MTH311 - Intermediate Analysis

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August 31st, 2020

1 Chapt.: Some properties of $\mathbb{R}$

Def.: Let $A \subseteq \mathbb{R}$ (i.e. A is a subset of $\mathbb{R}$)

1. b is said to be the upper bound of A (respectively the lower bound of A) if $x \leqslant b$ for all x in A (Respectively, $b \leqslant x$, for all x in A)

   Terminology:

   i. When b exists, A is said to be the bounded above (resp. bounded below)

   ii. A is said to be bounded if it is bounded both above and below.

2. A real number $\boldsymbol{\beta}$ is a least upper bound (lub) or the *supremum* of A if: $\boldsymbol{\beta}$ is an upper bound of A and if any other bound of A is greater or equal to $\boldsymbol{\beta}$.

3. A real number $\boldsymbol{\alpha}$ is a greatest lower bound (glb) or the *infimum* of A if: $\boldsymbol{\alpha}$ is a lower bound of A and if any other lower bound of A is less or equal to $\boldsymbol{\alpha}$.

Notation: $\boldsymbol{\alpha} = \inf(\mathrm{A}) = \mathrm{glb}(\mathrm{A})$

$\boldsymbol{\beta} = \mathrm{Sup}(\mathrm{A}) = \mathrm{lub}(\mathrm{A})$

<u>WARNING:</u> $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may not necessarily be in A. Their existence is not limited to the inclusion within the set, A.

<u>Rmk:</u> Consider $\mathbb{N} = \{1, 2, 3, \ldots\}$

Then: $\inf(\mathbb{N}) = 1$ because 1 is the greatest lower bound of our set, $\mathbb{N}$.

The Completeness Axiom:

If a *non-empty* set of $\mathbb{R}$ is bounded above, then it has a lub. i.e. if a set in $\mathbb{R}$ is not empty, then it has a *supremum*.

Theorem 1: The greatest lower bound property:

If A is *non-empty* and bounded below, then it has a glb. i.e. it has an *infimum*.

<u>Proof:</u> Let $L = \{a \in \mathbb{R} \quad s.t. a \,\text{is a lower bound of}\, A\}$

Since A is bounded below, L is NOT empty. i.e. $L \neq \varnothing$.

Let x be a random element of A. Since every element in L is a lower bound of A, $a \leqslant x$ for all $a \in L$. Thus, L is bounded above by x. Hence, the completeness axiom (which is applied to L) implies that L has a lub. Let us call this lub(L) $\boldsymbol{\alpha}$.

Now: Since x is an upper bound of L, $\boldsymbol{\alpha} \leqslant x$. $\longleftarrow$ This is because $\boldsymbol{\alpha}$ is a lub of L. Since x is chosen at random, this means that $\boldsymbol{\alpha} \leqslant x$ for any x in L. i.e. $\boldsymbol{\alpha}$ is a lower bound of A

Since $\boldsymbol{\alpha} = \mathrm{Sup}(L)$ and L is the set (By Construction) of all the lower bounds of A, we have that $a \leqslant \boldsymbol{\alpha}$ for every lower bound a of A. Therfore $\boldsymbol{\alpha} = \inf(A)$.

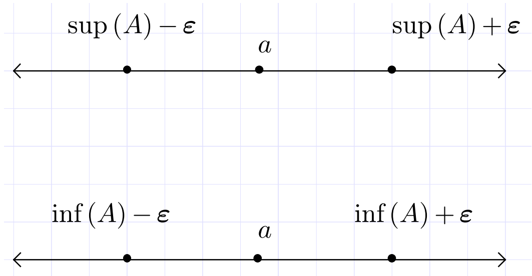
Theorem 2:

Let $A \subseteq \mathbb{R}$, A is a *non-empty* set and consider $\boldsymbol{\varepsilon} > 0$.

i. Suppose A has a lub. Then there is AT LEAST some $a\,\text{in}\,A$ s.t.: $\text{Sup}(A) - \boldsymbol{\varepsilon} < a \leqslant \text{Sup}(A)$

ii. Suppose A has a glb. Then there is AT LEAST some $a\,\text{in}\,A$ s.t. $\inf(A) \leqslant a < \inf(A) + \boldsymbol{\varepsilon}$

This is shown in the number lines below:



Proof: Since $\text{Sup}(A) - \boldsymbol{\varepsilon} < \text{Sup}(A)$, because $\boldsymbol{\varepsilon} > 0$ but $\boldsymbol{\varepsilon}$ is infinitesimally small, $\text{Sup}(A) - \boldsymbol{\varepsilon}$ cannot be an upper bound of A. Hence, there is at least an a in A s.t. $\text{Sup}(A) - \boldsymbol{\varepsilon} < a$. Finally, we conclude that there is at least an element a in A s.t. $\text{Sup}(A) - \boldsymbol{\varepsilon} < a < \text{Sup}(A)$. We use a similar argument for the proof of the inf(A).

Rmk: if a real number $\boldsymbol{\beta}$ is such that:

$\{x \leqslant \boldsymbol{\beta}\,\text{for all}\,x\,\text{in}\,A \wedge \forall \boldsymbol{\varepsilon} > 0, \exists x \quad s.t.\,\boldsymbol{\beta} - \boldsymbol{\varepsilon} < x \leqslant \boldsymbol{\beta}\}$, then: $\boldsymbol{\beta} = \text{Sup}(A)$. Similar argument for inf(A) except for the following change: $\boldsymbol{\beta} \leqslant x < \boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

Theorem 3:

Let $A \subseteq \mathbb{R}$, A is a *non-empty* set.

i. If A has a lub, then the lub is unique.

ii. If A has a glb, then this glb is unique.

Proof:

i. Suppose M and N are both lubs of A. Then M and N are both upper bounds of A, and $M \leqslant T$, for every T, upper bound of A. In particular, if $T = N$, then we have $M \leqslant N$.

ii. Similarly, $N \leqslant T$ for every T, upper bound of A. In particular, if $T = M$, then we have $N \leqslant M$.

Since $M \leqslant N$ and $N \leqslant M$, we have the logical conclusion that $M = N$.

Theorem 4:

Let $A, B \subseteq \mathbb{R}$, A and B are *non-empty* sets. Suppose that for all $a \in A$ and $b \in B$, we have $a \leqslant b$. Then:

i. A has a lub and B has a glb. Moreover, $\text{Sup}(A) \leqslant \inf(B)$.

ii. $\text{Sup}(A) = \inf(B)$ iff $\forall \boldsymbol{\varepsilon} > 0, \exists a \in A, b \in B$ s.t. $b - a < \boldsymbol{\varepsilon}$ (This is the distance between a and b. This means that for some a and b in the two sets, there is a pair where the distance between the two is infinitesimally small. If this pair exists, then the lub of A is the glb of B.

Proof: Since $A \neq \varnothing$ and $B \neq \varnothing$, every element of B is an upper bound of A and every element of A is a lower bound of B. Thus, the lub and glb properties (explained earlier) show that $\text{Sup}(A)\,\text{and}\,\inf(B)$ both exist.

To show that $\text{Sup}(A) \leqslant \inf(B)$, we proceed by contradiction: Assume $\text{Sup}(A) > \inf(B)$. We let:

$$\begin{aligned}
\text{Let } \boldsymbol{\mu} = \frac{\text{Sup}(A) - \inf(B)}{2} \quad & \boldsymbol{\mu} > 0 \\
\text{By } \textbf{THM2:}\ \exists a \in A \quad s.t.\ \text{Sup}(A) - \boldsymbol{\mu} < a \leqslant \text{Sup}(A) \quad & \text{Same follows for } b \\
\inf(B) < b < \inf(B) + \boldsymbol{\mu} \quad & \\
b < \inf(B) + \boldsymbol{\mu} \quad & \boldsymbol{\mu} = \frac{\text{Sup}(A) - \inf(B)}{2} \\
b < \inf(B) + \frac{\text{Sup}(A) - \inf(B)}{2} \quad & \text{We can algebraically manipulate this.} \\
b < \frac{\text{Sup}(A) + \inf(B)}{2} \quad & \text{We can further manipulate this algebraically.} \\
\frac{\text{Sup}(A) + \inf(B)}{2} = \text{Sup}(A) - \frac{\text{Sup}(A) - \inf(B)}{2} \quad & = \text{Sup}(A) - \boldsymbol{\mu} \\
b < \text{Sup}(A) - \boldsymbol{\mu} \quad & \\
\inf(B) < b < \text{Sup}(A) - \boldsymbol{\mu} \quad & \text{This is } a \text{ contradiction.}
\end{aligned}$$

Which means that $\text{Sup}(A) - \boldsymbol{\mu}$ is less than at least one element of A. This is a contradiction because it shows that $\inf(B)$ is smaller than at least one element of A, which goes against the initial setup of $a \leqslant b$.

September 2nd, 2020

No Gap Lemma:

i. A has a lub and B has a glb. Moreover, $\text{Sup}(A) \leqslant \inf(B)$.

ii. $\text{Sup}(A) = \inf(B)$ iff $\forall \boldsymbol{\varepsilon} > 0, \exists a \in A, b \in B$ s.t. $b - a < \boldsymbol{\varepsilon}$.

<u>Proof:</u>

$$\begin{aligned}
\text{Assume } \text{Sup}(A) = \inf(B) \quad & \\
\text{Let } \boldsymbol{\varepsilon} > 0.\ \textbf{Thm2} \quad \text{implies: } \exists a \in A \quad s.t.\ \text{Sup}(A) - \frac{\boldsymbol{\varepsilon}}{2} < a \leqslant \text{Sup}(A) \quad & \text{Same for } b. \\
\text{Sup}(A) - \frac{\boldsymbol{\varepsilon}}{2} < a \leqslant \text{Sup}(A) = \inf(B) \leqslant b < \inf(B) + \frac{\boldsymbol{\varepsilon}}{2} \quad & \\
i.e. \quad \text{Sup}(A) - \frac{\boldsymbol{\varepsilon}}{2} < a \leqslant b < \inf(B) + \frac{\boldsymbol{\varepsilon}}{2} \quad & \text{(assumption) } \inf(B) = \text{Sup}(A) \\
\text{thus: } b - a < \text{Sup}(A) + \frac{\boldsymbol{\varepsilon}}{2} - \left(\text{Sup}(A) - \frac{\boldsymbol{\varepsilon}}{2}\right) = \boldsymbol{\varepsilon} \quad & \\
i.e. \quad b - a < \boldsymbol{\varepsilon} \quad &
\end{aligned}$$

$$\begin{aligned}
\Leftarrow \text{(We proceed by contradiction)} \quad & \\
\text{Assume } \text{Sup}(A) \neq \inf(B) \quad & \\
\text{Let } \boldsymbol{\gamma} = \inf(B) - \text{Sup}(A) > 0 \quad & \text{(assumption) by } (i) \\
\text{Now: if } a \in A \text{ and } b \in B \quad \text{, then:} \quad & \\
a \leqslant \text{Sup}(A) \text{ and } b \geqslant \inf(B). \quad \text{so:} \quad & \\
b - a \geqslant \inf(B) - \text{Sup}(A) \quad & \\
\exists q \in \mathbb{Q} = \boldsymbol{\gamma} > 0 \quad & \\
\text{Hence, if we choose } \boldsymbol{\varepsilon} = \frac{\boldsymbol{\gamma}}{2} > 0 \quad \text{then we have:} \quad & \\
b - a > \frac{\boldsymbol{\gamma}}{2} (= \boldsymbol{\varepsilon}) \quad & \\
b - a > \boldsymbol{\varepsilon} \quad & \text{This is } a \text{ contradiction.}
\end{aligned}$$

Theorem 5: The Archimedean Property

Let $\varepsilon, b \in \mathbb{R}^+$. Then there is some $n \in \mathbb{N}$ s.t. $b < n\varepsilon$.

(Intuitively, it is possible to exceed any positive number no matter how large, by adding an arbitrary (small) positive number to itself sufficiently many times.

Proof: By contradiction.

Suppose $n\varepsilon \leqslant b$ for all $n \in \mathbb{N}$	
Let $A = \{n\varepsilon \quad s.t. \quad n \in \mathbb{N}\}$	Constructed set.
Then $A \subseteq \mathbb{R}, A \neq \varnothing$	This set is clearly in $\mathbb{R}$ and is non $-$ empty.
and b is an upper bound of A.	
Thus, TCA implies that A has a lub.	(TCA $=$ Completeness Axiom)
Let $m \in \mathbb{N}$ (arbitrary)	
$(m+1)\varepsilon \in A$	(A is the set of all numbers of the form $n\varepsilon$, where $n \in \mathbb{N}$. This is our construction)
$(m+1)\varepsilon \leqslant \mathrm{Sup}(A)$	Since they are both in $\mathbb{R}$
Therefore: $m\varepsilon + \varepsilon \leqslant \mathrm{Sup}(A)$	
$i.e.$ $m\varepsilon \leqslant \mathrm{Sup}(A) - \varepsilon$	Remember that ε is in $\mathbb{R}^+$ $i.e.$ $\varepsilon > 0$
Because m is chosen arbitrarily, we deduce that:	
$''\mathrm{Sup}(A) - \varepsilon''$ is another upper bound of A	
But since $\varepsilon > 0$ $\quad \mathrm{Sup}(A) - \varepsilon < \mathrm{Sup}(A)$	
This means that we found an upper bound of A	
that is less than Sup (A).	$\mathrm{Sup}(A)$ is the LEAST upper bound.
	This is a contradiction.

Corollary 6 (We use the Archimedean property):

Let $x \in \mathbb{R}$.

i. There is a unique integer $n \in \mathbb{Z}$ s.t. $n - 1 \leqslant x < n$. If $x \geqslant 0$, then $n \in \mathbb{N}$.

ii. if $x > 0$, there is some $m \in \mathbb{N}$ s.t. $\frac{1}{m} < x$.

(i) is simply the greatest integer function that we know from highschool or Calculus 1. Assume $x = -1.2$, then the floor of x is -2, and we can say $-2 < x < 1$. Remember that our n in (i) and $n - 1$ are two consecutive terms. Any real number is always between two consecutive integers.

(ii) is trivial for the case of bigger numbers. if $x = 1,000,000$, then almost any value of m would make the proposition true. However, this becomes interesting for cases where x is a small number. (ii) shows that we can always make a smaller number regardless of the value of x. We can always make x bigger than one over an integer.

Proof:

(i)

$''$Uniqueness$''$:

$$
\begin{array}{rcl}
\text{Suppose } \exists m, n \in \mathbb{Z} \quad s.t. n-1 \leqslant x < n & - & (*) \\
\text{and / or } m-1 \leqslant x < m & - & (**) \\
\text{We proceed by contradiction:} & - & \\
\text{Assume } m \neq n & - & \\
\text{Without the loss of generality, we can assume } n < m & - & \text{or } m < n \quad \text{(does not matter)} \\
\text{if } n < m, \text{then } n+1 \leqslant m & - & \\
x < n \leqslant m-1 & - & x < n \qquad \text{(from } * \text{)} \\
 & - & m-1 \leqslant x \quad \text{(from } ** \text{)} \\
x < n \leqslant m-1 \leqslant x & - & x < x \leftarrow \text{This is } a \text{ contradiction.}
\end{array}
$$

Thus, the uniqueness of our n is guaranteed.

$''$Existence$''$:

$$
\begin{array}{rl}
\text{First; if } x = 0 & \text{then} \\
\text{take} \quad n = 1 & 0 \leqslant x < 1 \\
 & \\
\text{Second; Suppose } x > 0 & \text{We can apply Archimedean Property } (*) \\
\varepsilon = 1 \wedge b = x & \\
\exists k \in \mathbb{N} \quad s.t. x < k & \\
\text{Let } B = \{m \in \mathbb{N} \quad s.t. x < m\} & \text{set is not empty (}k\text{ is inside)} \\
\text{Clearly } B \subseteq \mathbb{N} \text{ and } B \neq \varnothing & \\
\text{Also there is } n \in B \quad s.t. n \leqslant m \text{ for all } m \in B & B \text{ is bounded below (by } n\text{)} \\
x < n & \text{Because } n \in B \\
\text{Now, by minimality:} & \\
n-1 \notin B & \text{Because } n \text{ is the smallest in } B, \text{so } n-1 \notin B \\
\text{If } n-1 \in \mathbb{N} \wedge n-1 \notin B, \text{then} & \\
n-1 \leqslant x & \text{If } n-1 \notin \mathbb{N} \quad \text{then } n = 1 \quad (n-1=0) \\
\text{Since } x > 0 & \text{from } (*) \\
n-1 < x & \\
\text{Therefore} \quad n-1 \leqslant x & \\
n-1 \leqslant x < n & \text{We have proved the existence for } x > 0 \\
 & \\
\text{Third; if } x < 0, \text{then } -x > 0 & \\
\text{We apply the second point to } -x & \\
m-1 \leqslant -x < m & \text{Multiply by } -1 \\
i.e. \quad -m+1 \geqslant x > -m &
\end{array}
$$

Let us discuss the subcases of the third case:

1. if $x = -m+1$, then we let $n = -m+2$ and so $n-1 = -m+1 = x$; $x \geqslant -m+1$ (which is $n-1$)

   So we can see that x is because two consecutive cases, $-m+2 > x > -m+1$.

2. if x if $-m+1 > x > -m$, then we take $n = -m+1$; where, again, we have two consecutive terms around x

(ii)

Now let us proof (ii), which is stating that if $x>0$, there is some $m\in\mathbb{N}$ s.t. $\frac{1}{m}<x$. The proof is as follows:

$$\begin{aligned}
\text{Since } x>0, \frac{1}{x}>0. & \\
\text{By } (i), \exists n\in\mathbb{N} \quad s.t. & \\
n-1\leqslant\frac{1}{x}<n & \\
\frac{1}{x}<n \quad \text{so } x>\frac{1}{n} & \\
\text{This proves our initial statement} & \quad \text{(Using the first point)}
\end{aligned}$$

Theorem 7:

Let $p\in\mathbb{N}$. Suppose there is no $u\in\mathbb{Z}$ s.t. $p=u^2$ (i.e. p is NOT a perfect square). Then we have the following:

$$\sqrt{p}\notin\mathbb{Q} \qquad (i.e.\ \sqrt{p} \text{ is not rational})$$

<u>Proof</u>: By Corollary 6

$$\begin{aligned}
\exists! \quad \boldsymbol{\nu}\in\mathbb{Z} \quad s.t. & \\
\boldsymbol{\nu}-1\leqslant\sqrt{p}<\boldsymbol{\nu} & \quad \text{By our hypothesis, we know that } \sqrt{p}\notin\mathbb{Z} \\
\boldsymbol{\nu}-1<\sqrt{p}<\boldsymbol{\nu} & \quad \text{Because } \sqrt{p} \text{ is not an integer.} \\
\text{We ket } s=\boldsymbol{\nu}-1 & \quad s\in\mathbb{Z} \\
s<\sqrt{p}<s+1 & \\
\text{Since } p\geqslant 1, \text{then } \sqrt{p}>1 & \quad \text{Therefore } s\geqslant 1 \text{ (because of the inequality)}
\end{aligned}$$

$$\begin{aligned}
\text{We proceed by contradiction:} & \\
\text{Suppose } \sqrt{p}\in\mathbb{Q} & \quad i.e. \exists a,b\in\mathbb{N} \quad s.t.\ \sqrt{p}=\frac{a}{b} \\
\text{Consider the following set:} & \\
E=\left\{d\in\mathbb{N} \quad s.t. \exists c\in\mathbb{Z} \text{ for which } \sqrt{p}=\frac{c}{d}\right\} & \\
\text{Clearly } E\subseteq\mathbb{N} \quad \text{and } E\neq\varnothing & \quad \text{because } b \text{ (from above) is in } E \\
\text{Moreover, there is } n\in E \quad s.t. n\leqslant x \text{ for all } x\in E & \\
\text{By definition of } E, \exists m\in\mathbb{Z} \quad s.t.\ \sqrt{p}=\frac{m}{n} & \\
\text{Since } s<\sqrt{p}<s+1, \text{we have} & \\
s<\frac{m}{n}<s+1 & \quad \text{Now divide by } s \text{ and subtract } n s; \\
0<m-n s<n &
\end{aligned}$$

$$\begin{aligned}
\text{Now, one can verify that:} & \\
(n p-s m)^2-(m-\mathrm{ns})^2 p=(s^2-p)(m^2-n^2 p) & \quad \text{Little tricky, but we can check this by expanding} \\
(m^2-n^2 p)=0 \text{ because } \sqrt{p}=\frac{m}{n} & \\
\text{Therefore } (n p-s m)^2-(m-\mathrm{ns})^2 p=0 & \\
i.e. p=\frac{(n p-m s)^2}{(m-n s)^2} & \\
\text{So: } \sqrt{p}=\frac{(n p-m s)}{(m-n s)} & \quad \text{Numerator}\in\mathbb{Z} \text{ and Denom}\in\mathbb{N} \\
\text{Thus: } (m-n s)\in E & \quad (m-n s) \text{ plays the role of } d
\end{aligned}$$

This contradicts the fact that n is the smallest element in E. In fact, $(m - ns) < n$. Therefore $\sqrt{p} \notin \mathbb{Q}$. This proof essentially states that if our p is not a perfect square, then it is not possible for $\sqrt{p}$ to be a rational number. i.e. $\sqrt{p} \notin \mathbb{Q}$ for $p \neq$ perfect square.

September 7th, 2020

Theorem 8: Density of the Rational and Irrational Numbers

Let $a, b \in \mathbb{R}$ and suppose $a < b$. Then:

i. $\exists q \in \mathbb{Q}$ s.t. $a < q < b$ and

ii. $\exists r \in \mathbb{R} - \mathbb{Q}$ s.t. $a < r < b$

Proof:

(i) Since $b - a > 0$ (By **COROLLARY 6**), it implies the existence of $n \in \mathbb{N}$, s.t. $\frac{1}{n} < b - a$. Thus:

$$\begin{aligned} an + 1 &< bn \\ \text{Moreover, } \boldsymbol{C6} \text{ implies:} & \\ \text{the existence of } m \in \mathbb{Z} \quad s.t. m - 1 &\leqslant \underline{an - 1 < m} && (1) \\ \text{and } m - 1 + 1 &\leqslant an - 1 + 1 < m + 1 \\ \text{so:} \quad m &\leqslant an \leqslant m + 1 \\ \text{or:} \quad m &\leqslant an < bn && \text{From first line} \\ \textit{i.e.} \quad m &< bn && (2) \end{aligned}$$

$$\begin{aligned} \text{(1) and (2) imply that: } an < m < bn \\ \text{and hence:} \quad a < \frac{m}{n} < b \\ \frac{m}{n} \in \mathbb{Q} \text{ since } m \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{N} \end{aligned}$$

(ii) Since $\sqrt{2}$ >0, then $\frac{1}{\sqrt{2}} > 0$ and $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$.

$$\begin{aligned} \text{By } (i), \text{we know that } \exists q \in \mathbb{R} \\ s.t. \frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}, \text{and so} \\ a < \frac{q}{\sqrt{2}} < b \end{aligned}$$

Finally, since $\sqrt{2} \in \mathbb{R} - \mathbb{Q}$, and $q \in \mathbb{Q}$, $q \frac{1}{\sqrt{2}} \in \mathbb{R} - \mathbb{Q}$. This is by question 5, (ii) in the first homework.

Rmk: In the proof of (ii), we assumed that $q \neq 0$. Otherwise if $q = 0$, we can apply **THM8, (i)** to find a q', s.t.

$$\frac{a}{\sqrt{2}} < q' < q = 0 < \frac{b}{\sqrt{2}}$$

We then finish the proof with the q' instead of the q, as we did in the proof for (ii). This means that even if our initial $q = 0$, then we can find another rational number in between $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$ where this new value is no longer 0. Then we can proceed as we did above.

Terminology:

(i) of **THM8** is equivalent to saying $\mathbb{Q}$ is dense in $\mathbb{R}$.

(ii) of **THM8** is equivalent to saying $\mathbb{R} - \mathbb{Q}$ is dense in $\mathbb{R}$.

What does it mean to be dense in $\mathbb{R}$? **THM8** (i) states that every pair of real numbers has at least one rational number in between them. This means that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(ii) states that every pair of real numbers also has at least one irrational number in between the two of them.

Very important: $\mathbb{Q}$ does not satisfy the completeness axiom (Or the least upper bound (LUB) property). It is only $\mathbb{R}$ that satisfies this. Not every set of numbers satisfies this property.

That is: a set of rational numbers may be bounded above (by rational numbers) but not have a rational upper bound that is less than all the other rational upper bounds.

Let $S = \{q \in \mathbb{Q}, s.t. q^2 < 2\}$. $\sqrt{2}$ is an upper bound in this set, S.

THM8 implies that: $\forall \varepsilon > 0$, $\exists q_0 \in \mathbb{Q}$ s.t. $\sqrt{2} - \varepsilon < q_0 < \sqrt{2}$.

Thus: **THM2 (RMK)** implies that $\sqrt{2} = \sup(S)$. However, we know that $\sqrt{2}$ is irrational (**THM7**) For it is a non-perfect square.

Therefore, if q_1 is a rational upper bound of S, then:

$$\sqrt{2} < q_1 \quad (\text{Not} \leqslant)$$

Again, **THM8** implies $\exists q_2 \in \mathbb{Q}$ s.t. $\sqrt{2} < q_2 < q_1$. i.e. q_2 is another upper bound of S. We can keep going with this forever and we will never stop. We cannot find a rational number that is smaller than all the other upper bounds of S. we can keep going for $q_3, q_4, \ldots, q_n$. Therefore we have proven that we cannot find a least upper bound for $\mathbb{Q}$. S has no rational supremum.

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2 Chapt.: On the Topology of the $\mathbb{R}$ line

Defintion 1: Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. It does not have to be an interval, it can be union, intersection, etc.

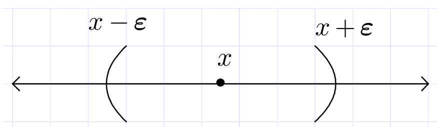
The complement of S, denoted S^c, is the set of all elements in $\mathbb{R}$ that are not in S.

Very important:

i. Every set, S, contains the empty set, $\varnothing$, because if $\varnothing$ is not contained in S, then we are saying there are elements in $\varnothing$ that are not in S. This is already a contradiction because the empty set $\varnothing$ has no elements within it. Therefore, all sets, S, s.t. $S \subseteq \mathbb{R}$, contain $\varnothing$.

ii. For ant $S \subseteq \mathbb{R}$, we have $(S^c)^c = S$. Furthermore, $S \cap S^c = \varnothing$ and $S \cup S^c = \mathbb{R}$.

Defintion 2: If $x \in \mathbb{R}$ and $\varepsilon > 0$, the open interval $(x - \varepsilon, x + \varepsilon)$ is called an ε-neighborhood of x. This neighborhood is small and very close to x.



Definition 3: If a set S contains an ε-neighorhood of a point, x, then S is called a neighborhood of x. The neighborhood does not have to be centered at the point, x. Moreover, x is said to be an interior point of S. That means we can find an ε-neighborhood for x that is totally included in S.

Definition 4: The set of all the interior points of S is called interior of S, and is denoted by $\dot{S}$.

Definition 5: A set $S \subseteq \mathbb{R}$ is said to be open if every point in S is an interior point.

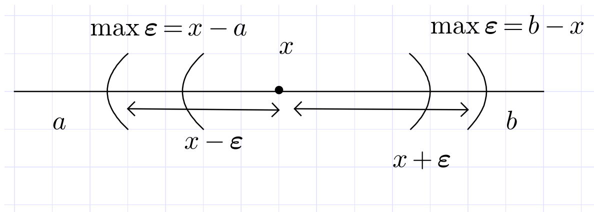
i.e. $\forall x \in S, \quad \exists \varepsilon > 0 \quad \text{s.t.} \ (x-\varepsilon, x+\varepsilon) \subset S$. This means that S is open.

Definition 6: A set S is said to be closed if S^c is open.

<u>Exp:</u>

1. Any open interval, (a,b) is an open set, because if for $x \in (a,b)$, we choose ε to be $\varepsilon \leqslant \min\{x-a, b-x\}$,

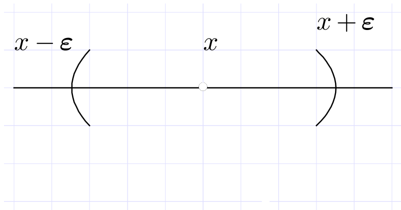
   then $(x-\varepsilon, x+\varepsilon) \subset (a,b)$.



2. The entire line, $\mathbb{R} = (-\infty, \infty)$, which is our set here, is open (obviously), and therefore this means that $\mathbb{R}^c$ is closed. $\mathbb{R}^c = \varnothing$. This leads us to the conclusion that $\varnothing$ is a closed set. However, $\varnothing$ is also open if not, that means $\varnothing$ contains at least one point that is NOT an interior point. This is absurd because $\varnothing$ contains no points.

But we must note that the complement of an open set, in our case $\varnothing$, is a closed set. This means that $\mathbb{R}$ is also closed. In $\mathbb{R}$, only $\mathbb{R}$ and the empty set, $\varnothing$ are both open and closed at the same time.

Definition 7: The deleted ε-neighborhood of x is the set that contains every point of the ε-neighborhood except for x itself.



Theorem 1:

i. The union of finite or infinite open sets is open (A countable union of opens is open).

ii. A countable intersection of closed sets is closed.

<u>Proof:</u>

(i) Let G be a collection (finite or infinite) of open sets, further let $S = \cup\{V \quad s.t. \quad V \in G\}$. This V is an open set. We need to show that S is open. That means every point in S must be an interior point. We will only use the definitions to show this. In other words, every point in S has at least one ε-neighborhood that is completely inside of S.

$$\begin{aligned}
\text{Let } x \in S \quad & x \text{ abritrary but fixed.} \\
\text{Since } S = V \cup G \text{, then:} \quad & \\
x \text{ is in at least one of } V \text{ or } G \quad & \\
\exists V_0 \in G \quad s.t. x \in V_0 \quad & \\
V_0 \text{ is assumed to be an open set} \quad & \text{(By construction)} \\
\text{Then } \exists \varepsilon > 0 \quad s.t. (x-\varepsilon, x+\varepsilon) \in V_0 \quad & \\
V_0 \subset S \quad & \\
i.e. \quad (x-\varepsilon, x+\varepsilon) \subset S \quad & \text{for some } \varepsilon > 0
\end{aligned}$$

Since x was chosen randomly, we conclude that every x in S is an interior point. Hence, S is an open set.

(ii) Let F be a collection (finite or infininte) of closed sets and let $T = \cap\{U \quad s.t. U \in F\}$. This U is a closed set.

$$T^c = \cup\{U^c \quad s.t. \quad U \in F\}$$
$$\text{We know that } U^c \text{ is an open set by } (i)$$
$$\text{Since } T^c \text{ is open, then } T \text{ is closed}$$

An example:

$$(-1, 3) \cup (7, 18)$$

Both $(-1, 3)$ and $(7, 18)$ are open sets, so their union is obviously going to be an open set. This is much easier to understand when demonstrated through actual number lines in $\mathbb{R}$.

September 9th, 2020

Definition 1: Limit point - Let $S \subseteq \mathbb{R}$, a set of $\mathbb{R}$, doesn't have to be interval. x is a limit point of S if every deleted neighborhood of x contains at least one point of S.

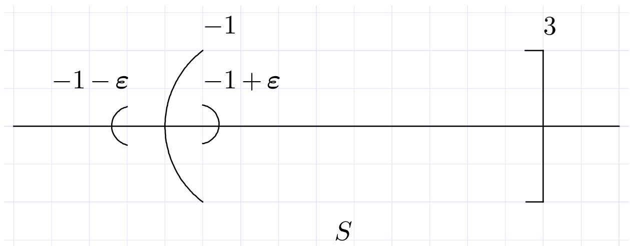
i.e. $\forall \varepsilon > 0$, $((x - \varepsilon, x + \varepsilon) - x) \cap S = \{x, \text{and at least another point of } S\}$

Definition 2: x is a boundary point of S if every ε-neighborhood of x contains at least one point from S and at least one point NOT in S.

i.e. $\forall \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \cap S \neq \varnothing \quad \text{and} \quad (x - \varepsilon, x + \varepsilon) \cap S^c \neq \varnothing$

We can show both these definitions on a number line.

$$\text{Let } S = (-1, 3]$$



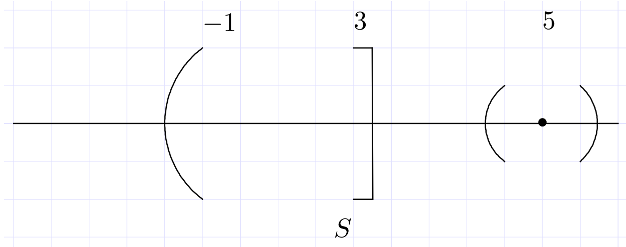
In this case, -1 can be a boundary point of S, because the neighborhood contains at least one point in S and one point outside of S. It can also be a limit point if we just consider -1 as a deleted point, and then we take the deleted neighborhood with the same argument.

Definition 3: The set of all boundary points of S is called the boundary of S and is denoted by $\boldsymbol{\vartheta} S$.

The closure of S, denoted $\bar{S}$, is: $\bar{S} = S \cup \boldsymbol{\vartheta} S$.

Definition 4: x is an isolated point of S if $x \in S$ and there is at least one ε-neighborhood of x that contains no points from S other than x itself. Unlike the first 3 definitions, we actually need x to be in S. It is not also for all ε-neighborhoods. It is only for at least one of them.

Let $S = (-1, 3] \cup \{5\}$



5 is clearly an isolated point, as its neighborhood contains no other elements of S. However, we also know that $5 \in S$, by definition.

Defintion 5: x is an exterior point of S if x is an interior point of the complement of S, S^c. Revisit the definition of an interior point to understand this definition better. The collection of all exterior points of S is called the exterior of S.

Exp:

1. $S = (-\infty, -1] \cup (1, 2) \cup \{3\}$. Then:

   The set of all limit points of $S = (-\infty, -1] \cup [1, 2]$

   $\boldsymbol{\vartheta} S = \{-1, 1, 2, 3\}$

   $\bar{S} = (-\infty, -1] \cup (1, 2) \cup \{3\}$

   $\{3\}$ is the only isolated point of S

   The exterior of $S = (-1, 1) \cup (2, 3) \cup (3, +\infty)$

2. For $n \geqslant 1$, $I_n = \left[\frac{1}{2n+1}, \frac{1}{2n}\right]$ and $S = \bigcup_{n=1}^{\infty} I_n$

   The set of all limit points of S is $S \cup \{0\}\}$

   $\boldsymbol{\vartheta} S = \left\{x \in \mathbb{R} \quad s.t. \quad x = 0 \quad \text{or} \quad x = \frac{1}{n}, \text{with}\, n > 2\right\}$

   $\bar{S} = \ S \cup \{0\}$

   S has no isolated points

   The exterior of S:

   $$(-\infty, 0) \cup \left[\bigcup_{n=1}^{\infty} \left(\frac{1}{2n+2}, \frac{1}{2n+1}\right)\right] \cup \left(\frac{1}{2}, \infty\right)$$

3. Let $S = \mathbb{Q}$. Then:

   Since any interval contains at least one rational number, every real number, $\mathbb{R}$, is a limit point of $\mathbb{Q}$. Thus the set of all the limit points of $\mathbb{Q} = \mathbb{R}$.

   Since every interval contains at least one irrational number, every real number, $\mathbb{R}$, is a boundary point for $\mathbb{Q}$. We know both these points because of the density of the rational numbers.

   $\boldsymbol{\vartheta} \mathbb{Q} = \mathbb{R}$

   $\bar{\mathbb{Q}}$: $\mathbb{Q} \cup \mathbb{R} = \mathbb{R}$

   $\mathbb{Q}$ has no isolated points

   $\mathbb{Q}$ is neither open nor closed. This is unlike $\mathbb{R}$ and $\varnothing$ that are BOTH closed AND open.

Theorem 2:

A set S is closed iff no points of S^c is a limit point of S (i.e. S is closed iff $\bar{S} = S$, and further i.e. a set is closed iff it contains all its limit points)

Proof: We need to show both directions since it is an iff relation.

$$\begin{aligned}
''\Rightarrow'' \quad \text{Assume } S \text{ is closed. Let } x \in S, x \text{ is arbitrary} \\
\text{Since } S \text{ is closed, } S^c \text{ is open} \\
\exists \varepsilon > 0 \quad s.t. \quad (x-\varepsilon, x+\varepsilon) \subseteq S^c \\
\text{This neighborhood does not contain any points of } S \\
\text{Since it is strictly inside } S^c \\
\text{Hence: } x \text{ cannot be a limit point of } S
\end{aligned}$$

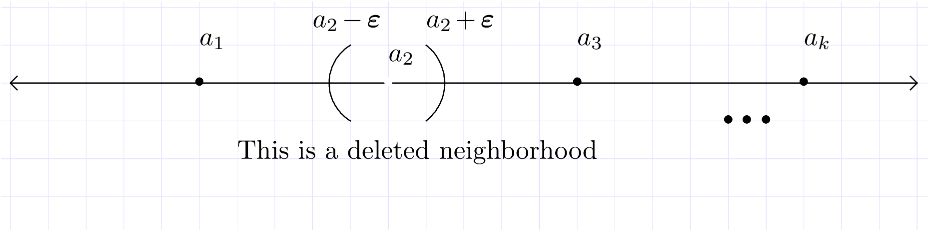
$$\begin{aligned}
''\Leftarrow'' \quad \text{If no points in } S^c \text{ is a limit point of } S \\
\text{every point in } S^c \text{ must have} \\
\text{at least one neighborhood that} \\
\text{does not contain any points from } S \\
\forall x \in S^c \quad \exists \varepsilon > 0 \quad s.t. \quad (x-\varepsilon, x+\varepsilon) \cap S = \varnothing \\
i.e. \forall x \in S^c \quad \exists \varepsilon > 0 \quad s.t. \quad (x-\varepsilon, x+\varepsilon) \subseteq S^c \\
\text{Therefore the complement of } S, S^c, \text{ is open} \\
\text{Hence } S \text{ is closed}
\end{aligned}$$

We have shown that it holds for both directions, therefore we conclude that S is closed iff no points in S^c is a limit point of S.

Rmk: It is NOT true that the closed set must have limit points. If S has no limit points, then the set of all limit points is $\varnothing$, and therefore contained in S ($\varnothing \in S$ for any S).

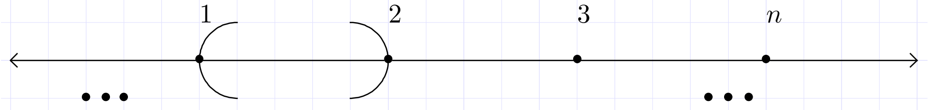
Hence, a set with no limit points is closed. For example, let us take S, a finite set of real numbers i.e.:

$$S = \{a_1, a_2, ..., a_k\}$$



We can find at least one deleted neighborhood that contains no other points of S. This set, shown above, is closed. i.e. S is closed.

The same argument is true if S was a countable infinite set (formed by isolated points). As an example, $\mathbb{Z}$ is closed.



$$\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n+1)$$

This is a union of open sets, and it is open. The complement of a closed set is open, therefore we can safely conclude that $\mathbb{Z}$ is open.

Open Covering:

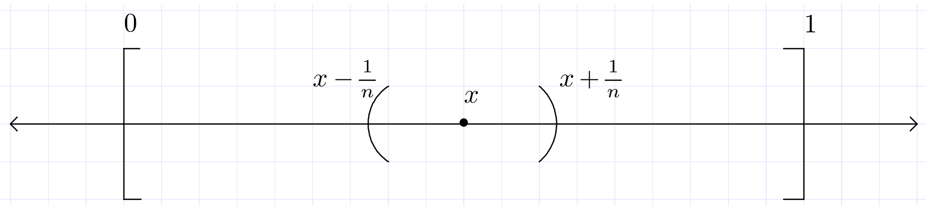
A collection, $\mathcal{V}$ of open sets is an open covering of a set $S \subseteq \mathbb{R}$ if every point of S is contained in an open set V of $\mathcal{V}$ i.e.:

$$S \subset \bigcup \{V \quad s.t. \quad V \in \mathcal{V}\}$$

Exp1:

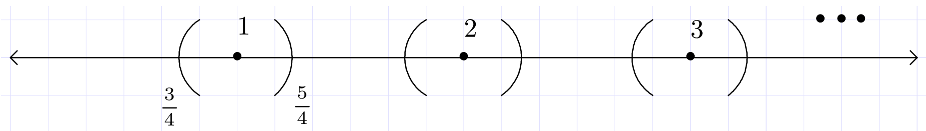
$$S = [0,1] \text{ is covered by the family of:}$$
$$V_x = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$$
$$\text{Where } 0 < x < 1$$
$$\text{and } n \in \mathbb{N}$$

This is similar to the characterization of the ε and sup and inf.



Exp2:

$$S = \{1, 2, 3, \ldots, n, \ldots\} \text{ is covered by the family of:}$$
$$\left(n - \frac{1}{4}, n + \frac{1}{4}\right), n \in S$$



Theorem 3: Heine-Borel Theorem

Let $C \subseteq \mathbb{R}$ be a close bounded interval and let $\mathcal{V} = \{V_i\}_{i \in I}$ be a collection of open intervals in $\mathbb{R}$. Suppose $C \subseteq \bigcup_{i \in I} V_i$ (i.e. $\mathcal{V}$ is an open covering of C).

$$\text{Then:} \quad \text{there are } n \in \mathbb{N} \text{ and } i_1, i_2, \ldots, i_n \in I \quad s.t. \quad C \subseteq \bigcup_{k=1}^{n} V_{i_k}$$

Explanation: If we start with a closed bounded interval of $\mathbb{R}$ and if we have a family of collections that covers this (in practice, this is an infinite family), in other words we have an open covering that is formed by infinitely many open sets, we can extract from this family finitely many opens (elements) that "do the job." In the beginning, we know that the family $\mathcal{V} = \{V_i\}_{i \in I}$ covers C. Then we can extract a finite set of opens that still counts as open covering for this set, C. We do not need infinitely many open sets to cover C.

September 14th, 2020

<u>Proof of Heine Borel Theorem:</u>

Let $C=[a,b]$

Let

$$S=\left\{ r\in[a,b] \quad s.t. \quad \text{there are } p\in\mathbb{N} \text{ and } i_1, i_2, \ldots i_p \text{ for which the interval } [a,r]\subseteq\bigcup_{k=1}^{p} V_{i_k} \right\}$$

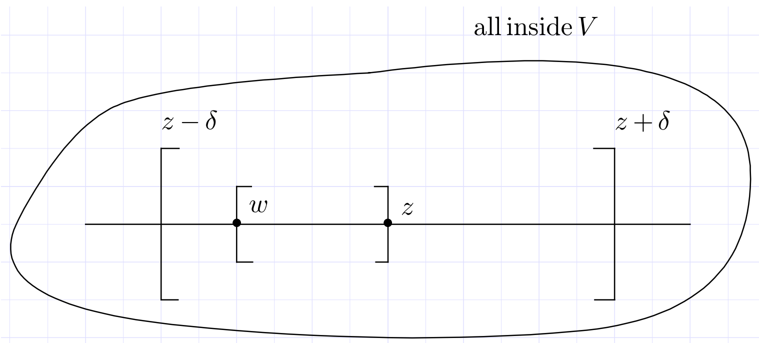
i.e. $[a,r]$ has finite open covering.

By construction, the set $S\subseteq[a,b]$. Thus, b is an upper bound of S. Also, $S\neq\varnothing$ because $a\in S$. In fact, when $r=a$, $[a,r]=\{a\}$ but a is at least in one of the $V_i's$, and one of the $V_i's$ is a finite covering. We therefore have a set that is bounded above and that is not empty. Thus, the completeness axiom implies that S has a least upper bound. Let $z=\sup(S)$.

Goal: First we show that $z\in S$ and then $z=b$. If $z=b$, then b is in S. If we can show that $b\in S$, then we are saying that the interval $[a,b]$ is included in the union of finitely many open coverings.

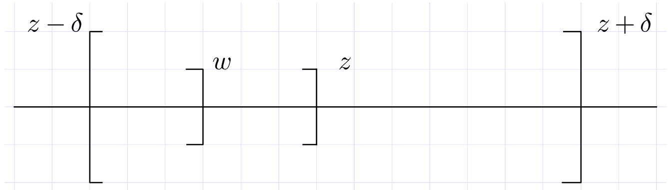
$$z\in S, z=b\Rightarrow b\in S$$

$$\begin{aligned}
\text{We know that } a\leqslant z\leqslant b & \\
\text{and so } z\in[a,b] & \\
\text{Thus } \exists m\in I \quad s.t. \quad z\in V_m & \\
\text{Now since } V_m \text{ is open,} & \\
\text{there exists } \delta>0 \quad s.t. & \\
[z-\delta, z+\delta]\subseteq V_m & \qquad (*) \\
\text{We now show that } z\in S\text{:} & \\
\text{If } z=a, \text{then } z\in S \text{ and we are done} & \\
\text{Suppose } z>a & \\
\text{Since } z=\sup(S) & \\
\text{There is some } w\in S \quad s.t. & \\
z-\delta<w\leqslant z & \qquad (**) \\
(*) \text{ and} (**) \text{ imply that:} & \\
[w,z]\subseteq V_m &
\end{aligned}$$



By definition of S, there are $p\in\mathbb{N}$ and $i_1, i_2, \ldots, i_p\in I$ s.t.:

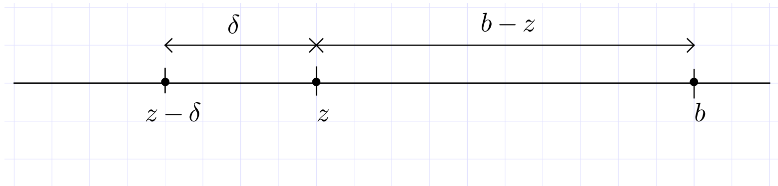
$$[a,w]\subseteq\bigcup_{k=1}^{p} V_{i_k} \quad \leftarrow \text{because } w\in S$$



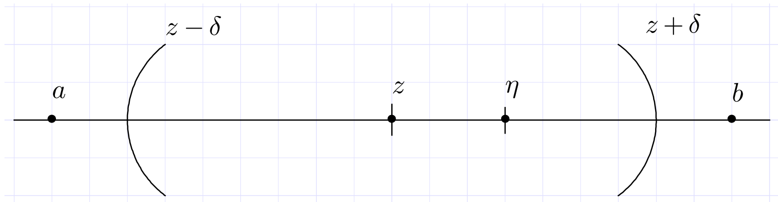
$$\begin{aligned}\text{Now:}\quad [a,z] &= [a,w]\cup[w,z]\\ &\subseteq \bigcup_{k=1}^{p} V_{i_k}\cup V_m\end{aligned}$$

We have $p+1$ open intervals
z satisfies the property of being included
in infinitely many open intervals
Hence, $z\in S$

Now, we show that $z=b$. We proceed by contradiction. First, we assume that $z\neq b$.



Let $\eta=\min\left\{\delta,\frac{b-z}{2}\right\}>0$, both $\delta>0$ and $\frac{b-z}{2}>0$. Then: $[z,z+\eta]\subseteq V_m$, and $z+\eta\in[a,b]$ because we know that $\eta\leqslant\frac{b-z}{2}\leftarrow$ by choice.



$[z,z+\eta]\subseteq V_m$ because $[z,z+\eta]\subset[z-\delta,z+\delta]$ and we know from earlier that $[z-\delta,z+\delta]\subseteq V_m$. Furthermore, we can clearly see from the number line that $z+\eta$ is included in the closed set $[a,b]$.

Notice that there is nothing special about us choosing to divide by 2. We could choose 3, 4, whatever we want. We only want to make sure that there is enough distance between z and b.

$$\begin{aligned}[a,z+\eta] &= [a,z]\cup[z,z+\eta]\\ &\subseteq \bigcup_{k=1}^{p} V_{i_k}\cup V_m\end{aligned}$$

We again have union of finitely many $V_i's$
Hence $z+\eta\in S$

This is because our $z+\eta$ satisfies the property required to be in S. If we recall, we can see that our r in this case is $z+\eta$ (See above definition for what the set S is).

This is a contradiction because $z=\sup(S)$, and we cannot have an element larger than $\sup(S)$, i.e. $\sup(S)+\delta$, that is still in S. Hence our assumption is wrong that $z\neq b$, and thus $z=b$.

Rmk (Important):

1. A closed bounded set of $\mathbb{R}$ is called a compact set. The only compacts of $\mathbb{R}$ are the closed bounded intervals (i.e. if C is a compact of $\mathbb{R}$, $\Leftrightarrow C = [a, b]$).

2. The theorem remains true if $C \neq \varnothing$ is a closed bounded subset of $\mathbb{R}$. i.e. we can drop the word interval and it would still be the same, with different construction of S.

Theorem 4: Bolzano-Weirstrass Theorem (*An application of Heine-Borel Theorem*)

Every bounded infinite set of real numbers, of course non-empty, has at least one limit point.

Proof: We shall show that a bounded non-empty set without a limit point can have only a finite number of points (Basically by contradiction).

$$\begin{aligned}
&\text{Assume } S \text{ has no limit points} \\
&\text{Then: } S \text{ is closed by } \textbf{THM2} && (*) \\
&\text{Every point of } x \in S \text{ has an open neighborhood} && (**) \\
&V_x \text{ that contains no points of } S \text{ other than } x \\
\\
&\text{Consider the collection of all neighborhoods} \\
&\mathcal{V} = \{V_x \quad s.t. \quad x \in S\} \\
&\text{Then } \mathcal{V} \text{ is an open covering of } S \\
&i.e. \quad S \subseteq \bigcup_{x \in S} V_x \\
&\text{Since } S \text{ is bounded and closed,} \\
&\text{Heine Borel theorem implies} \\
&\exists x_1, x_2, \ldots, x_n \\
&s.t. \quad S \subseteq \bigcup_{i=1}^{n} V_{x_i}
\end{aligned}$$

It follows that S is just formed by $x_1, x_2, \ldots, x_n$ which contradicts the fact that S is an infinite set. Therefore, by contradiction, we know that if S is an infinite set, then S must have at least one limit point.

---

3 Chapt.: Limits

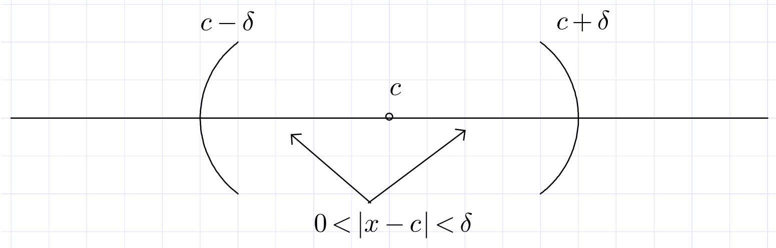
Let $I \subseteq \mathbb{R}$ be an open interval. let c be in I, and let $f: I \setminus \{c\} \to \mathbb{R}$ be a function. We do $\setminus\{c\}$ to show that the function does not necessarily have to be defined at c for it to have a limit at that point. c does not have to be in the domain of f, D_f.

Let $l \in \mathbb{R}$, where l is a finite real number. l is the limit of $f(x)$ as x goes to c, and written under the form $\lim_{x \to c} f(x) = l$ if $\forall \varepsilon > 0$, $\exists \delta$ (may depend upon ε) s.t.:

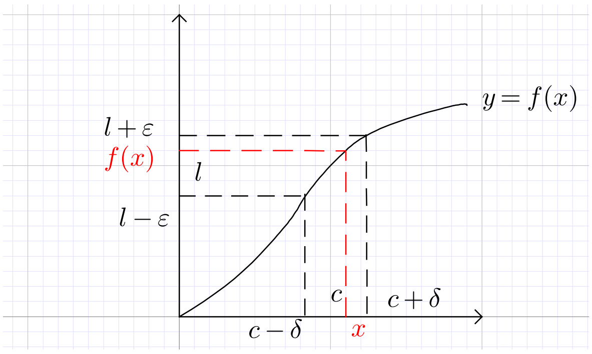
$$x \in I \setminus \{c\} \text{ and } |x - c| < \delta \text{ implies } |f(x) - l| < \varepsilon$$

$|f(x) - l| < \varepsilon$ means that $l - \varepsilon < f(x) < l + \varepsilon$

i.e. $0 < |x - c| < \delta$. We can get close to c, but never equal to c. This $|x - c|$ is a deleted δ-neighborhood of c.



Since this is a deleted δ-neighborhood and $x \neq c$, then $x \in (c - \delta, c)$ or $x \in (c, c + \delta)$. For every choice of ε (extremely small), we can find some δ and the deleted δ-neighborhood of c s.t. for every point inside that neighoborhood, the difference between the value of the image of x, $f(x)$, and the limit itself is less than our choice of ε.



September 16th, 2020

Exp. 1) Show that $\lim_{x\to 4}(5x + 1) = 21$.

$$\begin{aligned}
\text{Let } \varepsilon > 0. \text{ We want to find } \delta > 0 \\
s.t. \text{ if } x \in \mathbb{R}\backslash\{4\} \\
\text{and } |x - 4| < \delta, \text{ then} \\
|(5x + 1) - 21| < \varepsilon \\
\text{If} \quad |5x + 1 - 21| < \varepsilon \\
\text{then} \quad |5x - 20| = |5(x - 4)| \\
= 5|x - 4| < \varepsilon \\
\text{or} \quad |x - 4| < \frac{\varepsilon}{5} \\
\text{We choose } \delta = \frac{\varepsilon}{5} \\
\text{If } x \in \mathbb{R}\backslash\{4\} \text{ and } |x - 4| < \delta, \text{ then:} \\
|(5x + 1) - 21| < 5\delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon
\end{aligned}$$

Exp. 2) Show that $\lim_{x\to 3}(x^2 - 1) = 8$

$$\begin{aligned}
\text{Let } \varepsilon > 0. \text{ We shall find some } \delta > 0 \\
s.t. \text{ if } x \in \mathbb{R}\backslash\{3\} \text{ and } |x - 3| < \delta \\
\text{then} \quad |(x^2 - 1) - 8| < \varepsilon \\
|x^2 - 9| = |x - 3| \cdot |x + 3| \\
\text{So if:} \quad |(x^2 - 1) - 8| < \varepsilon
\end{aligned}$$

$$\text{then:} \quad |x-3| < \frac{\varepsilon}{|x+3|}$$

$$\text{This } \frac{\varepsilon}{|x+3|} \text{ cannot be } \delta \text{ since}$$

$$\delta \text{ does not depend on } x$$

$$\text{If } |x-3| < 1, \text{then } -1 < x-3 < 1$$

$$(x \text{ is very close to } 3)$$

$$\text{We do } +6 \text{ to all sides}$$

$$5 < x+3 < 7$$

$$\text{Then } 5 < |x+3| < 7$$

$$\frac{1}{|x+3|} > \frac{1}{7}$$

$$\frac{\varepsilon}{|x+3|} > \frac{\varepsilon}{7}$$

$$\text{Now:} \quad \text{We choose } \delta \text{ to be } \min\left\{1, \frac{\varepsilon}{7}\right\}$$

$$\text{Then:} \quad \text{If } x \in \mathbb{R}\backslash\{3\} \text{ and } |x-3| < \delta$$

$$|(x^2-1)-8| < \delta|x+3| < 7\delta$$

$$7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

$$\text{Therefore } |(x^2-1)-8| < \varepsilon$$

Exp. 3) Show that $\lim_{x\to 0}\frac{1}{x}$ does not exist (As a finite real number). We obviously have to proceed by contradiction.

$$\text{We will assume } \lim_{x\to 0}\frac{1}{x} = l \qquad l \in \mathbb{R}$$

$$\text{Then we shall find } \varepsilon > 0 \quad s.t. \forall \delta > 0$$

$$\text{there is some } x \in \mathbb{R}\backslash\{0\} \text{ for which}$$

$$x - 0 < \delta \text{ but } \left|\frac{1}{x} - l\right| \geqslant \varepsilon$$

$$\text{If } \left|\frac{1}{x} - l\right| \geqslant \varepsilon, \text{then:}$$

$$\frac{1}{x} - l \geqslant \varepsilon \quad \text{or} \quad \frac{1}{x} - l \leqslant -\varepsilon$$

$$\frac{1}{x} \geqslant l + \varepsilon \quad \text{or} \quad \frac{1}{x} \leqslant 1 - \varepsilon$$

It suffices to find x for which one of the inequalities hold. We will be discussing three cases: $l > 0$, $l = 0, l < 0$.

1. Assume $l > 0$

$$\text{Take } \varepsilon = \frac{l}{2}$$

$$\text{For every } \delta > 0, \text{we choose } x = \min\left\{\frac{\delta}{2}, \frac{1}{l+\varepsilon}\right\} \qquad \frac{1}{l+\varepsilon} > 0, \frac{\delta}{2} > 0$$

$$x \in (0, +\infty)$$

$$|x - 0| = |x| < \frac{\delta}{2} < \delta$$

$$\text{but } \left|\frac{1}{x} - l\right| \geqslant \varepsilon \text{ because}$$

$$x \leqslant \frac{1}{l+\varepsilon}\left(\text{and so } \frac{1}{x} \geqslant l + \varepsilon\right)$$

2. Assume $l = 0$

$$\begin{aligned}
\text{Take } \varepsilon &= 1 \\
\text{For every } \delta > 0, \text{we choose } x &= \min\left\{\frac{\delta}{2}, 1\right\} \qquad x \text{ is positive} \\
|x - 0| = |x| \leqslant \frac{\delta}{2} &< \delta, \text{but} \\
\left|\frac{1}{x} - 0\right| &\geqslant 1 (= \varepsilon)
\end{aligned}$$

3. Assume $l < 0$

$$\begin{aligned}
\text{Take } \varepsilon = \frac{|l|}{2} &> 0 \qquad \text{(There is nothing special about 2)} \\
\text{For every } \delta > 0, \text{we choose } x = \max\left\{-\frac{\delta}{2}, \frac{1}{l - \varepsilon}\right\} & \qquad \text{Both are} < 0 \\
\text{Thus:} \quad x \in (-\infty, 0), \text{and } |x - 0| = |x| < \delta & \\
\text{Because } 0 > x \geqslant -\frac{\delta}{2} & \\
\text{But } \left|\frac{1}{x} - l\right| \geqslant \varepsilon \text{ because } 0 > x \geqslant \frac{1}{l - \varepsilon} & \\
\text{and so } \frac{1}{x} \leqslant l - \varepsilon, \text{ie } \frac{1}{x} - l \leqslant -\varepsilon &
\end{aligned}$$

Lemma 1:

Let $I \subseteq \mathbb{R}$ be an open interval, $c \in I$ and $f: I \backslash \{c\} \rightarrow \mathbb{R}$ be a function.

$$\text{If } \lim_{x \rightarrow c} f(x) = l \text{ for some } l \in \mathbb{R}, \text{then } l \text{ is } \underline{\text{unique}}.$$

Proof: Assume $\lim_{x \rightarrow c} f(x) = l_1$ and $\lim_{x \rightarrow c} f(x) = l_2$ for some $l_1, l_2 \in \mathbb{R}$.

$$\begin{aligned}
\text{Let } \varepsilon > 0. \text{ Then} \\
\exists \delta_1, \delta_2 > 0 \qquad s.t. \\
x \in I \backslash \{c\} \text{ and } |x - c| < \delta_1, \text{implies} \\
|f(x) - l_1| < \varepsilon \\
\text{and} \\
x \in I \backslash \{c\} \text{ and } |x - c| < \delta_2, \text{implies} \\
|f(x) - l_2| < \varepsilon \\
\\
\text{We let } \delta = \min\{\delta_1, \delta_2\} \\
\text{If } x \in \mathbb{R} \backslash \{c\} \text{ and } |x - c| < \delta, \text{then:} \\
|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \\
\leqslant |l_1 - f(x)| + |f(x) - l_2| \\
\text{Both are} < \varepsilon \text{ because } |x - c| < \delta < \delta_1 \text{ or } \delta_2 \\
\text{Since both are less than } \varepsilon, \text{then } |l_1 - l_2| < \varepsilon \\
\text{Therefore, } |l_1 - l_2| = 0 \\
\text{Or:} \quad l_1 = l_2
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $l_1 - l_2 = 0$, or $l_1 = l_2$.

Theorem 2: *"Sign Preserving Propery for limits"*

Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$, and let $f: I \backslash \{c\} \to \mathbb{R}$ be a function.

Suppose that $\lim_{x \to c} f(x)$ exists.

i. If $\lim_{x \to c} f(x) > 0, \exists M > 0$ and $\exists \delta > 0$ s.t. $x \in I \backslash \{c\}$ and $|x - c| < \delta$ implies that $f(x) > M > 0$

ii. If $\lim_{x \to c} f(x) < 0$, $\exists N < 0$ and $\exists \delta > 0$ s.t. $x \in I \backslash \{c\}$ and $|x - c| < \delta$ implies that $f(x) < N < 0$

<u>Proof:</u>

(i) Let $l = \lim_{x \to c} f(x) > 0$. Let $M = \frac{l}{2}$. Moreover, $\exists \delta > 0$ s.t. $x \in I \backslash \{c\}$ and $|x - c| < \delta$. Then:

$$|f(x) - l| < \frac{l}{2}, \text{or} \quad -\frac{l}{2} < f(x) - l < \frac{l}{2}$$

$$i.e.\, f(x) > l - \frac{l}{2} = M$$

Proof of (ii) is very similar to (i), therefore we will not include it. We simply follow the same instructions.

Lemma 3:

If $\lim_{x \to c} f(x)$ exists, then $\exists \delta > 0$ s.t. the restriction of f to $(I \backslash \{c\} \cap (c - \delta, c + \delta))$ is bounded.

<u>Proof:</u> Let $l = \lim_{x \to c} f(x)$. Then $\exists \delta > 0$ s.t. $x \in I \backslash \{c\}$ and $|x - c| < \delta$ implies that $|f(x) - l| < 1$

$$\text{Then:} \quad -1 < f(x) - l < 1 \,\text{for all}\, c - \delta < x < c + \delta$$

i.e. $l - 1 < f(x) < l + 1$ for all $c - \delta < x < c + \delta$. This is enough for us to conclude that it is bounded.

<u>Rmk:</u> f is said to be bounded if $\exists M > 0$,

$$|f(x)| \leqslant M, \text{for}\, x \in I$$

To be bounded means: $m \leqslant f(x) \leqslant N$. Let $M = \max\{|N|, |m|\}$. Then $|f(x)| \leqslant M$.

With respect to our proof:

$$\begin{aligned} |f(x)| - |l| \leqslant |f(x) - l| &< 1 \\ \text{and so:} & \\ |f(x)| < 1 + |l| & \quad 1 + |l| \,\text{plays the role of}\, M \end{aligned}$$

September 21st, 2020

Lemma 4:

Let $I \subseteq \mathbb{R}$ be an open interval, $c \in I$ and let $f, g: I \backslash \{c\} \to \mathbb{R}$ be functions.

Suppose that $\lim_{x\to c} f(x) = 0$, and that g is founded. Then $\lim_{x\to c} f(x)g(x) = 0$.

Proof:

$$\begin{aligned}
\text{Since } g \text{ is bounded, } \exists M > 0 \\
\text{st } |g(x)| \leqslant M \text{ for all } x \in I \backslash \{c\} \\
\text{Let } \varepsilon > 0. \exists \delta > 0 \quad \text{st:} \\
x \in I \backslash \{c\} \text{ and } |x - c| < \delta \text{ implies } |f(x) - 0| < \frac{\varepsilon}{M} \\
\text{Thus: if } |x - c| < \delta, \text{then:} \\
|f(x)g(x) - 0| = |f(x)| \cdot |g(x)| \\
\leqslant |f(x)| \cdot M \\
< \frac{\varepsilon}{M} \cdot M = \varepsilon
\end{aligned}$$

Hence:

$$\lim_{x\to c} f(x)g(x) = 0$$

Theorem 5: Squeeze Theorem

Let $I \subseteq \mathbb{R}$ be an open interval and $c \in I$. Furhtermore, $f, g, h: I \backslash \{c\} \to \mathbb{R}$ are all functions. Suppose:

$$f(x) \leqslant g(x) \leqslant h(x) \, \forall x \in I \backslash \{c\}$$

Assume $\lim_{x\to c} f(x) = l = \lim_{x\to c} h(x)$ for some $l \in \mathbb{R}$. Then $\lim_{x\to c} g(x) = l$.

Proof:

$$\begin{aligned}
\text{Let } \varepsilon > 0, \exists \delta_1 > 0 \text{ and } \exists \delta_2 > 0 \quad \text{st} & \\
x \in I \backslash \{c\} \text{ and } |x - c| < \delta_1 \text{ implies } |f(x) - l| < \varepsilon & \quad \leftarrow l - \varepsilon < f(x) \\
\text{and} & \\
x \in I \backslash \{c\} \text{ and } |x - c| < \delta_2 \text{ implies } |f(x) - l| < \varepsilon & \quad h(x) < l + \varepsilon \\
& \\
\text{Now: let } \delta = \min \{\delta_1, \delta_2\} & \\
\text{If } x \in I \backslash \{c\} \text{ and } |x - c| < \delta, \text{then:} & \\
l - \varepsilon < f(x) \leqslant g(x) \leqslant h(x) < l + \varepsilon & \\
i.e. \, |g(x) - l| < \varepsilon &
\end{aligned}$$

Definitions:

Let $I \subseteq \mathbb{R}$ be an open interval, $c \in I$, $f: I \backslash \{c\} \to \mathbb{R}$ be a function.

Definition 1: A real number l is said to be a right-hand limit of f at c, written as:

$$\lim_{x\to c^+} f(x) = l$$

if for each $\varepsilon > 0, \exists \delta > 0$ st $x \in I \backslash \{c\}$ and $c < x < c + \delta$ implies $|f(x) - l| < \varepsilon$.

Definition 2: A real number l is said to be a left-hand limit of f at c, written as:

$$\lim_{x\to c^-} f(x) = l$$

if for each $\varepsilon > 0, \exists \delta > 0$ st $x \in I \setminus \{c\}$ and $c - \delta < x < c$ implies that $|f(x) - l| < \varepsilon$.

Lemma 6:

$$\lim_{x \to c} f(x)$$

exists iff ($\Leftrightarrow$) $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and are equal.

<u>Proof:</u>

$$\begin{aligned}
&\Rightarrow \\
&\text{Suppose } \lim_{x \to c} f(x) = l \text{ for some } l \in \mathbb{R} \\
&\text{Let } \varepsilon > 0. \text{ Then } \exists \delta > 0 \quad \text{st} \\
&x \in I \setminus \{c\} \text{ and } |x - c| < \delta \text{ implies } |f(x) - l| < \varepsilon \\
&\text{Now: if } x \in I \setminus \{c\} \text{ and } c < x < \delta + c \\
&\text{then } |x - c| < \delta \text{ and hence } |f(x) - l| < \varepsilon \\
&\text{Therefore:} \\
&\lim_{x \to c^+} f(x) = l
\end{aligned}$$

$$\begin{aligned}
&\text{Similarly, we show that } \lim_{x \to c^-} f(x) = l, \\
&\text{if we take } c - \delta < x < c
\end{aligned}$$

$$\begin{aligned}
&\Leftarrow \\
&\text{Assume } \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = M \\
&\text{for some } M \in \mathbb{R} \\
&\text{Let } \varepsilon > 0, \exists \delta_1 \text{ and } \delta_2 > 0 \quad \text{st} \\
&x \in I \setminus \{c\} \text{ and } c < x < c + \delta_1 \\
&\text{implies } |f(x) - M| < \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\text{and } x \in I \setminus \{c\} \text{ and } c - \delta_2 > x > c \\
&\text{implies } |f(x) - M| < \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\text{Now let } \delta = \min\{\delta_1, \delta_2\} \\
&\text{Then if } x \in I \setminus \{c\} \text{ and } |x - c| < \delta \\
&\text{We have the following situation:}
\end{aligned}$$

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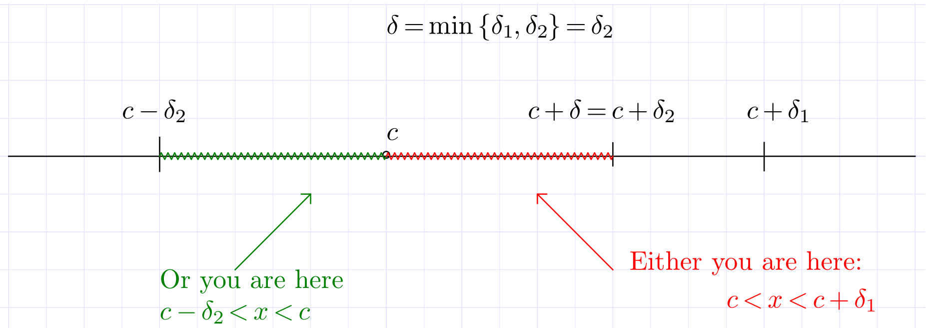


In both cases, we will have that:

$$|f(x)-M|<\varepsilon$$

Therefore, we have shown that:

$$\lim_{x\to c} f(x)=M$$

Theorem 7:

Suppose that f is defined on a bounded interval (a,b) and that $-\infty<\alpha=\inf_{a<x<b}f(x)$ and $\beta=\sup_{a<x<b}f(x)<+\infty$

i. If f is increasing on (a,b), then $\lim_{x\to a^+} f(x)=\alpha$ and $\lim_{x\to b^-} f(x)=\beta$

ii. If f is decreasing on (a,b), then $\lim_{x\to a^+} f(x)=\beta$ and $\lim_{x\to b^-} f(x)=\alpha$

iii. if $a<c<b$, then $\lim_{x\to c+} f(x)$ and $\lim_{x\to c^-} f(x)$ exis. Moreover, if f is increasing:

$$\lim_{x\to c^-} f(x)\leqslant f(c)\leqslant \lim_{x\to c+} f(x)$$

and if f is decreasing:

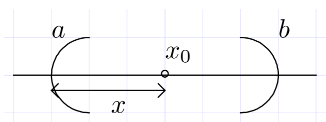
$$\lim_{x\to c^-} f(x)\geqslant f(c)\geqslant \lim_{x\to c+} f(x)$$

<u>Proof:</u>

(i)

We shall show that $\lim_{x\to a^+} f(x)=\alpha$

Important: f is increasing and well defined
If $M>\alpha$,
then $\exists x_0\in(a,b)$ st $\alpha\leqslant f(x_0)<M$
Since f is increasing,
we have $f(x)<M \quad \forall x\in(a,x_0)$



Let $\varepsilon>0$ and $M=\alpha+\varepsilon$
Then:

$$\alpha\leqslant f(x)<\alpha+\varepsilon \quad \forall a<x<x_0 \qquad (*)$$

Now we let $\delta=x_0-a$

$(*)$ is equivalent to:
$|f(x)-\alpha|<\varepsilon$ for all $a<x<a+\delta$,

$$\text{and hence } \lim_{x\to a^+} f(x) = \alpha$$

$$\text{We shall show } \lim_{x\to b^-} = \beta.$$
$$\text{If } M < \beta, \text{then } \exists x_0 \in (a,b) \text{ st}$$
$$M < f(x_0) \leqslant \beta$$
$$\text{Since } f \text{ is increasing,}$$
$$\text{we have } M < f(x) \leqslant \beta \text{ for all } x \in (x_0, b)$$

$$\text{let } \varepsilon > 0 \text{ and let } M = \beta - \varepsilon.$$
$$\text{Then:}$$
$$b - \varepsilon < f(x) \leqslant \beta \text{ for all } x_0 < x < \beta \qquad (**)$$
$$\text{Let we let } \delta = \beta - x_0$$

$$(**) \text{ is equivalent to:}$$
$$|f(x) - \beta| < \varepsilon \text{ for all } \beta - \delta < x < \beta$$
$$\text{and hence } \lim_{x\to b^-} f(x) = \beta$$

(ii) Similar to the proof of (i).

(iii)

Suppose f is increasing. Applying (i) to f on the interval (a,c) and the interval (c,b) shows that:

$$\lim_{x\to c^-} f(x) \text{ and } \lim_{x\to c^+} f(x) \text{ exists}$$

$$\text{if } x < c < x', \text{then:}$$
$$f(x) \leqslant f(c) \leqslant f(x') \qquad \text{Because } f \text{ is increasing}$$
$$\text{Passing to the limit:}$$

$$\lim_{x\to c^-} f(x) \leqslant f(c) \leqslant \lim_{x\to c^+} f(x') \qquad x' \text{ is a dummy variable}$$

Similar argument for when f is decreasing.

September 23rd, 2020

4 Chapt. Continuity

Let $A \subseteq \mathbb{R}$ be a set and let $f: A \to \mathbb{R}$ be a function. f is said to be continuous at $c \in A$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x \in A$ and $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. Our δ clearly depends on both ε and c. In other words, we can say:

$$\lim_{x\to c} f(x) = f(c)$$

Exp:

$$f:(0,+\infty)\to\mathbb{R}$$

$$x\mapsto f(x)=\frac{1}{x}$$

Show that f is continuous at every c in $(0,+\infty)$.

$$\text{if}\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon,\text{then:}\left|\frac{x-c}{cx}\right|<\varepsilon\quad i.e. |x-c|<\varepsilon cx\qquad \varepsilon cx \text{ cannot be } \delta$$

$$\begin{aligned}
\text{The idea is to find some lower bound for } x\\
\text{if } |x-c|<\frac{c}{2},\text{then:}\\
-\frac{c}{2}<x-c<\frac{c}{2}\\
c-\frac{c}{2}<x<\frac{3c}{2}\\
\frac{c}{2}<x<\frac{3c}{2}\qquad & \text{we take } \frac{c}{2}<x \text{ to be the lower bound}
\end{aligned}$$

$$\underline{\text{The real proof is as follows:}}$$

$$\begin{aligned}
\text{Let } \varepsilon>0 \text{ and let } \delta=\min\left\{\frac{c}{2},\frac{3c^2}{2}\right\}\\
\text{Suppose } x\in(0,+\infty) \text{ and } |x-c|<\delta.\text{ Then:}\\
\left|\frac{1}{x}-\frac{1}{c}\right|=\frac{|x-c|}{xc}<\frac{\varepsilon\frac{3c^2}{2}}{\frac{2}{c}c}=\frac{\varepsilon c^2}{4}=\varepsilon'
\end{aligned}$$

Theorem 1:

Let A be an open interval, $B\subseteq\mathbb{R}$ be a set, $g:A\to B$ and $f:B\to\mathbb{R}$ be functions.

$$\text{If } \lim_{x\to c} g(x) \text{ exists in } B \text{ and } f \text{ is continuous at } \lim_{x\to c} g(x),\text{ then:}$$

$$\lim_{x\to c} f(g(x))=f\Big(\lim_{x\to c} g(x)\Big)$$

Note that $c\in A$ but g does not necessarily have to be defined at c.

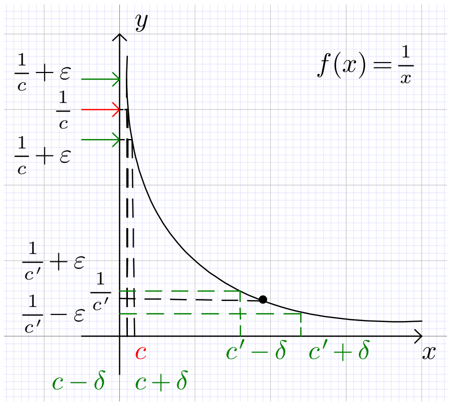
Proof:

$$\begin{aligned}
\text{Let } l=\lim_{x\to c} g(x)\in B\\
\text{Let } \varepsilon>0.\text{ Then } \exists\eta>0\quad s.t.\\
y\in B \text{ and } |y-l|<\eta \text{ implies } |f(x)-l|<\varepsilon\\
\\
\text{Also, } \exists\delta>0\quad s.t.\\
x\in A\backslash\{c\} \text{ and } |x-c|<\delta\\
\text{implies that } |g(x)-l|<\eta
\end{aligned}$$

$$\text{Now suppose } |x-c|<\delta. \text{ Then:}$$
$$|g(x)-l|<\eta$$
$$\text{So:}$$
$$|f(g(x))-f(l)|<\varepsilon$$

$$\text{Therefore } \lim_{x\to c} f(g(x)) = f\Big(\lim_{x\to c} g(x)\Big)$$

Uniform Continuity:



There are functions where regardless of our choice of δ, it would work for any point on our function, f.

Let $f\colon A\subseteq\mathbb{R}\to\mathbb{R}$ be a function. f is said to be uniformly continuous if $\forall\varepsilon>0, \exists\delta>0$ (δ depends on our ε only) such that:

$$x, y\in A \text{ and } |x-y|>\delta \text{ implies } |f(x)-f(y)|<\varepsilon$$

Consequence: Uniform continuity implies continuity. This means that it is essentially stronger. However, in general, the converse is not true.

Exp 1)

$$f_1\colon (1,+\infty)\to\mathbb{R}$$

$$x\mapsto f_1(x)=\frac{1}{x}$$

$$\text{Let } \varepsilon>0. \text{ Take } \delta=\varepsilon>0$$
$$\text{If } x, y\in(1,+\infty) \text{ and if}$$
$$|x-y|<\delta=\varepsilon, \text{ then:}$$
$$\left|\frac{1}{x}-\frac{1}{y}\right| = \frac{|y-x|}{xy} < \frac{\varepsilon}{1\cdot 1} = \varepsilon$$

Exp 2)

$$f\colon (0,+\infty)\to\mathbb{R}$$

$$x\mapsto f(x)=\frac{1}{x}$$

We shall show that f is not uniformly continuous. To do so, we will show its negation. This is the following:

$$\exists \varepsilon > 0 \quad \text{st}\, \forall \delta > 0, \exists x, y \in (0, +\infty) \,\text{with}\, |x - y| < \delta \,\text{but}\, \left|\frac{1}{x} - \frac{1}{y}\right| \geqslant \varepsilon$$

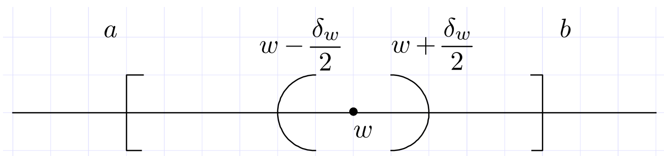
$$\begin{aligned}
\text{Let}\, \varepsilon = 1 \,\text{and let}\, \delta > 0\\
\text{Take}\, x = \sqrt{\delta} \,\text{and}\, y = \frac{\sqrt{\delta}}{1 + \sqrt{\delta}}\\
\text{Then:}\\
|x - y| = \left|\sqrt{\delta} - \frac{\sqrt{\delta}}{1 + \sqrt{\delta}}\right| = \left|\frac{\delta}{1 + \sqrt{\delta}}\right| < \delta\\
\\
\text{but:}\\
|f(x) - f(y)| = \left|\frac{1}{\sqrt{\delta}} - \frac{1 + \sqrt{\delta}}{\sqrt{\delta}}\right| = \frac{\sqrt{\delta}}{\sqrt{\delta}} = 1 \geqslant \varepsilon & \quad \text{Since}\, \varepsilon = 1
\end{aligned}$$

Theorem 2:

Let $C \subseteq \mathbb{R}$ be a closed bounded interval (i.e. C is a compact of $\mathbb{R}$), and let $f: C \rightarrow \mathbb{R}$ be a function. If f is continuous at every poing of C, then f is uniformly continuous. In other words, the continuity + compact domain gives us uniformly continuous. For the proof, we will use Heine-Borel theorem.

<u>Proof:</u>

$$\begin{aligned}
\text{Let}\, \varepsilon > 0.\\
\text{Since}\, f \,\text{is continuous, for each}\, w \in C\\
\exists \delta_w \quad \text{st}\, x \in C \,\text{and}\, |x - w| < \delta_w\\
\text{implies}\, |f(x) - f(w)| < \frac{\varepsilon}{2}\\
\\
\text{We then form the family}\\
\left\{\left(w - \frac{\delta_w}{2}, w + \frac{\delta_w}{2}\right)\right\}_{w \in C}\\
\text{of open intervals in}\, \mathbb{R} \,\text{st:}\\
C \subseteq \bigcup_{w \in C} \left\{\left(w - \frac{\delta_2}{2}, w + \frac{\delta_w}{2}\right)\right\}
\end{aligned}$$



$$\begin{aligned}
\text{This is an open covering of}\, C\\
\\
\text{The Heine Borel Theorem implies:}\\
\exists n \in \mathbb{N} \,\text{and}\, \exists w_1, w_2, \ldots, w_n \,\text{st} & \qquad w_k \,\text{are fixed}\\
C \subseteq \bigcup_{k=1}^{n} \left(w_k - \frac{\delta_{w_k}}{2}, w_k + \frac{\delta_{w_k}}{2}\right)\\
\\
\text{Let}\, \delta = \min_{1 \leqslant k \leqslant n} \left\{\frac{\delta_{w_k}}{2}\right\}
\end{aligned}$$

$$\begin{aligned}
\text{Now, suppose } x, y \in C \text{ and } |x-y| &< \delta \\
\text{Since } y \in C, \exists p \in \{1, \ldots., n\} \text{ st} \\
y \in \left(w_p - \frac{\delta_{w_p}}{2}, w_p + \frac{\delta_{w_p}}{2} \right)&, \text{and so:} \\
|y - w_p| < \frac{\delta_{w_p}}{2} &< \delta_{w_p} \qquad (*) \\
\text{Since } \delta \leqslant \frac{\delta_{w_p}}{2}&, \text{ we also have:} \\
|x-y| < \delta \leqslant \frac{\delta_{w_p}}{2}&. \text{ Thus:} \\
|x - w_p| = |x - y + y - w_p&| \\
\leqslant |x + y| + |y - w_p&| \\
< \frac{\delta_{w_p}}{2} + \frac{\delta_{w_p}}{2} = \delta_{w_p}& \qquad (**)
\end{aligned}$$

$$\begin{aligned}
(*) \text{ and } (**) \text{ imply:} \\
|f(x) - f(y)| = |f(x) - f(w_p) + f(w_p) - f(y)| \\
\leqslant |f(x) - f(w_p)| + |f(w_p) - f(y)| \\
|f(x) - f(w_p)| < \frac{\varepsilon}{2} \quad \text{By } (*) \\
|f(w_p) - f(y)| < \frac{\varepsilon}{2} \quad \text{By} (**) \\
< \varepsilon
\end{aligned}$$

Theorem 3:

Let $A \subseteq \mathbb{R}$ be a bounded non-empty set and let $f: A \to \mathbb{R}$ be a function. If f is uniformly continuous, then f is bounded (The only way a uniformly continuous function can be NOT bounded is if its domain is not bounded).

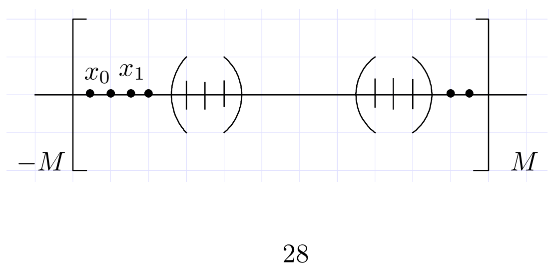
September 28th, 2020

Proof:

Since our set, A is bounded, $\exists M > 0$ s.t. $A \subseteq [-M, M]$.

$$\begin{aligned}
\text{Since } f \text{ is uniformly continuous,} \\
\exists \delta > 0 \quad \text{st } x, y \in A \text{ and } |x - y| < \delta \\
\text{imply } |f(x) - f(y)| < \varepsilon \; (\text{Let } \varepsilon = 1) \\
|f(x) - f(y)| < 1
\end{aligned}$$

$$\begin{aligned}
\text{By } \mathbf{CRLY6\ CHAPT1}: \\
\exists n \in \mathbb{N} \text{st} \\
\frac{1}{n} < \frac{\delta}{2M} \qquad (2M = \text{length of } [-M, M]) \\
\frac{2M}{n} < \delta
\end{aligned}$$



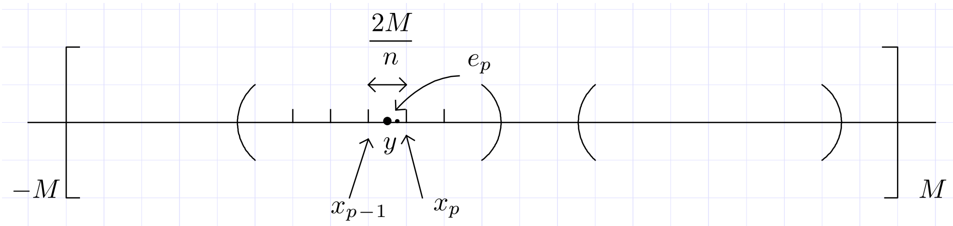
We divide $[-M, M]$ into n subintervals of equal width $\frac{2M}{n}$.

$$\begin{aligned}
&\text{Let } x_0, x_1, x_2, \ \ldots\ x_n \text{ be in } [-M, M] \\
&\text{st } x_0 = -M < x_1 < x_2 < \cdots < x_n = M \\
&\text{and } x_i - x_{i-1} = \frac{2M}{n} \text{ for } i = \{1, \ ....n\}
\end{aligned}$$

$$\begin{aligned}
&\text{For each } i \in \{1, \ ....n\}, \text{ we define the real number} \\
&E_i = \begin{cases} 0, & \text{if } A \cap [x_{i-1}, x_i] = \varnothing \\ f(e_i) + 1, & \text{if } A \cap [x_{i-1}, x_i] \neq \varnothing \end{cases} \\
&e_i \text{ is a fixed point in } A \cap [x_{i-1}, x_i]
\end{aligned}$$

$$\begin{aligned}
&\text{Next, we let } E = \max\{E_1, E_2, \ldots., E_n\} \\
&\text{Now, let } y \text{ be a random element in } A. \\
&\text{We shall show that } |f(y)| \leqslant E
\end{aligned}$$

$$\begin{aligned}
&\text{Since } y \in A, \exists p \in \{1, \ldots., n\} \text{ st} \\
&y \in A \cap [x_{p-1}, x_p], \text{ and so:}
\end{aligned}$$



$$\begin{aligned}
&|y - e_p| \leqslant |x_p - x_{p-1}| = \frac{2M}{n} < \delta \\
&\text{Using the uniform continuity of } f, \text{ we have:} \\
&|f(y) - f(e_p)| < 1 \\
&|f(y) - f(e_p)| > |f(y)| - |f(e_p)| \\
&|f(y)| < 1 + |f(e_p)| = E_p \\
&\text{Therefore:} \\
&|f(y)| < E, \text{ for any random } y \in A
\end{aligned}$$

Hence f is bounded on A.

Corollary 4:

Let $C \subseteq \mathbb{R}$ be a compact set (i.e. closed bounded interval). Let $f: C \to \mathbb{R}$ be a continuous function. Then f is bounded on C.

Proof:

$$\begin{aligned}
&\text{Since } f \text{ is continuous over } C, \\
&\textbf{THM2} \text{ implies that } f \text{ is uniformly continuous} \\
&\textbf{THM3} \text{ implies that } f \text{ is bounded}
\end{aligned}$$

Theorem 5: **(Extreme Value Theorem)**

Let $C \subseteq \mathbb{R}$ be a compact. If $f: C \rightarrow \mathbb{R}$ is continuous, $\exists x_{\min}$ and $x_{\max}$ in C st $f(x_{\min}) \leqslant f(x) \leqslant f(x_{\max}) \quad \forall x \in C$. In other words, the maximum and the minimum values of f occur somewhere in the closed bounded interval C. i.e.:

$$\inf_{x \in C} f(x) = f(x_{\min}) \text{ and } \sup_{x \in C} f(x) = f(x_{\max})$$

Proof:

$$\begin{aligned}
&\text{By } \mathbf{CRLY4}, f \text{ is bounded} \\
&\text{ie the set } f(C) := \{f(x) \text{ st } x \in C\} \\
&\text{is bounded. Also, } C \neq \varnothing \\
&\text{and so } f(C) \neq \varnothing \\
&\text{Thus the axiom of completeness and glb property} \\
&\text{implies:} \quad \sup f(C) \text{ and } \inf f(C) \text{ exist} \\
\\
&\text{Let } M = \sup f(C) \\
&\text{We shall show that } \exists x_{\max} \text{ in } C \text{ st } f(x_{\max}) = M \\
\\
&\text{Since } M = \sup f(C), \text{it is an upper bound of } f(C) \\
&f(x) \leqslant M \quad \forall x \in C \\
\\
&\text{Assume } f(x) < M \quad \forall x \in C \\
&\text{Define } g(x) = \frac{1}{M - f(x)} \qquad (\text{Well defined, } M \neq f(x)) \\
&\text{Clearly } g \text{ is continuous} \\
\\
&\text{Using } \mathbf{CLRY4}, g \text{ is bounded} \\
&\text{Thus } \exists P > 0 \text{ st} \\
&|g(x)| \leqslant P \quad \forall x \in C \\
\\
&\text{Since } g(x) > 0, \text{we have } g(x) = \frac{1}{M - f(x)} \leqslant P \\
&M - f(x) \geqslant \frac{1}{P} \\
&f(x) \leqslant M - \frac{1}{P} < M \quad \forall x \in C \\
\\
&\text{This contradicts the fact that } M = \sup f(C) \\
&\text{Hence } f(x) \leqslant M, \text{which means} \\
&\text{there is (at least) one } x_{\max} \in C \text{ st} \\
&f(x_{\max}) = M
\end{aligned}$$

We can use the same to show that there is at least one $x_{\min}$.

Theorem 6: (Intermediate Value Theorem)

Let $[a,b]$ be a compact of $\mathbb{R}$ and let $f\colon [a,b] \to \mathbb{R}$ be a continous function. Then for every real number r strictly between $f(a)$ and $f(b)$, there exists c st $f(c)=r$.

In other words, f takes on all values between $f(a)$ and $f(b)$.

Proof:

Without loss of generality, we may assume $f(a) < r < f(b)$. Consider the set

$$S = \{x \in [a,b] \text{ st } f(x) < r\}$$

Clearly $S \neq \varnothing$ because $a \in S$. Also,
S is bounded above by b.
Thus the completeness axiom implies
$\sup(S)$ exists as a finite real number
Let $c = \sup(S)$

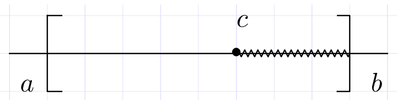
We want to show that $f(c) = r$
To do so, we show $f(c) \geqslant r$ and $f(c) \leqslant r$

*
By definition of S, the set
$f(S) := \{f(x) \text{ st } x \in S\}$ is bounded above by r
Also, $f(S) \neq \varnothing$ because $S \neq \varnothing$
Thus the completeness axiom implies that
$\sup f(S)$ exists

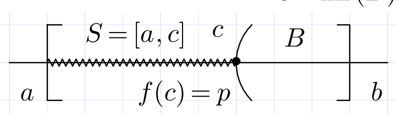
Hence:
$f(c) \leqslant \sup f(S) \leqslant r$
because r is an upper bound
and $f(\sup(S)) \leqslant \sup f(S)$

Therefore $f(c) \leqslant r$

**
Since $f(c) \leqslant r < f(b)$
$c \neq b$



Thus the interval $(a,b]$ is non degenerate
This means it contains more than one element
Let $B = (c,b]$
Then the set $f(B) \neq \varnothing$
$c = \inf(B)$



$$\begin{aligned}
\text{Moreover, because } c = \sup(S), \text{ it follows that:}\\
B \subseteq [a,b] \backslash S\\
\\
\text{Thus } f(x) \geqslant r \quad \forall x \in B\\
\text{Therefore, the set } f(B) := \{ f(x) \text{ st } x \in B \}\\
\text{is bounded below by } r.\\
\text{Thus the glb property implies } \inf(f(B)) \text{ exists}\\
f(c) \geqslant \inf f(B) \geqslant r\\
\text{because } f(\inf B) \geqslant \inf(f(B))\\
\text{and } r \text{ is a lower bound of } f(B)\\
\\
\text{Therefore } f(c) \geqslant r
\end{aligned}$$

We have therefore shown that $f(c) = r$. Because $r \neq f(a)$ and $r \neq f(b)$, c cannot be a or b. Hence we conlcude that $c \in (a,b)$.

Important: The image of a compact set C of $\mathbb{R}$ by a continuous function is compact. In other words,

$$\text{if } C = [a,b] \text{ is a compact of } \mathbb{R}$$

$$\text{and } f: C \rightarrow \mathbb{R} \text{ is continous, then:}$$

$$f(C) := \{ f(x) \text{ st } x \in C = [a,b] \} = f([a,b]) \text{ is compact.}$$

September 30th, 2020

5 Chapt.: Differentiability

Def: f is said to be differentiable at an interior point c of its domain if:

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{exists.}$$

In this case, this limit, denoted by $f'(c)$, is called the derivative of f at c.

Let us say D_f is the domain of f. If $c \in D_f^\circ$, then $\exists \varepsilon > 0$ s.t. $(c - \varepsilon, c + \varepsilon) \subseteq D_f$. This means that c cannot be a boundary point. Everytime we want to talk about the derivative of f at the point c, we have to make sure that f is defined on the ε-neighorhood of c. We need to make sure there is at least one neighborhood of c where the function is well-defined over this neighborhood.

Sometimes it is convenient to let $x = h + c$. Then we can write:

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

Lemma 1:

If f is differentiable at c, then $f(x) = f(c) + [f'(c) + E(x)](x - c)$, where the function E is defined in a neighborhood of c and:

$$\lim_{x \rightarrow c} E(x) = E(c) = 0$$

This means that E is continuous at c.

$$\begin{aligned} f(x) = f(c) + f'(c)(x-c) + E(x)(x-c) \qquad (*) \\ f(c) + f'(c)(x-c) \text{ is the linearization of the function} \\ \text{at the point } c \\ \text{and } E(x)(x-c) \text{ is the error} \\ \text{it goes to zero as we get closer to the actual value} \end{aligned}$$

<u>Proof</u>:

$$\text{Let } E(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} - f'(c) & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases} \qquad (**)$$

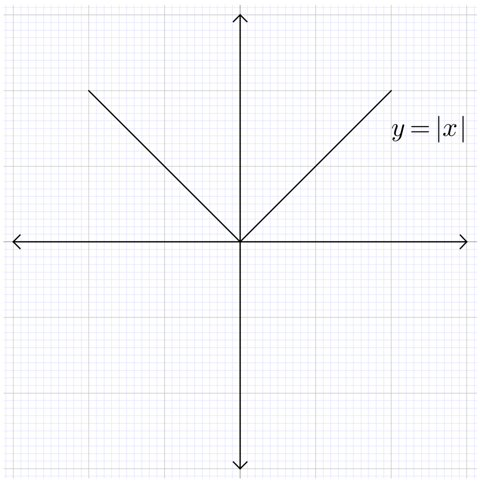
$$\lim_{x \to c} E(x) = \lim_{x \to c} \left[\frac{f(x)-f(c)}{x-c} - f'(c) \right] = 0$$

$$\text{(Because } f \text{ is differentiable at } c\text{)}$$

$$\text{Then } \lim_{x \to c} E(x) = 0 = E(c)$$

If we solve (**) for $f(x)$, we obtain the original, (*) by construction.

Theorem 2:



We can see that this function (well-known example) is continuous but it is not differentiable. What can we take from this?

If f is differentiable at a point c, then f is continuous at the point c. However, in general, the converse is not true. This means that if f is continuous at a point, then it isn't necessarily differentiable at that point.

<u>Proof</u>:

Since the RHS of (*) is continuous (all constants and continuous functions) at c, then the LHS ($f(x)$) is continuous at c. End of proof.

Theorem 3: *The chain rule*

Suppose that we have a function, g, that is differentiable at the point c. Suppose that f is differentiable at $g(c)$. Then the composite $h = f \circ g$ is differentiable at c and we know that $h'(c) = f'(g(c)) \cdot g'(c)$.

<u>Proof</u>:

Since f is differentiable at $g(c)$, ***LM1*** implies that:

$$f(t) = f(g(c)) + [f'(g(c)) + E(t)](t - g(c))$$

Where $\lim_{x \to g(c)} E(t) = E(g(c)) = 0$

We let $t = g(x)$. Then:

$$\frac{f(g(x)) - f(g(c))}{x - c} = [f'(g(c)) + E(g(x))] \cdot \frac{g(x) - g(c)}{x - c}$$

Since $h(x) = f(g(x))$, we have:

$$\frac{h(x) - h(c)}{x - c} = [f'(g(c)) + E(g(x))] \cdot \frac{g(x) - g(c)}{x - c}$$

Now, since E is continuous at $g(c)$. **THM1 of Chapt. 4** implies:

$$\lim_{x \to c} E(g(x)) = E\Big(\lim_{x \to c} g(x)\Big) = E(g(c)) = 0$$

g is continuous at c because g is differentiable at c

Hence:

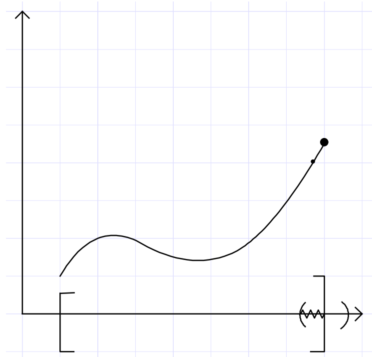
$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = f'(g(c)) \cdot g'(c)$$

Therefore, h is differentiable at c and $h'(c) = f'(g(c)) \cdot g'(c)$

Definition: *Extreme Values*

$f(c)$ is siad to be an extreme value of f if $\exists \delta > 0$ st $f(x) - f(c)$ does not change signs on

$$(c - \delta, c + \delta) \cap D_f$$



Terminology: Local maximum (respectively local minimum) points are called extreme points.

Theorem 4: If f is differentiable at a point c in D_f° (interior of the domain of f),and if c is the "local" extreme point, then we must have:

$$f'(c) = 0$$

Proof:

We will show that if $f'(c) \neq 0$, then c is not a local extreme point. Non-B implies Non-A. Simple as that.

By **LM1**, we have:

$$\frac{f(x)-f(c)}{x-c}=f'(c)+E(x)$$
$$\text{Where} \lim_{x\to c} E(x)=0$$

$$\text{If we are assuming } f'(c)\neq 0, \text{ and } \lim_{x\to c} E(x)=0,$$
$$\exists \delta>0 \text{ st } \forall x\in(c-\delta,c+\delta),$$
$$|E(x)|\leqslant|f'(c)|$$

$$\text{Thus } f'(x)+E(x) \text{ will not change signs on}$$
$$(c-\delta,c+\delta)$$
$$\text{it has the same sign of } f'(c) \text{ on } (c-\delta,c+\delta)$$

$$\text{But } (x-c) \text{ changes signs on } (c-\delta,c+\delta)$$
$$\text{So } f(x)-f(c) \text{ must change signs on } (c-\delta,c+\delta)$$

This contradicts the fact that c is a local extreme point.

Theorem 5: *Rolle's Theorem*

Suppose that f is continuous on a compact $[a,b]$ of $\mathbb{R}$, and suppose f is differentiable on the open interval (a,b).

If $f(a)=f(b)$, then $\exists$ (at least) c in (a,b) st $f'(c)=0$.

Proof:

Since f is continuous on the compact $[a,b]$, the **EVT** implies that f attains its maximum and minimum values on $[a,b]$. If these two extreme values are equal to each other, this means that the function is constant over the interval $[a,b]$. If a function is constant, then its derivative is equal to 0 for any point on the interval $[a,b]$.

If the two extreme values differ (so these extreme values cannot be attained at a and b because we are assuming that $f(a)=f(b)$), then at least one of them (of these extreme values) must be attaineed at some point c in (a,b). Hence, **THM4** implies that $f'(c)=0$, since c is an extreme point.

Theorem 6: *Cauchy's MVT (Application of Rolle's Theorem)*

Let $[a,b]\subseteq\mathbb{R}$ be a compact. Suppoaw f,g are continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c\in(a,b)$ st:

$$[f(b)-f(a)]\cdot g'(c)=[g(b)-g(a)]\cdot f'(c)$$

Proof:

$$\text{Let } h(x)=[f(b)-f(a)]\cdot g(x)-[g(b)-g(a)]\cdot f(x)$$
$$\text{Then } h \text{ is continuous on } [a,b]$$
$$\text{and differentiable on } (a,b)$$

Moreover:

$$h(a)=f(b)g(a)-g(b)f(a)=h(b)$$
$$\text{Hence, } \textbf{Rolle's Theorem} \text{ implies:}$$
$$\exists c\in(a,b) \text{ st } h'(c)=0$$
$$\text{ie } [f(b)-f(a)]\cdot g'(c)-[g(b)-g(a)]\cdot f'(c)=0$$
$$\text{ie } [f(b)-f(a)]\cdot g'(c)=[g(b)-g(a)]\cdot f'(c)$$

October 5th, 2020

Theorem 8: *IVT for derivatives*

Suppose f is differentiable on $[a,b]$.

This means that $f'_-(b)=\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$ and $f'_+(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ exist. Assume that the values are distinct, meaning that $f'_+(a)\neq f'_-(b)$. Let r be strictly between $f'_+(a)$ and $f'_-(b)$. Then $\exists c\in(a,b)$ st $f'(c)=r$

Proof:

$$\begin{aligned}
&\text{Without loss of generality,}\\
&\text{we may assume } f'_+(a)<r<f'_-(b)\\
&\text{Let } g(x)=f(x)-r\,x\\
&\text{We want to show that } \exists c\in(a,b) \text{ st } g'(c)\,f'(c)-r=0\\
&\text{ie } f'(c)=r
\end{aligned}$$

$$\begin{aligned}
&\text{Observe that } g'_+(a)=f'_+(a)-r<0\\
&\text{and } g'_-(b)=f'_-(b)-r>0
\end{aligned}$$

$$\begin{aligned}
&\text{By } \mathbf{LM1}\text{, we know that:}\\
&g(x)=g(a)+[f'_+(a)+E_1(x)](x-a)\\
&\text{with } \lim_{x\to a^+}E_1(x)=0\\
&\text{Thus } g(x)-g(a)=[g'_+(a)+E_1(x)](x-a) \qquad x-a>0\\
&g'_+(a)<0
\end{aligned}$$

$$\begin{aligned}
&\text{Since } \lim_{x\to a^+}E_1(x)=0,\\
&\exists\delta>0 \text{ st } |E_1(x)|\leqslant|g'_+(a)| \text{ on } (a,a+\delta),\\
&\text{and so:}\\
&[g'_+(a)+E_1(x)]<0 \text{ on } (a,a+\delta)
\end{aligned}$$

$$\text{Thus} \quad g(x)-g(a)<0 \text{ on } (a,a+\delta)$$

$$\begin{aligned}
&\text{Using the same argument, we can find}\\
&\delta'>0 \text{ st } g(x)-g(b)<0 \text{ on } (b-\delta',b)
\end{aligned}$$

$$\begin{aligned}
&\text{Let } \tilde{\delta}=\min\{\delta,\delta'\}\\
&\text{Then:}\\
&(*)\left\{\begin{array}{ll} g(x)-g(a)<0, \text{ie } g(x)<g(a) & \text{on } (a,a+\delta)\\ g(x)-g(b)<0, \text{ie } g(x)<g(b) & \text{on } (b-\delta',b)\end{array}\right.
\end{aligned}$$

Since g is continuous on $[a,b]$ because g is differentiable and since $[a,b]$ is a compact, g attains its minimum value at some point c in $[a,b]$. but $(*)$ implies that c cannot be a or b

$$\text{Hence } c \in (a,b), \text{ ie } c \in [a,b]^\circ$$

$$\text{Therefore, } \textbf{THM4} \text{ implies that } g'(c) = 0$$
$$\text{ie } f'(c) = r, \text{ with } c \in (a,b)$$

$$\text{Comment: if } f'_-(b) \leqslant f'_+(a), \text{ we redo the same proof}$$
$$\text{with } -f \text{ instead of } f$$

Lemma 9:

Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and let $f, g: I \to \mathbb{R}$ be two functions. Suppose that f and g are continuous on I and differentiable on I°.

i. if $f'(x) = 0$ for all $x \in I^\circ$, then f is constant on I

ii. if $f'(x) = g'(x)$ on I°, then $f(x) = g(x) + C$, for some constant C

Proof:

i.

$$\text{Suppose } f'(x) = 0, \forall x \in I^\circ$$
$$\text{Let } a \text{ and } b \text{ be any random points in } I.$$
$$\text{We want to show that } f(a) = f(b).$$
$$\text{Without loss of generality, assume } a < b$$
$$\text{Applying the } \textbf{MVT} \text{ to } f \text{ on } [a,b] \subseteq I \text{ implies:}$$
$$\exists c \in (a,b) \text{ st } f'(c) = \frac{f(b) - f(a)}{b - a}, c \in I^\circ$$

$$\text{Since } c \in I^\circ, f'(c) = 0 \text{ and so } f(b) - f(a) = 0$$
$$\text{ie } f(b) = f(a), \text{ and since } a \text{ and } b \text{ were random,}$$
$$f \text{ is constant on } I$$

ii.

$$\text{Apply } (i) \text{ to the function } f - g$$

Rmk: A direct consequence of **LM9** is that if F and $\tilde{F}$ are two antiderivatives of f, then $F(x) = \tilde{F}(x) + C$ for some constant $C \in \mathbb{R}$.

Thorem 10:

Let $I \subseteq \mathbb{R}$ be a non-degenerate interval. Suppose that $f: I \to \mathbb{R}$ is continuous on I and differentiable on I°.

i. $f'(x) \geqslant 0 \forall x \in I^\circ$ iff f is increasing

ii. $f'(x) \leqslant 0 \forall x, I^\circ$ iff f is decreasing

Proof:

i.

$$\Longrightarrow \text{Assume } f'(x) \geqslant 0 \forall x \in I^\circ$$
$$\text{Let } a < b \text{ be two random elements in } I$$
$$\text{We want to show that } f(a) \leqslant f(b)$$

$$\text{Since } a < b, [a,b] \subseteq I$$
$$\text{Apply the } \mathbf{MVT} \text{ to } f \text{ on } [a,b]$$
$$\text{to obtain: } \exists c \in (a,b) \text{ st } f'(c) = \frac{f(b)-f(a)}{b-a} \geqslant 0$$
$$b - a > 0 \text{ since } b > a$$

$$\text{Thus } f(b) - f(a) > 0$$
$$f(b) > f(a),$$
$$\text{and so } f \text{ is increasing}$$

$$\Longleftarrow \text{Assume } f \text{ is increasing on } I$$
$$\text{let } c \in I^\circ \text{ be a random point}$$
$$\text{We want to show that } f'(c) > 0$$

$$\text{Since } c \in I^\circ, \exists \delta > 0 \text{ st } (c-\delta, c+\delta) \subseteq I$$
$$\text{Let } x \in (c, c+\delta). \text{ Then:}$$
$$f(c) \leqslant f(x) \text{ and } \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \qquad x - c > 0$$
$$\text{and the limit exists (since it}'s \text{ a differentiable function)}$$
$$\text{and so the limit from right and left also exist}$$
$$\text{Since } \frac{f(x)-f(c)}{x-c} \geqslant 0 \text{ on } (c, c+\delta),$$
$$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = f'(c) \geqslant 0$$

$$\text{Since } c \text{ is randomly chosen in } I^\circ,$$
$$f'(x) \geqslant 0 \text{ for all } x \in I^\circ$$

ii. Same point applies, just the other way around.

Theorem 11: *First Derivative Test*

Let $I \subseteq \mathbb{R}$ be a non-degenerate open interval, $c \in I$, and $f: I \to \mathbb{R}$ be a function.

Suppose that c is a critical point of f (Either $f(c) = 0$ or f is not differentiable at c), and f is continuous on I and differentiable on $I \backslash \{c\}$.

i. Suppose that $\exists \delta > 0$ st:

   $x \in I$ and $c - \delta < x < c$ implies $f'(x) \geqslant 0$ and $x \in I$ and $c < x < c + \delta$ implies $f'(x) \leqslant 0$, then c is a local maximum.

ii. Suppose $\exists \delta > 0$ st:

   $x \in I$ and $c - \delta < x < c$ implies $f'(x) \leqslant 0$ and $x \in I$ and $c < x < c + \delta$ implies $f'(x) \geqslant 0$, then c is a local minimum.

Proof:

i.

Because $c \in I$, and I is open, we may take
some smaller value of δ so that
$[c - \delta, c + \delta] \subseteq I$

Let $a = c - \delta, b = c + \delta$, then $[a, b] \subseteq I$

Since f is continuous on I
and differentiable on $I \backslash \{c\}$,
f is continuous on $[a, b]$ and differentiable on
(a, c) and (c, b)

By hypothesis:
$f'(x) \geqslant 0$ for all $x \in (a, c)$ and
$f'(x) \leqslant 0$ for all $x \in (c, b)$

Thus: **THM10** implies that
f is increasing on $[a, c]$ and f is decreasing on $[c, b]$
Hence:
$f(x), f(c) \leqslant 0$ on $[a, c]$
and $f(x) - f(c) \geqslant 0$ on $[c, b]$

Therefore, c is a local maximum.

ii. Similar to (i)

Theorem 12: *Second Derivative Test*

Let $I \subseteq \mathbb{R}$ be an open interval, $c \in I$, and $f: I \rightarrow \mathbb{R}$ be a function. Suppose that f is differentiable on I, $f'(c) = 0$, and $f''(c)$ exists.

i. If $f''(c) > 0$ then c is a local minimum

ii. If $f''(c) < 0$, then c is a local maximum

Proof:

i.

By the definition of f'', we have:

$$f''(c) = \lim_{x \rightarrow c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \rightarrow c} \frac{f'(x)}{x - c} \qquad \text{Since } f'(c) = 0$$

Since $f''(c) > 0$, we have: $\lim_{x \rightarrow c} \frac{f'(x)}{x - c} > 0$
By **THM2 of Chapt. 3**, $\exists M > 0$ and $\delta > 0$ st
$\frac{f'(x)}{x - c} > M$ for $x \in I \backslash \{c\}$ and $|x - c| < \delta$

$$\text{if } x \in I \text{ and } c - \delta < x < c, \text{ we have:}$$
$$\frac{f'(x)}{x-c} > M > 0$$
$$\text{Thus } f'(x) \text{ must be} \leqslant 0, \text{ beacuse } x - c < 0$$
$$\text{for } x \in (c - \delta, c)$$

$$\text{if } x \in I \text{ and } c < x < c + \delta, \text{ we have:}$$
$$\frac{f'(x)}{x-c} > M > 0$$
$$\text{Then } f'(x) \text{ must be} \geqslant 0 \text{ on } (c, c+\delta)$$
$$\text{because } x - c > 0 \text{ on } (c, c+\delta)$$

$$\text{Hence by (ii) of } \mathbf{THM11}, c \text{ is a local min}$$

ii. Same argument, very similar to (i)

October 7th, 2020

6 Chapt. Riemann Integrals

Def: Let $[a,b] \subseteq \mathbb{R}$ be a compact.

1. A partition of $[a,b]$ is a set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

2. If $P = \{x_0, x_1, \ldots x_n\}$ is a partition of the interval $[a,b]$, then the norm of P, $\|P\|$ (or length or mesh) is defined by"

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\} = \max_{1 \leqslant i \leqslant n}\{x_i - x_{i-1}\}$$

In Calculus 1, to make things simple, we take the intervals to be of the same size.

3. A partition, Q of $[a,b]$ is called a refinement of P if $P \subseteq Q$. It is essentially if we take the P and add more points to the partition. This is why it is a refinement. Q contains P and the extra points that we add. Q is obtained by inserting additional points between those of P.

   Exp: Let us take $P = \left\{0, \frac{1}{2}.1\right\}$ is a partition of $[0,1]$. Then we take $Q = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ is a refinement of P. However, if we consider $R = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ is not a refinement of P because the points of P are not contained in R. i.e. $P \nsubseteq R$

Rmk 1) If P and Q are two partitions of $[a,b]$, then $P \cup Q$ is a refinement for both P and Q. This is because $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$.

Rmk 2) If Q is a refinement of P, then $\|Q\| \leqslant \|P\|$.

Def: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$.

1. The representative set of P is a set $T = \{t_1, t_2, t_3, \ldots, t_n\}$ such that each $t_i \in [x_{i-1}, x_i]\, \forall i = \{1, 2, 3, \ldots, n\}$. In Caclulus 1, we considered these to be the endpoints, again for simplicity, but in theory we can have our t_i be anywhere in the interval. These points are random points. When we pick them, they are the representatives, and you must note that there are infinitely many of them, because each interval contains infinitely many points.

2. Let $f\colon [a,b] \to \mathbb{R}$ be a function. The Riemann sum of f wrt the partition P and T, denoted and defined by:

$$S(f,P,T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

<u>Def</u>: *"Upper & lower sums"*

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition ofor $[a,b]$. For each $i \in \{1, \ldots, n\}$, we let:

$$M_i(f) = \sup f([x_{i-1}, x_i]) = \sup_{x_{i-1} \leqslant x \leqslant x_i} f(x)$$

$$m_i(f) = \inf f([x_{i-1}, x_i]) = \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x)$$

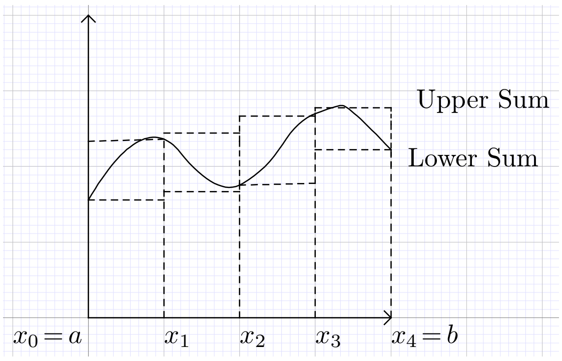
We know that $M_i(f)$ and $m_i(f)$ exist $\forall i$ because we said that f is bounded, meaning that the function is bounded both above and below.

The upper sum of f wrt P, denoted by $U(f,P)$ is defined by:

$$U(f,P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

This is different to our first Riemman sum because we are taking the highest values of f over that interval. Furthermore, the lower sum of f wrt P, denoted by $L(f,P)$ is defined by:

$$L(f,P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$



***Lemma 1*:**

Let $[a,b] \subseteq \mathbb{R}$ be a compact, non-degenerate. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function (We are sure that M_i and m_i exist). Let $P = \{x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$. Then, we have the following results:

1. If T is a representative set of P, then:

$$L(f,P) \leqslant S(f,P,T) \leqslant U(f,P)$$

This means that the Riemann sum is bounded below by the lower sum and bounded above by the upper sum.

2. If R is a refinement of P, then:

$$L(f,P) \leqslant L(f,R) \leqslant U(f,R) \leqslant U(f,P)$$

3. If Q is a partition of $[a,b]$, then:

$$L(f,P) \leqslant U(f,Q)$$

Proof:

1.

$$\begin{aligned}
&\text{Let } T = \{t_1, t_2, \ldots, t_n\}\\
&\text{ie } t_i \in [x_{i-1}, x_i]\ \forall i \in \{1, \ldots, n\}\\
&\text{Then for every random but fixed } t_i,\\
&\text{we have:}\\
&m_i(f) = \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x) \leqslant f(t_i) \qquad \text{(Automatically)}\\
&f(t_i) \leqslant \sup_{x_{i-1} \leqslant x \leqslant x_i} f(x) = M_i(x)\\
\text{Thus: } &\sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \leqslant \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leqslant \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})\\
&= L(f,P) \leqslant S(f,P,T) \leqslant U(f,P)
\end{aligned}$$

2.

$$\begin{aligned}
&\text{Let } R = \{y_0, y_1, \ldots, y_k\}, \text{refinement of } P\\
&P = \{x_0, x_1, \ldots, x_n\} \subseteq R\\
&\text{Note that } y_0 = x_0 \text{ and } y_k = x_n
\end{aligned}$$

$$\begin{aligned}
&\text{Let } i \in \{1, \ldots, n\}\\
&\text{There are } s, t \in \{1, \ldots, k\} \text{ st}\\
&x_{i-1} = y_{s-1} \text{ and } x_i = y_t\\
&\text{This is because } P \subseteq R \text{ and they are indexed}
\end{aligned}$$

$$\begin{aligned}
&\text{Hence:}\\
&[x_{i-1}, x_i] = [y_{s-1}, y_s] \cup [y_s, y_{s+1}] \cup \cdots \cup [y_{t-1}, y_t]\\
&\text{if } j \in \{s, \ldots, t\}, \text{then:}\\
&f([y_{j-1}, y_j]) \subseteq f([x_{i-1}, x_i]), \text{and so:}\\
&m_i^P(f) = \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x) \qquad (\text{wrt } P)\\
&\leqslant \inf_{x \in [y_{j-1}, y_j]} f(x) = m_j^R(f)
\end{aligned}$$

$$\begin{aligned}
&\text{Thus:}\\
&m_i^P(f)(x_i - x_{i-1}) = m_i^P(f)[(y_s - y_{s-1}) + (y_{s+1} - y_s) + \cdots + (y_t - y_{t-1})]\\
&\leqslant m_s^R(f)(y_s - y_{s-1}) + m_{s+1}^R(f)(y_{s+1} - y_s) + \cdots + m_k^R(f)(y_t - y_{t-1})\\
&\text{and hence:}\\
&L(f,P) = \sum_{i=1}^{n} m_i^P(f)(x_i - x_{i-1}) \leqslant \sum_{i=1}^{k} m_j^R(f)(y_j - y_{j-1}) = L(f,R)
\end{aligned}$$

A similar argument can be used for the upper sum.

3.

$$\begin{aligned}&\text{Since } P\cup Q \text{ is refinement for both } P \text{ and } Q\\&(2) \text{ implies that:}\\L(f,P)\leqslant L(f,P\cup Q)&\leqslant U(f,P\cup Q)\leqslant U(f,Q)\\&\text{We use (2) to see that } L(f,P)\leqslant U(f,Q)\\&\text{and } U(f,Q)\leqslant U(f,P)\end{aligned}$$

October 12th, 2020

Def: Let $[a,b]\subseteq\mathbb{R}$ be a compact, and let $f\colon[a,b]\to\mathbb{R}$ be a bounded function. Let $K\in\mathbb{R}$. The real number K is the Riemman integral of f, written:

$$\int_a^b f(x)\mathrm{dx}=K$$

if $\forall\varepsilon>0,\exists\delta>0$ st if P is a partition of $[a,b]$ with $\|P\|<\delta$ and if T is a representative set of P, then:

$$|s(f,P,T)-K|<\varepsilon$$

Terminology: If $\int_a^b f(x)\mathrm{dx}$ exists, ie K exists, then we say that f is Riemman integrable.

Lemma 2: If $f\colon[a,b]\to\mathbb{R}$ is Riemman integrable, then $\exists!K$ st $\int_a^b f(x)\mathrm{dx}=K$.

Proof: Suppose $\exists K_1\neq K_2$ st $\int_a^b f(x)\mathrm{dx}=K_1=K_2$. Then we proceed as follows:

$$\begin{aligned}&\text{Let } \varepsilon=\frac{|K_1-K_2|}{2}>0\\&\text{Then } \exists\delta_1 \text{ st if } P_1 \text{ is a partition of } [a,b]\\&\text{with } \|P_1\|<\delta_1, \text{ then:}\\&|S(f,P_1,T_1)-K_1|<\varepsilon\\[1em]&\text{Also, } \exists\delta_2 \text{ st if } P_2 \text{ is a partition of } [a,b]\\&\text{with } \|P_2\|<\delta_2, \text{ then:}\\&|S(f,P_2,T_2)-K_2|<\varepsilon\\[1em]&\text{Let } \delta=\min\{\delta_1,\delta_2\}\\&\text{Let } R \text{ be the partition of } [a,b] \text{ with}\\&\|R\|<\delta\\&\text{Let } V \text{ be a representative set of } R\\[1em]&\text{Then}\\|K_1-K_2|&=|K_1-S(f,R,V)+S(f,R,V)-K_2|\\&\leqslant|K_1-S(f,R,V)|+|S(f,R,V)-K_2|\\&<\varepsilon+\varepsilon=2\varepsilon=|K_1-K_2|\\&\text{This is a contradiction}\end{aligned}$$

Theorem 3: *Characterization of a Riemman Integrable function*

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then the following statements are equivalent:

a) f is Riemman integrable

b) $\forall \varepsilon > 0, \exists \delta > 0$ st if P is a partition of $[a,b]$, with $\|P\| < \delta$, then:

$$|U(f,P) - L(f,P)| = U(f,P) - L(f,P) < \varepsilon$$

c) $\forall \varepsilon > 0, \exists$ a partition, P, of the interval $[a,b]$ st:

$$U(f,P) - L(f,P) < \varepsilon$$

The three statements all of iff relations to one another. This is how we will proceed: a⇔b, c⇔b, and therefore c⇔a. This means that the two statements a and c are equivalent to one another.

$$a \Rightarrow b$$

Suppose f is Riemman Integrable

Let $\varepsilon > 0. \exists \delta > 0$ st if R is a partition of $[a,b]$ with $\|R\| < \delta$, then:

$$\left| S(f,R,T) - \int_a^b f(x)\mathrm{dx} \right| < \frac{\varepsilon}{4}$$

Let $P = \{z_0, z_1, z_2, \ldots, z_k\}$ be a partition of $[a,b]$ with $\|P\| < \delta$

Let $i \in \{1, \ldots, k\}$. By definitions of $m_i(f)$ and $M_i(f)$, there are $c_i \in [z_{i-1}, z_i]$ and $d_i \in [z_{i-1}, z_i]$ st

$$m_i(f) \leqslant f(d_i) < m_i(f) + \frac{\varepsilon}{4k\delta}$$

$$\text{and } M_i(f) - \frac{\varepsilon}{4k\delta} < f(c_i) \leqslant M_i(f)$$

$$\text{Hence } |m_i(f) - f(d_i)| < \frac{\varepsilon}{4k\delta}$$

$$\text{and } |M_i(f) - f(c_i)| < \frac{\varepsilon}{4k\delta}$$

Let $C = \{c_1, c_2, \ldots, c_k\}, D = \{d_1, d_2, \ldots, d_k\}$. Then: C and D are representative sets of the partition P. Now:

$$|U(f,P) - S(f,P,C)| = \left| \sum_{i=1}^{k} M_i(f)(z_i - z_{i-1}) - \sum_{i=1}^{k} f(c_i)(z_i - z_{i-1}) \right|$$

$$\leqslant \sum_{i=1}^{k} |M_i(f) - f(c_i)|\,(x_i - x_{i-1}) \leqslant \sum_{i=1}^{k} \frac{\varepsilon}{4k\delta}$$

Note that $(z_i - z_{i-1}) < \delta$ because $\|P\| < \delta$

$$\text{Therefore we have } \sum_{i=1}^{k} \frac{\varepsilon}{4k\delta}\delta$$

$$= \sum_{i=1}^{k} \frac{\varepsilon}{4k} = k\left(\frac{\varepsilon}{4k}\right) = \frac{\varepsilon}{4}$$

A similar argument shows that"

$$|L(f,P)-S(f,P,D)|<\frac{\varepsilon}{4}$$

Therefore,

$$U(f,P)-L(f,P)=|U(f,p)-L(f,P)|$$

$$\begin{aligned}
&=|U(f,P)-S(f,P,C)+S(f,P,C)-S(f,P,D)+S(f,P,D)-L(f,P)|\\
&\leqslant|U(f,P)-S(f,P,C)|+|S(f,P,D)-L(f,P)|+|S(f,P,C)-S(f,P,D)|\\
&=|U(f,P)-S(f,P,C)|+|L(f,P)-S(f,P,D)|\\
&+\left|S(f,P,C)-\int_a^b f(x)\mathrm{dx}+\int_a^b f(x)\mathrm{dx}-S(f,P,D)\right|\\
&<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\left|S(f,P,C)-\int_a^b f(x)\mathrm{dx}\right|+\left|\int_a^b f(x)\mathrm{dx}-S(f,P,D)\right|\\
&=\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\\
&=\varepsilon
\end{aligned}$$

$b\Rightarrow a$

Suppose $\forall\varepsilon>0,\exists\delta>0$ st if P is a partition of $[a,b]$ with $\|P\|<\delta$

$U(f,P)-L(f,P)<\varepsilon$

Let $\mathcal{U}=\{U(f,P)|P$ is a partition of $[a,b]\}$

Let $\mathcal{L}=\{L(f,P)|P$ is a partition of $[a,b]\}$

We want to show that $\sup(\mathcal{L})=\inf(\mathcal{U})$

Clearly $\mathcal{U}$ and $\mathcal{L}$ are both non empty sets of $\mathbb{R}$.

By **LM1**, we know that if $L(f,Q)\in\mathcal{L}$

and $U(f,R)\in\mathcal{U}$, then:

$L(f,Q)\leqslant U(f,R)$

Let $\mu>0$. By hypothesis

$\exists\beta>0$ st if P is a partition of $[a,b]$ with $\|P\|<\beta$

$U(f,P)-L(f,P)<\mu$

We see that $\mathcal{L}$ and $\mathcal{U}$ satisfy the hypothesis of the "No Gap Lemma"

and therefore: $\mathcal{L}$ has a least upper bound

$\mathcal{U}$ has a greatest lower bound, and $\sup(\mathcal{L})=\inf(\mathcal{U})$

Let $K=\sup(\mathcal{L})=\inf(\mathcal{U})$

By hypothesis, $\exists\delta>0$ st if W is a partition of $[a,b]$ and $\|W\|<\delta$,

$U(f,W)-L(f,W)<\varepsilon$

Let T be a representative set of W. By **LM1**, we have

$L(f,W)\leqslant S(f,W,T)\leqslant U(f,W)$

We also know that $L(f,W)\leqslant K\leqslant U(f,W)$

Thus:

$|S(f,W,T)-K|\leqslant U(f,W)-L(f,W)<\varepsilon$

$|S(f,W,T)-K|<\varepsilon$

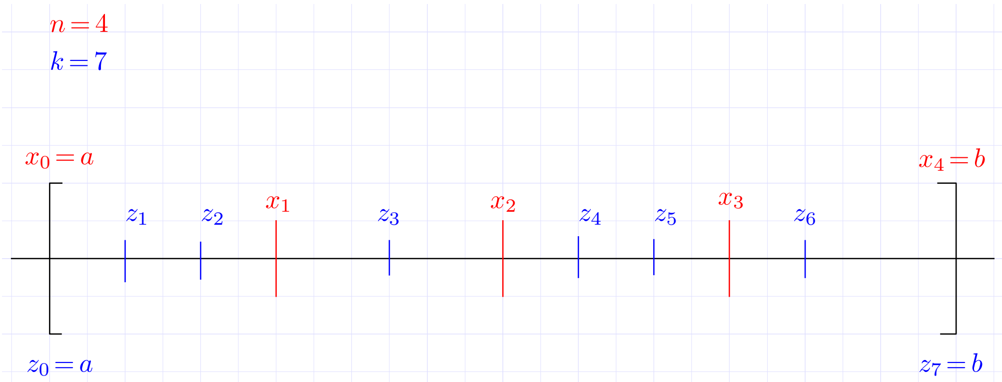
Hence: f is Riemman Integrable.

To show that b⇒c, it is trivial and will not be covered. However, c⇒b is not trivial and will be covered below:

$$c \Rightarrow b$$
$$\text{Suppose that } \forall \varepsilon > 0, \exists P \text{ partition of } [a, b] \text{ st } U(f, P) - L(f, P) < \varepsilon$$
$$\text{Let } \varepsilon > 0. \exists Q = \{x_0, x_1, x_2, \ldots., x_n\}$$
$$\text{st } U(f, Q) - L(f, Q) < \frac{\varepsilon}{2}$$
$$\text{Since } f \text{ is bounded}, \exists B > 0 \text{ st } |f(x)| \leqslant B \forall x \in [a, b]$$

$$\text{Let } \delta = \frac{\varepsilon}{2Bn}. \text{ Obviously, } \delta > 0$$
$$\text{Let } Z = \{z_0, z_1, \ldots., z_k\} \text{ be a partition of } [a, b]$$
$$\text{with } \|Z\| < \delta. \text{ We want to show that:}$$
$$U(f, Z) - L(f, Z) < \varepsilon$$

$$\text{Let } W = \{i \in \{1, \ldots., n\} | \, x_j \in (z_{i-1}, z_i) \text{ for some } j \in \{1, \ldots, n\}\}$$
$$\text{For each } j \in \{1, \ldots, n\}, \text{ consider:}$$
$$V_j = \{i \in \{1, \ldots, k\} \,|\, [z_{i-1}, z_i] \subseteq [x_{j-1}, x_j]\}$$
$$\text{Then } W \cup V_1 \cup V_2 \cup \cdots \cup V_k = \{1, \ldots., k\} \text{ and}$$
$$\text{the sets } W, V_1, V_2, V_3, \ldots., V_n \text{ pairwise disjoint}$$



$$W = \{3, 4, 6\}$$
$$V_1 = \{1, 2\}$$
$$V_2 = \varnothing$$
$$V_3 = \{5\}$$
$$V_4 = \{7\}$$
$$\text{Clearly the union of them all} = \{1, \ldots, k\}$$
$$\text{The intersection of each set} = \varnothing$$

$$\text{Because } x_0 = a \text{ and } x_n = b, \text{ the set } W \text{ contains at most } n - 1 \text{ elements.}$$
$$\text{For each } i \in \{1, \ldots., k\}, \text{ we have:}$$
$$M_i^Z(f) - m_i^Z(f) \leqslant 2B \qquad (**)$$
$$\text{Because } |f(x)| \leqslant B \forall x \in [a, b]$$

$$\text{For each } j \in \{1, \ldots, n\}, \text{ we have:}$$

$$\sum_{i \in V_j} (z_i - z_{i-1}) \leqslant x_{j-1} - x_j$$

$$\text{Because } [z_{i-1}, z_i] \subseteq [x_{j-1}, x_j] \text{ for } i \in V_j \qquad (*)$$

$$\text{and if } i \in V_j \text{, then:}$$

$$M_i^Z(f) - m_i^Z(f) \leqslant M_j^Q(f) - m_j^Q(f) \qquad (3*)$$

Thus:

$$\begin{aligned}
U(f, Z) - L(f, Z) &= \sum_{i=1}^{k} [M_i^Z(f) - m_i^Z(f)](z_i - z_{i-1}) \\
&\leqslant \sum_{i \in W} [M_i^Z(f) - m_i^Z(f)](z_i - z_{i-1}) + \sum_{j=1}^{n} \sum_{i \in V_j} [M_i^Z(f) - m_i^Z(f)](z_i - z_{i-1}) \qquad (**) \text{ and } (3*) \\
&\leqslant \sum_{i \in W} 2B(z_i - z_{i-1}) + \sum_{j=1}^{n} \sum_{i \in V_j} [M_i^Q(f) - m_i^Q(f)](z_i - z_{i-1}) \\
&< 2B(n-1)\delta + \sum_{j=1}^{n} [M_i^Q(f) - m_i^Q(f)] \sum_{i \in V_j} (z_i - z_{i-1}) \qquad (*) \\
&< 2B(n-1)\delta + \sum_{j=1}^{n} [M_i^Q(f) - m_i^Q(f)] (x_j - x_{j-1}) \\
&= 2B(n-1)\frac{\varepsilon}{2nB} + U(f, Q) - L(f, Q) \\
&< \left(\frac{n-1}{n}\right)\varepsilon + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

October 14th, 2020

<u>Def</u>: Let $[a, b] \subseteq \mathbb{R}$ be a compact and let $f: [a, b] \to \mathbb{R}$ be a <u>bounded</u> function.

1. The upper integral of f, is denoted and defined by:

$$\overline{\int_a^b} f(x)\mathrm{dx} = \inf \{U(f, p) \mid P \text{ is a partition of } [a, b]\}$$

   This means that it is the infimum of all the upper sums.

2. The lower integral of f, is denoted and defined by:

$$\underline{\int_a^b} f(x)\mathrm{dx} = \sup \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$$

   This means that it is the supremum of all the lower sums.

<u>Rmk</u>: Since f is bounded on $[a, b]$, $\exists \alpha, \beta \in \mathbb{R}$ st $\alpha \leqslant f(x) \leqslant \beta \quad \forall x \in [a, b]$. Thus, we can see the following: $\alpha(b - a) \leqslant L(f, P) \leqslant U(f, P) \leqslant \beta(b - a)$ for any partition P of $[a, b]$. Hence, the set of all the upper sums (and resp. all the lower sums) is non-empty and bounded below (resp. bouinded above). Therefore, the completeness axiom implies that:

$$\overline{\int_a^b} f(x)\mathrm{dx} \left(\text{resp. } \underline{\int_a^b} f(x)\mathrm{dx}\right) \text{exists.}$$

Exp: let $f:[a,b]\to\mathbb{R}$ defined by:

$$x\mapsto f(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\\ 1 & \text{if } x\in\mathbb{Q}\end{cases}$$

Let $P=\{x_0,x_1,\ldots,x_n\}$ be a random partition of $[a,b]$. By the density of rationals and irrationals in $\mathbb{R}$, we have that every interval $[x_{i-1},x_i]$ contains both rational and irrational numbers $\forall i=1,\ldots,n$. Therefore, $m_i(f)=0$ and $M_i(f)=1$ for all $i=1,\ldots,n$. Therefore:

$$U(f,P)=\sum_{i=1}^{n} M_i(f)(x_i-x_{i-1})=\sum_{i=1}^{n}(x_i-x_{i-1})=x_n-x_0=b-a$$

$$L(f,P)=\sum_{i=1}^{n} m_i(f)(x_i-x_{i-1})=0 \quad \text{since } m_i(f)=0$$

This is true for any partition, since we took a random P of the compact $[a,b]$. Hence:

$$\overline{\int_a^b f(x)\mathrm{dx}}=\inf(U(f,P))=b-a$$

$$\underline{\int_a^b f(x)\mathrm{dx}}=\sup(L(f,P))=0$$

Rmk: Using part (i) of the **NO GAP LEMMA**, we have that:

$$\underline{\int_a^b f(x)\mathrm{dx}}\leqslant\overline{\int_a^b f(x)\mathrm{dx}}$$

Theorem 4: *An Alternative Characterization of Integrable Functions*

Let $[a,b]$ be a compact of $\mathbb{R}$ and let $f:[a,b]\to\mathbb{R}$ be a bounded function. Then f is integrable iff:

$$\overline{\int_a^b f(x)\mathrm{dx}}=\underline{\int_a^b f(x)\mathrm{dx}}$$

And if this equality holds true, then the integral of f would be exactly equal to the upper integral which is equal to the lower integral. The proof is extremely simple and it will be shown in the homework. In other words:

$$\overline{\int_a^b f(x)\mathrm{dx}}=\underline{\int_a^b f(x)\mathrm{dx}}=\int_a^b f(x)\mathrm{dx}$$

Theorem 5: Let $[a,b]\subseteq\mathbb{R}$ be a compact and let $f:[a,b]\to\mathbb{R}$ be a function. If f is continuous on $[a,b]$, then f is also integrable on $[a,b]$.

Proof: Since f is continuous on the compact $[a,b]$, then we know by **THM2 Chpt.4** that f is uniformly continuous on $[a,b]$. So: $\forall\varepsilon>0, \exists\delta$ (δ depends only on ε) st if $x, y\in[a,b]$ and if $|x-y|<\delta$, then $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$.

$$\begin{aligned}
&\text{Let } P=\{x_0, x_1, \ldots, x_n\} \text{ be a partition of } [a,b]\\
&\text{with } \|P\|<\delta\\
&\text{For each } i=\{i, \ldots, n\}, \text{the restriction of } f \text{ to } [x_{i-1}, x_i]\\
&\text{is also continuous.}
\end{aligned}$$

$$\begin{aligned}
&\text{Thus, the } \mathbf{EVT} \text{ applied to } f \text{ on } [x_{i-1}, x_i] \text{ implies that:}\\
&\exists x^i_{\min} \text{ and } x^i_{\max} \text{ in } [x_{i-1}, x_i] \text{ st } f(x^i_{\min})\leqslant f(x)\leqslant f(x^i_{\max}) \qquad \forall x\in[x_{i-1}, x_i]
\end{aligned}$$

$$\begin{aligned}
&\text{Hence:}\\
&m_i(f)=f(x^i_{\min}) \text{ and } M_i(f)=f(x^i_{\max})\\
&\text{Since } \|P\|<\delta, |x_i-x_{i-1}|<\delta \text{ for each } i \text{ and hence:}\\
&|x^i_{\min}-x^i_{\max}|<\delta\\
&\text{Because } x^i_{\min}, x^i_{\max} \text{ is inside } [x_{i-1}, x_i]\\
&\text{Therefore:}\\
&|f(x^i_{\max})-f(x^i_{\min})|<\frac{\varepsilon}{b-a}
\end{aligned}$$

$$\begin{aligned}
&\text{Then, we obtain:}\\
&U(f,P)-L(f,P)\\
&=\sum_{i=1}^n M_i(f)(x_i-x_{i-1})-\sum_{i=1}^n m_i(f)(x_i-x_{i-1})\\
&=\sum_{i=1}^n [M_i(f)-m_i(f)](x_i-x_{i-1})\\
&=\sum_{i=1}^n [f(x^i_{\max})-f(x^i_{\min})](x_i-x_{i-1})\\
&<\sum_{i=1}^n \frac{\varepsilon}{b-a}(x_i-x_{i-1})\\
&=\frac{\varepsilon}{b-a}\sum_{i=1}^n (x_i-x_{i-1})\\
&=\frac{\varepsilon}{b-a}(x_n-x_0)=\frac{\varepsilon}{b-a}(b-a)=\varepsilon
\end{aligned}$$

Hence, by **THM3 (b)** implies that f is integrable.

Theorem 6: *Fundamental Theorem of Calculus (v1)*

Let $I\subseteq\mathbb{R}$ be a non-degenerate interval, and we take an element $a\in I$. Let $f: I\to\mathbb{R}$ be a function. Suppose that the restriction of f to any compact $C\subseteq\mathbb{R}$ is integrable.

$$\text{Let } F: I\to\mathbb{R} \text{ be defined as: } F(x)=\int_a^x f(t)\mathrm{d}t$$

This function, F, is well-defined because f is integrable over the compact with endpoints a and x. This means that it does not matter whether x is bigger than a or not.

i. if $c\in I$ and if f is continuous at c, then $F(x)$ is differentiable at c and $F'(c)=f(c)$

ii. if f is continuous on I (at every point $c\in I$), then F is differentiable on I and $F'=f$

Proof:

i. Assume $c \in I$ but c is not an endpoint of I. We will show that:

$$\lim_{h \to 0^+} \frac{F(c+h) - F(h)}{h} = f(c) = \lim_{h \to 0^-} \frac{F(c+h) - F(h)}{h}$$

$$\begin{aligned}
&\text{Let } \varepsilon > 0. \text{ Since } f \text{ is continuous at } c\\
&\exists \delta > 0 \text{ st if } w \in I \text{ and } |w - c| < \delta\\
&\text{then } |f(w) - f(c)| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\begin{aligned}
&\text{Since } c \text{ is not the right endpoint of } I, \text{ we may choose}\\
&\delta \text{ st } [c, c+\delta) \subseteq I
\end{aligned}$$

$$\begin{aligned}
&\text{Hence for } w \in [c, c+\delta), \text{ we still have:}\\
&|f(w) - f(c)| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\begin{aligned}
&\text{Let } h \in (0, \delta). \text{ Then:}\\
&t \in [c, c+h] \qquad \text{(Non degenerate interval)}\\
&\text{implies } |f(t) - f(c)| < \frac{\varepsilon}{2}
\end{aligned}$$

Now:

$$\begin{aligned}
\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| &= \left|\frac{1}{h}\left[\int_a^{c+h} f(t)\mathrm{dt} - \int_a^c f(t)\mathrm{dt}\right] - f(c)\right|\\
&= \left|\frac{1}{h}\left[\int_a^{c+h} f(t)\mathrm{dt} + \int_c^a f(t)\mathrm{dt}\right] - f(c)\right|\\
&= \left|\frac{1}{h}\int_a^{c+h} f(t)\mathrm{dt} - \frac{1}{h}\int_c^{c+h} f(c)\mathrm{dt}\right|\\
&= \left|\frac{1}{h}\int_c^{c+h} [f(t) - f(c)]\mathrm{dt}\right|\\
&\leqslant \frac{1}{h}\int_c^{c+h} |f(t) - f(c)|\mathrm{dt} \qquad |f(t) - f(c)| < \frac{\varepsilon}{2}\\
&< \frac{1}{h}\int_c^{c+h} \frac{\varepsilon}{2}\mathrm{dt}\\
&= \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

Therefore we have shown that $\lim_{h \to 0^+} \frac{F(c+h) - F(h)}{h} = f(c)$.

A similar argument is used to give us that $\lim_{h \to 0^-} \frac{F(c+h) - F(h)}{h} = f(c)$. Either we consider our h to be positive, and consider $F(c-h)$, or we simply take $h \in (-\delta, 0)$. Everything else is the same.

If c is the right endpoint of I (resp. the left endpoint of I), then we redo the same argument and we show that $F'_-(c) = f(c)$ (resp. $F'_+(c) = f(c)$)

ii. This is a consequence of (i). If f is continuous on I, then f is continuous at every point $c \in I$ and so $F'(c) = f(c) \quad \forall c \in I$. This is why we say that $F' = f$ on I.

Rmk: As a consequence of **FTC v1**, if we have a function that is continuous, then it has an antiderivative. Mathematically, $f: I \to \mathbb{R}$ is continuous $\Longrightarrow f$ has an antiderivative. An antiderivative is F.

Theorem 7: *Fundamental Theorem of Calculus (v2)*

Let $[a, b] \subseteq \mathbb{R}$ be a compact and $f: [a, b] \to \mathbb{R}$ be a function. Suppose that f is integrable and f has an antiderivative. We are assuming this. This does not necessarily have anything to do with continuity. If $F: [a, b] \to \mathbb{R}$ is an anti-derivative of f, then:

$$\int_a^b f(x)\mathrm{dx} = F(b) - F(a)$$

October 19th, 2020

Proof: “The **FTC v2**”

$$\int_a^b f(x)\mathrm{dx} = F(b) - F(a)$$

Since f is integrable, $\forall \varepsilon > 0$, $\exists \delta > 0$ st if P is a partitition of $[a, b]$ with $\|P\| < \delta$ and if T is a representative set of P, then $\left|S(f, P, T) - \int_a^b f(x)\mathrm{dx}\right| < \varepsilon$.

Let $\varepsilon > 0$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. Since F is an anti-derivative of f, F is differentiable $(F' = f)$. Thus for each $i = \{1, \ldots, n\}$, F is continuous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i). By applying the **MVT** to F on each $[x_{i-1}, x_i]$, we have that $\exists t_i \in (x_{i-1}, x_i)$ st:

$$F'(t_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = f(t_i),$$

and so: $F(x_i) - F(x_{i-1}) = f(x_i)(x_i - x_{i-1})$ for all $i = 1, \ldots, n$.

Let $T = \{t_1, t_2, \ldots, t_n\}$ be a representative set of P. Then:

$$S(f, P, T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

$$= F(x_n) - F(x_0) = F(b) - F(a)$$

$$\text{Hence} \left|S(f, P, T) - \int_a^b f(x)\mathrm{dx}\right|$$

$$= \left|F(b) - F(a) - \int_a^b f(x)\mathrm{dx}\right| < \varepsilon$$

Since this is true $\forall \varepsilon > 0$, we must have:

$$\int_a^b f(x)\mathrm{dx} = F(b) - F(a)$$

---

7 Chapt. Sequences

Def: A sequence is a mapping from $\mathbb{N}$ into $\mathbb{R}$. To each $i \in \mathbb{N}$ we associate a real $a_i \in \mathbb{R}$. The collection of all these real a_i is called the sequence $(a_n)_n$.

Def: “$\varepsilon - N$ definition of limits of sequences”

A finite real number $l, (l \in \mathbb{R})$ is said to be the limit of a sequence $(a_n)_n$, written:

$$\lim_{n \to +\infty} a_n = l$$

if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $n \in \mathbb{N}$ and $n \geqslant N$ implies $|a_n - l| < \varepsilon$. If our l is not a finite real number or it doesn’t exist, then we say that our sequence is divergent.

Exp. 1) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Let $\varepsilon > 0$. Then by **CLRY6 CHPT1**, there exists some $N \in \mathbb{N}$ st $\frac{1}{N} < \varepsilon$. For all $n \in \mathbb{N}$ and $n \geqslant N$, we have that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$. Hence we have shown that the limit as $n \to \infty$, the sequence $\frac{1}{n}$ goes to 0.

Exp. 2) Show that:

$$\{(-1)^n\}_n \,\text{diverges.}$$

First we assume that $\lim_{n\to\infty} (-1)^n = l$ for some $l \in \mathbb{R}$. We will proceed by contradiction. Let $\varepsilon = \frac{1}{2}$. Then $\exists N \in \mathbb{N}$ st $n \in \mathbb{N}$ and $n \geqslant N$ implies that $|(-1)^n - l| < \frac{1}{2}$.

Choose $n_1 \in \mathbb{N}, n_1 \geqslant N, n_1 \,\text{is odd, and}\, n_2 \in \mathbb{N}, n_2 \geqslant N, n_2 \,\text{is even.}$ Then we have the following:

$$\begin{aligned} |(-1)^{n_2} - (-1)^{n_1}| &= 2 \\ = |(-1)^{n_2} - l + (-1)^{n_2} - l| \leqslant |(-1)^{n_2} - l| + |(-1)^{n_2} - l| & \\ |(-1)^{n_2} - l| < \varepsilon, \text{and}\, |(-1)^{n_2} - l| &< \varepsilon \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

This is a contradiction, since we have that $|(-1)^{n_2} - (-1)^{n_1}| = 2$, and by this proof we are saying that $2 < 1$. Therefore we conclude that this limit does not exist and the sequence is divergent.

***Lemma 1*:**

If

$$\lim_{n\to\infty} a_n = l \,\text{for some}\, l \in \mathbb{R},$$

then we know that this l is unique (This lemma is the uniqueness of the limit).

Proof:

Assume $\exists l_1, l_2 \in \mathbb{R}$, where $l_1 \neq l_2$, st $\lim_{n\to\infty} a_n = l_1 = l_2$. Let $\varepsilon = \frac{|l_1 - l_2|}{2}$, then clearly $\varepsilon > 0$. Now $\exists N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ st $n \in \mathbb{N}$ and $n \geqslant N_1$ implies $|a_n - l_1| < \varepsilon$. Similarly, $n \in \mathbb{N}$ and $n \geqslant N_2$ implies that $|a_n - l_2| < \varepsilon$.

$$\begin{aligned} \text{Let}\, P &= \max\{N_1, N_2\} \\ \text{If}\, n \in N \,\text{and}\, n &\geqslant P, \text{then:} \\ |l_1 - l_2| &= |l_1 - a_n + a_n - l_2| \\ &\leqslant |l_1 - a_n| + |a_n - l_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{|l_1 - l_2|}{2} \\ |l_1 - a_n| + |a_n - l_2| &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \,\text{because:} \\ n \geqslant P \geqslant N_1 \,\text{and}\, n &\geqslant P \geqslant N_2 \end{aligned}$$

This is a contradiction because we are saying that $|l_1 - l_2| < \frac{|l_1 - l_2|}{2}$, which is not true. Therefore by contradiction we know that the limit of a sequence must be unique.

***Lemma 2*:**

If $(a_n)_n$ converges, then we know that $(a_n)_n$ is bounded. Of course, we know that generally, the converse is not true. In other words, if a sequence is bounded, then it does not necessarily mean that it is convergent.

Proof:

Let

$$\lim_{n\to\infty}(a_n) = l \text{ for some } l \in \mathbb{R}$$

Then $\exists N \in \mathbb{N}$ st $n \in \mathbb{N}$ and $n \geqslant N$ implies that $|a_n - l| < \varepsilon$. We will choose our $\varepsilon = 1$, and thus we have that: $|a_n - l| < 1$. So:

$$\begin{aligned} |a_n| - |l| \leqslant |a_n - l| &< 1 \\ \text{Thus } |a_n| &< 1 + |l| \text{ for all } n \geqslant N \end{aligned}$$

$$\begin{aligned} \text{Let } M = \max\{|a_1|, |a_2|, \ldots., |a_{N-1}|, 1 + |l|\} \\ \text{Then } |a_i| \leqslant M \text{ for all } i \geqslant 1. \end{aligned}$$

Thus we know that $(a_n)_n$ is bounded.

Lemma 3:

Let $(a_n)_n$ and $(b_n)_n$ be two sequences $\mathbb{R}$. If we have that:

$$\begin{aligned} \lim_{n\to\infty}(a_n) = 0 \text{ and if } (b_n)_n \text{ is bounded,} \\ \text{then } \lim_{n\to\infty}(a_n)(b_n) = 0 \end{aligned}$$

Proof:

Since $(b_n)_n$ is bounded, $\exists M > 0$ st $|b_n| \leqslant M$ for all n.

$$\begin{aligned} \text{Let } \varepsilon > 0. \\ \text{Since } \lim_{n\to\infty}(a_n) = 0, \exists N \in \mathbb{N} \\ \text{st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies:} \\ |a_n - 0| = |a_n| < \frac{\varepsilon}{M} \end{aligned}$$

$$\begin{aligned} \text{Now, for } n \in \mathbb{N} \text{ and } n \geqslant N, \text{ we have:} \\ |a_n b_n| = |a_n||b_n| \leqslant |a_n| M \\ < \frac{\varepsilon}{M} M \\ = \varepsilon \end{aligned}$$

Thus:

$$\lim_{n\to\infty}(a_n)(b_n) = 0$$

Theorem 4:

Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $\mathbb{R}$. Assume $\exists N \in \mathbb{N}$ st $a_n \leqslant b_n$ for all $n \geqslant N$. If $(a_n)_n$ and $(b_n)_n$ both converge, then:

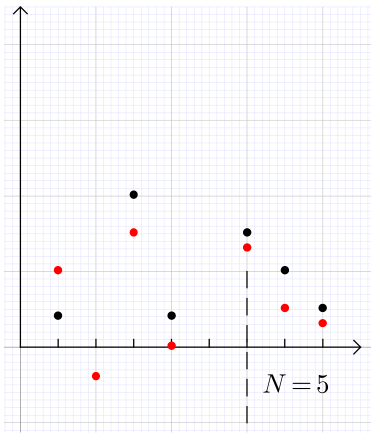
$$\lim_{n\to\infty}(a_n) \leqslant \lim_{n\to\infty}(b_n)$$

Note that even if we have the case where $a_n < b_n$, ie b_n is strictly bigger than a_n, when we pass it to the limit, we have $\leqslant$ and not just $<$. For example, let $a_n = \frac{1}{n} + 1$ and $b_n = 1, n \geqslant 1$ for both. Then we know that $b_n < a_n$ for all n. However, $\lim_{n\to\infty}(a_n) = 1 = \lim_{n\to\infty}(b_n)$, and thus we have equality beyond just $<$.

Proof:

$$\text{Let } l = \lim_{n\to\infty} (a_n) \text{ and } l' = \lim_{n\to\infty} (b_n)$$
$$\text{We will assume that } l > l'$$
$$\text{Let } \varepsilon = \frac{l - l'}{2}, \text{ then clearly } \varepsilon > 0$$
$$\exists N_1 \text{ and } N_2 \in \mathbb{N} \text{ st}$$
$$n \in \mathbb{N} \text{ and } n \geqslant N_1 \text{ implies } |a_n - l| < \varepsilon$$
$$\text{and}$$
$$n \in \mathbb{N} \text{ and } n \geqslant N_2 \text{ implies } |b_n - l'| < \varepsilon$$

$$\text{Let } P = \max\{N, N_1, N_2\}$$



Before the rank N, we don't care whether the images of the red sequence are below that of the black. However, after we pass N, then the image of the red sequence must always be less than that of the black sequence. This is the idea behind this N that we are choosing.

$$|a_P - l| < \varepsilon \text{ and } |b_P - l'| < \varepsilon$$
$$\text{This is because } P \geqslant N_1 \text{ and } P \geqslant N_2$$
$$\text{Hence}$$
$$l - \varepsilon < a_P < l + \varepsilon$$
$$\text{and}$$
$$l' - \varepsilon < b_P < l' + \varepsilon. \text{ Therefore:}$$
$$b_P < l' + \varepsilon = l' + \frac{l - l'}{2} = \frac{l + l'}{2}$$
$$= l - \frac{l - l'}{2}$$
$$= l - \varepsilon < a_P$$

This is a contradiction because $P \geqslant N$, and hence we have shown by contradiction that $m_{n\to\infty}(a_n) \leqslant \lim_{n\to\infty}(b_n)$ if we have $a_n < b_n$ for all n.

Theorem 5: *Squeeze Theorem for Sequences*

Let $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ be three sequences in $\mathbb{R}$. Suppose $\exists N \in \mathbb{N}$ st $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant N$. Then we have: if $(a_n)_n$ and $(c_n)_n$ converge and:

$$\lim_{n\to\infty} (a_n) = \lim_{n\to\infty} (c_n),$$
$$\text{then } (b_n)_n \text{ converges and}$$
$$\lim_{n\to\infty} (b_n) = \lim_{n\to\infty} (a_n) = \lim_{n\to\infty} (c_n)$$

Proof:

$$
\begin{aligned}
\text{Let } l = \lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n), \text{ and let } \varepsilon > 0 & \\
\text{Then } \exists N_1, N_2 \in \mathbb{N} \text{ st} & \\
n \in \mathbb{N} \text{ and } n \geqslant N_1 \text{ implies } |a_n - l| < \varepsilon & \qquad (*) \\
\text{and} & \\
n \in \mathbb{N} \text{ and } n \geqslant N_2 \text{ implies } |c_n - l| < \varepsilon & \qquad (**) \\
\text{Let } P = \max\{N_1, N_2, N\}. \text{ If } n \in \mathbb{N} \text{ and } n \geqslant P, \text{ then:} & \\
l - \varepsilon < a_n \leqslant b_n \leqslant c_n < l + \varepsilon & \qquad \text{from } (*) \text{ and } (**) \\
\text{This is equivalent to saying:} & \\
l - \varepsilon < b_n < l + \varepsilon & \\
\text{and thus } |b_n - l| < \varepsilon \text{ for all } n \geqslant P &
\end{aligned}
$$

Thus we have that:

$$\lim_{n\to\infty}(b_n) = \lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(c_n)$$

Theorem 6:

Let $(a_n)_n$ be a sequence in $\mathbb{R}$.

1. If $(a_n)_n$ is $\nearrow$ and bounded above, then $(a_n)_n$ converges and

$$\lim_{n\to\infty}(a_n) = \sup(a_n \mid n \in \mathbb{N})$$

2. If $(a_n)_n$ is $\nearrow$ but NOT bounded above, then:

$$\lim_{n\to\infty}(a_n) = +\infty$$

3. If $(a_n)_n$ is $\searrow$ and bounded below, then $(a_n)_n$ converges and

$$\lim_{n\to\infty}(a_n) = \inf(a_n \mid n \in \mathbb{N})$$

4. If $(a_n)_n$ is $\searrow$ and NOT bounded below, then:

$$\lim_{n\to\infty}(a_n) = -\infty$$

Proof:

1)

Let $A = \{a_n \mid n \in \mathbb{N}\}$. Clearly we know that $A \neq \varnothing$. Since $(a_n)_n$ is bounded above, then we know that our set A is bounded above. Thus the Completeness Axiom implies that $\sup(A)$ exists.

$$
\begin{aligned}
\text{Let } \varepsilon > 0. \text{ Then } \exists a_N \text{ st} & \\
\sup(A) - \varepsilon < a_N \leqslant \sup(A) & \\
\text{For all } n \in \mathbb{N}, n \geqslant N, \text{ we have:} & \\
\sup(A) - \varepsilon < a_N \leqslant a_n \leqslant \sup(A) < \sup(A) + \varepsilon & \qquad a_N \leqslant a_n \text{ because } (a_n)_n \text{ is } \nearrow \\
\text{Hence:} & \\
|a_n - \sup(A)| < \varepsilon \text{ for all } n \geqslant N &
\end{aligned}
$$

Hence we have shown that:

$$\lim_{n\to\infty} (a_n) = \sup (A) = \sup (a_n \mid n \in \mathbb{N})$$

2)

We have that:

$$\begin{aligned} \lim_{n\to\infty} (a_n) = +\infty \text{ iff } \forall M > 0, \exists N \in \mathbb{N} \text{ st} \\ n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies } a_n \geqslant M \end{aligned}$$

$$\begin{aligned} \text{Let } M > 0. \text{ Since } (a_n)_n \text{ is not bounded above,} \\ M \text{ is not an upper bound of } (a_n)_n \\ \text{ie } \exists N \in \mathbb{N} \text{ st } a_N > M \\ \\ \text{Since } (a_n)_n \text{ is } \nearrow, \text{ we have:} \\ a_n \geqslant a_N > M \text{ for all } n \geqslant N \end{aligned}$$

This exactly is the definition of a limit that goes to infinity, and thus we have shown that if $(a_n)_n$ is not bounded above, then the limit as $n \to \infty$ goes to infinity.

We follow the exact same proof for 3) and 4), but we obviously change to characterize the infimum, etc.

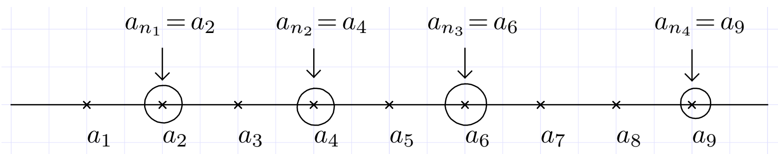
October 21st, 2020

Consequence of Theorem 6 and Lemma 2: *Monotone Convergence Theorem*

Let $(a_n)_n$ be a sequence in $\mathbb{R}$. Suppose that $(a_n)_n$ is monotone, meaning that it is eitehr $\nearrow$or $\searrow$. Then we have that $(a_n)_n$ converges iff $(a_n)_n$ is bounded.

Subsequences:

Let $(a_n)_n$ be a sequence in $\mathbb{R}$. Then if $g: \mathbb{N} \to \mathbb{N}$, $k \mapsto g(k) = n_k$ is strictly increasing, then the sequence $(a_{n_k})_k$ is called a subsequence of $(a_n)_n$.



We do not care about the sequence value of the sequence at that index. We only care about the index itself. This means that the index must be more than that of the previous. For example, a_{n_4} has to be equal to a of something strictly bigger than 4, in our case a_9. This order needs to be preserved with respect to the initial sequence.

Very Important: For all $k \geqslant 1$, then $n_k \geqslant k$.

Proof: We will use induction.

- $k = 1, n_1 \in \mathbb{N} \text{ and so: } n_1 \geqslant 1$

- Assume $n_k \geqslant k$, and we want to prove that $n_{k+1} \geqslant k+1$. Since g is strictly increasing: $n_{k+1} > n_k$, both $\in \mathbb{N}$. Then we can say that $n_{k+1} \geqslant n_k + 1 \geqslant k+1$.
- We have proven that $n_{k+1} \geqslant k+1$. Thus by induction For all $k \geqslant 1$, then $n_k \geqslant k$.

Exp. 1) Let $(a_n) = (-1)^n$. Then we have that $a_{2n} = (-1)^{2n} = 1$ is a subsequence of $(a_n)_n$.

Exp. 2) Let $(a_n)_{n \geqslant 1}$ be a sequence in $\mathbb{R}$. Then $(a_{2n+1})_{n \geqslant 1}$ is a subsequence of $(a_n)_{n \geqslant 1}$

Lemma 7:

If $(a_n)_n$ converges, then every subsequence $(a_{n_k})_k$ converge and:

$$\lim_{k \to \infty} a_{n_k} = \lim_{n \to \infty} a_n$$

Proof:

$$\text{Let } \lim_{n \to \infty} a_n = l \text{ for some } l \in \mathbb{R}$$

$$\begin{aligned}
\text{Then } \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies}\\
|a_n - l| < \varepsilon \ (\varepsilon > 0 \text{ is random})\\
\text{If } k \in \mathbb{N} \text{ and } k \geqslant N, \text{ then:}\\
n_k \geqslant k \geqslant N. \text{ Hence } |a_{n_k} - l| < \varepsilon\\
\text{ie } \lim_{k \to \infty} a_{n_k} = l
\end{aligned}$$

Lemma 8:

Let $(a_n)_n$ be a sequence in $\mathbb{R}$. Then $(a_n)_n$ has a monotone subsequence.

Proof:

$$\begin{aligned}
\text{For each } k \in \mathbb{N}, \text{ consider the set:}\\
T_k = \{a_n \mid n \geqslant k\}
\end{aligned}$$

For each k, T_k is the set of all the elements in the sequence after k. We will have two cases.

1. Suppose T_k has a greatest element for all $k \in \mathbb{N}$. We define the sequence $(n_k)_k$ using Definition by Recursion in the following way:

$$\begin{aligned}
\text{Let } n_1 \in \mathbb{N} \text{ be st } a_{n_1} \text{ is the greatest element of } T_1\\
\text{For example, let } T_1 = \{a_1, a_2, a_3, a_4\}\\
\text{Then } n_1 = 4\\
\text{There may be more than one } i \in \mathbb{N} \text{ st } a_i \text{ is a greatest element}\\
\text{of } T_1. \text{ In which case, we select the smallest possible } n_1 \text{ st } a_{n_1}\\
\text{is a greatest element of } T_1.
\end{aligned}$$

$$\begin{aligned}
\text{Let } n_2 \in \mathbb{N} \text{ be st } n_2 \geqslant n_1 + 1 \text{ (ie } n_2 > n_1)\\
\text{and } a_{n_2} \text{ is a greatest element of } T_{n_1+1}
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } \exists n_3 \in \mathbb{N} \text{ st } n_3 \geqslant n_2 + 1 \text{ (ie } n_3 > n_2)\\
\text{and } a_{n_3} \text{ is a greatest element of } T_{n_2+1}
\end{aligned}$$

Continuing in this way, we define a sequence $(n_k)_k$ that is strictly increasing, and st a_{n_k} is a greatest element of $T_{n_{k-1}+1}$ for all $k \geqslant 2$. Therefore, we have defined a subsequence $(a_{n_k})_{k \geqslant 1}$ of $(a_n)_n$. We are done extracting the subsequence.

We shall now show that $(a_{n_k})_k$ is monotone, and is in fact $\searrow$.

Now, let $i, j \in \mathbb{N}$. Suppose $i < j$.
Then automatically $j > i$ and $i \geqslant 1$ ie $j \geqslant 2$
We need to check the following two subcases:

1) $i = 1$
Then a_{n_i} is a greatest element of T_1.
Since $n_{j-1} + 1 > 1$, we have $T_{n_{j-1}+1} \subseteq T_1$.
Thus $a_{n_i} \geqslant a_{n_j}$
This means that the greatest element of $T_{n_{j-1}+1}$
must be less than the greatest element of T_1, since
$T_{n_{j-1}+1} \subseteq T_1$

2) $i > 1$
Since we are in $\mathbb{N}$, $i \geqslant 2$ and hence a_{n_i} is a greatest element
of $T_{n_{i-1}+1}$. Since $(n_k)_k$ is strictly increasing, $i < j$ implies
$n_{i-1} + 1 < n_{j-1} + 1$. Hence:
$T_{n_{i-1}+1} \supseteq T_{n_{j-1}+1}$ and therefore $a_{n_i} \geqslant a_{n_j}$

By putting together the two subcases, we deduce that the subseqence $(a_{n_k})_k$ is decreasing. We are done with this case.

2. Suppose $\exists r \in \mathbb{N}$ st T_r does not have a greatest element. We define a sequence $(m_k)_k$ in $\mathbb{N}$ using again, Definition by Recursion as follows:

Let $m_1 = r$. Then $a_{m_1} \in T_{m_1} = T_r$
a_{m_1} is not a greatest element of $T_{m_1} = T_r$
Hence, $\exists m_2 \in \mathbb{N}$ st $m_2 \geqslant m_1 + 1$ and $a_{m_2} > a_{m_1}$
as before, the choice of m_2 is not necessarily unique,
and so we select the smallest possible m_2 st
$m_2 \geqslant m_1 + 1$ (ie $m_2 > m_1$ and $a_{m_2} > a_{m_1}$)
Therefore $T_{m_2} \subseteq T_{m_1}$. Since:
$T_{m_1} - T_{m_2} = \{a_{m_1}, a_{m_1+1}, a_{m_1+2}, \ldots, a_{m_2-1}\}$
is a finite set and T_{m_1} has no greatest element,
we conclude that T_{m_2} has no greatest element either.
$T_{m_1} = T_{m_2} \cup \{a_{m_1}, \ldots, a_{m_2-1}\}$

Similarly, $\exists m_3 \in \mathbb{N}$ st $m_3 \geqslant m_2 + 1$ ie $m_3 > m_2$ and:
$a_{m_3} > a_{m_2}$
Continuing this way, we construct a sequence
$(m_k)_{k \geqslant 1} \in \mathbb{N}$ that is strictly $\nearrow$
st the subsequence $(a_{n_k})_{k \geqslant 1}$ of $(a_n)_n$ is increasing.

Since the subsequence is increasing, then it is monotone and thus we are done.

Theorem 9: *Bolzano-Weirstrass Theorem for Sequences*

Let $(a_n)_n$ be a sequence in $\mathbb{R}$. If $(a_n)_n$ is bounded, then $(a_n)_n$ has a convergent subsequence. This is only an existence proof. There is no indication as to what this subsequence is, whether it's unique or not, how to acquire it, etc.

Proof: By **LM8**, $(a_n)_n$ has a monotone subsequence $(a_{n_k})_k$. Since $(a_n)_n$ is bounded, the subsequence $(a_{n_k})_k$ is also bounded and hence by the monotone convergence theorem, $(a_{n_k})_k$ converges. We are done with the proof.

October 26th, 2020

Cauchy Sequences:

Def: A sequence $(a_n)_n$ is said to be a Cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st for every $m, n \in \mathbb{N}$ and $m, n \geqslant N$, we have $|a_m - a_n| < \varepsilon$. In street language, this means that the terms are getting closer and closer to each other.

Exp. 1) Let $a_n = \frac{1}{n}$. Then we know that $(a_n)_n$ is a Cauchy sequence. In fact, let $\varepsilon > 0$. Then by **CRLY6 CHPT1**, $\exists N \in \mathbb{N}$ st $\frac{1}{N} < \varepsilon$. Let $m, n \in \mathbb{N}$ with $m, n \geqslant N$. Without loss of generality, we assume that $m > n$. Then:

$$\begin{aligned} |a_m - a_n| &= \left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{n-m}{mn}\right| \\ &= \frac{m-n}{mn} \\ &\leqslant \frac{m}{mn} = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon \quad \text{because } n \geqslant N \end{aligned}$$

Exp. 2) Let $a_n = (-1)^n$. Then we know that $(a_n)_n$ is not a Cauchy sequence. Let $\varepsilon = 1$ and let $N \in \mathbb{N}$ be random. We choose $m, n \in \mathbb{N}$ st $m, n \geqslant N$ with m odd and n even. Then:

$$|a_m - a_n| = |(-1)^m - (-1)^n| = |-1 - 1| = 2 > \varepsilon$$

Therefore we have shown that $(a_n)_n$ is NOT a Cauchy sequence.

Theorem 10:

If $(a_n)_n$ is a convergent sequence in $\mathbb{R}$, then $(a_n)_n$ is a Cauchy sequence.

Proof:

$$\begin{aligned} &\text{Let } l = \lim_{n\to\infty} a_n \text{ for some } l \in \mathbb{R} \\ &\text{Let } \varepsilon > 0. \text{ Then } \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies:} \\ &|a_n - l| < \frac{\varepsilon}{2} \\ \\ &\text{Suppose } m \text{ and } n \in \mathbb{N} \text{ with } m, n \geqslant N. \text{ Then:} \\ &|a_m - a_n| = |a_m - l + l - a_n| \\ &\leqslant |a_m - l| + |l - a_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Theorem 11:

Let $(a_n)_n$ be a Cauchy sequence. If $(a_n)_n$ has a convergent subsequence, denoted $(a_{n_k})_k$, then $(a_n)_n$ converges, and:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_{n_k})$$

Proof:

$$\begin{aligned}
&\text{Let } l = \lim_{n\to\infty} a_{n_k} \text{ for some } l \in \mathbb{R}\\
&\text{Let } \varepsilon > 0. \text{ Since } (a_n)_n \text{ is a Cauchy seqence,}\\
&\exists N \in \mathbb{N} \text{ st } m, n \in \mathbb{N} \text{ and } m, n \geqslant N \text{ implies:}\\
&|a_m - a_n| < \frac{\varepsilon}{2}\\
&\text{Also, } \exists M \in \mathbb{N} \text{ st } k \in \mathbb{N} \text{ and } k \geqslant M \text{ implies:}\\
&|a_{n_k} - l| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\begin{aligned}
&\text{Let } P = \max\{M, N\}\\
&\text{If } n \in \mathbb{N} \text{ and } n \geqslant P, \text{ then:}\\
&|a_n - l| = |a_n - a_{n_p} + a_{n_p} - l|\\
&\leqslant |a_n - a_{n_p}| + |a_{n_p} - l|\\
&\text{Since } n_p \geqslant P = \max\{M, N\}, \text{ ie } P \geqslant M \text{ and } P \geqslant N\\
&\text{We have } |a_n - a_{n_p}| > \frac{\varepsilon}{2}\\
&\text{and } |a_{n_p} - l| > \frac{\varepsilon}{2}\\
&|a_n - a_{n_p}| + |a_{n_p} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Hence we have that $|a_n - l| < \varepsilon$ and:

$$\lim_{n\to\infty} a_n = l$$

Lemma 12:

If $(a_n)_n$ is a Cauchy sequence in $\mathbb{R}$, then $(a_n)_n$ is bounded.

Proof:

$$\begin{aligned}
&\text{Let } \varepsilon = 1. \text{ Then } \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies:}\\
&|a_n - a_N| < 1, \text{ where } m = M \text{ and } \varepsilon = 1\\
&|a_n| - |a_N| \leqslant |a_n - a_N| < 1\\
&|a_n| < 1 + |a_N| \text{ for all } n \geqslant N
\end{aligned}$$

$$\begin{aligned}
&\text{Now, let } M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}\\
&\text{Then } |a_n| \leqslant M \text{ for all } n \in \mathbb{N}\\
&\text{Thus } (a_n)_n \text{ is bounded.}
\end{aligned}$$

Theorem 13:

Every Cauchy sequence in $\mathbb{R}$ is convergent.

Proof:

If $(a_n)_n$ is a Cauchy sequence, then by **LM12**, $(a_n)_n$ is bounded. Thus Bolzano-Weirstrass theorem implies that $(a_n)_n$ has a convergent subsequence. Hence **THM11** implies that since we have a Cauchy sequence with a convergent subsequence, then $(a_n)_n$ converges.

Theorem: *Cauchy Completeness Theorem*

A sequence $(a_n)_n$ in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.

Proof: Combine **THM10** and **THM13**. By this we mean that every Cauchy sequence is convergent and every convergent sequence is a Cauchy sequence.

Theorem 14: *Sequential Characterization of Limits*

Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$, $f: I \backslash \{c\} \longrightarrow \mathbb{R}$ be a function and let $l \in \mathbb{R}$. Then:

$$\begin{aligned}\lim_{x \to c} f(x) = l \text{ iff}\\ \underline{\text{for every}} \text{ sequence } (c_n)_n \text{ in } I \backslash \{c\} \text{ st } \lim_{n \to \infty} c_n = c\\ \text{we have } \lim_{n \to \infty} f(c_n) = l\end{aligned}$$

Proof:

$$\begin{aligned}\Longrightarrow\\ \text{Suppose } \lim_{x \to c} f(x) = l\\ \text{Let } (c_n)_n \text{ be a sequence in } I \backslash \{c\}\\ \text{st } \lim_{n \to \infty} c_n = c\end{aligned}$$

$$\begin{aligned}\text{Let } \varepsilon > 0. \text{ Since } \lim_{x \to c} f(x) = l, \exists \delta > 0 \text{ st}\\ x \in I \backslash \{c\} \text{ and } |x - c| < \delta \text{ implies:}\\ |f(x) - l| < \varepsilon\end{aligned}$$

$$\begin{aligned}\text{Because } \lim_{n \to \infty} c_n = c, \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N\\ \text{implies } |c_n - c| < \delta\\ \text{Hence if } n \geqslant N, |f(c_n) - l| < \varepsilon, \text{ and therefore:}\\ \lim_{n \to \infty} f(c_n) = l\end{aligned}$$

$$\begin{aligned}\Longleftarrow\\ \text{Suppose } \lim_{n \to \infty} f(c_n) = l \text{ for every sequence } (c_n)_n \text{ in } I \backslash \{c\}\\ \text{st } \lim_{n \to \infty} c_n = c\end{aligned}$$

$$\begin{aligned}\text{Suppose } \lim_{x \to c} f(x) \neq l\\ \text{Then } \exists \varepsilon > 0 \text{ st for all } \delta > 0,\\ \exists x_n \in I \backslash \{c\} \text{ with } |x_n - c| < \delta \text{ but:}\\ |f(x_n) - l| \geqslant \varepsilon\\ \text{Let } \delta = \frac{1}{n}, n \geqslant 1\end{aligned}$$

$$\begin{aligned}\text{We therefore construct a sequence } (x_n) \text{ for which we have:}\\ |x_n - c| < \frac{1}{n} \text{ for all } n \geqslant 1, \text{ but}\\ |f(x_n) - l| \geqslant \varepsilon\end{aligned}$$

We have constructed a sequence that is approaching c, however, $f(x_n)$ is not.

$$\lim_{n\to\infty} x_n = c \,\text{by squeeze theorem}$$

But:

$$\lim_{x\to c} f(x) \neq l$$

Which is a contradiction, since we initially assumed that $\lim_{x\to c} f(x) = l$. Therefore we have that:

$$\lim_{x\to c} f(x) = l$$

Corollary 15: *Sequential Characterization of Continuity*

Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f: I \longrightarrow \mathbb{R}$ be a function. f is continuous at c iff:

$$\lim_{n\to\infty} f(c_n) = f(c) \,\text{for every sequence}\, (c_n)_n \in I \,\text{st}\, \lim_{n\to\infty} c_n = c$$

Proof:

$$\begin{aligned}
&\Longrightarrow \\
\text{Let } \varepsilon > 0.\ \text{Since } f \text{ is continuous at } c, \exists \delta > 0 \,\text{st} \\
x \in I \setminus \{c\} \text{ and } |x - c| < \delta \text{ implies:} \\
|f(x) - f(c)| < \varepsilon. \\
\text{Let } (c_n)_n \text{ be a sequence in } I \text{ st } \lim_{n\to\infty} c_n = c \\
\text{Then } \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies} \\
|c_n - c| < \delta.\ \text{Hence for all } n \geqslant N: \\
|f(c_n) - f(c)| < \varepsilon, \text{and therefore:} \\
\lim_{n\to\infty} f(c_n) = f(c)
\end{aligned}$$

$$\begin{aligned}
&\Longleftarrow \\
\text{Suppose } \lim_{n\to\infty} f(c_n) \neq f(c) \\
\text{Then } \exists \varepsilon > 0 \text{ st } \forall n \geqslant 1, \exists x_n \in I \text{ st } |x_n - c| < \delta \\
\text{but } |f(x_n) - f(c)| \geqslant \varepsilon \\
\text{Let } \delta = \frac{1}{n} \\
\\
\text{Thus we have constructed a sequence } (x_n)_n \\
\text{in } I \text{ st } \lim_{n\to\infty} x_n = c, \text{but } \lim_{n\to\infty} f(x_n) \neq f(c) \\
\text{Which is a contradiction. Hence:} \\
\lim_{x\to c} f(x) = f(c)
\end{aligned}$$

Theorem 16:

Let $[a,b]\subseteq\mathbb{R}$ be a compact, and let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a function. Suppose f is integrable on $[a,b]$, ie:

$$\int_a^b f(x)\mathrm{dx}\,\text{exists}$$

Let $(P_n)_n$ be a sequence of partitions of $[a,b]$ st:

$$\lim_{n\to\infty}\|P_n\|=0$$

For each n, let T_n be a representative set of P_n. Then:

$$\int_a^b f(x)\mathrm{dx}=\lim_{n\to\infty}S(f,P_n,T_n)$$

Proof:

$$\begin{aligned}&\text{Let } \varepsilon>0.\text{ Since } f \text{ is integrable,}\\ &\exists\delta>0 \text{ st if } Q \text{ is a partition of } [a,b]\\ &\text{with } \|Q\|<\delta \text{ and if } V \text{ is a representative set of } Q,\text{ then:}\\ &\left|S(f,Q,V)-\int_a^b f(x)\mathrm{dx}\right|<\varepsilon\end{aligned}$$

$$\begin{aligned}&\text{Since } \lim_{n\to\infty}\|P_n\|=0,\exists N\in\mathbb{N} \text{ st } n\in\mathbb{N} \text{ and } n\geqslant N \text{ implies:}\\ &|\|P_n\|-0|=\|P_n\|<\delta\\ &\text{Thus: if } n\in\mathbb{N} \text{ and } n\geqslant N,\text{ we have:}\\ &\left|S(f,P_n,T_n)-\int_a^b f(x)\mathrm{dx}\right|<\varepsilon\\ &\text{Therefore:}\\ &\lim_{n\to\infty}S(f,P_n,T_n)=\int_a^b f(x)\mathrm{dx}\end{aligned}$$

October 28th, 2020

8 Chapt. Series of Real Numbers

Def.:

$$\sum_{n\geqslant 1}a_n,\text{ where } (a_n)_n \text{ is a sequence in } \mathbb{R}$$

and a_n is the general term of the series.

1. For each $n\in\mathbb{N}$, we define the partial sum, or the nth sum, S_n, by:

   $$S_n=\sum_{i=1}^n a_i=a_1+\cdots+a_n$$

   $(S_n)_n$ is called the sequence of partial sums.

2. If:

   $$\lim_{n\to\infty}S_n=s\text{ for some } s\in\mathbb{R},$$

we say that $\sum_{n\geqslant 1} a_n$ converges to s and we write $\sum_{n\geqslant 1} a_n = s$. In this , s is called the sum of $\sum_{n\geqslant 1} a_n$. Otherwise, i.e. $\lim_{n\to\infty} S_n$ d.n.e., we say that $\sum_{n\geqslant 1} a_n$ diverges.

Theorem 1: *Divergence Theorem*

If:

$$\lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n\geqslant 1} a_n$ diverges.

Proof: We shall prove the contrapositive of the divergence test.

$$\begin{aligned}
\text{Assume} \sum_{n\geqslant 1} a_n \,\text{converges to}\, s \in \mathbb{R}\\
\text{Then}\, (S_n)_n \,\text{converges to}\, s,\\
\text{so}\, \forall\varepsilon > 0, \exists N \in \mathbb{N} \,\text{st}\, n \in \mathbb{N} \,\text{and}\, n \geqslant N \,\text{implies:}\\
|S_n - s| < \frac{\varepsilon}{2}
\end{aligned}$$

$$\begin{aligned}
\text{If}\, n \geqslant N + 1, \text{we have:}\\
|a_n - 0| = |a_n| = |S_n - S_{n-1}|\\
= |S_n - s + s - S_{n-1}|\\
\leqslant |S_n - s| + |s - S_{n-1}|\\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Hence:

$$\lim_{n\to\infty} a_n = 0$$

Theorem 2: *Cauchy's Convergence Criteria*

A series, $\sum_{n\geqslant 1} a_n$, converges iff $\forall\varepsilon > 0, \exists N \in \mathbb{N}\,\text{st}\, m, n \in \mathbb{N}\,\text{and}\, m \geqslant n \geqslant N$ implies:

$$|a_n + a_{n+1} + \cdots + a_{m-1} + a_m| < \varepsilon$$

Proof:

$$\begin{aligned}
|a_n + a_{n+1} + \cdots + a_{m-1} + a_m| = |S_m - S_{n-1}|\\
\forall\varepsilon > 0, \exists N \in \mathbb{N}\,\text{st}\, m, n \in \mathbb{N}\,\text{and}\, m \geqslant n \geqslant N\\
\text{implies}\, |S_m - S_{n-1}| < \varepsilon, \text{then}\, (S_n)_n \,\text{is a Cauchy sequence}
\end{aligned}$$

$$\begin{aligned}
\text{Therefore}\\
\sum_{n\geqslant 1} a_n \quad \text{converges iff}\\
(S_n)_n \,\text{converges, iff}\, (S_n)_n \,\text{is a Cauchy sequence}\\
\text{This is by the Cauchy completeness theorem.}
\end{aligned}$$

Corollary 3:

If:

$$\sum_{n\geqslant 1} a_n \,\text{converges, then}\, \lim_{n\to\infty} a_n = 0$$

We have already proved this, but we can see how we obtain this as a corollary of the previous theorem.

Proof:

$$\begin{aligned}\text{Take } m = n \text{ in } \textbf{THM2}\\ \text{Then we would have } |S_n - S_{n-1}| < \varepsilon\\ \text{which brings us back to the original proof}\end{aligned}$$

Notice that here, it is if $\longrightarrow$ then, whereas **THM2** showed an iff relationship. Where did we lose this? In the original theorem, we said for every $m \geqslant n \geqslant N$, but here, we are making a choice for m and n. Therefore, we lose the necessary and sufficient condition.

Lemma 4:

Suppose $a_n \geqslant 0$ for all $n \in \mathbb{N}$ or $a_n \leqslant 0$ for all $n \in \mathbb{N}$. Then:

$$\sum_{n \geqslant 1} a_n \text{ converges iff } (S_n)_n \text{ is bounded.}$$

Proof:

$$\begin{aligned}\text{if } a_n \geqslant 0 \text{ for all } n \in \mathbb{N}, \text{ then:}\\ (S_n)_n \text{ is increasing (ie monotone)}\\ \Longrightarrow \text{The Monotone Convergence Theorem}\\ \text{implies that:}\\ (S_n)_n \text{ converges iff } (S_n)_n \text{ is bounded}\end{aligned}$$

The same applies for when $(S_n)_n$ is decreasing, and this is because the sequence will still be monotone, and thus the same theorem applies.

Theorem 5: *Comparison Test*

Suppose $a_n \geqslant 0, b_n \geqslant 0 \quad \forall n \in \mathbb{N}$, and suppose that $\exists N \in \mathbb{N}$ st $a_n \leqslant b_n, \forall n \geqslant N$. Then we have the following:

1. If $\sum_n b_n$ converges, then $\sum_n a_n$ converges;
2. $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges.

Proof:

1.

$$\begin{aligned}\text{Let } (T_n)_n \text{ be the sequence of partial sums of } \sum_n b_n\\ \text{Since } \sum_n b_n \text{ converges, } \textbf{LM4} \text{ implies that } (T_n)_n\\ \text{is bounded, and thus } \exists M > 0 \text{ st } |T_n| \leqslant M \text{ for all } n \in \mathbb{N}\\ \text{Since } 0 \leqslant a_n \leqslant b_n \text{ for all } n \geqslant N, \text{ we have:}\\ 0 \leqslant S_n \leqslant T_n \text{ for all } n \geqslant N, \text{ and so:}\\ 0 \leqslant S_n \leqslant M, \forall n \geqslant N\\ \\ \text{Let } K = \max\{|S_1|, |S_2|, \ldots., |S_{N-1}|, M\}\\ \text{Then: } |S_n| \leqslant K \, \forall n \in \mathbb{N}\\ \text{ie } (S_n)_n \text{ is bounded, and hence } \sum_n a_n \text{ converges by } \textbf{LM2}\end{aligned}$$

2. This is the contrapositive of 1, so we will not go into its proof.

Theorem 6: *Limit Comparison Test*

Suppose $a_n \geqslant 0, b_n > 0 \quad \forall n \in \mathbb{N}$. If:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l \text{ for some } l \in (0, +\infty),$$
$$\text{then} \sum_n a_n \text{ converges iff} \sum_n b_n \text{ converges}$$

Proof:

$$\text{Since } l \in (0, +\infty), \frac{l}{2} > 0. \text{ Then:}$$
$$\exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies:}$$
$$\left| \frac{a_n}{b_n} - l \right| < \frac{l}{2}$$
$$\text{Thus:}$$
$$\frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2} \text{ for all } n \geqslant N$$
$$\frac{l}{2} b_n < a_n < \frac{3}{2} b_n \, l$$

$$\text{By the comparison test, we have that:}$$
$$\text{if} \sum_n a_n \text{ converges, then} \sum_n \frac{l}{2} b_n \text{ converges}$$
$$\text{and if} \sum_n a_n \text{ diverges, then} \sum_n \frac{3l}{2} b_n \text{ diverges}$$

$$\text{But } \frac{l}{2} \sum_n b_n \text{ and } \frac{3l}{2} \sum_n b_n \text{ converge iff} \sum_n b_n \text{ converges}$$
$$\text{Therefore:}$$
$$\sum_n a_n \text{ converges iff} \sum_n b_n \text{ converges}$$

Theorem 7:

Suppose $a_n \geqslant 0, b_n > 0$ for all $n \geqslant 1$. Then:

1. If:

$$\sum_n b_n \text{ converges and if } \overline{\lim_{n \to \infty}} \frac{a_n}{b_n} < +\infty, \text{ then} \sum_n a_n \text{ converges}$$

2. If:

$$\sum_n b_n \text{ diverges and if } \underline{\lim_{n \to \infty}} \frac{a_n}{b_n} > 0, \text{ then} \sum_n a_n \text{ diverges}$$

Proof:

1.

$$\begin{aligned}
\text{Since } \overline{\lim_{n\to\infty}} \frac{a_n}{b_n} < +\infty, \text{then:}\\
\text{The sequence} \left(\frac{a_n}{b_n}\right) \text{is bounded, and so}\\
\exists M > 0 \quad \text{st } 0 \leqslant \frac{a_n}{b_n} < M \text{ for all } n \geqslant 1\\
\text{ie } 0 \leqslant a_n \leqslant M b_n\\
\text{Since} \sum_n M b_n = M \sum_n b_n \text{ converges,}\\
\text{The Comparison Test implies that} \sum_n a_n \text{ converges}
\end{aligned}$$

2.

$$\begin{aligned}
\text{Since } \underline{\lim_{n\to\infty}} \frac{a_n}{b_n} > 0, \exists m > 0 \text{ and}\\
\exists K \in \mathbb{N} \text{ st } \frac{a_n}{b_n} \geqslant m \text{ for all } n \geqslant K\\
\text{So: } a_n \geqslant m\, b_n \text{ for } n \geqslant K\\
\text{Since} \sum_n b_n \text{ diverges, } m \sum_n b_n \text{ diverges, } m \neq 0\\
\text{and hence the Comparison Test implies:}\\
\sum_n a_n \text{ diverges}
\end{aligned}$$

Exp:

$$\sum_{n\geqslant 2} a_n = \sum_{n\geqslant 2} \frac{2 + \sin\left(n\frac{\pi}{6}\right)}{(n+1)^p (n-1)^q}$$

We want to find the values for p and q for which this series converges. We have our a_n, we simply need to find our b_n. That is the difficulty in this type of question. Take the following:

$$\sum_{n\geqslant 2} b_n = \sum_{n\geqslant 2} \frac{1}{n^{p+q}}$$

$$\begin{aligned}
\frac{a_n}{b_n} = \frac{2 + \sin\left(n\frac{\pi}{6}\right)}{\left(1+\frac{1}{n}\right)^p \left(1-\frac{1}{n}\right)^q}\\
\overline{\lim_{n\to\infty}} \frac{a_n}{b_n} = 3\\
\underline{\lim_{n\to\infty}} \frac{a_n}{b_n} = 1
\end{aligned}$$

$$\begin{aligned}
\text{Now:}\\
\sum_{n\geqslant 2} b_n \text{ converges iff } p + q > 1 \text{ and}\\
\text{diverges iff } p + q \leqslant 1\\
\text{Hence the same is true for} \sum_{n\geqslant 2} a_n
\end{aligned}$$

November 2nd, 2020

Corollary 8: *Limit comparison test using* ***THM7***

Suppose $a_n \geqslant 0, b_n > 0$ for all $n \geqslant 1$. If:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l, \text{where } l \in (0, +\infty), \text{then}$$
$$\sum_n a_n \text{ and} \sum_n b_n \text{ converge or diverge together}$$

<u>Proof</u>:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \text{ iff } \underline{\lim_{n\to\infty}} \frac{a_n}{b_n} = l = \overline{\lim_{n\to\infty}} \frac{a_n}{b_n}$$
$$\text{Hence, the result follows from } \textbf{THM7}$$

Theorem 9:

Suppose $a_n > 0, b_n > 0$ and $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$. Then:

1.

$$\text{If} \sum_n b_n \text{ converges, then} \sum_n a_n \text{ converges}$$

2.

$$\text{If} \sum_n a_n \text{ diverges, then} \sum_n b_n \text{ diverges}$$

<u>Proof</u>:

1.

$$\text{Since } \frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}, \text{we have that:}$$
$$0 < \frac{a_{n+1}}{b_{n+1}} \leqslant \frac{a_n}{b_n}, \text{and so}$$
$$\text{the sequence of positive terms } \left(\frac{a_n}{b_n}\right)_n \text{ is decreasing.}$$

$$\text{Therefore } \overline{\lim_{n\to\infty}} \frac{a_n}{b_n} < \infty, \text{and hence } \textbf{THM7.1} \text{ implies:}$$
$$\text{If} \sum_n b_n \text{ converges, then} \sum_n a_n \text{ converges}$$

2.

$$\text{Since } \left(\frac{a_n}{b_n}\right) \text{ is decreasing,}$$
$$\exists r > 0 \text{ and } N \in \mathbb{N} \text{ st}$$
$$b_n \geqslant r\, a_n \text{ for } n \geqslant N. \text{ Note that } r \text{ is a fixed constant}$$

$$\begin{aligned}
&\text{Now, we use the comparison test.}\\
&\text{Since} \sum_n a_n \text{ diverges,} \longrightarrow \sum_n r\, a_n \text{ diverges}\\
&\text{and by the comparison test}(\textbf{THM5})\\
&\Longrightarrow \sum_n b_n \text{ diverges}
\end{aligned}$$

Theorem 10: *Ratio Test*

Suppose $a_n > 0$ for all $n \in \mathbb{N}$. We could also say $\forall n \geqslant N$ Then we have the following results:

1.
$$\text{if } \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} < 1, \text{then} \sum_n a_n \text{ converges}$$

2.
$$\text{if } \underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} > 1, \text{then} \sum_n a_n \text{ diverges}$$

3.
$$\text{if } \underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} \leqslant 1 \leqslant \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}, \text{then the rest is inconclusive}$$

Proof:

1.

$$\begin{aligned}
&\text{If } \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} < 1, \text{then:}\\
&\exists 0 < r < 1 \text{ and } K \in \mathbb{N} \text{ st:}\\
&0 < \frac{a_{n+1}}{a_n} < r < 1\\
&(\text{Alternatively we take } K \geqslant N \text{ if we use the rank})\\
&\\
&\text{Now we write } r = \frac{r^{n+1}}{r^n} = \frac{b_{n+1}}{b_n}\\
&\sum_n b_n = \sum_n r^n \text{ converges as a geometric series}\\
&\text{with ratio } r \in (0,1), \textbf{THM9.1} \text{ implies that:}\\
&\sum_n a_n \text{ converges}
\end{aligned}$$

2.

$$\begin{aligned}
&\text{If } \underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} > 1, \exists r > 1 \text{ and } T \in \mathbb{N} \text{ st}\\
&\frac{a_{n+1}}{a_n} > r = \frac{r^{n+1}}{r_n} = \frac{b_{n+1}}{b_n}.\\
&\text{Since} \sum_n r^n \text{ diverges, } \textbf{THM9.2} \text{ implies that:}\\
&\sum_n a_n \text{ diverges}
\end{aligned}$$

3.

$$\text{Take} \sum_n a_n = \sum_{n \geqslant 1} \frac{1}{n^p}. \text{We know that the series}$$
$$\text{converges for } p > 1 \text{ and diverges for } p \leqslant 1. \text{ However,}$$
$$\overline{\lim_{n \to \infty}} \frac{a_{n+1}}{a_n} = 1 = \underline{\lim_{n \to \infty}} \frac{a_{n+1}}{a_n}$$
$$\text{And thus the test is inconclusive}$$

Theorem 11: *Rabee's Test*

Suppose $a_n > 0$ for some large n (ie $\exists N \in \mathbb{N}$ st $a_n > 0$ for all $n \geqslant N$). Let:

$$M = \overline{\lim_{n \to \infty}} n\left(\frac{a_{n+1}}{a_n} - 1\right) \text{and } m = \underline{\lim_{n \to \infty}} n\left(\frac{a_{n+1}}{a_n} - 1\right)$$

1.

$$\text{If } M < -1, \text{then} \sum_n a_n \text{ converges}$$

2.

$$\text{if } m > -1, \sum_n a_n \text{ diverges}$$

Proof:

1.

$$\text{By Taylor's Theorem, we have:}$$
$$\frac{1}{(1+x)^p} = 1 - p\,x + \frac{1}{2}\frac{p(p+1)}{(1+c)^{p+2}}\,x^2 \text{ for some } c \in (0, x)$$
$$\frac{1}{2}\frac{p(p+1)}{(1+c)^{p+2}}\,x^2 > 0 \text{ for } p > 0$$
$$\frac{1}{(1+x)^p} > 1 - p\,x \text{ for } x > 0 \text{ and } p > 0 \qquad (*)$$

$$\text{Now suppose } M < -p < -1. \text{ Then } \exists K \in \mathbb{N} \text{ st:}$$
$$n\left(\frac{a_{n+1}}{a_n}\right) < -p \text{ for } n \geqslant K$$
$$\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n} \quad \forall n \geqslant K$$
$$\text{Hence by letting } x = \frac{1}{n} \text{ in } (*), \text{ we obtain:}$$
$$\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n} < \frac{1}{(1+\frac{1}{n})^p} = \frac{n^p}{(n+1)^p}$$
$$= \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \forall n \geqslant K$$

$$\text{Since} \sum_n \frac{1}{n^p} \text{ converges for } p > 1, \text{because } -p < -1,$$
$$\textbf{THM9.1} \text{ implies that} \sum_n a_n \text{ converges by letting:}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}$$

2.

$$\text{By Taylor's Theorem, we have that:}$$
$$(1-x)^q = 1 - q\,x + q(q-1)(1-c)^{q-2}\frac{x^2}{2}$$
$$\text{for some } 0 < c < x < 0$$
$$q(q-1)(1-c)^{q-2}\frac{x^2}{2} < 0 \text{ if } q \in (0,1) \text{ and } x \in (0,1)$$
$$\text{Thus } (1-x)^q < 1 - q\,x \text{ for } c < q < 1 \text{ and } 0 < x < 1.$$

$$\text{Now suppose } -1 < -q < m. \text{ Then } \exists K \in \mathbb{N} \text{ st}$$
$$n\left(\frac{a_{n+1}}{a_n} - 1\right) > -q \quad \forall n \geqslant K$$
$$\frac{a_{n+1}}{a_n} \geqslant 1 - \frac{q}{n} > \left(1 - \frac{1}{n}\right)^q, \text{ by letting } x = \frac{1}{n}$$
$$\left(1 - \frac{1}{n}\right)^q = \left(\frac{(n-1)}{n}\right)^q = \frac{\frac{1}{n^q}}{\frac{1}{(n+1)^1}} = \frac{b_{n+1}}{b_n}$$
$$\text{Thus:}$$
$$\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n} \text{ for all } n \geqslant K$$
$$\text{Since } \sum_n \frac{1}{(n-1)^q} \text{ diverges for } q \in (0,1), \textbf{THM9.2} \text{ implies:}$$
$$\sum_n a_n \text{ diverges}$$

<u>Exp</u>:

$$\text{Take} \sum_n a_n = \sum_{n \geqslant 1} \frac{n!}{\alpha(\alpha+1)(\alpha+2)\ldots.\ (\alpha+n-1)}, \alpha > 0$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{\alpha+n} = 1$$
$$\text{The ratio test is thus inconclusive}$$
$$\text{However, } \lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+1}{\alpha+n} - 1\right)$$
$$= \lim_{n\to\infty} \frac{n(1-\alpha)}{\alpha+n} = 1 - \alpha$$

Hence, by using Rabee's test, the series converges when $\alpha > 2$ and diverges for $0 < \alpha < 2$.

<u>Rmk</u>: Rabee's test is inconclusive if $m \leqslant -1 \leqslant M$. In the situation provided above, when $\alpha = 2$, the rest is inconclusive. But when $\alpha = 2$, the series becomes:

$$\sum_n a_n = \sum_{n \geqslant 1} \frac{n!}{2(2+1)(2+2)\ldots.\ (2+n-1)} = \sum_{n \geqslant 1} \frac{n!}{2(3)(4)\ldots.\ (2+n-1)} = \sum_{n \geqslant 1} \frac{n!}{(n+1)!}$$

You can rewrite this as the following:

$$\sum_{n \geqslant 1} \frac{1}{n+1}$$

and thus sum is divergent, clearly.

Theorem 12: *Cauchy's Root Test*

Suppose $\exists K \in \mathbb{N}\,\text{st}\, a_n \geqslant 0\,\text{for all}\, n \geqslant K$. Then we have the following:

1.
$$\text{if}\, \overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} < 1, \text{then} \sum_n a_n \,\text{is convergent}$$

2.
$$\text{if}\, \overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} > 1, \text{then} \sum_n a_n \,\text{is divergent}$$

The test is inconclusive if $\overline{\lim}_{n\to\infty} a_n^{\frac{1}{n}} = 1$.

<u>Proof</u>:

1.
$$\begin{aligned}\text{if}\, \overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} < 1, \exists r \in (0,1)\,\text{and}\, \exists N \in \mathbb{N}\,\text{st}\\ N \geqslant K\,\text{and}\, a_n^{\frac{1}{n}} < r \quad \forall n \geqslant N\\ \text{Thus:}\\ a_n < r^n \quad \forall n \geqslant N\\ \text{Since} \sum_n r^n \,\text{converges, then} \sum_n a_n \,\text{converges}\\ \text{by the comparison test}\end{aligned}$$

2.
$$\begin{aligned}\text{if}\, \overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{n}} > 1, \text{then}\\ a_n^{\frac{1}{n}} > 1\,\text{for infinitely many values of}\, n\\ \text{and hence}\, a_n > 1\,\text{for infinitely many}\, n\\ \text{Thus we cannot have} \lim_{n\to\infty} a_n = 0\\ \text{Therefore, the series diverges by the divergence test}\end{aligned}$$

Proof of inconclusivity:

$$\begin{aligned}\text{Let} \sum_{n\geqslant 1} a_n = \sum_{n\geqslant 1} \frac{1}{n^p}\\ \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln\left(\frac{1}{n^p}\right)}\\ = \lim_{n\to\infty} e^{\frac{-p\ln(n)}{n}} = 1\\ \text{But we know that} \sum_{n\geqslant 1} \frac{1}{n^p} \,\text{converges for}\, p > 1\\ \text{and diverges for}\, p \leqslant 1\end{aligned}$$

Therefore the test is inconclusive.

November 4th, 2020

Theorem 13: *Dirichlet's Test for Series*

The series:

$$\sum_n a_n b_n \text{ converges if the following 3 conditions are satisfied:}$$

$$\lim_{n\to\infty} a_n = 0$$
$$\sum_n |a_{n+1} - a_n| < +\infty$$
$$|b_k + b_{k+1} + \cdots + b_n| \leqslant M \text{ for all } n \geqslant k, \text{ where } k \in \mathbb{N} \text{ and } M > 0$$

Proof:

We shall show that:

$$\lim_{n\to\infty} S_n < \infty \text{ where } (S_n)_n \text{ is the sequence of partial sums of}$$
$$\sum_n a_n b_n. \text{ To do so, we will use the so called summation of parts}$$

$$\text{Define } B_n = b_k + b_{k+1} + \cdots + b_n \text{ for } n \geqslant k$$
$$\text{Let } (S_n)_n \text{ be the partial sum of}$$
$$\sum_{i \geqslant k} a_i b_i$$

$$\begin{aligned} S_n = \sum_{i=k}^{n} a_i b_i &= a_k b_k + a_{k+1} b_{k+1} + \cdots + a_n b_n \\ &= a_k B_k + a_{k+1}(B_{k+1} - B_k) + \cdots + a_n(B_n - B_{n-1}) \\ &= (a_k - a_{k+1})B_k + (a_{k+1} - a_{k+2})B_{k+1} + \cdots + (a_{n-1} - a_n)B_{n-1} + a_n B_n \\ &= T_{n-1} + a_n B_n \end{aligned}$$
$$\text{Where } (T_n)_n \text{ is the sequence of partial sums of the series } \sum_{j \geqslant k} (a_j - a_{j+1})B_j$$

$$\text{Since } |(a_j - a_{j+1})B_j| \leqslant M|a_j - a_{j+1}| \text{ for all } j \geqslant M$$
$$\text{and since } \sum_{j \geqslant k} |(a_j - a_{j+1})| \text{ converges,}$$
$$\text{we deduce by comparison test that:}$$
$$\sum_{j \geqslant k} (a_j - a_{j+1})B_j \text{ is absolutely convergent} \longrightarrow \text{convergent}$$
$$\text{Hence the sequence of its partial sums, } (T_n)_n, \text{ converges.}$$

$$\text{Let } T = \lim_{n\to\infty} T_n. \text{ Now:}$$
$$S_n = T_{n-1} + a_n B_n$$
$$a_n \longrightarrow 0 \text{ and } B_n \text{ is bounded by } M$$
$$\text{Therefore by a previous result in } \textbf{CHPT7}, \lim_{n\to\infty} a_n B_n = 0$$

$$\text{Therefore:}$$
$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} T_{n-1} + \lim_{n\to\infty} a_n B_n \\ &= T + 0 = T \end{aligned}$$

Thus the series:

$$\sum_n a_n b_n \text{ converges}$$

Exp:

$$\sum_{n\geqslant 2} \frac{\sin(n\theta)}{n+(-1)^n}, \text{where } \theta \neq k\pi, k\in\mathbb{Z}$$

$$\text{We take } a_n = \frac{1}{n+(-1)^n} \text{ and } b_n = \sin(n\theta)$$

$$\text{Then } \lim_{n\to\infty} a_n = 0 \qquad 1^{\text{st}} \text{ condition}$$

$$|a_{n+1}-a_n| = \left|\frac{1}{n+1-1} - \frac{1}{n+1}\right| \leftarrow n \text{ is even}$$

$$= \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{n+1-n}{n(n+1)}\right| = \left|\frac{1}{n(n+1)}\right|$$

$$\text{clearly} \sum_{n\geqslant 2} \left|\frac{1}{n(n+1)}\right| \text{ converges} \qquad 2^{\text{nd}} \text{ condition}$$

$$B_n = \sin(2\theta) + \sin(3\theta) + \cdots + \sin(n\theta)$$

$$B_n = b_2 + b_3 + \cdots + b_k$$

We shall use the following trigonometric identity:

$$\sin(r\theta) = \frac{\cos\left(r-\frac{1}{2}\right)\theta - \cos\left(r+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)} \qquad \theta\neq k\pi, k\in\mathbb{Z}$$

$$B_n = \frac{\cos\left(\frac{3}{2}\theta\right) + \cos\left(\frac{5}{2}\theta\right) + \cos\left(\frac{5}{2}\theta\right) - \cos\left(\frac{7}{2}\theta\right) + \cdots + \cos\left(n-\frac{1}{2}\theta\right) + \cos\left(n+\frac{1}{2}\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\cos\left(\frac{3}{2}\theta\right) - \cos\left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}$$

$$\text{Hence } |B_n| \leqslant \frac{\left|\cos\left(\frac{3}{2}\theta\right)\right| + \left|\cos\left(n+\frac{1}{2}\right)\theta\right|}{2\left|\sin\left(\frac{\theta}{2}\right)\right|} \leqslant \frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|}$$

$$M = \frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|} \qquad 3^{\text{rd}} \text{ condition}$$

Therefore, since we have satisfied the three conditions, we know that our series converges.

Corollary 14: *Abel's Test*

The series:

$$\sum_n a_n b_n \text{ converges if}$$

$$a_{n+1} \leqslant a_n \text{ for } n \geqslant k \text{ and } k\in\mathbb{N}$$

$$\lim_{n\to\infty} a_n = 0$$

$$\text{and } |b_k + b_{k+1} + \cdots + b_n| \leqslant M \text{ for all } n \geqslant M$$

$$\text{for some } M\in\mathbb{R}, M>0$$

Proof:

$$\text{if } a_{n+1} \leqslant a_n \text{ for } n \geqslant k \text{, then:}$$
$$\sum_{n=k}^{m} |a_{n+1} - a_n| = \sum_{n=k}^{m} (a_n - a_{n+1}) = a_k - a_{m+1}$$
$$\text{Since } \lim_{n\to\infty} a_{m+1} = 0 \text{, the series}$$
$$\sum_{n\geqslant k} |a_{n+1} - a_n| = a_k < +\infty$$

Therefore all the hypotheses of the Dirichlet's test are satisfied, and thus the series converges by **THM13**. As we can see, this is just an extension of the previous, but for a decreasing sequence $(a_n)_n$ instead of an increasing series.

Corollary 15: *Alternating Series Test*

The series:

$$\sum_{n} (-1)^n a_n \text{ converges if:}$$

$$0 \leqslant a_{n+1} \leqslant a_n$$
$$\lim_{n\to\infty} a_n = 0$$

Proof:

$$\text{Let } b_n = (-1)^n$$
$$\text{Then } \{B_n = b_k + \cdots + b_n\}_n \text{ is bounded}$$
$$\text{because it is a sequence of } 0s \text{ and } 1s$$
$$\text{and hence } |B_n| \leqslant 1 \text{ for all } n$$

Therefore, the hypotheses of the Abel test are satisfied (2 are automatically satisfied by the test itself, we only showed the B_n condition), and thus we are done with the proof.

November 9th, 2020

9 Chapt. Sequence of Functions

Def.: Take $f_n: A \longrightarrow \mathbb{R}$. The sequence of functions, $(f_n)_n$, converges pointwise to a function, f, if we have the following:

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in A$$

In this case, we say that $(f_n)_n$ is pointwise convergent.

Exp: We take the following:

$$f_n\colon [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto f_n(x) = 1 - \frac{x^2}{n}, n \geqslant 1$$
$$\text{For every } x \in [0,1]\text{:}$$
$$\lim_{n \to \infty} \left(1 - \frac{x^2}{n}\right) = 1$$
$$\text{Hence } (f_n)_n \text{ converges pointwise to the constant function } 1$$

Exp: We have the following:

$$\text{Let } f_n\colon (-\infty, 1] \longrightarrow \mathbb{R}$$
$$x \longmapsto f_n(x) = \left(1 - \frac{n\,x}{n+1}\right)^{\frac{n}{2}}, n \geqslant 1$$

$$\lim_{n \to \infty} f_n(x) = \begin{cases} +\infty & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0,1] \end{cases}$$

$$\text{Hence } (f_n)_n \text{ converges pointwise on} [0,1] \text{ to:}$$
$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leqslant 1 \end{cases}$$

Def.: A sequence of functions $(f_n)_n$ converges uniformly to f if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ st } n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies } |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in A$$

Consequence: We can clearly see from the definition that uniform convergence implies pointwise convergence, but the other way around is not true. i.e. in general, the converse is not true.

Exp:

$$\text{Let } f_n\colon [-1,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto f_n(x) = \frac{x^n}{n}, n \geqslant 1$$

$$\text{Then } (f_n)_n \text{ converges uniformly to } f(x) = 0 \quad \forall x \in [-1,1]. \text{ In fact:}$$
$$0 \leqslant \left|\frac{x^n}{n}\right| \leqslant \frac{1}{n}, \text{ and by squeeze theorem, it goes to } 0$$

$$\text{Let } \varepsilon > 0. \text{ By } \textbf{CRLY6 Chapt1}, \exists \mathrm{N} \in \mathbb{N} \text{ st } \frac{1}{N} < \varepsilon$$
$$\text{Suppose } n \in \mathbb{N} \text{ and } n \geqslant N. \text{ Then:}$$
$$|f_n(x) - 0| = \left|\frac{x^n}{n}\right| \leqslant \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$

Hence, we have shown that our sequence of functions converges uniformly.

Exp:

$$\text{Let } g_n\colon [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto g_n(x) = x^n, n \geqslant 1$$

$$\text{If } x = 1, g_n(x) = 1 \text{ for all } n \geqslant 1$$
$$\text{Thus } \lim_{n\to\infty} g_n(1) = 1$$

$$\text{If } x \in [0,1), \text{then:}$$
$$\lim_{n\to\infty} g_n(x) = 0$$

$$\text{Hence } (g_n)_n \text{ converges pointwise to:}$$
$$g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

$$\text{We shall show that this convergence is not uniform.}$$
$$\text{To do so, we need to find } \varepsilon > 0 \text{ st}$$
$$\forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ and } n \geqslant N, \text{and } \exists x \in [0,1]$$
$$\text{st } |g_n(x) - g(x)| \geqslant \varepsilon$$

$$\text{Let } \varepsilon = \frac{1}{2}, \text{let } n \in \mathbb{N} \text{ and let } x = \frac{1}{\sqrt[n]{2}} \in (0,1)$$
$$\text{Then:}$$
$$|g_n(x) - g(x)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2} \geqslant \varepsilon$$

Therefore we cannot have uniform convergence, since we have proven the negation.

Theorem 1: *Cauchy's Criterion for Uniform Convergence*

$(f_n)_n$ converges uniformly iff $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $m, n \in \mathbb{N}$ and $m, n \geqslant N$ implies $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in A$.

Proof:

$$\Longrightarrow$$
$$\text{Suppose } (f_n)_n \text{ converges uniformly}$$
$$\text{to } f\colon A \longrightarrow \mathbb{R}$$

$$\text{Let } \varepsilon > 0. \text{ Then } \exists N \in \mathbb{N} \text{ st } r \in \mathbb{N} \text{ and } r \geqslant N \text{ implies}$$
$$|f_r(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in A$$
$$\text{Suppose } m, n \in \mathbb{N} \text{ and } m, n \geqslant N. \text{ Then:}$$
$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x)| \\ &\leqslant |f_m(x) - f(x)| + |f(x) - f_n(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$\Longleftarrow$

Let $y \in A$, random but fixed.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $m, n \in \mathbb{N}$ and $m, n \geqslant N$ implies:

$|f_n(y) - f_m(y)| < \varepsilon$

Thus the sequence of real numbers $(f_n(y))_n$ is a Cauchy sequence. By the Cauchy Completeness Theorem, $(f_n(y))_n$ converges.

Let $g(y) = \lim\limits_{n \to \infty} f_n(y)$

Let $\varepsilon > 0$. $\exists M \in \mathbb{N}$ st $m, n \in \mathbb{N}$ and $m, n \geqslant M$ implies:

$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}$ for all $x \in A$

Suppose $k \in \mathbb{N}$ and $k \geqslant M$. Let $z \in A$.

We know that $\lim\limits_{n \to \infty} f_n(z) = g(z)$ and hence $\exists P \in \mathbb{N}$ st $m \in \mathbb{N}$ and $m \geqslant P$ implies $|f_m(z) - g(z)| < \frac{\varepsilon}{2}$

Let $Q = \max\{M, P\}$. Then:

$$\begin{aligned}|f_k(z) - g(z)| &= |f_k(z) - f_Q(z) + f_Q(z) - g(z)| \\ &\leqslant |f_k(z) - f_Q(z)| + |f_Q(z) - g(z)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Theorem 2:

Suppose $(f_n)_n$ converges uniformly to f. If f_n is continuous for all $n \in \mathbb{N}$ (ie f_n is continuous at every point $c \in A$ for all $n \geqslant 1$), then f is continuous (ie f is continuous at every point in A).

Proof:

Let $\varepsilon > 0$

Since $(f_n)_n$ converges uniformly to f, $\exists N \in \mathbb{N}$ st $n \in \mathbb{N}$ and $n \geqslant N$ implies:

$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in A$

Let $c \in A$

Since f_N is continuous at c, $\exists \delta > 0$ st $x \in A$ and $|x - c| < \delta$ implies:

$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$

We shall show that f is continuous at c.

Suppose $x \in A$ and $|x - c| < \delta$. Then:

$$\begin{aligned}|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Rmk: If f_n is continuous for all n, and if $(f_n)_n$ converges pointwise to f but f itself is not continous, then we know that $(f_n)_n$ does not converge uniformly to f.

Theorem 3:

Let $[a,b]$ be a compact in $\mathbb{R}$, and let $(f_n)_n$ be a sequence of integrable functions on the compact, $[a,b]$. Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a function. If $(f_n)_n$ converges uniformly to f on $[a,b]$, then we say that f is integrable. In other words:

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{dx}=\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{dx}=\int_a^b f(x)\mathrm{dx}$$

November 11th, 2020

<u>Proof</u>: We shall prove the following

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{dx}=\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{dx}=\int_a^b f(x)\mathrm{dx}$$

We know that f_n is integrable for all n and $(f_n)_n$ converges uniformly to f.

$$\begin{aligned}
&\text{We first show that }\left(\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{dx}\right)_{n\geqslant 1}\\
&\text{is a Cauchy sequence}\\
&\text{Let }\varepsilon>0.\text{ Since }(f_n)_n\text{ converges uniformly to }f,\\
&\text{the Cauchy Criterion for Uniform Convergence}\\
&\Longrightarrow\exists N\in\mathbb{N}\text{ st }m,n\in\mathbb{N}\text{ and }m,n\geqslant N\text{ implies:}\\
&|f_n(x)-f_m(x)|<\frac{\varepsilon}{2(b-a)}\quad\forall x\in[a,b]\\
&\text{If }m,n\geqslant N,\text{ then:}\\
\left|\int_a^b f_n(x)\mathrm{dx}-\int_a^b f_m(x)\mathrm{dx}\right| &= \left|\int_a^b [f_n(x)-f_m(x)]\mathrm{dx}\right|\\
&\leqslant\int_a^b |f_n(x)-f_m(x)|\mathrm{dx}\\
&<\int_a^b \frac{\varepsilon}{2(b-a)}\mathrm{dx}=\frac{\varepsilon}{2(b-a)}(b-a)=\frac{\varepsilon}{2}\\
&\frac{\varepsilon}{2}<\varepsilon
\end{aligned}$$

$$\begin{aligned}
&\text{Since }\left(\int_a^b f_n(x)\mathrm{dx}\right)_n\text{ is Cauchy,}\\
&\text{The Cauchy Completeness Theorem implies that it converges.}
\end{aligned}$$

$$\text{Let } l=\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{dx}.\text{ We show that } l=\int_a^b f(x)\mathrm{dx}.$$

$$\begin{aligned}
&\text{Let }\eta>0.\text{ Since }(f_n)_n\text{ converges uniformly to }f,\\
&\exists M>0\text{ st }n\in\mathbb{N}\text{ and }n\geqslant M\text{ implies}\\
&|f_n(x)-f(x)|<\frac{\eta}{3(b-a)}\quad\forall x\in[a,b]
\end{aligned}$$

$$\begin{aligned}
&\text{Since } l=\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{dx},\ \exists K\in\mathbb{N}\text{ st }n\in\mathbb{N}\text{ and }n\geqslant K\text{ implies:}\\
&\left|\int_a^b f_n(x)\mathrm{dx}-l\right|<\frac{\eta}{3}
\end{aligned}$$

$$\text{Let } J = \max\{M, K\}, \text{because } f_J \text{ is integrable,}$$
$$\exists \delta > 0 \text{ st if } P \text{ is a partition of } [a,b] \text{ with } \|P\| < \delta$$
$$\text{and if } T \text{ is a representative set of } P, \text{then:}$$
$$\left| S(f_J, P, T) - \int_a^b f_J(x) \mathrm{dx} \right| < \frac{\eta}{3}$$

$$\text{Let } R \text{ be a partition of } [a,b] \text{ with } \|R\| < \delta$$
$$\text{and let } V = \{t_1, t_2, \ldots., t_n\} \text{ be a representative set of } R. \text{ Then:}$$
$$|S(f_J, R, V) - S(f, R, V)|$$
$$= \left| \sum_{i=1}^{n} (f_J(t_i) - f(t_i))(x_i - x_{i-1}) \right|$$
$$\leqslant \sum_{i=1}^{n} |f_J(t_i) - f(t_i)| \times |x_i - x_{i-1}|$$
$$< \sum_{i=1}^{n} \frac{\eta}{3(b-a)} |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{\eta}{3(b-a)} (x_i - x_{i-1})$$
$$= \frac{\eta}{3(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\eta}{3}$$

$$\text{Hence } |S(f, R, V) - l|$$
$$= \left| S(f, R, V) - S(f_J, R, V) + S(f_J, R, V) - \int_a^b f_J(x) + \int_a^b f_J(x) - l \right|$$
$$\leqslant |S(f, R, V) - S(f_J, R, V)| + \left| S(f_J, R, V) - \int_a^b f_J(x) \right| + \left| \int_a^b f_J(x) - l \right|$$
$$< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta$$

Hence we have shown that f is integrable and:

$$\int_a^b \lim_{n \to \infty} f_n(x) \mathrm{dx} = \int_a^b f(x) \mathrm{dx}$$

<u>Exp</u>: Let us take the following

$$f_n(x) = x^n \sin\left(\frac{1}{x^{n-1}}\right), \text{for } 0 < a \leqslant x \leqslant b < 1$$

This means that $[a, b]$ is a compact subset in $(0, 1)$. Then we clearly know that $(f_n)_n$ converges uniformly to the 0 function (constant). However,

$$f_n'(x) = n\, x^{n-1} \sin\left(\frac{1}{x^{n-1}}\right) - (n-1) \frac{x^n}{x^{n-2}} \cos\left(\frac{1}{x^{n-1}}\right)$$
$$= n\, x^{n-1} \sin\left(\frac{1}{x^{n-1}}\right) - (n-1) \cos\left(\frac{1}{x^{n-1}}\right)$$

Which has no limit as $n \longrightarrow +\infty$ for all $x \in (0, 1)$. This leads us into the next theorem, which is about the differentiability.

Theorem 4:

Let $(a,b)\subseteq\mathbb{R}$, and let $(f_n)_n$ be a sequence of functions on (a,b). Suppose f_n is differentiable for all n, and suppose that (f_n') converges uniformly. Finally, assume $\exists c\in(a,b)$ st the sequence of real numbers $(f_n(c))_n$ converges.

Then $\exists f\colon(a,b)\longrightarrow\mathbb{R}$ st f is differentiable, $(f_n)_n$ converges uniformly to f and f_n' converges uniformly to f'.

Proof:

We need the following trick:

$$\begin{aligned}&\text{Let } m,n\in\mathbb{N} \text{ and let } x\neq y\in(a,b)\\&\text{Applying the } \mathbf{MVT} \text{ to } f_n-f_m \text{ on the interval}\\&\text{of endpoints } x \text{ and } y \text{ implies the existence of}\\&d \text{ strictly between } x \text{ and } y \text{ st:}\\f_n'(d)-f_m'(d)=&\frac{[f_n(x)-f_m(x)]-[f_n(y)-f_m(y)]}{x-y}\end{aligned}$$

$$\begin{aligned}&\text{We now show that } (f_n)_n \text{ converges uniformly:}\\&\text{Ler } \varepsilon>0. \text{ Since } (f_n(c))_n \text{ converges, it is a Cauchy Sequence}\\&\text{Thus } \exists N\in\mathbb{N} \text{ st } m,n\in\mathbb{N} \text{ and } m,n\geqslant N \text{ implies:}\\&|f_m(c)-f_n(c)|<\frac{\varepsilon}{2}\\&\text{Since } (f_n')_n \text{ is uniformly convergent, } \mathbf{THM1} \text{ implies:}\\&\exists M\in\mathbb{N} \text{ st } n,m\in\mathbb{N} \text{ and } m,n\geqslant M \text{ implies:}\\&|f_m'(x)-f_n'(x)|<\frac{\varepsilon}{2(b-a)}\quad\forall x\in(a,b)\\&\text{Let } P=\max\{N,M\}. \text{ if } m,n\in\mathbb{N} \text{ and } m,n\geqslant P\\&\text{and } z\in(a,b), \text{ then we have two cases:}\end{aligned}$$

$$\begin{aligned}&1.\\&\text{if } z=c, \text{ then:}\\&|f_m(z)-f_n(z)|=|f_m(c)-f_n(c)|<\frac{\varepsilon}{2}<\\&2.\\&\text{if } z\neq c, \text{ then using our previous trick,}\\&\exists d \text{ strictly between } z \text{ and } c\text{st}\\f_n'(d)-f_m'(d)=&\frac{[f_n(z)-f_m(z)]-[f_n(c)-f_m(c)]}{z-c}\\&\text{Hence:}\\|f_n(z)-f_m(c)|=&|[f_n(z)-f_m(z)]-[f_n(c)-f_m(c)]+[f_n(c)-f_m(c)]|\\\leqslant&|[f_n(z)-f_m(z)]-[f_n(c)-f_m(c)]|+|[f_n(c)-f_m(c)]|\\=&|z-c|\times|f_n'(d)-f_m'(d)|+|[f_n(c)-f_m(c)]|\\<&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

Putting together the two cases, we conclude that $(f_n)_n$ converges uniformly on (a,b) using **THM1**, or the Cauchy Criterion for Uniform Convergence.

$$\begin{aligned}&\text{Let } f \text{ be this uniform limit.}\\&\text{Let } p \text{ be fixed but random in } (a,b)\\&\text{We will show that } \lim_{n\to\infty}f_n'(p)=f'(p)\\&\text{To do this, we use the following step:}\\&\text{For each } n\in\mathbb{N}, \text{ we define:}\end{aligned}$$

$$g_n(x) = \begin{cases} \frac{f_n(x) - f_n(p)}{x - p} & \text{if } x \neq p \\ f_n'(p) & \text{if } x = p \end{cases}$$

We shall show that $(g_n)_n$ is uniformly convergent on (a, b)

Let $\varepsilon > 0$. Since f_n' is uniformly convergent, **THM1** implies:

$\exists Q \in \mathbb{N}$ st $n, m \in M$ and $m, n \geqslant Q$ implies $|f_n'(x) - f_m'(x)| < \varepsilon$

Suppose $m, n \geqslant Q$. Let w be random in (a, b).

We have two cases here:

1.

if $w = p$, then:

$$|g_n(w) - g_m(w)| = |g_n(p) - g_m(p)| = |f_n'(p) - f_m'(p)| < \varepsilon$$

2.

if $w \neq p$, then:

$$\begin{aligned} |g_n(w) - g_m(w)| &= \left| \frac{f_n(w) - f_n(p)}{w - p} - \frac{f_m(w) - f_m(p)}{w - p} \right| \\ &= \left| \frac{[f_n(w) - f_m(w)] - [f_n(p) - f_m(p)]}{w - p} \right| \\ &= |f_n'(r) - f_m'(r)| \text{ for some } r \text{ strictly between } w \text{ and } p \\ &< \varepsilon \end{aligned}$$

Putting together both cases, we deduce that using **THM1**, $(g_n)_n$ converges uniformly on (a, b).

Let g be this uniform limit

Let $k \in \mathbb{N}$. Since f_k is differentiable at p, we have:

$$\lim_{x \to p} \frac{f_k(x) - f_k(p)}{x - p} = f_k'(p)$$

Therefore, g_k is continuous at p

$$\lim_{x \to p} g_k(x) = g_k(p)$$

Hence bt **THM2**, we deduce that g is continous at p

(because $(g_n)_n$ converges uniformly to g)

ie $\lim_{x \to p} g(x) = g(p)$

Since $(f_n')_n$ converges uniformly, $(f_n')_n$ converges pointwise

and thus $(f_n'(p))_n$ converges. Thus:

$$\begin{aligned} \lim_{x \to p} \frac{f(x) - f(p)}{x - p} &= \lim_{x \to p} \frac{\lim_{n \to \infty} (f_n(x) - f_n(p))}{x - p} \\ &= \lim_{x \to p} \left(\lim_{n \to \infty} \frac{f_n(x) - f_n(p)}{x - p} \right) \\ &= \lim_{x \to p} \left(\lim_{n \to \infty} g_n(x) \right) = \lim_{x \to p} g(x) = g(p) \\ &= \lim_{n \to \infty} g_n(p) = \lim_{n \to \infty} f_n'(p) \end{aligned}$$

We deduce that f is differentiable at any random point p of (a, b) and that $(f_n')_n$ converges pointwise to f'.

Since $(f_n')_n$ is uniformly convergent, and since uniform convergence implies pointwise convergence, we conclude that $(f_n')_n$ converges uniformly to f' by uniqueness of the limit.

November 16th, 2020

10 Chapt. Series of Functions

Def.:

1. Let $(f_n)_n$ be a sequence of functions from $A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. A series of functions from $A \longrightarrow \mathbb{R}$ is a formal sum $\sum_{n\geqslant 1} f_n = f_1 + f_2 + \cdots +$ where f_n is called a term of a series for each.

2. For each $n \in \mathbb{N}$, the n^{th} partial sum of the series $\sum_{n\geqslant 1} f_n$, denoted F_n, is defined by:

$$F_n = \sum_{i=1}^{n} f_i = f_1 + f_2 + \cdots + f_n$$

   The sequence $(F_n)_n$ is called the sequence of partial sums of the series $\sum_{n\geqslant 1} f_n$

3. The series $\sum_{n\geqslant 1} f_n$ converges pointwise if the sequence $(F_n)_n$ is pointwise convergent. If $(F_n)_n$ converges pointwise to a function $f: A \longrightarrow \mathbb{R}$, then we say the series $\sum_{n\geqslant 1} f_n$ is pointwise convergent to f and we write:

$$\sum_{n\geqslant 1} f_n = f$$

4. The series $\sum_{n\geqslant 1} f_n$ is uniformly convergent if $(F_n)_n$ converges uniformly. If $(F_n)_n$ convergers uniformly to $f: A \longrightarrow \mathbb{R}$, then we say that the series $\sum_{n\geqslant 1} f_n$ is uniformly convergent to f and we write:

$$\sum_{n\geqslant 1} f_n = f$$

Consequence:

1. If $\sum_{n\geqslant 1} f_n$ converges uniformly to f, then the sequence $(F_n)_n$ converges uniformly to f as well, and hence (by **CHAPT9**), $(F_n)_n$ is pointwise convergent to f. Therefore $\sum_{n\geqslant 1} f_n$ is also pointwise convergent to f.

2. If $\sum_{n\geqslant 1} f_n$ converges pointwise or uniformly to f, then we know that f is unique.

Exp: Consider the following series of functions

$$\sum_{n\geqslant 0} f_n(x) = \sum_{n\geqslant 0} x^n, \text{with } x \in \mathbb{R} = A$$

The n^{th} partial sum F_n is given by:

$$F_n(x) = \sum_{i=0}^{n} f_i(x) = \sum_{i=0}^{n} x^i = \begin{cases} \frac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1 \\ n + 1 & \text{if } x = 1 \end{cases}$$

$$\text{Clearly } \lim_{n\to\infty} f_n(x) = \begin{cases} \frac{1}{1 - x} & \text{if } |x| < 1 \\ \text{DNE} & \text{if } |x| \geqslant 1 \end{cases}$$

$$\text{Hence } (F_n)_n \text{ converges pointwise to } f(x) = \frac{1}{1 - x} \text{ on } (-1, 1)$$

For all $n \geqslant 0$ and all $x \in (-1, 1)$, we have:

$$|F_n(x) - f(x)| = \left| \frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x} \right| = \frac{|x|^{n+1}}{1 - x}$$

$$\text{Thus for each } n \geqslant 0, \text{ we see that } \lim_{x\to 1^-} |F_n(x) - f(x)| = +\infty$$

Hence $(F_n)_n$ does not converge uniformly to f on$(-1, 1)$

This is because the sequence of functions $(F_n)_n$ fails Cauchy's Criterion for Uniform Convergence. We can see that there exists some x for which the quantity $|F_n(x) - f(x)|$ is not less than some ε, which is the negation of the original statement for the uniform convergence.

$$\begin{aligned}
&\text{However, this convergence is uniform}\\
&\text{for every interval } [-r - r] \subset (-1, 1), \text{ where}\\
&0 < r < 1\\
&\text{In fact, for every } n \geqslant 0 \text{ and every } x \in [-r, r]\\
&|F_n(x) - f(x)| = \frac{|x|^{n+1}}{1-x} \leqslant \frac{r^{n+1}}{1-r}\\
&\text{Since } \lim_{n\to\infty} \frac{r^{n+1}}{1-r} = 0, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ st}\\
&n \in \mathbb{N} \text{ and } n \geqslant N \text{ implies:}\\
&|F_n(x) - f(x)| = \frac{r^{n+1}}{1-r} = \left|\frac{r^{n+1}}{1-r} - 0\right| < \varepsilon\\
&\text{for all } x \in [-r, r]
\end{aligned}$$

Therefore, $\sum_{n\geqslant 0} x^n$ converges uniformly to $f(x) = \frac{1}{1-x}$ over $[-r, r] \subset (-1, 1)$, where $0 < r < 1$. The idea is essentially to just stay away from the -1 and 1.

Theorem 1: *Cauchy Criterion for Uniform Convergence of Series of Functions*

A series $\sum_n f_n(x)$ convergences uniformly iff $\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ st } m, n \in \mathbb{N} \text{ and } m, n \geqslant N$ implies:

$$|f_m(x) + f_{m+1}(x) + \cdots + f_n(x)| < \varepsilon \text{ for all } x \in A$$

Proof:

$$\begin{aligned}
&\text{Apply } \textbf{THM1, CHPT9} \text{ to the sequence}\\
&\text{of partial sums } (F_n)_n, \text{ observing that:}\\
|F_n(x) - F_{m-1}(x)| &= \left|\sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{m-1} f_i(x)\right|\\
&= |f_m(x) + f_{m+1}(x) + \cdots + f_n(x)|
\end{aligned}$$

Corollary 2:

If $\sum_n f_n(x)$ converges uniformly on A, then:

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in A$$

This means that $(f_n)_n$ converges uniformly to the 0 function. If $(f_n)_n$ converges to 0, we cannot say that it is uniformly convergent. However, if it does not converge to 0, then we definitely know that it is not uniformly convergent.

Proof:

$$\begin{aligned}
&\text{Take } m = n \text{ in } \textbf{THM1}\\
&|f_n(x)| = |f_n(x) - 0|, \text{ and we are done}
\end{aligned}$$

Consequence: *Divergence test for series of functions*

1. If $(f_n)_n$ does not converge pointwise to the 0 function, then $\sum_n f_n(x)$ is not pointwise convergent.
2. If $(f_n)_n$ does not converge uniformly to the 0 function, then $\sum_n f_n(x)$ is not uniformly convergent.

Theorem 3: *Weierstrass M-Test*

Let $A \subseteq \mathbb{R}$ be a non-empty set and let $\sum_n f_n(x)$ be a series from $A \longrightarrow \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there exists M_n st:

$$|f_n(x)| \leqslant M_n \quad \text{for all } x \in A$$

Our M_n may depend on n, but it definitely does not depend on x. We are essentially saying that our f_n is bounded on A for every n.

If the series of real numbers $\sum_n M_n(x)$ is convergent, then our series of functions $\sum_n f_n(x)$ is uniformly convergent.

Proof:

$$\begin{aligned}
&\text{Let } (F_n)_{n\geqslant 1} \text{be the} \\
&\text{sequence of partial sums of} \sum_{n\geqslant 1} f_n(x) \\
&(S_n)_n \text{ be the} \\
&\text{sequence of partial sums of} \sum_{n\geqslant 1} M_n(x) \\
&\text{Let } \varepsilon > 0. \text{ Since} \sum_{n\geqslant 1} M_n(x) \text{ converges,} \\
&(S_n)_n \text{ is convergent, and hence the} \\
&\text{Cauchy Completeness Theorem implies that:} \\
&(S_n)_n \text{ is a Cauchy sequence.} \\
&\text{Thus } \exists N \in \mathbb{N} \text{ st } m, n \in \mathbb{N} \text{ and } m, n \geqslant N \text{ implies:} \\
&|S_n - S_m| < \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\text{Suppose } m, n \in \mathbb{N} \text{ and } m, n \geqslant N \\
&\text{let } x \in A, \text{ random but fixed.}
\end{aligned}$$

$$\begin{aligned}
&\text{If } n = m, \text{ then } |F_m(x) - F_n(x)| = |F_n(x) - F_n(x)| = 0 < \varepsilon \\
&\text{Without loss of generality, we may assume } n > m \geqslant N \\
&\text{Then:} \\
|F_n(x) - F_m(x)| &= \left|\sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{m} f_i(x)\right| \\
&= \left|\sum_{i=m+1}^{n} f_i(x)\right| \leqslant \sum_{i=m+1}^{n} |f_i(x)| \\
&= \sum_{i=m+1}^{n} M_i = \left|\sum_{i=m+1}^{n} M_i\right| = |S_n - S_m| < \varepsilon
\end{aligned}$$

It follows from the Cauchy Criterion for Uniform Convergence of sequences of functions that $(F_n)_n$ is uniformly convergent. Therefore, $\sum_n f_n$ converges uniformly.

Application of Weierstrass M-test:

Let $\sum_{n\geqslant 0} c_n(x-a)^n$ be a power series with radius of convergence R. Let us assume that $R>0$. This is to avoid the trivial case of the power series converging only for $x=a$.

If $P\in(0,R)$, then $\sum_{n\geqslant 0} c_n(x-a)^n$ is uniformly convergent on the compact $[a-P,a+P]$.

In fact, since $P\in(0,R)$, $a+P\in(a-R,a+R)$. Thus $\sum_{n\geqslant 0} c_n(x-a)^n$ is absolutely convergent for $x=a+P$. ie $\sum_{n\geqslant 0} c_n(a+P-a)^n=\sum_{n\geqslant 0} c_n(P)^n$ is absolutely convergent.

$$\begin{aligned}
\text{Hence} \sum_{n\geqslant 0} |c_n|P^n = \sum_{n\geqslant 0} |c_n|P^n \text{ converges}\\
\text{for every } n\in\mathbb{N}\cup\{0\} \text{ and for every } x\in[a-P,a+P],\\
\text{we have } |c_n(x-a)^n| = |c_n|\times|x-a|^n \leqslant |c_n|P^n\\
\text{Since} \sum_{n\geqslant 0} |c_n|P^n \text{ converges and } c_n(x-a)^n \text{ is bounded by it,}\\
\Longrightarrow \sum_{n\geqslant 0} c_n(x-a)^n \text{ converges uniformly on } [a-P,a+P]\\
\text{by the Weierstrass M-test}
\end{aligned}$$

Theorem 4:

Assume $\sum_n f_n$ converges uniformly to a function f on a non-empty subset $A\subseteq\mathbb{R}$. If for every n, f_n is continuous on A, then f is continuous on A.

Proof:

$$\begin{aligned}
\text{Let } (F_n)_n \text{ be}\\
\text{the sequence of partial sums of} \sum_n f_n\\
\text{Since} \sum_n f_n \text{ is uniformly convergent to } f,\\
(F_n)_n \text{ is uniformly convergent to } f\\
\text{For each } n, F_n \text{ is a continuous function}\\
\text{as a finite sum of a continuous function.}
\end{aligned}$$

Hence by **THM2, CHPT9**, f is continuous on A. We are done with the proof.

Theorem 5:

Let $[a,b]\subseteq\mathbb{R}$ be a compact. Assume the series $\sum_n f_n$ converges uniformly to f on $[a,b]$. If f_n is integrable on $[a,b]$ for all n, then f is also integrable on $[a,b]$ and:

$$\int_a^b f(x)\mathrm{dx} = \sum_n \left(\int_a^b f_n(x)\mathrm{dx}\right)$$

This is the same as integrating term by term, as we have gotten used to in Calculus. In other words, we are saying that:

$$\int_a^b \left(\sum_n f_n(x)\right)\mathrm{dx} = \sum_n \left(\int_a^b f_n(x)\mathrm{dx}\right)$$

We can interchange the sum and the integral.

Proof:

In the proof of **THM3, CHPT9**, replace the sequence of functions f_n by the sequence of partial sums $(F_n)_n$ of our series $\sum_n f_n$. We do not change anything else. Everything stays the same except the cahnge of f_n to $(F_n)_n$.

Exp:

We have seen in the first example of this chapter that $\sum_{n\geqslant 0} x^n$ converges uniformly to the function:

$$f(x) = \frac{1}{1-x} \text{ on every compact } [a,b] \subseteq (-1,1)$$

Hence:

$$\begin{aligned}\int_a^b \frac{1}{1-x}\mathrm{dx} &= \sum_{n\geqslant 0}\left(\int_a^b x^n \mathrm{dx}\right)\\ [-\ln(1-x)]_a^b &= \left[\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right]_a^b\\ = -\ln(1-b)+\ln(1-a) &= \sum_{n\geqslant 0}\left(\frac{b^{n+1}}{n+1}-\frac{a^{n+1}}{n+1}\right)\end{aligned}$$

In particular, if we take $b=0, a=x\in(-1,1)$, we obtain:

$$\begin{aligned}\ln(1-x) &= \sum_{n\geqslant 0} -\frac{x^{n+1}}{n+1}\\ &= -\sum_{n\geqslant 0}\frac{x^{n+1}}{n+1}\end{aligned}$$

This is simply the Maclaurin series for the function $\ln(1-x)$.

Theorem 6:

Let $(a,b)\subseteq\mathbb{R}$ be a non-empty open interval, and let $\sum_n f_n$ be a series of functions on (a,b). Suppose that f_n is differentiable on (a,b) for all n, $\sum_n f_n'$ is uniformly convergent and that $\exists c\in(a,b)$ st the series of real numbers

$$\sum_n f_n(c) \text{ converges.}$$

Then $\exists f\colon(a,b)\longrightarrow\mathbb{R}$ st f is differentiable on (a,b), $\sum_n f_n$ converges uniformly to f on (a,b) and $\sum_n f_n'$ converges uniformly to f'.

In other words, under the given hypotheses, we have:

$$\left(\sum_n f_n(x)\right)' = f'(x) = \sum_n f_n'(x)$$

Proof:

We once again follow the same proof as in the previous chapter, but we replace the sequence of functions $(f_n)_n$ be the sequence of partial sums $(F_n)_n$ of our series $\sum_n f_n$. This is all the changes we will make for this.

Exp:

$$\text{Consider the series} \sum_{n\geqslant 1} f_n(x) = \sum_{n\geqslant 1} (-1)^n \frac{1}{n} \cos\left(\frac{x}{n}\right)$$

$$\text{for } x \in (-r, r) \text{ where } r > 0$$

$$\text{For each } n \geqslant 1,\ f_n \text{ is differentiable on } (-r, r)$$

$$\text{For } x = 0\ (c \text{ playing role of } c)\text{, we have:}$$

$$\sum_{n\geqslant 1} f_n(0) = \sum_{n\geqslant 0} \frac{(-1)^n}{n} \text{ which is convergent.}$$

$$\text{For each } n \geqslant 1\text{, we have:}$$

$$f_n'(x) = \frac{(-1)^n}{n} \cdot \frac{-1}{n} \cdot \sin\left(\frac{x}{n}\right)$$

$$= (-1)^{n+1} \cdot \frac{1}{n^2} \cdot \sin\left(\frac{x}{n}\right)$$

$$\text{Thus for all } x \in (-r, r)\text{, we have:}$$

$$|f_n(x)| = \frac{\left|\sin\left(\frac{x}{n}\right)\right|}{n^2} \leqslant \frac{1}{n^2}$$

$$\text{Since} \sum_{n\geqslant 1} \frac{1}{n^2} \text{ converges, then by the Weierstrtass M-test,}$$

$$\text{The series} \sum_{n\geqslant 1} f_n' \text{ converges uniformly convergent on } (-r, r)$$

$$\text{Therefore:}$$

$$\left(\sum_{n\geqslant 1} (-1)^n \cdot \left(\frac{1}{n}\right) \cdot \cos\left(\frac{x}{n}\right)\right)' = \sum_{n\geqslant 1} (-1)^{n+1} \cdot \frac{1}{n^2} \cdot \sin\left(\frac{x}{n}\right)$$

Exp:

$$\text{Consider the series} \sum_{n\geqslant 1} \cos\left(\frac{x}{n}\right) \text{ for } x \in (-r, r), r > 0$$

$$\text{Differentiating term by term yields:}$$

$$\sum_{n\geqslant 1} \left(-\frac{1}{n}\right) \sin\left(\frac{x}{n}\right)$$

$$\text{Now } \left|-\frac{1}{n} \sin\left(\frac{x}{n}\right)\right| \leqslant \frac{1}{n} \frac{|x|}{n} = \frac{|x|}{n^2} \leqslant \frac{r}{n^2}$$

$$\text{Since} \sum_{n\geqslant 1} \frac{r}{n^2} \text{ converges, the Weierstrass M-test implies:}$$

$$\sum_{n\geqslant 1} \left(-\frac{1}{n}\right) \sin\left(\frac{x}{n}\right) \text{ converges uniformly}$$

However, we cannot conclude that the series $\sum_{n\geqslant 1} \cos\left(\frac{x}{n}\right)$ is uniformly convergent on $(-r, r)$, because in fact it is divergent since $\lim_{n\to\infty} \cos\left(\frac{x}{n}\right) = 1 \neq 0$ for all $x \in (-r, r)$.

November 23rd, 2020

11 Chapt. Weierstrass Approximation Theorem

Theorem: *Weierstrass Approximation Theorem*

Let $[a,b] \subseteq \mathbb{R}$ be a compact. Let f be a continuous function on $[a,b]$.
Then for each $\varepsilon > 0$, $\exists$ some polynomial, P_ε st

$$|f(x) - P_\varepsilon(x)| < \varepsilon \text{ for all } x \in [a,b]$$

Equivalently, we are saying there exists a sequence $(P_n)_n$ of polynomials that converges uniformly to f on $[a,b]$.

Rmk: f needs to be continuous on $[a,b]$, it does not have to be differentiable, etc. Continuity is the only condition that needs to be satisfied.

Bernstein Polynomial:

Let $f:[0,1] \longrightarrow \mathbb{R}$ be a function. The n^{th} Bernstein polynomial (of order or degree n) is defined by:

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Where we know already that: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Properties:

1.
$$B_n(1)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x+1-x)^n = 1$$

2.
$$\begin{aligned} B_n(x)(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n-1}{k} x^{k+1} (1-x)^{n-(k+1)} \\ &= x \sum_{k=0}^{n} \binom{n-1}{k} x^k (1-x)^{n-k} \\ &= x \cdot 1 = x \end{aligned}$$

We can conclude from 1 and 2 that:

$$\begin{aligned} B_n(a\,x+b) &= \sum_{k=0}^{n} \binom{n}{k} \left(a\frac{k}{n} + b\right) x^k (1-x)^{n-k} \\ &= a \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k} + b \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= a\,x + b \end{aligned}$$

In other words, Bernstein's polynomials reproduce polynomials of degree less or equal to 1. However, they do not produce polynomials greater or equal to 2.

$$\begin{aligned} B_n(x^2)(x) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k (1-x^{n-k}) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} (k+1)\, x^{k+1} (1-x)^{n-1-k} \\ &= \frac{x}{n} \left[\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} k\, x^k (1-x)^{n-1-k} + \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} x^k (1-x)^{n-1-k} \right] \end{aligned}$$

$$\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} x^k (1-x)^{n-1-k} = (x+1-x)^{n-1} = 1$$

$$= \frac{x}{n}\left[1 + \sum_{k=1}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} k\, x^k (1-x)^{n-1-k}\right]$$

$$= \frac{x}{n}\left[1 + \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-1-k)!} x^k (1-x)^{n-1-k}\right]$$

$$= \frac{x}{n}\left[1 + \sum_{k=0}^{n-2} \frac{(n-1)!}{k!(n-2-k)!} x^{k+1} (1-x)^{n-2-k}\right]$$

$$= \frac{x}{n}\left[1 + (n-1)\, x \sum_{k=0}^{n-2} \frac{(n-2)!}{k!(n-2-k)!} x^k (1-x)^{n-2-k}\right]$$

$$\sum_{k=0}^{n-2} \frac{(n-2)!}{k!(n-2-k)!} x^k (1-x)^{n-2-k} = 1, \text{ and so we get:}$$

$$= \frac{x}{n}[1 + (n-1)x] = \frac{n-1}{n} x^2 + \frac{1}{n} x$$

Bernstein Theorem:

If f is continuous on $[0,1]$ compact, then:

$$(B_n(f))_n \text{ converges uniformly to } f \text{ on } [0,1]$$

This means that $\forall \mathrm{E} > 0, \exists N_\varepsilon$ independent of x st $n \in \mathbb{N}$ and $n \geqslant N_\varepsilon$ implies $|f(x) - B_n(f)(x)| < \varepsilon$ for all $x \in [0,1]$.

Proof:

$$\text{Observe that:}$$
$$f(x) = f(x)(x+1-x)^n$$
$$= f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} f(x)\, x^k (1-x)^{n-k}. \text{ Thus we have:}$$
$$|f(x) - B_n(f)(x)|$$
$$= \left|\sum_{k=0}^{n} \binom{n}{k} f(x) x^k (1-x)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}\right|$$
$$= \left|\sum_{k=0}^{n} \binom{n}{k} \left[f(x) - f\left(\frac{k}{n}\right)\right] x^k (1-x)^{n-k}\right|$$
$$\leqslant \sum_{k=0}^{n} \binom{n}{k} \left|f(x) - f\left(\frac{k}{n}\right)\right| x^k (1-x)^{n-k}$$

Let $\varepsilon > 0$. Since f is continuous on $[0,1]$, f is uniformly continuous on $[0,1]$, and so $\exists \delta > 0$ st $x, y \in [0,1]$ and $|x - y| < \delta$ implies: $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Now we can see that:

$$|f(x) - B_n(f)(x)| \leqslant \sum_{\left|\frac{k}{n} - x\right| \geqslant \delta} \binom{n}{k} \left|f(x) - f\left(\frac{k}{n}\right)\right| x^k (1-x)^{n-1}$$

$$+\sum_{\left|\frac{k}{n}-x\right|<\delta}\binom{n}{k}\left|f(x)-f\left(\frac{k}{n}\right)\right|x^k(1-x)^{n-k}$$

$$<\frac{\varepsilon}{2}\sum_{\left|\frac{k}{n}-x\right|<\delta}\binom{n}{k}x^k(1-x)^{n-k}+\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}\left|f(x)-f\left(\frac{k}{n}\right)\right|x^k(1-x)^{n-1}$$

$$<\frac{\varepsilon}{2}(1)+\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}\left|f(x)-f\left(\frac{k}{n}\right)\right|x^k(1-x)^{n-k}$$

$$\text{Now if}\left|\frac{k}{n}-x\right|\geqslant\delta,\left(\frac{k}{n}-x\right)^2\geqslant\delta^2$$

$$\text{ie}\quad\frac{\left(\frac{k}{n}-x\right)^2}{\delta^2}\geqslant 1.\text{ Thus we have:}$$

$$\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}\left|f(x)-f\left(\frac{k}{n}\right)\right|x^k(1-x)^{n-k}$$

$$\leqslant\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}\left|f(x)-f\left(\frac{k}{n}\right)\right|\frac{\left(\frac{k}{n}-x\right)^2}{\delta^2}x^k(1-x)^{n-k}$$

$$\leqslant\frac{2M}{\delta^2}\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^k(1-x)^{n-k}$$

Where M is the maximum of f, attained on $[0,1]$

$$=\frac{2M}{\delta^2}\sum_{k=0}^{n}\binom{n}{k}\left[\frac{k^2}{n^2}-2\frac{k}{n}x+x^2\right]x^k(1-x)^{n-k}$$

$$=\frac{2M}{\delta^2}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{k}{n}x^k(1-x)^{n-k}-2x\sum_{k=0}^{n}\binom{n}{k}\frac{k}{n}x^k(1-x)^{n-k}+x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)$$

$$=\frac{2M}{\delta^2}(B_n(x^2)(x)-2x\,B_n(x)(x)+x^2)=\frac{2M}{\delta^2}\left(\frac{n-1}{n}x^2+\frac{1}{n}x-2x^2+x^2\right)$$

$$=\frac{2M}{\delta^2}\left(\frac{n-1}{n}x^2+\frac{1}{n}x-x^2\right)\leqslant\frac{2M}{\delta^2}\times\frac{x(1-x)}{n}\longleftarrow\text{because }x\in[0,1]$$

$$\leqslant\frac{2M}{n\,\delta^2}\longleftarrow\text{because }x\in[0,1]$$

Hence we can see that $|f(x)-B_n(f)(x)|<\frac{\varepsilon}{2}+\frac{2M}{n\delta^2}$. We choose our N_ε st when $n\geqslant N_\varepsilon$, we have:

$$\frac{2M}{\delta^2 n}\leqslant\frac{2M}{N_\varepsilon\delta^2}<\frac{\varepsilon}{2}\text{ ie }N_\varepsilon>\frac{4M}{\varepsilon\,\delta^2}$$

$$\text{For example, if we take }N_\varepsilon\geqslant\left[\!\left[\frac{4M}{\varepsilon\,\delta^2}\right]\!\right]+1.\text{ In this case:}$$

$$|f(x)-B_n(f)(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Rmk: The theorem remains true if f is continuous on $[a,b]$. In fact, define $h\colon[a,b]\longrightarrow[0,1]$, with $x\longmapsto h(x)=\frac{x-a}{b-a}$. h is continuous, intvertible with h^{-1} being well-defined.

Now:

$$f\circ h^{-1}\colon[0,1]\longrightarrow\mathbb{R}\text{ is continuous.}$$

Thus $\exists (p_n)_n$ st $(p_n)_n$ converges uniformly to $f \circ h^{-1}$ on $[0, 1]$. Finally, $q_n = p_n \circ h$ converges uniformly to f itself on $[a, b]$.