excitation photon is greater than energy of emitted photon
eg → ~~Ree~~ Ruby laser.

② **Electric discharge pumping**
In this method accelerated electrons produce by electric discharge give their energy to the atom so that atoms get excited to upper ~~limit~~ level.
e.g → helium neon laser

③ **Chemical pumping** :- In this process Chemical reaction is used to excite the atoms.
eg → Carbondioxide laser.

④ **X-Ray pumping** → X-ray Photons are use for pumping process.
eg → dye laser

Meta Stable State :-
this energy state is a ~~known~~ long lived energy state in which atom can stay for 10^{-3} sec. the atoms do not return to lower state imidiately thus the probability of spontaneous emission is quite negligible ~~active~~

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Active System

A material in which population inversion is achieve is called active system of laser. it contains callection of atom molecules or ions which is responsible for laser action it can be a liquid solid or gas.

eg → i) In ruby laser Chromium work as active system.

ii) In helium Neon laser, Neon works as active system

* Condition necessary to achieve laser action.

i) there must be population inversion.

ii) there must exist a metastable state. in which atom can stay for 10^{-3} sec

iii) we must use a Resonator (cavity).

→ there must be a pair of mirror at the ends of active system. One mirror is fully silvered and the other is partially silvered. the pair of mirrors is called resonator or cavity.

Relation between einstein ~~position~~ Cofficien

let N_1 be the no of atom in lower energy state E_1 and N_2 be the no. of atom in upper energy state E_2. let a radiation of frequency

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beam incident such that $h\nu = E_2 - E_1$

$u(\nu) \rightarrow$ energy density of radiation

① The no. of absorption per unit volume per unit time $\propto N_1 u(\nu)$

$$= B_{12} N_1 u(\nu)$$

$B_{12} \rightarrow$ Einstein coefficient for absorption

② the no. of spontaneous emission per unit volume per unit time $\propto N_2$

$$= A_{21} N_2$$

$A_{21} \rightarrow$ Einstein coefficient for spontaneous emission.

③ the no. of ~~time~~ Stimulated emission per unit time per unit volume $\propto N_2 u(\nu)$

$$= B_{21} N_2 u(\nu)$$

$B_{21} \rightarrow$ Einstein coefficient for stimulated emission.

In thermal equilibrium

rate of absorption = rate of emission

$$B_{12} N_1 u(\nu) = A_{21} N_2 + B_{21} N_2 u(\nu)$$

$$u(\nu) \left[B_{12} N_1 - B_{21} N_2 \right] = A_{21} N_2$$

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$$u(\nu) = \frac{A_{21} N_2}{B_{12} N_1 - B_{21} N_2} = \frac{A_{21} N_2}{N_2\left[B_{12}\frac{N_1}{N_2} - B_{21}\right]}$$

$$u(\nu) = \frac{A_{21}}{B_{12}\left[\frac{N_1}{N_2} - \frac{B_{21}}{B_{12}}\right]} \quad \text{---} \quad (4)$$

According to Boltzmann distribution la

$$N = N_0\, e^{-E/KT}$$

$N_0 \rightarrow$ Total no. of atoms.

$K \rightarrow$ Boltzmann Constant

$$N_1 = N_0\, e^{-E_1/KT}$$

$$N_2 = N_0\, e^{-E_2/KT}$$

$$\frac{N_1}{N_2} = \frac{N_0\, e^{-E_1/KT}}{N_0\, e^{-E_2/KT}} = e^{(E_2 - E_1)/KT}$$

$$\frac{N_1}{N_2} = e^{h\nu/KT} \quad \text{---} \quad (5)$$

Putting eq (5) in (4)

$$u(\nu) = \frac{A_{21}}{B_{12}\left[e^{h\nu/KT} - \frac{B_{21}}{B_{12}}\right]} \quad \text{---} \quad (6)$$

According to Planck's law of radiation

$$u(\nu) = \frac{8\pi h\nu^3}{c^3\left[e^{h\nu/KT} - 1\right]} \quad \text{---} \quad (7)$$

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Compairing ⑥ and ⑦

$$\frac{B_{21}}{B_{12}} = 1 \Rightarrow B_{12} = B_{21}$$

$$\boxed{\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3}}$$

Ruby laser (3-level laser)

Construction-1:- Active System

It consist a pink Ruby rod of cylindrical Shape. Ruby is a crystal of Aluminium oxide (Al_2O_3) doped with 0.05 percent Chromium oxide (Cr_2O_3)

Construction 2:- ~~Of~~ Pumping

Optical pumping used A ~~a~~ Coild lamp field with ~~X~~enon is bounded over the rod it gives light of wavelength 5500 Å in the form of flash.

Construction 3:- Resonator

Both ends of the rod are made optically flat and parallel. One ~~it~~ end ~~partially~~ is fully silvered and they other is partially Silvered.

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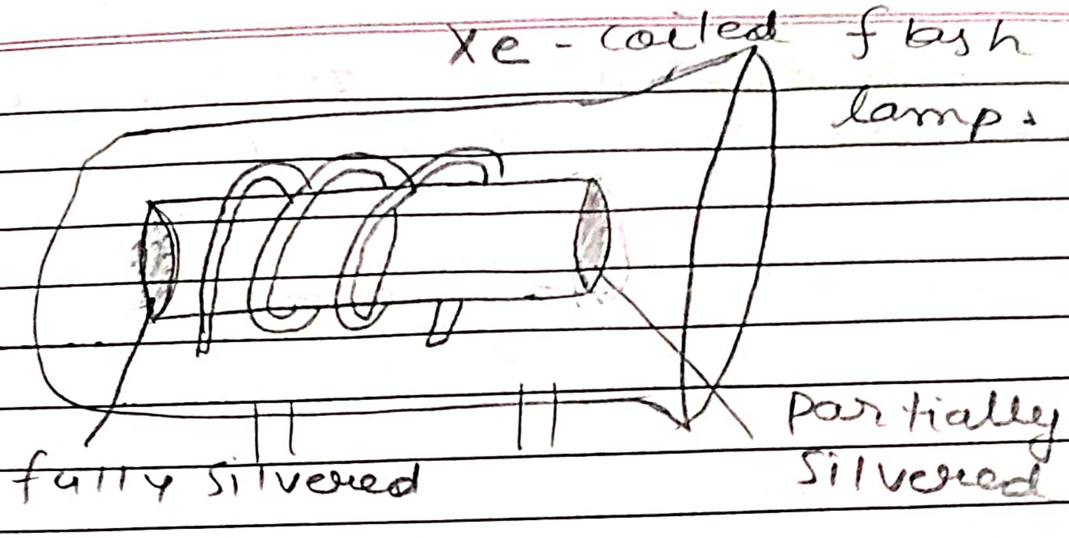

working

1. When flash of light from Xenon lamp falls upon ruby rod, chromium ions (Cr^{+++}) absorb photon of wavelength 5500 Å and move to excited State E_2. (Transition 1)

2. the excited ion gives there part of energy to the crystal and moves to metastable State E_3 (Transition 2).

3. One ion returns to ground State by spontaneous emission emitting a photon of wavelength 6943 Å (Transition 3)

4. this photon travels throug the rod and is reflected baking forth until its stimulate an ion in E_3 to stimulated emission. (Transition - 4). Hence two photons are emitted

5. Process 4 is repeated again and again

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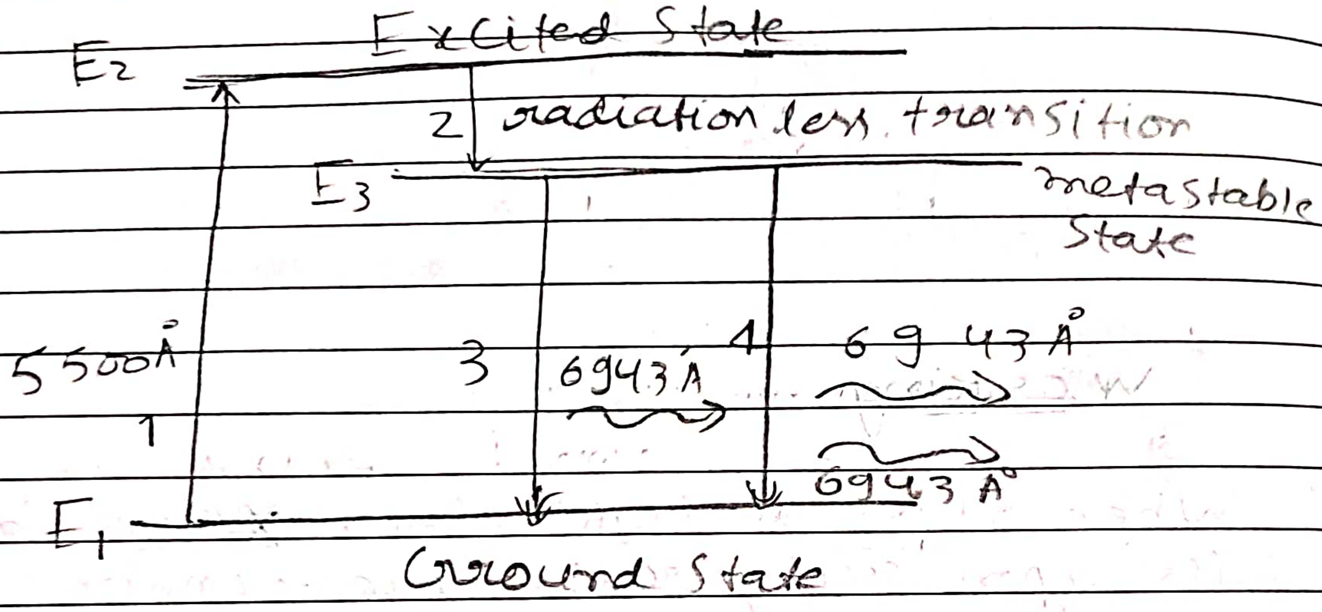
When heat becomes intense, laser comes out the partially on silvered end.



Drawback
the laser light emitted is in the form of pulses.

* Helium-Neon laser (4- level laser)

Construction 1:- Active System
it consist a narrow discharge tube of ~~Chords~~ quartz filled with Helium and Neon in the ratio 7:1 at low pressure.

Costruction 2:- ~~Inelastic ato~~ Pumping
Inelastic atom atoms is used two eléctrode are connected to high frequency alternating current.

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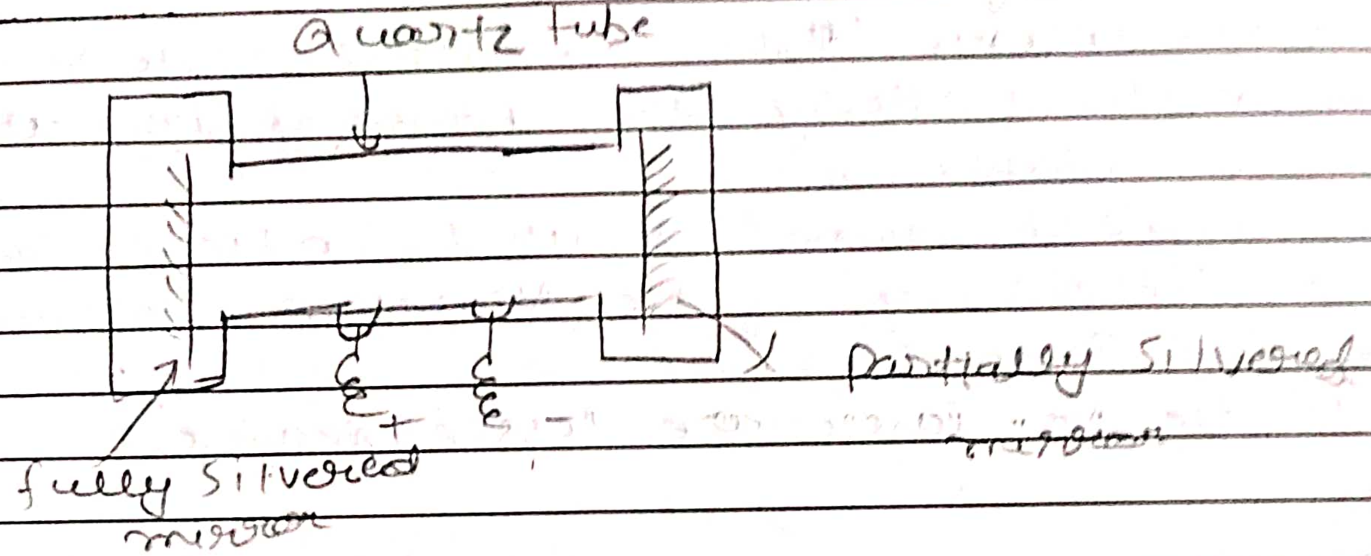
Construction 3 :- Resonator
Two plane parallel mirrors are attached with a tube. distance between mirrors is integral multiple of laser wavelength



Working

1. the helium atoms absorb energy from electrons produced by electric discharge ~~At~~ An move to excited state.
2. Excited helium collide with ground state neon and transfer there energy to neon. So that neon moves to metastable state.
3. When population inversion is achieve, laser action takes place between metastable state and excited state At $\boxed{18.70\,eV}$
4. Another transition due to spontaneous emission from excited to intermidiate state gives in coherent photon of wavelength 72 ~~nm~~ Nm
5. ~~the same~~

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5. the remaining exciting energy is lost in collision with tube's walls and neon returns to ground state.

Advantage :-
1. Continuous laser is emitted in He-Ne laser while in Ruby laser pulses of light are emitted.
2. Coolent is not required in He-Ne laser
3. efficiency of He-Ne is greater than ~~Ru~~ Ruby laser.
4. He-Ne laser are less expensive.