MTH418 - Homework II

by Dara Varam

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Question 1:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

i. For G_1:

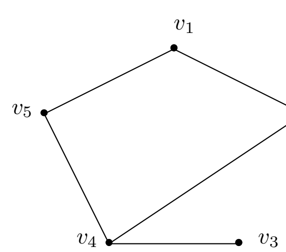
$$\deg(v_1) = 2, \deg(v_2) = 2, \deg(v_3) = 1, \deg(v_4) = 3, \deg(v_5) = 2$$

For G_2:

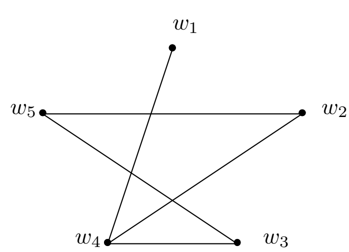
$$\deg(v_1) = 1, \deg(v_2) = 2, \deg(v_3) = 2, \deg(v_4) = 3, \deg(v_5) = 2$$

ii. Drawing G_1 and G_2:

G_1:



G_2:



iii. Construct a mapping from G_1 to G_2 to show isomorphism:

$$f: G_1 \longrightarrow G_2$$

$$\begin{aligned} f(v_1) &= w_5 \\ f(v_2) &= w_2 \\ f(v_3) &= w_1 \\ f(v_4) &= w_4 \\ f(v_5) &= w_3 \end{aligned}$$

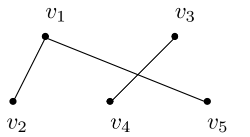
iv. Is G_1 or G_2 a $K_{m,n}$ for some $m, n \in \mathbb{Z}^+$? Draw them if so.

Assume G_1 is $K_{m,n}$ for some $m, n, \in \mathbb{Z}^+$. Then it has exactly $m + n$ vertices and $m \times n$ edges. Since we know that G_1 has 5 edges, $m \times n = 5$. This means that $m = 1, n = 5$ or $m = 5, n = 1$. In either case, we have that the total number of <u>vertices</u> is $m + n = 1 + 5 = 6$, but G_1 only has 5 vertices. A contradiction. Therefore G_1 is NOT $K_{m,n}$.

Similarly for G_2, we proceed by contradiction. Assume G_2 is $K_{m,n}$. Then $m \times n = 5 \Longrightarrow m = 1, n = 5$ or $m = 5, n = 1$. This implies that the number of vertices is $m + n = 6$, but we only have 6 vertices.

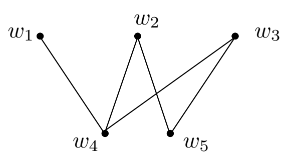
Another argument: Since we showed through the mapping of f that $G_1 \approx G_2$, then if G_1 is not $K_{m,n}$, automatically G_2 is not either.

For G_1:



We have a bipartite graph (can divide into set $A = \{v_1, v_3\}$ and $B = \{v_2, v_4, v_5\}$), but this is NOT a complete bipartite graph.

For G_2:



Once again we have a bipartite graph ($A = \{w_1, w_2, w_3\}$ and $B = \{w_4, w_5\}$) but we do not have a complete bipartite graph.

v. Find the permutation matrix p st $p\,A_1 = A_2\,p$

1. Take I_5:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. $R_1 \mapsto R_5$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. $R_3 \mapsto R_1$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. $R_5 \mapsto R_3$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the p we obtain that satisfies the equation $p\,A_1 = A_2\,p$ is:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

vi. We start with A_1 and perform the following operations:

1. $R_1 \mapsto R_5$
2. $R_3 \mapsto R_1$
3. $R_5 \mapsto R_3$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

4. Take the matrix you obtain here, and call it C. Now, replace the following <u>columns</u> in your new matrix C as follows:
5. $C_1 \mapsto C_5$
6. $C_3 \mapsto C_1$
7. $C_5 \mapsto C_3$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = A_2$$

You will end up with A_2 upon completing all the steps.

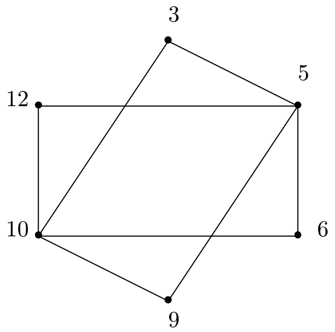
Question 2:

$$V = \{3, 5, 6, 9, 10, 12\}$$

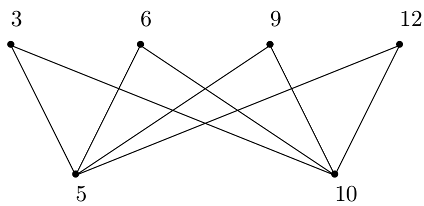
Two vertices a, b are connected by an edge iff $a \cdot b = 0 \in \mathbb{Z}_{15}$ (multiplication modulo 15). We proceed with the multiplication table to be able to draw our graph:

$\times_{15}$	3	5	6	9	10	12
3	9	0	3	12	0	6
5	0	10	0	0	5	0
6	3	0	6	9	0	12
9	12	0	9	6	0	3
10	0	5	0	0	10	0
12	6	0	12	3	0	9

Now we draw the graph:



1. Show that G is a $K_{m,n}$ for some $m,n\in\mathbb{Z}^+$



Choose the sets $A=\{3,6,9,12\}$ and $B=\{5,10\}$. We can see that this graph is a complete bipartite. This is because each of both 5 and 10 are connected to very vertex in the other set, A. Therefore, we can say that G is $K_{m,n}$ for $m=2$ and $n=4$. In other words;

$$G=K_{2,4}$$

2. Find the girth of G:

The shortest cycle in the graph: $3 — 5 — 9 — 10 — 3$. The other cycles in the graph are also of the same length, which is 4. Therefore;

$$\text{girth}(G)=4$$

Another argument: Since $G=K_{2,4}$ with $2,4\geqslant 2$, we have that the shortest cycle length is always 4 (by result introduced in the lecture).

3. Find the diameter of G:

The maximum distance between two vertices in our graph is 2. That means that each pair of vertices are at most 2 edges apart. Therefore;

$$\dim(G)=2$$

Another argument: Once again, by previous result introduced in the lecture, we know that for any complete bipartite graph $K_{m,n}$, $\text{diam}(K_{m,n})=2$.

4. Construct a minimum dominating set of G and determine the dominating number.

Since our graph is $K_{2,4}$, we take one vertex from each subset of vertices, say 10 and 9. Thus we have the dominating set $\{9,10\}$. Every vertex outside of this set is connected to one of the two. Since this set consists of two elements, we have that:

$$\gamma(G)=2$$

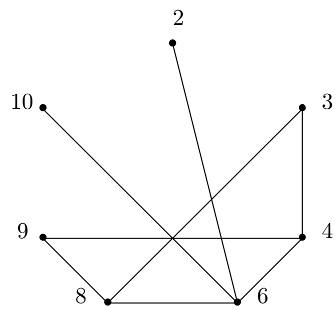
Note that any pair of vertices that from separate vertex subsets can be a dominating set. We could have chosen $\{3,5\}$ to be our dominating set, but $\gamma(G)$ would stay the same.

Question 3:

$$V = \{2,3,4,6,8,9,10\}$$

$\times_{12}$	2	3	4	6	8	9	10
2	4	6	8	0	4	6	8
3	6	9	0	6	0	3	6
4	8	0	4	0	8	0	4
6	0	6	0	0	0	6	0
8	4	0	8	0	4	0	8
9	6	3	0	6	0	9	6
10	8	6	4	0	8	6	4

We draw the graph:

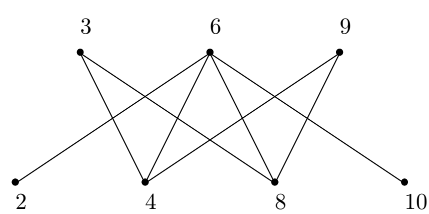


1. Show that G is NOT $K_{m,n}$ for some $m,n \in \mathbb{Z}^+$

   Assume G is $K_{m,n}$ for some $m,n \in \mathbb{Z}^+$. $\implies |E| = m \times n = 8$ (we know this from the graph drawn above).

   We could have $m=2, n=4$ or $m=4, n=2$. In either case, we know that $m+n=6$, but we have 7 edges. A contradiction.

   We could also have $m=8, n=1$ or $m=1, n=8$. $m+n=9 \neq 7$. Still a contradiction. Therefore it is impossible for us to have a complete bipartite graph. However, since we have no odd cycles, we can still construct a bipartite graph from G:



   We have produced a bipartite graph that consists of $A = \{3,6,9\}$ and $B = \{2,4,8,10\}$. This bipartite graph is NOT complete because the vertex 3 is not connected to every vertex in the set B, namely 2 and 10. Similarly, 9 is not connected to 2 and 10.

2. Find the girth of G:

We have two cycles within the graph: $3 — 4 — 6 — 8 — 3$ and $9 — 4 — 6 — 8 — 9$. Both of which are of length 4, and therefore:

$$\text{girth}(G) = 4$$

3. Find the diameter of G:

   The maximum distance between two vertices in our graph G is between the vertex 2 and 3, or 10 and 3, or 10 and 9 or 2 and 9. They all have the same length. We will use the distance between 2 and 3 as an example. The path is:

   $$2 — 6 — 4 — 3$$

   The rest of the pairs also follow in similar fashion. In each case, $d(a, b) = 3$. We do not have any distances longer than that in our graph. Therefore:

   $$\dim(G) = 3$$

4. Construct a minimum dominating set and determine the dominating number of G:

   Consider the set consisting of $\{4, 6\}$. Every vertex outside of this set is either connected to 6 through an edge, or connected to 4 through an edge. We could alternatively go with the dominating set $\{6, 8\}$. In both cases, the same principle applies.

   $$\gamma(G) = 2$$