# MTH320 - Abstract Algebra I

HW #2 (Solutions)

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*September 29th, 2020*

**Question 1:**

Let $A = \{1, 2, 3\}$ and $D$ be the power set of $A$, i.e., $D$ is the set of all subsets $A$ (note that $|D| = 2^3 = 8$). Define "$\cdot$" on $D$ to mean $a \cdot b = (a \cancel{b}) \cup (b \cancel{a})\, \forall a, b \in D$. Then $(D, \cdot)$ is an Abelian group.
Since $D$ is the set of all subsets of $A$, then:

$$D = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

The Caley's Table:

| $a \cdot b$ | $\varnothing$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|---|
| $\varnothing$ | $\varnothing$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
| $\{1\}$ | $\{1\}$ | $\varnothing$ | $\{1,2\}$ | $\{1,3\}$ | $\{2\}$ | $\{3\}$ | $\{1,2,3\}$ | $\{2,3\}$ |
| $\{2\}$ | $\{2\}$ | $\{1,2\}$ | $\varnothing$ | $\{2,3\}$ | $\{1\}$ | $\{1,2,3\}$ | $\{3\}$ | $\{1,3\}$ |
| $\{3\}$ | $\{3\}$ | $\{1,3\}$ | $\{2,3\}$ | $\varnothing$ | $\{1,2,3\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ |
| $\{1,2\}$ | $\{1,2\}$ | $\{2\}$ | $\{1\}$ | $\{1,2,3\}$ | $\varnothing$ | $\{2,3\}$ | $\{1,3\}$ | $\{3\}$ |
| $\{1,3\}$ | $\{1,3\}$ | $\{3\}$ | $\{1,2,3\}$ | $\{1\}$ | $\{2,3\}$ | $\varnothing$ | $\{1,2\}$ | $\{2\}$ |
| $\{2,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ | $\{3\}$ | $\{2\}$ | $\{1,3\}$ | $\{1,2\}$ | $\varnothing$ | $\{1\}$ |
| $\{1,2,3\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{1,3\}$ | $\{1,2\}$ | $\{3\}$ | $\{2\}$ | $\{1\}$ | $\varnothing$ |

**Table 1.**

(i) What is $e \in D$?

Obviously $e$ is the element where for some $a \in D$, $a \cdot e = a$. In other words, $(a - e) \cup (e - a) = a$. The only element with this property is $\varnothing$. For any $a$, $a \cdot \varnothing = a$. As an example:

$$\{1, 2, 3\} \cdot \varnothing = [\{1, 2, 3\} - \varnothing] \cup [\varnothing - \{1, 2, 3\}] = \{1, 2, 3\}$$

(ii) For each $a \in D$, find $a^{-1}$

Again, we will simply use the Caley's table to find the inverse of each of the 8 elements in $D$. We proceed as follows:

$$\begin{aligned}
\{1\}^{-1} &= \{1\} \qquad \text{Since } \{1\} \cdot \{1\} = \varnothing, \text{same argument for all} \\
\{2\}^{-1} &= \{2\} \\
\{3\}^{-1} &= \{3\} \\
\{1, 2\}^{-1} &= \{1, 2\} \\
\{1, 3\}^{-1} &= \{1, 3\} \\
\{2, 3\}^{-1} &= \{2, 3\} \\
\{1, 2, 3\}^{-1} &= \{1, 2, 3\} \\
\varnothing^{-1} &= \varnothing
\end{aligned}$$

As a matter of fact, each element is its own inverse (Again visible from the Caley's table).

(iii) For each $a \in D$, find $|a|$

A sample calculation is provided below as to how we get the order of each element. The rest is self explanatory.

$$\begin{aligned} \{1\}&: \\ \{1\}\cdot\{1\} &= \varnothing \\ \{1\}^2 &= \varnothing \\ \text{Therefore } |\{1\}| &= 2 \end{aligned}$$

$$\begin{aligned} |\{2\}| &= 2 \\ |\{3\}| &= 2 \\ |\{1,2\}| &= 2 \\ |\{1,3\}| &= 2 \\ |\{2,3\}| &= 2 \\ |\{1,2,3\}| &= 2 \\ |\varnothing| &= 1 \qquad \text{Since } \varnothing \text{ is the identity} \end{aligned}$$

(iv) The converse of the Lagrange theorem is correct when a group is finite and Abelian, i.e. if $D$ is an Abelian group, $|D| = n$, and $m|n$, Then $D$ has at least one subgroup with $m$ elements. Now the above group is Abelian and $|D| = 8$. Give a subgroup, say $H$, of $D$ with 4 elements. Verify that $H$ is a subgroup by doing the Caley's table. Does $D$ have an element of order 4?

(If $m|n$, then we must have a subgroup with $m$ elements, but not necessarily an element of order $m$)

Let us take $H = \{\varnothing, \{1,2\}, \{1,3\}, \{2,3\}\}$. This subset of $D$ is clearly a subgroup of $(D, \cdot)$. The Caley's table is shown below:

| $a \cdot b$ | $\varnothing$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ |
|---|---|---|---|---|
| $\varnothing$ | $\varnothing$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ |
| $\{1,2\}$ | $\{1,2\}$ | $\varnothing$ | $\{2,3\}$ | $\{1,3\}$ |
| $\{1,3\}$ | $\{1,3\}$ | $\{2,3\}$ | $\varnothing$ | $\{1,2\}$ |
| $\{2,3\}$ | $\{2,3\}$ | $\{1,3\}$ | $\{1,2\}$ | $\varnothing$ |

**Table 2.**

From the table we can see that $H$ is indeed a group. In fact, $H < D$. It satisfies all the properties of a group (Identity $e = \varnothing$, each element has an inverse, it is closed and associative). Furthermore, $H$ is an Abelian group since $\forall a, b \in H, b \cdot a = a \cdot b$.

Now we can see that $|H| = 4$, and $4|8$. However, it is evident that $\forall a \in H, |a| = 2$, except for the case of $a = \varnothing$, in which case $|\varnothing| = 1$. Therefore, we can conclude that if we have $m|n$, that does not necessarily imply that we can find a subgroup with $m$ elements that also has elements of order $m$.

**Question 2:**

Let $D = \{2, 4, 6, 8, 10, 12\}$. From HW1, we know that $D$ under multiplication modulo 14 is an Abelian group. Now $H = \{6, 8\}$ is a subgroup of $D$. Find all the left cosets of $H$. Since $D$ is Abelian, $H$ is a normal subgroup of $D$. Construct the Caley's table for the group $(D/H, *)$.

From HW1, we know that $e = 8$. We will take the binary operator to be $\cdot_{14}$. All the left cosets of of $H$ are as follows:

$$a \cdot H = \{a \cdot h \mid a \in D, h \in H\}$$

$$\begin{aligned}
2 \cdot H &= \{2 \cdot 6, 2 \cdot 8\} = \{12, 2\} \\
4 \cdot H &= \{4 \cdot 6, 4 \cdot 8\} = \{10, 4\} \\
6 \cdot H &= \{6 \cdot 6, 6 \cdot 8\} = \{8, 6\} = H \\
8 \cdot H &= \{8 \cdot 6, 8 \cdot 8\} = \{6, 8\} = H \\
10 \cdot H &= \{10 \cdot 6, 10 \cdot 8\} = \{4, 10\} \\
12 \cdot H &= \{12 \cdot 6, 12 \cdot 8\} = \{2, 12\}
\end{aligned}$$

Note that the identity here is:

$$e = 6 \cdot H = 8 \cdot H = H$$

We have 3 distinct left cosets of $H$. These are $2 \cdot H = \{2, 12\}, 4 \cdot H = \{4, 10\}$ and $6 \cdot H = \{6, 8\}$.

These are the elements of the set $D/H$.

$$D/H = \{2H, 4H, 6H\}$$

We define $*$, the binary operator on the set $D/H$ as the following:

$$\forall x, y \in D/H, x * y = (a \cdot b) \cdot H$$

$a, b$ are two left cosets of $H$.

Therefore, the Caley's table for $(D/H, *)$ would be:

| $x * y$ | $2H$ | $4H$ | $6H$ |
|---|---|---|---|
| $2H$ | $4H$ | $6H$ | $2H$ |
| $4H$ | $6H$ | $2H$ | $4H$ |
| $6H$ | $2H$ | $4H$ | $6H$ |

**Table 3.**

What is the identity of $(D/H, *)$? $6H$, since $\forall x \in D/H, x * 6H = x$. We can see from the Caley's Table that $(D/H, *)$ is closed, associative, each element has an inverse and it is closed. Furthermore, we can see that this group is Abelian because $\forall x, y \in D/H, x * y = y * x$.

**Question 3:**

Let $(D, \cdot)$ be a group, and $H, K$ are distinct subgroups of $D$ (i.e. $H \neq K$).

(i) Prove that $F = H \cap K$ is a subgroup of $D$ [Hint: Let $a, b \in F$. By class result, you only need to show that $a^{-1} \cdot b \in F$ for every $a, b \in F$].

$$
\begin{aligned}
F &= H \cap K \\
\\
\text{Firstly, since } H < D, \text{ we know that } \{e\} &\in H \\
\text{Similarly, since } K < D, \{e\} &\in K \\
\text{Therefore } H \cap K \text{ contains AT LEAST the identity} \\
\text{Or, in other words, } H \cap K &\neq \varnothing \\
\\
\text{Let } a, b &\in F \\
\text{This means that } a, b \in H \text{ and } a, b &\in K \\
\\
\text{Since } H \text{ and } K \text{ are both subgroups,} \\
\text{then } a^{-1} \cdot b \in H \text{ and } a^{-1} \cdot b &\in K \\
\text{and since } a^{-1} \cdot b \text{ is in both } H \text{ and } K, \\
\text{by definition of the intersection,} \\
a^{-1} \cdot b &\in F \\
\\
\text{Therefore } F = H \cap K \text{ is a subgroup of } D
\end{aligned}
$$

Since $F$ is a subgroup of $D$, and $F \subseteq H, F \subseteq K$, then we can also directly say that $F < H$ and $F < K$. Therefore $F$ is also a subgroup of both $H$ and $K$.

(ii) Assume that neither $K \subset H$ nor $H \subset K$. Prove that $H \cup K$ is never a subgroup of $D$.

We proceed by contradiction, i.e. we assume $F = H \cup K$ is a subgroup of $D$.

$$
\begin{aligned}
H \not\subset K \text{ and } K &\not\subset H \\
\text{we choose } a \in H \text{ and } b \in K, \text{ but } a \notin K \text{ and } b &\notin H \\
\\
\text{but since } F \text{ is a subgroup,} \\
a \cdot b &\in F \\
\text{Meaning that } a \cdot b \in H \text{ or } a \cdot b &\in K && \text{By definition of the union} \\
\\
a^{-1} \cdot a \cdot b \in H \rightarrow b &\in H && \text{Contradiction} \\
\text{OR} \\
a \cdot b \cdot b^{-1} \in K \rightarrow a &\in K && \text{Also a contradiction}
\end{aligned}
$$

In other words, if we assume the union to be a subgroup, then we would have that an element that cannot be in one of the subgroups $H$ and $K$ would be in them, which is a contradiction of the fact that $H \not\subset K$ and $K \not\subset H$.

Therefore, $H \cup K$ is never a subgroup of $D$.

(iii) Assume $|H| = |K| = m$, where $m$ is a prime positive integer. Prove that $H \cap K = \{e\}$

The intersection between $H$ and $K$ must be a subgroup, by the result proven in 3(i). This means that $H \cap K < D$. We can also say that $H \cap K < H$ and $H \cap K < K$. Now,

$$\begin{aligned}
&\text{Since } |H| = |K| = m \\
&\text{and } H \cap K < H \\
\\
&\text{Therefore, by Langrange}'s\text{ theorem:} \\
&|H \cap K| \,|m \\
&\text{The cardinality of } H \cap K \text{ divides } m, \\
&\text{which is the cardinality of } H \\
\\
&\text{But we know that } m \text{ is prime, meaning that:} \\
&\text{the only numbers that divide it are 1 and } m \\
&\text{So:} \\
&|H \cap K| = m \text{ or } |H \cap K| = 1 \\
\\
&\text{However:} \\
&\text{Since } H \text{ is not the same as } K \text{ and } m \text{ is prime,} \\
&|H \cap K| \neq m \\
&\text{So:} \\
\\
&|H \cap K| = 1 \\
\\
&\text{Since } H \cap K \text{ is a group with one element,} \\
&\text{then the only element it can contain is } e \\
\\
&\text{Therefore } H \cap K = \{e\}
\end{aligned}$$

We have proven that the intersection of two subgroups (which is itself a subgroup) of $D$ contains only the identity of $D$.

**Question 4:**

(a) **[CORRECTED]** Let $(D, \cdot)$ be a group, $H$ is a normal subgroup of $D$, and $K$ is a subgroup of $D$. Prove that $H \cdot K = \{h \cdot k | h \in H, k \in K\}$ is a subgroup of $D$. Note that $H$ is a subgroup of $H \cdot K$ and $K$ is a subgroup of $H \cdot K$ since $H \cdot e = H$ and $e \cdot K = K$ [Hint: Let $a, b \in H \cdot K$, by a class result, you only need to show that $a^{-1} \cdot b \in H \cdot K$ for every $a, b \in H \cdot K$].

$$\begin{aligned}
\text{Let } a, b &\in H \cdot K \\
a = h_1 \cdot k_1, b &= h_2 \cdot k_2 && h_1, h_2 \in H, k_1, k_2 \in K \\
a^{-1} \cdot b &= (h_1 \cdot k_1)^{-1} \cdot (h_2 \cdot k_2) \\
& k_1^{-1} \cdot h_1^{-1} \cdot h_2 \cdot k_2 \\
h_1^{-1} \cdot h_2 &\in H && \text{Since } H \text{ is a subgroup} \\
\text{Let } h_3 = h_1^{-1} \cdot h_2 &\in H \\
\text{Hence } a^{-1} \cdot b &= k_1^{-1} \cdot h_3 \cdot k_2
\end{aligned}$$

$$\begin{aligned}
\text{Since } H \text{ is normal, we have:}\\
k_1^{-1} \cdot h_3 \cdot k_2 = h_4 \cdot k_1^{-1} \cdot k_2\\
\text{For some } h_4 \in H\\
\\
\text{Let } k_3 = k_1^{-1} \cdot k_2\\
\text{meaning that } k_3 \in K\\
\\
\text{Therefore:}\\
a^{-1} \cdot b = h_4 \cdot k_3 \in H \cdot K
\end{aligned}$$

Therefore, we have proven that for every $a, b \in H \cdot K$, $a^{-1} \cdot b \in H \cdot K$. This condition is enough to satisfy the condition for subgroups, and therefore $H \cdot K$ is a subgroup of $D$.

(b) **[CORRECTED]** Consider $S_3$, the symmetric group of an equilateral triangle (As in HW1). Give a subgroup, say $H$ of $S_3$, that is not a normal subgroup of $S_3$.

$$\left\{ f_1\colon \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, f_2\colon \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, f_3 = e\colon \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, f_4\colon \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, f_5\colon \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, f_6\colon \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \right\}$$

This is the symmetric group of an equilateral triangle. Out of these 6 elements, we can form a subgroup, $H$ that is NOT a normal subgroup of $S_3$. This means that for some $a \in S_3, a \cdot H \neq H \cdot a$.

We need to note here that we mustn't fall into this trap: The condition for a normal subgroup is that we can find some $h, k \in H$ st $\forall a \in S_3$, $a \cdot h = k \cdot a$. $k$ and $h$ do not necessarily need to equal each other for the subgroup to be normal. With that in mind, let us take $H = \{e, f_4\}$:

$$H = \left\{ e\colon \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, f_4\colon \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \right\}$$

The Caley's table for this subset is:

| $\circ$ | $e$ | $f_4$ |
|---|---|---|
| $e$ | $e$ | $f_4$ |
| $f_4$ | $f_4$ | $e$ |

**Table 4.**

Clearly, from this Caley's table, we can see that the subset is a subgroup of $S_3$. Now, let us see if the subgroup is normal. Since being a normal subgroup means: $\forall a \in S_3, a \cdot H = H \cdot a$, the negation of the statement means that $\exists a \in D$ (at least one) where $a \cdot H \neq H \cdot a$.

Let us take some random element in $S_3$, which will serve as our $a$. Take $a = f_1$. Then:

$$\begin{aligned}
\text{We check to see if } a \cdot h = k \cdot a &\quad h, k \in H\\
f_1 \circ f_4 = f_6 &\quad \text{From Caley's Table in } \mathbf{HW1}\\
f_4 \circ f_1 = f_5 &\\
f_4 \circ f_1 \neq f_1 \circ f_4 &
\end{aligned}$$

Note that $H$ only has two elements, making it easy to see the other possibilities. Hence:

$$f_4 \cdot H \neq H \cdot f_4$$

And this shows that $H$ is NOT a normal subgroup of $S_3$.