0.1.3 Solution for HW III

MTH418 - Homework III

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Question 1: Let $G(V, E)$ be a connected graph of order n. Show that the size of G is $\geqslant n-1$.

Since G is a connected graph, there are two possibilities: It is either a graph with cycles or a graph with no cycles (a tree). Let $|E|$ be the number of edges (or the size) for G. We proceed as follows:

- Assume G is a tree (no cycles) We know by class result that $|E| = n-1$

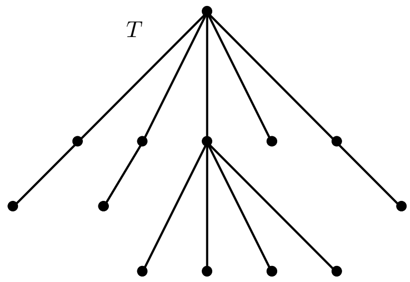
- Assume G contains cycles (at least one), then the path between some pair of vertices is not unique.

  In a tree, we know that the path between two vertices v_i and v_j is unique. However, since this graph contains cycles, there is at least some pair of vertices, v_f and v_g st. there is more than one path, formed by k edges. Thus $|E| = n - 1 + k$. Knowing that $|E|$ has increased by some constant k, we conclude that: $|E| > n - 1$.

If we combine the two cases, we can see that regardless of whether the graph G contains cycles or not, it will always be st. $|E| \geqslant n-1$, where n is the order of the graph.

Question 2: Let T be a tree of order 13. The degrees of the vertices of T are $1, 2$ and 5. If T has exactly 3 vertices of degree 2, how many end-vertices does it have?

All degrees of vertices in T are of order 1, 2 or 5, but we can only have 3 vertices of degree 2. Let us draw a tree as such to be able to better visualize the requirements:



In this tree, we can see that there are exactly 3 vertices st. $\deg(v) = 2$, we have 8 vertices st. $\deg(v) = 1$, and we have 2 vertices st. $\deg.(v) = 5$. Thus T (shown above) fits the requirements of the question.

To generalize this solution, we know that since we have exactly 3 vertices st. $\deg(v) = 2$, then we have 10 vertices of either order 5 or 1.

Let m be the size of T. We know through the class notes that the size of a tree is $n-1$. In this case, $m = 12$. Let E be the number of vertices st $\deg(v) = 1$. Thus we have that $(10 - E)$ is the number of vertices st. $\deg(v) = 5$. From class notes, we know that the sum of degrees is $2 \times m$. Thus if we sum the degrees:

$$\begin{aligned}
\sum_{i=0}^{n} \deg(v_i) &= 2m = 24 \\
3(2) + E(1) + (10 - E)(5) &= 24 \\
6 + E + 50 - 5E &= 24 \\
-4E &= -32 \\
\implies E &= 8
\end{aligned}$$

Therefore, the number of vertices st. $\deg(v_i) = 1$ is 8. There are 8 vertices with degree 1 regardless of how we draw the tree.

Question 3: Construct a minimum dominating set of C_{14} and P_{10}

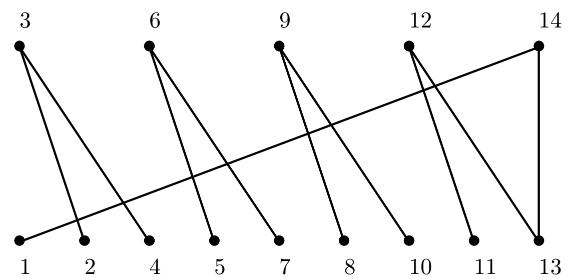
We can draw the graph for C_{14}:

$$1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10 — 11 — 12 — 13 — 14 — 1$$

Consider the following set:

$$\{3, 6, 9, 12, 14\}$$

It is easy to observe that every element outside of $\{3, 6, 9, 12, 14\}$ is connected by an edge to at least one of the 5 elements. We can draw this to further demonstrate:



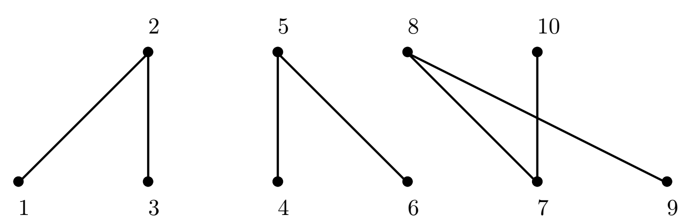
Therefore, $\gamma(C_{14}) = 5$, and our minimum dominating set:

$$\{3, 6, 9, 12, 14\}$$

For P_{10}, we first draw the graph:

$$1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10$$

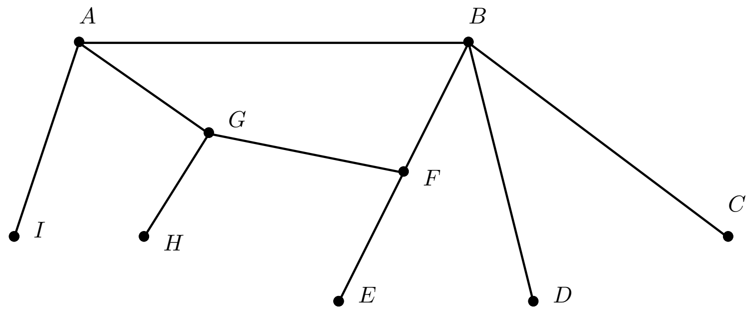
Consider the set $\{2, 5, 8, 10\}$. We can draw the graph to see the following:



Clearly, every element outside of $\{2,5,8,10\}$ is connected to one of those 4 elements, and therefore we know that $\gamma(P_{10}) = 4$ with our minimum dominating set:

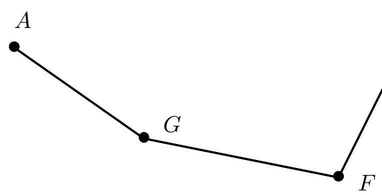
$$\{2,5,8,10\}$$

Question 4: Consider the graph below:



i. Is $A — G — F — B$ an induced subgraph of our graph?

   We can draw the graph for $A — G — F — B$:

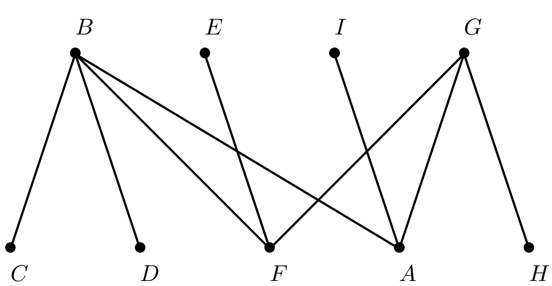


   The definition of an induced subgraph states that this new graph (let's call it $\boldsymbol{G'}(V_1, E_1)$) must be a subgraph of $\boldsymbol{G}$, and it must also be st. $e \in E_1$ iff $e \in E$.

   We can see that $V_1 := \{A, G, F, B\}$, and clearly $V_1 \subset V := \{A, B, C, D, E, F, G, H, I\}$. However, A and B are connected through an edge in the original graph but not in the subgraph. Therefore, since $A — B$ is not in the new graph, it is NOT an induced subgraph.

ii. Is our graph bipartite?

   The only cycle in the graph is $A — G — F — B — A$, which is of even length. Therefore we can construct a bipartite graph isomorphic to G:



   We can take the set $\alpha := \{B, E, I, G\}$ and $\beta := \{C, D, F, A, H\}$. There are no adjacent vertices in either α or β; the only vertices are between elements of α and elements of β. Therefore, G is bipartite.

iii. By staring, find diam(G)

The maximum distance between two vertices in G is 4, which can we obtained by taking $d(C, H), d(D, H)$ or $d(E, I)$. In either case, the length of the path is 4, which leads us to the conclusion:

$$\text{diam}(G) = 4$$

iv. Find the dominating set of G and thus find the dominating number.

Take the set $\{B, F, A, G\}$ or $\{B, E, I, G\}$. In both cases, every vertex outside of those 4 is connected to at least one of them. We cannot construct a set smaller than this, and therefore,

$$\gamma(G) = 4$$

Question 5: Let G be a connected graph, and let e be an edge that is a bridge. Show that e is an edge of every spanning tree of G.

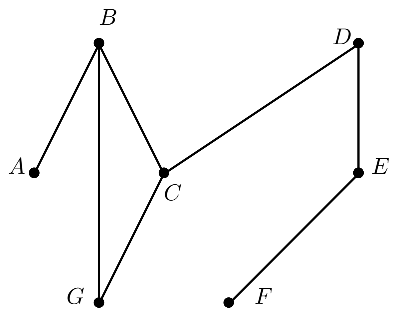
Let $V := \{v_1, v_2, \ldots, v_n\}$ be the vertices of G, and $E := \{e_1, e_2, \ldots, e, \ldots, e_n\}$ be the set of edges.

Since e is a bridge, then removing it will cause the graph to be disconnected. Let $T(V_1, E_1)$ be a spanning tree of of G. Since T is a spanning tree, then $V_1 = V$ (All vertices in G are also in T). Since T is also a tree, then there are no cycles, and the path between each pair of vertices is unique (from class notes).

Take two vertices, v_i and v_j st. $e = v_i — v_j$ (e is the edge that connects the two vertices). Since the path is unique, then e is the ONLY edge between the two vertices. If we were to remove e, then the graph would be disconnected, and thus we wouldn't have a spanning tree anymore (disconnections: no path between ALL vertices). Thus e has to be an edge between v_i and v_j.

Since v_i and v_j are ANY two vertices in the spanning tree, we know that this works for all edges. Therefore e is an edge of every spanning tree of G.

Question 6: Consider the graph below:



i. Find all cut-vertices of G

$$B, C, D \text{ and } E$$

The vertices B, C, D and E are all cut-vertices. Why is this the case? Because in each of the 4 cases, the removal of said vertex will cause the graph to be disconnected.

Removing B will cause the vertex A to be disconnected from the rest of the graph.

Removing C will cause the graph to split into two disconnected components ($A — B — G$ and $D — E — F$).

The same applies for removing D (disconnects E and F from the graph) and removing E (F is left by itself).

ii. Find all bridges of G

The edges you can remove to cause the graph to be disconnected are:

- $A - B$
- $C - D$
- $D - E$
- $E - F$

These are the only edges whose removals will cause the graph to be disconnected, and therefore are the bridges of G.

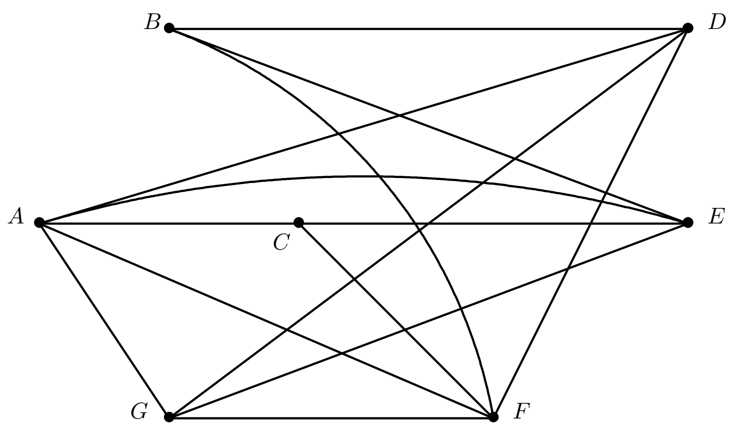
iii. By staring, find diam(G)

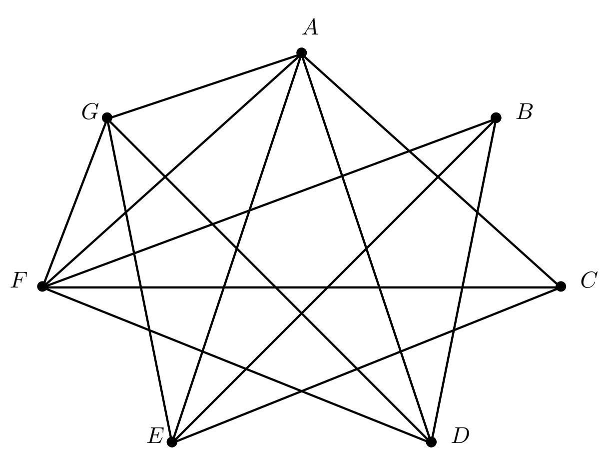
The maximum distance between two vertices is the distance between vertices A and F. The shortest path between the two is: $A - B - C - D - E - F$, which is a path of length 5. Therefore:

$$\text{diam}(G) = 5$$

iv. Draw the complement of G. Is $\bar{G}$ connected? How many edges does $\bar{G}$ have?

The below is two versions of the graph of $\bar{G}$ (One is slightly less ugly than the other, although both are exactly the same (isomorphic)):





We know that our original graph, G, has 7 edges (by staring). Since the graph has 7 vertices, we consider the size of the graph of K_7, which is 21. We subtract 7 from this quantity to get the size of $\bar{G}$, which is given by:

$$\text{size}(\bar{G}) = 21 - 7 = 14$$

We can double-check this with the graph we have drawn.

v. Draw 2 non-isomorphic spanning trees of G: