MTH312 - Advanced Calculus

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1 Chapt. Euclidean Space

Def.: The set of all ordered n-tuples of real numbers:

$$\mathbb{R}^n := \{x = (x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\}$$

This constitutes an n-dimensional Euclidean space.

An example of this would be the following: Let $\mathbb{R}^3 \longleftarrow X = (x, y, z) = x\, e_1 + y\, e_2 + z\, e_3$, where we have that $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Rmk)

1. We recall that the Cartesian product $A \times B$ of the sets A and B is by definition the set of all pairs (a, b) st $a \in A$ and $b \in B$. Thus $\mathbb{R}^n$ can be regarded as the Cartesian product of $\mathbb{R}$ with itself, n times.

   i.e. $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ exactly n times. This is the reason for the notation.

2. When $n = 1$, $\mathbb{R}$ is regarded as a line. When $n = 2$, we get $\mathbb{R}^2$, which is the $x - y$ plane. Finally, when we have $n = 3$, we get $\mathbb{R}^3$, which is the $x - y - z$ space.

   For any value of n higher than 3, we can no longer visualize.

3. Elements of $\mathbb{R}$, or $\mathbb{R}^1$ are called Scalars, and the elements of $\mathbb{R}^n$ for $n \geqslant 2$ are called vectors.

4. The vector $O = (0, 0, 0, \ldots, 0) \in \mathbb{R}^n$ is called the null (or the zero) vector. All the components are equal to the scalar 0.

Important: The Euclidean space $\mathbb{R}^n$ is endowed with 2 algebraic operations, namely: the vector addition and the scalar multiplication. With these two operations, $\mathbb{R}^n$ is a vector space. In other words, it satisfies the 10 axioms of a vector space.

Geometry in $\mathbb{R}^n$:

1. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. The dot product, which is the inner product between x and y, is defined by:

$$x \cdot y = \sum_{i=1}^{n} x_i\, y_i$$

   The dot product is simply a mapping of the following form: $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$. The result is always a real number (a scalar).

2. The dot product generates a norm in $\mathbb{R}^n$ (also called the modulus):

For $X=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, we define the norm of X by: $\|X\|=\sqrt{X\cdot X}$. This can also be written as the following:

$$\|X\|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

Very Important: We have that $\|X\|=0$ iff $X=O$. We need to have that the sum of the squares of each term is 0, and therefore X can only be the O vector.

3. ***Cauchy-Schwarz Inequality***: For all $x,y\in\mathbb{R}^n$, we have that $|x\cdot y|\leqslant\|x\|\times\|y\|$. The equality holds (i.e. $|x\cdot y|=\|x\|\times\|y\|$) iff one of x or y is a scalar multiple of the other. In other words, for example, we need to have something like $y=a\,x$, where a is a scalar.

   Proof:

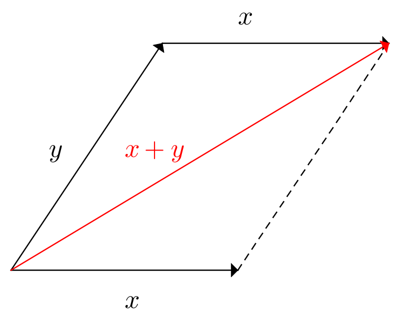
   Without loss of generality, we may assume that $x,y\neq O$, because the result is trivial of x or y is the null vector.

$$\begin{aligned}
&\text{For every } a\in\mathbb{R}, \text{we have that:}\\
0\leqslant(a\,x-y)\cdot(a\,x-y)&=a^2\|x\|^2-2a(x\cdot y)+\|y\|^2\\
&\text{Consider the quadratic polynomial:}\\
P(a)&=a^2\|x\|^2-2a(x\cdot y)+\|y\|^2\\
&\text{Since } P(a)\geqslant 0 \text{ for all } a, P \text{ has at most 1 root.}\\
&\text{Therefore the discriminant of } P \text{ must be} \leqslant 0\\
4(x\cdot y)^2-4\|x\|^2\|y\|^2&\leqslant 0\\
(x\cdot y)^2\leqslant\|x\|^2\|y\|^2, \text{ or } (x\cdot y)&\leqslant\|x\|\times\|y\|\\
\\
&\text{Finally, the equality holds when:}\\
&\text{the quadratic polynomial } P(a)=(a\,x-y)\cdot(a\,x-y)\\
&\text{has a root. ie } P(a)=0=(a\,x-y)\cdot(a\,x-y)\\
&\text{This implies } a\,x-y=0 \text{ and thus } y=a\,x.
\end{aligned}$$

4. ***Triangle Inequality:*** This is a consequence of the Cauchy-Schwarz inequality. For every $x,y\in\mathbb{R}^n$, we have that $\|x+y\|\leqslant\|x\|+\|y\|$. The equality holds if one of x or y is a non-negative scalar multiple of the other (ie $y=a\,x, a\geqslant 0$)

   Proof:

$$\begin{aligned}
\|x+y\|^2&=(x+y)\cdot(x+y)\\
&=\|x\|^2+2(x\cdot y)+\|y\|^2\leqslant|x\cdot y|\\
&\leqslant\|x\|^2+2\|x\|\|y\|+\|y\|^2 \qquad \text{By Cauchy-Schwarz Inequality}\\
&=(\|x\|+\|y\|)^2\\
\text{Hence } \|x+y\|&\leqslant\|x\|+\|y\|
\end{aligned}$$



5. ***Cosine of the angle between two vectors in $\mathbb{R}^n$:***

   The cosine of the angle θ bettwen x and y in $\mathbb{R}^n\backslash\{0\}$ is given by the following:

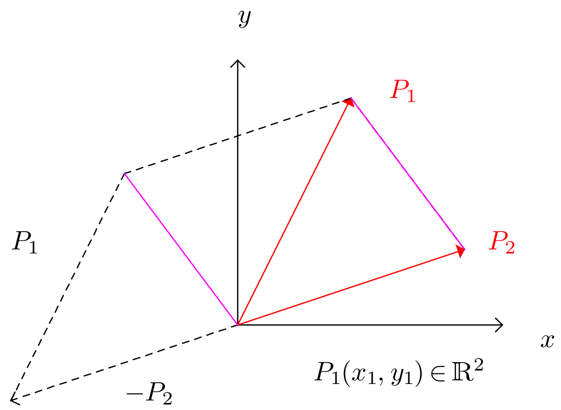
   $$\cos(\theta) = \frac{x \cdot y}{\|x\| \times \|y\|}$$

   Two vectors in $\mathbb{R}^n$ are said to be orthogonal if $\cos(\theta) = 0$, ie $x \cdot y = 0$

   Rmk) O is orthogonal to all vectors in $\mathbb{R}^n$.

6. We define the distance between two vectors in $\mathbb{R}^n$ by:

   $$d(x, y) = \|x - y\|$$



$P_1(x_1, y_1) \in \mathbb{R}^2$

$P_2(x_2, y_2) \in \mathbb{R}^2$

$d(P_1, P_2) = \|\overrightarrow{OP_1} - \overrightarrow{OP_2}\|$

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Recall that the distance between two vectors, x and y, is given by: $d(x, y) = \|x - y\|$. The following are the properties of the distance:

i. $d(x, y) \geqslant 0$ and $d(x, y) = 0$ iff $x = y$

ii. $d(x, y) = d(y, x)$

iii. $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}^n$

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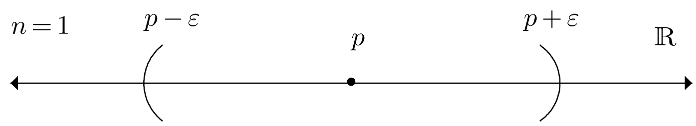
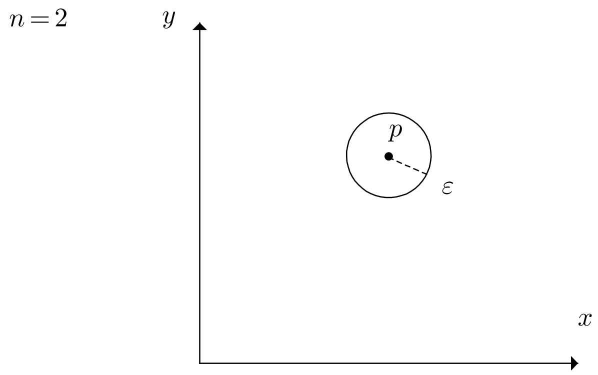
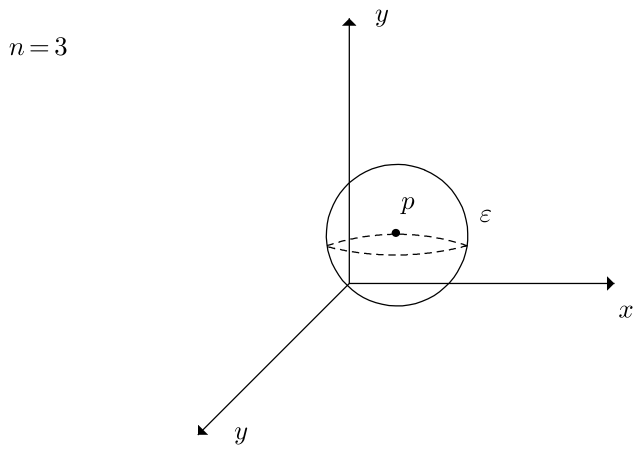
2 Chapt. On The Topology of $\mathbb{R}^n$

Def.: *ε-ball*: For every point $p \in \mathbb{R}^n$ and every real number $\varepsilon > 0$, where ε is the radius, we define the ε-ball, centered at p, by:

$$B(p,\varepsilon) := \{x \in \mathbb{R}^n \,|\, d(x,p) = \|x-p\| < \varepsilon\}$$



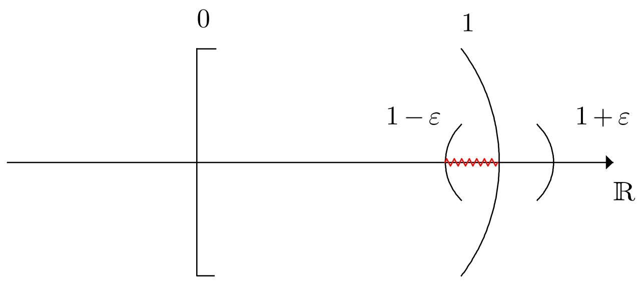




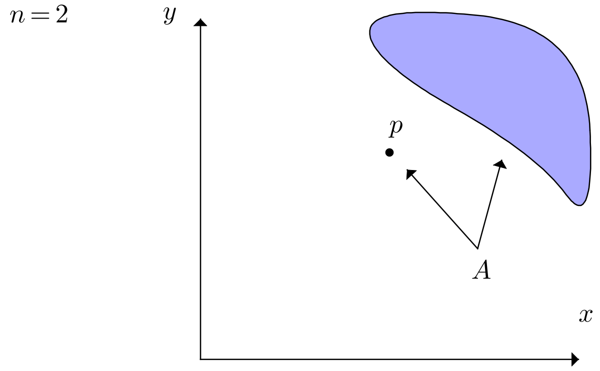
Def.: *Limit Point*: Let $A \subseteq \mathbb{R}^n$ be a subset and let $p \in \mathbb{R}^n$. We say that p is a limit point of A if for every $\varepsilon > 0, B(p,\varepsilon)$ contains some point $x \in A$ such that $x \neq p$. In other words,

$$B(p,\varepsilon) \cap A \neq \varnothing$$

The intersection contains at least one point x in A that is different from p.

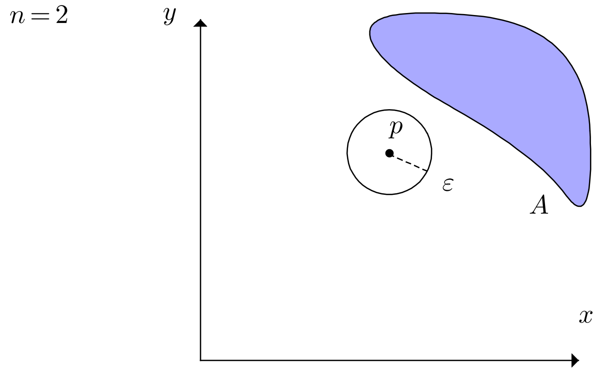


In this example, $A = [0, 1)$. We can see that we have a limit point at 1 since the ε-neighborhood of 1 contains at least one point x that is not 1 itself. Let us look at a more complicated example. Consider the set in $\mathbb{R}^2$ shown below.



In this case, A is the union of the set that is filled, along with a single point p indicated. The point p is NOT a limit point of A, because we can find some ε for which the ε-ball contains no points of A other than the point p itself.

Def.: *Isolated Point*: p is an isolated point of A if there exists $\varepsilon > 0$ st $B(p, \varepsilon) \cap A = \{p\}$.



Using the same example, we can see that p is an isolated point, because the ε-ball contains no points of A other than p itself.

Important:

i. An isolated point of A must be in A itself.

ii. A limit point of A does not have to be in A. In other words, a limit point of a set does not necessarily have to be in that set. Refer to the first example in $\mathbb{R}$. In that, the set is $A = [0, 1)$. Clearly 1 is not part of A, but it is still a limit point of A.

Sequences in $\mathbb{R}^n$: Let $A \subseteq \mathbb{R}^n$ be a subset. A sequence in A is an infinite set of vectors, denoted by the ordered set $\{x_1, x_2, \dots\}$. We shall use the following notation:

$$\{x_\nu\} \in \mathbb{R}^n$$

$$x_\nu = \{x_{1,\nu}, x_{2,\nu}, \dots, x_{n,\nu}\}$$

Note that this sequence x_ν is itself a vector in $\mathbb{R}^n$, with each of the $\{x_{1,\nu}, x_{2,\nu}, \dots, x_{n,\nu}\}$ being the components of x_ν.

Def.: *Null Sequence:* The sequence $\{x_\nu\} \in \mathbb{R}^n$ is null if for every $\varepsilon > 0$, $\exists$(there exists) some $\nu_0 \in \mathbb{N}$ st we have $\|x_\nu\| < \varepsilon$ for all $\nu > \nu_o$. In other words,

$$\lim_{\nu \to \infty} \|x_\nu\| = 0$$

Rmk 1)

i. If $\{x_\nu\}$ is a null sequence, and if $\{y_\nu\}$ is a sequence st $\|y_\nu\| \leqslant \|x_\nu\|$ for all ν, then we have that $\{y_\nu\}$ is a null sequence. This is standard by the squeeze theorem.

ii. If $\{x_\nu\}$ and $\{y_\nu\}$ are two null sequences, then $\{x_\nu + y_\nu\}$ is also a null sequence because:

$$0 \leqslant \|x_\nu + y_\nu\| \leqslant \|x_\nu\| + \|y_\nu\|$$

Lemma 2: *Component-wise nature of nullness*

The vector sequence $\{x_\nu = (x_{1,\nu}, x_{2,\nu}, \ldots, x_{n,\nu})\}$ is null iff each of its component is a scalar. This means that every element of $\{x_\nu\}$, $x_{i,\nu} \in \mathbb{R}$. We can also say that $\{x_{j,\nu}\}$ is null.

$$\lim_{\nu \to \infty} x_{j,\nu} = 0$$

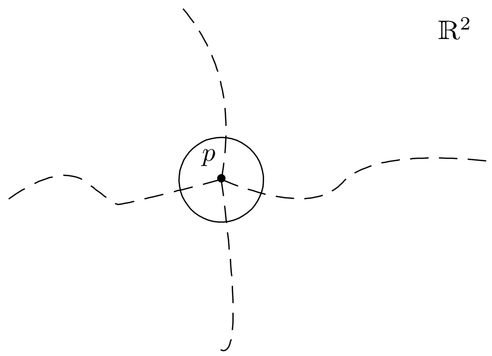
The sequence of scalars must go to 0. Let us look at the following example:

$$x_\nu = \left(\frac{2}{\nu}, \frac{3+\nu}{\nu!}\right) \in \mathbb{R}^2 \text{ where } \frac{2}{\nu} = x_{1,\nu}, \frac{3+\nu}{\nu!} = x_{2,\nu}$$
$$\left\{\frac{2}{\nu}\right\} \longrightarrow 0 \text{ and } \left\{\frac{3+\nu}{\nu!}\right\} \longrightarrow 0$$

Proof: By Rmk1, it is enough to show that $\{\|x_{1,\nu}, \ldots, x_{n,\nu}\|\}$ is null (ie goes to 0 as $\nu \longrightarrow \infty$)

$$\begin{aligned} &\sqrt{(x_{1,\nu})^2 + (x_{2,\nu})^2 + \cdots + (x_{n,\nu})^2} \\ \text{Since } &\{x_\nu = (x_{1,\nu}, x_{2,\nu}, \ldots, x_{n,\nu})\} \text{ is null,} \\ &\text{ie } x_\nu \longrightarrow 0 \text{ as } \nu \longrightarrow \infty, \\ &\text{then clearly } \|x_\nu\| \longrightarrow 0 \text{ as } \nu \longrightarrow \infty \end{aligned}$$

Def.: *Sequence Convergence:* Let $\{x_\nu\}$ be a sequence in $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$. We say that $\{x_\nu\}$ converges to p (or has a limit p) is the sequence $\{x_\nu - p\}$ is a null sequence.



$$d(x_\nu, p) = \|x_\nu - p\| \longrightarrow 0$$

Rmk 3)

i. If $\{x_\nu\}$ converges to p and to p', then we have that $p=p'$

ii. With respect to the definition of the null sequence, a null sequence in $\mathbb{R}^n$ is a sequence that converges to the null vector, O.

Prop. 4: *Component-wise Nature of Convergence*

The sequence of vectors $\{x_\nu=(x_{1,\nu},x_{2,\nu},\ldots,x_{n,\nu})\}$ converges to the vector $p=(p_1,p_2,\ldots,p_n)$ iff each component scalar sequence $\{x_{j,\nu}\}$, with $(j\in\{1,\ldots,n\})$ converges to the scalar p_j.

$$\lim_{\nu\to\infty} x_{j,\nu}=p_j$$

Exp)

$$x_\nu=\left(1+\frac{1}{\nu},e^{-\nu}\right)$$
$$\text{Then } \{x_\nu\} \text{ converges to } (1,0)$$
$$\text{because } \lim_{\nu\to\infty} 1+\frac{1}{\nu}=1 \ \text{ and } \ \lim_{\nu\to\infty} e^{-\nu}=0$$

Proof:

$$\begin{aligned}
\{(x_{1,\nu},x_{2,\nu},\ldots x_{n,\nu})\}&\text{ converges to }(p_1,p_2,\ldots,p_n)\\
\Leftrightarrow\{(x_{1,\nu}-p_1,x_{2,\nu}-p_2,&\ldots,x_{n,\nu}-p_n)\}\text{ is null}\\
\Leftrightarrow(\text{by } \mathbf{LM2})&\text{ each component scalar sequence}\\
\{x_{j,\nu}-p_j\}\text{ is a null sequence. ie }&\lim_{\nu\longrightarrow\infty} x_{j,\nu}-p=0\\
\text{for all } j=1,\ldots,n.\text{ In other words, }&\lim_{\nu\to\infty} x_{j,\nu}=p\\
\Leftrightarrow&\{x_{j,\nu}\}\text{ converges to } P
\end{aligned}$$

Lemma 5: *Sequential Characterization of the Limit Point*

$p\in\mathbb{R}^n$ is a limit point of a subset $A\subseteq\mathbb{R}^n$ iff p is the limit point of a sequence $\{x_\nu\}\in A$ st $x_\nu\neq p$ for $\nu\in\mathbb{N}$. p is a limit point of A if there exists a sequence $\{x_\nu\}\in A$ with $x_\nu\neq p$ for all ν st $\{x_\nu\}$ converges to P.

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Proof:

$$\begin{aligned}
&\Longrightarrow\\
\text{Let } p \text{ be the limit of } &\{x_\nu\}\in A \text{ with } x_\nu\neq p\\
&\text{and let } \varepsilon>0 \text{ (be random).}\\
\text{Then } \exists\nu_0\in\mathbb{N} \text{ st } d(x_\nu,p)&=\|x_\nu-p\|<\varepsilon \text{ for all } \nu>\nu_0\\
\text{This is equivalent to saying that }& B(p,\varepsilon) \text{ contains}\\
\text{infinitely many } x_\nu\in A. \text{ Hence }& p \text{ is a limit point of } A
\end{aligned}$$

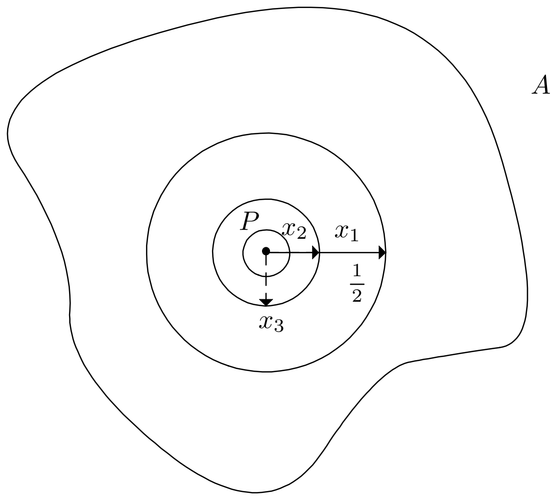
$$\begin{aligned}
&\Longleftarrow\\
\text{Let } p \text{ be a limit point of } A.&\text{ We shall construct a sequence}\\
\{x_\nu\}\in A \text{ with }& x_\nu\neq p \text{ st } \{x_\nu\}\longrightarrow p
\end{aligned}$$

$$\text{The } B\left(p, \frac{1}{2}\right) \text{ contains an element } x_1 \in A, x_1 \neq p$$

$$\text{Let } \varepsilon_2 = \frac{1}{2} d(x_1, p) = \frac{1}{2}\|x_1 - p\|. \text{ Then:}$$

$B(p, \varepsilon_2)$ contains $x_2 \neq p \in A.$ and continue defining the sequence $\{x_\nu\}$ in this fashion, with $\|x_\nu - p\| < \frac{1}{2^\nu}$.

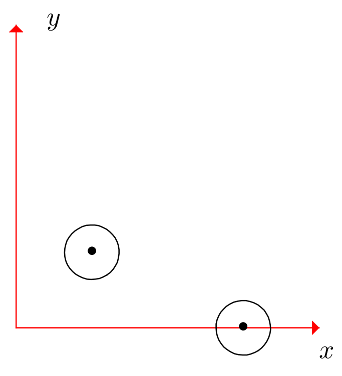
Clearly $\{x_\nu\} \longrightarrow p$, $\{x_\nu\}$ is in A and $x_\nu \neq p$ for all ν



Def.: *Closed Sets*

$A \subseteq \mathbb{R}^n$ (a subset) is said to be closed if it contains all its limit points. We are saying that **if** a set contains limit points, then it must contain it. However, if our set is for example a singleton, then it is automatically closed despite not having any limit points.

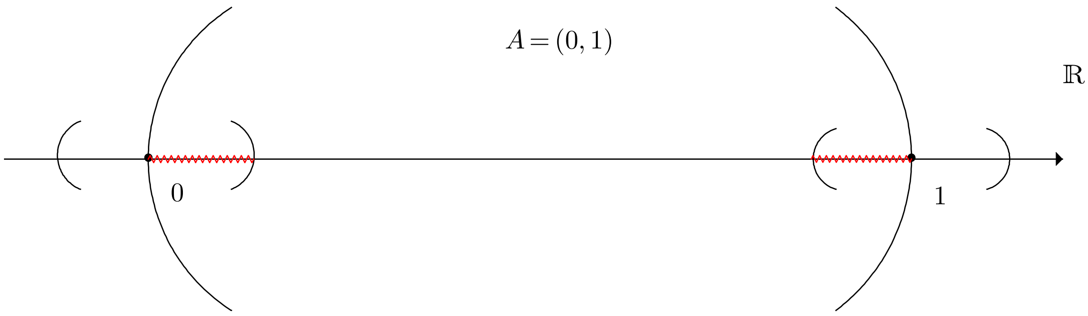
<u>Exp</u>: 1) Consider the following in $\mathbb{R}^2$ (the plane), the x-axis is a closed subset. Observe the graph below.



Clearly, we can see that while we are on the x-axis, we can find ε balls that contain points on the x-axis. However, as soon as we leave the x-axis, we can take some point in the plane which contains no points from the x-axis within its ε ball. Therefore, the x-axis is a closed subset of the $x - y$ plane of $\mathbb{R}^2$.

This is once again because every point off the x-axis is surrounded by a ball that miss the x-axis.

<u>Exp</u>: 2) The interval $(0, 1) \in \mathbb{R}$ is not closed in $\mathbb{R}$ because it does not contain the limit points 0 and 1.



Proposition 6: *Sequential Characterization of Closed Sets*

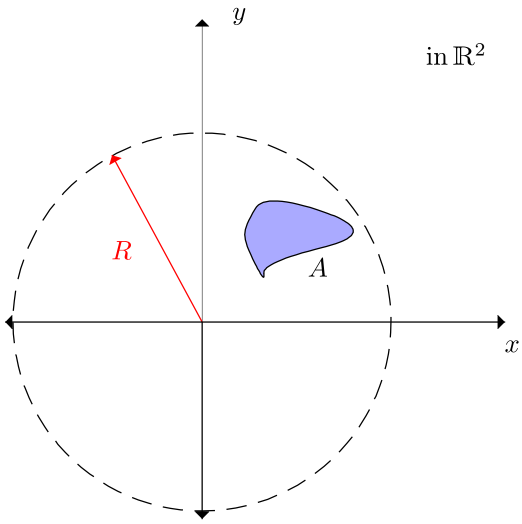
Let $A \subseteq \mathbb{R}^n$. A is closed iff every sequence in A that converges in $\mathbb{R}^n$ in fact converges in A. That means the sequence cannot converge to a point outside of A.

Proof:

$$\begin{aligned}
&\Longrightarrow \\
&\text{Suppose } A \text{ is closed} \\
&\text{Let } \{x_\nu\} \text{ be a sequence in } A \text{ that converges to } p \in \mathbb{R}^n \\
&\text{We shall show that } p \in A. \\
&\text{If } x_\nu = p \text{ for some } \nu, \text{ then } p \in A \\
&\text{If } x_\nu \neq p \text{ for all } \nu, \text{ by } \mathbf{LM5} \Longrightarrow, p \text{ is a limit point of } A \\
&\text{Since } A \text{ is closed, } p \in A
\end{aligned}$$

$$\begin{aligned}
&\Longleftarrow \\
&\text{Suppose that every convergent sequence in } A \\
&\text{converges to a point } p \in A. \\
&\text{Then } A \text{ contains all its limit points by } \mathbf{LM5} \Longleftarrow \\
&\text{Hence } A \text{ is closed}
\end{aligned}$$

Def.: *Bounded Sets*



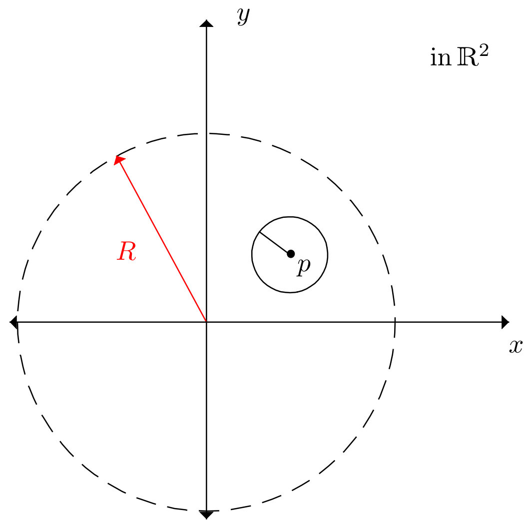
A subset $A \subseteq \mathbb{R}^n$ is bounded if $\exists R > 0$ st $A \subset B(O, R)$. This means that $d(x, O) = \|x - O\| = \|x\| < R$ for all $x \in A$. We are saying that A is completely included in a ball centered at the origin with radius R.

Proposition 7: *Convergence Implies Boundedness*

If a sequence $\{x_\nu\}$ converges in $\mathbb{R}^n$, then $\{x_\nu\}$ is bounded.

Proof:

$$\begin{aligned}\text{Assume } \{x_\nu\} \text{ converges to } p \in \mathbb{R}^n.\\ \text{Let } \varepsilon = 1. \text{ Then } \exists \nu_0 \in \mathbb{N} \text{ st } x_\nu \in B(p,1)\, \forall \nu \geqslant \nu_0\\ \text{Now, let } R > \max\left(\|x_1\|, \|x_2\|, \ldots, \|x_{\nu_0}\|, \|p\|+1\right)\\ \text{Then } x_\nu \in B(O,R) \text{ for } \nu = 1, \ldots, \nu_0\\ \text{and the triangle inequality shows that:}\\ d(x_\nu, p) \leqslant d(x_\nu, O) + d(O,p) = \|x_\nu\| + \|p\|\end{aligned}$$



This means that R definitely contains the points bigger than $\|p\|+1$. The ball with radius R contains the smaller ball, centerd at p. Mathematically:

$$\begin{aligned}B(p,1) \subset B(O,R)\\ \text{Hence } x_\nu \in B(O,R) \text{ for all } \nu \in \mathbb{N}\end{aligned}$$

Def.: *Subsequences*

A subsequence of a sequence $\{x_\nu\}$ is a sequence consisting of some (possibly all) of the original terms in ascending (increasing) order. If we have a sequence of the form: $x_1, x_2, x_3, \ldots, x_n$, we cannot have a change in the order. We can, however, have something of the form:

$$x_2, x_5, x_7, x_8, \ldots, x_m$$

This can be a subsequence because the order of the elements are maintained.

Notation: $\{x_{\nu_k}\}$ for a subsequence of $\{x_\nu\}$, where we have that $\nu_k \geqslant k$. Now, let us look at an example of a subsequence.

Exp:

$$\begin{aligned}\{x_\nu\} &= \{x_1, x_2, x_3, x_4, x_5, \ldots\}\\ \{x_{\nu_k}\} &= \{x_2, x_3, x_5, x_7, x_8, \ldots\}\end{aligned}$$

$$\begin{aligned}x_{\nu_1} &= x_2\\ x_{\nu_2} &= x_3\\ x_{\nu_3} &= x_5\\ x_{\nu_4} &= x_7\end{aligned}$$

Now, from this, we can see that (going from the beginning to the end) $x_{\nu_1} = x_2 \longleftarrow \nu_1 = 2 > 1$. Doing the same for the rest, we will recall the earlier definition of $\nu_k \geqslant k$:

$$\begin{aligned} x_{\nu_1} &= x_2 \longleftarrow \nu_1 = 2 > 1 \\ x_{\nu_2} &= x_3 \longleftarrow \nu_2 = 3 > 2 \\ x_{\nu_3} &= x_5 \longleftarrow \nu_3 = 5 > 3 \\ x_{\nu_4} &= x_7 \longleftarrow \nu_4 = 7 > 4 \end{aligned}$$

Lemma 8:

If a sequence $\{x_\nu\}$ converges to some $p \in \mathbb{R}^n$, then every subsequence of $\{x_\nu\}$ converges to p as well (same limit).

Proof:

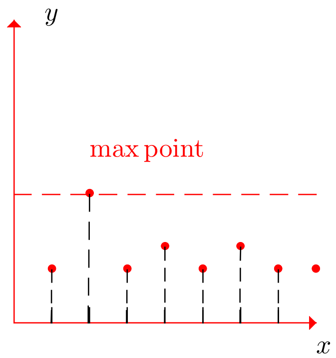
$$\begin{aligned} &\text{Let } \varepsilon > 0 \text{ be random.} \\ &\text{Then } \exists \nu_0 \in \mathbb{N} \text{ st } x_\nu \in B(p, \varepsilon)\, \forall \nu \geqslant \nu_0 \\ &\text{In particular, if } k \geqslant \nu_0, \text{then } \nu_k \geqslant \nu_0 \text{ and hence:} \\ &x_{\nu_k} \in B(p, \varepsilon) \longrightarrow \{x_{\nu_k}\} \text{ converges to } p \\ &(\text{Since } \{x_{\nu_k}\} \text{ is inside the } \varepsilon\text{-ball of } p) \end{aligned}$$

Theorem 9: *Bolzano Weierstrass Property in* $\mathbb{R}$

Let $A \subseteq \mathbb{R}$ non-empty subset. If A is bounded, then every sequence in A has a convergent subsequence.

Proof:

$$\begin{aligned} &\text{Let } \{x_n\} \text{ be a sequence in } A. \\ &\text{We call a term } x_n \text{ a max-point (or peak)} \\ &\text{if } x_n \geqslant x_m \text{ for all } m \geqslant n \end{aligned}$$

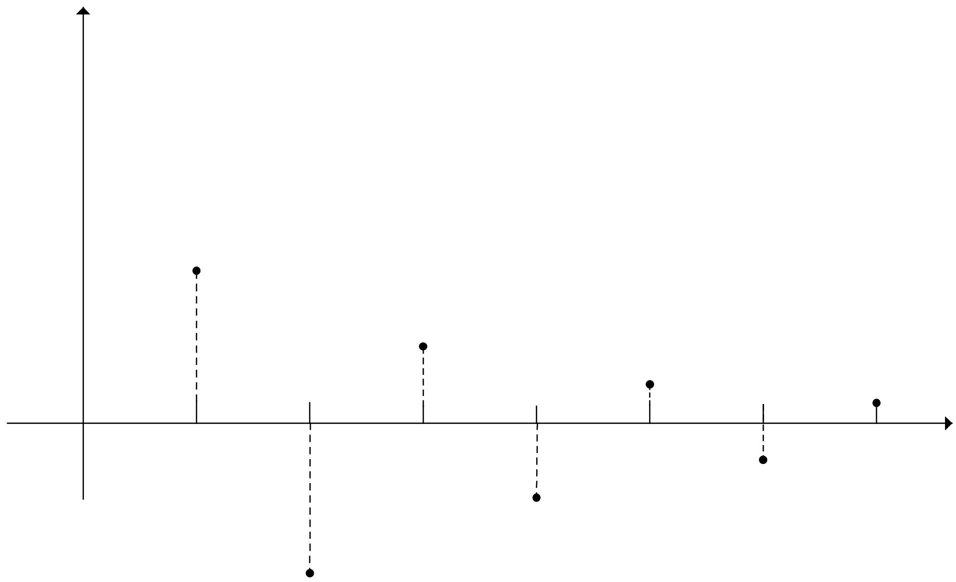


If there are infinitely many max-points in (x_n), then they form a decreasing subsequence of (x_n). In there are only finitely many peaks, then (x_n) has an increasing subsequence starting from the last max-point. In either cases, (x_n) has a monotone subsequence. Since A is bounded, every subsequence is bounded as well. Being monotonic and bounded will imply the convergence of the subsequence.

February 10th, 2021

Exp: 1)

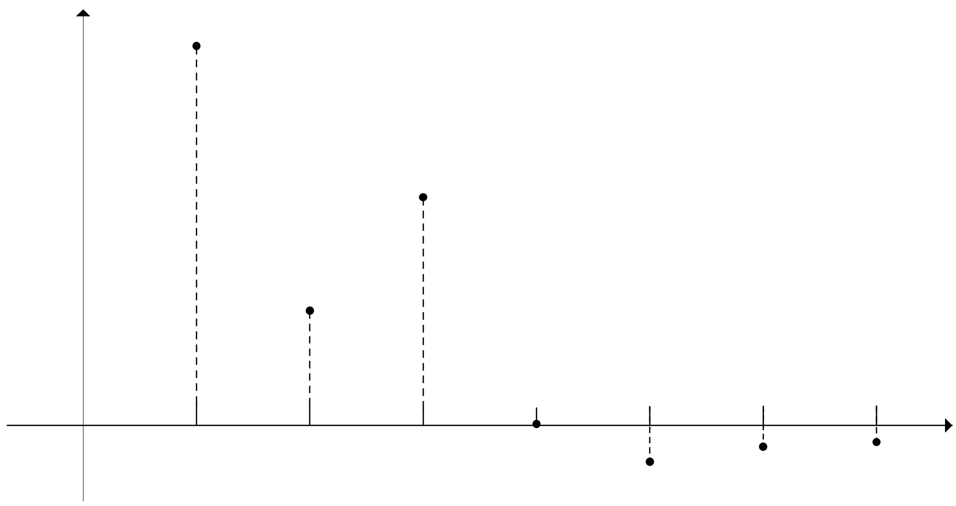
$$\{x_n\} = \left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots\right\}$$



All positive terms (they are $\frac{1}{n}, n \geqslant 1$) are peaks. Those peaks form a decreasing sequence.

Exp: 2)

$$\{x_n\} = \left\{43, 11, 27, 0, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \dots\right\}$$



In this situation, 43 is a peak, 27 is a peak and 0 is a peak. Therefore $\{x_n\}$ has only 3 (finitely many) peaks. The subsequence $\{x_{n_k}\} = \{-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \dots\}$ is convergent. This takes care of the second situation, where we have finitely many peaks. We can see that this convergent subsequence occurs only after the last peak.

These two examples demonstate the two possibilities. Recall the definition of a peak: x_n is a peak if $x_n \geqslant x_m$ for all $m > n$.

Theorem 10: *Bolzano-Weierstrass Property in* $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be a non-empty subset. Then A is bounded iff every sequence in A has a convergent subsequence that converges in $\mathbb{R}^n$. The sequence itself does not have to be convergent. However, we must be able to extract a converge subsequence for it to be bounded.

Proof:

$\Longrightarrow$

Suppose that A is bounded.
Consider any random sequence $\{x_\nu\} \in A$
Thus we can write:
$\{x_\nu = (x_{1,\nu}, x_{2,\nu}, x_{3,\nu}, \ldots, x_{n,\nu})\}$
The real sequence $\{x_{1,\nu}\}$ takes its value
in a bounded subset of $\mathbb{R}$. Why? Because A is bounded.
The components of any vector in A is bounded.
Thus by **THM9**, $\{x_{1,\nu}\}$ has a convergent subsequence,
$\{x_{1,\nu_k}\}$. The same happens for $\{x_{2,\nu}\}, \{x_{3,\nu}\}, \ldots \{x_{n,\nu}\}$.
This fashion exhibits a subsequence of $\{x_\nu\}$ that
converges at each component (component-wise convergence).

$$\{x_{\nu_k} = (x_{1,\nu_k}, x_{2,\nu_k}, \ldots, x_{n,\nu_k})\}$$

Each of which converges to some $(p_1, p_2, \ldots, p_n)$
Which are the limits of the subsequence $\{x_{\nu_k}\}$
Hence by **PROP4**, $\{x_{\nu_k}\}$ converges.

$\Longleftarrow$

Assume A is not bounded.
Then $\exists$ a sequence $\{x_\nu\} \in A$ st $\|x_\nu\| > \nu$ for all ν.
This sequence has no bounded subsequences. In fact,
for every subsequence $\{x_{\nu_k}\}$ of $\{x_\nu\}$, we have: $\|x_{\nu_k}\| > \nu_k \geqslant k$
for all $k \in \mathbb{N}$.
This sequence has no bounded subsequence.
No boundedness implies no convergence.
Thus every subsequence $\{x_\nu\}$ cannot be convergent
(By **PROP7**)

The negation: There exists a sequence in A st all the subsequences in this sequence do not converge in $\mathbb{R}^n$. This is what we proved in $\Longleftarrow$. In this proof, we first showed that (1) implies (2), and then non-(1) implies non-(2).

Def.: *Compacts* A subset K of $\mathbb{R}^n$ is said to be compact if it is both closed and bounded.

Theorem 11: *Sequential Characterization of Compact Subsets*

Let $K \subseteq \mathbb{R}^n$ be a subset. Then: K is compact iff every sequence in K has a convergent subsequence that converges in K.

Proof:

$\Longrightarrow$

Suppose K is compact and let $\{x_\nu\}$ be
any random sequence in K.
Since K is bounded, **THM10** implies that $\{x_\nu\}$
has a convergent subsequence. Also, since K is closed

PROP6 $'' \Longrightarrow ''$ implies that the limit of the subsequence
is in K (The limit of the subsequence is a limit point of K)

$\Longleftarrow$

Suppose every sequence in K
has a convergent subsequence.
Then in particular, every convergent sequence in K
that converges in $\mathbb{R}^n$ has a convergent subsequence in K
that converges in K.
By **LM8**, the limit of the sequence is the limit
of its subsequence, and so the sequence converges in K.
Hence, K is closed by **PROP6**.
Also, every sequence in K has a subsequence
that converges in K (in $\mathbb{R}^n$), and thus by **THM10**,
K is bounded.

Def.: *Continuous Mapping*

Let $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, where f is a mapping and A is the domain of f. Let $p \in A$. Then f is continuous at p if for every sequence $\{x_\nu\} \in A$ that converges to p, the sequence $\{f(x_\nu)\}$ converges to $f(p)$.

Rmk: The set of continuous mapping is a vector-space (Closed under addition, multiplication by scalar, etc.)

Important: Using the component-wise nature of convergence (**PROP4**), it is easy to see that if $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ have component functions $f_1, f_2, \ldots f_m$, then f is continuous at $p \in A$ iff each component f_i is continuous at p. f_i is a scalar-valued function, while f is a vector-valued function.

Exp:

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \text{ where } (x, y) \longmapsto (f_1(x, y) = x^2,\ f_2(x, y) = x\,y,\ f_3(x, y) = x + y)$$

Note that $f_1\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ st $(x, y) \longmapsto x^2$, and the same applies for the rest. These are all scalar-valued functions, but the entire function, f, itself, is a vector-valued function.

Theorem 12: *Continuous Impact of the Compact*

Let K be a compact of $\mathbb{R}^n$, and let $f\colon K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous mapping. Then we know that the set of all images of the elemnts of K, $f(K) := \{f(x) \text{ st } x \in K\}$ is a compact in $\mathbb{R}^m$.

Proof:

We shall use **THM11**
(Sequential characterization of the compact).
Let $\{y_\nu\}$ be any random sequence in $f(K)$.
Then for every $\nu \in \mathbb{N}, \exists x_\nu \in K$ st $y_\nu = f(x_\nu)$
Now consider the sequence $\{x_\nu\} \in K$.
Since K is compact, **THM11** $'' \Longrightarrow ''$ implies that
$\{x_\nu\}$ has a convergent subsequence $\{x_{\nu_k}\}$
that converges to a point $p \in K$.
Since f is continuous on K and since $p \in K$, then
f is continuous at p. Thus $\{f(x_{\nu_k})\}$ converges
to $f(p)$ be definition of a continuous mapping.
Since $\{f(x_{\nu_k})\}$ is a subsequence of $\{y = f(x_\nu)\}$,

THM11 implies that $f(K)$ is compact because
$f(p)$ is in $f(K)$, as $p \in K$.

February 15th, 2021

3 Chapt. Real / Vector-Valued Functions

Recall that $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a vector valued function. However, $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real valued function (we arrive to a result in $\mathbb{R}$).

Exp 1):

$$f(x, y) = \frac{\sqrt{x^2 - y}}{\sqrt{y}}$$

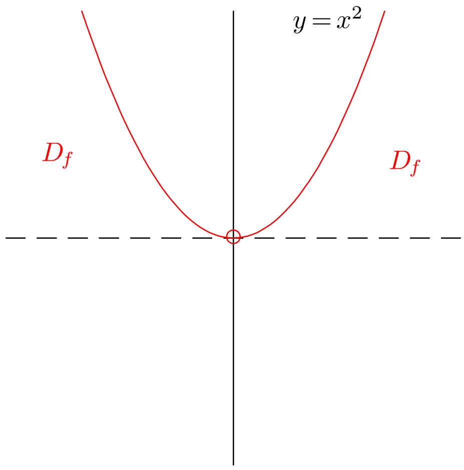
What is the domain of f? It is a subset of $\mathbb{R}^2$. f is well-defined iff the following two conditions are satisfied:

$$\begin{cases} y \leqslant x^2 \\ y > 0 \end{cases} \Longrightarrow 0 < y \leqslant x^2$$

Analytically, we can write that if the domain of f is denoted D_f, then we have:

$$D_f := \{(x, y) \in \mathbb{R}^2 | \, 0 < y \leqslant x^2\}$$

We can also draw this domain as follows:



The domain, shown on the graph, is the region between the $y = 0$ line and the parabola, not including the line $y = 0$. The origin is NOT included.

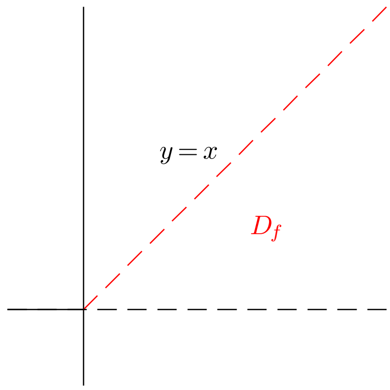
Exp 2):

$$f(x, y) = \frac{\ln(y)}{\sqrt{x - y}}$$

We know that f is well-defined iff the following conditions are held:

$$\begin{cases} y > 0 \\ y < x \end{cases} \Longrightarrow 0 < y < x$$

Thus we know that $D_f := \{(x, y) \in \mathbb{R}^2 | \, 0 < y < x\}$. If we want to draw this, we would have the following region over $\mathbb{R}^2$:



The domain is everything below the line $y = x$, exlcuding the line itself and excluding the $y = 0$ line.

Rmk: Let us say we have the following function, consider them and draw a conclusion regarding the component-wise nature

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto f(x, y) = (\ln(x), y\, e^x, \sin(x\, y)) \end{aligned}$$

We can see the three components of this function: $f_1(x, y) = \ln(x)$, $f_2(x, y) = y\, e^x$, $f_3(x, y) = \sin(x\, y)$. Looking at them individually:

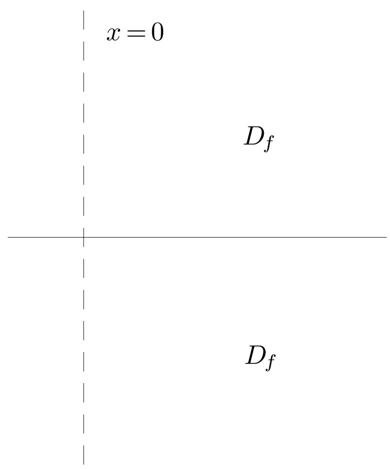
$$\begin{aligned} f_1\colon \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f_1(x, y) = \ln(x) \end{aligned}$$

$$\begin{aligned} f_2\colon \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f_2(x, y) = y\, e^x \end{aligned}$$

$$\begin{aligned} f_3\colon \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f_3(x, y) = \sin(x\, y) \end{aligned}$$

We look at the intersection of the domains of each of f_1, f_2 and f_3. This is the domain of f. We know that D_{f_3} and D_{f_2} is $\mathbb{R}$. However, for f_1, we know that $D_f := \{(x, y) \in \mathbb{R}^2 |\, x > 0\}$. The intersection of the three gives us the domain of f:

$$D_f := \{(x, y) \in \mathbb{R}^2 |\, x > 0\}$$

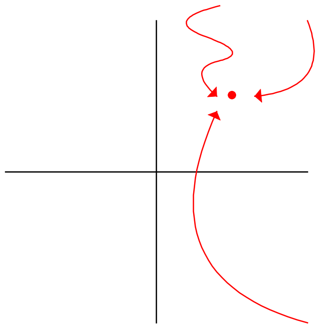


Let us consider limits in $\mathbb{R}$ and in $\mathbb{R}^2$. We know that in $\mathbb{R}$, the limit is defined as follows:

$$\lim_{x \longrightarrow a} f(x)$$

It looks as such on the number-line:

We either approach it from the left side or the right side. There is no other way of approaching the point a. However, let us see this in $\mathbb{R}^2$:



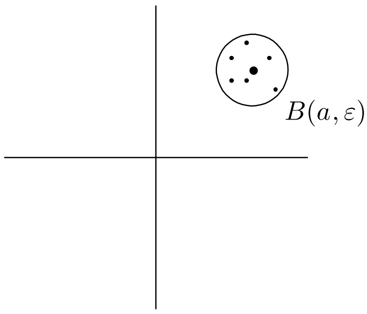
We can see that we can approach the point a from any random direction that we want. There are infinitely many ways we can approach it.

<u>Important</u>:

1. The straight line test can prove that a limit does not exist, or it can prove that the candidate value is the limit. This test is picking up one or two straight lines and approaching your point with it (no need to take crazy paths to the point to show the limit exists). If our limit exists, then it must be the candidate value.

2. When the straight line test determines the candidate value of the limit, approaching along a curve can further support the candidate or it can prove that the limit does not exist by determining a different candidate value.

3. If the limit along two distinct straight lines or two distinct curves are distinct, then the limit does NOT exist.

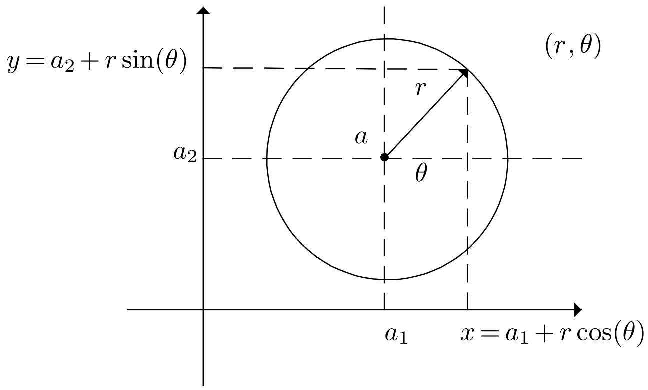
4. 

   Consider the function $f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ where $(x, y) \longmapsto f(x, y)$.We have an ε-ball in which infinitely many $f(x, y)$ exists. This is going to be our inspiration to find the general limit of f as we approach a.

$$\lim_{x \longrightarrow a} f(x, y)$$

   The size bounds (From the HW) can prove that a limit <u>does</u> exist.

5. Sometimes (when in $\mathbb{R}^2$), it is suitable to work with polar coordinates.



We are measuring r from a, not from the origin.

Exp 1)

$$\lim_{(x,y)\longrightarrow(0,0)}\frac{x^2-y^2}{x^2+y^2}$$

Along the $x=0$ line:

$$\lim_{(0,y)\longrightarrow(0,0)}\frac{-y^2}{y^2}=\lim_{(0,y)\longrightarrow(0,0)}-1=-1$$

Along the $y=0$ line:

$$\lim_{(x,0)\longrightarrow(0,0)}\frac{x^2}{x^2}=1$$

Since $1\neq-1$, the limit does not exist.

Exp 2)

$$\lim_{(x,y)\longrightarrow(0,0)}\frac{x\,y}{x^2+y^2}$$

Along the line $y=0$, we have:

$$\lim_{(x,0)\longrightarrow(0,0)}0=0$$

Along the line $y=x$, we have:

$$\lim_{(x,x)\longrightarrow(0,0)}\frac{x^2}{x^2+x^2}=\frac{1}{2}$$

Hence the limit does not exist.

Exp 3)

$$\lim_{(x,y)\longrightarrow(1,0)}\frac{y^3}{(x-1)^2+y^2}$$

By switching to polar coordinates, we have that $x=1+r\cos(\theta)$, and $y=r\sin(\theta)$. Thus:

$$\frac{y^3}{(x-1)^2+y^2}=\frac{r^3\sin^3(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)}=\frac{r^3\sin^3(\theta)}{r^2}=r\sin^3(\theta)$$

Therefore,

$$0\leqslant\left|\frac{y^3}{(x-1)^2+y^2}\right|=|r\sin^3(\theta)|\leqslant r$$

Since $\lim_{r\longrightarrow0}r=0$, then we have that:

$$\lim_{(x,y)\longrightarrow(1,0)}\frac{y^3}{(x-1)^2+y^2}=0$$

Exp 4)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x\ln(1+x^3)}{y(x^2+y^2)}$$

Along the line $y=x$, we have:

$$\lim_{(x,x)\longrightarrow(0,0)} \frac{x\ln(1+x^3)}{x(x^2+x^2)} = \lim_{(x,x)\longrightarrow(0,0)} \frac{\ln(1+x^3)}{2x^2}$$

$$= \lim_{x\longrightarrow 0} \frac{\frac{3x^2}{1+x^3}}{4x} = \lim_{x\longrightarrow 0} \frac{12x^3}{1+x^3} = 0$$

Along the curve $y=x^2$, we have:

$$\lim_{(x,x^2)\longrightarrow(0,0)} \frac{x\ln(1+x^3)}{x^2(x^2+x^4)}$$

$$= \lim_{(x,x^2)\longrightarrow(0,0)} \frac{\ln(1+x^3)}{x(x^2+x^4)}$$

$$= \lim_{x\longrightarrow 0} \frac{\frac{3x^2}{1+x^3}}{3x^2+4x^3}$$

$$= \lim_{x\longrightarrow 0} \frac{3}{(1+x^3)(3+4x)} = 1$$

Therefore, the limit does not exist.

Exp 5)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

We switch to polar coordinates:

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$\frac{x^3+y^3}{x^2+y^2} = \frac{r^3\cos^3(\theta)+r^3\sin^3(\theta)}{r^2} = r(\cos^3(\theta)+\sin^3(\theta))$$

$$0 \leqslant \left|\frac{x^3+y^3}{x^2+y^2}\right| \leqslant r(|\cos^3(\theta)|+|\sin^3(\theta)|) \leqslant 2r$$

$\lim_{r\longrightarrow 0} 2r = 0$, thus we have:

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$$

Exp 6)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2+y^2}{|x|+|y|}$$

It is easy to see that:

$$0 \leqslant \frac{x^2+y^2}{|x|+|y|} \leqslant \frac{(|x|+|y|)^2}{|x|+|y|}$$

Since $\lim_{(x,y)\longrightarrow(0,0)} |x|+|y| = 0$, we have that:

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2+y^2}{|x|+|y|} = 0$$

Exp 7)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2 y}{x^2+y^2}$$

$$\begin{aligned}
&\text{We have that } (|x|-|y|)^2 \geqslant 0\\
&(|x|-|y|)^2 = x^2 - 2|x|\,|y| + y^2, \text{ and so:}\\
&x^2+y^2 \geqslant 2|x|\,|y|\\
&\frac{1}{x^2+y^2} \leqslant \frac{1}{2|x|\,|y|}\\
&\text{Hence:}\\
&0 \leqslant \left|\frac{x^2 y}{x^2+y^2}\right| \leqslant \frac{x^2|y|}{2|x|\,|y|}\\
&= \frac{1}{2}|x|
\end{aligned}$$

$$\text{Since } \lim_{(x,y)\longrightarrow(0,0)} \frac{1}{2}|x| = 0, \text{ we conclude that}$$

$$m_{(x,y)\longrightarrow(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

February 17th, 2021

Derivatives of Multivariable Functions:

Def.: *Interior Points:* Let $A \subseteq \mathbb{R}^n$ be a subset. A point $a \in A$ is said to be an interior point if $\exists \varepsilon > 0$ st the ε-ball $B(a, \varepsilon)$ is strictly included in A.

Def.: *Partial Derivative:* Let $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and let $a = (a_1, a_2, \ldots, a_n)$ be an interior point of A. If:

$$\lim_{h\longrightarrow 0} \frac{f(a_1, a_2, \ldots, a_i + h, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h} \quad \text{exists,}$$

Then f is said to have a partial derivative wrt. the ith component x_i at a. In this case, we denote this derivative by:

$$\frac{\partial f(a)}{\partial x_i}, \partial_{x_i} f(a), f_{x_i}(a) \text{ or } D_i f(a)$$

Rmk: The partial differentiation wrt x_i is done in the usual fashion while treating all the remaining variables as constant.

In general, when $f = (f_1, f_2, \ldots, f_m)$, where each $f_i\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ for each $i = 1, \ldots, m$, if f_j is the jth component of f, then the terms:

$$\frac{\partial f_j(x)}{\partial x_i} \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m$$

constitutes an $m \times n$ matrix called the Jacobian of f at x, and is denoted $J_f(x)$.

Exp:

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$f(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)) = (x_1^2 \cos(x_2), x_2^2 + x_3^2 e^{x_1})$$

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \frac{\partial f_1(x)}{\partial x_3} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \frac{\partial f_2(x)}{\partial x_3} \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 \cos(x_2) & -x_1^2 \sin(x_2) & 0 \\ x_3^2 e^{x_1} & 2x_2 & 2x_3 e^{x} \end{pmatrix}$$

Def.: *Jacobian:* If $m=n$, (ie. $f\colon A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$), then the determinant of $J_f(x)$ (which in this case is a square matrix) is called the Jacobian. It is denoted either $\det[J_f(x)]$ or $|J_f(x)|$.

Higher Order Partial Derivatives: They are defined in a similar way. For example, the second-order partial derivative of f wrt x_i at a is given by the following:

$$\lim_{h\longrightarrow 0}\frac{f_{x_i}(a_1,a_2,\ldots,a_i+h,\ldots,a_n)-f_{x_i}(a)}{h}$$

only when the limit exists. In this case, we denote it by: $\frac{\partial^2 f(a)}{\partial x_i^2}$ or $f_{x_ix_i}(a)$. Also, the second-order derivative of f wrt x_i and x_j, with $i\neq j$, is given by:

$$\frac{f_{x_i}(a_1,a_2,\ldots,a_j+h,\ldots,a_n)-f_{x_i}(a)}{h}$$

It is denoted by: $\frac{\partial^2 f(a)}{\partial x_j\partial x_i}$ or $f_{x_jx_i}(a)$. It is similar to the format of the composition of functions.

Rmk: Under certain conditions, the order of differentiation is rrelevant, that is: $f_{x_jx_i}(a)=f_{x_ix_j}(a)$, $i\neq j$. What are these conditions?

Let $A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^m$ and let a be an interior point of A. Suppose in a neighborhood of a (in an ε-ball $B(a,\varepsilon)$), the followinf conditions are satisfied:

1. $\frac{\partial f(x)}{\partial x_i}$ and $\frac{\partial f(x)}{\partial x_j}$ exists and are finite for all $x\in B(a,\varepsilon)$.
2. Of the derivatives $\frac{\partial^2 f(x)}{\partial x_i\partial x_j}$ and $\frac{\partial^2 f(x)}{\partial x_j\partial x_i}$, one exists and is continuous at every x in $B(a,\varepsilon)$.

If these two are satisfied, then we know that $\frac{\partial^2 f(x)}{\partial x_i\partial x_j}=\frac{\partial^2 f(x)}{\partial x_j\partial x_i}$.

Exp:

$$f\colon\mathbb{R}^3\longrightarrow\mathbb{R}$$
$$f(x_1,x_2,x_3)\longmapsto x_1\,e^{x_2}+x_2\cos(x_1)$$

Then:

$$\frac{\partial f(x)}{\partial x_1}=e^{x_2}-x_2\sin(x_1)$$
$$\frac{\partial f(x)}{\partial x_2}=x_1\,e^{x_2}+\cos(x_1)$$

Obviously, we can see that both $\frac{\partial f(x)}{\partial x_1}$ and $\frac{\partial f(x)}{\partial x_2}$ exist and are finite for every $x=(x_1,x_2,x_3)\in\mathbb{R}^3$.

$$\frac{\partial f(x)}{\partial x_2\partial x_1}=e^{x_2}-\sin(x_1)$$

Which is continuous at any $x=(x_1,x_2,x_3)\in\mathbb{R}^3$

Now:

$$\frac{\partial f(x)}{\partial x_1\partial x_2}=e^{x_2}-\sin(x_1)=\frac{\partial f(x)}{\partial x_2\partial x_1}$$

Warning: The function $f\colon A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^m$ might have partial derivatives at an interior point a of A without necessarily being continuous at a. This is not the case at $\mathbb{R}$ (existence of derivative implies continuity). In higher orders of $\mathbb{R}$, derivatives do not imply continuity.

Exp:

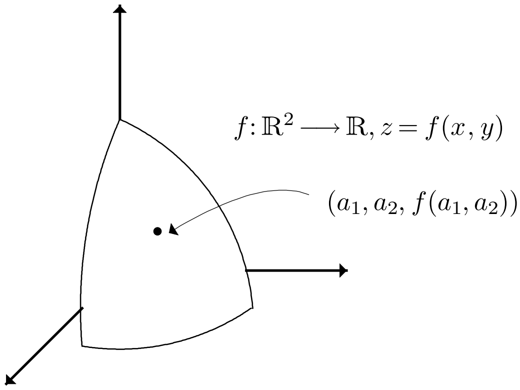
$$f(x,y)=\begin{cases}\frac{x\,y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

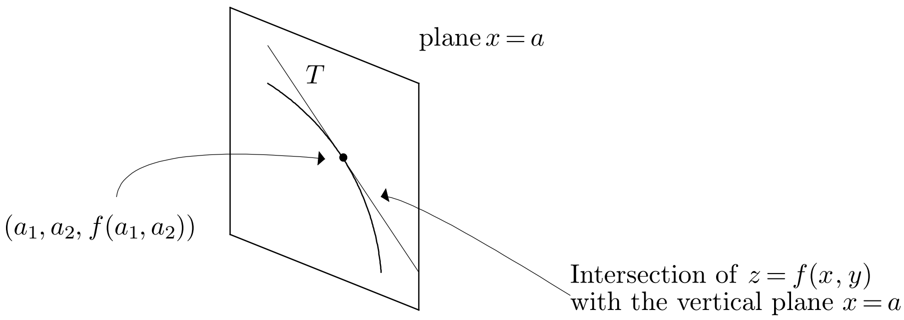
Clearly, f is not continuous at $(0,0)$, because: $\lim_{(x,x)\longrightarrow(0,0)} f(x,x) = \frac{1}{2} \neq f(0,0)$. Note that we are going along the line $y = x$. In fact, $\lim_{(x,y)\longrightarrow(0,0)} f(x,y)$ does not exist. However,

$$\lim_{h\longrightarrow 0} \frac{f(0+h,0)-f(0,0)}{h} = 0 \text{ and } \lim_{h\longrightarrow 0} \frac{f(0,0+h)-f(0,0)}{h} = 0$$

ie. we have that $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}$. Therefore the partial derivative exists, but f is not continuous at 0.

Geometric Interpretation of partial derivatives:





The slope of the tangent line on the plane (touching the point a) is the partial derivative of our function wrt to the plane. Mathematically,

$$\text{Slope}(T) = \frac{\partial f(a)}{\partial x}$$

Geometric Interpretation of the Jacobian:

The geometric meaning of the Jacobian is a local magnification of volume of a given function f. That is, if $B(a,\varepsilon)$ is an ε-ball centered at a (a is an interior point) of A and if $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then the ratio:

$$\frac{\text{Vol}(f(B(a,\varepsilon)))}{\text{Vol}(B(a,\varepsilon))} \xrightarrow[\varepsilon\longrightarrow 0]{} |\det[J_f(a)]|$$

<u>Exp</u>:

$$\text{Let } f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$f(x,y) = (x^2 + y, y^3 + x\,y)$$

$$\text{Let } a = 2, \text{ then we have that:}$$
$$J_f(1,2) = \begin{pmatrix} 2 & 1 \\ 2 & 13 \end{pmatrix}$$
$$\text{Thus } \det[J_f(1,2)] = 26 - 2 = 24$$

Therefore, we know that the area in $\mathbb{R}^2$ of the image $f(B(a,\varepsilon))$ is about 24 times as large as the area of $B(a,\varepsilon)$.

February 20th, 2021

Directional Derivative:

Take $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ (scalar / real-valued function). Let v be a unit vector (ie. $\|v\| = 1$) and let a be an interior point of A. We define the directional derivative of f at a in the direction of v by the following:

$$D_v f(a) = \lim_{h \longrightarrow 0} \frac{f(a + h\,v) - f(a)}{h}$$

Note that h is a scalar and v is a vector, so $h\,v$ is well-defined.

<u>Rmk</u>: In particular, if the vector $v = e_i$, then the directional derivative of f at a in the direction of e_i is just the partial derivative of f wrt. x_i, ie. $\frac{\partial f}{\partial x_i}(a)$. Note that $e_i = (0, 0, 0, \ldots, 1, \ldots, 0)$, which is the ith component.

The Gradient:

Let $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function. If all partial derivatives of f exist at an interior point of A, then the vector:

$$\left(\frac{\partial f(a)}{\partial x_1}, \frac{\partial f(a)}{\partial x_2}, \ldots, \frac{\partial f(a)}{\partial x_n} \right) \text{in } \mathbb{R}^n$$

is called the gradient vector of f at a and is denoted $\nabla f(a)$.

<u>Rmk</u>:

1. $D_v f(a) = \nabla f(a) \cdot v$, where $\cdot$ is the dot product

2. By the Cauchy-Schwartz Inequality, we have that:

$$\begin{aligned} |D_v f(a)| &= |\nabla f(a) \cdot v| \\ &\leqslant \|\nabla f(a)\| \rtimes \|v\| \\ &= \|\nabla f(a)\| \end{aligned}$$

$$\text{Therefore, we have the following:} \\ -\|\nabla f(a)\| \leqslant D_v f(a) \leqslant \|\nabla f(a)\|$$

   Thus the rate of change (i.e the rate of increase) of f in the direction of v varies from $-\|\nabla f(a)\|$ to $\|\nabla f(a)\|$, when v points in the opposite direction of $\nabla f(a)$ to when v points in the same direction as $\nabla f(a)$ respectively.

3. In particular, $\nabla f(a)$ points in the direction of greatest increase of f at a, and its modulus, which is $\|\nabla f(a)\|$ is precisely this greatest rate.

4. The directions orthogonal to $\nabla f(a)$ are the directions in which f neither increases nor decreases (because if $v \perp \nabla f(a)$, then $D_v f(a) = 0$).

<u>Exp</u>:

$$\begin{aligned} f &: \mathbb{R}^2 \longrightarrow \mathbb{R} \\ f(x, y) &= x^2 - y^2 \end{aligned}$$

Let $a = (1,2)$ and $v = (3,5)$. Find the value of $D_v f(1,2)$.

$$\nabla f(1,2) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)\right)$$
$$=(2,-4)$$

A unit vector in the direction of v is:

$$\frac{1}{\|v\|}v = \frac{1}{\sqrt{3^2+5^2}}(3,5) = \frac{1}{\sqrt{34}}(3,5)$$

Hence:

$$D_v f(1,2) = \frac{1}{\sqrt{34}}(6-20) = \frac{-14}{\sqrt{34}}$$

Chain Rule:

1st Situation.

$$g\colon \mathbb{R} \longrightarrow \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$
$$t \longmapsto (x(t), y(t)) \longmapsto f(x(t), y(t))$$
$$x\colon \mathbb{R} \longrightarrow \mathbb{R}, t \longmapsto x(t), y\colon \mathbb{R} \longrightarrow \mathbb{R}, t \longmapsto y(t)$$

Assume that $x(t)$ and $y(t)$ are differentiable (ie. $x'(t)$ and $y'(t)$ both exist). This is the usual derivative. We further assume that the partial derivatives of f wrt. x and y exist. Then:

$$g'(t) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t)$$

Note that $g'(t)$ is the usual derivative.

2nd Situation.

$$g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$
$$(s,t) \longmapsto (x(s,t), y(s,t)) \longmapsto f(x,y)$$
$$x\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \text{where } (s,t) \longmapsto x(s,t)$$
$$(\text{The same applies for } y)$$

In this case, we cannot consider the "usual" derivative of g, as opposed to the 1st situation. We will have:

$$\frac{\partial g}{\partial s}(s,t) = \frac{\partial f}{\partial x}(x,y)\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x,y)\frac{\partial y}{\partial s}(s,t)$$
$$\text{and}$$
$$\frac{\partial g}{\partial t}(s,t) = \frac{\partial f}{\partial x}(x,y)\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x,y)\frac{\partial y}{\partial t}(s,t)$$

3rd Situation.

$$\text{Let } f\colon D_1 \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\text{Let } g\colon D_2 \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^p$$

Let a be an interior point of D_1, and $f(a)$ be an interior point of D_2. If the $m \times n$ Jacobian matrix $J_f(a)$ and $p \times m$ Jacobian matrix $J_g(f(a))$ both exist, then the $p \times n$ (going from $\mathbb{R}^n$ to $\mathbb{R}^p$) Jacobian matrix $J_{g \circ f}(a)$ exists and is given by:

$$J_{g \circ f}(a) = J_g(f(a)) \times J_f(a)$$

Exp:

$$\text{Let } f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x_1, x_2) \longmapsto \begin{pmatrix} x_1^2 - x_2\cos(x_1) \\ x_1 x_2 \\ x_1^3 + x_2^3 \end{pmatrix}$$

$$\text{and let } g: \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(z_1, z_2, z_3) \longmapsto z_1 - z_2^2 + z_3$$

Then we have that:

$$J_f(x_1, x_2) = \begin{pmatrix} 2x_1 + x_2\sin(x_1) & -\cos(x_1) \\ x_2 & x_1 \\ 3x_1 & 3x_2^2 \end{pmatrix}$$

$$J_g(z_1, z_2, z_3) = \begin{pmatrix} 1 & -2z_2 & 1 \end{pmatrix}$$
$$\text{Thus } J_g(f(x_1, x_2)) = \begin{pmatrix} 1 & 2 - x_1 x_2 & 1 \end{pmatrix}$$

Hence:

$$J_{g\circ f}(x_1, x_2) = \begin{pmatrix} 1 & 2 - x_1 x_2 & 1 \end{pmatrix} \times \begin{pmatrix} 2x_1 + x_2\sin(x_1) & -\cos(x_1) \\ x_2 & x_1 \\ 3x_1 & 3x_2^2 \end{pmatrix}$$
$$= \begin{pmatrix} 2x_1 + x_2\sin(x_1) - 2x_1x_2^2 + 3x_1^2 & -\cos(x_1) - 2x_1^2x_2 + 3x_2^2 \end{pmatrix}$$

Differentiability of a real-valued (scalar) function:

Def.: Let $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and let a be an interior point of A. If the partial derivative of f exists at a, and if:

$$\lim_{H \longrightarrow O_{\mathbb{R}^n}} \frac{[f(a+h) - f(a)] - \nabla f(a) \cdot H}{\|H\|} = 0,$$

then we say that f is differentiable at a. H is a vector in $\mathbb{R}^n$.

Exp:

$$f(x, y) = x\, y^2 \text{ is differentiable at } (1, 3)$$

$$\text{In fact, } \frac{\partial f}{\partial x}(1, 3) = 9, \frac{\partial f}{\partial y}(1, 3) = 6$$
$$\text{Let } H = (h_1, h_2). \text{ Then:}$$
$$\frac{f(1 + h_1, 3 + h_2) - 9h_1 - 6h_2}{\sqrt{h_1^2 + h_2^2}}$$
$$= \frac{(1 + h_1)(9 + 6h_2^2 + h_2^2) - 9h_1 - 6h_2}{\sqrt{h_1^2 + h_2^2}}$$
$$= \frac{h_2^2 + 6h_1h_2 + h_1h_2^2}{\sqrt{h_1^2 + h_2^2}}$$
$$\text{Since } (h_1, h_2) \longrightarrow (0, 0), \text{ we can assume that:}$$
$$|h_1| < 1, \text{ and so:}$$

$$
\begin{aligned}
|h_2^2+6h_1h_2+h_1h_2^2| \leqslant h_2^2 + 6|h_1|h_2 + |h_1|h_2^2 &\leqslant h_2^2 + 3(2|h_1||h_2|) + h_2^2 \\
&= 2h_2^2 + 3(2|h_1||h_2|) \\
&\leqslant 2h_2^2 + 3(h_1^2+h_2^2) = 3h_1^2 + 5h_2^2 \leqslant 5(h_1^2+h_2^2) \\
&\text{Hence:} \\
0 \leqslant \frac{(1+h_1)(9+6h_2^2+h_2^2)-9h_1-6h_2}{\sqrt{h_1^2+h_2^2}} &\leqslant \frac{5(h_1^2+h_2^2)}{\sqrt{h_1^2+h_2^2}} \\
&= 5\sqrt{(h_1^2+h_2^2)} \xrightarrow{(h_1,h_2)\longrightarrow(0,0)} 0
\end{aligned}
$$

Thus we can see that $f(x,y)$ is differentiable at $(1,3)$.

February 22nd, 2021

Def.: *Function of class C^1*

A function $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be of class C^1 on A (denoted as $f \in C^1(A)$) if its partial derivatives exist and are continuous on A.

Very Important: Take $f\colon A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

$$
\begin{aligned}
f \in C^1(A) \Longrightarrow f \text{ is differentiable} \Longrightarrow \partial f \text{ exists} \\
f \text{ is differentiable} \Longrightarrow f \text{ is continuous} \\
\\
\text{However,} \\
\partial f \text{ exists does not necessarily imply continuity,} \\
\text{Or continuity does not necessarily imply } \partial f \text{ exists.}
\end{aligned}
$$

Counterexamples:

$$
\begin{gathered}
f\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \\
f(x,y) = \begin{cases} g(x,y) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}
\end{gathered}
$$

1. if $g(x,y) = \frac{x^2y^2}{x^2+y^2}$, then $f \in C(\mathbb{R}^2)$
2. if $g(x,y) = (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$, then f is not in $C^1(\mathbb{R}^2)$, but f is differentiable.
3. If $g(x,y) = \frac{x^2y}{x^2+y^2}$, then f is not differentiable but f is continuous and has partial derivatives.
4. If $g(x,y) = |x| + |y|$, then f is not differentiable, does not have partial derivatives, but it is continuous.
5. If $g(x,y) = \frac{y}{x^2+y^2}$, then f has no partial derivatives and f is not continuous.

Geometric Interpretation of Differentiability:

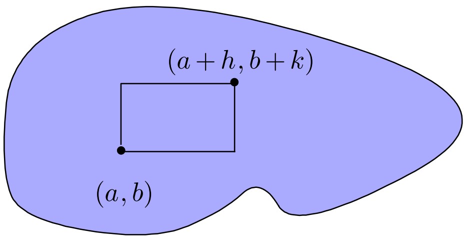
Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ and let $a \in \mathbb{R}$. If f is differentiable at a, then the graph $z = f(x,y)$ has a tangent plane at the point $(a, f(a))$. The equation of this tangent plane is:

$$
z = f(a) + \nabla f(a) \cdot (x - a)
$$

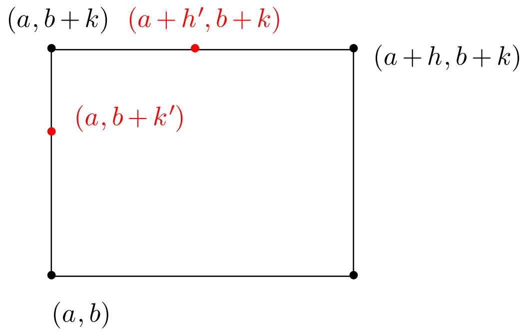
Mean Value Theorem:

Let $f\colon A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and let (a,b) be an interior point of A. Assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on a neighborhood of (a,b). Then for each (h,k) with $\|(h,k)\|$ very small, there are real numbers h' and k' where $a+h'$ lies between the real numbers a and $a+h$, and $b+k'$ that lies between b and $b+k$ st:

$$f(a+h,b+k)-f(a,b)=h\frac{\partial f}{\partial x}(a+h',b+k)+k\frac{\partial f}{\partial y}(a,b+k')$$



Neighborhood of (a,b)



<u>Proof</u>:

<u>We write</u>:

$$f(a+h,b+k)-f(a,b)$$
$$=f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b)$$

For $|h|$ and $|k|$ sufficiently small, all points on the sides of a small rectangle are in the neighborhood of (a,b).

As a single variable function, the function $f(x,b+k)$ $(f\colon x\longmapsto f(x,b+k))$ is differentiable on the open interval with endpoints $a, a+h$, and is continuous on the closed interval with endpoints $a, a+h$.

We can apply the MVT for the single variable function $f(x,b+k)$ to obtain the existence of $a+h'$ between a and $a+h$ st:

$$f(a+h,b+k)-f(a,b+k)$$
$$=h\frac{\partial f}{\partial x}(a+h',b+k) \qquad (1)$$

[Recall $f(b)-f(a)=(b-a)f'(c)$]

Similarly, the single variable function $f(a,y)$, $(y\longmapsto f(a,y))$ is differentiable on the open interval with endpoints $b, b+k$ and is continuous on the closed interval with endpoints $b, b+k$

Then by the MVT, there exists $b+k'$ between b and $b+k$ st

$$f(a,b+k)-f(a,b)=k\frac{\partial f}{\partial y}(a,b+k') \qquad (2)$$

$(1)+(2)\Longrightarrow$

$$f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b)$$
$$=h\frac{\partial f}{\partial x}(a+h',b+k)+k\frac{\partial f}{\partial y}(a,b+k')$$

Taylor's Theorem for Multivariable Functions:

Notation: Let $X=(x_1,x_2,\ldots,x_n)$. Then we have that $X\nabla$ denotes the 1st order differential operator of the form:

$$X\nabla=\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}$$

If m is a positive integer in $\mathbb{N}$, then $(x\nabla)^m$ denotes the mth order differential operator. For example, if $m=n=2$, then:

$$\begin{aligned}(x\nabla)^2=&\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x+2}\right)^2\\ =&x_1^2\frac{\partial^2}{\partial x_1^2}+2x_1x_2\frac{\partial^2}{\partial x_2\partial x_1}+x_2^2\frac{\partial^2}{\partial x_2^2}\end{aligned}$$

In general:

$$(x\nabla)^m=\sum_{k_1,k_2,\ldots,k_n}\frac{m!}{k_1!k_2!\ldots k_n!}(x_1^kx_2^kx_3^k\ldots x_n^k)\frac{\partial^m}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}}$$

Where the sum is taken over all the n-tuples $(k_1,k_2,\ldots,k_n)$ for which $\sum_{i=1}^n k_i=m$. If a real-valued function f has partial derivaties through order m, then:

$$(x\nabla)^m f(x)=\sum_{k_1,k_2,\ldots,k_n}\frac{m!}{k_1!k_2!\ldots k_n!}(x_1^kx_2^kx_3^k\ldots x_n^k)\frac{\partial^m f(x)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}}$$

....

Let $f\colon A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ and let a be an interior point of A. Assume there exists a neighborhood $N(a)$ of A with $N(a)\subset A$, over which all partial derivatives of order $\leqslant r$ exist and are continuous. Then for any $x\in N(a)$, we have:

$$f(x)=f(a)+\sum_{i=1}^{r-1}\frac{[(x-a)\nabla]^i f(a)}{i!}+\frac{[(x-a)\nabla]^r f(z)}{r!}$$

where z is a point on the line segment from a to x.

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Rmk: if $f(x)$ has partial derivatives of all orders in $N(a)$, then we have the series expansion of $f(x)$, which is given by:

$$f(x)=f(a)+\sum_{i=1}^{\infty}\frac{[(x-a)\nabla]^i f(a)}{i!}$$

In this case, the term $\frac{[(x-a)\nabla]^r f(z)}{r!}$ (the last term in Taylor's theorem) serves as a remainder of Taylor's series.

Exp:

$$\begin{gathered}f\colon\mathbb{R}^2\longrightarrow\mathbb{R}\\ f(x_1,x_2)=x_1x_2+x_1^2+e^{x_1}\cos(x_2)\end{gathered}$$

Clearly f has partial derivatives of all order (because the function is infinitely times differentiable). Thus in the neighborhood of $a=(0,0)$, we can write:

$$f(x_1,x_2)=f(0,0)+\sum_{i=1}^{2}\frac{[((x_1,x_2)-(0,0))\nabla]^i f(0,0)}{i!}$$
$$+\frac{1}{3!}[((x_1,x_2)-(0,0))\nabla]f(\xi x_1,\xi x_2),\text{ where } 0<\xi<1$$

$$f(0,0)=1$$
$$[(x_1,x_2)\nabla]f(0,0)=x_1\frac{\partial f(0,0)}{\partial x_1}+x_2\frac{\partial f(0,0)}{\partial x_2}=x_1$$
$$[(x_1,x_2)\nabla]^2 f(0,0)=3x_1^2+2x_1x_2-x_1^2$$
$$[(x_1,x_2)\nabla]^3 f(\xi x_1,\xi x_2)=x_1^3\,e^{\xi x_1}\cos(\xi x_2)-3x_1^2\,x_2\,e^{\xi x_1}\sin(\xi x_2)$$
$$-3x_1x_2^2\,e^{\xi x_1}\cos(\xi x_2)+x_2^3\,e^{\xi x_1}\sin(\xi x_1)$$

Hence:

$$f(x_1,x_2)=1+x_1+\frac{1}{2!}(3x_1^2+2x_1x_2-x_2^2)+\frac{1}{3!}[(x_1^3-3x_1x_2^2)e^{\xi x_1}\cos(\xi x_2)$$
$$+(x_2^3-3x_1^2x_2)e^{\xi x_1}\sin(\xi x_2)]$$

$1+x_1+\frac{1}{2!}(3x_1^2+2x_1x_2-x_2^2)$ serves as a 2nd order approximation of $f(x_1,x_2)$, and

$\frac{1}{2!}(3x_1^2+2x_1x_2-x_2^2)+\frac{1}{3!}[(x_1^3-3x_1x_2^2)e^{\xi x_1}\cos(\xi x_2)+(x_2^3-3x_1^2x_2)e^{\xi x_1}\sin(\xi x_2)]$ is the remainder.

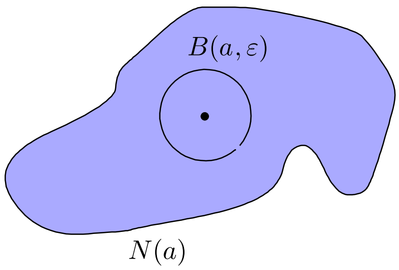
Critical Points:

Let $f: A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ and let $a\in A$.

1. f is said to have an absolute maximum (resp. absolute minimum) at a if for all $x\in A$, we have that $f(x)\leqslant f(a)$ (resp. $f(x)\geqslant f(a)$).

2. f is said to have a local maximum (resp. local minimum) if $\exists$ an ε-ball $B(a,\varepsilon)$ st:

$$\forall x\in B(a,\varepsilon),\ f(x)\leqslant f(a)\ (\text{resp. } f(x)\geqslant f(a))$$



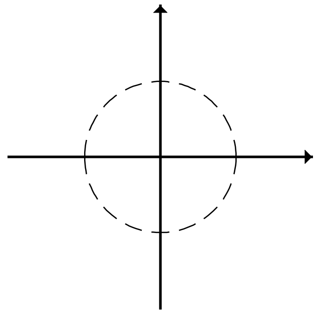
Terminology: Abs. max or abs. min (resp. local max or local min) are called extrema points.

Def.: *Critical Points:* An interior point a of A is called a critical point of f (or stationary point) if we have that $\nabla f(a)=0$.

Fermat's Theorem: Assume f is of class C^1 (ie. the partial derivatives at a exist and are continuous). Then f has a local extremum at a (ie. a is a local max. or local min) if $\nabla f(a)=0$ (ie. if a is a critical point of f).

Exp:

$$A := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$



$$f: A \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto f(x,y) = x^2 + y^2$$
$$\frac{\partial f}{\partial x} = 2x \text{ and } \frac{\partial f}{\partial y} = 2y$$

Thus:

$$\nabla f(x,y) = O_{\mathbb{R}^2} \text{ iff } x = 0 = y$$

Hence $(0,0)$ is the only critical point.

By definition, f has an abs. min at $(0,0)$ because $f(x,y) = x^2 + y^2 \geqslant f(0,0) = 0$ for all $(x,y) \in A$.

How do we find the abs. extremum of a continuous function f on a compact A?

- Find all critical points of f that are interior points of A (Those points are not on the boundary of $A \longleftarrow \partial A$)

  (a is in ∂A if $\forall \varepsilon > 0, B(a, \varepsilon)$ contains points in A and points not in A)

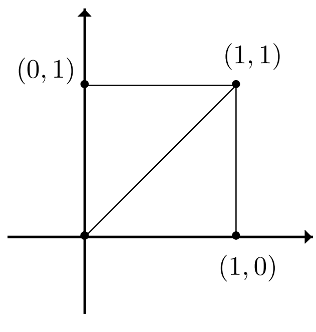
- Find all the points on ∂A where $\nabla f(a) = 0$.

- Evaluate f at the ponits where the partial derivatives (at least one) do not exist or the points where f is not differentiable.

  The highest value obtained (resp. lower value) in the previous steps will be the abs. max value (resp. abs. min. value)

Exp:

$$A = [0,1] \times [0,1] \longleftarrow \text{compact}$$
$$f: A \longrightarrow \mathbb{R}$$
$$f(x,y) = (x-y)^2$$



For every $(x,y)\in(0,1)\times(0,1)$ (interior points of A), we have that:

$$\frac{\partial f}{\partial x}=2(x-y) \text{ and } \frac{\partial f}{\partial y}=-2(x-y)$$
$$\text{Thus } \nabla f(x,y)=(0,0) \text{ iff } x=y$$
$$\text{In this case, } f(x,x)=0 \text{ (let } y=x)$$

Now, we evaluate f at the critical points on the boundary of A:

$$\text{Along } y=0\text{:}$$
$$\text{Consider the single variable function:}$$
$$g\colon(0,1)\longrightarrow\mathbb{R} \text{ where } x\longmapsto g(x)=f(x,0)$$
$$\text{Then } g(x)=x^2 \text{ and } g'(x)\neq 0 \text{ for all } x\in(0,1)$$

$$\text{Along } y=1\text{:}$$
$$\text{Let } g\colon(0,1)\longrightarrow\mathbb{R} \text{ where } x\longmapsto g(x)=f(x,1)$$
$$\text{So } g(x)=(x-1)^2 \text{ and } g'(x)=2(x-1)\neq 0 \text{ for all } x\in(0,1)$$

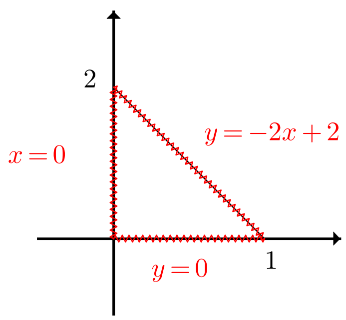
$$\text{Along } x=0\text{:}$$
$$\text{Let } g\colon(0,1)\longrightarrow\mathbb{R} \text{ where } y\longmapsto g(y)=f(0,y)$$
$$\text{So } g(y)=y^2 \text{ and } g'(y)=2y\neq 0 \quad \forall y\in(0,1)$$

$$\text{Along } x=1\text{:}$$
$$\text{Let } g\colon(0,1)\longrightarrow\mathbb{R} \text{ where } y\longmapsto g(y)=f(1,y)$$
$$\text{So } g(y)=(1-y^2) \text{ and } g'(y)=-2(1-y)\neq 0 \quad \forall y\in(0,1)$$

$$f(0,0)=0,\ f(0,1)=1,\ f(1,0)=1 \text{ and } f(1,1)=0$$

The only candidates to be abs. extrema are: $(0,0),(0,1),(1,0),(1,1)$ and (x,x) for $x\in(0,1)$. Hence each (x,x) is an abs. min. for all x and the only abs. max. are $(1,0)$ and $(0,1)$.

Another visual example:



We know that the boundary points are along the lines $y=0, x=0$ and $y=-2x+2$. We follow in the same fashion as the previous example and find the extrema.

March 1st, 2021

4 Chapt. Implicit / Inverse Function Theorem

Implicit Function Theorem *(Case of Two variables):*

Let $f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and let (x_0, y_0) be an interior point of A. Assume f is of class C^1 in a neighborhood of (x_0, y_0). If $f(x_0, y_0) = k$ (k is some constant in $\mathbb{R}$), and if $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then there exists an open interval I centered at x_0 (I is a neighborhood of x_0), and there exists a function:

$$\varphi: I \longrightarrow \mathbb{R}$$

of class C^1 on I (ie. φ is differentiable on I and its derivative φ' is continuous on I) st: $\varphi(x_0) = y_0$ for all $x \in I$ (ie. x is very close to x_0). We have:

$$f(x, \varphi(x)) = f(x_0, y_0) = k \text{ and } \varphi'(x) = -\frac{\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))}$$

Moreover, the equation of the tangent line to the curve $y = \varphi(x)$ at x_0 is given by:

$$y = \varphi'(x_0)(x - x_0) + y_0$$

Exp:

The function $y^3 - y = 3x$ defines implicitly on a neighborhood of $(2, -2)$ a function $y = \varphi(x)$. In fact, the function $f(x, y) = y^2 - y - 3x$ satisfies the hypotheses of the theorem and $f(2, -2) = 0 = k$, where $(2, -2) = (x_0, y_0)$.

We have that:

$$\frac{\partial f}{\partial y}(2, -2) = [2y - 1]_{(2,-2)} = -4 - 1 = -5 \neq 0$$

The tangent line to the curve $y = \varphi(x)$ at the point $(2, -2)$ is given by:

$$y = \varphi'(2)(x - 2) + (-2) = -\frac{\frac{\partial f}{\partial x}(2, -2)}{\frac{\partial f}{\partial y}(2, -2)}(x - 2) = 2$$

$$= -\frac{-3}{-5}(x - 2) - 2$$
$$= -\frac{3}{5}x - \frac{4}{5}$$

Rmk: If we differentiate (the usual derivative), the equation $f(x, \varphi(x)) = k$ wrt x, we obtain:

$$\frac{\partial f}{\partial x}(x, \varphi(x)) + \varphi'(x)\frac{\partial f}{\partial y}(x, \varphi(x)) = 0$$
$$\text{and hence: } \varphi'(x) = --\frac{\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))}$$

Implicit Function Theorem *(Case of Three variables):*

Let $f: A \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ and let (x_0, y_0, z_0) be an interior point of A. Assume f is of class C^1 in a neighborhood of (x_0, y_0, z_0). If $f(x_0, y_0, z_0) = k$ and if $\frac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$, then there exists a neighborhood (ε-ball centered at (x_0, y_0) $I \subset \mathbb{R}^2$ of (x_0, y_0) and there exists a function $\varphi: I \longrightarrow \mathbb{R}$ of class C^1 on I st:

- $\varphi(x_0, y_0) = z_0$,

- For every $(x, y) \in I$, we have:

$$f(x, y, \varphi(x, y)) = k$$

-

$$\frac{\partial \varphi}{\partial x}(x, y) = -\frac{\frac{\partial f}{\partial x}(x, y, \varphi(x, y))}{\frac{\partial f}{\partial z}(x, y, \varphi(x, y))}$$
$$\frac{\partial \varphi}{\partial y}(x, y) = -\frac{\frac{\partial f}{\partial y}(x, y, \varphi(x, y))}{\frac{\partial f}{\partial z}(x, y, \varphi(x, y))}$$

For the second point above, we are basically saying that the points (x, y, z) are sufficiently close to the points (x_0, y_0, z_0) that satisfy $f(x, y, z) = f(x_0, y_0, z_0) = k$ and are of the form $(x, y, \varphi(x, y))$.

Exp:

Consider the point $(1, 2, 1)$ ont he level surface $x y z + z^3 = 3$ of the function $f(x, y, z) = x y z + z^3$, ie. $f(1, 2, 1) = 3$.

$$\text{Since } \frac{\partial f}{\partial z} = x y + 3z^2, \text{ we have that:}$$
$$\frac{\partial f}{\partial z}(1, 2, 1) = 5 \neq 0$$

Now, by the implicit function theorem, we can solve for z as a function of x and y near the point $(1, 2)$. That is, there is a differentiable function φ defined on an ε-ball centered at $(1, 2)$, with $\varphi(1, 2) = 1$ st $f(x, y, \varphi(x, y)) = 3$ for all (x, y) in the ε-ball centered at $(1, 2)$.

$$\text{Also, } \frac{\partial \varphi}{\partial x}(1, 2) = -\frac{\frac{\partial f}{\partial x}(1, 2, 1)}{\frac{\partial f}{\partial z}(1, 2, 1)} = -\frac{2}{5}$$
$$\text{and } \frac{\partial \varphi}{\partial y}(1, 2) = -\frac{\frac{\partial f}{\partial y}(1, 2, 1)}{\frac{\partial f}{\partial z}(1, 2, 1)} = -\frac{1}{5}$$

An equation of the tangent plane to the surface $z = \varphi(x, y)$ at $(1, 2, 1)$ is given by:

$$-\frac{2}{5}(x - 1) - \frac{1}{5}(y - 2) = z - 1$$
$$-\frac{2}{5}(x - 1) - \frac{1}{5}(y - 2) - (z - 1) = 0$$

Rmk: The implicit function theorem says that the level surface S, which is $f(x, y, z) = k$, of a class C^1 function f is locally the graph of a differentiable function, say $z = \varphi(z, y)$ if the point (a, b, c) is on S (ie. $f(a, b, c) = k$) and if $\frac{\partial f}{\partial z}(a, b, c) \neq 0$, then:

$$\left(\frac{\partial \varphi}{\partial x}(a, b), \frac{\partial \varphi}{\partial y}(a, b), -1\right) \cdot (x - a, y - b, z - c) = 0 \qquad (*)$$

is the equation of the tangent plane to S at (a, b, c).

Since:

$$\frac{\partial \varphi}{\partial x}(a, b) = -\frac{\frac{\partial f}{\partial x}(a, b, c)}{\frac{\partial f}{\partial z}(a, b, c)} \text{ and } \frac{\partial \varphi}{\partial y}(a, b) = -\frac{\frac{\partial f}{\partial y}(a, b, c)}{\frac{\partial f}{\partial z}(a, b, c)}$$

Equation $(*)$ is the same as:

$$\nabla f(a, b, c) \cdot (x - a, y - b, z - c) = 0$$

In fact, it is the same as:

$$\left(-\frac{\frac{\partial f}{\partial x}(a,b,c)}{\frac{\partial f}{\partial z}(a,b,c)},-\frac{\frac{\partial f}{\partial y}(a,b,c)}{\frac{\partial f}{\partial z}(a,b,c)},-\frac{\frac{\partial f}{\partial z}(a,b,c)}{\frac{\partial f}{\partial z}(a,b,c)}\right)\cdot(x-a,y-b,z-c)=0$$

$$-\frac{1}{\frac{\partial f}{\partial z}(a,b,c)}\left(\frac{\partial f}{\partial x}(a,b,c),\frac{\partial f}{\partial y}(a,b,c),\frac{\partial f}{\partial z}(a,b,c)\right)\cdot(x-a,y-b,z-c)=0$$

$$\left(\frac{\partial f}{\partial x}(a,b,c),\frac{\partial f}{\partial y}(a,b,c),\frac{\partial f}{\partial z}(a,b,c)\right)\cdot(x-a,y-b,z-c)=0$$

$$=\nabla f(a,b,c)\cdot(x-a,y-b,c-z)=0$$

March 3rd, 2021

Inverse Function Theorem:

Let A be an open set of $\mathbb{R}^n$ (All the points of A are interior points). Let $f\colon A\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$. Assume f is of class C^1 on A. Let $a\in A$. If the Jacobian matrix $J_f(a)$ is invertible, ie. $\det[J_f(a)]\neq 0$, then there exists an open set $V\subset A$ containing a (V is a neighborhood of a, ie. V could be an ε-ball centered at a) and an open set $W\subset\mathbb{R}^n$ containing $f(a)$ (ie. W is a neighborhood of $f(a)$) st:

$$f\colon V\longrightarrow W \text{ has an inverse } f^{-1}\colon W\longrightarrow V$$

that is of class C^1 on W. Moreover, for every $y=f(x)\text{ in }W,\text{where }x\in V$, we have:

$$J_{f^{-1}}(y)=[J_f(x)]^{-1}$$

Exp:

$$f\colon\mathbb{R}^2\longrightarrow\mathbb{R}^2$$
$$(x,y)\longmapsto f(x,y)=(x^3-2x\,y^3,x+y)$$

If f locally invertible at $(1,-1)$? Clearly we can see that f is of class $C^1\text{ on }\mathbb{R}^2$.

$$J_f(1,-1)=\begin{pmatrix}3x^2-2y^2 & -4x\,y\\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix}$$

$$\det[J_f(1,-1)]=3\neq 0$$

$$\text{Thus }J_f(1,-1)\text{ is locally invertible at }(1,-1),$$

$$J_{f^{-1}}(1,-1)=-\frac{1}{3}\begin{pmatrix}1 & -4\\ -1 & 1\end{pmatrix}$$

Therefore, f is locally invertible at the point at $(1,-1)$.

What is the best affine approximation to the inverse near $f(1,-1)$? The affine approximation of f^{-1} is an approximation of 1st order. near $f(1,-1)=(-1,0)$. It is given by:

$$f^{-1}(-1+h,0+k)\approx f^{-1}(-1,0)+J_{f^{-1}}(-1,0)\times\begin{pmatrix}h\\ k\end{pmatrix}$$

Note that $\|(h,k)\|\text{ is very small, and that }J_{f^{-1}}(-1,0)=(J_{f^{-1}}(1,-1))^{-1}$

$$=\begin{pmatrix}1\\ -1\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1 & -4\\ -1 & 1\end{pmatrix}\begin{pmatrix}h\\ k\end{pmatrix}$$

$$=\left(1-\frac{1}{3}h+\frac{4}{3}k,-1+\frac{1}{3}h-\frac{1}{3}k\right)$$

Rmk: The actual f^{-1} about $(-1,0)$ may not be clear, but the Inverse function theorem guarantees its existence, and its affine approximation is easy to find. We don't need to see that f^{-1} is. This is only of the 1st order.

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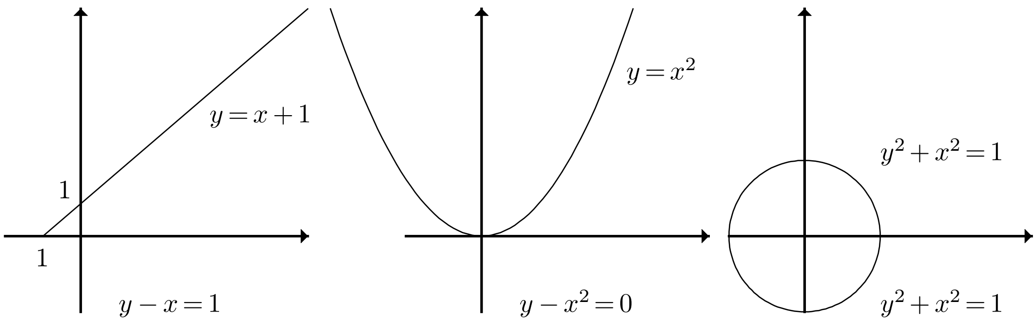
5 Chapt. Curves / Surfaces in Planes & Space

What is a curve?



All three curves are described by the means of their Cartesian equation: $f(x, y) = c$ for some constant $c \in \mathbb{R}$. From this point of view, a curve is a set of points, namely:

$$C := ((x, y) \in \mathbb{R}^2 \mid f(x, y) = c)$$

These are essentially level curves. But a curve can also be viewed as the path traced out by a moving point. Thus, if $\gamma(t)$ is the position of the point at time t, the curve is described by the function γ and a scalar parameter t with vector values in $\mathbb{R}^2$ for a plane curve and in $\mathbb{R}^3$ for a curve in space.

Def.: *Parameterization:* A parameterized curve in $\mathbb{R}^n$ is a map $\gamma\colon (\alpha, \beta) \longrightarrow \mathbb{R}^n$, where (a, β) is an interval in $\mathbb{R}$, $-\infty \leqslant \alpha < \beta \leqslant \infty$.

Exp: The parameterization of the parabola $y = x^2$ can be given by:

$$\gamma\colon (-\infty, \infty) \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \gamma(t) = (t, t^2)$$

But thhis is not the only parameterization of the parabola! Another parameterization could be the same domains but $t \longmapsto \gamma(t) = (t^3, t^6)$. This is a disadvantage, because proofs cannot depend on parameterization (choice of). So the parameterization of a given curve is not unique.

Def.: *Smooth Curve:* A function $f\colon (\alpha, \beta) \longrightarrow \mathbb{R}$ is said to be smooth if all its derivatives (usual derivatives)

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \text{exist for all } n \geqslant 1$$

(ie. f is of class C^1 on (α,β)). In our case, if:

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$$

where each γ_i is a real-valued function, is smooth if γ_i is smooth.

Def.: *Tangent Vector:* If γ is a parameterized curve, its first derivative $\dot{\gamma}(t)$ is called the tangent vector of γ at the point $\gamma(t)$ at t.

Proposition:

If the tangent vector of a parameterized curve is constant (constant vector), then the image of the curve is (part of) a straight line.

March 8th, 2021

<u>Proof</u>:

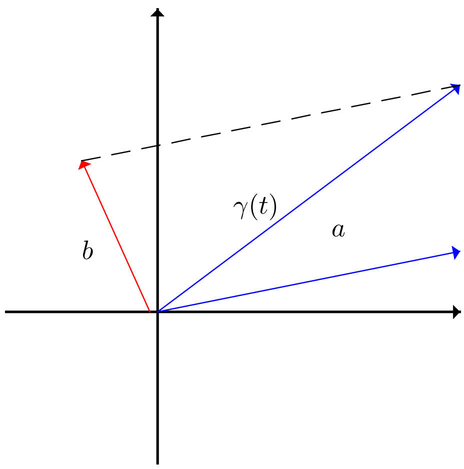
If $\dot{\gamma}(t) = a$, where a is a constant vector, then by integrating component wise wrt t, we obtain:

$$\gamma(t) = \int \dot{\gamma}(t)\mathrm{dt} = \int a\,\mathrm{dt} = t(a) + b$$

$$\text{where } b \text{ is another constant vector.}$$

$$= (a_1 t + b_1, a_2 t + b_2) = t(a_1, a_2) + (b_1, b_2)$$

If $a \neq 0$ (not the zero vector), then $t\,a + b$ is the parametric equation of the line parallel to a and passing thorugh the point b:



Arc-Length:

Def.: *Arc-Length:* The arc-length of a curve γ starting at the point $\gamma(t_0)$ is the function $s(t)$ given by:

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}\,(u)\|\mathrm{d}u$$

Thus $s(t_0) = 0$ and $s(t)$ is positive if $t \geqslant t_0$, and is negative if $t < t_0$. If we choose a different starting point $\gamma(\tilde{t_0})$, the resulting arc length $\tilde{s}$ differs from s by the constant:

$$\int_{t_0}^{\tilde{t_0}} \|\dot{\gamma}(u)\|\mathrm{d}u$$

This is because of the fact that:

$$\int_{t_0}^{t} \|\dot{\gamma}\,(u)\|\mathrm{d}u = \int_{t_0}^{\tilde{t_0}} \|\dot{\gamma}(u)\|\mathrm{d}u + \int_{\tilde{t_0}}^{t} \|\dot{\gamma}(u)\|\mathrm{d}u$$

$$s(t) = \mathrm{csnt} + \tilde{s}(t)$$

Rmk: By the FTC, $s(t)$ is differentiable. In fact:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = s'(t) = \frac{\mathrm{d}t}{\mathrm{d}t}\|\dot{\gamma}(t)\| + \frac{\mathrm{d}t_0}{\mathrm{d}t}\|\dot{\gamma}(t_0)\| = (1)\|\dot{\gamma}(t)\| + (0)\|\dot{\gamma}(t_0)\| = \|\dot{\gamma}(t)\|$$

Def.: *Speed:* If $\gamma\colon(\alpha,\beta)\longrightarrow\mathbb{R}^n$ is a parameterized curve, its speed at the point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$, and γ is said to be a unit-speed cuve if $\dot{\gamma}(t)$ is a unit vector (ie. $\|\dot{\gamma}(t)\|=1$ for all $(t\in(\alpha,\beta))$.

Prop. 1: Let $n(t)$ be a unit vector that is a smooth function of the parameter t. Then we have that:

$$\dot{n}(t)\cdot n(t) = 0 \text{ for all } t$$

ie. $\dot{n}(t)$ is the zero vector or is orthogonal to $n(t)$ for all t. In particular, if γ is a unit-speed curve, then $\ddot{\gamma}$ is a zero vector or is orthogonal to $\dot{\gamma}$ for all t.

Proof:

We recall the following "product formula":

$$\begin{aligned}\text{if } a\colon(\alpha,\beta)\longrightarrow\mathbb{R}^n \text{ and}\\ b\colon(\alpha,\beta)\longrightarrow\mathbb{R}^n \text{ are smooth functions}\\ \text{of } t, \text{then:}\\ \frac{\mathrm{d}}{\mathrm{d}t}(a\bullet b) = \frac{\mathrm{d}a}{\mathrm{d}t}\bullet b + a\bullet\frac{\mathrm{d}b}{\mathrm{d}t}\end{aligned}$$

Since n is a unit vector, we have: $n(t)\bullet n(t)=1$. We differentiate both sides wrt t:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(n(t)\bullet n(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(1)\\ \dot{n}(t)\bullet n(t) + n(t)\bullet\dot{n}(t) &= 0\\ 2\dot{n}(t)\bullet n(t) &= 0\\ \dot{n}(t)\bullet n(t) &= 0, n(t)\neq 0\end{aligned}$$

The last part follows by taking $n(t)=\dot{\gamma}(t)$

Def.: *Reparameterization:* A parameterized curve $\tilde{\gamma}\colon(\alpha,\beta)\longrightarrow\mathbb{R}^n$ is a reparameterization of a parametric curve $\gamma\colon(\alpha,\beta)\longrightarrow\mathbb{R}^n$ if there is a smooth bijective map: $\varphi\colon(\tilde{\alpha},\tilde{\beta})\longrightarrow(\alpha,\beta)$ (the reparameterization map) such that the inverse map: $\varphi^{-1}\colon(\alpha,\beta)\longrightarrow(\tilde{\alpha},\tilde{\beta})$ is also smooth and:

$$\tilde{\gamma}(\tilde{t}) = \gamma(\varphi(\tilde{t})) \text{ for all } \tilde{t}\in(\tilde{\alpha},\tilde{\beta})$$

$$\begin{aligned}(a,\beta) \xrightarrow{\varphi^{-1}} (\tilde{\alpha},\tilde{\beta}) \xrightarrow{\tilde{\gamma}} \mathbb{R}^n\\ (a,\beta) \xleftarrow{\varphi} (\tilde{\alpha},\tilde{\beta}) \longrightarrow \mathbb{R}^n\end{aligned}$$

The map is shown below:

$\tilde{\gamma}: \tilde{t} \longmapsto \tilde{\gamma}(\tilde{t})$

$$
\begin{tikzcd}
\tilde{\gamma}: (\tilde{\alpha}, \tilde{\beta}) \arrow[r] \arrow[d, "\varphi"'] & \mathbb{R}^n \\
\gamma: (\alpha, \beta) \arrow[u, bend right, "\varphi^{-1}"'] \arrow[ur] &
\end{tikzcd}
$$

$\tilde{\gamma}(\tilde{t}) = \gamma(\varphi(\tilde{t}))$

Exp: The circle $x^2 + y^2 = 1$ has a parameterization $\gamma(t) = (\cos(t), \sin(t))$. Another parameterization of the unit circle is $\tilde{\gamma}(t) = (\sin(t), \cos(t))$, because we still have $\sin^2(t) + \cos^2(t) = 1$. To see that $\tilde{\gamma}$ is a reparameterization fo γ, we have to find a reparameterization map φ st:

$$(\cos(\varphi(t)), \sin(\varphi(t))) = (\sin(t), \cos(t))$$

One solution is: $\varphi(t) = \frac{\pi}{2} - t$.

Def.: *Regular Points / Singular Points:* A point $\gamma(t)$ of a parameterized curve γ is called a regular point if $\dot{\gamma}(t) \neq 0$ (not the zero vector). Otherwise, $\gamma(t)$ is a singular point (or critical point). A curve is said to be a regular curve if all its points are regular.

Prop. 2: Any reparameterization of a regular curve is regular as well (reparameterizations do not affect the geometry of the curve itself).

Rmk: Two curves that are reparameterizations of each other have the same image, so they should have the same geometric properties.

Proof:

Assume γ and $\tilde{\gamma}$ are st. $\tilde{\gamma}(\tilde{t}) = \gamma(\varphi(\tilde{t}))$ as in the previous definition. Since $\varphi(\varphi^{-1}(t)) = t$, differentiating both sides implies:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}} \cdot \frac{\mathrm{d}\varphi^{-1}}{\mathrm{d}t} = 1$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}} \text{ is never 0 since } \tilde{\gamma}(\tilde{t}) = \gamma(\varphi(\tilde{t}))$$

Another application of the chain rule gives:

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}\tilde{t}}$$

This shows that

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} \text{ is never 0 if } \frac{\mathrm{d}\gamma}{\mathrm{d}t} \text{ is never 0, ie. if } \gamma \text{ is regular.}$$

Prop. 3: If γ is a regular curve, then its arc-legnth starting at any point of γ is a smooth function of t.

Proof:

We have already seen that s is differentiable and:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\gamma}(t)\|$$

To simplify the notation, assem that γ is in the plane, say $\gamma(t) = (u(t), v(t))$, where u and v are smooth functions of t, so that:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\dot{u}^2(t) + \dot{v}^2(t)}$$

The function $f(x) = \sqrt{x}$ on $(0, +\infty)$. Indeed one can show by induction on n that:

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n} x^{-\frac{2n+1}{2}}$$

Since u and v are smooth, $\dot{u}$ and $\dot{v}$ are also smooth, hence $\dot{u}^2(t) + \dot{v}^2(t)$ is also smooth. Since γ is regular,

$$\begin{gathered} \dot{u}^2(t) + \dot{v}^2(t) > 0 \text{ for all } t \\ \text{so } f(\dot{u}^2(t) + \dot{v}^2(t)) = \frac{\mathrm{d}s}{\mathrm{d}t} \end{gathered}$$

is a smooth function of t.

March 10th, 2021

Theorem 4:

A parameterized curve has a unit-speed reparameterization iff it is regular.

Proof:

$$\begin{aligned} &\Longrightarrow \\ &\text{Suppose that a parameterized curve:} \\ &\gamma: (\alpha, \beta) \longrightarrow \mathbb{R}^n \\ &\text{has a unit-speed reparameterization} \\ &\tilde{\gamma} \text{ with reparameterization map } \varphi \\ &\text{Letting } t = \varphi(\tilde{t}), \text{ we have:} \\ &\tilde{\gamma}(\tilde{t}) = \gamma(\varphi(\tilde{t})) = \gamma(t), \text{ and so:} \\ &\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} \\ &\text{Thus } \left\| \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} \right\| = \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}} \right\| \end{aligned}$$

Since $\tilde{\gamma}$ is unit-speed, we have that $\|\dot{\gamma}(\tilde{t})\| = \left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}}\right\| = 1$, and so $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ cannot be zero for all t. Hence γ is regular.

$$\begin{aligned}
&\Longleftarrow\\
&\text{Suppose } \gamma \text{ is regular}\\
&\text{ie. } \frac{\mathrm{d}\gamma}{\mathrm{d}t} \text{ is never zero for all } t.\\
&\text{Since } \frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\gamma}(t)\|, \text{ we have } \frac{\mathrm{d}s}{\mathrm{d}t} > 0 \text{ for all } t\\
&\text{where } s \text{ is the arc-length function of } \gamma
\end{aligned}$$

Thus s is one-to-one (injective). Since γ is a regular curve, **PROP3** implies that s is a smooth function of t. It follows from the inverse function theorem that $s\colon (\alpha, \beta) \longrightarrow s((\alpha, \beta)) = (\tilde{\alpha}, \tilde{\beta})$ is invertible and its inverse:

$$s^{-1}\colon (\tilde{\alpha}, \tilde{\beta}) \longrightarrow (\alpha, \beta) \text{ is smooth}$$

We take $\varphi = s^{-1}$ and let $\tilde{\gamma}$ be the corresponding reparameterization of γ:

$$\begin{tikzcd}
\gamma\colon (\alpha, \beta) \arrow[rr] \arrow[d] & & \mathbb{R}^n \\
\tilde{\gamma}\colon (\tilde{\alpha}, \tilde{\beta}) \arrow[u, bend right, "s^{-1}"'] \arrow[urr] & &
\end{tikzcd}$$

$$\begin{aligned}
\tilde{\gamma}(\tilde{t}) &= \gamma(s^{-1}(\tilde{t}))\\
\tilde{t} = s(t), s^{-1}(\tilde{t}) &= t\\
\text{ie. } \tilde{\gamma}(s(t)) &= \gamma(t)
\end{aligned}$$

Thus we have that:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \text{ and so:}\\
\left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}\right\| \cdot \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| = \|\dot{\gamma}\| = \frac{\mathrm{d}s}{\mathrm{d}t}\\
\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| = \frac{\mathrm{d}s}{\mathrm{d}t} > 0\\
\\
\text{Hence } \left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}\right\| = 1\\
\text{ie. } \tilde{\gamma} \text{ is a unit-speed reparameterization of } \gamma
\end{aligned}$$

Corollary 5:

Let γ be a regular vector and let $\tilde{\gamma}$ be a unit-speed reparameterization of γ:

$$\tilde{\gamma}(u(t)) = \gamma(t) \text{ for all } t$$

where u is a smooth function of t.

$$\begin{tikzcd}
\gamma\colon(\alpha,\beta) \arrow[rr] \arrow[d, "u"'] & & \mathbb{R}^n \\
\tilde{\gamma}\colon(\tilde{\alpha},\tilde{\beta}) \arrow[u, bend right, "u^{-1}"'] \arrow[urr] & &
\end{tikzcd}$$

$$\tilde{\gamma}(\tilde{t}) = \gamma(u^{-1}(\tilde{t}))$$
$$\tilde{t} = (u(t)), u^{-1}(\tilde{t}) = t$$

Then, if s is the arc-length of γ (starting at any point), we have:

$$u = \pm s + c, c \,\text{is constant} \quad (*)$$

Conversely, if we have $u = \mp s + c$, for some constant c and with either $+$ or $-$ sign ($u = s + c$ or $u = -s + c$), then $\tilde{\gamma}$ is a unit-speed reparameterization of γ.

Proof:

From the calculation in $\Longrightarrow$ of **THM4**, u gives a unit-speed reparameterization of γ iff:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \pm\left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| = \pm\left\|\frac{\mathrm{d}s}{\mathrm{d}t}\right\|$$

By integrating both sides, we obtain $u = \pm s + c$.

Rmk:

1. **CRLY5** says that the arc-length is essentially the only unit-speed parameter on a regular curve.

2. Although every regular curve has a unit-speed reparameterization, this may be very complicated, or even impossible to write down explicitly.

   Exp: The logarithmic spiral:

   $$\gamma(t) = (e^{kt}\cos(t), e^{kt}\sin(t))$$

   $$\|\dot{\gamma}\|^2 = (k^2+1)e^{2kt}, \text{which is never } 0$$
   $$\text{Therefore } \gamma \text{ is regular.}$$

   The arc-length function $s(t)$ starting from the point $(1,0)$ is given by:

   $$s(t) = \sqrt{k^2+1}\,\frac{e^{kt}-1}{k}$$

We solve this for t:

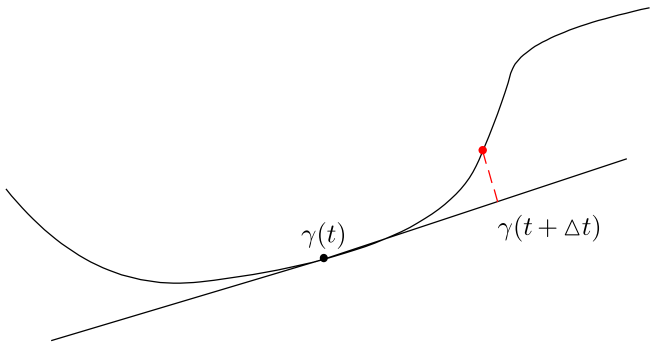
$$t=\frac{1}{k}\ln\left(\frac{k\,s}{\sqrt{k^2+1}}+1\right)$$

Hence, a unit-speed reparameterization of γ is given by:

$$\tilde{\gamma}(s)=\left(\left[\frac{k\,s}{\sqrt{k^2+1}}+1\right]\cos\left(\frac{1}{k}\ln\left(\frac{k\,s}{\sqrt{k^2+1}}+1\right)\right),\left[\frac{k\,s}{\sqrt{k^2+1}}+1\right]\sin\left(\frac{1}{k}\ln\left(\frac{k\,s}{\sqrt{k^2+1}}+1\right)\right)\right)$$

As you can see, this is very complicated. Sometimes, it can even be impossible to represent it explicitly.

Def.: *Curvature*



Let γ be a unit-speed curve, with parameter t. Assume the curve γ in $\mathbb{R}^n$ $(n\geqslant 2)$. Its curvature $\kappa(t)$ at the point $\gamma(t)$ is defined by:

$$\kappa(t)=\|\ddot{\gamma}(t)\|$$

Rmk: If $\ddot{\gamma}(t)=0$ for all t, then $\dot{\gamma}(t)=c$ for all t, where c is a constant vector, and thus the curve γ is a line (or part of it). The curvature would thus be 0.

Exp: What is the curvature of a circle? Is it going to depend on a point? No. We will see it in this example.

Let $\gamma(t)=(\cos(t),\sin(t))$ be a parameterization of the unit circle $x^2+y^2=1$. Then:

$$\begin{aligned}\dot{\gamma}(t)&=(-\sin(t),\cos(t))\\ \text{and thus }\|\dot{\gamma}(t)\|&=1\text{ for all }t\end{aligned}$$

Thus γ is unit-speed.

$$\begin{aligned}\|\ddot{\gamma}(t)\|&=(-\cos(t),-\sin(t))\\ \|\ddot{\gamma}(t)\|&=1\text{ for all }t\\ \text{Hence }\kappa(t)&=1\text{ for all }t\end{aligned}$$

Prop. 6:

Let γ be a regular curve in $\mathbb{R}^3$ (a curve in $\mathbb{R}^2$ can be see as a curve in the $x\,y-$plane and so in $\mathbb{R}^3$). Then the curvature is given by:

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Proof:

Let s be a unit-speed parameter of γ. Then, by the chain rule, we have:

$$\dot{\gamma} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma}{\mathrm{d}s} \cdot \frac{\mathrm{d}s}{\mathrm{d}t}$$

Thus we have:

$$\kappa = \left\|\frac{\mathrm{d}^2\gamma}{\mathrm{d}s^2}\right\| = \left\|\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right)\right\|$$

$$= \left\|\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\frac{\mathrm{d}\gamma}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}}\right)\right\| = \left\|\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\gamma/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t}\right)}{\frac{\mathrm{d}s}{\mathrm{d}t}}\right\|$$

$$= \left\|\frac{\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} - \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^3}\right\| \quad (*)$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \|\dot{\gamma}\|^2 = \dot{\gamma} \bullet \dot{\gamma}$$

By differentiating both sides:

$$2\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = 2\ddot{\gamma} \bullet \dot{\gamma}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = \ddot{\gamma} \bullet \dot{\gamma}$$

Using $(*)$, we obtain:

$$\kappa = \left\|\frac{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\ddot{\gamma} - \frac{\mathrm{d}^2s}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\dot{\gamma}}{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^4}\right\|$$

$$= \frac{\|(\dot{\gamma} \bullet \dot{\gamma})\ddot{\gamma} - (\ddot{\gamma} \bullet \dot{\gamma})\dot{\gamma}\|}{\|\dot{\gamma}\|^4}$$

We recall the triple product identity: $a \times (b \times c) = (a \bullet c)b - (a \bullet b)c$. Then we will have:

$$\kappa = \frac{\|\dot{\gamma} \times (\ddot{\gamma} \times \dot{\gamma})\|}{\|\dot{\gamma}\|^4}$$

$\dot{\gamma}$ and $(\ddot{\gamma} \times \dot{\gamma})$ are orthogonal to each other. We recall that if a is orthogonal to b, then: $\|a \times b\| = \|a\|\,\|b\|$. Hence κ becomes:

$$\kappa = \frac{\|\dot{\gamma}\|\,\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^4} = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

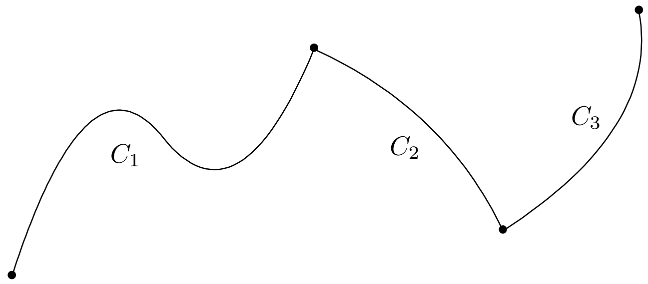
March 15th, 2021

6 Chapt. Line and Surface Integrals

Def.: Let f be a continuous function on a smooth curve C parameterized by $\gamma(t)=(x(t),y(t),z(t))$, where $a\leqslant t\leqslant b$. Note that a could be $-\infty$ and b could be ∞. The integral of f over C with respect to the arc length is:

$$\int_C f\,\mathrm{d}s=\int_a^b f(\gamma(t))\|\dot{\gamma}(t)\|\mathrm{d}t$$

For a piece-wise smooth curve C, we define the line integral to be the sum of the line integrals over smooth pieces.



$$\int_C f\,\mathrm{d}s=\int_{C_1\cup C_2\cup C_3} f\,\mathrm{d}s=\int_{C_1} f\,\mathrm{d}s+\int_{C_2} f\,\mathrm{d}s+\int_{C_3} f\,\mathrm{d}s$$

Exp: A wire lies along a helical curve C given by $\gamma(t)=(\cos(t),\sin(t),t)$, where $0\leqslant t\leqslant 4\pi$. Its density is $f(x,y,z)=z$. To find the mass of the wire, we need to integrate f along C:

$$\int_C f\,\mathrm{d}s=\int_0^{4\pi} f(\gamma(t))\|\dot{\gamma}(t)\|\mathrm{d}t$$

$$=\int_0^{4\pi} f(\cos(t),\sin(t),t)\|(-\sin(t),\cos(t),1)\|\mathrm{d}t$$

$$=\int_0^{4\pi} t\sqrt{\sin^2(t)+\cos^2(t)+1}\,\mathrm{d}t=\int_0^{4\pi}\sqrt{2}t\,\mathrm{d}t$$

$$=\frac{\sqrt{2}}{2}[t^2]_0^{4\pi}=8\sqrt{2}\pi^2$$

Def.: Let F be a continuous vector field (a function from $\mathbb{R}^n$ to $\mathbb{R}^n$), on a smooth curve C parameterized by $\gamma(t)=(x(t),y(t),z(t))$, with $a\leqslant t\leqslant b$. The unit tangent vector to C in the direction of increasing t is given by:

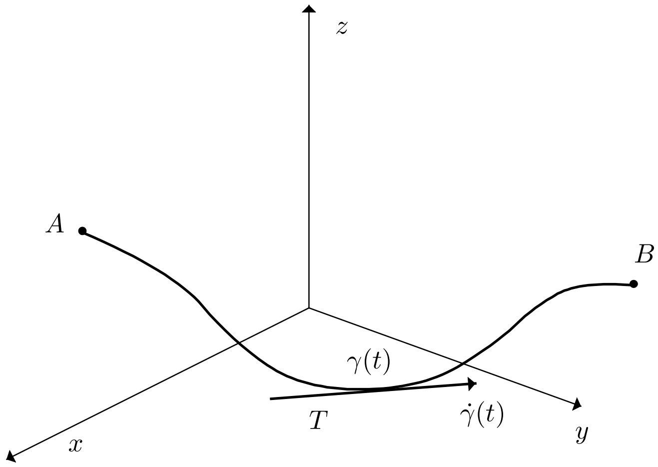
$$T=\frac{1}{\|\dot{\gamma}\|}\dot{\gamma}(t)$$

T orients C from $A=\gamma(a)$ to $B=\gamma(b)$. The integral of F along C (in the tangential direction) from $A=\gamma(a)$ to $B=\gamma(b)$ is given by:

$$\int_C F\bullet T\,\mathrm{d}s=\int_a^b F(\gamma(t))\bullet T(\gamma(t))\,\|\dot{\gamma}(t)\|\mathrm{d}t$$

$$\text{Note that } T(\gamma(t))=\frac{1}{\|\dot{\gamma}(t)\|}\,\dot{\gamma}(t),\ \text{and thus:}$$

$$\int_a^b F(\gamma(t))\bullet T(\gamma(t))\,\|\dot{\gamma}(t)\|\mathrm{d}t=\int_a^b F(\gamma(t))\bullet\dot{\gamma}(t)\,\mathrm{d}t$$



Exp: Find the work done by some force $F(x, y, z) = (y, -x, z)$ in moving from $(1, 0, 0) = \gamma(0)$ to the point $(1, e, 2\pi) = \gamma(2\pi)$ along the helical curve given by:

$$\gamma(t) = (\cos(t), \sin(t), t), 0 \leqslant t \leqslant 2\pi$$

At a point $\gamma(t)$ on C, we have $F(\gamma(t)) = F(\cos(t), \sin(t), t) = (\sin(t), -\cos(t), t)$, and:

$$\dot{\gamma}(t) = (-\sin(t), \cos(t), 1)$$

Thus:

$$\|\dot{\gamma}(t)\| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$$

Also:

$$T(\gamma(t)) = \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)$$

The work is given by:

$$\int_C F \bullet T \,\mathrm{d}s = \int_0^{2\pi} (\sin(t), -\cos(t), t) \bullet \frac{1}{\sqrt{2}}(-\sin(t), \cos(t), 1)\,\sqrt{2}\,\mathrm{d}t$$

$$= \int_0^{2\pi} (-\sin^2(t) - \cos^2(t) + t)\,\mathrm{d}t = \int_0^{2\pi} (t - 1)\,\mathrm{d}t$$

$$= \left[\frac{1}{2}t^2 - t\right]_0^{2\pi} = 2\pi(\pi - 1)$$

Evaluation of Line Integrals: For a piecewise smooth curve C parameterized by:

$$\gamma(t) = (x(t), y(t), z(t)), a \leqslant t \leqslant b,$$

the integral of the tangential component of $F = (f_1, f_2, f_3)$ over C is:

$$\int_C F \bullet T \,\mathrm{d}s = \int_a^b F(\gamma(t)) \bullet \dot{\gamma}(t)\,\mathrm{d}t$$

If we denote $T\,\mathrm{d}s = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, then:

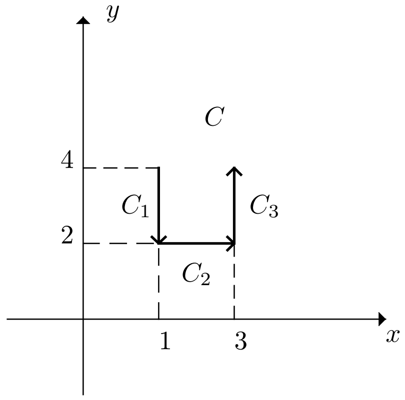
$$\int_C F \bullet T \,\mathrm{d}s = \int_C (f_1, f_2, f_3) \bullet (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$$

$$= \int_C f_1\,\mathrm{d}x + f_2\,\mathrm{d}y + f_3\,\mathrm{d}z$$

$$= \int_C f_1 \, \mathrm{d}x + \int_C f_2 \, \mathrm{d}y + \int_C f_3 \, \mathrm{d}z$$
$$= \int_a^b f_1(\gamma(t)) x'(t) \, \mathrm{d}t + \int_a^b f_2(\gamma(t)) y'(t) \, \mathrm{d}t + \int_a^b f_3(\gamma(t)) z'(t) \, \mathrm{d}t$$

The benefit of this approach is that we may use different parameterizations of C for evaluating the intergrals of different components of F.

Exp: Let $F(x, y) = (f_1, f_2) = (x^2 + 3, 2)$



$$\int_C F \bullet T \, \mathrm{d}s = \int_C (x^2 + 3, 2) \bullet (\mathrm{d}x, \mathrm{d}y)$$
$$= \int_C (x^2 + 3) \, \mathrm{d}x + \int_C 2 \, \mathrm{d}y$$
$$\int_C (x^2 + 3) \, \mathrm{d}x = \int_{C_1 \cup C_2 \cup C_3} (x^2 + 3) \, \mathrm{d}x$$
$$= \int_{C_1} (x^2 + 3) \, \mathrm{d}x + \int_{C_2} (x^2 + 3) \, \mathrm{d}x + \int_{C_3} (x^2 + 3) \, \mathrm{d}x$$

On C_1: No variation in x, thus $x'(t) = 0$

Hence $\int_{C_1} (x^2 + 3) \, \mathrm{d}x = 0$

On C_2:

C_2 can be parameterized by:

$\gamma(t) = (t, 2)$ for $1 \leqslant t \leqslant 3$. Then:

$x(t) = t$ and $\mathrm{d}x = \mathrm{d}t$. Thus:

$$\int_{C_2} (x^2 + 3) \, \mathrm{d}x = \int_1^3 (t^2 + 3) \, \mathrm{d}t = \frac{1}{3} t^3 + 3t |_1^3 = \frac{44}{3}$$

On C_3: $\mathrm{d}x = 0$ because there is no variation.

Hence $\int_{C_3} (x^2 + 3) \, \mathrm{d}x = 0$

Similarly, we will show that:

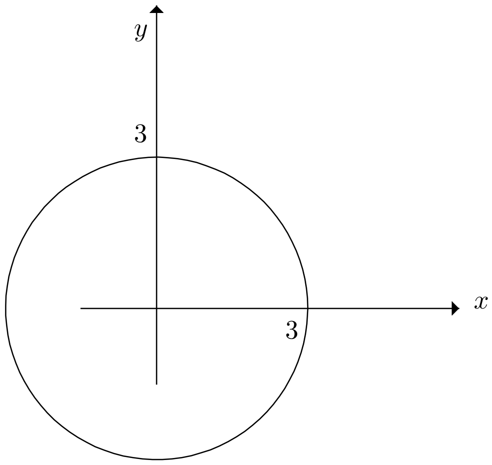
$$\int_C 2 \, \mathrm{d}y = 2 \left(\int_{C_1} \mathrm{d}y + \int_{C_2} \mathrm{d}y + \int_{C_3} \mathrm{d}y \right)$$
$$= 2((2 - 4) + 0 + (4 - 2)) = 0$$

Hence:
$$\int_C F \bullet T \,\mathrm{d}s = \frac{44}{3}$$

Def.: Let C be a smooth curve parameterized by $\gamma(t) = (x(t), y(t)), a \leqslant t \leqslant b$, and let $F = (f_1, f_2)$ be a continuous vector field on C. The flux of F across C in the direction of the normal vector:
$$N = N(x(t), y(t)) = \frac{1}{\|\dot{\gamma}(t)\|}(-y'(t), x'(t))$$
is given by:
$$\int_C F \bullet N \,\mathrm{d}s = \int_a^b F(\gamma(t)) \bullet \frac{1}{\|\dot{\gamma}(t)\|}(-y'(t), x'(t)) \|\dot{\gamma}(t)\| \,\mathrm{d}t$$
$$= \int_a^b F(\gamma(t)) \bullet (-y'(t), x'(t)) \,\mathrm{d}t$$
$$= \int_a^b -f_1\, y'(t) \,\mathrm{d}t + \int_a^b f_2\, x'(t) \,\mathrm{d}t = \int_a^b -f_1 \,\mathrm{d}y + f_2 \,\mathrm{d}x$$

Exp: Find the flux of $F(x, y) = (x, y)$ outward accross the circle C centered at the origin O and radius 3.

C can be parameterized by:
$\gamma(t) = (3\cos(t), 3\sin(t)), 0 \leqslant t \leqslant 2\pi.$
Then we have that $\dot{\gamma}(t) = (-3\sin(t), 3\cos(t))$



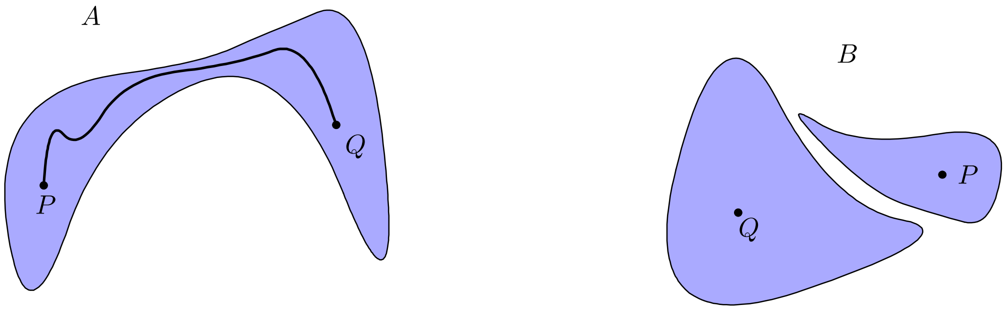
At every point $\gamma(t)$, there are two normal vectors:
$$-N(t) = \frac{1}{\|\dot{\gamma}(t)\|}(-3\cos(t), 3\sin(t)) = \frac{1}{3}(-3\cos(t), -3\sin(t)) = (-\cos(t), -\sin(t))$$
and we also have the normal vector $N(t) = (\cos(t), \sin(t))$. The flux of F across C is given by the following:
$$\int_C F \bullet N \,\mathrm{d}s = \int_0^{2\pi} F(\gamma(t)) \bullet N \;\|\dot{\gamma}(t)\| \,\mathrm{d}t$$
$$= \int_0^{2\pi} (3\cos(t), 3\sin(t)) \bullet (\cos(t), \sin(t))\, 3 \,\mathrm{d}t$$
$$= 9 \int_0^{2\pi} (\cos^2(t) + \sin^2(t)) \,\mathrm{d}t = 18\pi$$

Def.: *Conservative Vector Fields:* We say that a vector field F is conservative if it is the gradient of a class C^1 function g, ie. $F = \nabla g$. In this case, g is called a potential of F.

$$F(P) = \nabla g(P), \text{for every point } P \text{ in the domain of } F$$

March 17th, 2021

Def.: A set $A \subset \mathbb{R}^n$ is said to be connected if for every points P and Q in the set A, there is a curve <u>in A</u> with endpoints P and Q. We will see two examples:



In A, we can see that we can find a path between P and Q in the set. In the example for B, there is no curve that connects the 2 points <u>in B</u>.

Theorem 1: Suppose F is a continuous vector field from an open connected set D in $\mathbb{R}^n$ to $\mathbb{R}^n$, with $n \geqslant 2$. Then the following statements are equivalent:

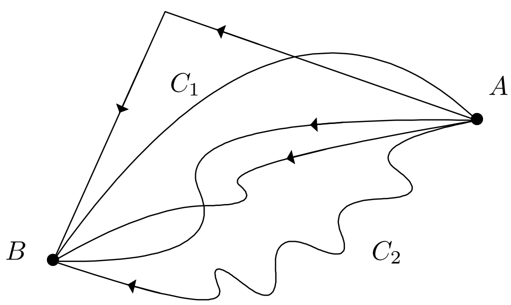
a) F is conservative

b) For every piecewise smooth closed curve C in D, we have:

$$\int_C F \bullet T \, \mathrm{d}s = 0$$

c) For any two points A and B in D, and for any two piecewise smooth curves C_1 and C_2 in D that start at A and end at B, we have:

$$\int_{C_1} F \bullet T \, \mathrm{d}s = \int_{C_2} F \bullet T \, \mathrm{d}s$$



$\int_{C_1} F \bullet T \, \mathrm{d}s = \int_{C_2} F \bullet T \, \mathrm{d}s$

<u>Proof</u>:

$$a \Longrightarrow b:$$

$$\text{Suppose } F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \text{conservative vector field}$$
$$\exists g \text{ st.} F(x,y,z) = (\partial_x g(x,y,z), \partial_y g(x,y,z), \partial_z g(x,y,z))$$
$$\text{For some smooth curve } C \text{ in D given by } \gamma(t), a \leqslant t \leqslant b,$$

we have:

$$\int_C F \bullet T\,\mathrm{d}s = \int_a^b F(\gamma(t)) \bullet \dot{\gamma}(t)\,\mathrm{d}t = \int_a^b \nabla g(\gamma(t)) \bullet \dot{\gamma}(t)\,\mathrm{d}t$$

By the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}(g(\gamma(t))) = \nabla g(\gamma(t)) \bullet \dot{\gamma}(t).$$

Thus:

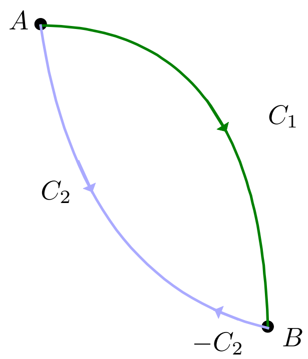
$$\int_C F \bullet T\,\mathrm{d}s = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma(t))\,\mathrm{d}t = g(\gamma(b)) - g(\gamma(a))$$

Since C is closed, we have that $\gamma(a) = \gamma(b)$ and hence:

$$\int_C F \bullet T\,\mathrm{d}s = 0$$

$b \Longrightarrow c$:

Let A and B be in D and let C_1, C_2 be two piecewise smooth curves in D that start at A and end at B. Consider the curve $C = C_1 \cup (-C_2)$



Then C is a piecewise smooth closed curve, and so by b), we have:

$$\int_C F \bullet T\,\mathrm{d}s = 0$$

$$= \int_{C_1 \cup (-C_2)} F \bullet T\,\mathrm{d}s = \int_{C_1} F \bullet T\,\mathrm{d}s - \int_{C_2} F \bullet T\,\mathrm{d}s$$

$$\int_{C_1} F \bullet T\,\mathrm{d}s = \int_{C_2} F \bullet T\,\mathrm{d}s$$

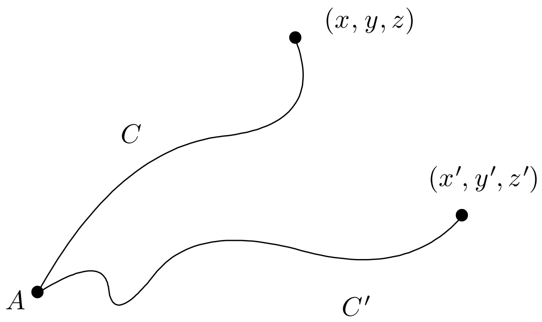
$c \Longrightarrow a$:

Since D is an open connected set, every two points in D can be joined by a piecewise smooth curve in D. Let A be a point in D. For each $(x, y, z) \in D$, there is a piecewise smotoh curve C in D from A to the point (x, y, z).

$$\text{Define } g(x, y, z) = \int_C F \bullet T\,\mathrm{d}s$$

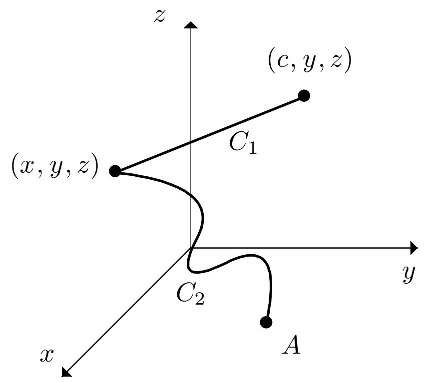
$$g = \int_A^{(x,y,z)} F \bullet T\,\mathrm{d}s$$

$$\nabla g = F$$



$$F(x) = \int_a^x f(t)\, \mathrm{d}t \longrightarrow F'(x) = f(x)$$

We will show that $\nabla g = F$. Since c) is true, any curve we choose from A to (x, y, z) will result in the same number $g(x, y, z)$. Since D is open and connected, there is a point (c, y, z) with $c < x$ so that the straight line C_1 from (c, y, z) to (x, y, z) is in D and can be combined with a curve C_2 in D from A to (c, y, z).



Using these curves, we have:

$$g(x, y, z) = \int_{C_1 \cup C_2} F \bullet T\, \mathrm{d}s = \int_{C_1} F \bullet T\, \mathrm{d}s + \int_{C_2} F \bullet T\, \mathrm{d}s$$

$$\int_{C_2} F \bullet T\, \mathrm{d}s \text{ does not depend on } x$$

We parameterize C_1 by:

$$\gamma(t) = (t, y, z) \text{ for } 0 \leqslant t \leqslant x$$

Writing components of $F = (f_1, f_2, f_3)$, we obtain:

$$\int_{C_1} F \bullet T\, \mathrm{d}s = \int_c^x F(\gamma(t)) \bullet \dot{\gamma}(t)\, \mathrm{d}t$$

$$= \int_c^x F(t, y, z) \bullet (1, 0, 0)\, \mathrm{d}t = \int_c^x f_1(t, y, z)\, \mathrm{d}t \quad (*)$$

We differentiate $(*)$ wrt. x:

$$\partial_x g(x, y, z) = f_1(x, y, z) + 0$$

$$\text{ie. } \partial_x g(x) = f_1$$

Similar argument shows that:

$$\partial_y g = f_2 \text{ and } \partial_z g = f_3$$

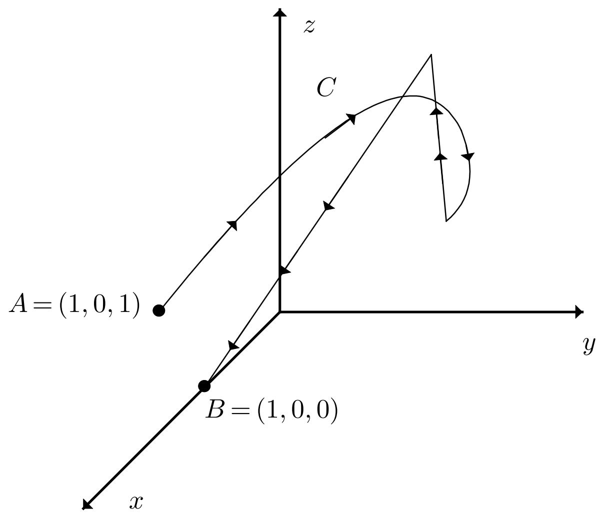
Hence $\nabla g = F$ and so F is conservative

Theorem 2: *Fundamental Theorem of Line Integrals:*

If C is a piecewise smooth curve from $A\,\text{to}\,B$ in the domain of a class C^1 function g, then:

$$\int_C \nabla g \bullet T\,\mathrm{d}s = g(B) - g(A), \text{where}\,\nabla g\,\text{is a conservative vector field.}$$

Exp:



$$\text{Let}\,F(x,y,z) = (x,y,z).\,\text{Find}\int_C F \bullet T\,\mathrm{d}s$$

It is easy to see that $F(x,y,z) = \nabla\left(\frac{1}{2}(x^2+y^2+z^2)\right)$.

$$\left(\frac{1}{2}(x^2+y^2+z^2)\right) = g$$

$$\begin{aligned}\text{Thus}\int_C F \bullet T\,\mathrm{d}s &= g(1,0,0) - g(1,0,1)\\ &= \frac{1}{2} - 1 = -\frac{1}{2}\end{aligned}$$

Curl of a Vector Field: Assume $F = (f_1, f_2, f_3)$ is a differentiable vector field from $\mathbb{R}^3\,\text{to}\,\mathbb{R}^3$. Then we have that:

$$\operatorname{curl}(F) = \nabla x\,F = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)i - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)k$$

$$\operatorname{curl}(F) = \left(\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right), -\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right), \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right)$$

Theorem 3: *Criterion for a Vector Field in* $\mathbb{R}^3$ *or* $\mathbb{R}^2$ *to be Conservative:*

a) If F is a conservative vector field of class C^1 from $\mathbb{R}^3\,\text{to}\,\mathbb{R}^3$, then $\operatorname{curl}(F) = O_{\mathbb{R}^3}$.

b) If F is a conservative C^1 vector field from $\mathbb{R}^2$ to $\mathbb{R}^2$, then $\operatorname{curl}(F) = O_{\mathbb{R}^2}$

Proof:

$$\begin{aligned}
&(a)\\
\text{If } F = \nabla g, \text{then } F = (f_1, f_2, f_3) &= (\partial_x g, \partial_y g, \partial_z g)\\
&\text{Thus:}\\
\operatorname{curl}(F) &= (\partial_y f_3 - \partial_z f_2, -\partial_x f_3 + \partial_z f_1, \partial_x f_2 - \partial_y f_1)\\
&= (\partial_{zy} g - \partial_{yz} g, -\partial_{xz} g + \partial_{zx} g, \partial_{xy} g - \partial_{yx} g)\\
&\text{They are all equal to each other.}\\
&\text{Since } F \text{ is of class } C^1 \text{ and since } F = \nabla g,\\
&g \text{ is of class } C^1 \text{ and so its second mixed partial}\\
&\text{derivatives do not depend on the order of differentiation.}\\
&\text{Thus:}
\end{aligned}$$

$$\operatorname{curl}(F) = O_{\mathbb{R}^3}$$

The same is applied for $\mathbb{R}^2$. It is exactly the same, with one less variable.

March 22nd, 2021

Exp: Let $F(x, y, z) = (y, -x, z)$. Then we know that:

$$\operatorname{curl}(F) = (0, 0, -2) \neq O_{\mathbb{R}^3}$$

Thus we know that F is not conservative.

Exp:

$$\text{Let } F(x, y, z) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right)$$

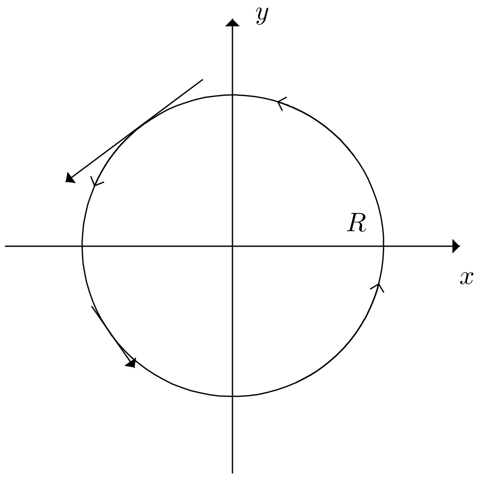
This is defined for $(x, y) \neq (0, 0)$. Then we know that $\operatorname{curl}(F) = (0, 0, 0)$. Let C be the circle in the xy-plane of center the origin and radius R, traversed counterclockwise. Observe that at any point along C:

$$\|F(x, y, 0)\| = \frac{1}{R},$$

and F is pointing in the same direction as the unit tangent vector $T(x, y, 0)$. Hence:

$$\begin{aligned}
F \bullet T = \|F\| &= \frac{1}{R}, \text{ and therefore:}\\
\int_C F \bullet T \, \mathrm{d}s &= \frac{1}{R} \times \text{"Length of } R\text{"}\\
&= \frac{1}{R} \times 2\pi R = 2\pi \neq 0
\end{aligned}$$

Even though $\operatorname{curl}(F) = O_{\mathbb{R}^3}$, F is not conservative, since the line integral along the closed curve C is not zero.



Exp: Show that $F(x,y)=(x+y,x+y^2)$ is conservative, by finding a potential function g so that $F=\nabla g$.

$$\begin{aligned}\text{We need to find } g \text{ st.}\\ \partial_x g = x+y \;\text{ and }\; \partial_y g = x+y^2\\ \text{Integrating } \partial_x g = x+y \text{ wrt } x\text{:}\\ g(x,y)=\frac{1}{2}x^2+y\,x+h(y)\\ \text{We differentiate this wrt } y\text{:}\\ \partial_y g = x+h'(y)=x+y^2\\ \text{Thus we conlcude that } h'(y)=y^2, h(y)=\frac{1}{3}y^3\\ \text{Hence:}\\ g(x,y)=\frac{1}{2}x^2+y\,x+\frac{1}{3}y^3\end{aligned}$$

Def.: *Surfaces and Surface Integrals:* Let γ be a class C^1 function from a smooth bounded set D $\mathbb{R}^2$ to $\mathbb{R}^3$, denoted:

$$\gamma(u,v)=(x(u,v),y(u,v),z(u,v))$$

Suppose γ satisfies the following conditions on the interior of D:

a) γ is one-to-one.

b) The partial derivatives of the component functions of γ are bounded.

c) The partial derivatives:

$$\gamma_u(u,v)=(\partial_u x,\partial_v y,\partial_v z) \;\text{ and }\; y_v(u,v)=(\partial_v x,\partial_v y,\partial_v z)$$

are line-independent, so that $\gamma_u(u,v)\times\gamma_v(u,v)\neq 0$. The range S of γ is called a smooth surface parameterized by γ. The plane that contains the point $\gamma(u,v)$ and has normal vector $\gamma_u(u,v)\times\gamma_v(u,v)$ is called the tangent plane to S at $\gamma(u,v)$.

Exp: Let $\gamma(u,v)=(3\cos(v)\sin(u),3\sin(v)\sin(u),3\cos(u))$ with $0\leqslant u\leqslant\pi$ and $0\leqslant v\leqslant 2\pi$.

Since $\|\gamma(u,v)\|=3^2(\cos^2(v)+\sin^2(v))\sin^2(u)+3^2\cos^2(u)=3^2=9$. The points $\gamma(v,u)$ lie on the sphere of radius 3 centered at the origin. γ is one-to-one on the interior of the domain where $0<u<\pi$ and $0<v<2\pi$. The partial derivatives are:

$$\begin{aligned}\gamma_u(u,v)&=(3\cos(v)\cos(u),3\sin(v)\cos(u),-3\sin(v)),\\ \gamma_v(u,v)&=(-3\sin(v)\sin(u),3\cos(v)\sin(u),0)\end{aligned}$$

These two are linearly independent where $\sin(u) \neq 0$. A normal vector to the tangent plane is:

$$\gamma_u(u,v) \times \gamma_v(u,v) = 9\sin(u)(\cos(v)\sin(u), \sin(v)\sin(u), \cos(u))$$

and its norm is:

$$\| \gamma_u(u,v) \times \gamma_v(u,v)\| = 9\sin(u)$$

Rmk: One way to parameterize the graph of a C^1 function $f(x,y) = z$ is to define a new function as such:

$$\gamma(x,y) = (x, y, f(x,y))$$

from the x-y plane to $\mathbb{R}^3$.

Exp: Let $f(x,y) = \sqrt{R^2 - x^2 - y^2}, R > 0$. Define a new function, $\gamma(x,y) = \left(x, y, \sqrt{R^2 - x^2 - y^2}\right)$ for $x^2 + y^2 \leqslant a^2 < R^2$. The partial derivatives are:

$$\gamma_x(x,y) = \left(1, 0, -\frac{x}{\sqrt{R^2 - x^2 - y^2}}\right)$$

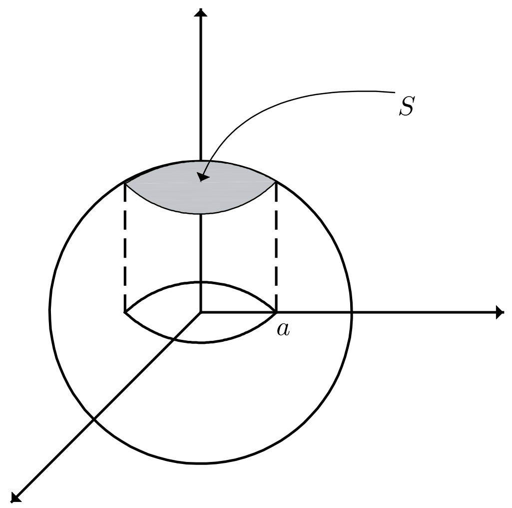
$$\gamma_y(x,y) = \left(0, 1, -\frac{y}{\sqrt{R^2 - x^2 - y^2}}\right)$$

These two are linearly independent. Since $0 \leqslant x^2 + y^2 \leqslant a^2 < R^2$, the derivatives are bounded. In fact, we can observe that:

$$\left|-\frac{x}{\sqrt{R^2 - x^2 - y^2}}\right| \leqslant \frac{R}{\sqrt{R^2 - x^2 - y^2}} \leqslant \frac{R}{\sqrt{R^2 - a^2}}, \text{ and}$$

$$\left|-\frac{y}{\sqrt{R^2 - x^2 - y^2}}\right| \leqslant \frac{R}{\sqrt{R^2 - a^2}}$$

The range S of γ is the part of the upper hemisphere of radius R centered at the origin, that sits above the disk of radius $a > 0$ centered at the origin of the x-y plane.



A normal vector to the tangent plane to S at a point $\gamma(x,y)$ is:

$$\gamma_x(x,y) \times \gamma_y(x,y) = \left(\frac{x}{\sqrt{R^2 - x^2 - y^2}}, \frac{y}{\sqrt{R^2 - x^2 - y^2}}, 1\right)$$

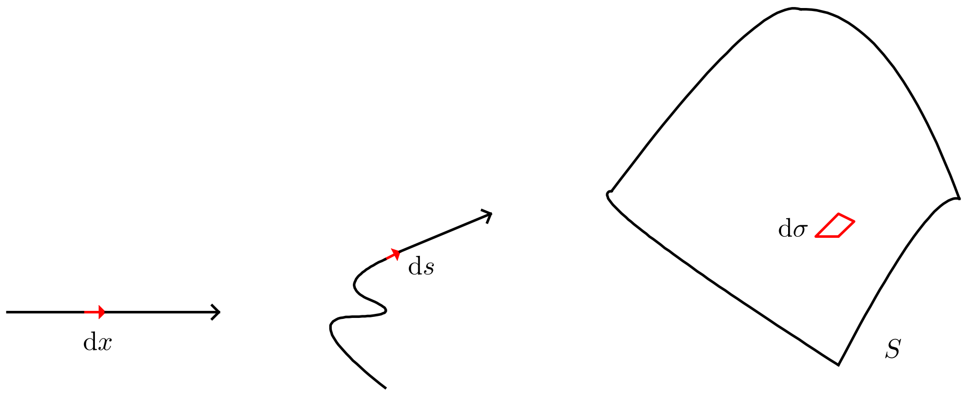
Rmk: If $\gamma(x,y) = (x, y, f(x,y))$ is a parameterization of the graph of f, then:

$$\begin{aligned} \gamma_x(x,y) &= (1, 0, f_x(x,y)) \\ \gamma_y(x,y) &= (0, 1, f_y(x,y)) \\ \text{and } \gamma_x \times \gamma_y &= (-f_x, -f_y, 1) \end{aligned}$$

This result is the normal vector at the point $(x, y, f(x,y))$. We can check this.

***Def.**: Area:* Suppose S is a smooth surface parameterized by γ from a smoothy bounded set D in the u, v plane to $\mathbb{R}^3$. The area of S is defined as:

$$\text{Area}(S) = \int_S \mathrm{d}\sigma = \int_D \|\gamma_u(u,v) \times \gamma_v(u,v)\| \mathrm{d}v$$



Exp: Find the area of the sphere S of radius R using the parameterization:

$$\gamma(\theta, \varphi) = (R\cos(\varphi)\sin(\theta), R\sin(\varphi)\sin(\theta), R\cos(\theta)) \text{ for } 0 \leqslant \theta \leqslant \pi \ \text{ and } \ 0 \leqslant \varphi \leqslant 2\pi$$

$$\begin{aligned} & \gamma_\theta(\theta, \varphi) \times \gamma_\varphi(\theta, \varphi) \\ = & R^2 \sin(\theta)(\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)) \\ & \text{Thus } \gamma_\theta(\theta, \varphi) \times \gamma_\varphi(\theta, \varphi) = R^2 \sin(\theta). \text{ Hence} \\ & \text{Area}(S) = \int_S \mathrm{d}\sigma = \int_0^\pi \int_0^{2\pi} R^2 \sin(\theta) \mathrm{d}\theta \mathrm{d}\varphi \\ = & R^2 \int_0^\pi \sin(\theta) \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi = R^2[-\cos(\theta)]_0^\pi \, 2\pi \\ = & 4\pi R^2 \end{aligned}$$

March 24th, 2021

Rmk: Assume the surface S is part of a level curve $g(x, y, z) = c$, where c is some constant in $\mathbb{R}$ with $\nabla g \neq 0$. By the implicit function theorem, locally near any point of S, one or more of the variables is a function of the other two. Suppose we have:

$$z = \varphi(x, y) \text{ with } x, y \in D$$

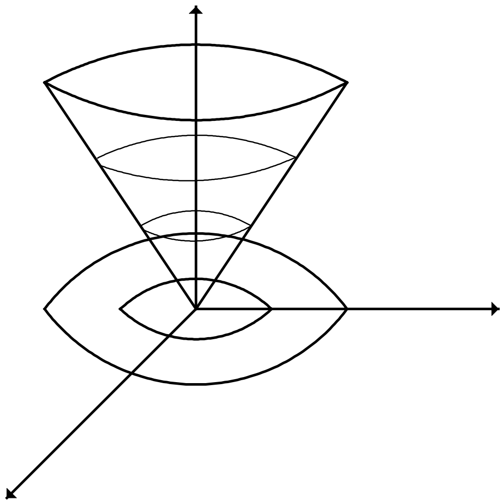
Then:

$$\begin{aligned} \varphi_x &= -\frac{g_x}{g_z} \\ \varphi_y &= -\frac{g_y}{g_x} \end{aligned}$$

And the area of S is given by the following:

$$\int_D \sqrt{\varphi_x^2 + \varphi_y^2 + 1}\,\mathrm{d}x\mathrm{d}y$$

Exp: Find the area of the part of the surface $g(x,y,z) = x^2 + y^2 - z^2 = 0 = c$ that lies over the set in the x-y plane where $1 \leqslant x^2 + y^2 \leqslant 9$.



By the implicit function theorem for $z = \varphi(x,y)$, we have:

$$\varphi_x = -\frac{2x}{-2z} = \frac{x}{z}$$
$$\varphi_y = -\frac{2y}{-2z} = \frac{y}{z}$$

Thus the area is given by:

$$\int_D \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}\,\mathrm{d}x\mathrm{d}y$$
$$= \int_D \sqrt{\frac{x^2 + y^2}{z^2} + 1}\,\mathrm{d}x\mathrm{d}y$$
$$\int_D \sqrt{2}\mathrm{d}x\mathrm{d}y = \sqrt{2}\int_D \mathrm{d}x\mathrm{d}y$$
$$= \sqrt{2}\,\mathrm{Area}(D)$$
$$= \sqrt{2}(9\pi - \pi) = 8\sqrt{2}\pi$$

Change of Variable Theorem:

Let $F(u,v) = (x(u,v), y(u,v))$ denote a smooth change of variables that maps a smoothly bounded set C onto a smoothly bounded set D so that the boundary of C is mapped to the boundary of D. Denote by f a continuous function on D. Then:

$$\int_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_C f(x(u,v), y(u,v))\,|J_F(u,v)|\mathrm{d}u\mathrm{d}v$$

Recall that $|J_F(u,v)|$ is the determinant of the Jacobian matrix.

Exp: Let $C = \{(u,v) \mid u^2 + v^2 \leqslant r^2\}$ and $D = \left\{(x,y) \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 \leqslant r^2\right\}$.

The mapping $F(u,v) = (a\,u, \text{bv}) \longrightarrow \text{ie.}\, x(u,v) = a\,u$ and $y(u,v) = b\,v$ sends points (u,v) in C one-to-one onto the points of D.

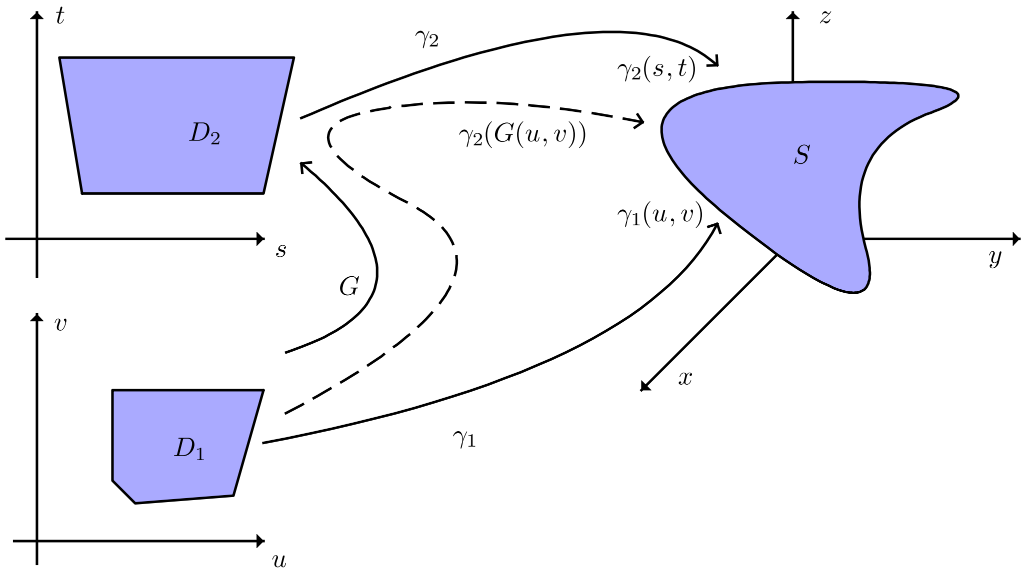
$$|J_F(u,v)| = \begin{vmatrix} a & 0 \\ c & b \end{vmatrix} = a\,b$$

Then:

$$\text{Area}(D) = \int_D 1 \mathrm{d}x\mathrm{d}y = \int_C 1 a\,b\,\mathrm{d}u\mathrm{d}v$$
$$= a\,b \int_C \mathrm{d}u\mathrm{d}v = a\,b\,\text{Area}(C) = a\,b\,\pi r^2$$

Important: Suppose S has two parameterizations, one $\gamma_1(u,v)$ from D_1 onto S, and another $\gamma_2(s,t)$ from D_2 onto S, and a differentiable "change of parameterization" G such that:

$$(s,t) = G(u,v) \text{ and } \gamma_2(s,t) = (G(u,v)) = \gamma_1(u,v)$$



By the chain rule, we have:

$$J_{\gamma_1}(u,v) = J_{\gamma_2 \circ G}(u,v) = J_{\gamma_2}(G(u,v))\, J_G(u,v)$$

This implies (in the Homework) that:

$$\det[J_G(u,v)]\,(\partial\gamma_2(u,v) \times \partial_t \gamma_2(u,v))$$
$$= \partial_u \gamma_1(u,v) \times \partial_v \gamma_1(u,v)$$

By the change of variables theorem, the area of S is given by:

$$\text{Area}(S) = \int_{D_2} \|\partial_s \gamma_2(s,t) \times \partial_t \gamma_2(s,t)\| \mathrm{d}s\mathrm{d}t$$

$$= \int_{D_1} \|\partial_s \gamma_2(G(u,v)) \times \partial_t \gamma_2(G(u,v))\|\, |\det[J_G(u,v)]| \mathrm{d}u\mathrm{d}v$$
$$= \int_{D_1} \|\partial_u \gamma_1(u,v) \times \partial_v \gamma_1(u,v)\| \mathrm{d}u\mathrm{d}v$$

Hence the area of S is independent of the parameterization of S.

Exp: With respect to our previous example (example of the cone), suppose that S can be expressed as the graph of $y=h(x,z)$ for (x,z) in some set E. Then:

$$h_x=-\frac{g_x}{g_y} \text{ and } h_z=-\frac{g_z}{g_y}$$

WE want to show that:

$$\int_D \sqrt{\varphi_x^2+\varphi_y^2+1}\mathrm{d}x\mathrm{d}y=\int_E \sqrt{h_x^2+h_z^2+1}\mathrm{d}x\mathrm{d}z$$

There is a mapping F from E onto D st.:

$$F(x,h(x,z))=(x,y)\in D$$

The Jacobian of F is given by:

$$\det(J_F(x,z))=\begin{vmatrix} 1 & 0 \\ h_x & h_z \end{vmatrix}=h_z=-\frac{g_z}{g_y}$$

The change of variables theorem implies:

$$\begin{aligned}\int_D \sqrt{\varphi_x^2+\varphi_y^2+1}\mathrm{d}x\mathrm{d}y &= \int_E \sqrt{\left(\frac{g_x}{g_y}\right)^2+\left(\frac{g_z}{g_y}\right)^2+1}\left|-\frac{g_z}{g_y}\right|\mathrm{d}x\mathrm{d}z \\ &= \int_E \sqrt{\left(\left(\frac{g_x}{g_y}\right)^2+\left(\frac{g_z}{g_y}\right)^2+1\right)\left(\frac{g_z}{g_y}\right)^2}\mathrm{d}x\mathrm{d}z \\ &= \int_E \sqrt{\left(\left(\frac{g_x}{g_y}\right)^2+1+\left(\frac{g_z}{g_y}\right)^2\right)}\mathrm{d}x\mathrm{d}z \\ &= \int_E \sqrt{h_x^2+h_z^2+1}\mathrm{d}x\mathrm{d}z\end{aligned}$$

Def.: *Surface Integrals:* Suppose S is a smooth surface parameterized by γ from a smooth bounded set D in the u-v plane to $\mathbb{R}^3$, and suppose f is continuous on S. The surface integral of f over S is defined as:

$$\int_S f\mathrm{d}\sigma=\int_D f(\gamma(u,v))\,\|\partial_u\gamma\times\partial_v\gamma\|\mathrm{d}u\mathrm{d}v$$

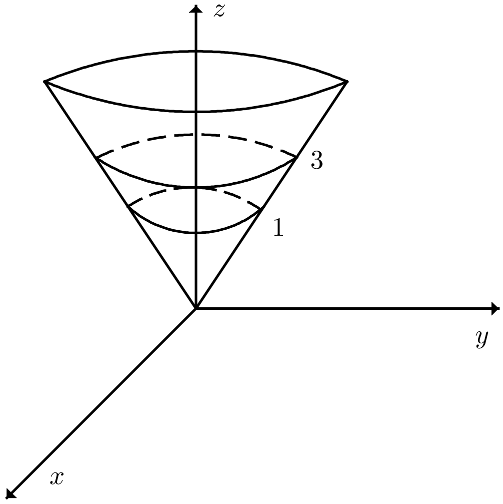
Rmk: When $f=1$ (constant function 1), then we have that:

$$\int_S f\mathrm{d}\sigma=\mathrm{Area}(S)$$

April 5th, 2021

Exp: Let S be the surface $z^2=x^2+y^2, 1\leqslant z\leqslant 3$. Suppse the charge density at each point (x,y,z) of S is $f(x,y,z)=e^{-x^2-y^2}$. Find the total charge on S. We parameterize S by:

$$\gamma(r,\theta)=(r\cos(\theta),r\sin(\theta),r)$$
$$\text{with } 1\leqslant r\leqslant 3 \text{ and } 0\leqslant\theta\leqslant 2\pi$$



The total charge is:

$$\begin{aligned}\int_S f\,\mathrm{d}\sigma =& \int_D f(\gamma(r,\theta))\|\partial_r\gamma \times \partial_\theta\gamma\|\mathrm{d}r\mathrm{d}\theta\\ &\int_0^{2\pi}\int_1^3 e^{-r^2\cos^2(\theta)-r^2\sin^2(\theta)}\sqrt{2}\,\mathrm{d}r\mathrm{d}\theta\\ \int_0^{2\pi}\int_1^3 e^{-r^2}\sqrt{2}\,\mathrm{d}r\mathrm{d}\theta =& 2\pi\sqrt{2}\left[-\frac{1}{2}e^{-r^2}\right]_1^3\\ =&\pi\sqrt{2}(e^{-1}-e^{-9})\end{aligned}$$

Properties:

1.
$$\int_S c\,f\,\mathrm{d}\sigma = c\int_S f\,\mathrm{d}\sigma$$

2.
$$\int_S (f+g)\,\mathrm{d}\sigma = \int_S f\,\mathrm{d}\sigma + \int_S g\,\mathrm{d}\sigma$$

3. If $m \leqslant f \leqslant M$, then we have that:
$$\int_S m\,\mathrm{d}\sigma = m\int_S \mathrm{d}\sigma \leqslant \int_S f\,\mathrm{d}\sigma \leqslant \int_S M\mathrm{d}\sigma = M\int_S \mathrm{d}\sigma$$
where $\int_S m\,\mathrm{d}\sigma = m\,\mathrm{Area}(S)$ and $\int_S M\mathrm{d}\sigma = M\,\mathrm{Area}(S)$

Flux of a vector field across a surface:

Suppose F is a vector field whose domain comtains a smooth surface S, parameterized by γ from the u-v plane to $\mathbb{R}^3$. By the definition of a parameterization of a smooth surface, we know that $\partial_u\gamma$ and $\partial_v\gamma$ are linearly independent and determine a tangent plane to S at the point $\gamma(u,v)$. The vector:

$$N(\gamma(u,v)) = \frac{1}{\|\partial_u\gamma \times \partial_v\gamma\|}\,\partial_u\gamma \times \partial_v\gamma$$

is a unit-vector normal to the tangent plane at $\gamma(u,v)$. We call:

$$F(\gamma(u,v)) \bullet N(\gamma(u,v)) = F(\gamma(u,v)) \bullet \frac{1}{\|\partial_u\gamma \times \partial_v\gamma\|} \partial_u\gamma \times \partial_v\gamma$$

The nromal component of F at $\gamma(u,v)$.

Def.: *Flux:* Let S be a smooth surface parameterized by γ from a smoothly bounded set D in the u-v plane to $\mathbb{R}^3$. Let $F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ be a continuous vector field on S. The integral of the normal component of F over S is called the flux of F across S in the direction of N:

$$\begin{aligned}\int_S F \bullet N \,\mathrm{d}\sigma &= \int_D F(\gamma(u,v)) \bullet N(\gamma(u,v)) \,\|\partial_u\gamma \times \partial_v\gamma\| \mathrm{d}u\mathrm{d}v \\ &= \int_D F(\gamma(u,v)) \bullet (\partial_u\gamma \times \partial_v\gamma) \mathrm{d}u\mathrm{d}v\end{aligned}$$

Exp: Let S be the surface $z^2 = x^2 + y^2$ with $1 \leqslant z \leqslant 3$. Find the flux of $F(x,y,z) = (-y,x,z)$ across S.

We already know (wrt. the previous example) that:

$$\gamma(r,\theta) = (r\cos(\theta), r\sin(\theta)) \text{ with } 1 \leqslant r \leqslant 3 \text{ and } 0 \leqslant \theta \leqslant 2\pi$$

is a parameterizxation of S.

$$\begin{aligned}\partial_r\gamma(r,\theta) &= (\cos(\theta), \sin(\theta), 1) \\ \partial_\theta\gamma(r,\theta) &= (-r\sin(\theta), r\cos(\theta), 0) \\ (\partial_r\gamma \times \partial_\theta\gamma) &= (-r\cos(\theta), -r\sin(\theta), r) \\ \text{and } \|(\partial_u\gamma \times \partial_v\gamma)\| &= r\sqrt{2}\end{aligned}$$

$$\begin{aligned}N(r,\theta) &= \frac{1}{r\sqrt{2}}(-r\cos(\theta), -r\sin(\theta), r) \\ &= \frac{1}{\sqrt{2}}(-\cos(\theta), -\sin(\theta), 1)\end{aligned}$$

This is the unit-vector that is normal to S. We will evaluate the flux of F across S in the direction N away from the z-axis. To get the flux of F across S in the direction away from the z-axis, we use the opposite of N, i.e.

$$\tilde{N}(r,\theta) = \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), -1)$$

Thus we will have the following:

$$\begin{aligned}\int_S F \bullet N \,\mathrm{d}\sigma &= \int_0^{2\pi}\int_1^3 (-r\sin(\theta), r\cos(\theta), r)\, \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), -1)\, r\sqrt{2}\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_1^3 -r^2 \,\mathrm{d}r\mathrm{d}\theta = -\frac{58}{3}\pi\end{aligned}$$

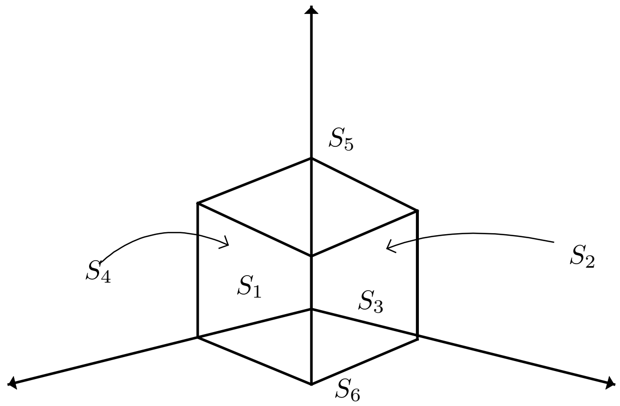
Orientation of S: An important thing to notice in the definition of the flux integral $\int_S F \bullet N \,\mathrm{d}\sigma$ is that the sign of the integrand depends on the order in which the cross product of $\partial_u\gamma$ and $\partial_v\gamma$ is taken. The result unit normal vectors:

$$\frac{1}{\|\partial_u\gamma \times \partial_v\gamma\|} \partial_u\gamma \times \partial_v\gamma$$
$$\frac{1}{\|\partial_v\gamma \times \partial_u\gamma\|} \partial_v\gamma \times \partial_u\gamma$$

can be used to disntinguish two sides of a surface. When we choose a side for the normal vectors, we say the surface is oriented.

April 7th, 2021

Exp: Find the flux of $F(x,y,z)=(x^2,x+1,z-y)$ outward across S, where S is the surface of the box $[0,2]\times[0,3]\times[0,4]$.



$$\int_S F\bullet N\,\mathrm{d}\sigma=\int_{S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6} F\bullet N\,\mathrm{d}\sigma$$
$$=\int_{S_1} F\bullet N\,\mathrm{d}\sigma+\int_{S_2} F\bullet N\,\mathrm{d}\sigma+\int_{S_3} F\bullet N\,\mathrm{d}\sigma+\int_{S_4} F\bullet N\,\mathrm{d}\sigma$$
$$+\int_{S_5} F\bullet N\,\mathrm{d}\sigma+\int_{S_6} F\bullet N\,\mathrm{d}\sigma$$

$$\text{Parameterizing } S_1 \text{ by: } \gamma_1(y,z)=(2,y,z)$$
$$\text{If we are in } S_1, x \text{ is always } 2\ ([0,2]\times[0,3]\times[0,4])$$
$$\text{Where } 0\leqslant y\leqslant 3 \text{ and } 0\leqslant z\leqslant 4$$

An outward unit normal vector to S_1 is $N=(1,0,0)$. Thus:

$$\int_{S_1} F\bullet N\,\mathrm{d}\sigma=\int_0^4\int_0^3 (4,3,z-y)\bullet(1,0,0)\,\mathrm{d}y\mathrm{d}z$$
$$=\int_0^4\int_0^3 4\mathrm{d}y\mathrm{d}z=4\times4\times3=48$$

Similarly, we have the following for the other surfaces:

$$\int_{S_2} F\bullet N\,\mathrm{d}\sigma=\int_0^4\int_0^3 (0,y+1,z-y)\bullet(-1,0,0)\,\mathrm{d}y\mathrm{d}z=0$$
$$\int_{S_3} F\bullet N\,\mathrm{d}\sigma=\int_0^4\int_0^2 (x^2,4,z-3)\bullet(0,1,0)\,\mathrm{d}x\mathrm{d}z=32$$
$$\int_{S_4} F\bullet N\,\mathrm{d}\sigma=\int_0^4\int_0^2 (x^2,1,z)\bullet(0,-1,0)\,\mathrm{d}x\mathrm{d}z=-8$$
$$\int_{S_5} F\bullet N\,\mathrm{d}\sigma=\int_0^3\int_0^2 (x^2,y+1,4-y)\bullet(0,0,1)\,\mathrm{d}x\mathrm{d}y=15$$
$$\int_{S_6} F\bullet N\,\mathrm{d}\sigma=\int_0^3\int_0^2 (x^2,y+1,0-y)\bullet(0,0,-1)\,\mathrm{d}x\mathrm{d}y=9$$

Therefore, we can get the final result:

$$\int_S F \bullet N \, \mathrm{d}\sigma = \int_{S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6} F \bullet N \, \mathrm{d}\sigma = 96$$

Alternative way to find the flux of F across S:

If S is the graph of C^1 function $z = f(x, y)$ over D ((x, y) are in D), we can also think of S as the level set of C^1 function g from $\mathbb{R}^3$ to $\mathbb{R}$:

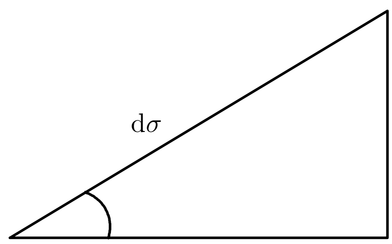
$$g(x, y, z) = z - f(x, y) = 0 \longleftarrow c \, \text{constant in} \, \mathbb{R}$$

Since ∇g is nromal at each point of S: $\nabla g = (-f_x, -f_y, 1)$, the two possibilities for a unit normal vector to S at the point $(x, y, f(x, y))$ are:

$$N = \pm \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}} (-f_x, -f_y, 1)$$

The <u>upward</u> flux of F across S is:

$$\int_S F \bullet N \, \mathrm{d}\sigma \overset{?}{=\!=} \int_D F(x, y, f(x, y)) \bullet (-f_x, -f_y, 1) \, \mathrm{d}x \mathrm{d}y$$



The cosine of that angle is equivalent to:

$$\frac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$$

D is the domain of f and S is the graph of f over D.

<u>Exp</u>: Let S be the part of the graph of $f(x, y) = x^2 + y^2$ above the rectangle $[0, 1] \times [0, 3]$ (this rectangle is our D and $x^2 + y^2$ plays the role of our z) with normal vector N having a negative component z. The flux of $F(x, y, z) = (-x, 3, z)$ across S is:

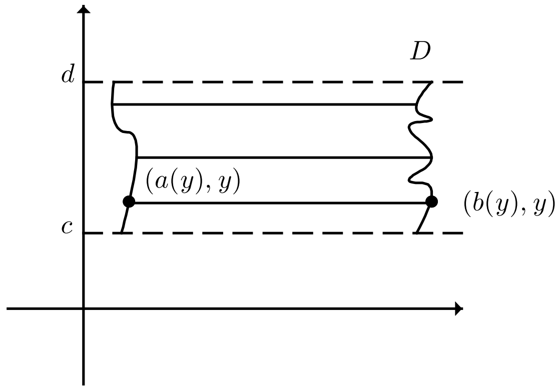
$$\int_S F \bullet N \, \mathrm{d}\sigma = \int_D F(x, y, f(x, y)) \bullet (f_x, f_y, -1) \, \mathrm{d}x \mathrm{d}y$$

$$= \int_0^3 \int_0^1 (-x, 3, x^2 + y^2) \bullet (2x, 2y, -1) \, \mathrm{d}x \mathrm{d}y$$
$$= \int_0^3 \int_0^1 (-2x^2 + 6y - x^2 - y^2) \, \mathrm{d}x \mathrm{d}y = 15$$

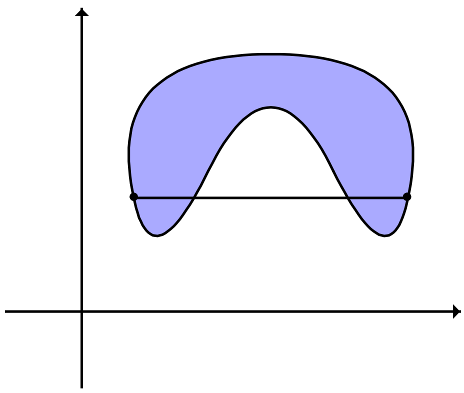
In previous examples, we had to use the parameterization to get to this stage. However, using this, we do not need to further parameterize, since $(x, y, f(x, y))$ is already a parameterization.

Chapt. 7: Green's / Divergence / Stokes' Theorems

Def.: *x-simple:* A bounded set D (in the x-y plane) is called x-simple if it has the following property: The set of all points in D whose second coordinate is y is an interval $D(y)$ parallel to the x-axis, whose end-points $(a(y), b(y))$ are continuous functions of y: $a(y) \leqslant y \leqslant b(y)$ and $c \leqslant y \leqslant d$. Below is a visualization of this:



The line connecting the two end-points has constant y. However, notice that the line must be entirely contained in D. Let us see an example:



Is this region D x-simple? Take a parallel segment line $(a(y), y)$ — $(b(y), y)$. We can see that there is a section of the line segment that is not within the domain D. Remember that the line segment has to be totally inside the domain. Therefore, this region D is NOT x-simple.

The integral of a continuous function $f(x, y)$ wrt. x over $D(y)$ is:

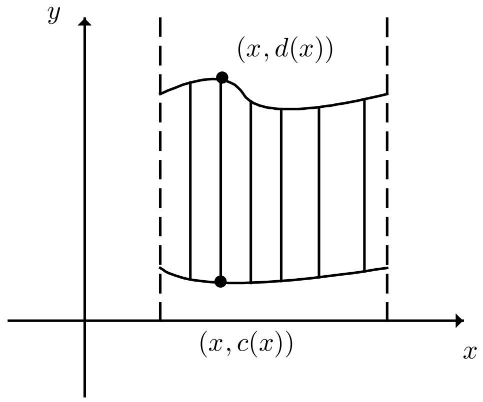
$$\int_{D(y)} f(x, y)\,\mathrm{d}x = \int_{x=a(y)}^{x=b(y)} f(x, y)\,\mathrm{d}x$$

which is (ie. the result of that integral) a continuous function y. We integrate this function wrt. y between c and d, and we obtain:

$$\int_c^d \left(\int_{x=a(y)}^{x=b(y)} f(x, y)\,\mathrm{d}x \right) \mathrm{d}y$$

We call this an interated integral.

Similarly, D is said to be -simple if the set of all points in D whose first coordinate x is an interval $D(x)$ is parallel to the y-axis and whose end-points $c(x)$ and $d(x)$ are continuous function sin x.



The iterated integral of a continuous function $f(x, y)$ is:

$$\int_a^b \left(\int_{y=c(x)}^{y=d(x)} f(x, y)\, \mathrm{d}y \right) \mathrm{d}x$$

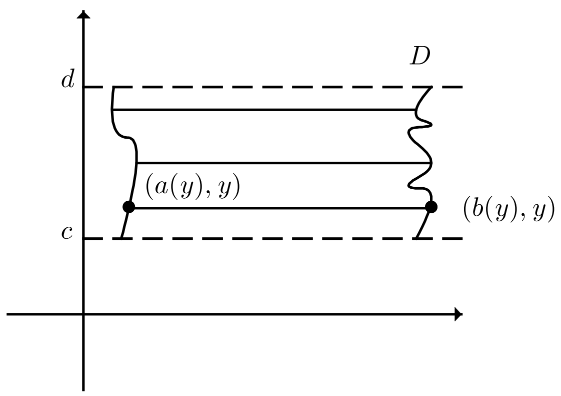
This integral is a continuous function in x.

April 12th, 2021

Divergence Theorem:

Let D be a smoothly bounded set in the x-y plane that is both x-simple and y-simple. Let f be a function of calss C^1 (ie. f_x and f_y are continuous in D). Fixing y and applying the FTC in one variable implies:

$$\int_{a(y)}^{b(y)} f_x(x, y)\, \mathrm{d}x \overset{(*)}{=\!=} f(b(y), y) - f(a(y), y)$$



Integrating (*) wrt. y from c to d implies:

$$\int_c^d \left(\int_{a(y)}^{b(y)} f_x(x, y)\, \mathrm{d}x \right) \mathrm{d}y = \int_c^d [f(b(y), y) - f(a(y), a)]\, \mathrm{d}y \,(**)$$

The iterated integral on the left is the integral of f_x over D. ie.:

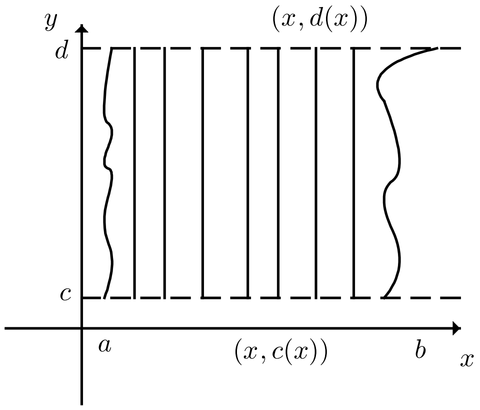
$$\int_D f_x \, \mathrm{d}A$$

Now: Observe that the integral on the right is equal to the line integral $\int_{\partial D} f \, \mathrm{d}y$, where ∂D is the boundary of D, taken counter-clockwise from c to d, using the parameterization $\gamma_1(y) = (b(y), y)$ on the graph of $b(y)$ with y running from c to d, and $\gamma_2(y) = (a(y), y)$ on the graph of $a(y)$ with y going down from d to c.

Therefore, (**) can be writen as:

$$\int_D f_x \, \mathrm{d}x\mathrm{d}y = \int_{\partial D} f \, \mathrm{d}y \quad (1)$$

Let g be another C^1 function on D. Wee obtain an analogous formula for the integral of g_y over D.



$$\int_a^b \left(\int_{c(x)}^{d(x)} g_y(x, y) \, \mathrm{d}y \right) \mathrm{d}x = \int_a^b [g(x, d(x)) - g(x, c(x))] \mathrm{d}x$$

The iteratived integral on the left sum is equal to the surface integral:

$$\int_D g_y \, \mathrm{d}A$$

The integral on the right side is the line integral of g taken over the boundary of D, ∂D, in the clockwise direction. Taking it in the counter-clockwise direction, we obtain:

$$\int_D g_y \, \mathrm{d}A = -\int_{\partial D} g \, \mathrm{d}x \quad (2)$$

$$(1) + (2) =$$

$$\int_D (f_x + g_y) \, \mathrm{d}x\mathrm{d}y = \int_{\partial D} f \, \mathrm{d}y - g \, \mathrm{d}x \quad (3)$$

In vector language, let $\gamma(s) = (x(s), y(s))$ be the arc-length parameterization of ∂D, increasing in the counter-clockwise direction. s is between 0 and the length of ∂D. Since the tangent vector $\left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s} \right)$ has length 1, at each point of ∂D,

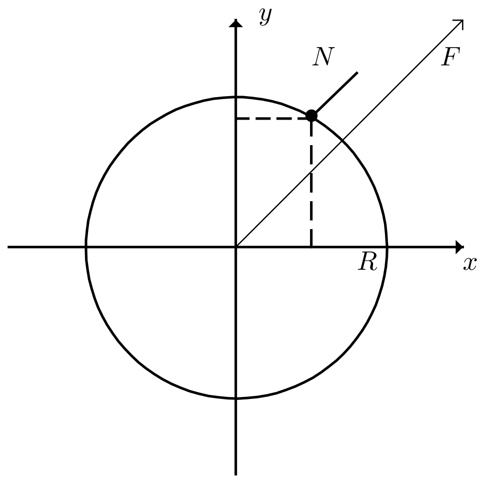
$$N = \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s} \right)$$

is the outward-pointing normal unit vector. For $F=(f,g)$, we recall that: $\operatorname{div}(F)=f_x+g_y$. Then (3) can be written as:

$$\int_D \operatorname{div}(F)\,\mathrm{d}x\mathrm{d}y=\int_D \operatorname{div}(F)\,\mathrm{d}A=\int_{\partial D} F\bullet N\,\mathrm{d}s$$

This is the so-called Divergence Theorem.

Exp: Let $F(x,y)=(x,y)$ and let D be a disk of radius R centered at O. At each point on ∂D, the outward ponting unit normal vector to ∂D is parallel to F.



Thus at each point of ∂D, we have the following:

$$\begin{aligned} F\bullet N=\|F\|=\sqrt{x^2+y^2}=R \\ \text{This is because } (x,y) \text{ are on } \partial D \\ \|N\|=1 \end{aligned}$$

$$\begin{aligned} \int_{\partial D} F\bullet N\,\mathrm{d}s=\int_{\partial D}\|F\|\,\mathrm{d}s=\int_{\partial D} R\,\mathrm{d}s \\ =R\int_{\partial D}\mathrm{d}s=R(2\pi R)=2\pi R^2 \end{aligned}$$

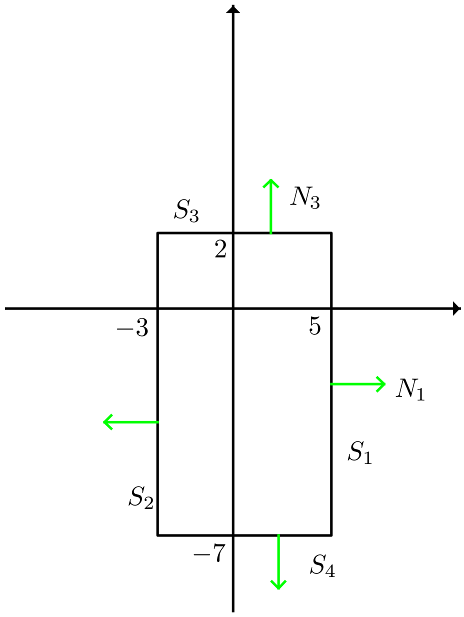
On the other hand, we have the following:

$$\begin{aligned} \operatorname{div}(F)=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)=1+1=2 \\ \Longrightarrow \int_D \operatorname{div}(F)\,\mathrm{d}A=\int_D 2\,\mathrm{d}A=2\,\mathrm{Area}(D)=2\pi R^2 \end{aligned}$$

Exp: Let $F(x,y)=(x,y)$ and let D be the rectangle region where $-3\leqslant x\leqslant 5$ and $-7\leqslant y\leqslant 2$. Find the flux outward across ∂D. Since $\operatorname{div}(F)=2$, the Divergence Theorem implies:

$$\int_{\partial D} F\bullet N\,\mathrm{d}s=\int_D \operatorname{div}(F)\,\mathrm{d}A=\int_D 2\,\mathrm{d}A=2\,\mathrm{Area}(D)=144$$

But what if we don't want to use the Divergence Theorem?

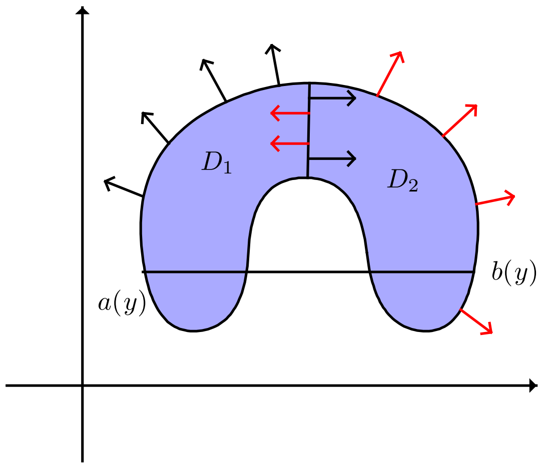


We consider each of the S_i separately and then take the following:

$$\int_{\partial D} F \bullet N \, \mathrm{d}s = \sum_{i=1}^{4} \int_{S_i} F \bullet N \, \mathrm{d}s$$

since we know that $\partial D = S_1 \cup S_2 \cup S_3 \cup S_4$. This is a lot more computation since we need to parameterize each S_i.

Def.: *Regular:* We call a smoothly bounded set D regular if it is the union of a finite number of smoothly bounded subsets, each of which is x-simple and y-simple and any two have only a boundary arc in common.

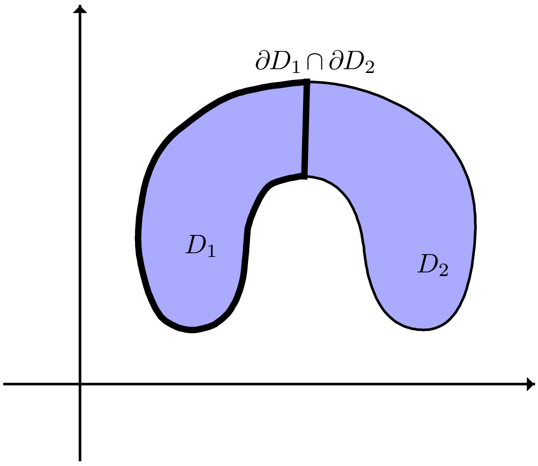


We know that D is not x-simple by itself, but if we cut the region st. $D = D_1 \cup D_2$, then each of D_1 and D_2 is x-simple and y-simple. Being regular essentially means that we can cut up a domain into nice pieces that are x and y-simple. The boundary arc we are talking about is what we have "cut" between the two regions.

The red arrows represent the normal vectors to ∂D_2 and the black arrows represent the normal vectors to ∂D_1. We know that ∂D is affected now that we have split the region into 2, but at the boundary arc, we can observe that the normal vectors cancel one another out.

April 14th, 2021

Recall the example from the last lecture:



We have that $D = D_1 \cup D_2$. Using that, we will go through as such:

$$\int_{D_1} \mathrm{div}\,(F)\,\mathrm{d}A = \int_{\partial D_1} F \bullet N\,\mathrm{d}s, \text{ and :}$$

$$\int_{D_2} \mathrm{div}\,(F)\,\mathrm{d}A = \int_{\partial D_2} F \bullet N\,\mathrm{d}s$$

$$\text{Thus} \int_{D} \mathrm{div}\,(F)\,\mathrm{d}A = \int_{D_1 \cup D_2} \mathrm{div}\,(F)\,\mathrm{d}A$$

$$= \int_{D_1} \mathrm{div}\,(F)\,\mathrm{d}A + \int_{D_2} \mathrm{div}\,(F)\,\mathrm{d}A = \int_{\partial D_1} F \bullet N\,\mathrm{d}s + \int_{\partial D_2} F \bullet N\,\mathrm{d}s$$

In fact, the normal vector to the connecting curve $C = \partial D_1 \cap \partial D_2$ that is outward wrt. D_1 is the negative of the normal vector to C that is outward wrt. D_2. Therefore, in the sum of the integrals $F \bullet N$ over C cancel.

$$\int_C F \bullet N\,\mathrm{d}s - \int_C F \bullet N\,\mathrm{d}s = 0$$

Theorem 1: *The Divergence Theorem:*

If $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is C^1 on a regular set D, then:

$$\int_D \mathrm{div}\,(F)\,\mathrm{d}A = \int_{\partial D} F \bullet N\,\mathrm{d}s$$

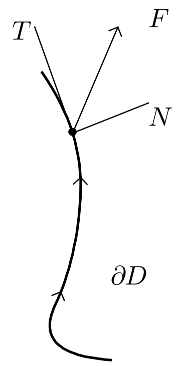
Exp: Let $F(x,y) = (x, -y)$. The outward flux of F acorss ∂D of a regular set D is given by:

$$\int_{\partial D} F \bullet N\,\mathrm{d}s = \int_D \mathrm{div}\,(F)\,\mathrm{d}A$$

$$= \int_D \left[\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}y}(-y)\right] \mathrm{d}A = \int_D (1-1)\,\mathrm{d}A = 0$$

Consequence: We will use the Divergence Theorem to find the integral of the tangential component of F along ∂D:

$$\int_{\partial D} F \bullet T \,\mathrm{d}s$$



Supose F is of class C^1 on a regular set D and ∂D is transversed in a counter-clockwise direction. If the outward unit normal vector at a point on ∂D is $N = (n_1, n_2)$, then the unit tangent vector there (at the same point) is:

$$T = (t_1, t_2) = (-n_2, n_1)$$

For $F = (f, g)$, we have:

$$\int_{\partial D} F \bullet T \,\mathrm{d}s = \int_{\partial D} (-f n_2 + g n_1) \,\mathrm{d}s$$

By the Divergence Theorem and the definition of Curl, we have that:

$$\begin{aligned}\int_{\partial D} [g n_1 - f n_2] \,\mathrm{d}s &= \int_D \left[\frac{\partial g}{\partial x} + \frac{\partial}{\partial y}(-f)\right] \mathrm{d}A \\ &= \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}A = \int_D \operatorname{curl}(F) \,\mathrm{d}A\end{aligned}$$

Theorem 2: *Green's Theorem:*

If $F = (f, g)$ is of class C^1 on a regular set D in $\mathbb{R}^2$, then:

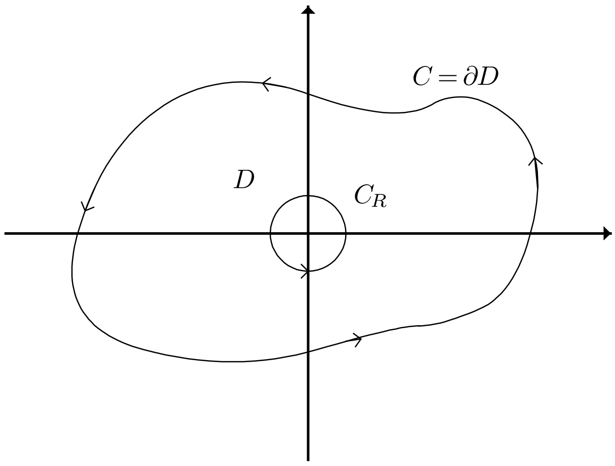
$$\int_D \operatorname{curl}(F) \,\mathrm{d}A = \int_{\partial D} F \bullet T \,\mathrm{d}s$$

As we have demonstrated, it is simply a consequence of the Divergence Theorem.

Rmk:

$$\begin{aligned}\int_{\partial D} (g n_1 - f n_2) \,\mathrm{d}s &= \int_{\partial D} H \bullet N \,\mathrm{d}s \\ (g n_1 - f n_2) = H \bullet N &\longrightarrow H = (g, -f) \\ \int_{\partial D} H \bullet N \,\mathrm{d}s &= \int_D \operatorname{div}(H) \,\mathrm{d}A \\ = \int_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] \mathrm{d}A &= \int_D \operatorname{curl}(F) \,\mathrm{d}A\end{aligned}$$

Exp: Let $F(x,y)=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$. We can see the region of D:

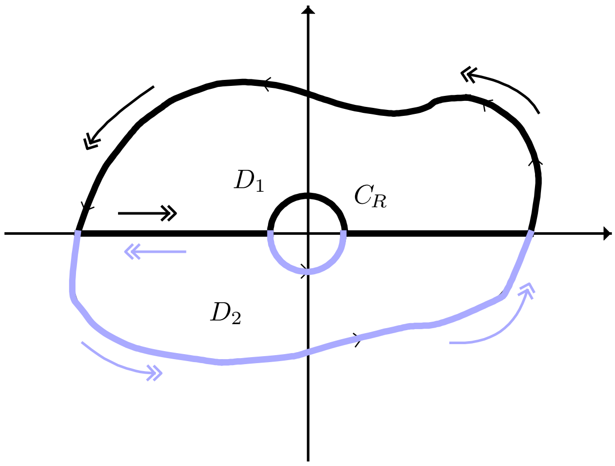


We know that F is not of class C^1 on D because F is not defined at the origin $(0,0)$. We wil make a circle at the origin of radius R, and cut the domain into two pieces that avoid the origin.

C_R is the circle of cetner $(0,0)$ and radius R, with $C_R\subseteq D$. Also, we have that C_R does not touch ∂D. For every $(x,y)\in D$ but $(x,y)\neq(0,0)$, we have:

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)=0$$

We will split D into two regions, D_1 and D_2.

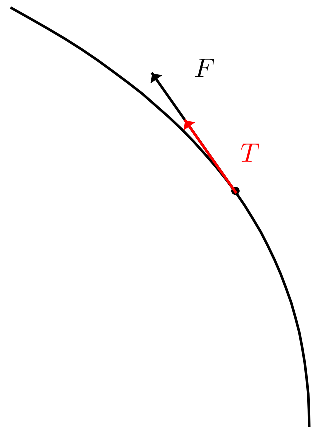


$$\begin{aligned}\int_{\partial D}F\bullet T\,\mathrm{d}s&=\int_{\partial D_1}F\bullet T\,\mathrm{d}s+\int_{\partial D_2}F\bullet T\,\mathrm{d}s\\&=\int_{\partial D}F\bullet T\,\mathrm{d}s+\int_{C_R}F\bullet T\,\mathrm{d}s=0\quad(*)\end{aligned}$$

This is because by the Green's Theorem, we have that:

$$\int_{\partial D}F\bullet T\,\mathrm{d}s=0\,\text{because}\,\mathrm{curl}(F)=0$$

Now, since F and the unit normal vector has the same direction at every point in $C=\partial D$,



Thus: $F \bullet T = \|F\|$. On C_R, we have the following:

$$
\begin{aligned}
-\int_{C_R(\text{clockwise})} F \bullet T\, \mathrm{d}s &= -\int_{C_R(\text{clockwise})} \|F\|\, \mathrm{d}s \\
&= -\int_{C_R(\text{clockwise})} \frac{1}{R}\, \mathrm{d}s \\
= -\frac{1}{R}\int_{C_R(\text{clockwise})} \mathrm{d}s &= -\frac{1}{R}(2\pi R) = -2\pi
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
(*)\text{ implies that} \int_{C=\partial D} F \bullet T\, \mathrm{d}s &= -\int_{C_R(\text{clockwise})} F \bullet T\, \mathrm{d}s \\
&= -2\pi
\end{aligned}
$$

Theorem 3: *Divergence Theorem in* $\mathbb{R}^3$

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a C^1 vector field on a regular set D in $\mathbb{R}^3$ (ie. D is a finite union of smoothly bounded sets in $\mathbb{R}^3$ that are x-y-z simple and have only boundary points in common) and let N be the unit normal vectors to ∂D that point out of D. Then:

$$\int_D \operatorname{div}(F)\, \mathrm{d}V = \int_{\partial D} F \bullet N\, \mathrm{d}\sigma, \text{where } \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

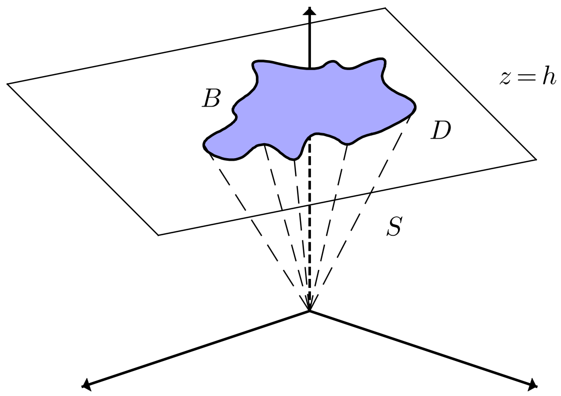
Exp: Let $F(x, y, z) = (x, y, z)$ and let D be the solid rectangular box described by: $2 \leqslant x \leqslant 4$, $7 \leqslant y \leqslant 10$ and $1 \leqslant z \leqslant 5$. Find the flux of F outward ∂D.

By the Divergence Theorem in $\mathbb{R}^3$, we have the following:

$$
\begin{aligned}
\int_{\partial D} F \bullet N\, \mathrm{d}\sigma &= \int_D \operatorname{div}(F)\, \mathrm{d}V \\
= \int_D \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] &= \int_D 3\mathrm{d}V \\
= 3\,\text{Volume}(D) = 3 \times 2 \times 3 \times 4 = 72
\end{aligned}
$$

April 19th, 2021

Exp:



This is a cone, but the base is not a disk anymore, but rather a region B on the plane $z=h$. Let the area of the region B be A (ie. $\text{Area}(B)=A$). We shall use the vector field $F(x,y,z)=(x,y,z)$ to show that the volume of the solid D is $\frac{1}{3}h\,A$.

Since $\operatorname{div}(F)=3$, the Divergence Theorem implies:

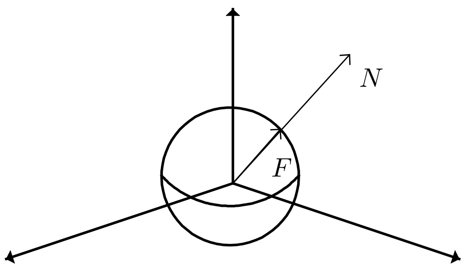
$$\begin{aligned}\int_{\partial D} F\bullet N\,\mathrm{d}\sigma &= \int_D \operatorname{div}(F)\,\mathrm{d}V\\ &=3\int_D \mathrm{d}V = 3\,\text{Volume}(D)\end{aligned}$$

The unit normal vector to the surface B in the plane $z=h$ is: $(0,0,1)$, and the normal vectors to the surface S are orthogonal to F at every point (x,y,z). Therefore:

$$\begin{aligned}\int_{\partial D} F\bullet N\,\mathrm{d}\sigma &= \int_B F\bullet N\,\mathrm{d}\sigma + \int_S F\bullet N\,\mathrm{d}\sigma\\ &=\int_B (x,y,h)\bullet(0,0,1)\,\mathrm{d}\sigma + \int_S 0\,\mathrm{d}\sigma\\ &=\int_B h\,\mathrm{d}\sigma = h\,\text{Area}(B) = h\,A\\ 3\,\text{Volume}(D) &= h\times A\\ \Longrightarrow \text{Volume}(D) &= \frac{1}{3}h\,A\end{aligned}$$

Exp: Find the flux og $F(x,y,z)=\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ outward across a sphere S_R of center the origin and radius R.

This is another example where we have a problem at the origin, since F is not defined at $(0,0,0)$. So we compute the surface integral. At each point on the sphere S_R, the direction of F is radial and normal to the surface (ie. F and the normal vectors N have the same direction).



$$F\bullet N = \|F\|$$

$$\text{But at each point of } S_R,\ \|F\| = \frac{1}{R^2}$$

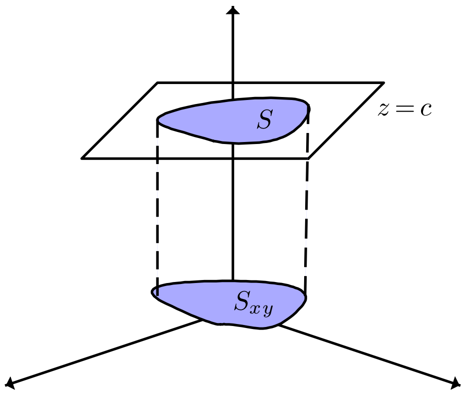
$$\int_{S_R} F \bullet N\,\mathrm{d}\sigma = \int_{S_R} \|F\|\mathrm{d}\sigma = \int_{S_R} \frac{1}{R^2}\,\mathrm{d}\sigma$$
$$= \frac{1}{R^2}\int_{S_R} \mathrm{d}\sigma = \frac{1}{R^2}(4\pi R^2) = 4\pi$$

Stokes' Theorem:

Green's Theorem says that for a class C^1 vector field $F(x,y) = (f_1(x,y), f_2(x,y))$ on a regular region D in the plane, we have:

$$\int_D \mathrm{curl}(F)\,\mathrm{d}A = \int_{\partial D} F \bullet T\,\mathrm{d}s$$

Suppose we have a flat surface S that lies in the plane $z = c$ in $\mathbb{R}^3$. Denote by S_{xy} the "shadow" of S in the x-y plane.



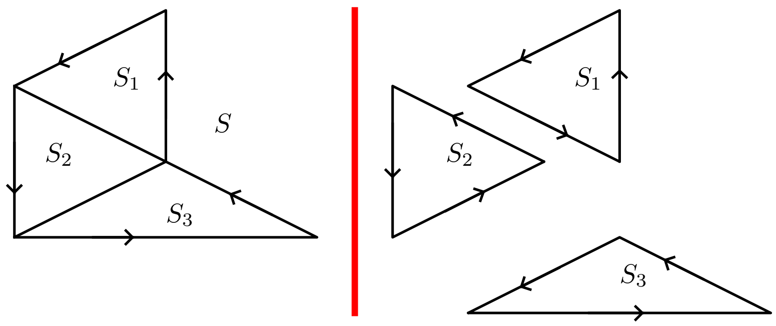
In other words, S_{xy} is the set of points $(x,y,0)$ st. (x,y,c) are in S.

For $G(x,y) = (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$, we have:

$$\int_S (\mathrm{Curl}(G)) \bullet N\,\mathrm{d}\sigma = \int_S (\mathrm{Curl}(G)) \bullet (0,0,1)\,\mathrm{d}\sigma$$
$$= \int_{S_{xy}} \left(\frac{\partial g_2}{\partial x}(x,y,c) - \frac{\partial g_1}{\partial y}(x,y,c)\right)\mathrm{d}\sigma$$
$$= \int_{\partial S_{xy}} g_1(x,y,c)\mathrm{d}x + g_2(x,y,c)\mathrm{d}y = \int_{\partial S} G \bullet T\,\mathrm{d}s$$
$$\Longrightarrow \int_S \mathrm{curl}(G) \bullet N\,\mathrm{d}\sigma = \int_{\partial S} G \bullet T\,\mathrm{d}s \quad (*)$$

(*) also holds for surfaces in every plane, not just in planes parallel to the x-y plane. Hence if our surface is the union of finitely many plane polygon faces S_i that meet along their edges, then we can use the additivity of surface integrals and line integrals to add formulaes for the face:

$$\int_S \mathrm{curl}(G) \bullet N\,\mathrm{d}\sigma$$
$$\text{where } S = \bigcup_i S_i$$



We can see from the separated surfaces that the following holds:

$$\int_{E_{12}} G \bullet T_1 \,\mathrm{d}s = -\int_{E_{12}} G \bullet T_2 \,\mathrm{d}s$$

In other words, along the edge between S_1 and S_2, the two tangent vectors are exactly opposites, and thus cancel out. This is why we can see in in the following that we can proceed from line 1 to line 2.

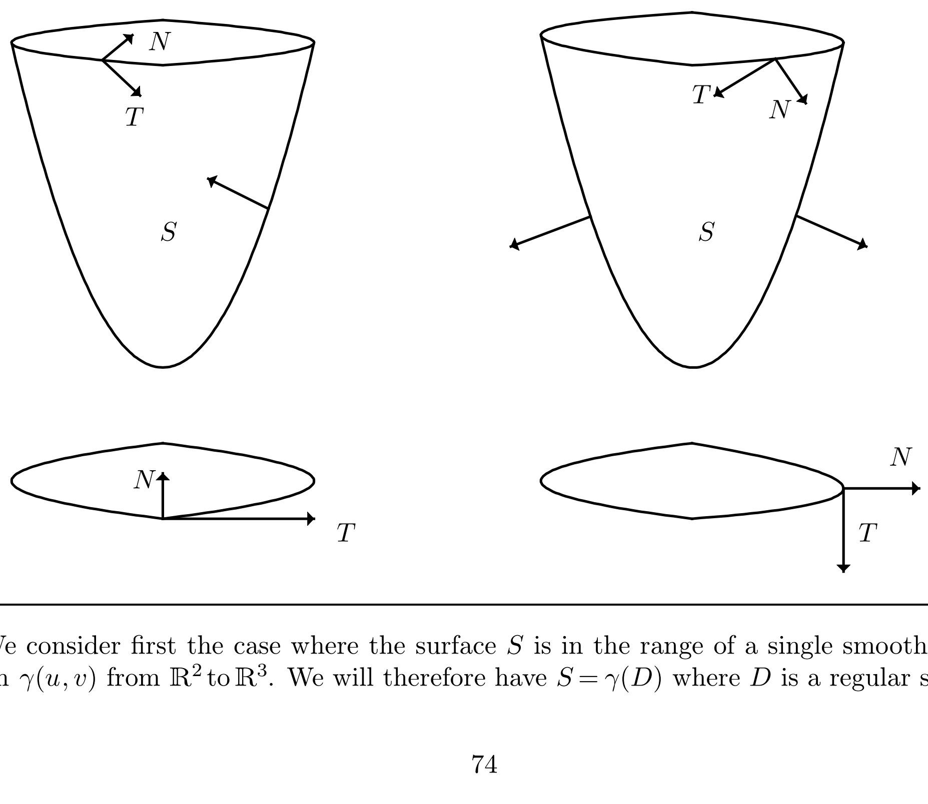
$$\begin{aligned} = \sum_{i=1}^{n} \int_{S_i} (\mathrm{curl}(G)) \bullet N \,\mathrm{d}\sigma = \sum_{i=1}^{n} \int_{\partial S_i} G \bullet T_i \,\mathrm{d}s \\ = \int_{\partial S} G \bullet T \,\mathrm{d}s \end{aligned}$$

Theorem 4: *Stokes' Theorem:*

Let G be a vector field that is C^1 on a piecewise smooth oriented surface S in $\mathbb{R}^3$ whose boundary ∂S is a piecewise smooth curve, and that the domains of parameterization of S are regular sets in $\mathbb{R}^2$. Then we have:

$$\int_S \mathrm{curl}(G) \bullet N \,\mathrm{d}\sigma = \int_{\partial S} G \bullet T \,\mathrm{d}s$$

where the orientation of the unit normal vector N to S and of the unit tangentvector T are chosen as in what follows:



Proof: We consider first the case where the surface S is in the range of a single smooth parameterization $\gamma(u, v)$ from $\mathbb{R}^2$ to $\mathbb{R}^3$. We will therefore have $S = \gamma(D)$ where D is a regular set in $\mathbb{R}^2$, ie. (u,v) in D and $\gamma(\partial D)=\partial S$. Then:

$$\int_S \mathrm{curl}(G)\bullet N\,\mathrm{d}\sigma=\int_D(\mathrm{curl}(G))(\gamma(u,v))\bullet(\partial_u\gamma\times\partial_v\gamma)\,\mathrm{d}u\mathrm{d}v$$

It is not hard to check (using the chain rule) that:

$$\begin{aligned}&(\mathrm{curl}(G))(\gamma(u,v))\bullet(\partial_u\gamma\times\partial_v\gamma)\\ =&\left[\frac{\partial}{\partial u}G(\gamma(u,v))\bullet\partial_v\gamma\right]-\left[\frac{\partial}{\partial v}G(\gamma(u,v))\bullet\partial_u\gamma\right]\quad(*)\end{aligned}$$

Let F be a vector field on D in $\mathbb{R}^2$ defined by:

$$F(u,v)=(G(\gamma(u,v))\bullet\partial_u\gamma,G(\gamma(u,v))\bullet\partial_v\gamma)$$

Then we have that:

$$\begin{aligned}\mathrm{curl}(F(u,v))=&\left[\frac{\partial}{\partial u}G(\gamma(u,v))\bullet\partial_v\gamma\right]-\left[\frac{\partial}{\partial v}G(\gamma(u,v))\bullet\partial_u\gamma\right]\\ &=G(\gamma(u,v))\bullet\partial_{uv}\gamma+\partial_v\gamma\circ\frac{\partial}{\partial u}G(\gamma(u,v))\\ &-\partial_{vu}\gamma\bullet G(\gamma(u,v))-\partial_u\gamma\bullet\frac{\partial}{\partial v}G(\gamma(u,v))\\ &=\mathrm{curl}(G)\gamma(u,v)\bullet(\partial_u\gamma\times\partial_v\gamma)\quad(**)\end{aligned}$$

$$\begin{aligned}\int_S\mathrm{curl}(G)\bullet N\,\mathrm{d}\sigma=&\int_D\mathrm{curl}(G)\gamma(u,v)\bullet(\partial_u\gamma\times\partial_v\gamma)\\ &=\int_D\mathrm{curl}(F)\mathrm{d}u\mathrm{d}v\\ &\text{By }(*)\text{ and }(**)\end{aligned}$$

By the Green's Theorem, we know that:

$$\int_D\mathrm{curl}(F)\,\mathrm{d}u\mathrm{d}v=\int_{\partial D}F\bullet T\,\mathrm{d}s$$

Now, let $R(t)=(u(t),v(t))$, with $a\leqslant t\leqslant b$ that parameterizes ∂D. Then:

$$\begin{aligned}&\int_{\partial D}F\bullet T\,\mathrm{d}s=\int_a^b F(R(t))\bullet R'(t)\,\mathrm{d}t\\ =&\int_a^b[G(\gamma(R(t)))\bullet\partial_u\gamma(R(t)),G(\gamma(R(t)))\bullet\partial_v\gamma(R(t))]\bullet R'(t)\\ &\gamma(R(t))=(x,y,z),G=(g_1,g_2,g_3)\\ =&\int_a^b(g_1x_u+g_2y_u+g_3z_u,g_1x_v+g_2y_v,g_3z_v)\bullet R'(t)\,\mathrm{d}t\\ &R'(t)=(u'(t),v'(t))\\ =&\int_a^b[(g_1x_u+g_2y_u+g_3z_u)u'(t)+(g_1x_v+g_2y_v,g_3z_v)v'(t)]\,\mathrm{d}t\\ =&\int_a^b g_1\mathrm{d}x+g_2\mathrm{d}y+g_3\mathrm{d}z=\int_{\partial S}G\bullet T\,\mathrm{d}s\end{aligned}$$

The range of $\gamma(R(t))$ is ∂S. In the beginning of the proof, we wrote that $\gamma(\partial D)=\partial(S)$.

This concludes the case where S has a single parameterization. Now suppose that S is the union of such surfaces that meet pairwise on common edges. We assume they are oriented so that the line integrals associated to adjoining parts cancel on these common edges. The edges that are not common to two parts constitute ∂S, so Stokes' formula for S is obtained by adding Stokes' formulas for the parameterized parts of S, Rmk: It is easy to show taht if f is of class C^2 in $\mathbb{R}^3$, or a subset of $\mathbb{R}^3$, and if $F = \nabla f = (f_x, f_y, f_z)$, then:

$$\begin{aligned}
\mathrm{curl}(F) &= \mathrm{O}_{\mathbb{R}^3} \\
\text{In fact: if } F &= (F_1, F_2, F_3), \text{then:} \\
\mathrm{curl}(F) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \\
= (0,0,0) \text{ since } f \text{ is of class } C^2
\end{aligned}$$

we shall show that under an additional condition on the domain of F, the converse holds: Suppose that every closed loop C in the domain of F is the boundary ∂S of a smooth surface S in D. By Stokes' theorem, if $\mathrm{curl}(F) = 0$, we have:

$$\int_{C = \partial S} F \bullet T \, \mathrm{d}s = \int_S \mathrm{curl}(F) \bullet N \, \mathrm{d}\sigma = 0$$

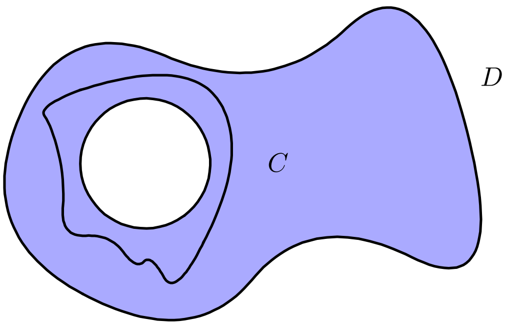
By **THM1 in Chapt. 6**, there is a potential g st. $\nabla g = F$. Domains that have the property that we need are called simply connected.

Def.: *Simply Connected:* We say that a set in $\mathbb{R}^n$ is simply conneced if it is connected and if every simple closed curve in the set can be shrunk continuously within the set to a point.

Comments:

i. Intuitively, a simply connected domain is a domain with no holes that pass all te way through it.

   Exp:

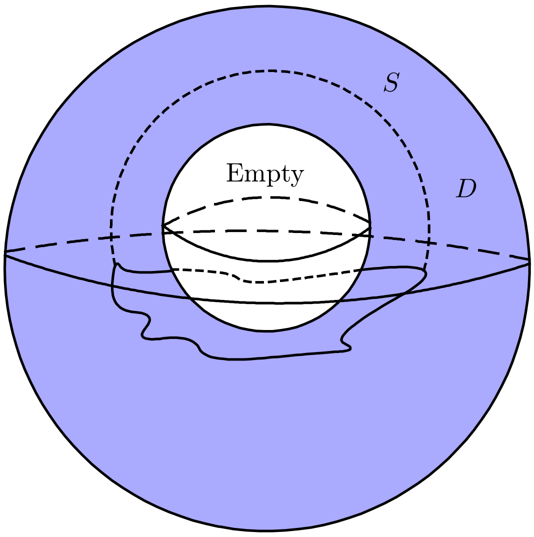


   We can see that D is a region with a hole in it, and C is the boundary that covers a surface S. In other words, $C = \partial S$. However, we can see that S contains a region inside it that is NOT in D. Therefore, D is not simply connected.

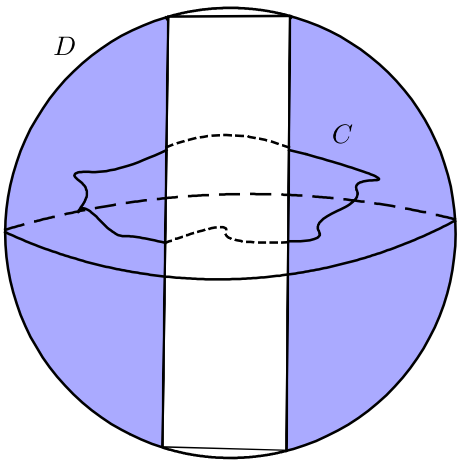
   If we want to connect this to our definition, notice that we cannot shrink the region to a point inside D. This means that for continually smaller regions covered by the boundary C, we will eventually "fall" into the hole, and thus would be outside of D. This is why it is not simply connected.

April 26th, 2021

Exp: in $\mathbb{R}^3$:



The trick is to take some simple closed path C that surrounds the empty sphere in the bigger region, D. We take a region that can wear a "hat" that contains the sphere but is also still in D. This hat is S. We can think of this as closing the dome inside D. In other words, C shrinks upwards up to the top of S, which is where we are taking the "point" to be. Thus we are avoiding the hole or the empty sphere inside D. However, consider the following region:



As we can see, in this case we cannot take the hat that would lead to C shrinking to a point inside D. Intuitively, this is also what we mean when we say that the hole should not go through the entirety of D.

Another example can be $D = \mathbb{R}^3 - \{z\text{-axis}\}$. Again, we cannot take a contour C that would shrink inside of D, since the z-axis is not included. The word simple means that there are no knots in the region. We can draw it as such:

i. Covered in last lecture

ii. If a set is simply connected, then every piecewise smooth simple closed curve in the set is the boundary of a piecewise smooth surface in the set.

Theorem 5:

Suppose F is a class C^2 vector field on an open simply connected set U in $\mathbb{R}^2$ on which $\operatorname{curl}(F) = \mathrm{O}$. Then there is a function g from U to $\mathbb{R}$ st. $\nabla g = F$.

Exp: The set D_1 of points in $\mathbb{R}^3$ with $\|x\| \neq 0$ (ie. $\mathbb{R}^3 - \{\mathrm{O}\}$) is simply connected, but the set D_2 of points in $\mathbb{R}^3$ not on the z-axis, as shown in the last lecture, is not simply connected.

So if F is a class C^1 vector field on D_1 with $\operatorname{curl}(F) = \mathrm{O}$, then F has a potential ie. $\exists g\colon D_1 \longrightarrow \mathbb{R}$ st. $\nabla g = F$, where g is of class C^2. Since $D_2 \subset D_1$, F is of class C^1 on D_2. But F has no potential on D_2, by the theorem just introduced.

Exp:

$$F(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right)$$

Observe, firstly, that $\operatorname{curl}(F) = \mathrm{O}$.

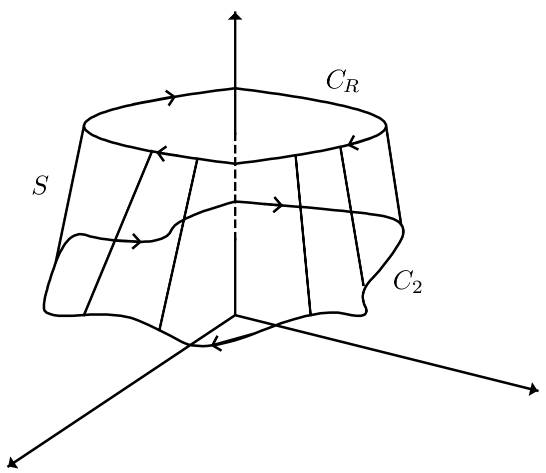
a) Find the circulation of F around a piecewise smooth simple closed curve C_1 that does not encircle or intersect the z-axis, where $x = 0$ and $y = 0$.

   Let S be a piecewise smooth surface with $\partial S = C_1$. By Stokes' Theorem, we have:

$$\int_{C_1 = \partial S} F \bullet T \, \mathrm{d}s = \int_S \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma = 0$$

b) Find the circulation of F around a piecewise smooth simple closed cuve C_2 that encircles the z-axis once but does not interset it.

   Since F is not defined on the z-axis, $\operatorname{curl}(F)$ is not defined there either. Every surface with boundary C_2 would include a point on the z-axis, and so Stokes' Theorem cannot be applied on such surface. Conside the circle C_R of radius R cented on the z-axis in a horizontal plane $z = k$ that does not intersect C_2.



Let S be a piecewise smooth oriented surface whose bonudary is $C_R \cup C_2$. Stokes' Theorem applied to S gives:

$$\int_{C_2} F \bullet T \, \mathrm{d}s + \int_{C_R} F \bullet T \, \mathrm{d}s = \int_{\partial S = C_1 \cup C_R} F \bullet T \, \mathrm{d}s = \int_S \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma$$

$$= \int_S \mathrm{O} \, \mathrm{d}\sigma = 0$$

Now, we can parameterize C_R by:

$$\gamma(t) = (R\cos(t), R\sin(t), k) \text{ with } 0 \leqslant t \leqslant 2\pi$$

$$0 = \int_{C2} F \bullet T \, \mathrm{d}s + \int_0^{2\pi} \left(\frac{-R\sin(t)}{R^2}, \frac{R\cos(t)}{R^2}, 0 \right) \bullet (-R\sin(t), -R\cos(t), 0) \, \mathrm{d}t$$

$$= \int_{C_2} F \bullet T \, \mathrm{d}s + \int_0^{2\pi} (-\sin^2(t) - \cos^2(t)) \, \mathrm{d}t$$

$$= \int_{C_2} F \bullet T \, \mathrm{d}s - \int_0^{2\pi} \mathrm{d}t \Longrightarrow \int_{C_2} F \bullet T \, \mathrm{d}s = \int_0^{2\pi} \mathrm{d}t = 2\pi$$

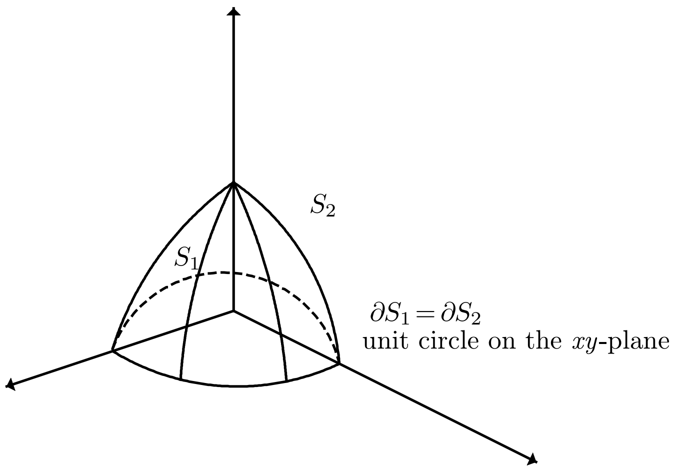
Rmk: If S_1 and S_2 are two smooth surfaces that have the same oriented boundary, ie. $\partial S_1 = \partial S_2$, and if F is a class C^1 vector field on both surfaces S_1 and S_2, then Stokes' Theorem implies that the fluxes of F across S_1 and S_2 are equal because:

$$\int_{\partial S_1} F \bullet T \, \mathrm{d}s = \int_{\partial S_2} F \bullet T \, \mathrm{d}s$$

$$\int_{S_1} \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma = \int_{S_2} \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma$$

Where the normal vectors N are consistent with the boundary orientation.

Exp: Let $F(x, y, z) = (z^2, -2x, y^5)$, and let S_1 be the disk of radius 1: $x^2 + y^2 \leqslant 1$ in the x-y plane, and let S_2 be the upper unit hemisphere (ie. of radius 1 and centered at the origin).

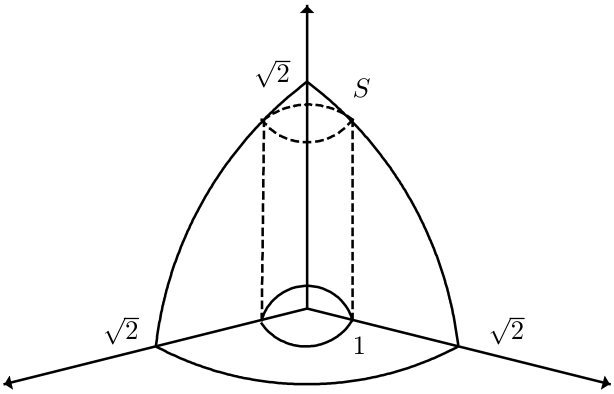


$$\int_{S_2} \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma = \int_{S_1} \operatorname{curl}(F) \bullet N \, \mathrm{d}\sigma$$

$$= \int_{S_1} \operatorname{curl}(F) \bullet (0, 0, 1) \, \mathrm{d}\sigma$$

$$= -2 \int_{S_1} \mathrm{d}\sigma = -2 \operatorname{Area}(S_1) = -2\pi$$

Exp: Let $F(x, y, z) = (z, x, y)$ and let $S = \{(x, y, z) \in \mathbb{R}^3 \text{ st } x^2 + y^2 + z^2 = 2, x^2 + y^2 \leqslant 1 \text{ and } z \geqslant 0\}$. Verify Stokes' Theorem:



A parameterization of S is given by: $\gamma(r,\theta)=\left(r\cos(\theta), r\sin(\theta), \sqrt{2-r^2}\right)$, with $0\leqslant r\leqslant 1$ and $0\leqslant\theta\leqslant 2\pi$,

$$\gamma_r(r,\theta)\times\gamma_\theta(r,\theta)=\left(\frac{r^2}{\sqrt{2-r^2}}\cos(\theta), \frac{r^2}{\sqrt{2-r^2}}\sin(\theta), r\right)$$

$$\text{On the other hand, } \operatorname{curl}(F)=(1,1,1)$$

Therefore, using the definition of a surface integral (taken from **Chapt. 6**), we have:

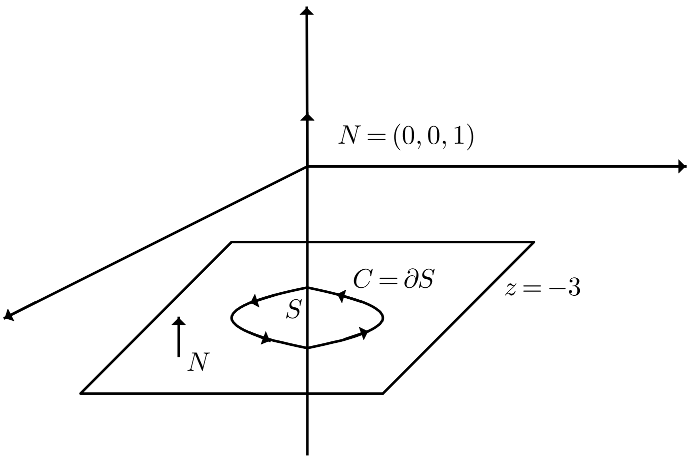
$$\int_S \operatorname{curl}(F)\bullet N\,\mathrm{d}\sigma=\int_0^1\int_0^{2\pi}(1,1,1)\bullet\left(\frac{r^2}{\sqrt{2-r^2}}\cos(\theta), \frac{r^2}{\sqrt{2-r^2}}\sin(\theta), r\right)\mathrm{d}\theta\mathrm{d}r$$
$$=2\pi\int_0^1 r\,\mathrm{d}r=\pi$$

A parameterization of ∂S is: $\gamma(1,\theta)=(\cos(\theta),\sin(\theta),1)$, with $0\leqslant\theta\leqslant 2\pi$. This is the same as saying $r=1$. Then:

$$\int_{\partial S}F\bullet T\,\mathrm{d}s=\int_0^{2\pi}(1,\cos(\theta),\sin(\theta))\bullet(-\sin(\theta),\cos(\theta),0)\,\mathrm{d}\theta$$
$$=\int_0^{2\pi}(-\sin(\theta)+\cos^2(\theta))\mathrm{d}\theta=\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\mathrm{d}\theta$$
$$=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\theta=\pi=\int_S\operatorname{curl}(F)\bullet N\,\mathrm{d}\sigma$$

Exp: (*Evaluation of a line integral by Stokes' Theorem*) Evaluate the line integral:

$$\int_C F\bullet T\,\mathrm{d}s$$



where C is the circle $x^2+y^2=4$ in the plane $z=-3$, oriented counter-clockwise and finally, we have that $F(x,y,z)=(y,x\,z^3,-z\,y^3)$ which is a class C^1 vector field on $\mathbb{R}^3$.

Let S be the disk in the plane $z=-3$ of equation $x^2+y^2\leqslant 4$. Then $\partial S=C$. The unit normal vectors to S are $N=(0,0,1)$. At every point in S, we have $F(x,y,z)=(y,-27x,3y^3)$. Thus on S, we have:

$$\operatorname{curl}(F)\bullet N=\frac{\partial}{\partial x}(-27x)-\frac{\partial}{\partial y}(y)=-27-1=-28$$
$$\int_C F\bullet T\,\mathrm{d}s=\int_{\partial S}F\bullet T\,\mathrm{d}s=\int_S \operatorname{curl}(F)\bullet N\,\mathrm{d}\sigma=-28\int_C \mathrm{d}\sigma$$
$$=-28\,\mathrm{Area}(S)=(-28)\,(4\pi)=-112\pi$$

Rmk: *Green's Theorem in $\mathbb{R}^2$ as a special case of Stokes' Theorem:*

Let $F=(F_1,F_2)$ be a class C^1 vector field on a regular set D in $\mathbb{R}^2$ whose ∂D is a piecewise smooth simple closed curve oriented counter-clockwise. Then:

$$\int_D \operatorname{curl}(F)\,\mathrm{d}A=\int_D\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathrm{d}A$$
$$=\int_D \operatorname{curl}(F)\bullet(0,0,1)\,\mathrm{d}A\overset{\text{Stokes}'\text{Theorem}}{=\!=\!=\!=}\int_{\partial D}F\bullet T\,\mathrm{d}s$$
$$=\int_{\partial D}F_1\mathrm{d}x+F_2\mathrm{d}y$$

April 28th, 2021

Chapt. 8: Differential Forms

1-Forms: A differential 1-form (or simply 1-form) on an open subset of $\mathbb{R}^2$ is an expression:

$$F(x,y)\mathrm{d}x+G(x,y)\mathrm{d}y$$

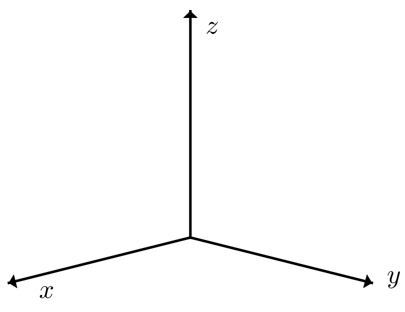
where $F,G\colon\mathbb{R}^2\longrightarrow\mathbb{R}$ are real-valued functions.

Exp: The total differential (called also the exterior derivative), denoted as such:

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y\,\text{is a 1-form}$$

Rmk: A 1-form is very similar to a vector field. In fact, we can set up a correspondence:

$$F\,i+G\,j+H\,k\longleftrightarrow F\mathrm{d}x+G\mathrm{d}y+H\mathrm{d}z$$
$$\text{Vector field in }\mathbb{R}^3\longleftrightarrow\text{1-form derivative}$$



$$\nabla f=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}k\longleftrightarrow\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\mathrm{d}z$$

May 3rd, 2021

Def.: *Exact and Closed:* A 1-form $F\mathrm{d}x + G\mathrm{d}y$ on $\mathbb{R}^2$ with C^1 coefficients (ie. F and G are C^1) is said to be exact if $\exists$ some C^2 function $f(x, y)$ with the total differentian $\mathrm{d}f = F\mathrm{d}x + G\mathrm{d}y$, ie. $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$. A 1-form is said to be closed if $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$

Rmk:

1. In terms of differential form, the vector field $F = (F_1, F_2)$ is conservative when $F_1\mathrm{d}x + F_2\mathrm{d}y$ is exact.

2. If a 1-form exact, then it is automatically closed (ie. Exact $\Longrightarrow$ Closed). In fact, if $\exists f$ function of class C^2 st. $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$, then we will have:

$$\frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial G}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

   But since the function is of class C^2, then we know that they are equal. In other words, we have that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, or alternatively, $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$.

Important Exp: Let A be an open subset of $\mathbb{R}^n$ and let $f\colon A \longrightarrow \mathbb{R}$ be a smooth function. We associate to f the 1-form given by:

$$\mathrm{d}f = f_{x_1}\mathrm{d}x_1 + f_{x_2}\mathrm{d}x_2 + \cdots + f_{x_n}\mathrm{d}x_n$$

If $\gamma\colon [a, b] \longrightarrow A$ which maps $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$ is a parameterization of A, then:

$$\begin{aligned}\int_A \mathrm{d}f &= \int_a^b (f_{x_1} \circ \gamma_1)\gamma_1' + \cdots + (f_{x_n} \circ \gamma_n)\gamma_n' \\ &= \int_a^b (f \circ \gamma)' = (f \circ \gamma)|_a^b\end{aligned}$$

This is by the chain rule in the coordinates. Hence, the 1-form $\mathrm{d}f$ measures the changes in f along curves.

In general: let $F = (F_1, F_2, \ldots F_n)$ be a vector field on A. We consider the 1-form:

$$\begin{aligned}&\omega = F_1\mathrm{d}x_1 + F_2\mathrm{d}x_2 + \cdots + F_n\mathrm{d}x_n \\ \Longrightarrow &\int_A \omega = \int_{t=a}^{t=b} \left(\sum_{i=1}^n (F_i \circ \gamma_i)\gamma_i'\right)(t)\end{aligned}$$

which is the flow of F along the curve A.

Exp: Let $\omega = x\mathrm{d}y - y\mathrm{d}x$ be a 1-form on $\mathbb{R}^2$ and let:

$$\gamma: [-1, 1] \longrightarrow \mathbb{R}^2$$
$$t \mapsto \gamma(t) = (t^2 - 1, t^3 - t)$$

$$\int_{\gamma([-1,1])} \omega = \int_{-1}^{1} -(t^3 - t)2t\,\mathrm{d}t + \int_{-1}^{1} (t^2 - 1)(3t^2 - 1)\mathrm{d}t$$

$$(F_1 \circ \gamma)(t) = -(t^3 - t) \text{ and } \gamma_1'(t) = 2t$$
$$\text{and}$$
$$(F_2 \circ \gamma)(t) = (t^2 - 1) \text{ and } \gamma_2'(t) = (3t^2 - 1)$$

$$= \int_{-1}^{1} (t^4 + 2t^3 - 4t^2 + 1)\mathrm{d}t$$

Def.: *2-forms:* A 2-form is an expression built using the "wedge product" of pairs of 1-forms. The wedge product, denoted $\wedge$, satisfies:

- $u \wedge v = -v \wedge u$
- $u \wedge u = 0$
- $c(u \wedge v) = (c\,u) \wedge v = u \wedge (c\,v)$
- $(u + v) \wedge w = (u \wedge w) + (v \wedge w)$

Exp:

$$(3\mathrm{d}x + \mathrm{d}y) \wedge (e^x \mathrm{d}x + 2\mathrm{d}y)$$

$$3e^x \mathrm{d}x \wedge \mathrm{d}x + 6\mathrm{d}x \wedge \mathrm{d}y + e^x \mathrm{d}y \wedge \mathrm{d}x + 2\mathrm{d}y \wedge \mathrm{d}y$$
$$= (6 - e^x)\mathrm{d}x \wedge \mathrm{d}y$$

Rmk: A vector field $F = F_1 i + F_2 j + F_3 k$ can be converted to a 2-form and back:

$$F_1 i + F_2 j + F_3 k \longleftrightarrow F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y$$
$$i = j \times k \longleftrightarrow \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z$$

Important: Consider the 1-form $\omega = F(x, y, z)\mathrm{d}x + G(x, y, z)\mathrm{d}y + H(x, y, z)\mathrm{d}z$. We want to find the exterior derivative $\mathrm{d}\omega$ which will be a 2-form. TO do so, we use the following rules: Let α, β be 1-forms and let f be a function. Then:

$$\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$$
$$\mathrm{d}(f\alpha) = (\mathrm{d}f) \wedge \alpha + f\mathrm{d}\alpha$$
$$\mathrm{d}(\mathrm{d}x) = 0 = \mathrm{d}(\mathrm{d}y) = \mathrm{d}(\mathrm{d}z)$$

Then, we will have the following:

$$\begin{aligned}
\mathrm{d}\omega =& \mathrm{d}(F\mathrm{d}x)+\mathrm{d}(G\mathrm{d}y)+\mathrm{d}(H\mathrm{d}z)\\
=&(\mathrm{d}F)\wedge\mathrm{d}x+F\mathrm{d}(\mathrm{d}x)+(\mathrm{d}G)\wedge\mathrm{d}y+G\mathrm{d}(\mathrm{d}y)\\
&+(\mathrm{d}H)\wedge\mathrm{d}z+H\mathrm{d}(\mathrm{d}z)\\
\\
=&(\mathrm{d}F)\wedge\mathrm{d}x+(\mathrm{d}G)\wedge\mathrm{d}y+(\mathrm{d}H)\wedge\mathrm{d}z\\
=&\left(\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y+\frac{\partial F}{\partial z}\mathrm{d}z\right)\wedge\mathrm{d}x\\
&+\left(\frac{\partial G}{\partial x}\mathrm{d}x+\frac{\partial G}{\partial y}\mathrm{d}y+\frac{\partial G}{\partial z}\mathrm{d}z\right)\wedge\mathrm{d}y\\
&+\left(\frac{\partial H}{\partial x}\mathrm{d}x+\frac{\partial H}{\partial y}\mathrm{d}y+\frac{\partial H}{\partial z}\mathrm{d}z\right)\wedge\mathrm{d}z\\
=&\left(-\frac{\partial F}{\partial y}\mathrm{d}z+\frac{\partial F}{\partial z}\mathrm{d}y\right)+\left(\frac{\partial G}{\partial x}\mathrm{d}z-\frac{\partial G}{\partial z}\mathrm{d}x\right)+\left(\frac{\partial H}{\partial y}\mathrm{d}x-\frac{\partial H}{\partial x}\mathrm{d}y\right)
\end{aligned}$$

Important: If f is a C^2 real-valuef function on an open subset of $\mathbb{R}^3$, then $\mathrm{d}(\mathrm{d}f)=0$. Similarly, if a 1-form $\omega=F\mathrm{d}x+G\mathrm{d}y+H\mathrm{d}z$ where F,G,H are C^2, then $\mathrm{d}(\mathrm{d}\omega)=0$. In fact:

$$\begin{aligned}
\mathrm{d}f =& f_x\mathrm{d}x+f_y\mathrm{d}y+f_z\mathrm{d}z\\
\mathrm{d}(\mathrm{d}f)=&\mathrm{d}(f_x\mathrm{d}x)+\mathrm{d}(f_y\mathrm{d}y)+\mathrm{d}(f_z\mathrm{d}z)\\
\\
&\text{For } f_x\mathrm{d}x:\\
(\mathrm{d}f_x)\wedge\mathrm{d}x+f_x\mathrm{d}(\mathrm{d}x)=&(\mathrm{d}f_x)\wedge\mathrm{d}x\\
=&-\frac{\partial^2 f}{\partial y\partial x}\mathrm{d}x\wedge\mathrm{d}y+\frac{\partial^2 f}{\partial z\partial x}\mathrm{d}z\wedge\mathrm{d}x
\end{aligned}$$

The same applies for f_y and f_z. Thus:

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f)=&(f_{yx}-f_{xy})\mathrm{d}x\wedge\mathrm{d}y+(f_{zy}-f_{yz})\mathrm{d}y\wedge\mathrm{d}z+(f_{xz}-f_{zx})\mathrm{d}z\wedge\mathrm{d}x\\
=&0
\end{aligned}$$

Also:

$$\mathrm{d}(\mathrm{d}\omega)=[G_{xz}-F_{yz}+H_{yx}-G_{zx}+F_{zy}-H_{xy}]\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=0$$

In terms of vector notation, this is equivalent to:

$$\begin{aligned}
\nabla_x(\nabla f)=&\operatorname{curl}(\nabla f)=O_{\mathbb{R}^3}\\
&\text{and}\\
&\operatorname{div}(\operatorname{curl}(V))=0
\end{aligned}$$

May 5th, 2021

Poincaré's Lemma for 1-forms:

If $\omega=F\mathrm{d}x+G\mathrm{d}y+H\mathrm{d}z$ is a closed 1-form on $\mathbb{R}^3$ with C^1 coefficients, then ω is exact. In fact:

$$\text{if } f(x_0,y_0,z_0)=\int_C\omega$$

where C is any piecewise C^1 curve connecting $(0,0,0)$ to (x_0, y_0, z_0), then $\mathrm{d}f = \omega$.

Poincaré's Lemma for 2-forms:

If ω is a 2-form on $\mathbb{R}^3$ st. $\mathrm{d}\omega = 0$, then $\exists$ a 1-form ς st. $\mathrm{d}\varsigma = \omega$. In terms of vector notation, this is equivalent to saying that: if a vector field $F = \nabla g$, then $\mathrm{curl}(F) = O_{\mathbb{R}^3}$.

Surface Integrals: Let S be a smooth parameterized surface in $\mathbb{R}^3$ given by:

$$x = x(u,v), y = y(u,v), z = z(u,v)$$

and (u,v) in $D \subseteq x$-y plane, ie. $D \subseteq \mathbb{R}^3$. Then we have the following:

$$\begin{aligned}\mathrm{d}x &= \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\\ \mathrm{d}y &= \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\end{aligned}$$

Then:

$$\begin{aligned}\mathrm{d}x \wedge \mathrm{d}y &= \left(\frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v\right) \wedge \left(\frac{\partial y}{\partial u}\mathrm{d}u + \frac{\partial y}{\partial v}\mathrm{d}v\right)\\ &= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathrm{d}u \wedge \mathrm{d}v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u \wedge \mathrm{d}v\\ &= \frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u \wedge \mathrm{d}v\end{aligned}$$

In this way it is possible to convert any 2-form to $u - v$ coordinates.

Def.: The integral of a 2-form on S is given by:

$$\begin{aligned}&\int_S F\mathrm{d}x \wedge \mathrm{d}y + G\mathrm{d}y \wedge \mathrm{d}z + H\mathrm{d}z \wedge \mathrm{d}x\\ &= \int_D \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)}\right]\mathrm{d}u\mathrm{d}v\end{aligned}$$

Important: In practice, the integral of a 2-form can be calculated by first converting it to the form $f(u,v)\mathrm{d}u \wedge \mathrm{d}v$ and then evaluate:

$$\int_D f(u,v)\mathrm{d}u\mathrm{d}v$$

wrt. the previous definition of $\omega = F\mathrm{d}x \wedge \mathrm{d}y + G\mathrm{d}y \wedge \mathrm{d}z + H\mathrm{d}z \wedge \mathrm{d}x$,

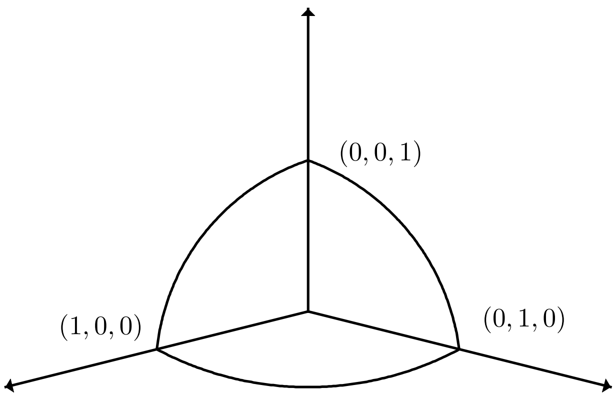
$$\begin{aligned}&= F\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u \wedge \mathrm{d}v + G\frac{\partial(y,z)}{\partial(u,v)}\mathrm{d}u \wedge \mathrm{d}v + H\frac{\partial(z,x)}{\partial(u,v)}\mathrm{d}u \wedge \mathrm{d}v\\ &= \left[F\frac{\partial(x,y)}{\partial(u,v)} + G\frac{\partial(y,z)}{\partial(u,v)} + H\frac{\partial(z,x)}{\partial(u,v)}\right]\mathrm{d}u \wedge \mathrm{d}v\\ &= f(u,v)\mathrm{d}u \wedge \mathrm{d}v\end{aligned}$$

Exp: Let S be the oriented (N is pointing outward S) upper half of the unit sphere in $\mathbb{R}^3$, parameterized using the spherical coordinates:

$$\begin{aligned}
x = x(\varphi,\theta) &= \sin(\varphi)\cos(\theta)\\
y = y(\varphi,\theta) &= \sin(\varphi)\sin(\theta)\\
z = z(\varphi,\theta) &= \cos(\varphi)\\
\text{with } 0 \leqslant \varphi \leqslant \frac{\pi}{2} \text{ and } &0 \leqslant \theta \leqslant 2\pi
\end{aligned}$$

Then:

$$\begin{aligned}
\mathrm{d}x \wedge \mathrm{d}y &= \frac{\partial(x,y)}{\partial(\varphi,\theta)} = \cos(\varphi)\sin(\varphi)\mathrm{d}y \wedge \mathrm{d}\theta\\
\int_S \mathrm{d}x \wedge \mathrm{d}y &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\varphi)\sin(\varphi)\mathrm{d}\varphi\mathrm{d}\theta\\
&= \int_0^{2\pi}\left[\left[\frac{1}{2}\sin^2(\varphi)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\right]\mathrm{d}\theta
\end{aligned}$$



Rmk: If an oriented surface S has two different smooth C^1 parameterizations, then for any 2-form ω, the surface integral of ω over S calculated wrt. both parameterizations agree.

Important:

1. Let $F = (F_1, F_2, F_3)$ be a vector field in $\mathbb{R}^3$. Then the flux of F across S can be seen as follows:

$$\int_S F \bullet N \mathrm{d}\sigma = \int_S F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y$$

In fact, we first convert F to a 2-form:

$$F = F_1 i + F_2 j + F_3 k \longleftrightarrow F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y$$

and then we integrate.

$$\begin{aligned}
N\mathrm{d}\sigma &= (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)\\
F \bullet N\mathrm{d}\sigma &= F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}$$

2. ***Generalized Stokes' Theorem:***

Let S be an oriented smooth surface with smooth boundary ∂S. Then for every C^1 1-form ω on an open set of $\mathbb{R}^3$ containing S, we have:

$$\int_S \mathrm{d}\omega = \int_{\partial S} \omega$$

Applications:

1. In $\mathbb{R}^3$:

$\omega = F_1 \mathrm{d} \wedge \mathrm{d}y + F_2 \mathrm{d}y \wedge \mathrm{d}z + F_3 \mathrm{d}x \wedge \mathrm{d}y$ is a 2-form. Thus $\int_S \omega$ can be seen as the surface integral of a vector field $F = F_1 i + F_2 j + F_3 k$ over S.

$$\int_{\partial S} \omega = \int_{\partial S} F \bullet N \mathrm{d}\sigma$$

On the other hand:

$$\begin{aligned}\int_{\partial S} \mathrm{d}\omega &\overset{(*)}{=\!=} \int_S \mathrm{d}[F_1 \mathrm{d}y \wedge \mathrm{d}z + F_2 \mathrm{d}z \wedge \mathrm{d}x + F_3 \mathrm{d}x \wedge \mathrm{d}y] \\ &\int_S \left(\frac{\partial F_1}{\partial x}\mathrm{d}x + \frac{\partial F_1}{\partial y}\mathrm{d}y + \frac{\partial F_1}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &+ \left(\frac{\partial F_2}{\partial x}\mathrm{d}x + \frac{\partial F_2}{\partial y}\mathrm{d}y + \frac{\partial F_2}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\ &+ \left(\frac{\partial F_3}{\partial x}\mathrm{d}x + \frac{\partial F_3}{\partial y}\mathrm{d}y + \frac{\partial F_3}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= \int_S \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \overset{(**)}{=\!=} \int_S \operatorname{div}(F) \mathrm{d}V \\ (*) \text{ and } (**) &\Longrightarrow \int_{\partial S} F \bullet N \,\mathrm{d}s = \int_S (\operatorname{div}(F)) \mathrm{d}V\end{aligned}$$

2. In $\mathbb{R}^2$:

$\omega = F_1 \mathrm{d}x + F_2 \mathrm{d}y + F_3 \mathrm{d}z$

$$\int_{\partial S} \omega = \int_{\partial F} F_1 \mathrm{d}s + F_2 \mathrm{d}y + F_3 \mathrm{d}z$$

On the other hand:

$$\begin{aligned}\int_S \mathrm{d}\omega &= \int_S \mathrm{d}(F_1 \mathrm{d}x + F_2 \mathrm{d}y + F_3 \mathrm{d}z) \\ &= \int_S \left(\frac{\partial F_1}{\partial x}\mathrm{d}x + \frac{\partial F_1}{\partial y}\mathrm{d}y + \frac{\partial F_1}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \\ &+ \left(\frac{\partial F_2}{\partial x}\mathrm{d}x + \frac{\partial F_2}{\partial y}\mathrm{d}y + \frac{\partial F_2}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\ &+ \left(\frac{\partial F_3}{\partial x}\mathrm{d}x + \frac{\partial F_3}{\partial y}\mathrm{d}y + \frac{\partial F_3}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\ &= \int_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x \\ &+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right) \mathrm{d}z \wedge \mathrm{d}y = \int \operatorname{curl}(F) \bullet N \mathrm{d}\sigma\end{aligned}$$

3. In $\mathbb{R}$:

Let ω be a 0-form ie. $\omega = f$, and S is the interval $S = [a, b]$

$$\int_S \mathrm{d}\omega = \int_{\partial S} \omega$$

$$\text{If } \omega = f \longleftarrow \text{0-form:}$$

$$\mathrm{d}\omega = \mathrm{d}f = f'(x)\mathrm{d}x. \text{ Thus:}$$

$$\int_{[a,b]} f'(x)\mathrm{d}x = \int_{\{a,b\}} f = \int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$$

It is the sum of the values of f at the endpoints.
