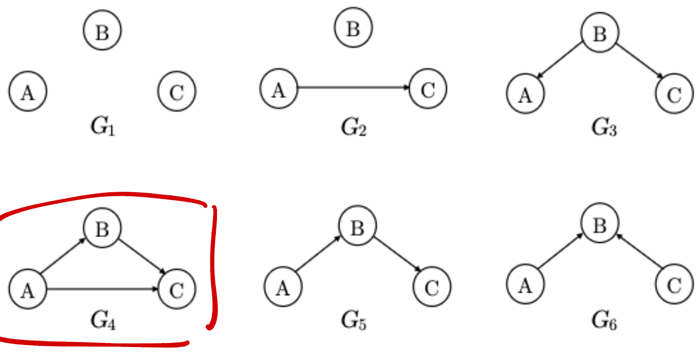
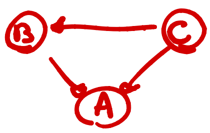
Question 1:

a. Given random variables A,B,C, assume all we know about the joint can be represented by the product P(A|B,C)P(B|C)P(C). Which of the six graphs are guaranteed to be able to represent P(A,B,C)?

P(A|B,C)
P(B|C)
P(C)



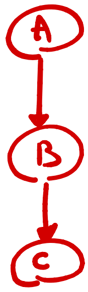


This is correct.

But the arrows are pointing the wrong way?

b. Now assume all we know about the joint distribution P(A,B,C) is that it can be represented by the product P(C|B)P(B|A)P(A). Which of the six graphs are guaranteed to be able to represent P(A,B,C)?

P(A)
P(B|A)
P(C|B)

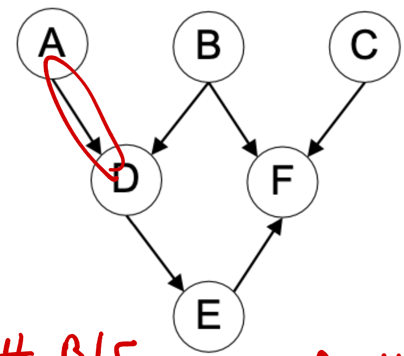


So it's G5.

Apparently it's G3 & G4 also.

How?

Question 2: For the graphs below, what is the minimal set of edges that must be removed such that the corresponding independence relations are guaranteed to be true?



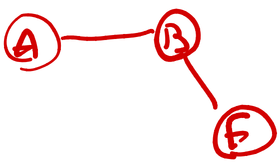
cannot be determined.

$A \perp\!\!\!\perp B|F$ — no need to remove anything.

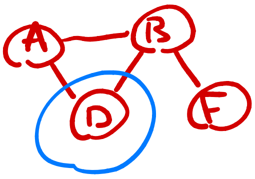
$A \perp\!\!\!\perp F|D$ — remove edge AD

$B \perp\!\!\!\perp C$ — no need to remove anything. Already disconnected.

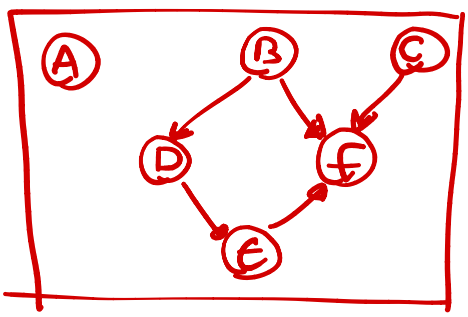
A ⫫ B|F



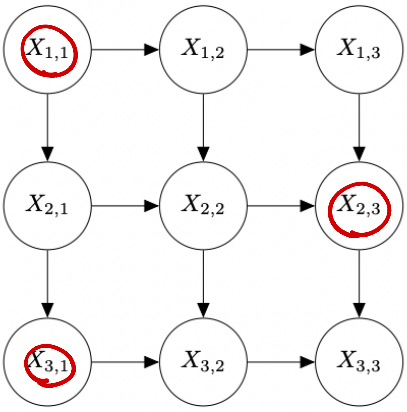
A ⫫ F|D



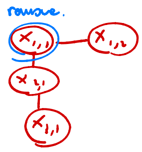
remove.



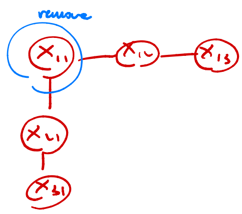
Question 3. Consider the Bayes' Net below with 9 variables. Tick all the true statements below.



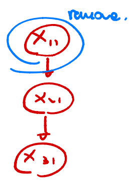
$X_{3,1} \perp\!\!\!\perp X_{1,2} \mid X_{1,1}$



$X_{3,1} \perp\!\!\!\perp X_{1,3} \mid X_{1,1}$



$X_{3,1} \perp\!\!\!\perp X_{2,1} \mid X_{1,1}$



a. $X_{3,1}$ is independent of $X_{1,2}$ given $X_{1,1}$ ☑

b. $X_{3,1}$ is independent of $X_{2,1}$ given $X_{1,1}$ ☒

c. $X_{3,1}$ is independent of $X_{1,3}$ given $X_{1,1}$ ☑

d. $p(X_{2,3}|X_{1,1}, X_{3,1}) = p(X_{2,3}|X_{1,1})$ ☒

$P(X_{23}|X_{11} X_{31})$

$= P(X_{23}|X_{11})$

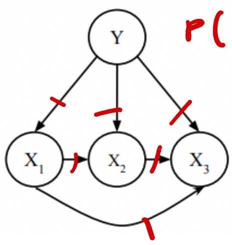
Check: $X_{31} \perp\!\!\!\perp X_{23} \mid X_{11}$

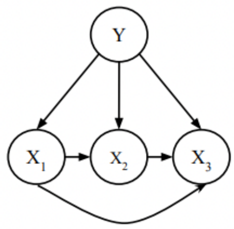
Not guaranteed.

Question 4: Consider the following Bayesian Network where all variables are binary. You want to know how much memory is required to store the Bayes Net on disk. The amount of memory depends on the number of values you would need to store in the conditional probability tables for that Bayes Net. Give the least number of parameters you would need to store to completely specify the Bayes Net.

$P(Y)\,P(X_1|Y)P(X_2|X_1,Y)$

$P(X_3|X_1,X_2,Y)$





Question 5
We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 30%, 60%, and 75%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X1, X2, and X3.

a) Draw the Bayesian network corresponding to this setup and define the necessary CPTs.

b) Calculate which coin was most likely to have been drawn from the bag if the observed flips come out [X1=tail, X2=heads, X3=head]

Question 4:

$$P(Y)\,P(X_1|Y)\,P(X_2|X_1,Y)\,P(X_3|X_1,X_2,Y)$$

Y	$P(Y)$
0	
1	

$2^1 = 2$

X_1	Y	$P(X_1\|Y)$
0	0	
0	1	
1	0	
1	1	

$2^2 = 4$

X_2	X_1	Y	$P(X_2\| X_1, Y)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

$2^3 = 8$

X_3	X_1	X_2	Y	$P(X_3\|X_1,X_2,Y)$

$2^4 = 16$

$2 + 4 + 8 = 14$

$Y = 1$

$X_1 = 2$

$X_2 = 4$

$X_3 = 8$

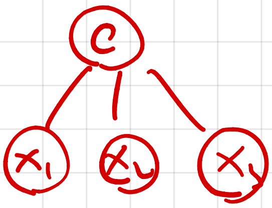
$2^0 + 2^1 + 2^2 + 2^3$

$= 1 + 2 + 4 + 8 = \boxed{15}$

$$P(X_1 = T, X_2 = H, X_3 = T)$$

$$= P(c)\,P(X_1 | c)\,P(X_2 | c)\,P(X_3 | c)$$

$$= \propto \begin{bmatrix} P(a) & P(x_1 = T | a)\,P(x_2 = H | a)\,P(X_3 = H | a) \\ P(b) & P(x_1 = T | b)\,P(x_2 = H | b)\,P(X_3 = H | b) \\ \underbrace{P(c)}_{1/3} & P(x_1 = T | c)\,P(x_2 = H | c)\,P(X_3 = H | c) \end{bmatrix}$$

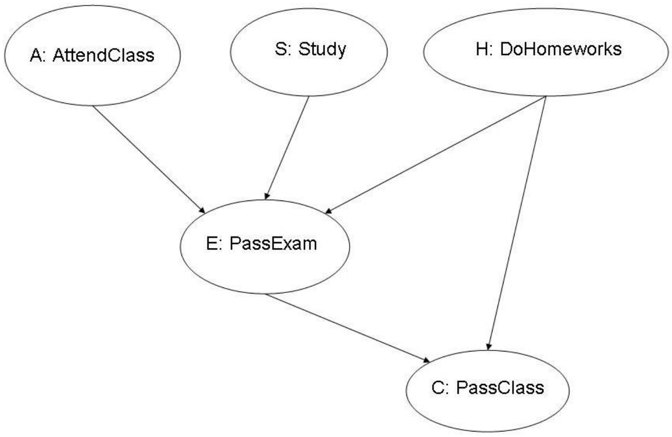


c	P(c)
X_1	1/3
X_2	1/3
X_3	1/3

c	X_1	$P(X_1 \mid c)$
a	H	0.3
b	H	0.6
c	H	0.75

$$\propto \begin{bmatrix} \frac{1}{3}(0.7)(0.3)(0.3) \\ \frac{1}{3}(0.4)(0.6)(0.6) \\ \frac{1}{3}(0.25)(0.75)(0.75) \end{bmatrix} = \propto \begin{bmatrix} 0.021 \\ 0.048 \\ 0.046 \end{bmatrix} \rightarrow \text{normalize this.}$$

Question 6



1. Write down the expression for the joint distribution as it factorizes according to the Bayes Net below.

$$P(A)\,P(S)\,P(H)\,P(E \mid A,S,H)\,P(C \mid E,H)$$

2. Write down the expression for the probability of passing the class, given that you attend class and study but don't do the homeworks.

$P(+c \mid +a,+s,-h)$ — E is not observed.

$$P(C \mid +a,+s,-h) = \alpha \sum_{E} P(C,+a,+s,-h,e)$$

$$= \alpha \left[P(C,+a,+s,-h,+e) + P(C,+a,+s,-h,-e) \right]$$

$$\alpha \left[\begin{array}{l} \quad P(+a)P(+s)P(-h)P(+e \mid +a,+s,-h)\,P(+c \mid +e,-h) \\ + P(+a)P(+s)P(-h)P(-e \mid +a,+s,-h)\,P(+c \mid -e,-h) \\ \quad P(+a)P(+s)P(-h)P(+e \mid +a,+s,-h)\,P(-c \mid +e,-h) \\ + P(+a)P(+s)P(-h)P(-e \mid +a,+s,-h)\,P(-c \mid -e,-h) \end{array} \right]$$

Question 7:

$$X \to Y \to Z$$

X	$\mathbf{P}(X)$
$+x$	$2/5$
$-x$	$3/5$

Y	X	$\mathbf{P}(Y \mid X)$
$+y$	$+x$	2/3
$-y$	$+x$	1/3
$+y$	$-x$	3/4
$-y$	$-x$	1/4

Z	Y	$\mathbf{P}(Z \mid Y)$
$+z$	$+y$	1/3
$-z$	$+y$	2/3
$+z$	$-y$	1/5
$-z$	$-y$	4/5

(a) Rejection Sampling

Your task is to estimate $P(+y \mid +x, +z)$ using rejection sampling. Below are some samples that have been produced by prior sampling (i.e., the rejection stage in rejection sampling hasn't happened yet). Which of the following samples would be rejected by rejection sampling?

1. $+x, +y, +z$ keep
2. $-x, +y, +z$ reject
3. $-x, -y, +z$ reject
4. $+x, -y, -z$ reject
5. $+x, -y, +z$ keep.

$\Big\}$ keep 1,5.

(b) Estimating $P(+y \mid +x, +z)$ Using Rejection Sampling

Using rejection sampling, provide an estimate of $P(+y \mid +x, +z)$ from the five prior samples, or explain why it cannot be computed. 0.5 one case +y, one case -y.

(c) Estimating $P(+y \mid +x, +z)$ Using Likelihood Weighting

Using the following samples (which were generated using likelihood weighting), estimate $P(+y \mid +x, +z)$ using likelihood weighting, or explain why it cannot be computed.

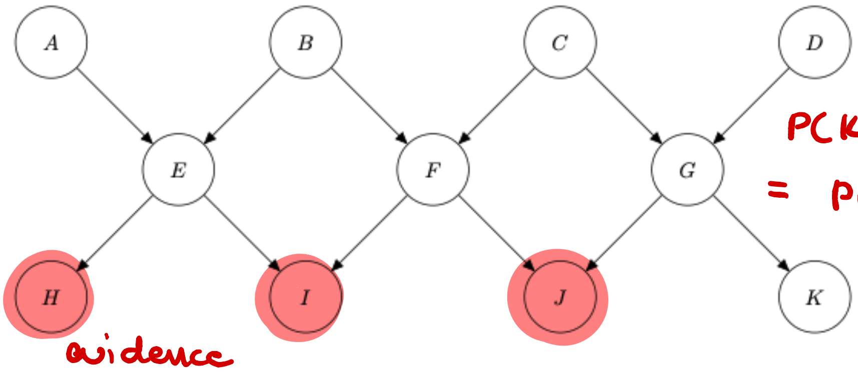
1. $+x, +y, +z$ $\rightarrow w_1 = 2/5 \cdot 1/3 = 2/15$
2. $+x, -y, +z$ $\rightarrow w_2 = 2/5 \cdot 1/5 - 2/25$
3. $+x, +y, +z$ $\rightarrow w_3 = 2/15$

2/3 true.

$$P(+y \mid +x, +z) = \frac{w_1 + w_2}{w_1 + w_2 + w_3} = 0.615$$

Question 8:

We are running MCMC sampling in the Bayes net shown below for the query P(B,C|+ h,+i,+j). The current state is +a,+b,+c,+d,+e,+f,+g,+h,+i,+j,+k.Write out an expression for the MCMC sampling distribution for each of A, F, and K in terms of conditional probabilities available in the network.



$MB(K) = G$

$P(K \mid MB(K))$
$= P(K \mid G)$
$= \alpha P(K \mid +g)$

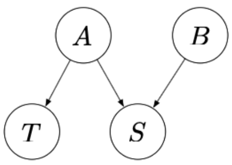
$P(A \mid MB(A)) = \alpha P(A) \cdot P(E \mid A, B)$
$= \alpha P(A) \cdot P(+e \mid A, +b)$

$MB(F) = B, C, I, J, E, G$ $= \alpha P(F \mid B, C) P(I \mid E, F) P(J \mid F, G)$
$= \alpha P(F \mid +b, +c) P(+i \mid +e, F) P(+j \mid F, +g)$

Question 9:

Suppose that a patient can have a symptom (S) that can be caused by two different diseases (A and B). Disease A is much rarer, but there is a test T that tests for the presence of A. The Bayes net and corresponding conditional probability tables for this situation are shown below

$\mathbf{P}(A)$	
$+a$	0.1
$-a$	0.9



$\mathbf{P}(B)$	
$+b$	0.5
$-b$	0.5

$\mathbf{P}(T \mid A)$		
$+a$	$+t$	1
$+a$	$-t$	0
$-a$	$+t$	0.2
$-a$	$-t$	0.8

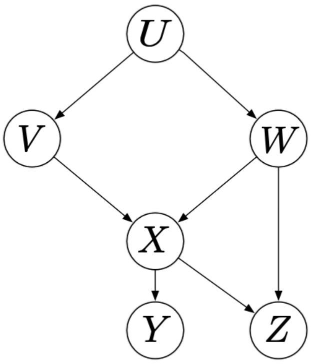
$\mathbf{P}(S \mid A, B)$			
$+a$	$+b$	$+s$	1
$+a$	$+b$	$-s$	0
$+a$	$-b$	$+s$	0.80
$+a$	$-b$	$-s$	0.20
$-a$	$+b$	$+s$	1
$-a$	$+b$	$-s$	0
$-a$	$-b$	$+s$	0
$-a$	$-b$	$-s$	1

a. Compute the entry $P(-a, -t, +b, +s)$ from the joint distribution.
b. What is the probability that a patient has disease A given that they have disease B?
c. What is the probability that a patient has disease A given that they have symptom S, disease B, and test T returns positive?

d. What is the probability that a patient has disease A given that they have symptom S and test T returns positive?

e. Suppose that both diseases A and B become more likely as a person ages. Add any necessary variables and/or arcs to the Bayes net to represent this change. Also, state one independence or conditional independence assertion that is removed due to your changes.

f. Based only on the structure of the Bayes net below, decide whether the following conditional independence assertions are guaranteed to be true, guaranteed to be false, or cannot be determined by the structure alone.



(i) $V \perp\!\!\!\perp W$
(ii) $V \perp\!\!\!\perp W \mid U$
(iii) $V \perp\!\!\!\perp W \mid U, Y$
(iv) $V \perp\!\!\!\perp Z \mid U, X$
(v) $X \perp\!\!\!\perp Z \mid W$

Answers:

(a) $P(-a, -t, +b, +s)$

$$P(A) \cdot P(T \mid A) \cdot P(B) \cdot P(S \mid A, B)$$

$$\underbrace{P(-a)}_{0.9}\ \underbrace{P(-t \mid -a)}_{0.8}\ \underbrace{P(+b)}_{0.5}\ \underbrace{P(+s \mid -a, +b)}_{1}$$

$$= \underline{\underline{0.36}}$$

(b) $P(A \mid B) = P(A)$

$$= P(+a)$$

$$= 0.1$$

(c) $P(A \mid S, T, B)$

$$P(A)\, P(T \mid A)\, P(S \mid A, B)\, P(B)$$

$$= P(A)\, P(+t \mid A)\, P(+s \mid A, +b)\, P(+b)$$

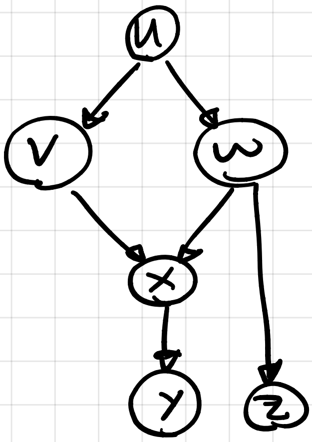
$$\propto \begin{bmatrix} \overbrace{P(+a)}^{0.1}\ \overbrace{P(+t \mid +a)}^{1}\ \overbrace{P(+s \mid +a, +b)}^{1}\ \overbrace{P(+b)}^{0.5} \\ \underbrace{P(-a)}_{0.9}\ \underbrace{P(+t \mid -a)}_{0.2}\ \underbrace{P(+s \mid -a, +b)}_{1}\ \underbrace{P(+b)}_{0.5} \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1/20 \\ 0.09 \end{bmatrix} = \begin{bmatrix} 0.357 \\ 0.643 \end{bmatrix}$$

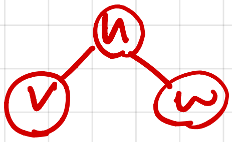
(d) $P(A \mid +s, +t)$

$$\alpha \begin{bmatrix} \overbrace{P(+a)}^{0.1}\ \overbrace{P(+t \mid +a)}^{1}\ \overbrace{P(+s \mid +a, +b)}^{1}\ \overbrace{P(+b)}^{0.5} + \overbrace{P(+a)}^{0.1}\ \overbrace{P(+t \mid +a)}^{1}\ \overbrace{P(+s \mid +a, -b)}^{0.8}\ \overbrace{P(-b)}^{0.5} \\ \underbrace{P(-a)}_{0.9}\ \underbrace{P(+t \mid -a)}_{0.2}\ \underbrace{P(+s \mid -a, +b)}_{1}\ \underbrace{P(+b)}_{0.5} + \underbrace{\underbrace{P(-a)}_{0.9}\ P(+t \mid -a)\ P(+s \mid -a, -b)\ \underbrace{P(-b)}_{0.5}}_{c} \end{bmatrix}$$

$$\alpha \begin{bmatrix} 1/20 + \frac{1}{25} \\ 9/100 \end{bmatrix} = \alpha \begin{bmatrix} 0.09 \\ 0.09 \end{bmatrix} \rightarrow \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

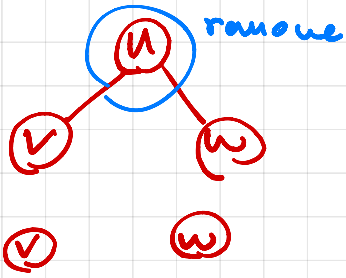


$V \perp\!\!\!\perp W$



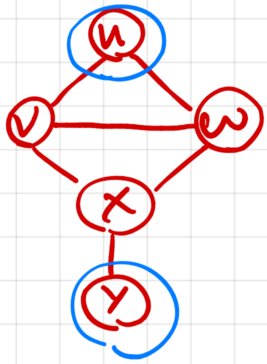
cannot be determined

$V \perp\!\!\!\perp W \mid U$



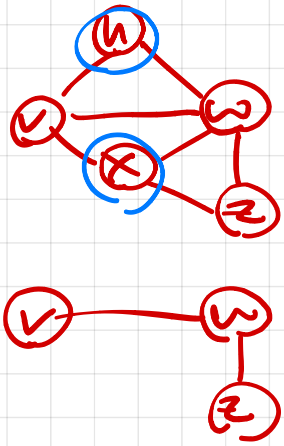
True.

$V \perp\!\!\!\perp W \mid U, Y$



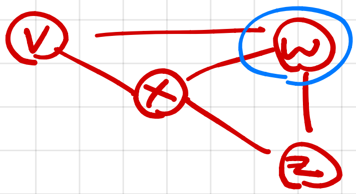
Cannot be determined.

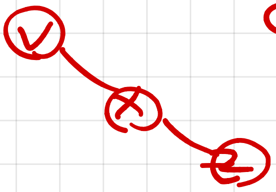
$V \perp\!\!\!\perp Z \mid U, X$



Cannot be determined.

$X \perp\!\!\!\perp Z \mid W$



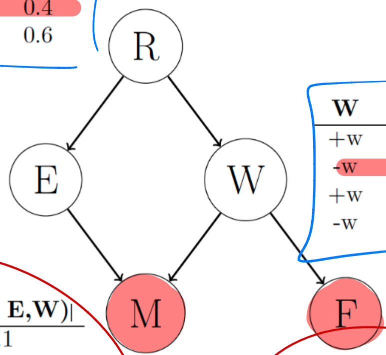


Cannot be determined.

Question 10

Consider the Bayes net and corresponding probability tables shown below. Fill in the following table with the probabilities of drawing each respective sample given that we are using each of the following sampling techniques. For rejection sampling, we say that a sample has been drawn only if it is not rejected. Consider $M = +m$ and $F = +f$ as evidence and $P(f, m) = 0.181$

R	P(R)
+r	0.4
-r	0.6



E	R	P(E | R)
+e	+r	0.3
-e	+r	0.7
+e	-r	0.6
-e	-r	0.4

W	R	P(W | R)
+w	+r	0.9
-w	+r	0.1
+w	-r	0.2
-w	-r	0.8

M	E	W	P(M | E,W)
+m	+e	+w	0.1
-m	+e	+w	0.9
+m	+e	-w	0.45
-m	+e	-w	0.55
+m	-e	+w	0.35
-m	-e	+w	0.65
+m	-e	-w	0.9
-m	-e	-w	0.1

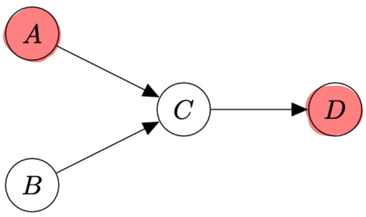
F	W	P(F | W)
+f	+w	0.15
-f	+w	0.85
+f	-w	0.75
-f	-w	0.25

$P(sample\|method)$	(+r,+e,-w,+m,+f)	(+r,-e,+w,-m,+f)
Prior method	0.00405	
Rejection sampling	0.00405/0.181 = 0.02238	
Likelihood weighting		

This makes no sense.

Question 11

Assume you are given the Bayes net and the corresponding CPTs shown below



$P(A)$	
+a	0.1
-a	0.9

$P(B)$	
+b	.7
-b	.3

$P(C\|A,B)$			
+c	+a	+b	.25
-c	+a	+b	.75
+c	-a	+b	.6
-c	-a	+b	.4
+c	+a	-b	.5
-c	+a	-b	.5
+c	-a	-b	.2
-c	-a	-b	.8

$P(D\|C)$		
+d	+c	.5
-d	+c	.5
+d	-c	.8
-d	-c	.2

a. Assume we receive evidence that A = + a. If we were to draw samples using rejection sampling, what is the expected fraction of samples that will be rejected?

b. Next, assume we observed both A = + a and D = + d. What are the weights for the following samples under likelihood weighting sampling?

  i. (+a,−b,+c,+d)

  ii. (+a,−b,−c,+d)

  iii. (+a,+b,−c,+d)

c. Given the samples in the previous question, estimate $P(-b| +a, +d)$

(a) $A = +a$

$\Longrightarrow$ $P(-a) = 0.9$

$\therefore$ 90% rejected.

(b) $A = +a$, $D = +d$

(i) $P(+a, -b, +c, +d) \longrightarrow w = 0.1 \times 0.5 = 0.05$

(ii) $P(+a, -b, -c, +d) \longrightarrow w = 0.1 \times 0.8 = 0.08$

(iii) $P(+a, +b, -c, +d) \longrightarrow w = 0.1 \times 0.2 = 0.08$

(c) $P(-b \mid +a, +b)$

$$= \frac{0.05 + 0.08}{0.05 + 0.08 + 0.08} = \frac{13}{21} = 0.619 \approx \underline{\underline{0.62}}$$