Scales in Physics

Debaiudh Das
Niser
debaiudh.das@niser.ac.in

Abhishek Majhi
Indian Statistical Institute
INSPIRE Faculty (Project Guide)
abhishek.majhi@gmail.com

1 Backreaction in Central Force Problem

In treating specific details of actual central force problems, a change in the orientation of our discussion is desirable. Hitherto solving a problem has meant finding r and θ as functions of time with E, l, etc., as constants of integration. But most often what we really seek is the equation of the orbit, i.e., the dependence of r upon θ, eliminating the parameter t. For central force problems, the elimination is particularly simple, since t occurs in the equations of motion only as a variable of differentiation. Indeed, one equation of motion, simply provides a definite relation between a differential change dt and the corresponding change $d\theta$:

$$ldt = mr^2 d\theta$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

We can define the Lagrangian as :

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 + \frac{1}{2}mr^2\dot{\vec{\theta}}^2 - V(r) = \frac{1}{2}m(\dot{\vec{r}}^2 + r^2\dot{\vec{\theta}}^2) - V(r)$$

The canonical momentum is thus evaluated to be:

$$p_\theta = \frac{d}{dt}(mr^2\dot{\vec{\theta}}) = 0 => l = mr\dot{\vec{\theta}}(constant)$$

Hence by writing Euler Lagrange's equation for the system we get :-

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\vec{q}}_j}) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$=> \frac{d}{dt}(m\dot{\vec{r}}) - mr\dot{\vec{\theta}}^2 = -\frac{\partial V}{\partial r} = f(r)$$

$$=> m\ddot{v} - \frac{l^2}{mr^3} = f(r)$$

Hence the total energy becomes:

$$E = \frac{1}{2}m(\dot{\vec{r}}^2 + r^2\dot{\vec{\theta}}^2) + V(r)$$

$$=> m\ddot{r} = -\frac{d}{dr}(v + \frac{l^2}{2mr^2})$$

Hence,

$$m\ddot{r}\dot{r} = \frac{d}{dt}(\frac{1}{2}m\dot{r}^2)$$

Now we try to formulate an expression for the angle of perihelia for a planetary system from the following equations:

$$\frac{d}{dt}(\frac{1}{2}m\dot{r}^2) = -\frac{d}{dt}(V + \frac{1}{2}\frac{l^2}{mr^2})$$

$$\frac{d}{dt}(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2})$$

$$\frac{d}{dt}(\frac{1}{2}m\dot{r}^2) + \frac{1}{2}\frac{l^2}{mr^2} + V) = 0$$

as $l = mr^2\dot{\theta}$

$$\dot{\theta} = \frac{l}{mr^2}$$

$$d\theta = l\frac{dt}{mr^2} = \frac{ldr}{\sqrt{2mr^4(E - V - \frac{l^2}{2mr^2})}}$$

Hence,

$$\theta = \theta_0 - \int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

Note: there was a change of variable involved, particularly $u = \frac{1}{r}$. We later use this method to calculate the perihelion angle of mercury for a given central force expression and find it out to be very small, this is arising from the back reaction of the central force problem.

Sir derived a modified expression for Newton's central force equation of gravitation. We test wether this expression gives the backreaction corrections of Newtonian gravitational force. The expression is:

$$F = \frac{c^4}{G}\frac{s_1 s_2}{x_1 x_2}$$

where x_1 = $(N_1 + 1_1)s_1, x_2 = (N_2 + 1_2)s_2$ and $1_1, 1_2$ corresponds to unity with respect to the length scales s_1, s_2 respectively.

Being computationally extensive, Mathematica was used to numerically calculate the angle of perihelion precision of mercury arising from this equation.The data used are:

$$G = 6.67408 \times 10^{-11} m_3 kg_{-1} s_{-2}$$

$$R = 57.91 \times 10^9 m$$

$$M_1 = 3.285 \times 10^{23} kg$$

$$M_2 = 1.989 \times 10^{30} kg$$

For the modified central force problem, we have the potential to be:

$$V = GM[\frac{1}{r} - \frac{GM}{2c^2r^2} + \frac{G^2M^2}{3c^4r^3}]$$

After calculation, the angle of precession comes in the order of 10^{-5} degrees which is much less than the precession angle of perihelia calculated from general relativity.

2 Readings

2.1 Post Newtonian Approximation-Precession of Perihelia

Post Newtonian formalism can be used to calculate the precession of planets in the solar system, taking into account other planets, solar rotation and solar oblateness. The potential $\phi + \psi$ is over overwhelmingly dominated by the spherically symmetric part $-\frac{GM_0}{r}$ of the sun's contribution. So,

$$\phi + \psi \equiv -\frac{GM_{\text{O}}}{r} + \varepsilon(\mathbf{x}, t)$$

(ϵ not only includes the Newtonian potentials of the other planets but also any quadrapole moment or higher terms of the sun's contribution.)

$$\frac{d\mathbf{v}}{dt} = -\frac{GM_O\mathbf{x}}{r^3} + \eta + \bigcirc\left(\overline{v}^6\right)$$

(η is defined as a small perturbation)

$$\eta = -\nabla\left(\varepsilon + 2\phi^2\right) - \frac{\partial\zeta}{\partial t} + \mathbf{v} \times (\nabla \times \zeta)\frac{\partial\phi}{\partial t} + 4\mathbf{v}(\mathbf{v}\cdot\mathbf{v})\phi - \mathbf{v}^2\mathbf{v}\phi$$

The precession of perihelia is calculated by the rate of change of the Runge-Lenz vector.

$$\mathbf{A} = -M_\odot G\frac{\mathbf{x}}{r} + (\mathbf{v} \times \mathbf{h})$$

If the perturbation η would have been absent, then the orbit would have been an ellipse.

$$r = \frac{L}{1 + e\cos(\varphi - \varphi_0)}$$

$$\frac{d\varphi}{dt} = \frac{\sqrt{LM_\odot G}}{r^2}$$

$$\frac{dr}{dt} = e\sqrt{\frac{M_\odot G}{L}}\sin(\varphi - \varphi_0)$$

e is the eccentricity and L is the semilatus rectum. If $\theta = \frac{\pi}{2}$, with perihelia at an azimuthal angle ϕ_0 then h would be a constant vector normal to the orbit.

$$|\mathbf{h}| = \sqrt{L}M_\odot G$$

The rate of change of perihelia $\frac{dL_0}{dt}$ caused by any perturbation is just the component of the change $\frac{dA}{dt}$ in the unit vector $A = \frac{A}{|A|}$ along a direction perpendicular to both A and h.

$$\frac{d\varphi_0}{dt} = (\hat{\mathbf{h}} \times \hat{\mathbf{A}}) \cdot \frac{d\hat{\Lambda}}{dt} = (\mathbf{h} \times \mathbf{A}) \cdot \frac{\frac{d\mathbf{A}}{dt}}{|\mathbf{h}|\mathbf{A}^2}$$

Hence,

$$\frac{d\mathbf{A}}{dt} = \eta \times \mathbf{h} + \mathbf{v} \times (\mathbf{x} \times \eta)$$

Both $\frac{dA}{dt}$ and $\frac{dL_0}{dt}$ are linear in η, hence adding up precessions by η. The largest term in η is the part of the $-\Delta\epsilon$ arising from the Newtonian potentials of the other planets. The next largest term is obtained from the relativistic corrections setting up ϕ and ς equal to the values they would have for a spherically non-rotating sun.

$$\phi_\odot = -\frac{GM_\odot}{r} \quad \zeta_\circ = 0$$

Thus, the precession is

$$\frac{d\varphi_0}{dt} = 8M_{\bigcirc}GhL^{-3}\left[1+e\cos\left(\varphi-\varphi_0\right)\right]^3\sin^2\left(\varphi-\varphi_0\right) - M_{\odot}Ge^{-1}hL^{-3} \times\left\{7\left[1+e\cos\left(\varphi-\varphi_0\right)\right]^2 + 4\left[1+e\cos\left(\varphi-\varphi_0\right)\right]^3 + \left[1+e\cos\left(\varphi-\varphi_0\right)\right]^4\right\}\cos\left(\varphi-\varphi_0\right)$$

Since ϕ_0 changes slowly, the change in one revolution can be determined by integrating it with respect to time over one period. This gives :

$$\Delta\varphi_0 = \int_0^{2\pi}\frac{d\varphi_0}{dt}\frac{dt}{d\varphi}d\varphi = \frac{L^2}{h}\int_0^{2\pi}\frac{d\varphi_0}{dt}\left[1+e\cos\left(\varphi-\varphi_0\right)\right]^{-2}d\varphi$$

Hence we are left with:

$$\Delta\varphi_0 = 6\pi\frac{M_{\odot}G}{L}$$

2.2 Accelerated observers in special relativity

Gravity can be mocked up by acceleration, focussing on a region far from attracting matter and so it is free of disturbance, such that spacetime there can be considered to have a lorentz geometry. Special relativity was developed precisely to predict the physics of accelerated objects.

$$\Delta\tau \equiv \begin{pmatrix} timeintervaltickedoff \\ byobserver'sclocksashe \\ movesavectordisplacement \\ \xi alonghisworldline \end{pmatrix} = \left[-g(\xi,\xi)\right]^{1/2}$$

$$l \equiv \begin{pmatrix} spatialdimensions \\ oflattice \end{pmatrix}\begin{pmatrix} theaccelerationmeasured \\ byaccelerometershecarries \end{pmatrix}^{-1} \equiv \frac{1}{g}$$

Here accelerations differ and clocks get out of step by a fractional amount gl.

A coordinate independent frame in geometric terms is defined and then the results are projected onto the basis vectors of its accelerated frame. This leads to complicated fractional differences if gl¡¡1 and a coriolis kind of force, including inertial forces due to observer's acceleration.

3 Dimensions and fundamental constants

In most theoretical aspects, we treat gravitational constant, speed of light and plancks constant as "God-given" and fundamental. This section sheds some light on the same topic. According to the updated draft of the international standard units of measurement, the units of measurement are redefined according to higher precision, particularly:

- The speed of light c is exactly 299792458 metres per second (ms1);
- The ground state hyperfine structure transition frequency of the caesium-133 atom $\Delta\nu_{Cs}$ is exactly 9192631770 hertz (Hz);
- The luminous efficacy K_cd of monochromatic radiation of frequency 54010^{12} Hz (540 THz) a frequency of green-colored light at approximately the peak sensitivity of the human eye is exactly 683 lumens per watt (lmW1).

By some elementary substitution into the newton's gravitation equation we can show that this leads to the expression:

$$\frac{\delta G}{G(\Delta\nu_{cs})} = -\frac{2\Phi}{c^2}$$

This questions the use of fundamental constants such as G and also leads to conclusions of change in G due to lorentz transformations which leads to:

$$G` = \gamma^2 G$$

where G‘ is the observed gravitational constant on a different lorentz frame of reference and G is the gravitational constant on the observer’s frame of reference. This leads us to think that there is a variation of fundamental constants due to special relativity.