Homogeneous function:-

A function $f(x,y)$ is said to be homogeneous function in which the power of each term is the same.

A function $f(x,y)$ is a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n. Thus

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n.$$

is a homogeneous function of order n.

it can also written as

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_n\left(\frac{y}{x}\right)^n\right] = x^n\,\phi\left(\frac{y}{x}\right)$$

e.g.:- (i) $x^3\left[1 + \frac{y}{x} + 3\left(\frac{y}{x}\right)^2 + 6\left(\frac{y}{x}\right)^3\right]$, order $= 3$

(ii) $\dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \dfrac{\sqrt{x}}{x^2}\,\dfrac{1+\sqrt{y/x}}{1+(y/x)^2} = x^{-3/2}\,\phi(y/x)$

order $= -3/2$

(iii) $\dfrac{x^2+1}{x^2+y^2}$ is not homogeneous function.

Euler's theorem on homogeneous function:

Statement:- If z is a homogeneous function of x, y of order n, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$

Proof:- Since z is a homogeneous function of x, y of order n. then z can be written as

$$z = x^n f\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

Differentiating (1) partially w.r.to 'x' we have

$$\frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) - x^{n-2}y\,f'\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1}y\,f'\left(\frac{y}{x}\right) \quad \text{--- (2)}$$

Differentiating (1) partially w.r. to 'y', we have

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\Rightarrow y\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{y}{x} \quad \text{—} (3)$$

Adding (2) and (3), we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n \cdot x^n f\left(\frac{y}{x}\right)$$

$$\Rightarrow \left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz\right]$$

Proved

Probm: {Euler's Deduction formula- I}

If z is a homogeneous function of x, y of degree n, and $z = f(u)$, then

$$\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}\right]$$

Proof: Since z is a homogeneous function of x,y of order n, then by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{—} (1)$$

Now $z = f(u)$,

$$\therefore \frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Substituting in (1), we get

$$x f'(u)\frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} = n f(u)$$

$$\Rightarrow \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}\right]$$

Proved

Q:- Verify Euler's theorem for $u = \dfrac{x^3y^3}{x^3+y^3}$ — (1)

Solution: here. $u = \dfrac{x^6\,(y/x)^3}{x^3\left(1+(y/x)^3\right)} = x^3\,\dfrac{(y/x)^3}{1+(y/x)^3}$

u is a homogeneous function of degree 3 so by Euler's theorem $\left[x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u\right]$

from (1),

$$\frac{\partial u}{\partial x} = \frac{(x^3+y^3)(3x^2y^3) - x^3y^3(3x^2)}{(x^3+y^3)^2} = \frac{\cancel{3x^5y^3} + 3x^2y^6 - \cancel{3x^5y^3}}{(x^3+y^3)^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{\cancel{} + 3x^3y^6 \cancel{}}{(x^3+y^3)^2} = \frac{3x^3y^6}{(x^3+y^3)^2} \quad \text{— (2)}$$

and

$$\frac{\partial u}{\partial y} = \frac{(x^3+y^3)\cdot(3x^3y^2) - x^3y^3(3y^2)}{(x^3+y^3)^2} = \frac{3x^6y^2 + \cancel{3x^3y^5} - \cancel{3x^3y^5}}{(x^3+y^3)^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{3x^6y^3}{(x^3+y^3)^2} \quad \text{— (3)}$$

adding (2) and (3), we have, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3x^3y^6 + 3x^6y^3}{(x^3+y^3)^2}$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{3x^3y^3(x^3+y^3)}{(x^3+y^3)^2}$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\,\dfrac{x^3y^3}{x^3+y^3}$

or $\quad \left[x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u\right]$

hence, Euler's theorem is verified.

Q:- If $u = \log_e\left(\dfrac{x^4+y^4}{x+y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Solution. We have, $\quad u = \log\left(\dfrac{x^4+y^4}{x+y}\right)$

here u is not homogeneous function but

if $z = e^u = \dfrac{x^4+y^4}{x+y} = \dfrac{x^4\left(1+(y/x)^4\right)}{x\,(1+y/x)} = x^3\,\phi\left(\dfrac{y}{x}\right)$ is

homogeneous equation of order 3. hence by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

and $\quad z = e^u \Rightarrow \dfrac{\partial z}{\partial x} = e^u\,\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial z}{\partial y} = e^u\,\dfrac{\partial u}{\partial y}$

$\Rightarrow e^u x\dfrac{\partial u}{\partial x} + e^u y\dfrac{\partial u}{\partial y} = 3\cdot e^u \Rightarrow \boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3}$ Proved

Q:- If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Solution: here, we have
$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$
is not a homogeneous function but if $z = \cos u$ then
$$z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$ is homogeneous equation of order $1/2$. then by Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$$

$$x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{1}{2}z$$

$$\Rightarrow x\frac{\partial u}{\partial x}(-\sin u) + y\frac{\partial u}{\partial y}(-\sin u) = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

or $\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0\right]$ Proved

Q:- If z be a homogeneous function of degree n, show that

(i) $x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = (n-1)\frac{\partial z}{\partial x}$ (ii) $x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}$

(iii) $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$.

Solution:- By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ —(1)

Diff (1) partially w.r.to 'x' we get

$$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = n\frac{\partial z}{\partial x} \Rightarrow x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = (n-1)\frac{\partial z}{\partial x}$$

—(2) (i) Proved

Diff (1) partially w.r.to 'y', we get

$$x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n\frac{\partial z}{\partial y} \Rightarrow x\frac{\partial^2 z}{\partial x\partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}$$

(3) Proved (ii)

Now, multiplying (2) by x and (3) by y, and adding them, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)\left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right\}$$

$$= (n-1)nz$$

Proved

Euler's deduction formula - II

Prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$

where, $g(u) = n \frac{f(u)}{f'(u)}$

Proof:- By Euler's deduction formula I

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = g(u) \qquad \left(\text{given } n \frac{f(u)}{f'(u)} = g(u)\right) \quad \text{--- (1)}$$

Differentiating (1) partially w.r. to 'x' we have

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot 1 + y \frac{\partial^2 u}{\partial x \partial y} = g'(u) \cdot \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = [g'(u) - 1] \frac{\partial u}{\partial x} \quad \text{--- (2)}$$

Similarly on differentiating (1) partially w. r. to 'y', we have

$$y \frac{\partial^2 u}{\partial y^2} + x \frac{\partial^2 u}{\partial y \partial x} = [g'(u) - 1] \frac{\partial u}{\partial y} \quad \text{--- (3)}$$

multiplying (2) by x, (3) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = [g'(u) - 1] \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = [g'(u) - 1]\, g(u)$$

Proved

Ques:- If $u = \log\left(\frac{x^4 - y^4}{x - y}\right)$, Prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -3$

Proof

here, $u = \log\left(\frac{x^4 - y^4}{x - y}\right)$

let $z = e^u = \frac{x^4 - y^4}{x - y}$

here z is homogeneous function of order 3.

$$\Rightarrow \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3z$$

or

$$x \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + y \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = 3e^u$$

$$\Rightarrow \quad x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = 3e^u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3$$

so here $g(u) = 3$

By deduction formula II, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= 3\,[0 - 1] = -3$$

Proved.

Q→ If $u = \tan^{-1}(x^2 + 2y^2)$

then show that (i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \sin u \cdot \cos 3u$

Proof:- Here, we have,

$u = \tan^{-1}(x^2 + 2y^2)$ is not homogeneous function

let $\quad z = \tan u = x^2 + 2y^2$

here z is homogeneous function of order 2.

$$\Rightarrow \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow \quad x \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + y \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

$$\Rightarrow \quad x (\sec^2 u) \frac{\partial u}{\partial x} + y (\sec^2 u) \frac{\partial u}{\partial y} = 2 \tan u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = 2 \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \sin u \cos u = \sin 2u$$

Proved.

(ii) By Euler's II deduction formula.

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

here $g(u) = \sin 2u$

$\Rightarrow g'(u) = 2 \cos 2u$

$$= \sin 2u\,[2 \cos 2u - 1]$$

$$= 2 \sin u \cos u\,[2(2\cos^2 u - 1) - 1]$$

$$= 2 \sin u \cos u\,[4 \cos^2 u - 3]$$

$$= 2 \sin u\,[4 \cos^3 u - 3 \cos u]$$

$$= 2 \sin u \cos 3u$$

Proved.

Q:- If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$

Q:- If $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u.$$

Solution:- Here, u is not a homogeneous function.

Now, let

$$v = \sin u = \frac{x^3+y^3+z^3}{ax+by+cz} = \frac{x^3\left(1+(y/x)^3+(z/x)^3\right)}{x\left(a+b\,y/x+c\,z/x\right)}$$

$$= x^2\,\phi(y/x)$$

so that v is a homogeneous function of order 2.

Then by Euler's theorem, $\cancel{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}}$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 2v$$

or $$x\frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial z} = 2v$$

$$\Rightarrow x(\cos u)\frac{\partial u}{\partial x} + y(\cos u)\frac{\partial u}{\partial y} + z(\cos u)\frac{\partial u}{\partial z} = 2\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2\sin u}{\cos u}$$

or $$\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u\right]$$ Proved.

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Prob^m. If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$, Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$$

Prob^m Verify Euler's theorem for the function

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$

Proof! We have, $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ — ①

here u is a homogeneous function of degree zero

so by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Verification!

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— ②}$$

Again Diff ① partially w.r.to 'y' we get

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{-x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— ③}$$

adding ② and ③, we get

$$\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\right]$$

Theorem is verified