## 4.9 Final Exam Solution

Question 1: $f = (1\,3\,2\,4) \circ (1\,2\,3) \circ (4\,5)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}$$

(i) Is $f \in A_5$? $f = (1\,4\,5) = (1\,4) \circ (1\,5)$

We have an even number of 2-cycles, thus $f$ is an even permutation

$\therefore$ $\underline{f \in A_5.}$

(ii) $|f|$: Since $f = (1\,4\,5) \rightarrow |f| = 3.$

(iii) Determine $f^{-1}$: $f^{-1} = (5\,4\,1)$

Question 2: All non-cyclic Abelian, 36 elements = $2^2 \cdot 3^2$ elements.

| Partition of 2: | Order $2^2$ | Order $3^2$ |
|---|---|---|
| 0 + 2 | $\mathbb{Z}_4$ | $\mathbb{Z}_9$ |
| 1 + 1 | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ |

4 groups total.

We want non-cyclic, with order 9 element (unique).

All: $\mathbb{Z}_4 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{36}$ since $\gcd(4,9) = 1$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{18}$

$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6 \oplus \mathbb{Z}_6$

Since there is a unique subgroup of order 9;

$\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ is non-cyclic
$\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$ is non-cyclic
$\mathbb{Z}_6 \oplus \mathbb{Z}_6$ is non-cyclic

Of these, they each have order 9 element. $\Longrightarrow$ $\operatorname{lcm}(|a|, |b|) = 9$

Since they are Abelian & non-cyclic, converse of Lagrange implies uniqueness for all three structures.

<u>Question 3:</u> $f: \mathbb{Z}_5 \oplus \mathbb{Z}_5 \to \mathbb{Z}_5$ ; $f(a,b) = a^{-1} + 2b$.

(i) Let $a,b,c,d \in \mathbb{Z}_5 \oplus \mathbb{Z}_5$. Then: $f(a+b, c+d) =$

$$= (a+b)^{-1} + 2(c+d)$$
$$= b^{-1} + a^{-1} + 2 \cdot c + 2 \cdot d$$
$$= [a^{-1} + 2 \cdot c] + [b^{-1} + 2 \cdot d]$$
$$= f(a,c) + f(b,d).$$

$\therefore$ $f$ is a group homomorphism.

(ii) $Ker(f)$: $a,b$ s.t. $f(a,b) = e$ in $\mathbb{Z}_5 = 0$.

By observation: consider $a^{-1} + 2 \cdot b$ where $a \in \mathbb{Z}_5, b \in \mathbb{Z}_5$

$f(0,0) \to 0$

| a \ b | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 1 | 3 |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |

$2^{-1} + 2(1) = 3 + 2 = 5 \mod 5 \equiv 0$

$\therefore (2,1)$.

$4^{-1} = 1$; $2(2) = 4$; $\Rightarrow (4,2)$ works.

$(1)^{-1} = 4$; $2(3) = 6$; $4 + 6 = 0 \mod 5$; $\Rightarrow (1,3)$ works.

$3^{-1} = 2$, $2(4) = 8$; $2 + 8 = 0 \mod 5$.

$\therefore$ $ker(f) = \{(0,0), (4,2), (1,3), (2,1), (3,4)\}$.

Question 3: $a^{-1} + 2b = f(a,b)$

(iii) Union of all left cosets make up $(\mathbb{Z}_5, +) \oplus (\mathbb{Z}_5, +)$

$\ker(f) = \{(0,0), (4,2), (1,3), (2,1), (3,4)\}$

$1 + \ker(f) = \{(1,1), (0,2), (2,3), (3,2), (4,0)\}$

$2 + \ker(f) = \{(2,2), (1,3), (3,4), (4,3), (0,1)\}$

$3 + \ker(f) = \{(3,3), (2,4), (4,0), (0,4), (1,2)\}$

$4 + \ker(f) = \{(4,4), (3,0), (0,1), (1,0), (2,3)\}$

$$\left.\begin{array}{l} f(0,0) = f(4,2) = f(1,3) = f(2,1) = f(3,4) = e. \\ f(1,1) = f(0,2) = f(2,3) = f(3,2) = f(4,0) = 1 \\ f(2,2) = \ldots = 2 \\ f(3,3) = \ldots = 3 \\ f(4,4) = \ldots = 4 \end{array}\right\}$$

<u>Question 4:</u> $(Aut(\mathbb{Z}_{24}), \circ) \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$.

(i) We know by class result: $(Aut(\mathbb{Z}_{24}), \circ) \cong (U(\cancel{24}^{24}), \times)$

$$U(24) = U(2^3 \cdot 3) \cong U(2^3) \oplus U(3).$$

$$\Rightarrow \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

$$\therefore (Aut(\mathbb{Z}_{24}), \circ) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2; \quad \left\{ \begin{array}{l} m_1 = 2 \\ m_2 = 2 \\ m_3 = 2 \end{array} \right\}$$

(ii) Subgroup, $H$, with $\Gamma H = 4$. Can $H$ be cyclic?

Construct ~~F: (Z_4 ...~~

Construct $K : (U(20), \cdot) \to (Aut(\mathbb{Z}_{24}), \circ)$

$K(a) = f_a$ for every $a \in U(24)$ & $f_a \in Aut(\mathbb{Z}_{24})$.

Let $f_a : (\mathbb{Z}_{24}, +) \to (\mathbb{Z}_{24}, +)$, $f_a(b) = ab \pmod{24}$.

$U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\}$.

~~the U(24) st. H = {1,5} is a subgroup order.~~

Take $\{1, 5, 7, 11\}$

| | 1 | 5 | 7 | 11 |
|---|---|---|---|---|
| 1 | 1 | 5 | 7 | 11 |
| 5 | 5 | 1 | 11 | 7 |
| 7 | 7 | 11 | 1 | 5 |
| 11 | 11 | 7 | 5 | 1 |

$$\therefore H \leqslant U(24) \Rightarrow \{f_5, f_7, f_{11}, e\} < Aut(\mathbb{Z}_{24}).$$

However, $H$ cannot ~~bg~~ cyclic because we cannot find elements that could form subgroups $L$, $M$ st. $|L| \neq 2$ & $|M| \neq 2$.

All subgroups are of order 2 and thus $H$ would never ~~bg~~ be cyclic.

Question 5: $(D, \cdot)$ group, $H \lhd D$ s.t. $D/H$ cyclic but $D$ is not Abelian.

Take some ~~$\mathbb{Z}$~~ $n \in \mathbb{Z} \geq 5$. We know by class result that $A_n \lhd S_n$. We have a group, $S_n$, and a normal subgroup, $A_n$.

Now: $|S_n / A_n| = \dfrac{|S_n|}{|A_n|} = \dfrac{n!}{\frac{n!}{2}} = 2$, 2 is a prime.

$\therefore$ since every group of prime order is cyclic (by result),

$S_n / A_n$ cyclic.

But we know that $S_n$ is <u>not</u> Abelian. Refer to HW1 ~~problem~~ problem for counter example.

Question 6: All Abelian with 72 elements:

$72 = 2^3 \times 3^2$

| Partition of 3 | Partition of 2 | Order $2^3$ | Order $3^2$ |
|---|---|---|---|
| 0+3 | 0+2 | $\mathbb{Z}_8$ | $\mathbb{Z}_9$ |
| 1+2 | 1+1 | $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ | $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. |
| 1+1+1 | | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ | |
| | | 3 | 2 |

We have $3 \cdot 2 = \underline{6 \text{ total}}$

i.e. There are 6 Abelian groups with $\underline{72 \text{ elements}}$.

Question 7: $U(360) = U(2^3 \cdot 3^2 \cdot 5)$

$\cong U(2^3) \oplus U(3^2) \oplus U(5)$

$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$

since $\gcd(3,4) = 1 \Rightarrow$ combine $\mathbb{Z}_4 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{12}$.

$\therefore \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$. $\Longrightarrow$ invariant factors.

$\therefore m_1 = 2, m_2 = 2, m_3 = 2, m_4 = 12$

Q8. Assume that D has a subgroup H such that [H :D] = n, where 2 <= n <=4. Then there is a nontrivial group homomorphism F : D ---> S_n. Since D is simple, Ker(F) = {e} or Ker(F) = D. Since F is nontrivial, Ker(F) not = D. Thus Ker(F) = {e}. Thus by the first-isomorphism Theorem, D is isomorphic to Range(F) = subgroup of S_n, which is impossible, since |D| >= 60 and |S_n| <= 24. Thus D does not have a subgroup H such that 1 < [H:D] <= 4.

Question 9. $f: D \to L$ group homomorphism, $H < \text{range}(f)$.

$$K = \{ a \in D \mid f(a) \in H \} < D \quad , \quad \ker(f) \subseteq K.$$

Since $H < \text{range}(f)$, $\longrightarrow$ $|H| \mid |\text{range}(f)|$

Let $a, b \in K$. Show that $a^{-1} \cdot b \in K$.

$$\left. \begin{array}{l} a \in D \text{ s.t. } f(a) \in H \\ b \in D \text{ s.t. } f(b) \in H \end{array} \right\}$$

We know $H$ group, so: $[f(a)]^{-1} \in H \longrightarrow f(a^{-1}) \in H$.

$$f(a^{-1} \cdot b) = [f(a)]^{-1} \times [f(b)] \Longrightarrow f(a^{-1}) \times f(b)$$

$f(a^{-1}) \in H$, $f(b) \in H$; $\Longrightarrow$ closed, $\therefore$ $K < D$.

Since $H < \text{range}(f)$; $\longrightarrow$ the identity element is in $H$.

Since $K$ consists of all the elements that map to $f(a) \in H$, this means $K$ maps to some elements in the range of $F$, and $e$ is in the range of $F$.

$\therefore$ The elements that map to $e$ must be in $K$, and thus $\ker(f) \subseteq K$.

Done.