MTH418 - Graph Theory

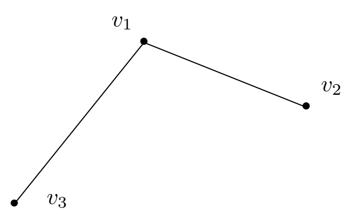
by Dara Varam

Instructor: Dr. Ayman Badawi

February 1st, 2021

Graphs: A graph consists of the following. $G=(V,E)$ where V is the set of vertices and E is the set of edges. Most of the semester, we will be dealing with undirected simple graphs.

We usually refer to vertices by dots, such as the following: $\bullet$. Every graph consists of both vertices and edges. Let us look at an example of a graph.



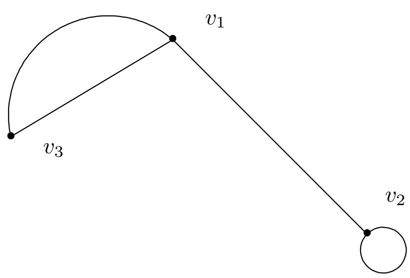
A vertex is each of the v_1, v_2, v_3 shown on the graph above. On the other hand, an edge is a line segment that connects two vertices. In the graph above, we have three vertices and two edges, and they are denoted as follows:

$$E = \{v_1 - v_2, v_1 - v_3\}$$

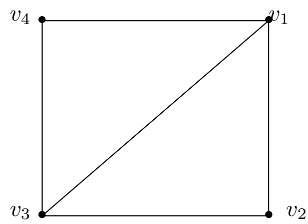
We could also use the following notation to denote edges: $E = \{\{v_1, v_2\}, \{v_1, v_3\}\}$. In our case, we have that $|V| = 3$ and $|E| = 2$.

What do we mean when we say that a graph is undirected? It means that there essentially is no arrow. There is no difference between $v_1 - v_2$ and $v_2 - v_1$. Later on, we will see examples of graphs that are directed. In that case, the aforementioned edges are distinct.

What do we mean when we say that a graph is simple? Vertices do not have loops, meaning that they do not go to themselves, and there is only at most <u>one</u> edge between any two vertices. Let us see an example of a graph that is NOT simple.



Clearly v_2 goes to itself and there are two edges between v_3 and v_1. Therefore, our graph is not simple, although it is still undirected. Consider the following graph:

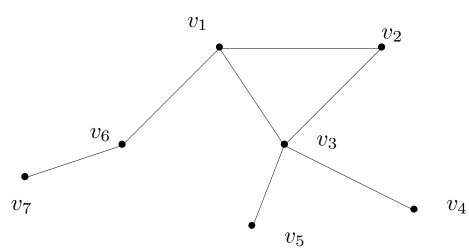


Clearly we can see that we have $G=(V,E)$. By staring, we have that this graph is both undirected and simple. The sets are given as follows:

$$V=\{v_1,v_2,v_3,v_4\}$$
$$E=\{v_1-v_2,v_2-v_3,v_3-v_4,v_1-v_4,v_1-v_3\}$$

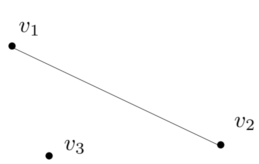
Clearly it is obvious that $|V|=4$ and $|E|=5$.

Consider the graph shown below:



In general, the degree of a vertex v_i is the number of edges that are connected to it. For example, we have the following: $\deg(v_1)=3$, $\deg(v_2)=2=\deg(v_6)$. We also have that $\deg(v_4)=\deg(v_5)=\deg(v_7)=1$. Finally, $\deg(v_3)=4$. These are the degrees both each of the 7 vertices in our graph. Note that, once again, the graph is both simple and undirected.

Now, look at the example provided:



Then we have that $\deg(v_3)=0$. Note that 0 is an even number.

Fact: The sum of the degrees of each vertex in a graph is equal to 2 times the number of edges. Mathematically:

$$\sum_{i=1}^{n}\deg(v_i)=2\times|E|$$

Similarly, we can rearrange this to get the number of edges in a graph: $|E|=\frac{\sum_{i=1}^{n}\deg(v_i)}{2}$.

Proof: Since each edge is counted twice and calculating all the degrees, then we can divide by 2 to get the number of edges. This should be common knowledge in graph theory.

Question: Let K be the number of vertices that have odd degrees. Convince me that K is an even number. In other words, what we are trying to say is that we cannot have a graph with 6 vertices, where $\deg(v_1) = 1, \deg(v_2) = 3, \deg(v_3) = 1, \deg(v_4) = 2, \deg(v_5) = 4, \deg(v_6) = 2$. We cannot have such a graph. Why?

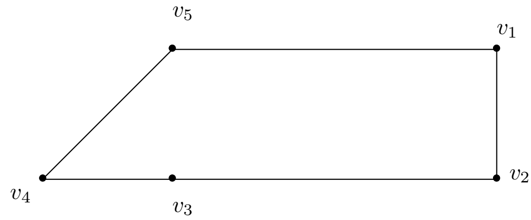
Clearly we can see that our K in the example provided is 3. If our claim is true, then this example cannot result in a graph.

Solution: Since the sum of the degrees is $2 \times |E|$, then it must be an even number. However, if we have an odd number of vertices with odd degrees, then we cannot have an even number as the sum. Let us take $O = \{\text{set of all vertices with odd degrees}\}$ and $N = \{\text{set of all vertices with even degrees}\}$. Mathematically, we have that:

$$\sum_{v \in O} \deg(v) + \sum_{v \in N} \deg(v) = \sum_{i=1}^{n} \deg(v_i) = 2 \times |E|$$

Thus the sum of the degrees of vertices from O and N must produce an even number, and there we have our solution.

Look at the graph below:



Our question is to find the distance between the vertices v_1 and v_4. i.e. Find $d(v_1, v_4)$. Note that this is an unweighted graph, meaning that the edges do not have a numerical value associated to their "weights." In that case, what exactly do we mean by the distance?

$$d(v_1, v_4) = \text{length of shortest path}$$

Paths: Let us take an arbitrary example, between two vertices V and W.

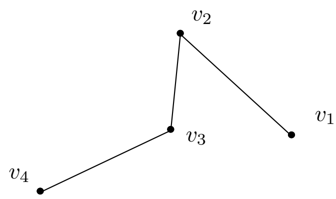
$$V — v_1 — v_2 — W$$

A path is a sequence of edges from V to W. Every edge is a path, but not every path is an edge. For example, in the above graph, we can see that $v_1 — v_5 — v_4$ is a path, but it is not an edge, since it is not between **2** vertices. The length of a path is the number of edges you use to go from one edge to another.

With that being made clear, we can see that there are two different paths between v_1 and v_4, but the distance is the length of the <u>shortest</u> path, which would be 2.

$$v_1 — v_5 — v_4$$

Look at the following graph:



We can see that $d(v_1, v_4) = 3$. Between every two vertices, there is only one path. This means that there is only one direction you can take. This is not the same as the graph before this one, where there were multiple paths to take.

February 3rd, 2021

Recall from the last lecture that:

$$\sum_{i=1}^{n} \deg(v_i) = 2 \times |E|$$

Question: Can we construct a graph with the following degrees? 5, 6, 4, 4, 5, 3, 2

Solution: No, we have 3 vertices that have odd degrees (5, 5, 3). From the previous lecture, we know that we cannot have an odd number of vertices with odd degrees. In other words, we need to have an even number of vertices with odd degrees.

Question: Can we construct a graph with the following degrees? 4, 4, 6, 2, 2, 4, 2, 2

Solution: We have an algorithm that we can use with any graph to see that if we can construct it. We can use it for any question of the above type. This means that we can use it for the first question as well, even though we knew that we cannot construct that graph because of the fact that we had an odd number of vertices with odd degrees. This is the *Hakimi-Havel Algorithm*:

1. Arrange the degrees in descending order. In this example, it would be: 6, 4, 4, 4, 2, 2, 2, 2

2. We select the next 6 degrees after the first one and reduce the degree of each of them by 1. This would result in the following:

$$6, 3, 3, 3, 1, 1, 1, 2$$

3. If we can figure it out here, then we stop. If not, we remove the vertix with highest degree, and repeat the process (arrange in descending order). First, we take the next 3 degrees (after removing the first), and so on.

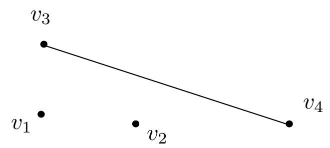
$$3, 3, 3, 2, 1, 1, 1$$

$$2, 2, 1, 1, 1, 1$$

$$1, 1, 1, 1, 0$$

$$1, 1, 0, 0$$

Clearly, at this stage, we can see that we can construct a graph with degrees 1, 1, 0, 0. It would look like the following:



Therefore, we can conclude that since we have created a valid graph of the reduced form, we can create a graph with vertices of degree 6, 3, 3, 3, 1, 1, 1, 2. This is the idea behind our algorithm. This algorithm works for simple, undirected graphs.

Where do we stop? When we see a number become negative, then we can quickly see that we have to stop the algorithm. By another logic, we can also stop when we have something like a vertex of degree 1 and every other vertex has degree 0. In that case, it would be illogical and we obviously have to stop.

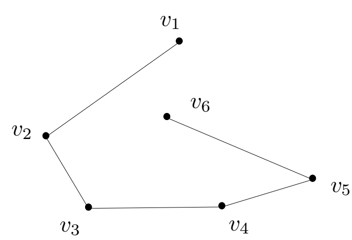
Question: Can we construct a graph with the following degrees? 4, 2, 2, 0, 2

Solution: We apply the algorithm to the degrees:

$$\begin{gathered} 4, 2, 2, 2, 0 \\ 1, 1, 1, -1 \end{gathered}$$

We immediately stop because we have a vertex with negative degree. Therefore, the answer is no. There is no graph with the mentioned degrees, by the algorithm we have used.

Def.: *Connected Graphs:* A graph, $G = (V, E)$ is connected iff there is a path between every two vertices. Consider the following example:



Our graph is clearly connected because there is a path between each of the 6 vertices. However, note that this doesn't mean there is an edge between them. Recall that every edge is a path but not every path is an edge. Consider the following two paths:

$$\begin{gathered} v_2 — v_3 — v_4 — v_5 — v_6 \\ v_2 — v_3 — v_4 — v_3 — v_4 — v_5 — v_6 \end{gathered}$$

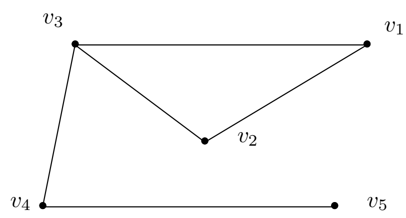
The difference between the two (in the example shown above) is that the second one has repeated vertices, while the first does not. This is the difference between a walk and a path.

Path: $v_1 — v_2 — \ldots — v_n$ is a path. All the vertices are distinct except for v_1 and v_n (They could be the same, which would make it a cycle). This means that we do not go through a vertex more than once in a path.

Walk: There is no restriction in terms of the vertices we visit. Vertices may appear more than once. In other words, a walk is a path in which vertices can appear more than once.

Def.: *Cycles:* Consider the path $v_1 — v_2 — \ldots — v_n$. This path is a cycle if we have that $v_1 = v_n$. In other words, the path starts and ends at the same vertix. Note that this means v_1 is a repeated vertex, although this is not an issue. It is still a path.

Consider the following graph:



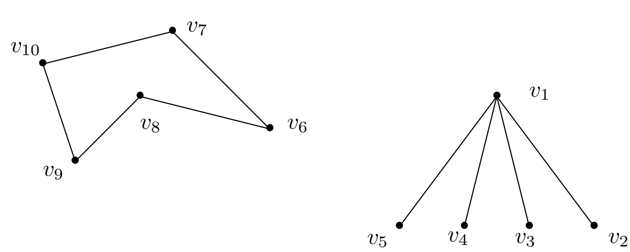
In this case, the following path: $v_1 — v_2 — v_3 — v_1$ is a cycle. Now consider the sequence:

$$v_1 — v_2 — v_3 — v_4$$

This is obviously a path. But it is also a walk, because every path is a walk, but not every walk is a path. Now, to demonstrate a walk, consider the sequence shown below:

$$v_1 — v_3 — v_2 — v_1 — v_3 — v_4 — v_5$$

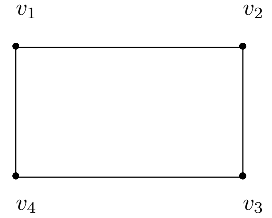
Since we have repeated vertices, this sequence is clearly a walk and NOT a path. Now, consider the graph given below:



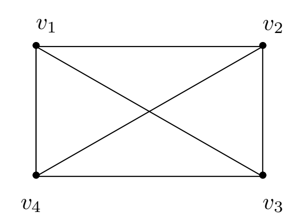
This graph is NOT connected, because we don't have a path between each two vertices. However, you can observe that there are two components, each of which are connected graphs. In other words, this graph actually consists of two connected subgraphs.

Def.: *Complete Graphs:* A connected graph is called complete iff every two vertices are connected by an *edge* (Not to be mistaken with a path).

The difference between a complete and connected graph is that the complete graph has an edge between every pair of vertices, while a connected graph does not necessarily have this. Furthermore, every complete graph is connected, but not every connected graph is complete. Observe the following examples.



Connected but not complete



Complete graph with n vertices

Notation: A complete graph with n vertices is denoted by K_n. For example, K_4 is a complete graph with 4 vertices. This is the same as the graph shown on the right.

We know the following fact: In a complete graph, K_n, each vertex has degree equal to $n-1$, where $n \geqslant 2$.

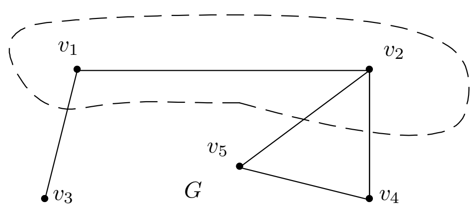
February 8th, 2021

Trail: Every trail is a walk, but not every walk is a trail. In a trail, you have to visit every edge once, but we cannot visit the same edge more than once. In walks, we can (obviously) visit edges more than once.

Recall the *Hakimi-Havel* algorithm: Where we check to see whether a sequence of positive integers form a simple, undirected graph.

Def.: *Subgraphs*

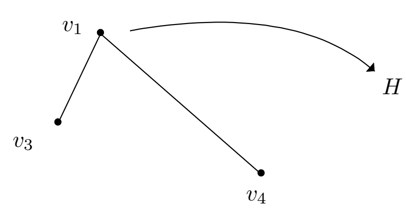
Consider a graph $G = (V, E)$, and another graph $H = (V_1, E_1)$. We say that H is a subgraph of G iff $V_1 \subseteq V$ and $E_1 \subseteq E$. Consider the following example:



Consider H to be the part of the graph consisting of v_1 and v_2. Is H a subgraph of G? Yes, because:

$$\{v_1, v_2\} \subseteq \{v_1, v_2, \ldots, v_5\} \text{ and } E_1 = \phi \subseteq E$$

Now, look at the following graph:



Is H a subgraph of the original graph? Let us look at the two conditions.

$$v_1 = \{v_1, v_3, v_4\} \subseteq V, \text{but } E_1 \nsubseteq E$$

Therefore, H is NOT a subgraph of G.

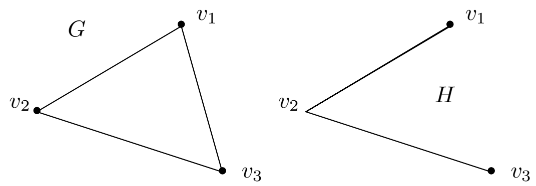
Def.: *Induced Subgraphs*

Consider the graph $G=(V,E)$. We say that $H=(V_1,E_1)$ is an induceed subgraph of G if the two conditions hold:

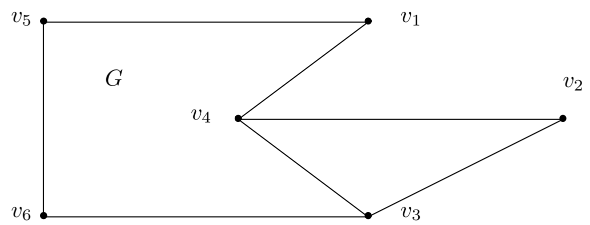
1. H is a subgraph of G

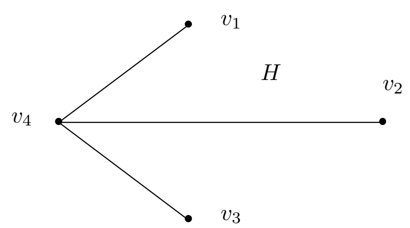
2. $e \in E_1$ iff $e \in E$, where e is an edge.

Consider the following example to understand the second condition:



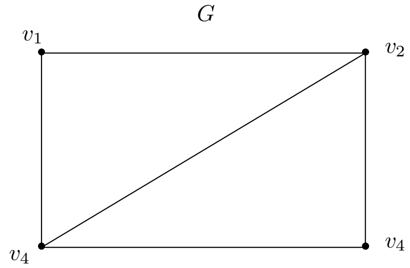
We have that H is a subgraph of G but it is NOT an induced subgraph. Why? If v_1 and v_3 are connected in the original graph, then they must be connected in the induced subgraph. Clearly in our example, H is not induced because v_1 and v_3 are not connected through an edge.





We have that (by staring) H is a subgraph of G. However, H is NOT an induced subgraph because in G we have an edge between v_3 and v_2. This edge does not exist in H. If we wanted H to be an induced subgraph, we would have to remove v_2 and the edge $v_4 — v_2$.

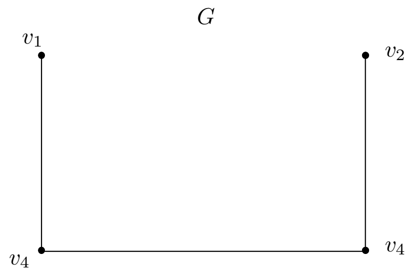
One way to think of an induced subgraph is to think of the same graph, with some of the vertices removed. Let us look at another example:



If we remove the edge $v_4 — v_2$, then we would have a subgraph H, but it would not be induced bceause of the fact that we have an edge missing.

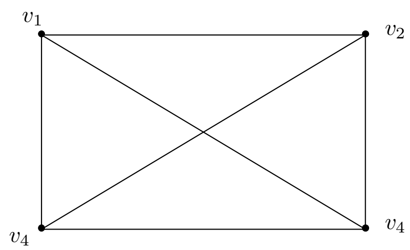
Def.: *Spanning Subgraph*

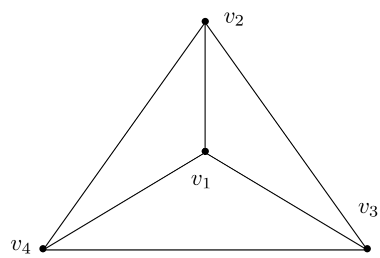
Consider the graph $G = (V, E)$ and another graph $H = (V_1, E_1)$. H is called a spanning subgraph iff $V_1 = V$. This means that we have the same vertices, but the edges can be removed. The set of vertices in the subgraph is the same set as the original, but this is not necessarily the case for the set of edges. Look at the same example as the previous:



This is clearly a subgraph, but it is NOT induced. However, it IS a spanning subgraph because we still have all the vertices v_1, v_2, v_3 and v_4. Spanning subgraphs and general subgraphs are easy, but the only one we need to be careful about is the induced subgraphs.

Recall the definition of a complete graph: A connected graph in which every two vertices are connected by an edge. This means that every pair of vertices are connected by an edge. Notation: K_n where n is the number of vertices. For example, K_4:





Note that both of these are examples of complete graphs with 4 vertices. We can consider both of them as K_4. There is however, more than one way of drawing these graphs.

Recall the definition of a connected graph: There exists a path between any two vertices within the graph. It does not necessarily have to be complete to be connected. We can look at the following graph:

We can see that the graph is connected, but because of the missing diagonals, it is NOT complete.

Fact: Let E be the total edges in K_n, with $n \geqslant 2$. Then we have that:

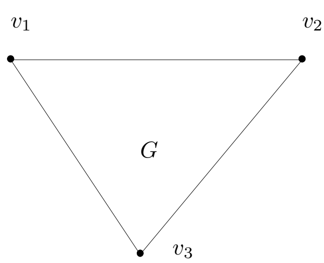
$$|E(K_n)| = \frac{n(n-1)}{2}$$

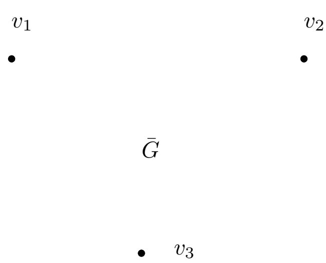
Why is this the case? The degree of each vertex is $n-1$. So the sum of the degrees is $n(n-1)$. We apply this to the earlier correlation between the number of edges and the number of vertices to get the above formula.

Def.: *Complement of a Graph*

$G = \overline{(V, E)}$. We say that $\bar{G} = (V_1, E_1)$ is the complement of G. Two vertices in $\bar{G}$ are connected by an edge iff they are not connected by an edge in the original graph, G. However,

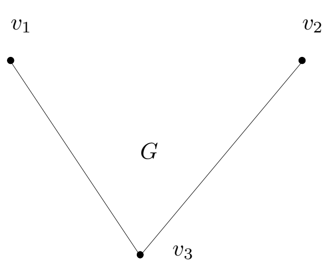
$$V_1 = V \text{ but } E_1 = \text{every edge NOT in } E$$

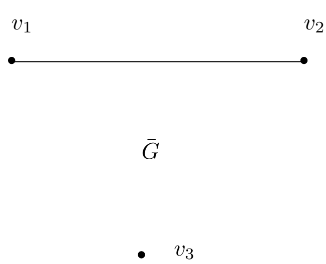




Where $V_1 = V$ and $E_1 = \varnothing$

This is also a spanning subgraph because all the vertices are present and $\varnothing \subseteq E$. Similarly, if we had the following graph, the complement would be:

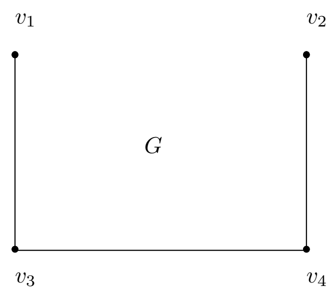


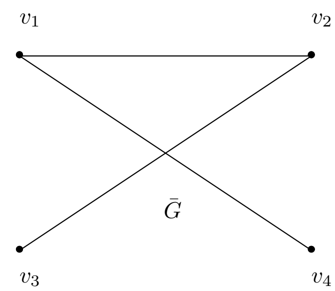


Where $V_1 = V$ and $E_1 = \{v_1 — v_2\}$

This is NOT a spanning subgraph because it is not a subgraph at all. The edge $v_1 — v_3$ is not an edge in the original graph, or mathematically: $E_1 \nsubseteq E$.

Let us take another example:





Fact: Let $G=(V,E)$ be a graph, and let $\bar{G}=(V,\bar{E})$ be the complement of G.

$$|E|+|\bar{E}|=\frac{n(n-1)}{2}$$

In other words, the number of edges in the graph and the number of edges in the complement we have the total number of edges for K_n. If we combine the edges in G and its complement, we will have a complete graph. That is what this fact is saying.

Clearly we have that $E\cap E_1=\varnothing$, and $E\cup E_1=\text{set of all edges of } K_n \Longrightarrow |E|+|\bar{E}|=\frac{n(n-1)}{2}$.

Question: Is there a graph with n vertices st $|\bar{E}|=10$?

Solution: There are a few ways to do this. First of all, we could have a graph with n vertices and no edges. Alternatively, we can follow the following formula:

$$G=K_n-|\bar{E}|$$

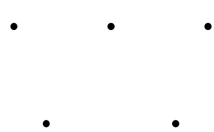
Using this, we would have exactly $|\bar{E}|$ edges in the complement of our graph. Let us look at the two ways with an example: If we want a graph st $|\bar{E}|=10$, choose any n where $\frac{n(n-1)}{10}\geqslant 10$. We can choose 6 for this case. Then:

$$G=K_6-10\,\text{edges}$$
$$\text{Therefore } \bar{G}=(V,\bar{E}) \text{ with } |\bar{E}|=10$$

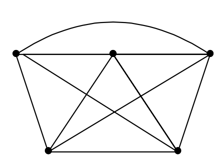
The complement of the graph consists of the 10 edges that are missing from K_6.

February 10th, 2021

Let us detail one of the solutions proposed for the problem in the previous lecture. We want a graph such that $G=(V,E)$ and $\bar{G}=(V,\bar{E})$, with $|\bar{E}|=10$. Look at the following solution:



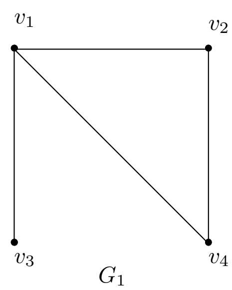
$G=(V,E)$
$V=\{v_1,v_2,v_3,v_4,v_5\}$
$E=\{\varnothing\}$



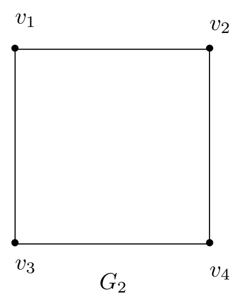
$\bar{G}=(V,\bar{E})$
$V=\{v_1,v_2,v_3,v_4,v_5\}$
$|\bar{E}|=10$

Definition of Isomorphism of Graphs: In the street language, let us consider the question. What does it mean when a graph, G_1, is isomorphic to another graph, G_2? This may be the fact that we draw them differently but both have the same graph properties. For example, if G_1 has 3 vertices of degree 1, then G_2 has exactly 3 vertices of degree 1.

In the official language: Consider $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. G_1 and G_2 are graph-isomophic iff $\exists$ a bijective fucntion $f: V_1 \longrightarrow V_2$ st $\forall a, b \in V_1$, if $a — b \in E_1$, then we have that $f(a) — f(b) \in E_2$. Let us look at an example.



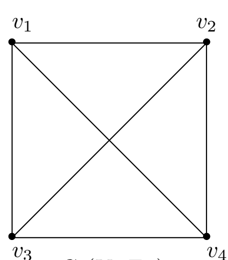
G_1



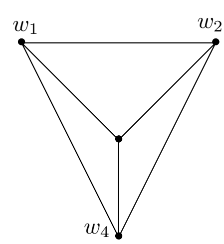
G_2

Are G_1 and G_2 isomorphic? Both have 4 edges, and both have 4 vertices. However, in G_1, v_1 has exactly degree 3, while G_2 has no vertices of degree 3. Therefore, they are not isomorphic to each other. Another reason why they are not graph-isomorphic is that G_1 has a cycle of length 3, while G_2 does not.

Let us look at another graph:



$G_1(V, E_V)$

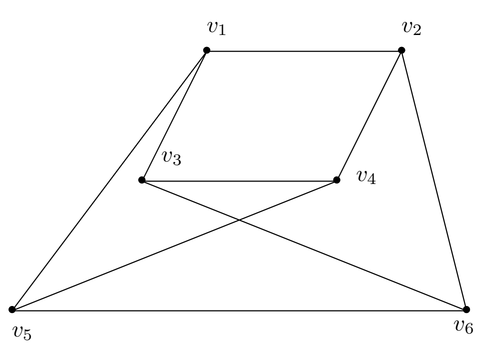


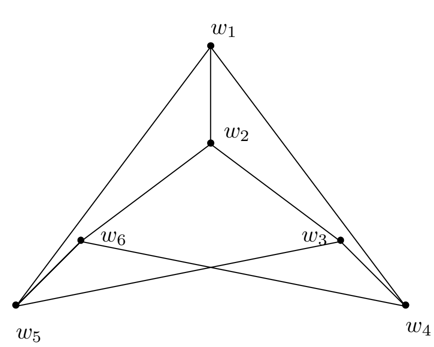
$G_2(W, E_W)$

Are G_1 and G_2 isomorphic? Let us construct the mapping.

$$f: V \longrightarrow W \text{ where } f(v_1) = w_1$$

We know that both these graphs are representations of K_4. The structures of G_1 and G_2 are exactly the same. Let us think of another pair of graphs.



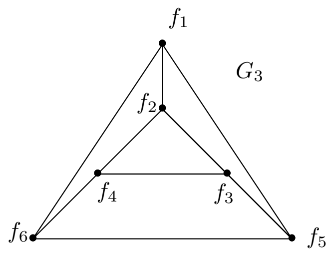


These are both graphs of order 6, but are they the same graph? Firstly, they should have the same number of vertices and edges. Both have 9 edges and 6 vertices. Every vertex in both are of degree 3. Since all of them have the same degree, this is one of those special cases where our mapping can be each vertex to the other.

$$f\colon V_1 \longrightarrow V_2$$

$$\begin{aligned}
v_1 &\longrightarrow w_1\\
v_2 &\longrightarrow w_2\\
v_3 &\longrightarrow w_4\\
v_4 &\longrightarrow w_3\\
v_5 &\longrightarrow w_5\\
v_6 &\longrightarrow w_6
\end{aligned}$$

We make the mapping also based on whether or not the corresponding vertices have edges in between them as well. For example, v_2 maps to w_2 because there exists an edge $v_2 — v_6$, and also an edge in G_2: $w_2 — w_6$. Look at the following graph:



Our claim is that this graph is NOT isomorphic to G_1 and G_2. Why is this the case? Because in this graph, we have a cycle of length 3 ($f_2 — f_3 — f_4$), while we do not have any 3-cycles in G_1 and G_2.

....

In general, it is very hard to see whether two graphs are isomoprhic to each other. It is often not enough to each whether they have the same number of edges, vertices, degrees, etc... If you can find a way to do it, you don't need to do your PhD anymore. You'll get a Fields Medal.

....

Def.: *K-Regular Graphs:* A graph is called K-regular if each vertex has degree equal to K.

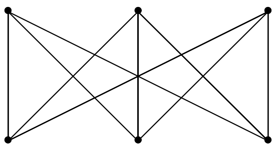
Question: Assume G_1 and G_2 are of order n, and both are K-regular for some value K. Is G_1 isomorphic to G_2?

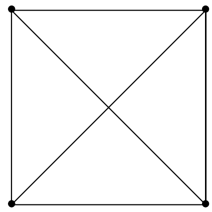
Solution: Not necessarily. It is not always the case, although it is possible. We will look at a counter-example to show this. Consider the graphs shown above. Clearly we know that G_3 in 3-regular and so is G_1 and G_2. Furthermore, they have the same number of edges and vertices. However, we saw that they are NOT isomorphic because of the existence of the 3-cycle in G_3. Therefore, by counter-example, we know that this is not always true.

Assume we have a graph, $G(V, E)$ where G is 5-regular. What can we say about $|V|$? Remember that the sum of the degrees of a graph has to be an even number $(2 \times |V|)$. Therefore, we know that $|V|$ is an even number bigger or equal to 6.

Fact: Assume $G(V,E)$ is K-regular, where K is an odd integer. Then $|V|$ is an even integer $\geqslant K+1$.

Look at the following 3-regular graphs, used to demonstrate this fact:

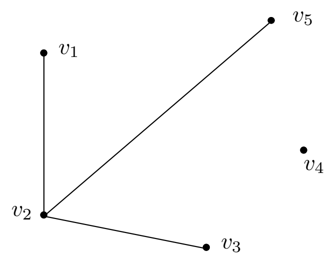




The number of vertices on the first graph is 6, while the number of vertices on the second graph (K_4) is 4. Both are even numbers $\geqslant 3$, since the graphs are 3-regular.

February 15th, 2021

Question: Imagine we have the following graph $G(V,E)$. Find the adjacency matrix of G.



Solution: The adjecency matrix is simply a matrix in which if there is an edge between two vertices, we put a 1. If there is no edge, we put a 0. Moreover, if we are allowing loops in our graph, then we put a 2 instead of a 1 in a loop with the vertex itself.

$$\begin{array}{c|ccccc|} & v_2 & v_5 & v_4 & v_3 & v_1 \\ v_2 & 0 & 1 & 0 & 1 & 1 \\ v_5 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 1 & 0 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Why did we arrange it like this and not the natural why? We should. But it would be a different matrix to what we have above. Let us look at it.

$$\begin{array}{c|ccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 0 & 0 \end{array}$$

But why did we do it like that the first time? How many different adjacency matrices do we have? There are finitely many adjacency matrices for the same graph, but there are a lot. It simply depends on how we decide to write our vertices. How many different ways are there? 5! different ways.

Theorem: Consider two graphs, G_1 and G_2 that are of the same order. $G_1 \approx G_2$ iff they have a <u>common</u> adjacency matrix. This means that out of all the different adjacency matrices that they have, one should be common between the two of them. G_1 and G_2 can have many different adjacency matrices. This is a bit difficult to do, however, since we need to consider all the adjacency matrices of both the graphs.

This is more something that is easier to do with the help of computer programs and algorithms. Let us look at an example of an adjacency matrix for the sake of displaying some of the properties:

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \left|\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right| \end{array}$$

If the above is $\mathrm{adj}(G)$, then consider $[\mathrm{adj}(G)]^T$. Clearly we can see that:

$$\mathrm{adj}(G) = [\mathrm{adj}(G)]^T$$

There is no playing around here. We cannot perform row / column operations on the adj. matrices in order to get something that is common between two graphs. We need to go through each one and compare.

...

Unsolved Problem: Consider 2 graphs, $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, both of the same order. Then we say that $G_1 \approx G_2$ iff:

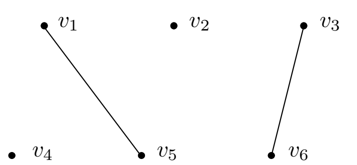
$$\forall 1 \leqslant i \leqslant n \longrightarrow (G_1 - v_i) \approx (G_2 - v_i)$$

We have to try this for all i between 1 and n. This is actually a conjecture that has not yet been proved using our current knowledge of mathematics, but we also cannot find a counter-example to disprove it.

...

Def.: *Bipartite Graphs*

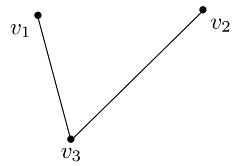
A graph $G(V, E)$ is called a bipartite graph iff $V = A \cup B$, where $A \cap B = \varnothing$, every two vertices in A are NOT adjacent (not connected by an edge), and every two vertices in B are not connected by an edge (not adjacent). Consider the following graph:



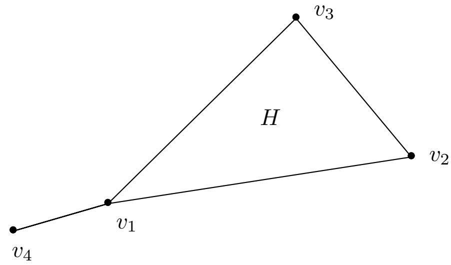
$$A = \{v_1, v_2, v_3\}$$
$$B = \{v_4, v_5, v_6\}$$

Clearly we can see that this graph is bipartite. Why is this true? Because there is no intersection between the two sets of A and B, and each pair of vertices in A has no edge with each other (resp. in B).

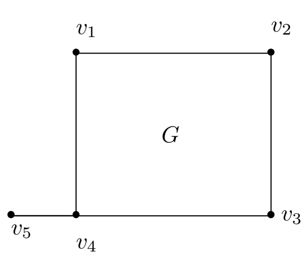
Consider the following graph:

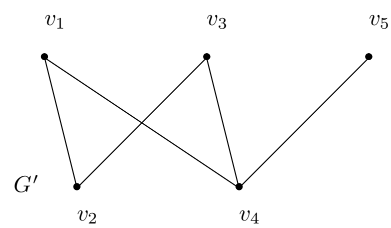


Is this graph bipartite? Yes. You can select $A = \{v_1, v_2\}$ and let $B = \{v_3\}$. Then clearly we can see that this fits the conditions for a bipartite graph. Now, how about the following graph?



If you spend the rest of your life and the next, you cannot show that this graph is bipartite. However, look at the following graph:





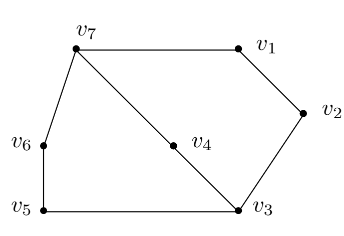
Although we would not originally be able to see, upon redrawing the graph (maintaining the same properties), we can see that $A = \{v_1, v_3, v_5\}$ and $B = \{v_2, v_4\}$. This graph is clearly bipartite. Are the two graphs isomoprhic? Of course, they are the same graph. This means that they have a common adj. matrix.

What is the difference between the graphs G and H? First of all, there is one more vertex and also one more edge in G.

However, the big observation here is that in H, we have a cycle in $v_1 — v_2 — v_3 — v_1$, which is a cycle of length 3, while we have a cycle in G with $v_1 — v_2 — v_3 — v_4$. This is a cycle of length 4. We will now see the theorem.

Theorem: A graph $G(V, E)$ is bipartite iff it has no odd cycles. This means that if our graph has even a single odd cycle, then we definitely cannot say that it is bipartite.

Look at the following graph:



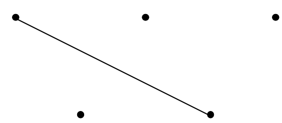
Is this graph bipartite? No. The reason for this is that we have a cycle:

$$v_1 — v_2 — v_3 — v_4 — v_7 — v_1$$

This is a cycle of length 5, which is odd. Therefore we know that we don't have a bipartite graph by the theorem just introduced. We don't need to waste our time splitting the vertices into two sets. This is where we stop.

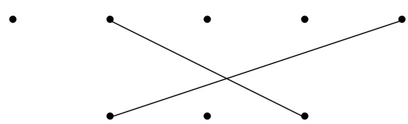
February 17th, 2021

Fact: A graph is bipartite iff it has no odd cycles.



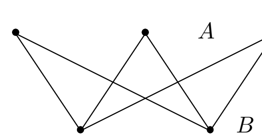
This graph is of order 5. This means that there are 5 vertices. There is a set A, containing the three vertices on the top, and a set B, consisting of the 2 vertices on the bottom. This graph is bipartite with the notation: $B_{3,2}$. This means that the set A contains 3 vertices, and the other set contains 2 vertices.

Let us draw $B_{5,3}$. This is a graph of order 8 (8 vertices).



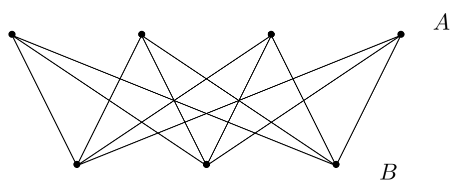
Clearly there are many different ways of drawing $B_{5,3}$. In other words, there are many different graphs that can be made to be $B_{5,3}$.

Def.: A bipartite graph is called a complete bipartite graph iff every vertex in A is connected to every vertex in B. Consider the following graph:



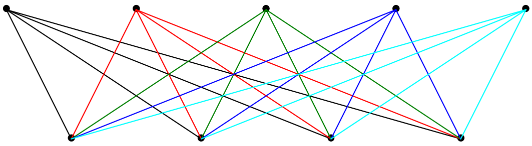
Now, look at the this graph:

$B_{4,3}$



This is a graph representing $B_{4,3}$. This is also a complete graph of the form $K_{4,3}$.

Reminder: When we say that a graph is $K_n, n \geqslant 1$, this is a complete graph. On the other hand, when we say $K_{m,n}$, we have a *complete bipartite graph*. This is not the same as K_n. For example, if we consider $K_{5,4}$:



Fact: $K_{m,n}$ has exactly $m \times n$ edges. If we assume $|A| = m$ and $|B| = n$, then each vertex in set A has degree n, and each vertex in set B has degree m.

Proof: We use the trivial result:

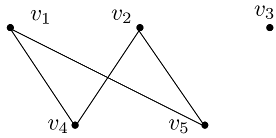
$$\begin{aligned}\sum \text{degrees} &= \sum_{v \in A} \deg(v) + \sum_{w \in B} \deg(w) \\ &= m\,n + m\,n = 2\,m\,n = 2|E| \\ \Longrightarrow |E| &= \frac{2m\,n}{2} = m\,n\end{aligned}$$

Def.: *Girth:* Consider the graph $G(E, V)$. The girth of the graph, denoted as $\operatorname{girth}(G)$, is the length of the shortest cycle. Recall that a cycle is a path which the first vertex is the same as the terminating vertex. If a graph has no cycles, then we say that it has girth ∞.

What is $\operatorname{girth}(K_n)$, for $n \geqslant 3$? Since there is an edge between every pair of vertices, we know that the cycle with shortest length is 3. Thus $\operatorname{girth}(K_n) = 3 \text{ for } n \geqslant 3$.

Proof: Since $n \geqslant 3$, then $v_1 — v_2 — v_3 — v_1$ is a cycle of length 3 in K_n.

What is the girth of $K_{m,n}$, where $m = 1$ or $n = 1$? $\operatorname{girth}(K_{m,n}) = \infty$, since there are no cycles in the graph. On the other hand, if we have $K_{m,n}$ with $m, n \geqslant 2$, then $\operatorname{girth}(K_{m,n}) = 4$. This is always the case. Why is this true?

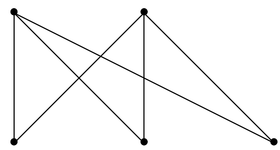


Consider the cycle $v_1 — v_4 — v_2 — v_5 — v_1$. This is a cycle of length 4. The girth of $K_{m,n}$ will never be 3, or 5, or 7... This is because the graph would not be bipartite otherwise.

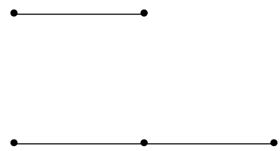
Proof:

$$\begin{aligned}A\colon v_1, v_2, v_3, \ldots, v_m &\text{ with } m \geqslant 2 \\ B\colon w_1, w_2, w_3, \ldots, w_n &\text{ with } n \geqslant 2 \\ \text{Since the graph is } K_{m,n}&\text{, then:} \\ v_1 — w_1 — v_2 — w_2 — v_1 &\text{ is a cycle.}\end{aligned}$$

Look at the graph below:

$K_{2,3} =$ 

The complement of the graph, $\overline{K_{2,3}}$:



Can we calculate the number of edges in the complement of $K_{m,n}$? Is there a formula? Recall that the graph $K_{m,n}$ has order $m+n$. Also recall that:

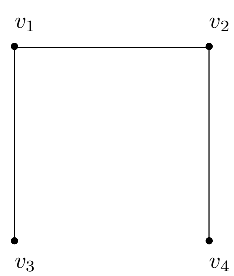
$$|E(K_{m,n})| + |\bar{E}(K_{m,n})| = \frac{(n+m)(n+m-1)}{2}$$

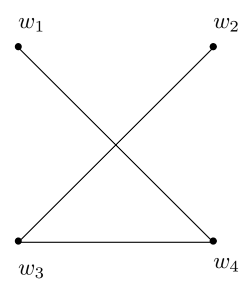
Finally, remember that the number of edges in a complete graph, K_w, is: $\frac{w(w-1)}{2}$. This is linked to the above formula. We will use this information to derive the number of edges in the complement of the complete bipartite graph:

$$\begin{aligned}
m\,n + |\bar{E}_{K_{m,n}}| &= \frac{(n+m)(n+m-1)}{2} \\
|\bar{E}_{K_{m,n}}| &= \frac{(n+m)(n+m-1)}{2} - m\,n \\
&= \frac{n^2 + 2m\,n + m^2 - n - m - 2m\,n}{2} \\
&= \frac{n^2 + m^2 - (n+m)}{2}
\end{aligned}$$

Will the complement of $K_{m,n}$ be connected? No. There will be no edges connecting the two sets, A and B. This is because a complete bipartite graph has edges between the two sets <u>only</u>.

Def.: *Self Complement Graph:* A graph whose complement is itself. We can demonstrate a self complementing graph in the example below. Another way of saying this is that the graph G is isomorphic to its complement, $\bar{G}$.





Are these two graphs (where the right graph is the complement of the left) isomorphic? Consider the mapping:

$$\begin{aligned} f: G &\longrightarrow \bar{G} \\ v_1 &\longrightarrow w_3 \\ v_2 &\longrightarrow w_4 \\ v_3 &\longrightarrow w_2 \\ v_4 &\longrightarrow v_1 \end{aligned}$$

This graph is a self-complement. How about we have a graph with 3 vertices? Can we have a self-complement graph with 3 vertices? No. This is never the case. In fact, if we have a graph that is a self-complement, it always has to have 4 or more vertices.

Let G be a graph of order n st the graph is isomorphic to its complement. In mathematical terms, we have that $G \approx \bar{G}$. We know that $|E| + |\bar{E}| = \frac{n(n-1)}{2}$.

$$\begin{aligned} \text{Since } G \approx \bar{G}, |E| = |\bar{E}| = m \\ \Longrightarrow m + m = \frac{n(n-1)}{2} = 2m \\ 4m = n(n-1) \\ n(n-1) \text{ must be a multiple of } 4 \\ \text{ie } 4|n \;\text{ or }\; 4|(n-1) \\ \Longrightarrow n = 4K \text{ for some } K \geqslant 1 \in \mathbb{Z} \\ \text{or } n = 4K + 1 \end{aligned}$$

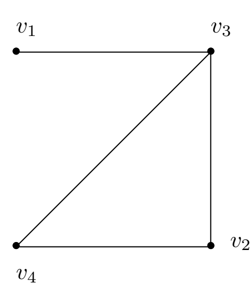
For example, if we have a graph of order 7, then we cannot have that it is isomorphic to its complement. This is because $7 \neq 4K$ or $4K + 1$. However, we can order with a graph of order 5, because $5 = 4(1) + 1$. Therefore, a graph of order 5 can be isomoprhic to its complement, ie self-complement graph.

February 20th, 2021

Consider the following adjacency matrix:

$$\mathrm{Adj}(G) = A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The graph that would correspond to this adj. matrix would be:



Can this graph be bipartite? No. The cycle $v_3 — v_2 — v_4 — v_3$ is a cycle of length 3 (odd). A graph with an odd cycle cannot be bipartite.

Recall that two graphs can be isomorphic even if they don't have the same adj. matrices. They can be rearranged and manipulated through row operations. As long as they have a common adj. matrix, they can be isomorphic.

Def.: *Permunation Matrix:* A permutation matrix is an $n \times n$ st each row has "1" exactly once. All other entries in a row are 0. For example, consider the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

This is a permutation matrix that is not necessarily obtained from I_n. Therefore we

To apply this to the above, consider the following result.

Result: Let A_1 be an adj. matrix of a graph $G_1(V_1, E)$. Consider A_2, adj. matrix for G_2. The result states that $G_1 \approx G_2$ iff $\exists p$ (permutation matrix) st $p\,A_1 = A_2\,p$. If we can find some p st the equation holds, then the two graphs are isomorphic.

Consider:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Definitely we can see that as matrices, $A_1 \neq A_2$. Now consider our matrix p:

$$p = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \text{permutation matrix}$$

We can see that p contains a single "1" on each row (obtained from I_5). In this question, A_1 and A_2 are indeed isomorphic. Can we verify this through $p\,A_1 = A_2\,p$? Since p is invertible, we know that:

$$A_2 = p\,A_1\,p^{-1}$$

Def.: *Diameter of a Graph:* The diameter of a graph, denoted as $\text{diam}(G)$, is given by the following set:

$$\text{diam}(G) = \max\,\{d(a,b) | a, b \in V \text{ and } a \neq b\}$$

This means that $\dim(G)$ is the maximum distance between two vertices of a graph. For example, if we are given that $\dim(G) = 4$, then we know that $\forall a, b \in V \longrightarrow d(a,b) \leqslant 4$. Note that $d(a,a) = 0$. For a graph $K_{m,n}$ (complete bipartite), what is the diameter?

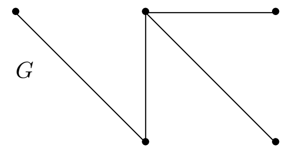
$$\dim(K_{m,n}) = 2$$

This is always the case. It is trivial. However, we can see the formal proof below.

Proof:

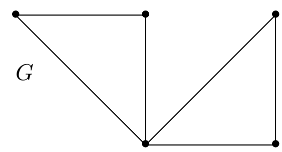
We have that $A = \{v_1, v_2, \ldots, v_m\}$ and $B = \{w_1, w_2, \ldots, w_n\}$. Choose some $v \in A$ and some $w \in B$. Clearly for $K_{m,n}$, $d(v, w) = 1$. Now, to go the other way around, choose $v \in A$ and $w' \in B$. Clearly we can see that $v — w' — w$ is a path of length 2 for any pair of vertices, v and w.

Similarly, consider K_n. We know trivially that $\text{diam}(K_n) = 1$. Now consider the graph below:



$$\text{diam}(G) = 3 = \max\{d(a, b) | a, b \in V\}$$

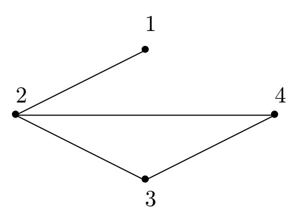
Now consider another example, with the graph below:



$$\text{diam}(G) = 2$$

February 22nd, 2021

Given the following graph, we can produce an adj. matrix.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Question: For each vertex, find the degree.

$$\deg(1) = 1$$
$$\deg(2) = 3$$
$$\deg(3) = 2$$
$$\deg(4) = 2$$

We can do this by just looking at the adj. matrix. The sum of the numbers in the row and column for the given vertex should be the same. This is a simple observation.

Look at the following two graphs:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Our claim is that $G_1 \approx G_2$. Show this, and also show p st $p\,A_1 = A_2\,p$. Then, using words, show how we can get A_2 from A_1.

$$f : G_1 \longrightarrow G_2$$

$$\begin{aligned} f(1) &= 4 \\ f(2) &= 1 \\ f(3) &= 3 \\ f(4) &= 2 \end{aligned}$$

This mapping will work st $p\,A_1 = A_2\,p$.

In another example, we could take some mapping $K : G_2 \longrightarrow G_1$ where we would have $p\,A_2 = A_1\,p$. f and K are the same, but opposites.

Now, let us try to obtain p. Take I_4. We will do the following steps:

1. $R_1 \mapsto R_4$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2. $R_2 \mapsto R_1$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3. $R_4 \mapsto R_2$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is our p. We will use a calculator to see whether or not the equality $p\,A_1 = A_2\,p$. It indeed holds.

How did we come up with this? Look at the mapping of f. We can see that $f(1)=4$, so therefore we take $R_1 \mapsto R_4$ from the identity matrix I_4. Similarly, we can see that $f(2)=1$, so we interchange the rows R_1 and R_2. We continue in this fashion.

Now, we will get A_2 from A_1 by interchanging rows and columns.

Start with A_1:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

1. $R_1 \mapsto R_4$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2. $R_2 \mapsto R_1$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3. $R_4 \mapsto R_2$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that when we say $R_4 \mapsto R_2$, this means that we replace the current R_2 with R_4 from the original matrix, A_1.

Let us call this matrix C. Now, let's do the same thing but with columns. In other words, do the same mapping, but on columns. 1st column with 4th, 1st with 2nd, etc.

Start with C

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. $C_1 \mapsto C_4$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2. $C_2 \mapsto C_1$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

3. $C_4 \mapsto C_2$

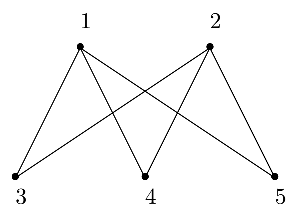
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We can see (after the column replacements) that this matrix is the same as A_2. For verification:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_2$$

Def.: *Dominating Set:* Take a graph $G(V, E)$. A subset B of V is called a dominating set if every vertex in $V - \{B\}$ is connected to at least one vertex in B.

Consider the following graph of $K_{2,3}$:



Our claim is that $B = \{3, 4, 5\}$ is a dominating set. This is because every vertex in $V - \{B\}$, which is the set $\{1, 2\}$, is connected by an edge to at least one of $\{3, 4, 5\}$. We can also see that $L = \{1, 2\}$ is also a dominating set of $K_{2,3}$.

Another example of a dominating set in $K_{2,3}$ is $K = \{2, 4\}$. Every vertex in $K_{2,3}$ exlcuding 2 and 4 is connected to one of either 2 or 4.

Def.: *Dominating Number:* The dominating number, denoted $\gamma(G)$, is the size of a smallest dominating set.

Let us try to understand this better through the use of an example. Consider the graph $K_{3,7}$. What is the dominating number of this graph?

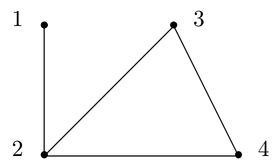
$$\gamma(K_{3,7}) = 2$$

Why is this the case? Take one vertex from each set (similar to the above example with taking $\{2, 4\}$). Then we know that the size of this set is 2.

What is $\gamma(K_{1,n})$? Clearly this would be 1. This is because the 1 vertex at the top of the graph is connected to every vertex (n in total) in the second set.

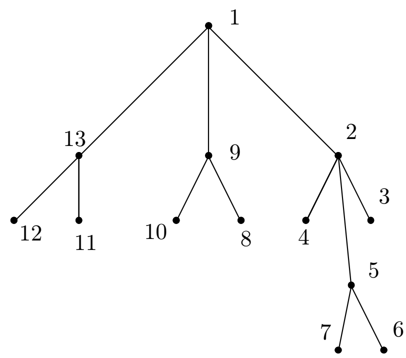
In fact, the general case is that for $m, n \geqslant 2$, then $\gamma(K_{m,n}) = 2$. To demonstrate the idea of the dominating number further, note that $\gamma(K_n) = 1$ for $n \geqslant 2$, since every vertex is connected to every other vertex in a complete graph.

Consider the following graph:



We can see that $\gamma(G) = 1$. This is because we can choose a dominating set, $B = \{2\}$, and every other vertex is connected to 2. Therefore trivially we can see that $\gamma(G) = 1$.

Another graph:



What is $\gamma(G)$? Consider the set of vertices $\{2, 5, 9, 13\}$. These are all nodes in the tree that fall in between the root and the leaves. Within this set, we can see that every other vertex is connected through an edge to one of these 4. Therefore, since $|\{2, 5, 9, 13\}| = 4$, we have that $\gamma(G) = 4$.

February 24th, 2021

Def.: *Size:* Given a graph $G(V, E)$, we know that the order n means that $|V| = n$. On the other hand, if we say that a graph G has size m, this means that the number of edges is m. In other words, a graph with order n and size $m \Longrightarrow |V| = n$ and $|E| = m$.

Def.: *Tree:* We call a connected graph a tree iff G has no cycles.

Fact: A connected graph is a tree iff between every two distinct vertices, there is a unique path. There is only one way to go from one vertex to another. There is no other way.

Sketch Proof:

We can see that if we want to go from a to b in our two graphs, there is only one way to go in the first one but more than one way in the second one. Why is this interesting? Clearly we know that a tree contains no cycles. If the path between two vertices in a graph is not unique, we automatically know that we can create a cycle. Therefore, $\Longleftrightarrow$.

$$\begin{aligned}\Longrightarrow \text{Assume } G \text{ is a tree. Let } a, b \in V\\ \text{We shall show that } \exists! \text{ path from } a \text{ to } b\\ \text{Deny:}\\ \text{Assume } p_1, p_2 \text{ are 2 diff paths from } a \text{ to } b\\ \text{It is clear that the graph will have a cycle}\end{aligned}$$

(contradiction)

$\Longleftarrow$

Assume $\exists! p$ between a, b. Show G is a tree

Deny:

Since G is connected and not a tree,

$\exists$some cycle $v_1 — v_2 — \ldots — v_n$

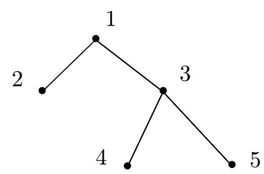
which is a path from v_1 to v_n

But $v_1 — v_n$ is also a path from v_1 to v_n

Therefore we have more than one path

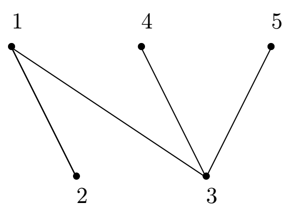
(contraction)

Consider the graph $K_{1,5}$. This is clearly a tree, because there is no cycle within the graph. Is every tree a $K_{1,n}$ for some n? No. This is not the case. This would only work if our tree has 1 level. Look at the following graph (tree):



We say that the tree is $B_{n,m}$ for some n, m, where it is a bipartite graph. What is our set A and what is our set B? Since this graph has no cycles, then it definitely cannot have any odd cycles, which by definition makes it a bipartite graph.

$$A := \{1, 4, 5\}$$
$$B := \{2, 3\}$$



This makes our graph (tree) $B_{3,2}$. Now, is every $B_{m,n}$ a tree? No. We can easily produce bipartite graphs that contain cycles, which renders trees out of the possibilities.

1. We know that $K_{1,n}$ is a tree, but not every tree is $K_{1,n}$;

2. Wc also know that every tree is $B_{m,n}$, but not every graph of the form $B_{m,n}$ is a tree.

Def.: *End-Vertex:* A vertex v in a graph is called an end-vertex iff $\deg(v) = 1$. It is clear that every tree has at least 1 end-vertex.

Fact: A connected graph of order n is a tree iff it is of size $n - 1$. This means that the number of edges in the graph is $n - 1$.

Proof:

$$\Longrightarrow$$

Assume G is a tree, we show that $|E| = n-1$

If $n=2$, then it is clear.

Assume the result is true for some $n=k$, $k \geqslant 2$

We prove it for $n=k+1$

Assume G is a graph of order $k+1$

We show that $|E| = k$

Since G is a tree, G has an end vertex, say v

Now $G-\{v\}$ is some tree order k

By assumption, for $G-\{v\}$, $|E| = k-1$

$\Longrightarrow |E| = k$ for G

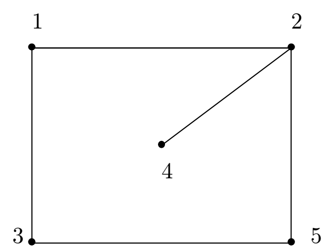
$$\Longleftarrow$$

Construct an argument, etc...

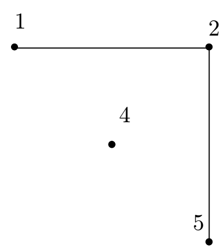
Question: Can we have a tree of order 8 and size 6? No. This is because the size has to be $n-1$, which is 7 in our case.

Fact: Every connected graph G has a spanning subgraph that is a tree. This is called a spanning tree of G. Recall that if we have a graph, $G(V,E)$, a subgraph $H(V_1,E_1)$ is a spanning subgraph iff $V = V_1$. This means that the set of vertices is the same (not that $V_1 \subseteq V$).

Also recall that H is an induced subgraph of G iff $V_1 \subseteq V$ and $a \text{ — } b$ is an edge of H iff $a \text{ — } b$ is an edge of G.



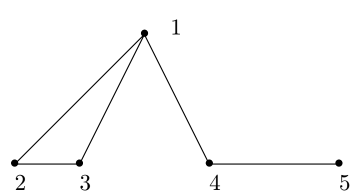
G



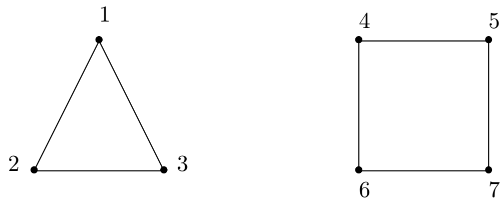
H

We can see that H is a subgraph of G, clearly, but it is not an induced subgraph. Why? Because we have an edge between 2 and 4 in the original graph, but there is no edge between 2 and 4 in H.

March 1st, 2021

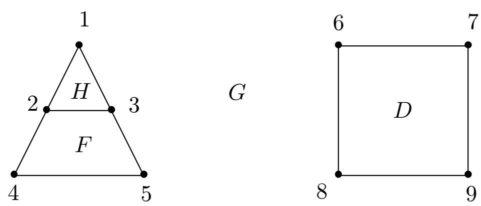


This is a connected graph. We say that a connected graph consists of a single component. In other words, the graph above is 1-component. Now look at the following graph:



This graph is not connected, because there is no path between the vertices $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$. Each one of the individual sets are, however, connected. Note that $\{1, 2, 3\}$ is an induced subgraph of G and so is the set $\{4, 5, 6, 7\}$. We can say that G has 2 components.

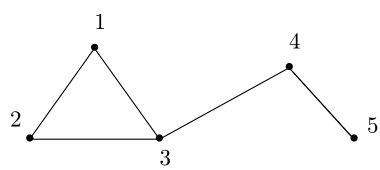
We say that D is a component of a graph G if D is a connected induced subgraph of G and D is not a subgraph of a connected subgraph of G.



Is H a component of the original graph, G? Note that H consists of $1 — 2 — 3 — 1$, and is definitely an induced subgraph of G. However, it is not a component, since H is a subgraph of a larger subgraph of G. Therefore, it cannot be a component. G in our case has 2 components.

Def.: *Eccentricity:* Assume that our graph $G(V, E)$ is connected. Choose some $v \in V$. The eccentricity of v is denoted and defined by the following:

$$e(v) = \max\{d(v, u) | u \in V\}$$



$$\begin{gathered} e(1) = \max\{d(1,2), d(1,3), d(1,4), d(1,5)\} \\ = \max\{1, 1, 2, 3\} = 3 \\ \text{Therefore we have that } e(1) = 3 \end{gathered}$$

$$e(2) = 3, e(3) = 2, e(4) = 2, e(5) = 3$$

What can we connect eccentricity to? The diameter of a graph.

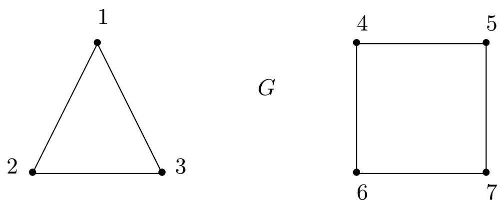
$$\operatorname{diam}(G) = \max\{e(v) | v \in V\}$$

We define the radius as the minimum eccentricity of all the vertices in a graph. Mathematically, we say that:

$$\operatorname{rad}(G) = \min\{e(v) | v \in V\}$$

In the example of the graph provided above, we have that the set of $\{e(v) | v \in V\} = \{3, 3, 2, 2, 3\}$. We take the minimum of this to obtain: $\operatorname{rad}(G) = 2$.

The natural follow up question would be: If a graph is not connected, how would be calculate the eccentricity?



$$e(i) = \infty \quad \forall 1 \leqslant i \leqslant 7$$

This is because you cannot get from (for example) vertex 1 to vertex 7.

Def.: *Path Graph:* Consider the graph $v_1 — v_2 — \ldots — v_n$ where $v_1, \ldots, v_n$ are all distinct vertices. Such a graph is called a path-graph of order n, denoted P_n. This graph is clearly also a tree since it does not contain any cycles.

Question: Let $n \geqslant 2$. What is the size of P_n?

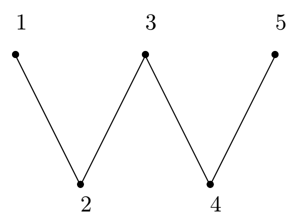
Solution: Since we know that a path-graph is a tree (of order n), then clearly, from previous result, we know that the size of P_n is $n - 1$.

Another approach to the proof:

$$\begin{aligned}
v_1 — v_2 — v_3 — \ldots — v_n& \\
\deg(v_1) = 1 &= \deg(v_n) \\
\deg(v_i) &= 2 \quad \forall 1 < i < n \\
\sum \text{degrees} &= 2\,|E| \\
2(n-2) + 2 &= 2\,|E| \\
2n - 4 + 2 &= 2\,|E| \\
2n - 2 &= 2\,|E| \\
|E| &= n - 1
\end{aligned}$$

Is P_n a bipartite graph? Consider $P_5 = 1 — 2 — 3 — 4 — 5$. Then we can split the vertices into two sets:

$$A = \{1, 3, 5\} \text{ and } B = \{2, 4\}$$

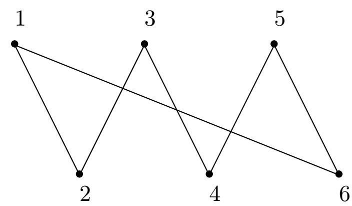


What is the dominating number of P_5, denoted $\gamma(P_5)$? The smallest dominating set is $\{2, 4\}$, and thus $\gamma(P_5) = 2$.

Def.: *Cycled Graph:* Assume we have a graph $1 — 2 — 3 — \ldots — n — 1$. This is a cycle, for $n \geqslant 3$. A graph in this form is called a cycled graph, denoted by C_n. This means we have a cycled graph of order n. For example, C_5: $1 — 2 — 3 — 4 — 5 — 1$. C_n cannot be a tree (because it is literally a cycle).

Is C_5 a bipartite graph? No, because it contains an odd cycle. What about C_6? Yes. This leads us to the result: C_6 is a bipartite graph iff n is even.

$$C_6 = 1 — 2 — 3 — 4 — 5 — 6 — 1$$



What is $\gamma(C_6)$? 2. Choose $\{1,4\}$ or any pair of vertices not in the same subset for the bipartite graph representation. Clearly we can see that every vertex outside of $\{1,4\}$ is connected to either 1 or 4. Generally, dominating number problems are considered hard in Graph Theory. The first thing that you may think in that graph is that $\gamma(C_6) = 3$. However, since 6 is connected to 1, this changes everything.

In general, $\gamma(P_n) = \lfloor \frac{n}{2} \rfloor$. How do we calculate the dominating number for C_5? There is no formula for this. However, look at the graph for C_5:

$$1 — 2 — 3 — 4 — 5 — 1$$

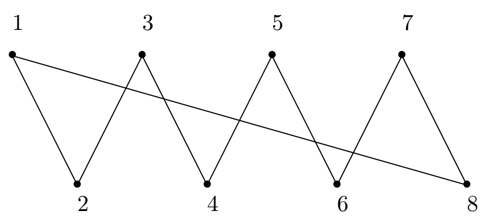
Take the set $\{2,4\}$. Every vertex outside of this set is connected to either vertex 2 or vertex 4. Thefore, we know that $\gamma(C_5) = 2 = \lfloor \frac{5}{2} \rfloor$.

What about $\gamma(C_7)$?

$$1 — 2 — 3 — 4 — 5 — 6 — 7 — 1$$

Take the set of vertices $\{2, 4, 6\}$, every vertex is connected to one of these three. Therefore, $\gamma(C_7) = 3$.

For even n, this idea of the floor of $\frac{n}{2}$ would not work. Consider C_8:



A dominating set for this graph: $\{1,4,7\}$. Every vertex outside of $\{1,4,7\}$ is connected to one of the three vertices. Can we make a smaller dominating set? No. Do we have a formula for finding $\gamma(C_n)$, where n is even and $n \geqslant 4$?

$$\text{for } n \geqslant 4, \text{even}, \gamma(C_n) = \frac{n}{2} - 1$$

March 3rd, 2021

Recall the concept of a dominating set: Assume we have a graph of order n. A set of vertices, $D = \{v_1, v_2, \ldots, v_m\}$, where $m < n$ st every vertex of the graph, G, outside of D is connected by an edge to at least one vertex in D.

Furthermore, the dominating number is the size of the smallest dominating set. This is all explained in street language for ease of understanding.

Consider the graph, P_9:

$$1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9$$

What is $\gamma(P_9)$? It is the smallest dominating set of the graph. Consider the following set:

$$\{2, 5, 8\}$$

This set is the smallest dominating set of the graph P_9. We can see that everything outside of the set is connected to at least one of these three vertices. Therefore, since the size of this set is 3, then $\gamma(P_9) = 3$.

Now look at the graph for P_{15}:

$$1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11 — 12 — 13 — 14 — 15$$

The smallest dominating set for this graph is: $\{2, 5, 8, 11, 14\}$. This means that $\gamma(P_{15})$. We can form the general case formula for P_n:

$$\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$$

What do we expect the value for $\gamma(P_7)$? We take $\left\lceil \frac{7}{3} \right\rceil = 3$. Another question: Find $\gamma(P_{11})$ and construct the smallest dominating set of it:

$$\gamma(P_{11}) = \left\lceil \frac{11}{3} \right\rceil = 4 \text{ and } \{2, 5, 8, 11\}$$

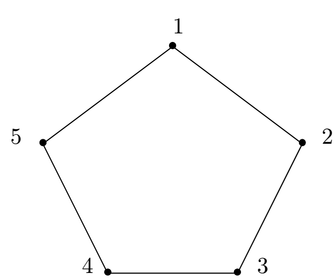
$$1 — \mathbf{2} — 3 — 4 — \mathbf{5} — 6 — 7 — \mathbf{8} — 9 — 10 — \mathbf{11}$$

....

Application: Imagine we have a computer station, we want to hire hackers. What is the minimum number of hackers we need to be able to hack all of the computers in the work-station? Where do we place them in order to connect to everyone else? This is a very good way of explaining how the concept of the dominating number and dominating set works. We can use any other example of this line of thought.

....

Consider the graph C_n. Would the dominating set be the same as P_n, or would it be different? Look at the graph for C_4:

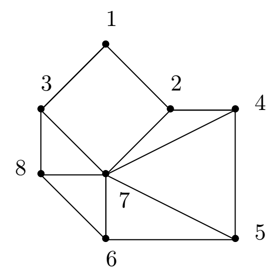


What is going to be $\gamma(C_5)$? It will be 2, because if we look at $1 — 2 — 3 — 4 — 5 — 1$, we can see that by selecting the set $\{2, 5\}$, everything outside the set of vertices will be connected by an edge to either 2 or 5. The general formula:

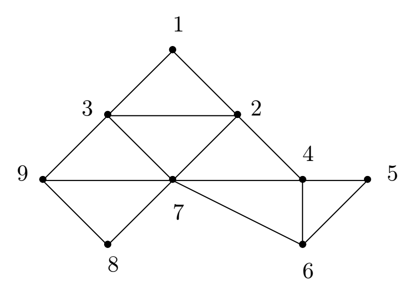
$$\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

Def.: *Strongly Dominating Set:* The set is a dominating set, and every vertex within the set should be connected to at least one other vertex in the set through an edge. This is more complicated and is a rather new area of research.

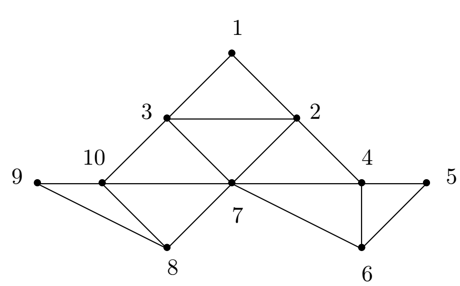
Consider the following graph:



What is $\gamma(G)$? Clearly, we don't need a formula for this example. We can see that every vertex in the graph is connected to either 7 or 1, meaning that we select the set: $\{1, 7\}$ as our dominating set. Therefore, $\gamma(G) = 2$. Another example:

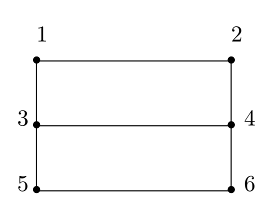


What is the smallest dominating set of this graph? Choose $\{1, 5, 7\}$. Done. We can consider more examples:



In this case, we can choose $\{1, 4, 10\}$. Can we find a dominating set with the vertex 7? We can see that the vertex 7 has the highest degree, but we cannot find a <u>minimum</u> dominating set with it. This goes to show that the vertex with highest order is not necessarily the vertex that would produce a minimum dominating set. A dominating set with 7: $\{1, 7, 5, 9\}$.

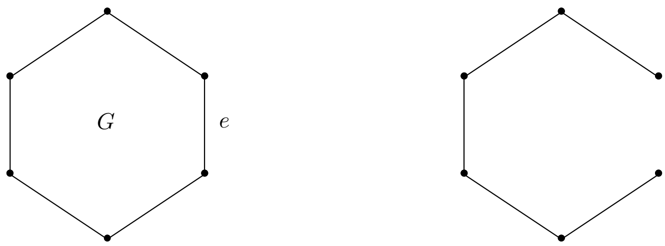
One last example:



We know that $\gamma(G) = 2$. Choose any of the following dominating sets: $\{1, 6\}, \{3, 4\}, \{2, 5\}, \ldots$

Result: Every connected graph has a spanning tree.

Let us sketch the idea before we move on to the actual proof: We can start with a cycle.



If we have a cycle and we remove an edge, what kind of graph will we have? We will have a P_n graph. In a cycle, $C_n - e = P_n$. The graph will, however, stay connected. We will have the same number of vertices but the cycle will become a path. This is what makes it a spanning tree. Recall that spanning means that we have the same vertices, and tree means that we have no cycles. That is clearly visible through our sketch.

We can see that by removing e_1 and e_2 from the graph on the left, we have removed all possible cycles from the graph, but we are clearly keeping the same vertices. Therefore, we have constructeed a spanning tree. Thus: $G - \{e_1, e_2\}$ is a spanning subgraph, which is a tree.

Is that the only spanning tree, or can we find others? Remove the edges e_2 and e_3.

We can see that the two removals produce graphs that are not isomorphic to each other. In the example just shown, we can see that we have vertices of degree 3, but none of those is the former.

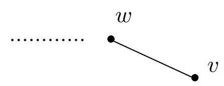
Def.: *Cut-Vertex*: For a graph, $G(V, E)$, consider the vertex $v \in V$. We say that v is a cut-vertex of G if $G - v$ is disconnected. This means that when we remove a vertex, in our case v, from the graph, then we also remove all the edges that are connected to v.

Look at the following example:

If we remove the vertex 1, the graph is still connected. Therefore, 1 is NOT a cut-vertex of G. In fact, there is no vertex in G that is a cut-vertex. The graph will remain connected regardless of which vertex you remove.

March 8th, 2021

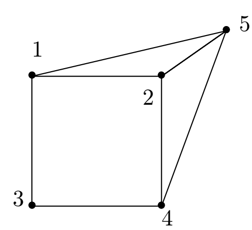
Let us go back to the concept of cut-vertices. Consider a graph $G(V,E) \longrightarrow$ connected, order n, size m. Take a vertex, $v \in V$ st $\deg(v) = 1$. Will it be possible that $G - v$ is disconnected? No. Why is this the case? Let us visualize.



The vertex v is not connected to anything other than w, since $\deg(v) = 1$. If we remove the vertex, then we only remove the edge w — v. Therefore, no matter what happens on "the other side" of w, the graph cannot be disconnected (worst-case: w and v are the only vertices of G, removing v automatically leaves us with a single vertex w, which is connected).

By removing v, we have the graph $G - v$, which is connected and of order $n - 1$ and of size $m - 1$.

Fact: If v is a cut-vertex of a graph $G(V,E)$, then $\deg(v) \geqslant 2$. Note that this does NOT mean every vertex of degree 2 is a cut-vertex. Recall the square from last lecture: each vertex is of degree 2, but none of them are cut-vertices. Let us look at another graph:



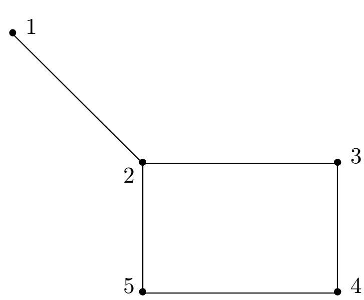
Is the vertex 2 a cut-vertex? No. If we remove it, the graph is still connected. What can we observe about vertex 2? Look at the graph of P_4:

$$1 — 2 — 3 — 4$$

If we remove the vertex 3, then it will be disconnected. Therefore 3 is a cut-vertex, and $\deg(3) = 2$. The vertex 2 is also a cut-vertex, by the same principle. This will lead us to the following result:

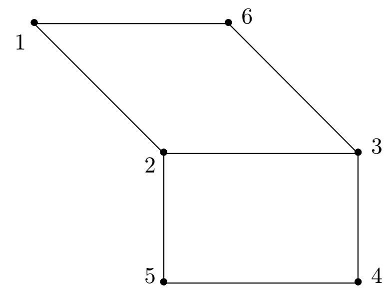
Result: Let $G(V,E)$ be a connected graph. $v \in V$ is a cut-vertex iff $\exists w, z \in V$ st every path from w to z passes through the vertex v.

Consider the example graph shown below:



Is 2 a cut-vertex? You can observe that every path from 3 to 1, from 4 to 1 and from 5 to 1 passes through 2. We only need to find ONE pair of vertices (note that the result says THERE EXISTS, not for every). Therefore, 2 is a cut-vertex. Another way of looking at it: Can we find a path from 3 to 1 without passing through vertex 2? No. Therefore 2 is a cut-vertex.

One more example:



Is 2 a cut-vertex now? No. Because we can find a path from 3 to 1 that does not pass through 2. In fact, the method to proving that it is not a cut-vertex is to remove vertex 2 and show that we can still traverse between any pair of vertices. i.e. $G-2$ is connected, and hence 2 is not a cut-vertex.

Sketch: $\Longrightarrow$Assume v is a cut-vertex. Show that $\exists w, z \in V$ st every path from w to z passes through v.

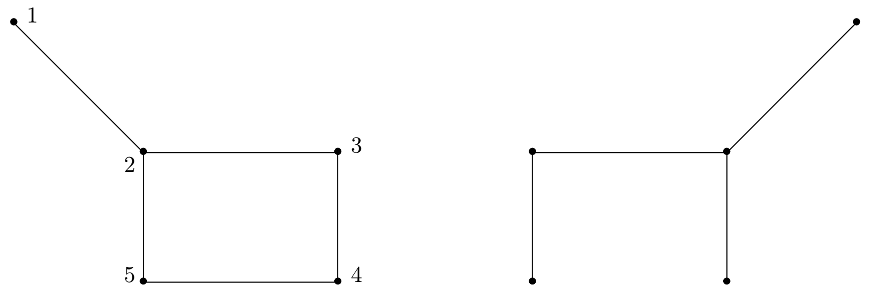
Proof: Since v is a cut-vertex, $G-v$ is disconnected. This means that there exists at least some w and $z \in V$ which are not connected through a path, by the definition of a disconnected graph. Therefore, every path from w to z must pass through v.

$\Longleftarrow$ Assume $\exists w, z \in V$ st every path from w to z passes through v. Show that v is a cut-vertex. This is trivial.

Def.: *Bridge*: An edge, e, is called a bridge iff $G-e$ is disconnected.

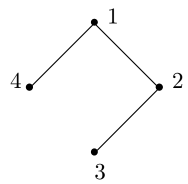
Rmk:

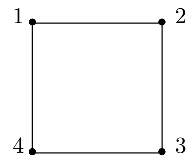
- If the graph is of order n and size m, and if v is a cut-vertex, then $G-v$ is of order $n-1$ and size $m-\deg(v)$
- If e is a bridge, then $G-e$ is of order n and size $m-1$. We can see this through the following example:



We can see that the graph on the right is the same as the one on the left, except we have removed the edge $5 — 4$. We can see that $G-\{3 — 4\}$ is of order 5 and of size 4. Our claim is that the only bridge here is $1 — 2$. Why is this the case? Because that is the only edge we can remove that would result in the graph being disconnected.

Let us look at the following graph:





What can we say about the two graphs? If we remove an edge from the one on the left, then it is a bridge. On the right, however, that is not the case. The graph stays connected regardless of what you remove.

Fact: Let $G(V, E)$ is a connected graph. An edge e is a brige iff we cannot form a cycle in G where e is an edge within such cycle.

Sub-Fact: We know that C_n has no bridges, because it is a cycled graph itself. This is trivial. On the other hand, for P_n, every edge is a bridge.

Sketch: Assume that e is a bridge. Show that every cycle of the graph, G (if such cycle exists), does not contain e has an edge.

$\Longleftarrow$ Assume C is a cycle of G st e is an edge of C. Hence $G - e$ is connected since $C - e$ stays connected. A contradiction. Thus our denial is invalid. We conclude that every cycle of G does not have e as an edge.

The converse: Assume G does not have a cycle C, where e is an edge of C. Show that $G - e$ is disconnected (i.e. e is a bridge). We know that since e is an edge of C, then if we remove it, it is no longer a cycle. Therefore e is the only path between some two vertices and thus it is a bridge.

March 10th, 2021

1. If we have a graph, $G(V, E)$m with $v \in V$, then v is a cut-vertex iff $\exists w, z \in V$ st every path from w to z must pass through v.

2. Consider $e \in E$. Then e is a bridge (cut-edge) iff e is not an edge of any cycle of G.

Consider the two sets, A and B. Then we have that the Cartesian product is defined by:

$$A \times B = \{(a, b) |\, a \in A \text{ and } b \in B\}$$

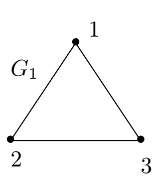
Now, what would this look like with graphs?

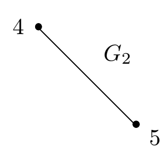
Def.: *Cartesian Product between two Graphs:* Imagine you have two graphs, $G_1(V_1, E_1), G_2(V_2, E_2)$. The notation: $G_1 \square G_2$ defines the Cartesian product of G_1 with G_2, where:

$$V = \{(a, b) |\, a \in V_1, b \in V_2\}$$

Two distinct vertices of V, say (a_1, b_1) and (a_2, b_2), are adjacent (connected by an edge) iff $a_1 = a_2$ and $b_1 — b_2 \in E_2$ OR $a_1 — a_2 \in E_1$ and $b_1 = b_2$.

Let us look at an example to be able to show this:





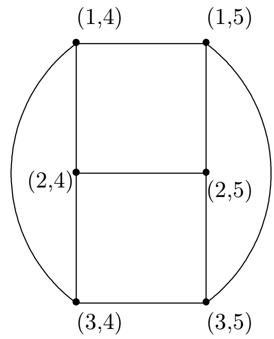
We say that the vertices of $G_1 \square G_2$ is $V_1 \times V_2 = V$, defined by the following pairs of vertices:

$$V = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\} \text{ and } |V| = |V_1| \times |V_2|$$

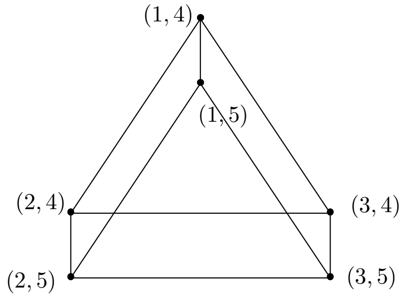
Let us look at them in another way.

$$\begin{matrix} (1,4) & (1,5) \\ (2,4) & (2,5) \\ (3,4) & (3,5) \end{matrix}$$

Is $(1,4)$ connected to $(1,5)$? Yes, beacuse $a_1 = a_2$ and $5 — 4$ is an edge in G_2. We continue in this fashion.



Another way of drawing this:



The graph above shows all the possible edges between the vertices of $G_1 \square G_2$.

Is this graph a tree? No, bceause there are cycles. Is the graph bipartite? No, because we can have a cycle: $(1,5) — (2,5) — (3,5) — (1,5)$, which is of odd degree. Therefore, it is not bipartite.

Since every edge is in a cycle, then we do not have any bridges within the graph. Are there any cut-vertices? No, because the graph remains connected regardless of any single removal of a vertex. We can choose any two vertices and find more than one path between them.

March 15th, 2021

Def.: Take two graphs, $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. Then we say that $G_1 \square G_2$ is an undirected, simple graph with vertex set $V = V_1 \times V_2 = \{(a,b) \mid a \in V_1 \text{ and } b \in V_2\}$ st. two vertices $(a_1, b_1), (a_2, b_2)$ are connected by an edge iff:

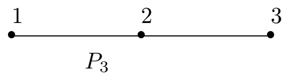
1. $a_1 = a_2$ and $b_1 — b_2 \in E_2$, or:

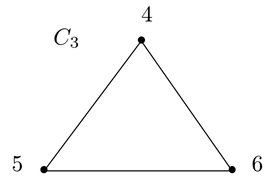
2. $a_1 — a_2 \in E_1$ and $b_1 = b_2$

We know that $|V| = |V_1| \times |V_2|$, and that if G_1 is of order n, with G_2 being of order m, then $G_1 \square G_2$ is of order $m\,n$.

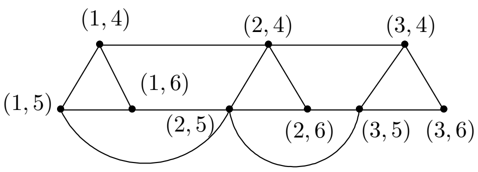
<u>How do we visualize the Cartesian Product, $G_1 \square G_2$?</u> Let us see if we can draw $P_3 \times C_3$.

<u>Solution</u>: We know how to draw C_3 and P_3, they will be drawn below:



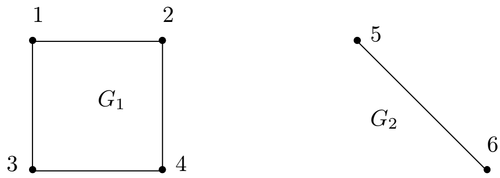


The steps are as follows. We will draw them to be able to visualize it at each step.

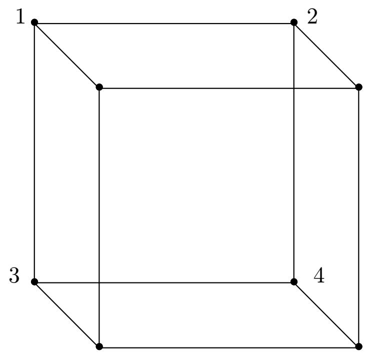


Since 5 is connected to 6 in C_3, then we must have that $(1,5)$ is connected to $(2,5)$. And similarly, $(2,5)$ is connected to $(3,5)$. Furthermore, we know that $(1,6)$ is connected to $(2,6)$, which in turn is connected to $(3,6)$.

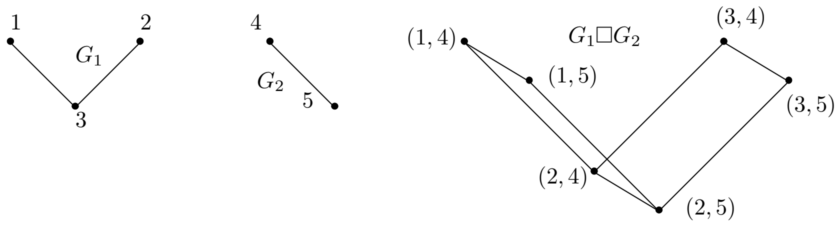
Now let us look at the following two graphs:



How would we draw the graph for $G_1 \square G_2$? At each vertex in the first graph, we put a copy of G_2. It will look as such:



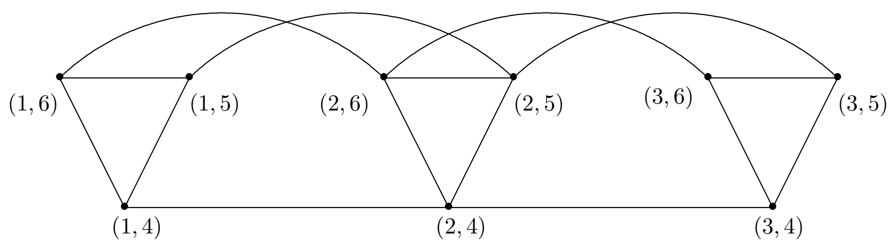
Another example:



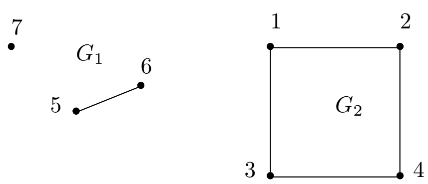
How to visualize $G_1 \square G_2$:

1. At each vertex of G_1, draw a copy of G_2
2. if $u, v \in V_1$ and $u — v \in E_1$, then connect tthe corresponding vertices with an edge.

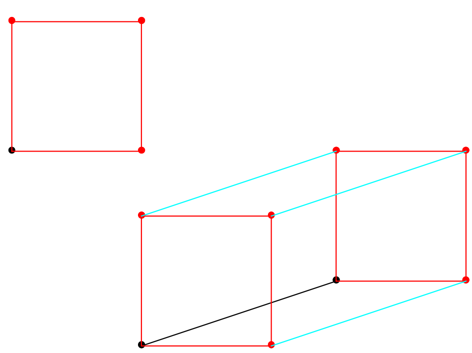
Let us try to re-visualize $P_3 \times C_3$, with an easier graph to see:



Question:



What will the graph of $G_1 \square G_2$ look like?



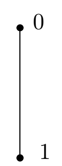
What can we observe from the graph above? If one of either G_1 or G_2 are disconnected, then the Cartesian Product is also disconnected.

Hypercube (n-cube):

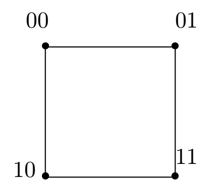
$$Q_1 = K_2 \text{ and } Q_2 = K_2 \times K_2$$

We will take them to be in the binary base. By this, we mean that K_2 and $K_2 \times K_2$ are drawn as such:

$Q_1 = K_2$

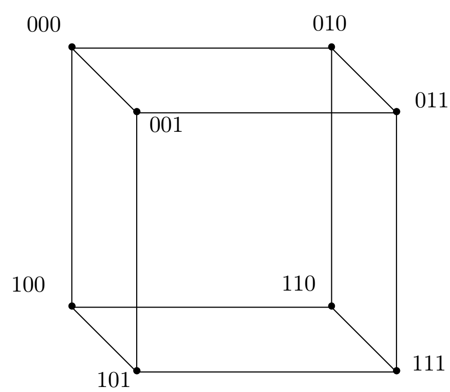


$Q_2 = K_2 \times K_2$



Continuing in this fashion, we take $Q_n = Q_{n-1} \times K_2$. Thus we know that:

$$Q_3 = Q_2 \times K_2 = K_2 \times K_2 \times K_2$$



There is an easy way to draw Q_n. We already know that:

1. $Q_n = Q_{n-1} \times K_2$

2. We have that $|V| = 2^n$ for the graph Q_n. Each vertex is an n-string of 0s and 1s.

3. Two vertices in Q_n are connected by an edge iff they differ in one and only one bit.

4. If $v \in V$, then $\deg(v) = n$. This implies that Q_n is always n-regular.

   Let us take the vertex 010 for example. We know that 010 is conneted to 110, 000, and 011. These are the three bit differences in 010. We also know that these vertices belong in Q_3.

5. $|E| = n\,2^{n-1}$. What is the proof of this?

$$\begin{aligned} \sum \deg(v_i) &= 2\,|E| \\ |V| = 2^n, \text{each of degree } n \\ \sum \deg(v_i) = n\,2^n &= 2|E| \\ |E| &= n 2^{n-1} \end{aligned}$$

6. $\operatorname{girth}(Q_n) = 4$ for all $n \geqslant 2$. There is no cycle of length 3 in the graph.

$$000 — 100 — 110 — 010 — 000$$

7. We can sketch Q_n is a bipartite graph. Why? Because it contains no odd cycles.

March 17th, 2021

Recall the concept of the hypercube, which is $Q_n = Q_{n-1} \square K_2$. What is the diameter of Q_n?

Question: Consider Q_4. Find the distance $d(0101, 0010)$.

Solution:

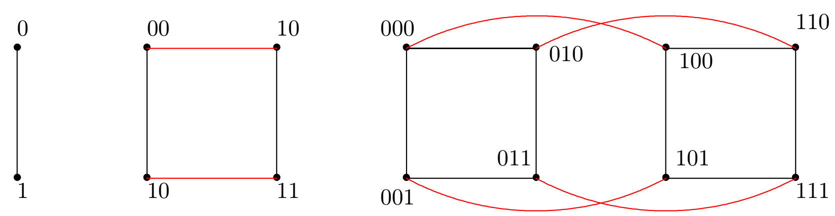
$$0101 \text{ --- } 0001 \text{ --- } 0000 \text{ --- } 0010$$

By changing only one bit at a time, we can see that there exists a path of length 3. This is the shortest possible path between the two vertices. Therefore, $d(0101, 0010) = 3$. Can we find a path of length 2? No. This is because the vertices differ in 3 bits. Essentially, we can see that the length of shortest path is the same as the Hamming distance.

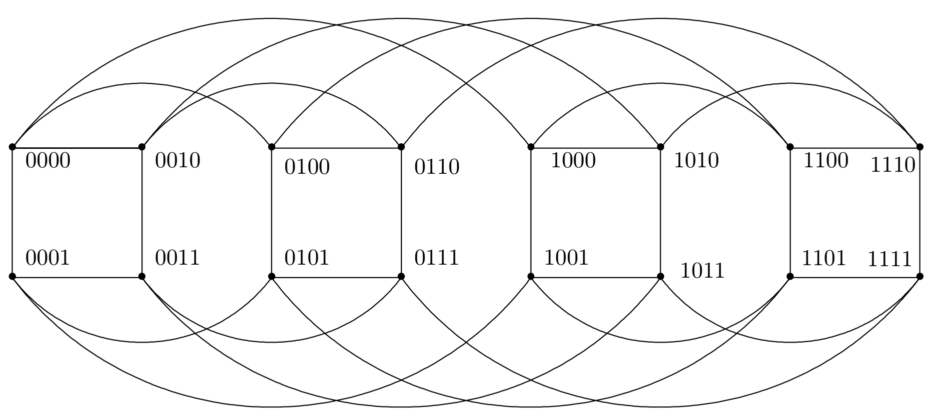
Now, what is $\text{diam}(Q_n)$? It is n. Why? $d(000\ldots00, 111\ldots11) = n$. The maximum number of bit changes is if we have to change every single one, which in a Q_n graph is equal to n.

In general, $d(v, w) =$ no. of differences in bits. There are many examples of this. It is trivial.

There is another way of constructing Q_n.

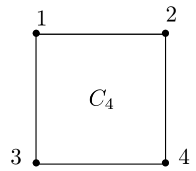


The idea is to replicate the previous layer and add a 0, 1 to the front. This is much nicer and easier than constructing through the hypercube. Now, let us see Q_4:

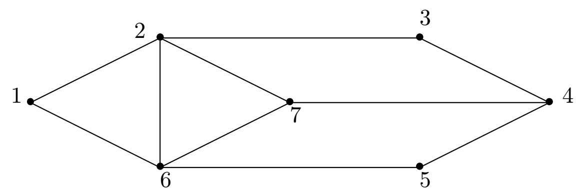


Def.: *Independent set of vertices:* Given a graph $G(V, E)$, the subset $I \subset V$ is called an independent set of vertices iff every two vertices in I are not adjacent (every two vertices in I are not connected by an edge).

Maximum Independent Set: The maximum number of vertices in a graph that are non-adjacent. Let us see the graph below to visualize this:



What is a maximum independent set of C_4? Considet the set of vertices $\{1,4\}$ or $\{2,3\}$. They are not adjacent to one another. Is $\{1,4,3\}$ an independent set? No. This is because $3 — 4$ is an edge. Now, let us see another graph:



What is a maximum independent set of our graph, G? We know that G is of order 7. There is more than one maximum independent set. However, they all share the same number of vertices.

$$\{1,3,5,7\}$$

In this question, this is the only maximum independent set. However, for example, $\{2,4\}$ is also an independent set, just not the maximum.

This is a maximum independent set. If a graph is complete bipartite, then we have $K_{m,n}$. Trivially, the maximum independent set is the bigger one of m, n.

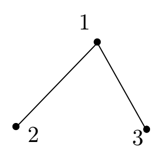
Let I be a maximum independent of vertices. $\alpha(G) = |I|$. In words, this is the size of the maximum independent set. If we say that $\alpha(I) = 4$, then every maximum independent set must have 4 elements (Similar fashion to dominating numbers & dominating sets).

We know that $\gamma(G) = 2$. Take the dominating set $\{2,4\}$. Is there another dominating set? Take $\{4,6\}$.

Def.: *Vertex-Cover:* Take a graph, $G(V,E)$ A subset $C \subset V$ is called a vertex cover of the graph iff every edge of the graph has a a terminal or initial vertex in C.

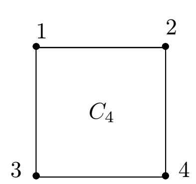
Look at the graph:

G



What is the vertex-cover of G? It cannot be $\{2\}$, because $1 — 3$ is an edge and therefore 1 is not a terminal vertex. The vertex-cover of G is $\{1\}$.

Another example:



View vertex-cover: If $a — b$ is an edge of G, then either $a \in C$ or $b \in C$. Thus we can see that $\{1,4\}$ is a vertex-cover, but $\{1,2\}$ is not. Why? Because the edge $\{3,4\}$ does not terminate at either 1 or 2. However, $\{1,4\}$ is a vertex-cover because every edge in C_4 terminates at either 1 or 4. $\{2,3\}$ is another example

What is a minimum dominating set of C_4? We can take $\{1,4\}$ or $\{2,3\}$. Is there a connection between the vertex-cover and the minimum dominating set?

March 22nd, 2021

Recall the independent set: A subset of vertices, I, where every two vertices in I are not connected through an edge.

Independence number: $\alpha(G) = |M|$, where M is a maximum independent set of vertices.

Vertex-cover (C): A subset of vertices st. whenever $a — b \in E$, then either $a \in C$ or $b \in C$.

Vertex-cover number: $\beta(G) = |C|$ where C is a minimum vertex-cover of G.

Result: For a graph $G(V,E)$, let C be a subset of V. Then C is a vertex-cover of G iff $V - C$ is an independent set. This means that the set of vertices not including the vertices in the vertex-cover are all non-adjacent.

Proof:

$$\begin{aligned}
&\Longrightarrow\\
&\text{Assume } C \text{ is a vertex-cover of } G\\
&\text{Show that } V - C \text{ is an independent set.}\\
&\text{Let } a, b \in V - C. \text{ Show } a — b \notin E.\\
&\text{Deny: } a — b \in E\\
&\text{Hence either } a \in C \text{ or } b \in C.\\
&\text{Contradiction, since } a, b \in V - C.\\
&\text{Hence } a — b \notin E.\\
&\text{Thus } V - C \text{ is an independent set.}
\end{aligned}$$

$$\begin{aligned}
&\Longleftarrow\\
&\text{Assume } V - C \text{ is an independent set.}\\
&\text{Show that } C \text{ is a vertex-cover.}\\
&\text{Assume } a — b \in E \text{ for some } a, b \in V\\
&\text{Show } a \in C \text{ or } b \in C\\
&\text{Since } a — b \in E, \text{ and } V - C \text{ indep., we conclude that:}\\
&a \text{ or } b \notin V - C\\
&\text{Why? Because if both } a, b \in V - C,\\
&\text{we cannot have the edge } a — b\\
&a \notin V - C \Longrightarrow a \in C\\
&b \notin V - C \Longrightarrow b \in C
\end{aligned}$$

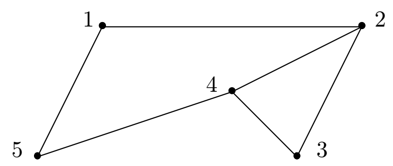
Result: Assume C is a vertex-cover. Then: $|C| + |V - C| = |V|$. This is trivial, and clear from the previous argument.

Result: Let $G(V,E)$ be a graph of order n. Then we have that $\alpha(G) + \beta(G) = n$.

Proof:

$$\begin{aligned}
\text{We know that } |V-C|+|C| = |V| = n \\
\text{This is true for any vertex-cover } C. \\
\text{Assume } C \text{ is a minimum vertex-cover.} \\
\text{Then } V-C \text{ is a maximum independent set.} \\
\Longrightarrow |V-C| = \alpha(G), |C| = \beta(G) \\
|V-C|+|C| = \alpha(G)+\beta(G) = n
\end{aligned}$$

Consider the following graph as an example:

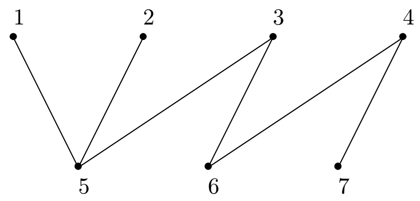


Give a minimum vertex-cover of G. Consider $\{1,2,4\}$. This is a minimum vertex-cover. Most likely, if you take the vertex with the highest degree, it works well as the vertex-cover.

$$V-C = \{3,5\}$$

This is a maximum independent set of G.

Another example:



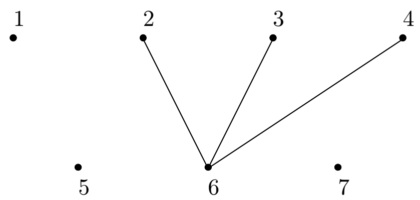
This is a bipartite graph (not complete bipartite). What is a minimum vertex-cover of the graph? Another way of denoting this graph is $B_{4,3}$.

$$C = \{5,6,7\}, \text{ and thus } |\beta| = 3$$

This means that the maximum independent set of G is:

$$V-C = \{1,2,3,4\}, \text{ and thus } \alpha(G) = 4$$

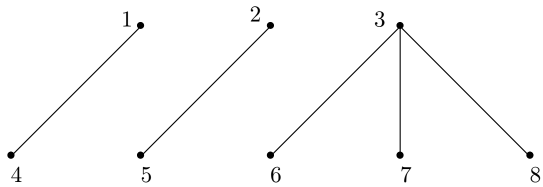
Let us look at another $B_{4,3}$, with diffferent edges:



We can see that the minimum vertex-cover of the graph is $\{6\}$, because all edges in the graph terminate at v_6. Thus the maximum independent set is $V-C = \{1,2,3,4,5,7\}$. Then $\beta(G) = 1$ and $\alpha(G) = 6$.

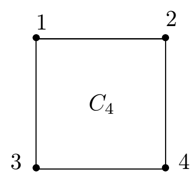
Result: Assume $B_{m,n}$ is connected. Then $\beta(B_{m,n}) = \min\{m,n\}$ and $\alpha(B_{m,n}) = \max\{m,n\}$. This is trivial since the graph is connected, and thus each vertex from the upper set is connected to some vertex in the lower set.

Consider the graph:



This graph is not connected. However, we can see that $C = \{1,2,3\}$ and $M - C = \{4,5,6,7,8\}$ is the maximum independent set. Thus $\alpha(G) = 5$ and $\beta(G) = 3$. This goes to show that the graph does not necessarily have to be connected for the result to hold.

Note: The domination set need not be the vertex-cover. Last lecture, we saw the example of C_4, where the dominating set was the same as the vertex-cover:



However, we will show that this is not always the case. Take P_4:

$$1 — 2 — 3 — 4$$

We know that $\{1,4\}$ is a minimum dominating set, but $\{1,4\}$ is not a vertex-cover. Why? Because $2 — 3$ is an edge that does not terminate at 1 or 4. The minimum vertex-cover is $\{2,3\}$, which is another dominating set. Can we prove that every vertex-cover is a dominating set? Yes, but the converse is not true.

March 24th, 2021

Fact: Let $G(V,E)$ be a connected graph and C be a set of vertices. If C is a minimum vertex-cover, then C is a dominating set. However, it need not be a minimum dominating set.

Proof:

$$\begin{aligned}
&\text{Let } C \text{ be a vertex-cover of } G\\
&\text{We will show that } C \text{ is a dominating set.}\\
&\text{Let } a \in V - C. \text{ We show } \exists b \in C \text{ st. } a — b \in E\\
&\text{Since } C \text{ is a vertex-cover, and } a — b \in E,\\
&b \in C\\
\\
&\text{Thus } C \text{ is a dominating set.}
\end{aligned}$$

Fact: Assume your graph $G(V,E)$ is connected of order n. Then $\alpha(G) + \gamma(G) = n$

Proof:

$$\begin{aligned}
&\text{Let } C \text{ be a minimum vertex-cover of } G\\
&\text{Then } \beta(G) = c_1 = \gamma(G) \text{ (By previous result)}\\
&\text{Let } M \text{ be a maximum independent set}\\
&\text{Hence } \alpha(G) = |M|\\
&\text{From last lecture, } \alpha(G) + \beta(G) = n\\
&\Longrightarrow \alpha(G) + \gamma(G) = n
\end{aligned}$$

If we find the maximum independent set of G, we can automatically find the vertex-cover and a dominating set.

Question: $G(V, E)$ is connected and of order n. Say M is a maximum independent set st. $|M| = m$, with $m < n$. Find a minimum dominating set and find $\gamma(G)$.

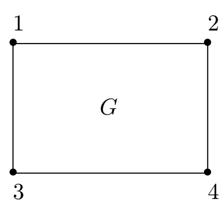
Solution: $C = V - M$, which is the minimum vertex-cover. But since the graph is connected, C is a minimum dominating set. We know that $\alpha(G) + \gamma(G) = n$, and thus $\gamma(G) = n - m$.

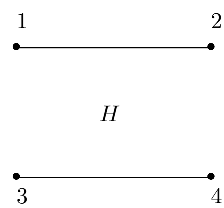
End of Content for Exam I

Def.: *Matching Subgraphs:* Consider the graph $G(V, E)$. A subgraph $H(V_1, E_1)$ of G is called matching iff for every $w \in V_1$, $\deg(w) = 1$. This is the degree of w in H. To make it more clear, we can say that:

$$\deg_H(w) = 1$$

Look at the following example:





$H = \{1 — 2, 3 — 4\}$

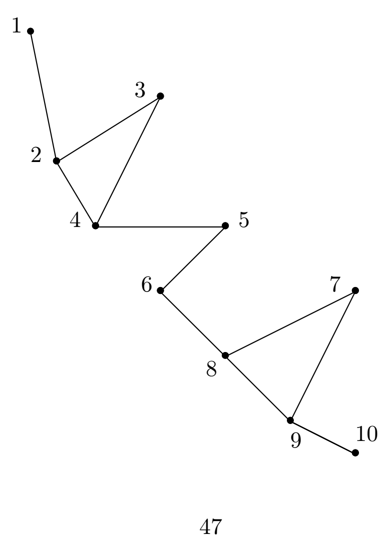
It is clear that H is a subgraph of G, but it is not an induced subgraph (The edges $1 — 3$ and $2 — 4$ are not present in H). However, H is a spanning subgraph of G, because $V_1 = V$ and $E_1 \subset E$.

Now, note the following: $\deg_H(1) = 1, \deg_H(2) = 1, \deg_H(3) = 1, \deg_H(4) = 1$. Since every vertex of H is of degree 1, then we conclude that H is a matching subgraph of G.

Equivalent Definition of Matching Subgraphs: A subgraph $H(V_1, E_1)$ is a matching subgraph of $G(V, E)$ iff every edge in E_1 has no common vertex with every other edge in E_1.

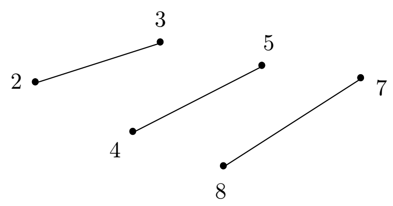
Common language: If $a — b$ and $c — d \in E_1$, then a, b, c, d are all distinct vertices.

One more way of saying it: $H(V_1, E_1)$ is a matching subgraph of G if every two edges in E_1 have no common vertex. Now, let us look at some examples.

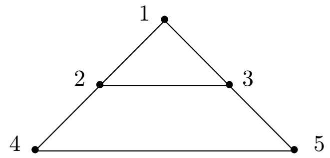


We claim that this graph, G, has a matching subset of size 3 (meaning that the set of edges of the subgraph has 3 elements).

Consider the graph: $H = \{2 — 3, 4 — 5, 7 — 8\}$. This is a matching subgraph of G. If we draw it, it would simply look like this:



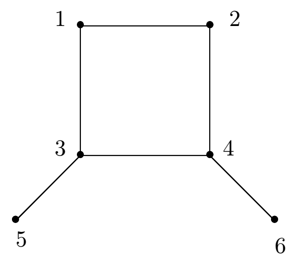
What are we interested in by looking at this? Look at this example of a graph:



A maximum matching subgraph of this would be: $H = \{1 — 2, 3 — 5\}$. Another one would be $F = \{2 — 3, 4 — 5\}$.

Def.: *Matching Number:* Let H be a matching of maximum size, say m. Then the matching number is equal to m.

Look at the following graph:



The maximum matching of this would be $H = \{1 — 2, 3 — 5, 4 — 6\}$. It is clear to see that by selecting the wrong edges, we can easily be mistaken. Notice that $\{1 — 2, 3 — 4\}$ is a matching subgraph, but it is not the maximum. Can we make a matching of size 4? No. It is impossible since we do not have 4 distinct pairs of vertices.

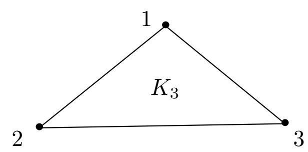
April 5th, 2021

Recall the definition of a matching set: Take $G(V, E)$, with $M \in E$. M is called a matching subgraph if whenever $a — b, c — d \in E$, then a, b, c, d are distinct vertices. Another way of saying this is: Every two edges in E have no common vertex.

$$m(G) = |M|, \text{where } M \text{ is maximum matching}$$

Example:

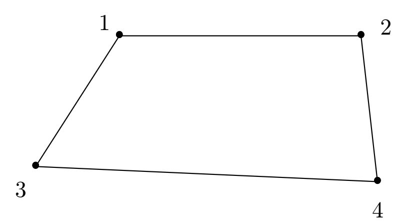
1)



In this case, $M = \{1 — 3\}$, or $M = \{2 — 3\}$, or $M = \{1 — 2\}$. Therefore, we know that $m(K_3) = 1$, which is the cardinality of the maximum matching set.

What if we take a square instead?

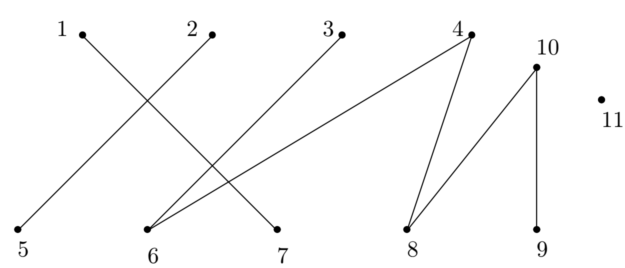
2)



Note that this graph is not K_4. Do not forget this. Now, let us see the possible maximum matching sets: $M = \{1 — 2, 3 — 4\}$ or $M = \{1 — 3, 2 — 4\}$. In both cases, we can lead to the conclusion that:

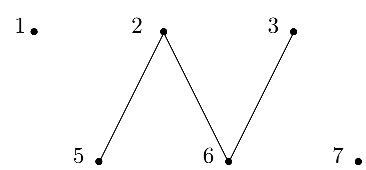
$$m(G) = 2 = |M|$$

3)



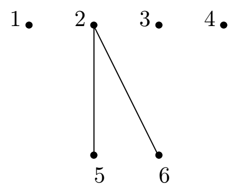
$$M = \{1 — 7, 2 — 5, 3 — 6, 4 — 8, 10 — 9\}, m(G) = 5$$

Look at the following example of a bipartite graph that will lead to a result about the matching number:

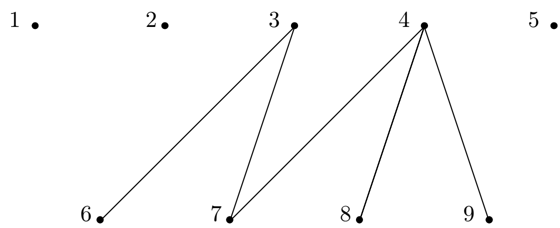


We know that $M = \{1 — 5, 3 — 6\}$, and thus $m(B_{4,3}) = 2$.

Result: Assume your graph G is $B_{m,n}$ st. $|A| = m$ and $|B| = n$. Assume $m > n$. Let h be the number of vertices in A that are connected by an edge to some vertices in B, and let k be the number of vertices in B that are connected to some vertices in A. Then $m(G) = \min\{h, k\}$.



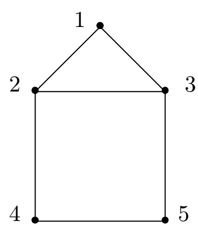
This is $B_{4,2}$, where $B = \{5, 6\}$. We can take another example:



This is $B_{5,4}$. Note that $k = 2$, and $h = 4$. Thus we know that $m(B_{5,4}) = \min\{4, 2\} = 2$. We can use this information to construct the minimum matching set:

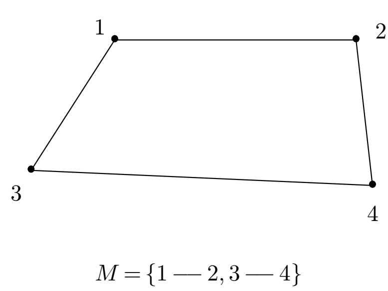
$$M = \{3 — 6, 4 — 9\} \text{ or } M = \{3 — 7, 8 — 4\}$$

If a graph has no odd cycles, then we know $m(G)$, because we can draw the graph as a bipartite. The problem arises when the graph has odd cycles. Let us demonstrate:



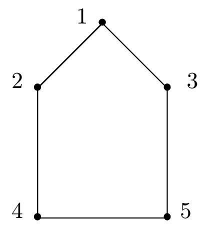
This graph contains an odd cycle $(1 — 2 — 3 — 1)$, which is of length 3. Therefore we cannot make a bipartite graph out of this. Thus we have to manually check to see what the maximum matching set is. We can come up with $M = \{1 — 3, 5 — 4\}$ or $M = \{3 — 5, 2 — 4\}$. These sets are of cardinality 2, which means that $m(G) = 2$.

Def.: *Perfect Matching:* Let M be a matching set of a graph $G(V, E)$, say $M = \{a_1 — b_1, a_2 — b_2, \ldots, \}$. Let us take the set $V_1 = \{a, b \,|\, a — b \in M\}$. If $V_1 = V$, then we say that M is a perfect matching set. In other words, if we take the vertices of all the edges in the match, then the set of vertices should be the same as the set of vertices in the original graph G. We are essentially using all the vertices in the graph.



$$M = \{1 — 2, 3 — 4\}$$

This is a perfect matching set, because it uses all 4 of the vertices that are in the graph G. Another example:



This is a graph representing C_5. We claim that this graph has no perfect matching sets. We can find a matching set for this graph: $M = \{1 — 2, 4 — 5\}$ or $M = \{1 — 2, 3 — 4\}$. These are maximum matching. However, they do not include all the vertices, and thus there is no perfect matching set.

Consider P_6:

$$1 — 2 — 3 — 4 — 5 — 6$$

What is a maximum matching set for P_6? Is there a perfect match for it?

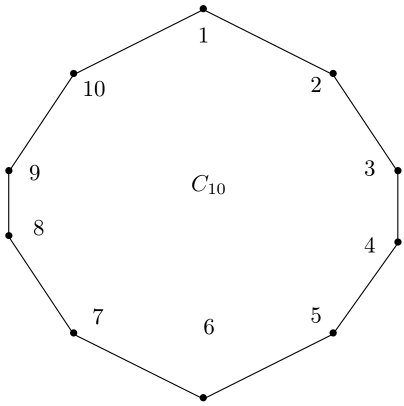
$$M = \{1 — 2, 3 — 4, 5 — 6\}$$

We can see that this matching set includes all the vertices in P_6, and thus M is a perfect matching set for P_6. On that note, $m(G) = 3$.

Note that evert perfect matching set is a maximum matching set, but it is not true the other way around. In other words, not every maximum matching set is a perfect matching set.

Result: A graph C_n or P_n has perfect matching set iff n is even. Furthermore, $m(C_n \text{ or } P_n) = \frac{n}{2}$.

We will take the example of C_{10} to demonstrate this result:



$$M = \{1 — 2, 3 — 4, 5 — 6, 7 — 8, 9 — 10\}, \text{ and } m(C_{10}) = 5 = \frac{10}{2}$$

<u>Proof</u>: It is trivial.

We know from this result that $m(C_n \text{ or } P_n) = \frac{n}{2}$ as long as n is even. But what can we say about $m(C_n \text{ or } P_n)$ if n is odd instead?

$$m(C_n \text{ or } P_n) = \frac{(n-1)}{2} = \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{for } n \text{ odd}$$

When we have a tree, we have to redraw it as a bipartite graph, and we apply the earlier result taking the minimum between the two sets' connections to each other.

Result: We say that $K_{m,n}$ has a perfect matching set iff $m = n$.

Proof:

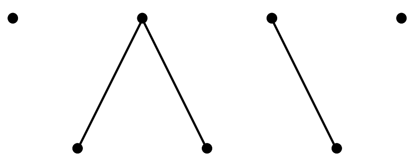
$$\text{We have that } m(K_{m,n}) = \min\{m, n\}$$

This is because every perfect matching set is a maximum, and by the first result, we know we have to include all the vertices for it to be maximum, and this implies that $m = n$. Otherwise, there is no way to choose the perfect matching set.

April 7th, 2021

Def.: *Edge-Cover*: Consider a graph $G(V, E)$. A subset of G, denoted $E_C \subset E$ is called an edge-cover of G iff $\forall a \in V, \exists$ some edge a — $b \in E_C$, for some $b \in V$.

Exp:

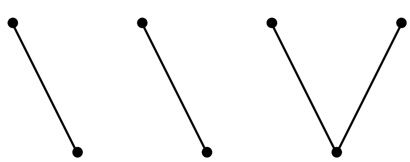


What is a maximum matching for this graph? $\{2 — 5, 3 — 7\}$. In other words,

$$M = \{2 — 5, 3 — 7\}$$

However, this graph has no edge-cover, because this graph is not connected, and has vertices of degree 0. Let us look at another example:

Exp:



We can see that this graph is also not connected. However, this graph does NOT have isolated vertices (vertices of degree 0). We will proceed: $M = \{1 — 5, 2 — 6, 3 — 7\}$. But what would be the edge-cover?

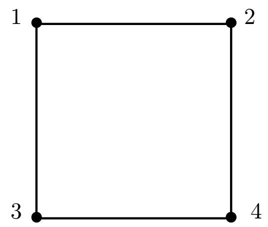
$$E = \{1 — 5, 2 — 6, 3 — 7, 4 — 7\}$$

This is the minimum edge-cover, because we cannot come up with a smaller set.

$$\beta_e(G) = |E_C| \,\text{st.}\, E_C \,\text{is minimum}$$

We know that $m(G) = 3, \beta_e(G) = 4 \Longrightarrow |V| = 7$.

<u>Exp</u>: Consider the graph of C_4:



$$\begin{aligned} M = \{1 — 2, 3 — 4\}, m(C_4) = 2 \\ E_C = \{1 — 2, 3 — 4\} \\ m(C_4) + \beta_e(C_4) = |V| = 4 \end{aligned}$$

Result: Consider $G(V, E)$, graph with no isolated vertices (no vertices of degree 0). Then we can say that:

$$m(G) + \beta_e(G) = n = |V|,$$

where n is the order of G.

<u>Proof</u>:

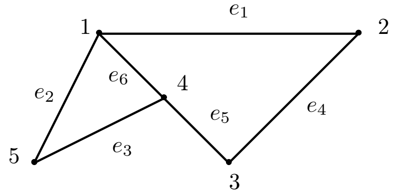
Assume M is a maximum matching set. Asssume that $\beta_e(G) \leqslant m(G)$.

This proof was left incomplete and will be revisited in a later lecture.

Def.: *Incident:* Given a graph $G(V, E)$, where $e \in G \,\text{st.}\, e = a — b$ for some $a, b \in V$. Then we say that e is incident at a and e is incident at b. When we have that e is incident at a vertex a, then e could be one of two things: $e = a — b$, or $e = b — a$.

This can lead to another definition of the degree of a vertex: The number of edges that are incident at the vertex, say a.

Question: Assume we have a labeled graph $G(V, E)$, where labeled means that all edges and vertices have labels.

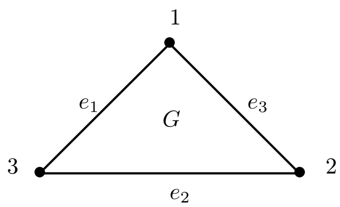


The incidence matrix:

	e_1	e_2	e_3	e_4	e_5	e_6
1	1	1	0	0	0	0
2	1	0	0	1	0	0
3	0	0	0	1	1	0
4	0	0	1	0	1	1
5	0	1	1	0	0	0

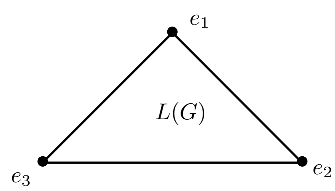
The sum of the numbers in each row is the degree of the vertex, and the sum of the numbers in each column is always 2, because each edge connects only 2 vertices.

Line Graphs: Let us demonstrate what a line graph is through an example. Consider the following graph G, which we will use to construct $L(G)$.

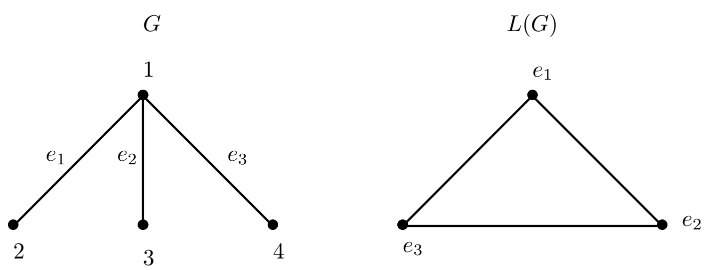


We swap out the vertices with the edges, which are labeled in G.

$e_m, e_n \in V(L(G))$ are connected by an edge in $L(G)$ iff they have a common vertex in G. This means that they are incident at the same vertex in G.



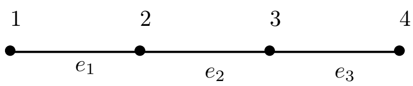
We can see that $L(G) \approx G$, and in fact the example shown is K_3. In other words, $L(K_3) \approx K_3$. We can see another graph:



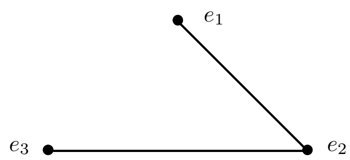
Assume that we have $L(G_1) \approx L(G_2)$. Does this necessarily mean that $G_1 \approx G_2$? No. This is not the case. We can see that in the examples we just provided. We showed in the examples that $L(K_3) = K_3$, but also that $L(K_{1,3}) \approx K_3$. Howwever, we know that $K_3 \not\approx K_{1,3}$.

Exp:

P_4:



$L(P_4)$:

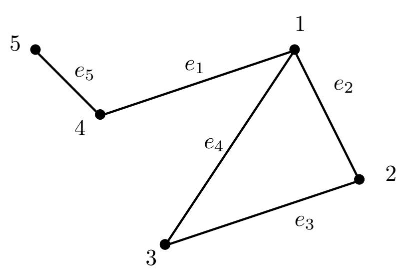


We can see that $L(P_4) \not\cong P_4$, because in fact $L(P_4) = K_{1,2}$.

We can also observe that if we have a graph G of size m and order n, then the line graph $L(G)$ will be of order m.

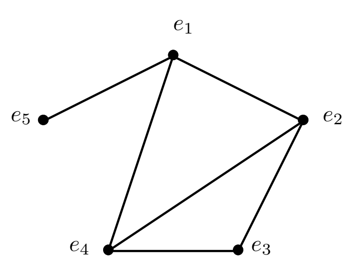
April 12th, 2021

Consider the following graph:



$$V_L = \{e_1, e_2, e_3, e_4, e_5\}$$

We can draw $L(G)$ as such:



The order of $L(G)$ is equal to the size of G.

Result: Assume that G is of order n and size m. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of the original graph. Assume $d_1, d_2, \ldots, d_n$ are degrees of the vertices of V respectively, ie. d_1 is the degree of vertex v_1, ...

Then we can say that size$(L(G)) = \frac{d_1^2 + d_2^2 + d_3^2 + \cdots + d_n^2 - 2m}{2}$

Sketch: The idea is to choose a vertex v_i, with $\deg(v_i)=d_i$, where we have $1 \leqslant i \leqslant n$. d_i's edges are connected to v_i.

The number of edges in $L(G)$ that connect the d_i's edges, or d_i's vertices in $L(G)$. The number of edges in $L(G) = d_1C2 + d_2C2 + \cdots + d_nC2$, where C is the combinational choice. Thus we will have the following:

$$\begin{aligned}\frac{d_1(d_1-1)}{2}+\frac{d_2(d_2-1)}{2}+\cdots+\frac{d_n(1-d_n)}{2}\\ =\frac{d_1^2-d_1+d_2^2-d_2+\cdots+d_n^2-d_n}{2}\\ =\frac{d_1^2+d_2^2+\cdots+d_n^2-(d_1+d_2+\cdots+d_n)}{2}\\ (d_1+d_2+\cdots+d_n)=2m=2|E|\end{aligned}$$

Question: Assume the degrees of the vertices of a graph of order 5 are: $3, 2, 1, 1, 1$. Find the order and the size of $L(G)$.

Solution: The order of $L(G) = \frac{\sum \text{degrees of } G}{2}$. Thus we will have order$(L(G)) = 4$. To find the size, we will use the formula:

$$\frac{9+4+1+1+1-2(4)}{2}=4$$

Result: Let w be a vertex in $L(G)$, in other words w is an edge of the graph G. Then $\deg(w)$ would be: $\deg(a)+\deg(b)-2$, where $w = a - b$, an edge of G st. $a, b \in V_G$.

Assume h is adjacent to w in the line graph $L(G)$. Then h and w have either a as a common vertex or b are a common vertex. Thus:

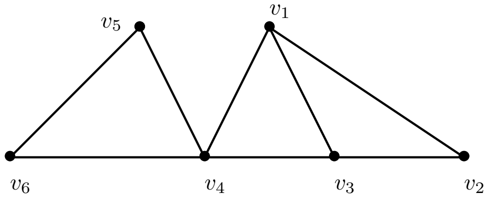
$$\begin{aligned}\deg(w) &= [\deg(a)-1]+[\deg(b)-1]\\ &=\deg(a)+\deg(b)-2\end{aligned}$$

Def.: *Eulerian Graph:* F_m ("Fake cycle") has m edges of order $n \leqslant m$, but vertices are allowed to be repeated, but the edges are not. Formal definition: A graph of order n and size m is called Eulerian iff it is connected and F_m is a subgraph of G. In common language:

$$a - a_1 - a_2 - \ldots - a$$

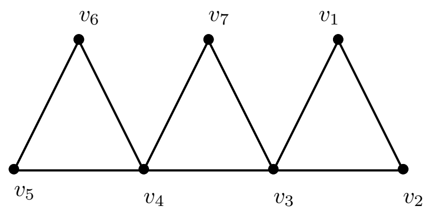
This cycle contains all distinct edges of the original graph, but $a_1, a_2, \ldots a_n$ need not be distinct. In other words, in the cycle, we can visit each edge exactly once, but vertices can be visited more than once.

Exp:



This graph is not Eulerian, because we cannot have any cycle that would contain all the edges and is visited only once. Remember that a cycle means that we have to start and finish at the same vertex.

We can see another example:



We claim that G is Eulerian. Then we can construct F_9:

$$2 — 1 — 3 — 4 — 5 — 6 — 7 — 5 — 3 — 2$$

The edges are all distinct, but we can see that we visited some vertices more than once, such as 3 and 5.

Note that a fake cycle is the same as a circuit. Fake cycle is the Dr. Ayman Badawi term for it.

Result: A connected graph $G(V, E)$ is Eulerian iff $\deg(v)$ is an even integer for evert $v \in V$.

Def.: *Semi-Eulerian:* A connected graph $G(V, E)$ is called semi-Eulerian if there is a fake path, or a trail: $a — b_1 — b_2 — \ldots — b_k — b$ with $a \neq b$. The vertices need not be distinct, but it has all edges of G.

April 14th, 2021

Recall ***Def.:*** *Eulerian Graph:* A graph is called Eulerian if it is connected and it has some F_m, circuit, that contains all edges distinctly in G.

Result: A connected graph is Eulerian iff the degree of every vertex is an even integer.

Sketch: First we prove that a graph G st. the degree of each vertex is $\geqslant 2$, contains a cycle.

mini-Sketch: Assume G is of order n, and we have that $v_1 — v_2 — v_3$. If we have that $v_3 — v_1$ is an edge, then we automatically have a cycle. Therefore, we assume that $v_3 — v_1$ is not an edge. We can continue with $v_1 — v_2 — v_3 — v_4$. If $v_4 — v_1$, then we have a cycle, and so on and so forth. This process must terminate because the graph is of order $n < \infty$. Hence at some point we must have $v_k — v_i$ an an edge for some $1 \leqslant i \leqslant k - 2$.

Now to prove the Eulerian result:

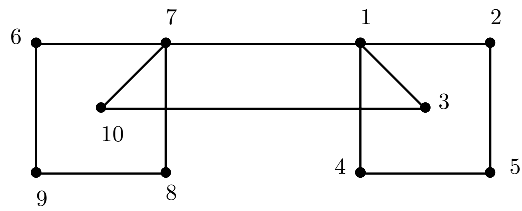
$\Longrightarrow$ Assume the graph is Eulerian. We show that the degree of each vertex is an even integer $\geqslant 2$. G has order n and size m. It should have a circuit or a fake cycle F_m, denoted as:

$$F_m: v_1 — v_2 — v_3 — \ldots — v_k — v_1$$

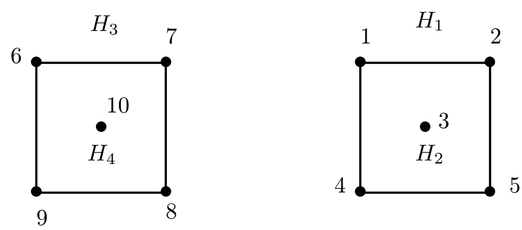
This cycle has exactly m distinct edges. Note that not all vertices need to be distinct, but once again, all edges are. Everytime we visit a vertex v_i in F_m, there will be two edges connected to it. Since the edges of F_m are distinct, we conclude that $\deg(v_i) = 2K$ for some $K \geqslant 1$.

$\Longleftarrow$ Assume each vertex of G is an even integer $\geqslant 2$. We will show that G is Eulerian. Since the degree of each vertex is $\geqslant 2$, we have already proved that the graph G must have a cycle C. If C cotains all edges in G, then we are done. Assume C does not contain all edges of G. We prove the converse by induction. Assume every connected graph with even degree-vertices and of order $<m$ is Eulerian.

We first remove all edges from C. Consider the following graph:



Take C to be: $1 — 3 — 10 — 7 — 1$. We can see that all edges have even degree. If we remove these edges, the order of the graph will remain to be n, but the new graph will look as such:



When we remove the edges in the cycle, then we will have a disconnected graph. Let $H_1, H_2, \ldots, H_k$ be the components of G. In this example, we have 4 components in total, but the order is the same. The degree of each vertex of every component is either 0 or an even integer. This is because if we remove the edges in the cycle, we reduce the degree by 2. Also, note that each component must have at least one vertex of C.

H_1 must contain a vertex of C, say v_1. Size of $H_1 < m$, and the degree of each vertex of H_1 is even and it is definitely connected by the definition of the components. $\Longrightarrow$ H_1 has a circuit (In our example, it is $1 — 2 — 5 — 4 — 1$.

H_2 must contain a vertex of the cycle C. In our case, it is 3. We can go from 1 to 3, and from 3 we can go to the next component, H_4 with the vertex 10, and so on.

The idea is to remove the edges of a cycle, because the number of edges of each component will be less then m. Each component will also have an even degree. We then keep track of the vertices of the components to form a new cycle. Each component will be Eulerian.

Recall ***Def.:*** *Semi-Eulerian:* A connected graph $G(V, E)$ is called semi-Eulerian if there is a fake path, or a trail: $a — b_1 — b_2 — \ldots — b_k — b$ with $a \neq b$. The vertices need not be distinct, but it has all edges of G. The initial vertex and the terminal vertex cannot be the same.

Result: A connected graph is semi-Eulerian iff exactly 2 vertices in the graph are of one degree.

Proof: Assume your graph is semi-Eulerian. We will show that G has exactly 2 vertices of odd degree. We can take our fake path (trail):

$$v_1 — v_2 — v_i — \ldots — v_1 \neq v_k$$

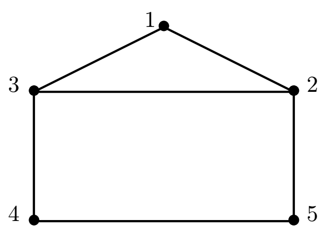
and it contains all edges of G. This means that the degree of every vertex in the trail is even except v_1 and v_k. If v_1 is a repeated vertex, then it will have even degree, which is a contradiction. The degree of v_1 and v_k have to both be odd.

Is an Eulerian graph also semi-Eulerian? No. It will never be the case.

Def.: *Hamiltonian Graph:* A connected graph of order n and size m is Hamiltonian iff C_n is a subgraph of G

Def.: *Hamiltonian Path:* A connected graph G of order n and size m is called a Hamiltonian path iff P_n is a subgraph of G.

Exp:



Is this graph Eulerian? No. It is not. Is the graph semi-Eulerian? Yes, because there are exactly two vertices that are of odd degree (2 and 3).

We can construct a trail:

$$2 — 5 — 4 — 3 — 1 — 2 — 3$$

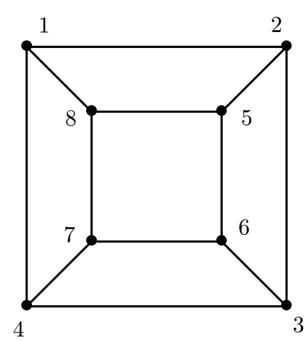
This graph is Hamiltonian, because C_5: $1 — 2 — 5 — 4 — 2 — 1$ is in the graph. It is also a Hamiltonian path because it contains P_5: $1 — 2 — 5 — 4 — 3$. In fact, we can conclude that every graph that is Hamiltonian also contains a Hamiltonian path.

April 19th, 2021

Recall ***Def.:*** *Hamiltonian and Hamiltonian Path:* A connected graph $G(V,E)$ of order n is Hamiltonian iff C_n is a subgraph of G. $G(V,E)$ is a Hamiltonian path iff P_n is a subgraph. Clearly a Hamiltonian graph is a Hamiltonian path, but the converse is not true.

Result: Assume that $G(V,E)$ is connected and of order n. Assume that $\deg(x)+\deg(y) \geqslant n$ for every non-adjacent pair of vertices, x and y. The conclusion is that G is a Hamiltonian graph.

Exp: Construct a Hamiltonian graph of order 7. We will look at the trivial case: C_7. Now, look at the following graph:

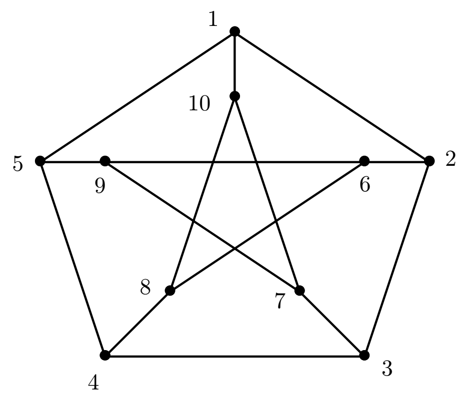


This graph is definitely not Eulerian nor is it semi-Eulerian. However, is this graph Hamiltonian? In other words, can we find C_8 as a subgraph of this graph? Consider the following:

$$1 — 2 — 5 — 8 — 7 — 6 — 3 — 4 — 1$$

Therefore, since C_8 is a subgraph of G, then it is a Hamiltonian graph.

Def.: *Petersen Graph:* Connected of order 10 and of size 15, and has the following shape:

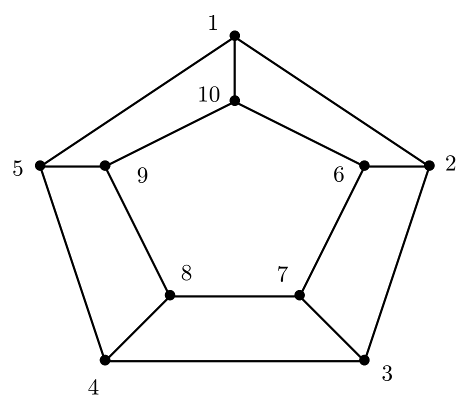


It is clear that the Petersen graph is 3-regular. Therefore it is definitely not Eulerian. However, is it Hamiltonian? No, it is not. However, it is a Hamiltonian path. We consider the following:

$$1 — 2 — 3 — 4 — 5 — 9 — 6 — 8 — 10 — 7$$

This is P_{10}, and therefore we conclude that it is a Hamiltonian path. It is interesting to note, however, that if we remove one vertex from this graph, then it will always be Hamiltonian. In other words, $G - \{v\}$ is Hamiltonian for any vertex $v \in V_G$.

Consider the following graph:



This is a graph of order 10 and size 15. However, this graph $G_1 \not\approx$ Petersen graph. This graph is, unlike the Petersen graph, Hamiltonian. We can construct:

$$C_{10}: 1 — 2 — 3 — 4 — 5 — 9 — 8 — 7 — 6 — 10 — 1$$

Def.: *Chromatic Number:* The minimum number of colors needed to color the vertices of a graph st. every two adjacent vertices have different colors. It is denoted as $\chi(G)$

Def.: *Chromatic Index:* The minimum number of colors needed to color the edges of a graph st. every two incident edges (every pair of edges that share a vertex) have different colors. It is denoted as $\chi'(G)$

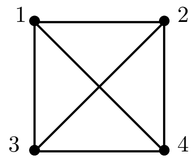
Exp: Consider the graphs for K_n.

For K_3:

$$\chi(K_3) = 3$$

$$\chi'(K_3) = 3$$

For K_4:



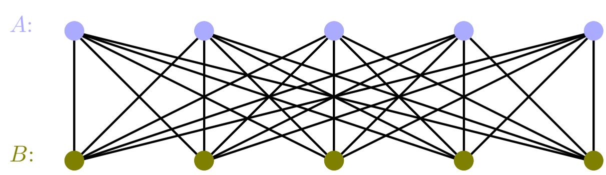
$$\chi(K_4) = 4$$

$$\chi'(K_4) = 4$$

We can see that for every n, the graph of K_n would result in $\chi(K_n) = n$ and $\chi'(K_n) = n$. However, what is the chromatic number of a complete bipartite graph?

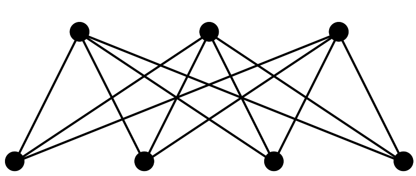
$$\chi(K_{n,m}) = 2$$

Why is this the case?

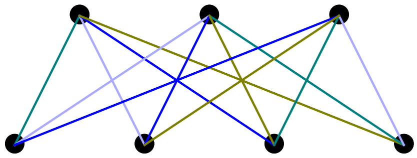


Since each set A, B contains vertices that are non-adjacent, then we only need two colors. The same can be applied for $\chi(B_{n,m})$ iff not all vertices are isolated. It follows the same principle, as the sets contain vertices that are non-adjacent. Completion is not a requirement.

Now, let us consider the graph of $K_{3,4}$:



What is $\chi'(K_{3,4})$? Our claim is that it is going to be 4. All of the degrees of the above set are 4, and therefore the maximum number of incident edges is going to be 4. We can draw it as such to see:



We can see that we have 4 distinct colors, and we can see the formula:

$$\chi'(K_{n,m}) = \max\{n, m\}$$

What about the cyclic graph? We can see that:

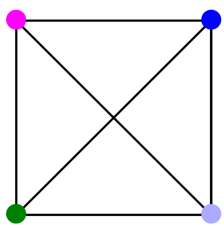
$$\chi(C_n) = 2 \text{ for } n \text{ even}$$
$$\chi(C_n) = 3 \text{ for } n \text{ odd}$$

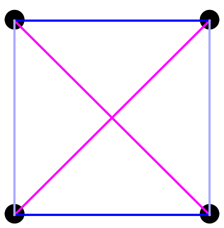
April 21st, 2021

Recall ***Def.:*** *Chromatic Number:* The minimum number of colors needed to color the vertices of a graph st. every two adjacent vertices have different colors. It is denoted as $\chi(G)$

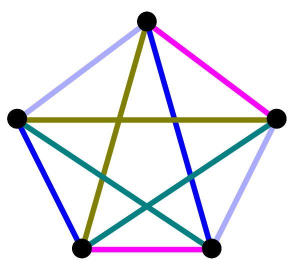
Recall ***Def.:*** *Chromatic Index:* The minimum number of colors needed to color the edges of a graph st. every two incident edges (every pair of edges that share a vertex) have different colors. It is denoted as $\chi'(G)$

Also, we recall that $\chi(K_n) = n$ and $\chi'(K_n) = n$. We will visualize with the graph for K_4:





Now, let us see for K_5:



We can see in this case that we had to use 5 distinct colors, so for n odd, we have $\chi'(K_n) = n$.

$$\chi(K_n) = n$$
$$\chi'(K_n) = n - 1 \text{ for } n \text{ even}$$
$$\chi'(K_n) = n \text{ for } n \text{ odd}$$

$$\chi(K_{n,m}) = 2$$
$$\chi'(K_{n,m}) = \max\{n, m\}$$

$$\chi(P_n) = 2$$
$$\chi'(P_n) = 2$$

$$\chi(C_n) = 2 \text{ for } n \text{ even}$$
$$\chi(C_n) = 3 \text{ for } n \text{ odd}$$
$$\chi'(C_n) = 2 \text{ for } n \text{ even}$$
$$\chi'(C_n) = 3 \text{ for } n \text{ odd}$$

Is there a relation between the edge-coloring of a graph and another type of graph? The line-graph! We can see that $\chi'(G) = \chi(L(G))$. In other words, the edge-coloring of a graph is equal to the chromatic number of the line graph.

Notation: We say that $\Delta(G)$ is the maximum degree of a vertex. This will lead to our result:

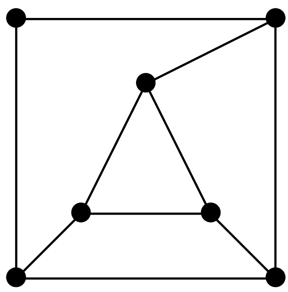
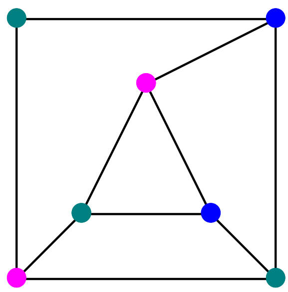
Result: If $G(V, E)$ is bipartite, then $\chi'(G) = \Delta(G)$. In other words, the edge-coloring index of a graph is equal to the maximum degree of the vertices in the graph. This is another way of saying that $\chi'(G) = \max(m, n)$ for $G = K_{m,n}$.

Brook's Theorem: Let G be a graph st. $G \neq K_n$ and $G \neq C_m$ for some odd integer m. Then we can say that $\chi(G) \leqslant \Delta(G)$.

If we take K_4 for example, we know that $\Delta(K_4) = 4$, and $\chi(K_4) = \Delta + 1$. Furthermore, we can make the following observations:

$$\Delta(C_n, n \text{ odd}) = 2$$
$$\chi(C_n, n \text{ odd}) = 3 = \Delta + 1$$

Exp: Consider the following graph:

By the theorem, we know that $\chi(G) \leqslant 3 = \Delta(G)$. We can see from the graph on the right that $\chi(G) = 3$. Can we find an example of a graph where $\Delta(G) \neq \chi(G)$? Consider $G = K_{10,10}$. Then we know that $\Delta(K_{10,10}) = 10$, but since it is a biparite, then automatically $\chi(K_{10,10}) = 2$.

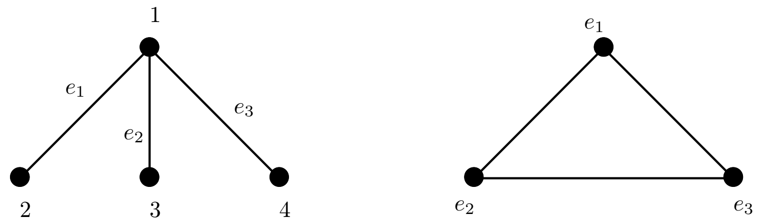
On the other hand, we can say that the chromatic index is always bigger or equal the maximum degree of the vertices in the graph. Mathematically, we say that $\chi'(G) \geqslant \Delta$.

For any graph, we know that the maximum possible chromatic number is $\chi(G) = \Delta + 1$. Furthermore, we have that $\chi'(G) = \chi(L(G)) \leqslant \Delta + 1$. From this, we conclude:

$$\chi'(G) = \Delta \text{ on } \Delta + 1$$

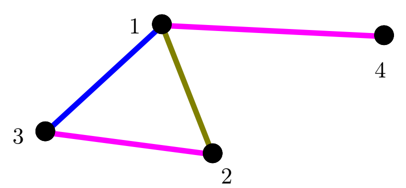
Question: When will $\chi'(G)$ be $\Delta + 1$? iff $L(G) = K_n$ or $L(G) = \text{odd cycle}$, and this is by Brook's theorem.

What is $\chi'(K_{1,3})$? We know that it is 3. However, consider $L(K_{1,3})$:



It is easy to see that $L(K_{1,3}) \approx K_3$. Thus we can connect the results: $\chi'(K_{1,3}) = \chi(K_3) = 3$.

Exp: Look at the following graph:



Then we have that $\chi'(G) = 3$.

We need to construct a graph where the line graph is an odd cycle in order to find a graph st. $\chi'(G) = \Delta$. Is it true that $\chi'(G) = \Delta + 1$ iff $G = K_n$ or odd cycle?

April 26th, 2021

Recall that:

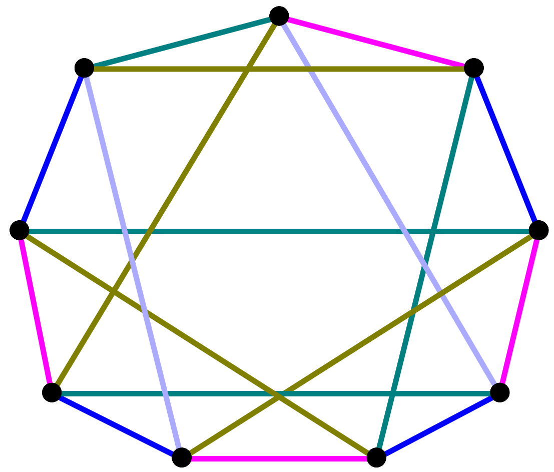
$$\chi'(K_n) = \Delta(K_4) = n - 1 \text{ for } n \text{ even}$$
$$\chi'(K_n) = \Delta + 1 = n \text{ for } n \text{ odd}$$

So far, we have dealt with graphs that are connected for the sake of understanding the chromatic index and number. However, what do we do if the graph is not connected? Then we say that the chromatic index is the maximum of the chromatic index of each component of the graph, and the same applies for the chromatic number.

Recall ***Result:*** If a graph G is bipartite, then we have that $\chi'(G) = \Delta$.

We can also recall that $\chi'(C_n) = \Delta + 1 = 3$ if n is odd. This will lead us into the next result, which is given as such:

Result: Assume G is connected and k-regular of order n, where n is odd. Then we can conclude that $\chi'(G) = \Delta + 1 = k + 1$. This result is a special case of the above fact that $\chi'(C_n) = 3$ for n odd. We can look to the following graph, where we have 9 vertices, 4-regular:



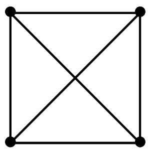
We can see that the graph has chromatic index 5, ie. $\chi'(G) = 5 = \Delta + 1$.

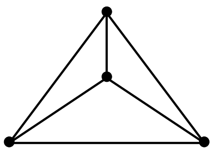
This is based off of Brook's theorem:

Recall ***Brook's Theorem:*** G is connected, then $\chi(G) \leqslant \Delta$ except for K_n and C_n for n odd.

Def.: *Planar Graphs:* A connected graph is called planar if it can be drawn on a piece of paper st. the edges intersect only at the vertices.

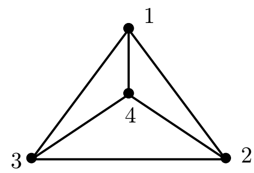
Exp:



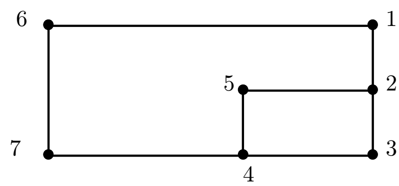


This is the graph for K_4, is it planar? Not drawn like the first, but we can see from the second drawing of it that is planar. It is important to see that the condition for a graph to be planar is that it can be drawn like that.

Def.: *Faces of Planar:* Consider the same graph for K_4:



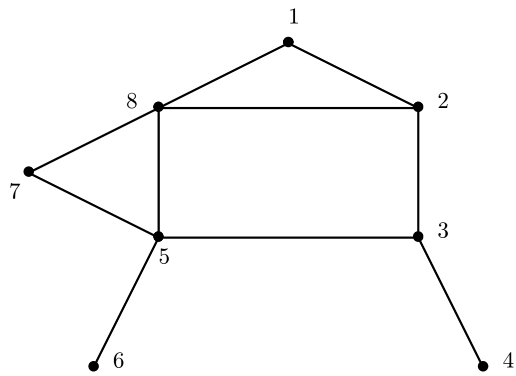
How many faces does this graph have? We claim that the faces are $4 — 1 — 2 — 4, 1 — 3 — 4 — 3$, and finally $3 — 4 — 2 — 3$. These are the three faces of this graph. We think of it as taking scissors and cutting out of the graph without changing anything. Another face is the whole table itself. This is the trivial case. A face cannot be partitioned into smaller faces. Therefore, K_4 has 4 faces. Let us look at another graph:



By staring, we can see that this graph is planar.

$$1 — 2 — 5 — 4 — 7 — 6 — 1$$
$$2 — 3 — 4 — 5 — 2$$

In total, we have 3 faces for this graph. Another example:



How many faces does this graph have? $3+1=4$. They are trivial to see. The order of this graph is 8, and it has 10 edges. Notice that $8-10+4=2$. This leads to our result:

Result: Let G be a connected planar of order n and size m. Then $n-m+f=2$, where f is the number of faces.

Sketch: Since the graph is connected and planar, we can start from C_3 and build the graph from there. We add one vertex each time, which means that n goes up by 1 and so does m, since we cannot just add edges outside of the vertices. This means that the number will never change.

April 28th, 2021

Recall ***Result:*** If G is connected, of order n and size m, then $n-m+f=2$. This leads to a second result:

Result: Assume G is a connected planar graph of order n and size m. Then $m \leqslant 3n-6$. We will also have another result using this result:

Result: Assume G is a connected planar graph of order n and size m. Then $3f \leqslant 2m$.

Sketch: If we assume that each face consists of C_3 (3 edges for each face). Note that the default face has all edges. If we put these two pieces of information together, we will have that $3f \leqslant 2m$. Now, we can return to $n-m+f=2$. From the above result, we have that $f \leqslant \frac{2m}{3}$. Thus:

$$\begin{aligned} n-m+f &= 2 \\ n-m+\frac{2m}{3} &\geqslant 2 \\ 3n-3m+2m &\geqslant 6 \\ 3n-m \geqslant 6 \Longrightarrow m &\leqslant 3n-6 \end{aligned}$$

Question: Convince me that K_5 is non-planar. This means that we cannot draw it st. the edges do not cross.

Solution: For K_5, $m=10$ and $n=5$. Can we see that $m \leqslant 3n-6$? $10 \nleqslant 3(5)-6=9$. Therefore, we know that K_5 is not planar.

Does this also mean that K_6 is non-planar? Since K_5 is a subgraph of K_6, and thus it cannot be planar. This leads to this fact:

Fact: K_n is planar iff $2 \leqslant n \leqslant 4$.

Note that we can have a connected graph where $m \leqslant 3n-6$, but this does not necessarily mean that G is planar. This relationship is not iff.

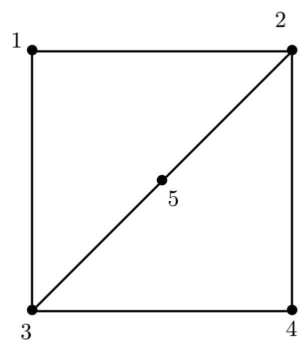
Exp: Consider the graph for $K_{3,3}$

$$m \leqslant 3n-6$$
$$9 \leqslant 3(6)-6 \Longrightarrow \text{true}$$

Assume $K_{3,3}$ is planar. Then $n-m+f=2$. Thus $6-9+f=2 \Longrightarrow f=5$. What is the girth of $K_{3,3}$? The length of the shortest cycle in $K_{3,3}$ is 4, thus $\text{girth}(K_{3,3})=4$. Hence $4f \leqslant 2m \Longrightarrow f \leqslant \frac{18}{4}$, but this is never equal to 5. Therefore we have a contradiction. Despite the fact that $m \leqslant 3n-6$, we can see that $K_{3,3}$ is non-planar.

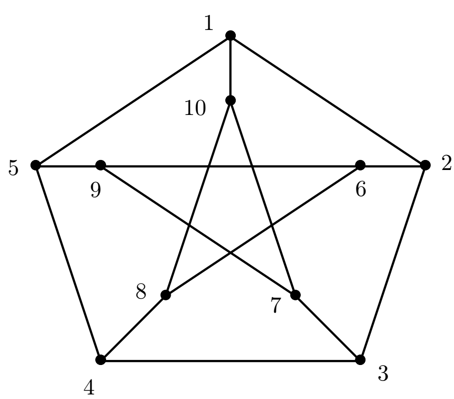
Remark: Assume a connected graph has girth $=k$, $k \geqslant 3$. Then $kf \leqslant 2m$, where m is the number of edges. Note that $k \neq \infty$.

Is $K_{3,2}$ planar? Yes. This means that $n-m+f=2$ and $m \leqslant 3n-6$. Let us try to draw this graph. First, we find f to make this easier. $f=3$.



We can see that we have 3 faces in total, and that this graph is isomorphic to $K_{3,2}$. We can also see that $K_{n,m}$ where $n \geqslant 3$ and $m \geqslant 3$ is non-planar, and the simple explanation for this is that $K_{3,3}$ is always a subgraph of this.

Recall the Petersen graph:



Properties:

1. This graph is non-planar;

2. It is a Hamiltonian path;

3. It is not Hamiltonian, unless we remove exactly one vertex;

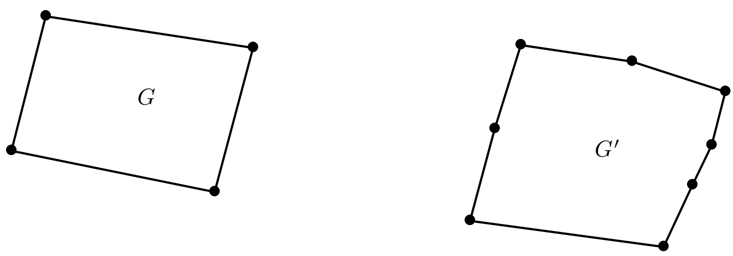
4. The Petersen graph is 3-regular, of order 10 and size 15. The chromatic index,

$$\chi'(G) = 4 = \Delta + 1$$

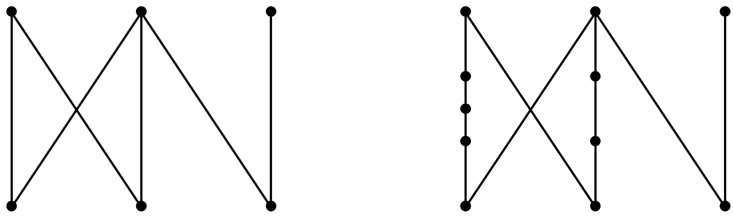
Why is it non-planar? The $m \leqslant 3n - 6$ holds. However, if the Petersen graph is planar, then $f = 7$. we have that $\text{girth}(G) = 5$, meaning that $5f \leqslant 30 \Longrightarrow f \leqslant 6$, which is a contradiction.

Recall the n-cube or Q_n. We know that Q_3 has 8 vertices and 12 edges. We have that Q_2, Q_3 are planar, while Q_n for $n \geqslant 4$ are non-planar. We simply have to show that Q_4 is non-planar, because everything else contains it is a subgraph.

Def.: *Subdivision Graph:*



We take the original edges and divide them into further "fragments." The graph on the right is a subdivision of the graph on the left. Consider the example of:



Again, we can see that the graph on the right is a subdivision of the graph on the left, because the edges are fragmented into smaller edges that are connecting other vertices.

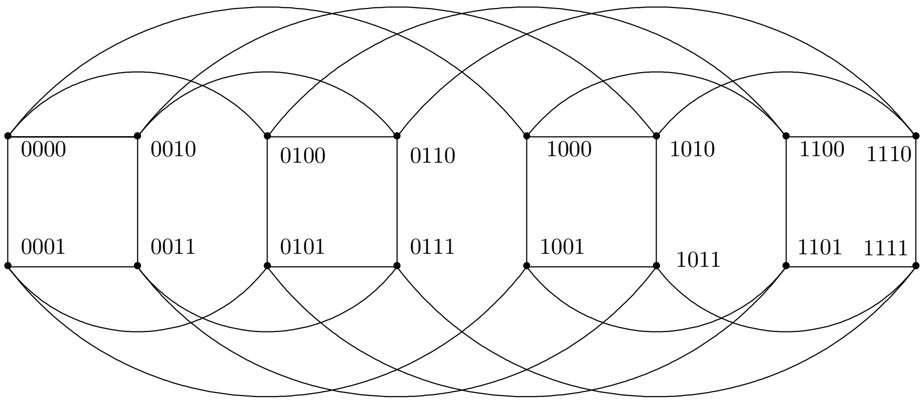
Big Result: A connected graph G is planar iff one of the following condition holds: G does not have a subgraph that is a subdivision of $K_{3,3}$ or K_5.

May 3rd, 2021

Recall the big result from last lecture:

Kuratowski's Theorem: Consider $G(V, E)$, a connected graph. Then G is planar iff it does not have a subgraph that is a subdivision of $K_{3,3}$ or K_5.

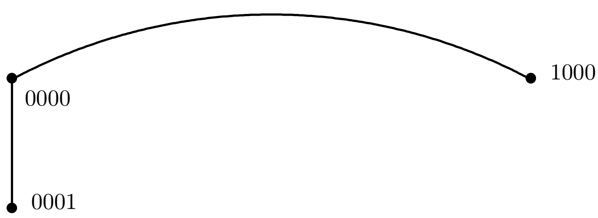
We will use this theorem to convince ourselves that the 4-cube or Q_4 is not planar. This means that we will show that Q_4 must have a subgraph that is a subdivision of $K_{3,3}$ or K_5.



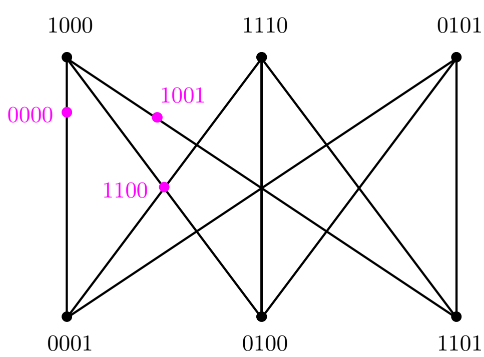
We select 6 total vertices in the graph of Q_4:

1000	1110	0101
0001	0100	1101

We can see that this is somewhat similar to the graph of $K_{3,3}$, since we have 3 vertices in the top set and 3 in the bottom. Out of these 6, none are connected to one another. However, can we, for example, find some vertex st. it connects to both 1000 and 0001? Yes, it is the vertex 0000.

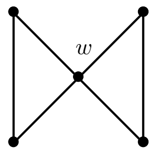


Now, we know that 1000 and 0100 are not connected through an edge, so we find a vertex that is connected to both of them: 1100. We proceed in the same fashion to connect the vertices highlighted above:



By doing so, we can see that Q_4 contains a subdivision of $K_{3,3}$, and thus the graph of Q_4 is non-planar.

Exp: Consider the graph of $K_{2,2}$



In this case, is the graph of $K_{2,2}$ with the new vertex w a subdivision of $K_{2,2}$? Yes, does this mean that we can share an edge within a subdivision of a graph? This is the question at hand.

We can also go through another method to show that Q_4 is not planar. We were previously shown that $m \leqslant 3n-6 \Longleftrightarrow 3f \leqslant 2m$, and this is based on the assumption that the girth of a graph is 3. This new formula:

Fact: If girth$(G)=4$, and G is a connected planar, then we have that $m \leqslant 2n-4$.

Sketch:

$$\begin{aligned} 4f \leqslant 2m &\Longrightarrow f \leqslant \frac{m}{2} \\ n-m+f=2 &\Longrightarrow n-m+\frac{m}{2} \geqslant 2 \\ &2n-2m+m \geqslant 4 \\ &\Longrightarrow m \leqslant 2n-4 \end{aligned}$$

Using this fact, we can show that Q_4 is non-planar. In Q_4, we have $n=16$ and $m=32$. Therefore, we come up with the equality:

$$\begin{aligned} &(\text{Recall girth}(Q_n)=4 \text{ for } n \geqslant 4) \\ m \leqslant 2n-4 &\Longrightarrow 32 \leqslant 2(16)-4 \\ &32 \leqslant 28\text{: False} \end{aligned}$$

Therefore, this is another way of showing that Q_4 is non-planar.

What if the girth of our graph is 7 (For a connected planar graph)? Then we proceed as follows:

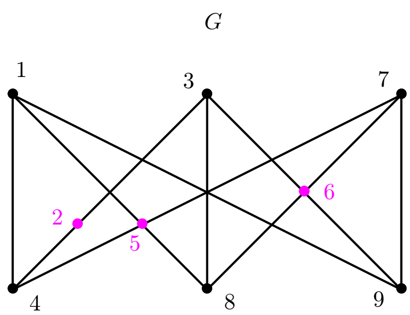
$$\begin{aligned} 7f \leqslant 2m &\Longrightarrow f \leqslant \frac{2m}{7} \\ &n-m+f=2 \\ &7\left[n-m+\frac{2m}{7} \geqslant 2\right] \\ &7n-7m+2m \geqslant 14 \\ &\Longrightarrow m \leqslant \frac{7}{5}n-\frac{14}{5} \end{aligned}$$

If we know the girth of a graph, then we can play around and change the relationship between the number of edges and the number of vertices.

Fact: Q_k is planar iff $K=2,3$. It is not planar for any other value.

Fact: $K_{n,2}$ is a planar. Why is this the case? It will never have a subgraph that is a subdivision of $K_{3,3}$ or K_5. Therefore, trivially, it cannot be non-planar.

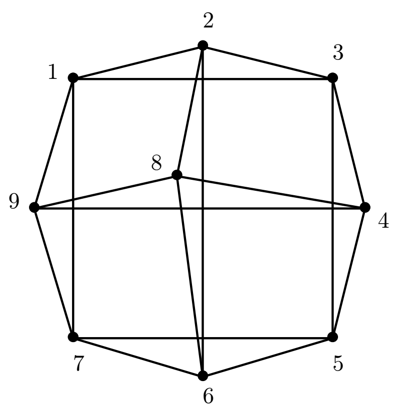
Exp:



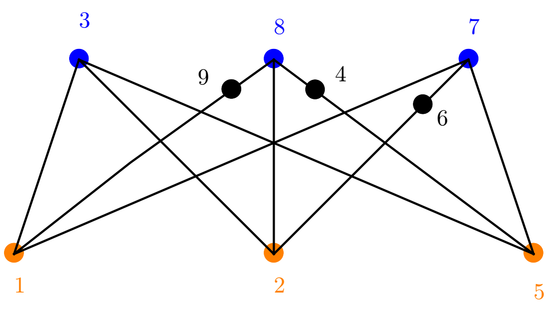
Show that G is not planar. Since the graph itself is a subdivision of the graph of $K_{3,3}$, then we know by default that it cannot be planar. This is by Kuratowski's theorem, which states that a graph is planar iff it does not contain a subdivision of $K_{3,3}$ or K_5.

May 5th, 2021

Exp: Show that the following graph is non-planar:



We can see that the graph has a subgraph that is a subdivision of $K_{3,3}$. Let us construct this subgraph:



By construction, we can see that we have a subdivision of $K_{3,3}$ in the graph, and therefore it is non-planar. Now, let us try the formulas to prove the same:

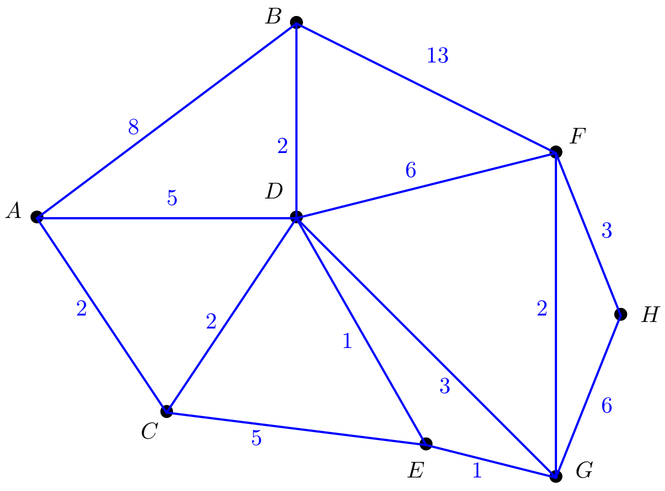
$$\begin{aligned} m &\leqslant 3n-6 \\ m &= \frac{4\times 9}{2} = 18 \\ 18 \leqslant 3(9)-6 &\Longrightarrow 18 \leqslant 21 \end{aligned}$$

Therefore, it satisfies this condition. We can look at another method / formula:

$$\begin{aligned} 9-18+f=2 &\Longrightarrow f=11 \\ 3f\leqslant 2m &\Longrightarrow 3(11)\leqslant 36 \\ & 33 \leqslant 36 \end{aligned}$$

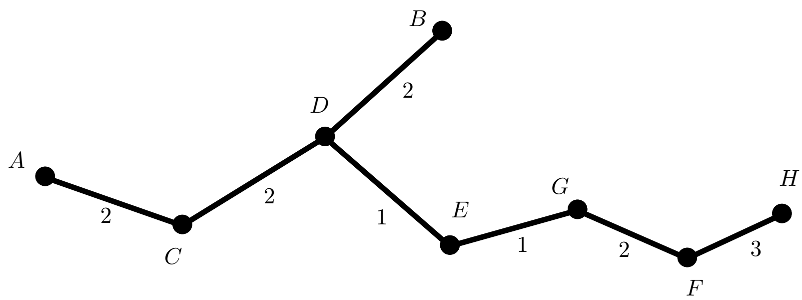
This condition is also satisfied. This means that regardless of what formula we try to use, we end up having to construct the subdivision of $K_{3,3}$.

Dijkstra's Algorithm: We construct a tree so that the weighted path between every two vertices is minimum. Consider the graph below:



How do we construct the tree st. the weighted path between each two vertices is a minimum?

	A	B	C	D	E	F	G	H
A	$\underline{0_A}$	8_A	2_A	5_A	∞	∞	∞	∞
C	$-$	8_A	$\underline{2_A}$	4_C	7_C	∞	∞	∞
D	$-$	6_D	$-$	$\underline{4_C}$	5_D	10_D	7_D	∞
E	$-$	6_D	$-$	$-$	$\underline{5_D}$	10_D	6_E	∞
B	$-$	$\underline{6_D}$	$-$	$-$	$-$	10_D	6_E	∞
G	$-$	$-$	$-$	$-$	$-$	8_G	$\underline{6_E}$	12_G
F	$-$	$-$	$-$	$-$	$-$	$\underline{8_G}$	$-$	11_F
H	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$\underline{11_F}$



The tree shown above is that of the least weighted path, according to the algorithm that is highlighted in the table above. In words, this is how the algorithm works:

1. Take the first vertex and look at all the adjacent vertices, look at the weight / distance between the first vertex and the others;

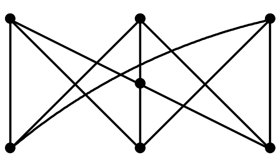
2. Take the minimum distance, this will be the first vertex connected. Then we move on to the second vertex and consider the distances between that vertex and the rest, excluding the first vertex;

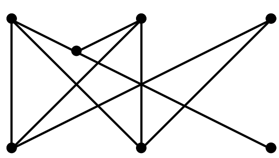
3. If the distance between that vertex and the others is less than the sum of the distance of the first vertex and the new additional vertex, replace it with that. From here, we again take the minimum, and that will be the next vertex;

4. Continue in this fashion until we reach the end of the set of vertices. Based on the indexed weight between two vertices, we can decide where we want the vertex to go in the construction of the tree.

May 17th, 2021

Recall the idea of subdivisions:

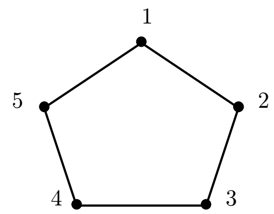




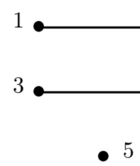
The one on the left is a subidivision of $K_{3,3}$, while the one on the right is not. This is because you cannot share the same "path" to get from one vertex to the other, but you can share the same added vertex to get from one to the other.

Def.: *K-factor* Let $G(V, E)$ be a connected graph. A spanning subgraph H (using all vertices) that is K-regular is called the K-factor of the original graph, G.

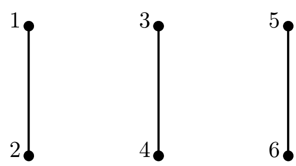
Exp: Does C_5 have a 1-factor subgraph?



No. We cannot have a spanning subgraph of C_5 where each vertex is of degree x, which in our case could only be 1.



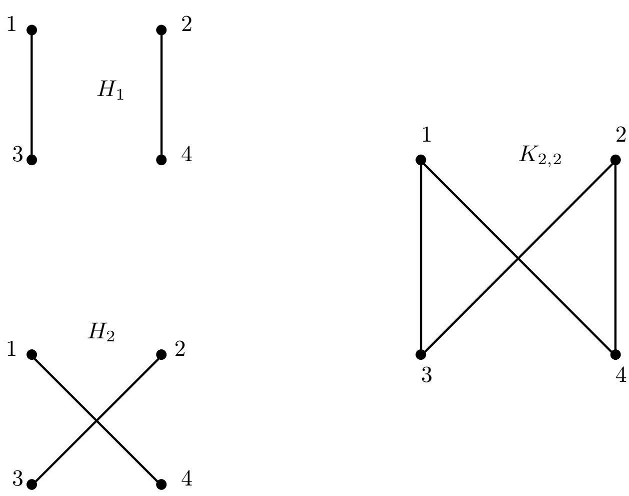
However, we know that C_6 is a K-fold graph because of the fact that we can draw it as follows:



This is a spanning subgraph of C_6 that is 1-factor. Note that the subgraph, H, is a perfect matching of C_6.

Result: A connected graph $G(V, E)$ of order n has a 1-factor spanning subgraph iff it has a perfect matching set. This also means that we cannot have an odd order, since a perfect matching set needs to be of even order anyway.

Idea behind K-factor: This is like a puzzle, we take the pieces and when we put them together, we have the graph. Consider the graph of $K_{2,2}$:

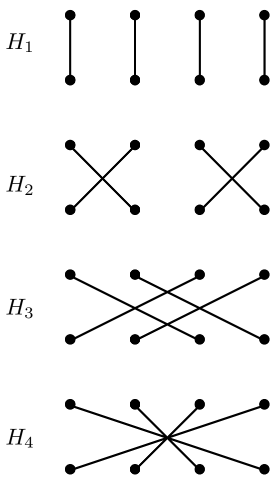


We can see that both H_1 and H_2 are two spanning subgraphs of $K_{2,2}$ that are 1-factors. If we both the two together, then we clearly get $K_{2,2}$. Recall the Cartesian product (similar to the idea i n Abstract Algebra):

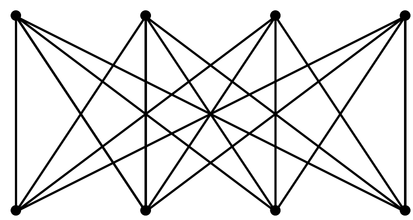
$$K_{2,2} = H_1 \oplus H_2$$

Consider $K_{4,4}$. Can we write it as a composition of some K-factor? Yes, we can write it as 4 1-factors.

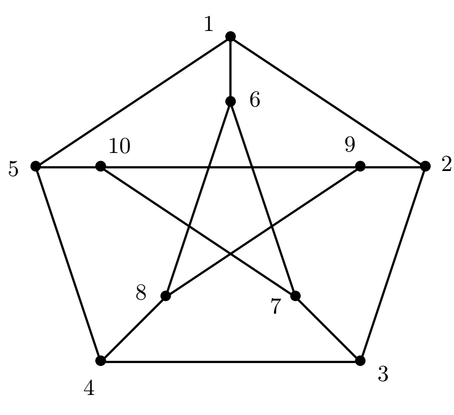
$$K_{4,4} = H_1 \oplus H_2 \oplus H_3 \oplus H_4$$
$$\text{where each } H_i \text{ is 1-factor}$$



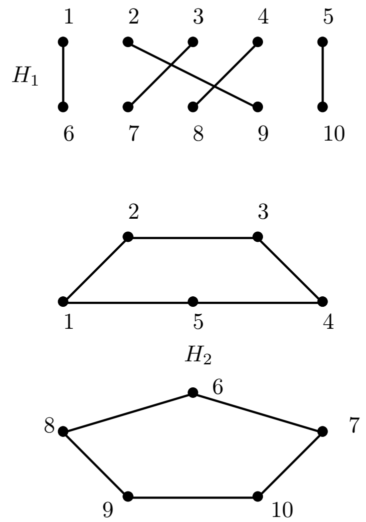
$K_{4,4}$



Let us now consider the Petersen graph: Recall that it is 3-regular, not planar and the chromatic index, $\chi' = \Delta + 1 = 4$. Can we draw a composition of the Petersen graph into some K-factors?

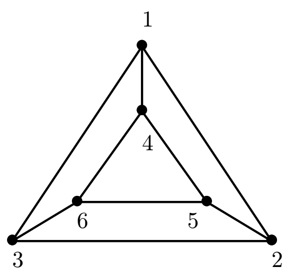


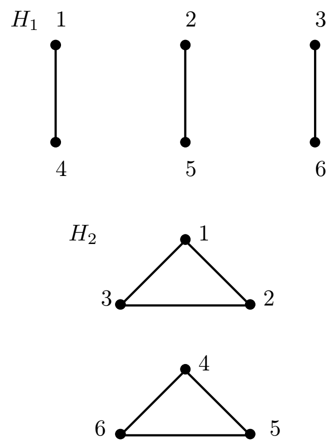
There is no way that we can draw this graph as some Petersen $= H_1 \oplus H_2 \oplus H_3 \oplus \cdots \oplus H_n$ where each H_i is some K-factor. However, what if $H_1, \ldots, H_n$ are not of the same K-factor? We can draw the Petersen graph as $H_1 \oplus H_2$ where H_1 is 1-factor and H_2 is 2-factor.



We can see that if we "combine" H_1 and H_2, then we will get the Petersen graph. Also, it is clear that the Petersen graph has a perfect matching, and we would expect at the beginning for it to work as a composition of some K-factor graphs. The problem arises because the pentagon and the star in the middle both have an odd number of vertices.

Now, consider the following graph:





We can see that the graph on the left is 3-regular and of order 6. Howwever, we cannot split it into some H_i that are K-factor, unless they are of different factors. The two graphs on the right show the composition, showing that $G = H_1 \oplus H_2$ where H_1 is a 1-factor and H_2 is a 2-factor. In the final, we might get a graph that we are familiar with and see whether or not we can factor it. There is, however, no theorem on how we can actually do it. It is mostly trial and error.

Consider the graph of $K_{3,2}$. Can we do some partition for this? It has no 1-factor. But does it have a 2-factor? No. What about $K_{4,2}$? It is of order 6, which is even, but not every even ordered graph has a vertex match. $K_{4,2}$ has no perfect matching so it cannot be 1-factor. It also cannot be any K-factor, as we can easily see through an example of checking for 2-factor. There will always be a repeated vertex.

What about $K_{4,3}$? Can we construct a 2-factor of this graph? We can prove that $K_{6,5}$ does not have a spanning subgraph that is K-regular, and then generalize.

Proof: Assume H is a spanning subgraph that is K-regular. Then:

$$\sum_{i=1}^{n} \deg(v_i) = K(6+5) = K(11) = 2|E_H|$$
$$\text{But } K \text{ is odd and } K(11) = \text{odd.}$$
$$\Longrightarrow K \text{ cannot be odd. Contradiction}$$

Therefore, $K_{6,5}$ cannot have a spanning subgraph that is 1, 3, or 5-regular, or any odd number, But we still need to check to see if it has a spanning subgraph that is 2-regular. In the next lecture, we will try to generalize this for $K_{m,n}$.

19th May, 2021

Def.: *K-factorable*: A connected graph $G(V, E)$ is called K-factorable, $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, where each H_i is a K-factor of the original graph G. Recall that each H_i is a K-regular spanning subgraph of the original G. When a graph is K-factorable and we have n compositions, that means that our graph G is $(n \times K)$-regular.

....

Open Problem: *(Conjecture)*

Assuume G is connected, K-regular of order $n = 2h$.

1. If h is odd, and $K \geqslant h$, then our graph G is 1-factorable.

2. If h is even, and $K \geqslant h - 1$, then our graph G is 1-factorable.

We do not have a mathematical proof for this. However, using programs and straight computation, we can get the feel that this is correct. Let us come up with some examples where this is right. Consider $K_{2,2} \longrightarrow$ 2-regular, $n = 2(h)$ where $h = 2$ and $K = h = 2$. Then:

$$K_{2,2} = H_1 \oplus H_2$$

where H_1, H_2 are both 1-factors. Now, consider $K_{n,n}$:

$$K_{n,n} = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

where each H_i is again, 1-factor. We have generalized this for any $K_{m,n}$ where $m = n$.

....

Result: Let $G(V, E)$ be a connected graph of order n. G has a 2-factor subgraph iff G has a Hamiltonian cycle.

Proof:

$\Longrightarrow$Assume G has a spanning 2-regular subgraph, H. Then $H = 1 — 2 — 3 — 4 — \ldots — n-1$. This implies that the graph is Hamiltonian, where $H = C_n$.

$\Longleftarrow$ Assume that G is Hamiltonian. This implies that $C_n = H$ is a spanning 2-regular subgraph of the original graph. This is exactly what we mean when we say that C_n is a 2-regular subgraph of G.

Now, when does the graph of $K_{m,n}$ have a 2-regular spanning subgraph? This graph is Hamiltonian iff $m = n$. $K_{3,2}$, for example, is not Hamiltonian. When we try to do it, we will never have enough edges to go back to the first.

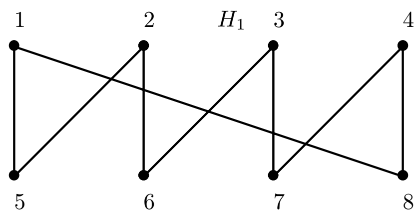
Sub-result: $K_{m,n}$ is Hamiltonian iff $m = n$.

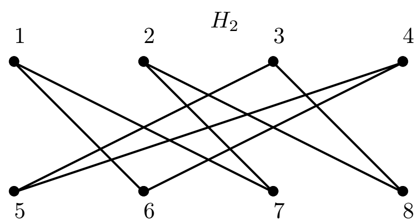
Thus, we can see that, as an example, $K_{6,5}$ does not have a 2-factor spanning subgraph because it is not Hamiltonian. This leads us to the conclusion: $K_{m,n}$ has a 2-factor subgraph when $m = n$.

What about the case of K_n, with $n \geqslant 3$? It has a 2-factor because we can write it as:

$$1 — 2 — 3 — 4 — \ldots — n - 1$$

Is $K_{4,4}$ 2-factorable? We are asking to see if we can write $K_{4,4} = H_1 \oplus H_2$ where each H_i is a 2-factor.





Yes, $K_{4,4}$ is 2-factorable.

Let us look at some Linear Algebra. Take any graph of the form K_n, and look at its adjacency matrix. We know that the adjacency matrix for any graph is alway symmetrical, and from a result in Linear Algebra we have that if a matrix is symmetrical then all its eigenvalues are real. Thus, all eigenvalues of an adjacency matrix of a graph G are real.

Reminder: Take $A, n \times n$ and α is an eigenvalue of A. Then we conclude quickly that:

$$\exists \quad \text{some point} \neq (0, 0, \ldots, 0)$$
$$A \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \neq 0$$

Look at K_4 and its adjacency matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$\Longrightarrow 3 \text{ is an eigenvalue of adj}(K_4)$$

What if we take K_5 instead of K_4? Then this implies that 4 is an eigenvalue of $\mathrm{adj}(K_5)$. In general, the sum of the rows (or columns) of the adjacency matrix (should all be equal) is an eigenvalue for the adjacency matrix. Thus $n-1$ is an eigenvalue of $\mathrm{adj}(K_n)$. However, this is not the only eigenvalue of $\mathrm{adj}(K_n)$.

How do we calculate eigenvalues in general?

$$\text{Set } |X I_n - \mathrm{adj}(K_n)| = 0$$
$$\text{find } X$$

$$\text{Look at the matrix:}$$
$$X I_n - \mathrm{adj}(K_n) = \begin{bmatrix} X & -1 & -1 & \dots & -1 \\ -1 & X & -1 & \dots & -1 \\ \vdots & -1 & \ddots & -1 & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & -1 & \dots & X \end{bmatrix}$$
$$\text{We want} \left\| \begin{bmatrix} X & -1 & -1 & \dots & -1 \\ -1 & X & -1 & \dots & -1 \\ \vdots & -1 & \ddots & -1 & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & -1 & \dots & X \end{bmatrix} \right\| = 0$$
$$\text{If } X = -1 \Longrightarrow |X I_n - \mathrm{adj}(K_n)| = 0$$
$$\text{Thus } -1 \text{ is also an eigenvalue of } \mathrm{adj}(K_n).$$

These are the only two eigenvalues of the adjacency matrix. The characteristic polynomial of $\mathrm{adj}(K_n) = (X - (n-1))(X+1)$. Let us calculate the eigenspace of -1, and we will show that it will have dimension $n-1$.

$$(-1)\, I_n - \mathrm{adj}(K_n) \begin{bmatrix} -1 & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \\ \vdots & -1 & \ddots & -1 & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & -1 & \dots & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$-x_1 - x_2 - x_3 - \dots - x_n = 0$$
$$x_1 = -x_2 - x_3 - x_3 - \dots - x_n$$

We have $n-1$ free variables, which means that the dimension of the eigenspace of -1 is $n-1$. Thus the characteristic polynomial of $\mathrm{adj}(K_n)$ is:

$$(X+1)^{n-1}(X-(n-1))$$

This means that the eigenvalue -1 is repeated $n-1$ times, and the eigenvalue $n-1$ is repeated once.

For $K_{m,n}$, the eigenvalues are 0, repeated $n+m-2$ times , and $\sqrt{n\,m}$ and $-\sqrt{n\,m}$ each repeated once.

$$(X^2 - n\,m)X^{n+m-2}$$

This part of the course will not be included in the final exam.