Black Holes in String Theory

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Date

DEDICATION

To my mother

DECLARATION

I hereby declare that I am the sole author of this thesis in partial fulfillment of the requirements for a postgraduate degree from National Institute of Science Education and Research (NISER). I authorize NISER to lend this thesis to other institutions or individuals for the purpose of scholarly research.

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Date:

The thesis work reported in the thesis entitled Black Holes in String Theory was carried out under my supervision, in the school of physical sciences at NISER, Bhubaneswar, India.

Signature of the thesis supervisor
School:
Date:

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ABSTRACT

Black holes are interesting physical objects analogous to thermodynamical systems, this report covers a mathematical overview of the analogy and further tries to extend it to the case of quantum black holes. We address the problems encountered here, namely - the information loss paradox and the violation of unitarity. We seek string theory construction of higher dimensional objects known as branes and how they provide valuable insights towards addressing these issues and attempt to partially resolve them, here the AdS_3/CFT_2 correspondence is helpful and plays a crucial role in our study. We seek solutions of string/brane actions and hence get accustomed to solution generating techniques from such actions. We provide a neat recipe of constructing black holes from these solutions and further calculate factors like entropy, absorption and emission of these objects. We observe a precise consistency in the entropy and a connection of these objects with their classical counterparts.

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Chapter 1

Introduction

General Relativity is a classical theory of gravitation, which reduces to Newtonian gravity in the non-relativistic weak field limit. Black holes are a consequence of general relativity and are characterised by objects with an event horizon which covers/censors some type of singularity inside. These singularities contribute to an incomplete picture of such objects as any physical theory would break down at a singularity. Hence, this hints towards the existence of some modified or new theory which would describe black holes in the microscopic limit. It can be seen that black holes have a deep connection with thermodynamic systems, so much so that there exists an analogy between the laws of black hole mechanics and the laws of thermodynamics. It is found that the formulation of quantum black holes and its connection to thermodynamics yield unphysical results of information loss and breakdown of unitarity.

It is known that if we naively try to formulate a quantum field theory of gravity, it will be a non-renormalizable theory. Hence, one needs to look in string theory constructions of higher dimensional objects known as branes, which provide a good description of a viable quantum theory of gravity. In this thesis we plan to investigate the construction of black holes in string theory and how it can partially resolve the problems of quantum black holes.

Chapter 2

Black Holes in General Relativity

General Relativity is a theory of gravitation which connects the geometrical property of the space-time (the curvature tensor [1]) with its physical nature (the stress energy tensor). This is demonstrated by the Einstein's field equation for the source free case ($T_{\mu\nu} = 0$), which is given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.1}$$

Here $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature. Broadly, we can say that the Ricci curvature tensor acts as a measure of the degree to which the geometry of a given metric tensor differs locally from that of ordinary Euclidean space or pseudo-Euclidean space. While the scalar curvature represents the amount by which the volume of a small geodesic ball in a Riemannian manifold deviates from that of the standard ball in Euclidean space.

2.1 The Schwarzschild Black Hole

The Schwarzschild geometry is the unique source-free solution of Einstein's equation with spherical symmetry that approaches ordinary Minkowski space at large distances. By itself the Schwarzschild geometry does not quite describe a black hole in the astrophysical sense, but understanding it is a necessary prerequisite for studying them. Explicitly the spacetime metric for the Schwarzschild black hole is given by

$$ds^2 = -(1 - \frac{r_H}{r}) dt^2 + (1 - \frac{r_H}{r})^{-1} dr^2 + r^2 d\Omega_2^2 \tag{2.2}$$

[1]A tensor can be thought of as a generalization of a vector.

The Schwarzschild metric is a solution of the action in 4 dimensions:

$$S = \frac{1}{16\pi G_4} \int d^4 \sqrt{-g} R[g] \tag{2.3}$$

In the above equation G is Newton's gravitational constant in 4 dimensions. Ω_2 is the metric of a 2-sphere. In equation (2.2) we can see that r=0 and $r = r_H$ are two very special cases where the metric in this form diverges. This is termed as a singularity. It must be noted that while r=0 is the real singularity, $r = r_H$ is only a coordinate artifact. The pathology can be described in a way such that it is coordinate invariant by studying the divergence of the fully contracted Riemann tensor which is $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. The curvature tensor represents the tidal force experienced by a rigid body moving along a geodesic [2].

Specifically, r_H is the event horizon of the black hole where:

$$r_H = 2GM \tag{2.4}$$

Here G is the Newton's gravitational constant and M is the mass parameter. This is also known as the Schwarzschild radius.

Now we try to construct a coordinate system where we can successfully remove the coordinate singularities. A good choice for this is the Kruskal-Szekeres coordinates. These coordinates are motivated by observing that radial null geodesics in the Schwarzschild geometry can be parametrized as follows:

$$t = \pm r_* + C \tag{2.5}$$

Here C is some constant and r_* is the new radial direction given by:

$$r_* = r + \log(r - 1) \tag{2.6}$$

[2]A curve representing the shortest distance between two points on a surface

r_* is termed as the the "tortoise" coordinate as it fits an infinite coordinate range into a finite geodesic distance. We can define the Kruskal-Szekeres coordinates as:

$$U \equiv -e^{\frac{r_*-t}{2}} \tag{2.7}$$

$$V \equiv e^{\frac{r_*+t}{2}} \tag{2.8}$$

We see that the lines of constant U or V are radial null geodesics. These coordinates have the convenient feature that

$$UV = (1-r)e^r \tag{2.9}$$

We note that the singularity is when $UV = 1$ while the horizon is when either U or V is zero. The metric takes the form:

$$ds^2 = -\frac{2}{r}e^{-r}(dUdV + dVdU) + r^2 d\Omega_2^2 \tag{2.10}$$

The off-diagonal nature of the metric can be easily removed by defining yet another set of coordinates

$$U = T - X \tag{2.11}$$

$$V = T + X \tag{2.12}$$

in these new coordinates, the metric is:

$$ds^2 = \frac{4}{r}e^{-r}\left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2 \tag{2.13}$$

We see that now there is no singularity of any kind at $r = 1$.

Now that we have extracted a form of the Schwarzschild metric needed for us to study the near horizon physics, we would start defining the thermal properties of a

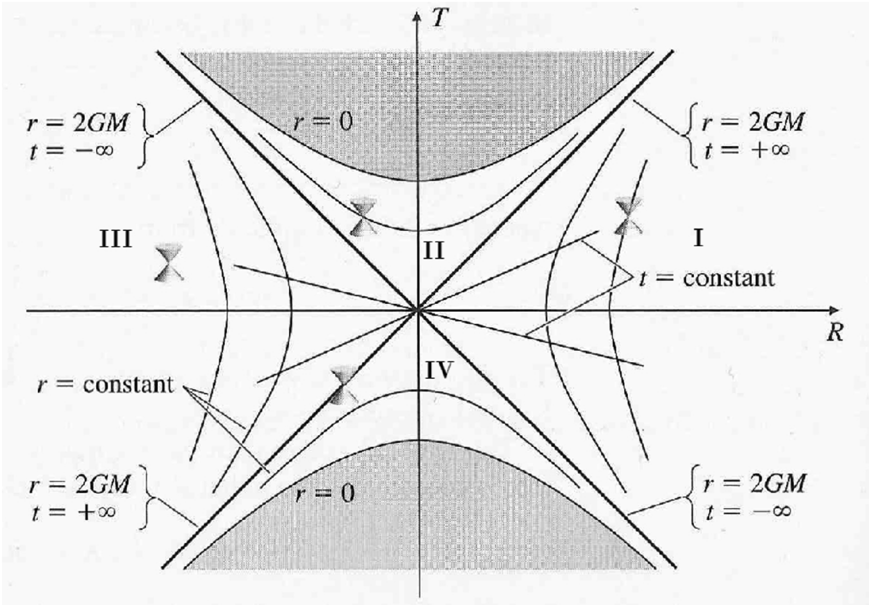


Figure 2.1: The Kruskal extension of the Schwarzschild Black Hole

black hole. From quantum field theory, we know that a field theory can be defined by a local path integral with appropriately setup boundary conditions. After some methods involving time rescaling and Wick rotation, if we consider the Schwarzschild black hole to be the background spacetime of a quantum field theory and evaluate the Euclidean Feynman path integral from that scenario, then we can relate it with a statistical mechanical partition function and derive the temperature for the case. This is the Hawking temperature of the black hole.

One way of deriving the Hawking temperature is when we consider the original Schwarzschild metric again (we wick rotate t=iτ) :

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.14}$$

We have taken G=1. As seen earlier, this metric is singular for the case of r=2M. Now, we redefine:

$$r - 2M = \frac{x^2}{8M} \tag{2.15}$$

Now it is interesting to note that the metric takes the form:

$$ds_E^2 = (\kappa x^2)d\tau^2 + dx^2 + \frac{1}{4\kappa^2}d\Omega^2 \tag{2.16}$$

Here $\kappa = \frac{1}{2r}$, so we get the metric which is the product of the metric on S^2 and the Euclidean Rindler spacetime, which is:

$$ds_E^2 = dx^2 + x^2 d(\kappa\tau)^2 \tag{2.17}$$

Now, we make a periodic identification as:

$$\tau \sim \tau + \frac{2\pi}{\kappa} \tag{2.18}$$

Not only can the cordinate singularity can be avoided with this periodic property, but also we can take an Euclidean functional integral over fields in this spacetime $\Phi(x, \tau)$, which will be periodic in τ.

Once, the Euclidean functional integral defined by :

$$Z = \int [D\Phi] e^{-S_E[\Phi]} \tag{2.19}$$

is taken over [3] the fields Φ, being periodic in imaginary time with period $\hbar\beta$, then we can write the partition function as:

$$Z = tre^{-\beta H} \tag{2.20}$$

So, for a Schwarzschild black hole, we see that $\hbar\beta = \frac{2\pi}{\kappa}$ where $\beta = (k_B T)^{-1}$. Hence we get the temperature as:

$$T_H = \frac{\kappa\hbar}{2\pi k_B} \tag{2.21}$$

When we impose the conditions of $\hbar = k_B = 1$ and the form of κ then:

$$T_H = \frac{1}{8\pi G_4 M} \tag{2.22}$$

[3]here $\mathrm{S}_E = \int dt(-ip\dot{q} + H)$

This is identified as the Hawking temperature of a Schwarzschild Black Hole. We see that the black hole radiates with a thermal spectrum and the Hawking temperature is the physical temperature felt by an observer at infinity.

2.2 The Reissner-Nordström Black Hole

Once we have looked into Schwarzschild Black holes, we can say that they are characterised by the mass parameter. This distinguishing property is known as a hair of a black hole. A Black hole can have multiple hairs including mass, charge and angular momentum. Now we study the Reissner-Nordström Black Hole, which is a static spherical black hole metric with charge as a characterizing property.Here Einstein gravity is coupled to a U(1) gauge field, here both the metric and the gauge field can be turned on. It is a black hole with mass and electric charge, but no spin. The metric looks like:

$$ds^2 = -\Delta_+(\rho)\Delta_-(\rho)dt^2 + \Delta_+(\rho)^{-1}\Delta_-(\rho)^{-1}d\rho^2 + \rho^2 d\Omega_2^2 \tag{2.23}$$

where:

$$F_{t\rho} = \frac{Q}{\rho^2} \tag{2.24}$$

$$\Delta_\pm(\rho) = (1 - \frac{r_\pm}{\rho}) \tag{2.25}$$

$$r_\pm = G_4(M \pm \sqrt{M^2 - Q^2}) \tag{2.26}$$

In the above equations, G is the Newton's Gravitational Constant in 4 dimensions, M is the mass parameter and Q is the charge parameter in the (t,ρ, Ω) coordinate system. here Ω has the same definition as mentioned in the previous section. In the RN case, there are two horizons, located at $r = r_+$ and $r = r_-$

For a black hole, a singularity is always associated and covered by an event horizon (which is a coordinate artefact), as violation of this will result in unphysical scenarios where any physical process throughout the universe might be influenced by the true singularity where known physical laws break down. This is known as the cosmic censorship hypothesis.

Cosmic censorship would require that the true singularity is hidden behind a horizon and hence we will have the relation (for constraining $r_\pm$)

$$M \geq |Q|$$

The Hawking temperature here will be:

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi G_4 \left(M + \sqrt{M^2 - Q^2}\right)^2} \tag{2.27}$$

Now we want to derive an important property of the Reissner-Nordström spacetime. We start by considering the extremal geometry and the double horizon at r_0 ($r \sim \rho - r_0$) then,

$$\Delta_\pm = 1 - \frac{r_0}{\rho} = (1 + \frac{r_0}{r})^{-1} \equiv H(r)^{-1} \tag{2.28}$$

$$\rho^2 = r^2(1 + \frac{r_0}{r})^2 \tag{2.29}$$

$$ds^2_{ext} = -H(r)^{-2}dt^2 + H(r)^2(dr^2 + r^2 d\Omega_2)^2 \tag{2.30}$$

This has a manifested SO(3) symmetry in isotropic coordinates.

The extremal Reissner-Nordström black hole has a near horizon geometry near r=0 as:-

$$ds^2 = -\frac{r^2}{r_0^2}dt^2 + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\Omega^2 \tag{2.31}$$

We introduce another new coordinate to see the interesting property of the metric:

$$z \equiv \frac{r_0^2}{r} \tag{2.32}$$

$$\frac{dz}{z} = \frac{dr}{r} \tag{2.33}$$

$$ds^2 \to \frac{r_0^2}{z^2}(-dt^2 + dz^2) + r_0^2 d\Omega_2^2 \tag{2.34}$$

We can see that this metric becomes a direct product of an AdS_2 and a S_2. Hence the RN spacetime interpolates between two maximally symmetric and asymptotically flat spacetimes.

In the extremal limit the Reissner-Nordström black hole possesses an extra symmetry called supersymmetry.

Chapter 3

Black Hole Thermodynamics

From the previous chapter, we have got an idea that black holes indeed look like thermal objects with a finite temperature in some cases. While integrating general relativity, quantum mechanics and thermodynamics into a comprehensive description of black holes is quite complicated, but the basic properties of black hole mechanics and its analogy to thermodynamics can be expressed as a fairly simple set of rules known as black hole thermodynamics. Essentially these are the laws of thermodynamics re-expressed in terms of properties of black holes.

3.1 Classical Black Holes

We begin formulating the thermodynamic properties of classical black holes, like the ones described in the previous chapter. For this we need to first geometrically setup our system of observer and the background spacetime.

We can represent an observer in a spacetime by an inextendible timelike curve (say ν). If we say that the chronological past of ν is determined by $I^-(\nu)$ and the future horizon (say h^+) of ν is defined to be the boundary of $I^-(\nu)$ then we can say that:

> The points p ϵ h^+ will lie on a null geodesic segment which is entirely contained within h^+ (future inextendible). We also note that the convergence of the null geodesics that generate h^+ cannot diverge at a point on h^+ itself.

We can similarly define

- A past horizon h^-

- h^+ and h^- for families of observers

3.1.1 Black Holes and Event Horizons

As hinted by the cosmic censorship hypothesis in the previous chapter, we will be required to have a geometrical formulation of the event horizon which should always be associated with a singularity for a black hole. When we consider an asymptotically flat space-time (M,g_{ab}), the notion of the spacetime being asymptotically flat can be precisely achieved by the notion of a conformal null infinity.

We can also consider the family of observers Γ who can escape to arbitrarily very large distances at large times. We can now say that if the past of these considered observers (that is $I^-(\Gamma)$) will fail to be the entire spacetime, then a black hole is present ($B \equiv M - I^-(\Gamma)$). The horizon here (h^+) will be termed as the future horizon of the black hole. It should be noted here that this mathematical formulation still allows the existence of "naked singularities" to be present in our theory.

3.1.2 The Cosmic Censorship Hypothesis

The Cosmic Censorship Hypothesis suggests that there must not exist any naked singularities in nature, instead the black hole singularities will be accompanied by an event horizon (which is denoted by the mathematical formulation of an event horizon from the previous section). This should always hold true as naked singularities are unphysical and might affect any physical system at any part of the universe it is present in.

We define a Cauchy surface (C) in a time-orientable spacetime to be a set with the characteristic property that every timelike inextendible curve in the manifold will intersect C in precisely one point. The spacetime is said to be globally hyperbolic if it possesses such a Cauchy surface C. This naturally implies the fact that the manifold will have a topology $R \times C$

An asymptotically flat spacetime with a black hole is termed predictable if one can prove the existence of a region of the manifold which contains the entire exterior region and the event horizon (that is globally hyperbolic). This mathematically illustrates the idea that no "naked singularities" shall be present.

> The maximal Cauchy evolution (globally hyperbolic) of an asymptotically flat initial dataset will generically yield an asymptotically flat spacetime with a complete null infinity.

We note that the validity of this hypothesis will assure us that any observer which stays outside the black hole cannot be causally influenced by the singularity.

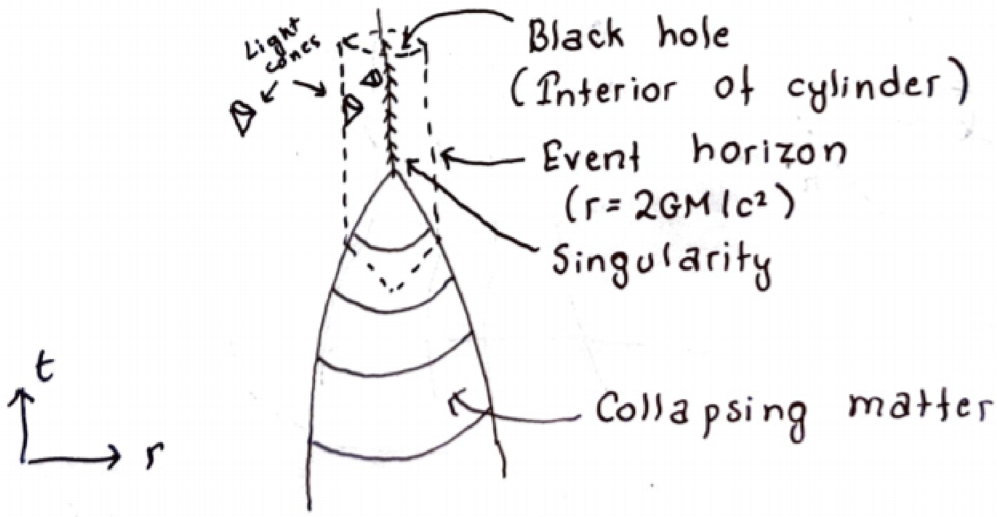


Figure 3.1: Spacetime diagram of a gravitational collapse

3.1.3 Null Geodesics and the Raychaudhari Equation

The Raychaudhari Equation is a fundamental and important equation in general relativity which relates and describes motion of a nearby collection of matter.

The event horizon is defined as the boundary of the causal past of null infinity.

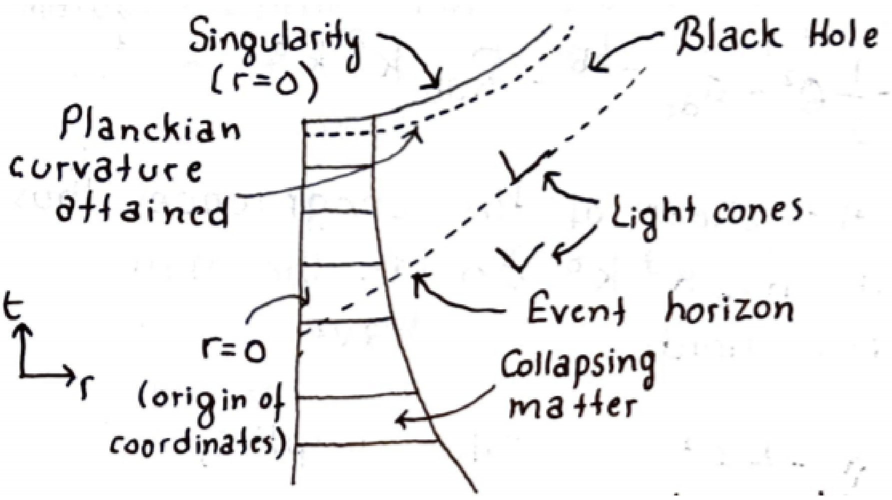


Figure 3.2: Spacetime diagram of a gravitational collapse, here angular directions are suppressed so the light cones become straightened.

Such boundaries are generated by null geodesics. The affine parameter goes to infinity as we approach null infinity, and no caustics form until then. So, the expansion of the event horizon has to be nonnegative. As the expansion gives the rate of change of the logarithm of the area density, this means the event horizon area can never go down, at least classically, assuming the null energy condition.

To derive it formally, we can define the expansion θ for a congruence [1] of null geodesics with λ as the affine parameter, by:

$$\theta = \nabla_a k^a \tag{3.1}$$

It follows that the area of an infinitesimal area element which is transported along the null geodesics will vary as:

$$\frac{d(\ln A)}{d\lambda} = \theta \tag{3.2}$$

It should be noted here that for the case of null geodesics generating a null hypersurface (like the event horizon of a black hole), the twist term (ω_{ab}) will vanish.

[1]a congruence of curves is the set of integral curves of a vector field in a four-dimensional Lorentzian manifold which is interpreted physically as a model of spacetime.

The Raychaudhari equation will yield :

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b \tag{3.3}$$

Here σ_{ab} is the shear of the congruence. Provided that the null energy condition holds ($R_{ab}k^a k^b \geq 0$),we get:

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2 \tag{3.4}$$

hence,

$$\frac{1}{\theta(\lambda)} \leq \frac{1}{\theta_0} + \frac{1}{2}\lambda \tag{3.5}$$

As a consequence of this, if $\theta_0 < 0$ then $\theta(\lambda_1) = -\infty$ at some point $\lambda_1 < \frac{2}{|\theta_0|}$, given that we can extend the geodesic that far.

3.1.4 The Area Theorem

If a horizon (h^+) is generated by future inextendible null geodesics, then it cannot have $\theta = -\infty$ at any point of the horizon. Hence, $\theta \geq 0$ should be a necessary condition for the horizon generators to be complete. Although, for a predictable black hole, we can show that $\theta \geq 0$ without assuming that the generators of the event horizon will be future complete. This is achieved by a clever argument where we deform the horizon outwards at a point where $\theta < 0$

We let S_1 be a Cauchy surface for the globally hyperbolic region which appears in the definition of out predictable black hole. We let S_2 be another Cauchy surface which lies to the future of S_1. As we know the generators of h^+ are future, there are three conditions:

- Null geodesics will have positive expansion
- There may be new null geodesics
- There should not be any missing null geodesics

3.1.5 Killing Vector Fields

We define an isometry to be a diffeomorphism that will leave the metric invariant. While, a Killing vector field, ξ^a, will be the infinitesimal generator of a one-parameter group of isometries. It will satisfy the equation :

$$\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)} = 0 \tag{3.6}$$

For a Killing vector field, we can let

$$F_{ab} = \nabla_a \xi_b = \nabla_{[a}\xi_{b]}$$

Then we can say that ξ^a will be uniquely determined by its value and the value of the defined quantity F_{ab} at an arbitrarily chosen point p.

3.1.6 Bifurcate Killing Horizons

In two dimensions:

Suppose we have a Killing field, ξ^a, which vanishes at a point p. We can then say that ξ^a is determined by F_{ab} at p, as mentioned earlier. In two dimensions we see that $F_{ab} = \alpha\epsilon_{ab}$. So. ξ^a will be unique upto a scaling factor.

We note that for a Riemannian metric, the orbits of the generated isometries by ξ^a must be closed in nature (near p) and the structure of the orbit will be like a rotation in flat space.

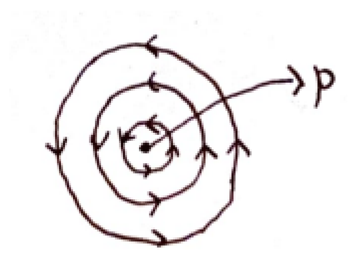


Figure 3.3: Orbit structure in 2 dimensions

Similarly, if we have a Lorentzian metric, the isometries must have to carry the null geodesics through the point p into themselves and the orbit structure near p would be like a Lorentz boost in 2d Minkowski spacetime.

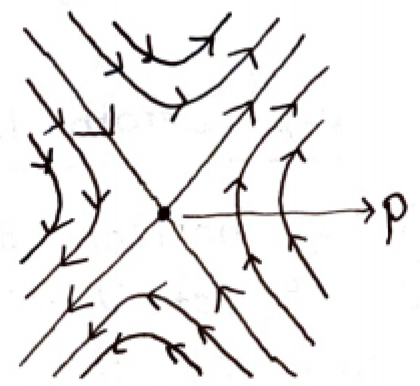


Figure 3.4: Orbit structure in 2d Minkowski spacetime with Lorentzian metric

In four dimensions:

In four dimensions we see that similar results (to the 2d case) holds true only if ξ^a vanishes on a 2d surface Σ. Here, if the metric is Lorentzian and Σ is spacelike then the orbit structure of ξ^a will look like a lorentz boost in 4d Minkowski spacetime, near Σ. *The pair of intersecting null surfaces h_A and h_B at Σ generated by the null geodesics which are orthogonal to Σ are called the bifurcate Killing horizons.*

It consequently follows that ξ^a is normal to both h_A and h_B. We can say that any null surface (say h) having the property of a Killing field being normal to it is called a Killing horizon.

3.1.7 Surface Gravity and the Zeroth law

We suppose h to be a Killing horizon which has a corresponding Killing field (ξ^a). If U denotes an affine parameter of the null geodesic generators of h and k^a denotes the associated tangent, we will get the expression:

$$\xi^a = fk^a \tag{3.7}$$

Here f is the partial derivative of U with respect to the Killing parameter along the null generator of h. We can then proceed to precisely define the surface gravity κ by:

$$\kappa = \xi^a \nabla_a \ln f = \frac{\partial \ln f}{\partial u} \tag{3.8}$$

Equivalently, we can say that on h:

$$\xi^b \nabla_b \xi^a = \kappa \xi^a \tag{3.9}$$

It is natural to see that κ will be a constant along each generator of h.

In general, we can see a variation of κ from generator to generator of h but still we can draw out the following three versions of the Zeroth Law from here:

If h is a connected Killing Horizon for a spacetime (in which we can say that Einstein's equation will hold, with the matter fields obeying the energy condition), then κ will be a constant on the given h.

For a connected killing horizon, h, we can suppose that :

- ξ^a to be a hypersurface which is orthogonal in the static case
- there exists another Killing field, ϕ^a which will commute with ξ^a and will satisfy the condition on h: $\nabla_a(\psi^b \omega_b) = 0$, here ω_a is the twist of the first hypersurface.

In this case, κ has to be a constant on h.

If h_A and h_B are two null surfaces which comrpise a bifurcate Killing horizon (connected), Tthen κ will be a constant on the given h_A and h_B.

3.1.8 Constancy of κ and Bifurcate Killing Horizons

As we have just seen, according to the zeroth law, κ will be a constant over a bifurcate Killing horizon. Conversely, we can also show that if the κ is a non-zero constant over a given Killing horizon,h, then the horizon can be extended locally so as to form one of the null surfaces (h_A or h_B) of a bifurcate Killing horizon.

3.1.9 Hawking's Rigidity Theorem

Let us suppose we have a stationary and asymptotically flat solution of Einstein's equation which has a black hole, then Hawking's Rigidity Theorem says that the event horizon (h^+) of the black hole will be a Killing horizon.

One of the important conditions should be that the Killing field ξ^a must be a tangent to h^+.We can show that ξ^a is hypersurface orthogonal if ξ^a is a normal to h^+, hence indicating that the spacetime is static. If the normality condition does not suffice then there must be another Killing field that is normal to the horizon given (suppose χ^a). It can be further shown that we can get a linear combination of ξ^a and χ^a, termed as ψ^a, such that it has closed spacelike orbits, indicating that the spacetime is axisymmetric.

A stationary black hole must be static and axisymmetric.

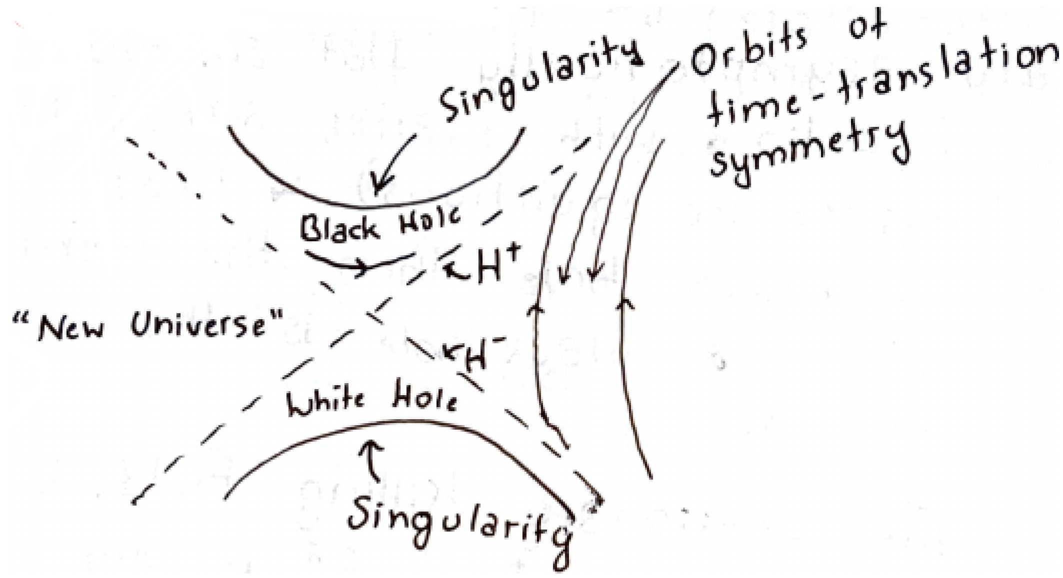


Figure 3.5: Idealized Black Hole equilibrium state

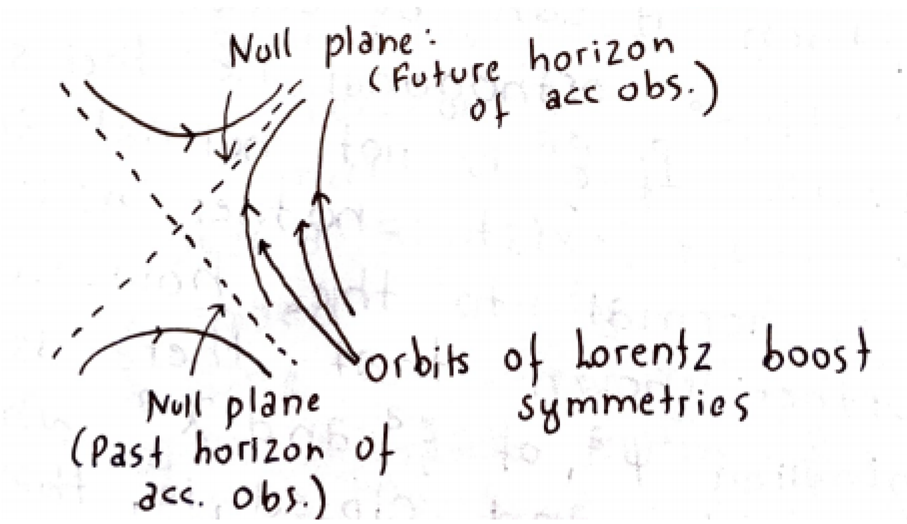


Figure 3.6: Analog: Lorentz Boost in Minkowski spacetime

3.1.10 Summary

- If we say that the cosmic censorship will hold, then it is evident that starting from nonsingular (good) initial conditions, gravitational collapse will form a black hole which will be predictable.

- The surface area of the event horizon of a black hole will be a non-decreasing function of time.

We can naturally expect that a black hole will asymptotically approach a stationary final state quickly.Then, the event horizon of this state:

- Will be a Killing Horizon
- Will have a constant surface gravity.
- Will have the bifurcate Killing horizon structure if $\kappa \neq 0$

3.2 First Law of Black Hole Mechanics and Black Hole entropy

3.2.1 Lagrangian and Hamiltonian in Classical Field Theory

Lagrangian and Hamiltonian formulations are important as they play a major role in quantization of field theories. The existence of such a lagrangian or hamiltonian will provide us information about the essential auxiliary structure to the classical field theory, providing us with many key properties to the theory.

3.2.2 Lagrangian and Hamiltonian in Particle Mechanics

We get the relation below for particle paths q(t) and L being a function of $(q, \dot{q})$:

$$\delta L = [\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}]\delta q + \frac{d}{dt}[\frac{\partial L}{\partial \dot{q}}\delta q] \tag{3.10}$$

We call L as the Lagrangian of the system, provided the equations of motion are:

$$0 = E \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{3.11}$$

Here, the boundary term is of the form:

$$\Theta(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}}\delta q = p\delta q \tag{3.12}$$

this term is usually discarded, here $p \equiv \frac{\partial L}{\partial \dot{q}}$. We obtain the following quantity by taking a variation of Θ (antisymmetrized) at a given time:

$$\Omega(q, \delta_1 q, \delta_2 q) = [\delta_1 \Theta(q, \delta_2 q) - \delta_2 \Theta(q, \delta_1 q)]|_{t_o} = [\delta_1 p \delta_2 q - \delta_2 p \delta_1 q]|_{t_o} \tag{3.13}$$

We know that Ω here will be independent of time t_0 if the varied paths,i.e. $\delta_1 q(t)$ and $\delta_2 q(t)$ satisfy the linearized equations of motion with q(t). We can obtain a finite-dimensional phase space Γ on a non-degenerate Ω by factoring the space of all paths by the degeneracy subspaces of Ω. We say that a Hamiltonian H will be a function on this Γ, and its pullback to F will satisfy the equation:

$$\delta H = \Omega(\dot{q}; q; \delta q) \tag{3.14}$$

the above relation should be true for all provided values of δq, such that q(t) will satisfy the equations of motion. Equivalently, one can say the EOM are:

$$\dot{q} = \frac{\partial H}{\partial p} \tag{3.15}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{3.16}$$

3.2.3 Diffeomorphism covariant theories

A diffeomorphism covariant theory has a Lagrangian constructed entirely out of dynamical fields, hence there would be no background structure in the theory, apart from the underlying manifold structure of the spacetime. The lagrangian for such a theory takes the form given below, where the dynamical fields are ϕ, the metric is $_{ab}$ and the tensor fields are ϕ:

$$L = L(g_{ab}, R_{bcde}, ..., \nabla_{(a_1...)} \nabla_{a_m)} R_{bcde}; \psi, ..., , \nabla_{(a_1...)} \nabla_{a_l)} \psi) \tag{3.17}$$

3.2.4 The First Law of Black Hole Mechanics

In a diffeomorphism covariant theory, we can say for any solution ϕ, any $\delta\phi$ and any ξa we will have:

$$\Omega(\phi, \delta\phi, \mathcal{L}_{xi}\phi) = \int_{\mathcal{C}} \xi^a \delta C_a + \int_{\delta\mathcal{C}} [\delta \mathcal{Q}_\epsilon - \xi.\textcircled{H}] \tag{3.18}$$

If ϕ is a stationary horizon and has a Killing horizon with a bifurcation surface Σ then we done ξ^a as the horizon killing field with vanishing boundary,

$$\xi^a = t^a + \Omega_H \phi^a \tag{3.19}$$

if we let $\mathcal{L}_\xi \phi = 0$ and we assume that $\delta\phi$ satisfy the linearized equation, then we can say the following (assuming $\mathcal{C}$ to be a hypersurface which extends from Σ to ∞)

$$0 = \int_\infty [\delta \mathcal{Q}_\epsilon - \xi.\textcircled{H}] - \int_\Sigma \delta Q_\xi \tag{3.20}$$

Furthermore, we can say using the properties of Killing Horizons that:

$$\delta \int_\Sigma Q_\xi = \frac{\kappa}{2\pi} \delta S \tag{3.21}$$

Here S can be expanded by the binomial to Σ and thus we get the exact formula for black hole entropy in an arbitrary diffeomorphism covariant theory of gravity, hence capturing the essence of the first law of Black Hole mechanics :

$$\frac{\kappa}{2\pi} \delta S = \delta\epsilon - \Omega_H \delta J \tag{3.22}$$

3.2.5 Black Holes and Thermodynamics

There is an interesting correspondence between a stationary black hole and a body in thermal equilibrium, as evident from the thermodynamic connections. We can characterize bodies which are in thermal equilibrium by ”state parameters” similar to uniquely characterizing a stationary black hole with M,J and Q.

Black Holes	Thermodynamics
0th Law	
The surface gravity is a constant all over the event horizon if stationary.	The temperature is constant all over a body if in thermal equillibrium.
1st Law	
$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H \delta J + \Phi_H \delta Q$	$\delta E = T\delta S - P\delta V$
2nd Law	
$\delta A \geq 0$	$\delta S \geq 0$

3.3 Summary and Comparison

Hence, we can draw a relation between the laws of black hole mechanics and the laws of thermodynamics in the following manner:

- 0th Law: The surface gravity of a black hole is a constant all over the event horizon if its stationary, while in thermodynamics, the temperature is constant all over a body if its in thermal equillibrium. Indeed this is the Hawking temperature for the black hole case.

- 1st Law: For the black hole:

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H \delta J + \Phi_H \delta Q \tag{3.23}$$

  While in the thermodynamic picture:

$$\delta E = T\delta S - P\delta V \tag{3.24}$$

- 2nd Law : The area of a Black hole is always increasing (≥ 0) while the entropy of a thermodynamical system is always increasing (≥ 0)

Chapter 4

Quantum Black Holes

Now we consider the case of quantum black holes where we have to investigate various quantum processes involved with the black hole geometry in the near horizon limit, which include particle creation,

4.0.1 Particle creation by black holes

We must note that throughout our theory, we take a semi-classical treatment of gravity, i.e, we approximate the complete theory of quantum gravity by treating the matter fields in our theory as of a quantum nature while the gravitational fields as a classical theory. This is done for reasons which will become apparent gradually.

Black holes form excellent black bodies and hence a distant observer can see a thermal flux of species of particles being emitted from a black hole, due to the particle creation properties in quantum field theory. The temperature of the emitted radiation will be as derived for the Hawking temperature relation:

$$kT = \frac{\hbar\kappa}{2\pi} \tag{4.1}$$

If we have a Schwarzschild Black Hole, then we have $\kappa = \frac{c^3}{4GM}$ and hence,

$$T \sim 10^{-7}\frac{M_{\odot}}{M} \tag{4.2}$$

We get a corresponding mass loss with this process for the black hole:

$$\frac{dM}{dt} \sim AT^4 \propto M^2\frac{1}{M^4} = \frac{1}{M^2} \tag{4.3}$$

We can therefore see that a black hole which is completely isolated will evaporate completely in a time given by:

$$\tau \sim 10^{73}(\frac{M}{M_{\odot}})^3 seconds \tag{4.4}$$

4.1 Analogous quantities

We can draw an analogy of conventional thermodynamical quantities with quantities emerging from black hole physics:

- Mass with Energy
- $\frac{\kappa}{2\pi}$ with temperature (But it is really the Hawking temperature of black holes)
- A/4 with entropy (due to validity of the generalised second law)

4.1.1 The Generalized Second Law

According to the conventional definition of the second law of thermodynamics, δS should always be greater than equal to 0, while the modified black hole area theorem tells us that δA should be greater than equal to 0. Although classically, the area of a black hole never decreases, it does decrease during a quantum particle creation process. So, we may have a modified equation, first suggested by Bekenstein, that reads $\delta S' \geq 0$ where we have S' as:

$$S' = S + \frac{1}{4}\frac{c^3}{G\hbar}A \tag{4.5}$$

We note that here S stands for the entropy of matter which is outside the black hole.

4.1.2 Quantum Entanglement

Now we consider quantum effects like quantum entanglement and question how will a system which is entangled will evolve in a black hole geometry. This results in the information problem of quantum black holes.

We can describe a joint quantum system with the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, given that the quantum system consists of two subsystems having $\mathcal{H}_1$ and $\mathcal{H}_2$ as their Hilbert spaces. The product space will contain not only simple product states like $|\Psi_1\rangle \otimes |\Psi_2\rangle$ but also linear combination of such product states (not being able to expressed as simple products). Now, if this is the case and the joint subsystem cannot be expressed as a simple product, the consisting subsystems are said to be entangled with states of each subsystem mixed.

In quantum field theory, at small spacelike seperations, we can say that the field will always be strongly entangled with itself. We can see from the following formula for a massless Klein Gordon field on Minkowski spacetime :

$$\langle 0| \phi(x)\phi(y) |0\rangle = \frac{1}{4\pi^2}\frac{1}{\sigma(x,y)} \tag{4.6}$$

For no presence of entanglement, the above expression would become 0.

4.2 The Information Problem

In our case, we take a semi classical treatment of gravity as mentioned before. This eases treatment of hawking radiation but leads to the information loss problem. In this picture, if the black hole evaporates completely, we will get a mixed final state, and hence we will have a dynamic evolution of a pure initial state to a mixed final state, and hence have an irreversible information loss into black holes. In the forthcoming sections, the possibilities of the semi-classical picture going wrong in few ways is discussed.

We see that physical information could permanently disappear in a black hole, allowing many physical states to devolve into the same state. This is controversial because it violates a core precept of modern physics—that, in principle, the value

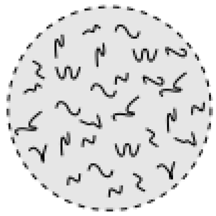


Figure 4.1: Proposed figure of a fuzzball- state information distributed throughout

of a wave function of a physical system at one point in time should determine its value at any other time. A fundamental postulate of the Copenhagen interpretation of quantum mechanics is that complete information about a system is encoded in its wave function up to when the wave function collapses. The evolution of the wave function is determined by a unitary operator, and unitarity implies that information is conserved in the quantum sense.

4.3 Possible Resolutions

4.3.1 Fuzzballs

The first possibility of resolving the information loss problem is that maybe no black hole is ever formed. One possibility may be that the black hole does not have a singularity in the center but there may exist state information distributed throughout the black hole, with microscopic string like hairs all over the black hole. If this is true, both the semi-classical notion of general relativity and quantum field theory would have to go wrong and break in a low curvature/energy scale.

Moreover, if the fuzzball structure does not form at the appropriate moment, there will always be a violation of causality and locality in the low curvature limit too.

4.3.2 Firewalls

A black hole firewall is a hypothetical phenomenon where an observer falling into a black hole encounters high-energy quanta at (or near) the event horizon. This may be a major departure from our treatment of gravity as a semi-classical theory, during evaporation. This alternative will also be very radical in the sense that the destruction of the entanglement that may exist between the inside and the outside of such a black hole during its evaporation will involve a breakdown of quantum field theory in a low curvature limit. In this case, objects such as "Firewalls" need to come into existence at the horizon, in order to ensure the destruction of the entanglement.

It should be noted that although there is no robust theory of firewalls, they would require the breakdown of physical laws near the horizon and would also violate notions of causality and locality, so that it can bring the entanglement from deeper inside the black hole to the outside of the event horizon.

4.3.3 Remnants

As a result of Hawking evaporation of non-singular black holes, there may remain stable remnants with vanishing Hawking temperature. The possibility of a black hole forming very small remnants is not a very physical law breaking alternative because the breakdown of our semi-classical picture only happens near the Planck scale. However, invoking remnants to maintain a pure state is severely problematic since if the remnants do not interact with the external world, it is not clear how it would resolve the problem as the information would still remain inaccessible and in some manner lost. Whereas if the remnants do interact with the outside world then the consequences would have severe thermodynamic problems, as they will contain arbitrarily many states at that tiny Planck scale energy, it should thus be favoured over all other forms of matter thermodynamically.

4.3.4 A final burst

It can also be theorized that an arbitrarily large amount of information may be released from an object of Planck mass and size, thus resolving the information problem. Infact in the work by Hotta, Schutzhold and Unruh, they considered a model of an accelerating mirror in 1+1 spacetime dimensions which emit a moving mirror radiation - similar to the Hawking radiation. These partner particles are indistinguishable from vacuum fluctuations and can carry the information at no energy cost.

But one major caveat to this alternative is that in higher spacetime dimensions, a similar treatment of entanglement correlated with vacuum fluctuations emanating from a small spatial region is not possible.

4.4 Arguments against information loss

4.4.1 Violation of Unitarity

We can say that the phrase "obeying Unitarity" has two connotations:

- Probability being conserved
- Pure states evolving to pure states

The failure of the first condition would imply a serious problem and a breakdown of the quantum theory, but the semi-classical picture does not involve non conservation of probability.

While, the failure of the second condition might occur in any situation where we don't have the final time as a Cauchy surface, this is not a bad condition.

As an example, if we consider the evolution of a massless Klein Gordon field in Minkowski spacetime, we will get an evolution of pure states to mixed states if we choose the final time as a hyperboloid. This per say is not a violation of quantum mechanics and the semiclassical analysis involves a similar treatment.

4.4.2 Failure of Energy and momentum conservation

Banks, Peskin and Susskind showed in their work that the evolution of pure states to mixed states would lead to a violation of the conservation of energy and momentum, however their work considered evolution of a Markovian nature (Lindblad equation). We suspect that this would not actually be a preferable model for the evaporation of a black hole, as the black hole must clearly have a memory of the energy it previously emitted.

According to Wald, a quantum mechanical decoherence process would not require energy exchange and would still sustain the conservation laws of energy if the environment system is taken to be a spin bath. Unruh in his work showed a quantum mechanical system interacting with a hidden spin system in a way so as to have an evolution of pure states to mixed states for the quantum system, with the sustainability of the exact energy conservation.

It is theorized that there might not be much of a problem to maintain the exact conservation of energy and momentum in quantum mechanics when we consider an evolution of pure states to mixed states.

4.5 AdS/CFT

The argument of AdS/CFT against everything discussed in the previous few sections is that if we can draw out an exact correspondence of asymptotically AdS spacetimes and one dimension lower conformal field theories, then since the latter theory would not admit a non unitary evolution of states, such an evolution must not also be possible in the gravity counterpart, if we consider the full quantum gravity picture

(upto the string corrections.)

Although being a robust correspondence, AdS/CFT has not been formulated yet with the degree of precision required to tackle the argument against information loss.

AdS/CFT arguments against the information loss assumes implicitly conditions such as:

- The theorized correspondence will be sufficiently local so that we have a one to one correspondence of late time bulk observables in the gravity side to the late time observables in the CFT side.
- There is a deterministic nature of evolution in the conformal system.

Although not understood clearly, these assumptions would imply that the solutions to Einstein's equation which we get in general relativity will be determined uniquely by the behaviour of spacetime near infinity at some time, this is in contradiction with the gluing theorems of GR.

4.6 Conclusion

Although AdS/CFT appears to be the most viable and promising candidate for giving an explanation of this information loss, more work needs to be done here to completely resolve the problems arising from it.

Chapter 5

Black Holes and branes

String theory is arguably the most compelling candidate for being a consistent theory of quantum gravity. It provides us a successful microscopic description of "certain" black holes, giving a correct reproduction of the Bekenstein Hawking entropy from the explicit sum over states, up to the last numerical factors. This indicates that the right microscopic degrees of freedom are identified within the theory. Moreover, it reduces to conventional general relativity at low energy, providing us a theory which encompasses both the microscopic and macroscopic regimes. So, in this chapter we consider higher dimensional objects known as branes and we see how they are relevant for the study of black holes.

5.1 The metric of branes

For an infinitely extended object, we can consider an ansatz metric :

$$ds^2 = f(x) \sum_{a=1}^{q} dy_a dy_a + h(x) \sum_{i=1}^{p} dx_i dx_i \tag{5.1}$$

in the above equation p+q gives the total number of dimensions of the metric and for simplification we take y_a to be the timelike directions with Euclidean signature. A Lorentz symmetry among all the brane world-volume directions is implied if a solution exists for this ansatz, as the time direction will be symmetrically related to the other directions of the same brane world-volume.

5.1.1 The stress tensor

We can denote a q-form field as $A_{\mu_1 \dots \mu_q}$, so as A will be totally antisymmetric in its indices. The field strength from this gauge field will be:

$$\begin{aligned} F_{\mu_1\mu_2\dots\mu_{0+1}} \equiv & \partial_{\mu_1} A_{\mu_2\dots\mu_0\mu_{q+1}} + (-1)^q \partial_{\mu_2} A_{\mu_3\dots\mu_{0+1}\mu_1} + \partial_{\mu_3} A_{\mu_4\dots\mu_{v+1}\mu_1\mu_2} \\ & + \dots + (-1)^q \partial_{\mu_{q+1}} A_{\mu_1\dots\mu_q} \end{aligned} \tag{5.2}$$

We can write an action similar to what we did in electromagnetism, F being antisymmetric in its indices too:

$$S = -\frac{1}{2} \int d^D \xi \frac{1}{(q+1)!} \sqrt{-g} F^2 \tag{5.3}$$

Similarly, we can write the stress energy tensor as:

$$T_{\mu\nu} = \frac{1}{q!} F_{\mu\mu_1\dots\mu_0} F_{\nu}^{\mu_1\dots\mu_0} - \frac{1}{2} g_{\mu\nu} \frac{1}{(q+1)!} F^2 \tag{5.4}$$

Now, we consider a compatible ansatz for our gauge field (with the ansatz metric), let it be:

$$A_{12\dots q} = A(x) \tag{5.5}$$

Hence, A will only be a function of x, the ansatz are chosen for reasons of symmetry and such that the metric is only dependent on the x coordinate, and the coefficients or the gauge does not depend on the transverse y coordinate. The field strength will be:

$$F_{i12\dots q} = A_{,i} \tag{5.6}$$

Here i is the index for the x coordinates, running up till p.

$$T_{ab} = \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} - \frac{1}{2} \delta_{ab} f h^{-1} f^{-q} A_{,i} A_{,i} = \frac{1}{2} \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} \tag{5.7}$$

$$T_{ij} = A_i A_{,j} f^{-q} - \frac{1}{2} \delta_{ij} h h^{-1} f^{-q} A_{,k} A_{,k} = f^{-q} \left[A_{,i} A_{,j} - \frac{1}{2} \delta_{ij} A_{,k} A_{,k} \right] \tag{5.8}$$

$$T = \frac{q+2-p}{2} f^{-q} h^{-1} A_{,k} A_{,k} \tag{5.9}$$

and we get :

$$\begin{aligned} T_{ab} - \tfrac{1}{p+q-2} g_{ab} T &= \delta_{ab} h^{-1} f^{-(q-1)} A_{,i} A_{,i} \tfrac{p-2}{p+q-2} \\ T_{ij} - \tfrac{1}{p+q-2} g_{ij} T &= f^{-q} \left[A_{i,i} A_{,j} - \tfrac{q}{p+q-2} \delta_{ij} A_{,k} A_{,k} \right] \end{aligned} \tag{5.10}$$

5.1.2 Coefficient comparison

In order to find a solution, we choose an ansatz for it as:

$$h = f^{\nu} \tag{5.11}$$

$$A = C f^{\frac{\nu}{2}} \tag{5.12}$$

We know that the Einstein's equation is as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \tag{5.13}$$

In terms of T and the dimensions, we can write:

$$R\left(1 - \frac{D}{2}\right) = T, \quad R = \frac{2T}{2-D} \tag{5.14}$$

So, the Einstein's equation can also be written as:

$$R_{\mu\nu} = T_{\mu\nu} + \frac{1}{2}\frac{2T}{2-D} = T_{\mu\nu} - \frac{1}{D-2} T \tag{5.15}$$

We have,

$$h_{,ij} = f^{\nu}_{,ij} = \nu \left[f^{\nu-1} f_{,i} \right]_{,j} = \nu f^{\nu-1} f_{,ij} + \nu(\nu-1) f^{\nu-2} f_{,i} f_{,j} \tag{5.16}$$

Now we compare the respective coefficients from our stress tensor equation and the equation given above.

For $f_{,ij}$

In our ansatz, we see that only the first term in lhs can be comparable to the einstein tensor, the term is:

$$f^{-1}[-\frac{q}{2}+\frac{2\nu}{2}-\frac{p\nu}{2}] \tag{5.17}$$

We do not get any more coefficients of such a form as the Einstein's equations $G_{ab} = T_{ab}$ does not have such a term.

$f_{,kk}$

We follow the same treatment and find that the RHS does not have such a term but the LHS has the coefficient:

$$\delta_{ij}f^{-1}\left[-\frac{1}{2}\nu-\frac{1}{2}\left[-\frac{q}{2}-\frac{q}{2}\nu(2-p)-\nu\frac{p}{2}\right]\right]=\delta_{ij}f^{-1}\left[\frac{q}{4}+\frac{\nu}{4}[-2+q(2-p)+p]\right] \tag{5.18}$$

Unlike being identically 0, we require the following condition:

$$f=v^{\mu}, \quad f_{,k}=\mu v^{\mu-1}v_{,k}, \quad f_{,kk}=\mu(\mu-1)v^{\mu-2}v_{,k}v_{,k}+\mu v^{\mu-1}v_{,kk} \tag{5.19}$$

Also,

$$v_{kk}=0 \tag{5.20}$$

From the above relations, we get:

$$f_{kk}=\frac{\mu-1}{\mu}f^{-1}f_{,k}f_{,k}\equiv\lambda f^{-1}f_{,k}f_{,k} \tag{5.21}$$

We get from $h=v^{\nu\mu}$, that

$$h_{,kk}=\bar{\lambda}h^{-1}h_{,k}h_{,k} \tag{5.22}$$

Where

$$\bar{\lambda}=\frac{\mu\nu-1}{\mu\nu} \tag{5.23}$$

From here too, we get no such new relation.

$f_{,k}f_{,k}$

By the similar treatment, we do not get other equations from the $f_{,i}f_{,j}$ term except:

$$C^2 = -\frac{p+q-2}{q(p-2)} \tag{5.24}$$

Now we look into the equation:

$$R_{ab} = T_{ab} - \frac{1}{q+p-2}T \tag{5.25}$$

and we similarly get:

$$\frac{\nu}{2} - \frac{\lambda}{2} - \frac{q-2}{4} - \frac{p\nu}{4} = C^2\frac{q^2}{4}\frac{p-2}{p+q-2} \tag{5.26}$$

$$\frac{-q\nu}{4} + \frac{\nu^2}{2} - \frac{\bar{\lambda}\nu^2}{2} - \frac{(p-2)\nu^2}{4} = -\frac{q}{p+q-2}C^2\frac{q^2}{4} \tag{5.27}$$

These equations could be solved to get :

$$C^2 = -\frac{q+p-2}{q(p-2)} \tag{5.28}$$

$$\lambda = \frac{q}{2} + 1, \quad \rightarrow \quad \mu = -\frac{2}{q} \tag{5.29}$$

$$\bar{\lambda} = \frac{4-p}{2}, \quad \rightarrow \quad \bar{\mu} = \mu\nu = \frac{2}{p-2} \tag{5.30}$$

5.1.3 The final solution

After getting all the coefficient terms by comparison with the Einstein's equation in p+q=D dimensions, we can write explicitly the form of the brane metric in terms of p,q and ν :

$$ds^2 = v^{-\frac{2}{9}}dy_a dy_a + v^{\frac{2}{p-2}}dx_i dx_i \tag{5.31}$$

The gauge potential here will be:

$$A_{a_1\ldots a_0} = \left[-\frac{p+q-2}{q(p-2)}\right]^{1/2} v^{-1} \tag{5.32}$$

Although an imaginary coefficient of A may look strange, it is important to note that a Wick rotation had been performed to continuation to Euclidean signature.

We can also examine some metrics of our importance in string theory, the first being from 11-d supergravity, where the sources arise from a 2-brane and a 5-brane (p=8,q=3). The metric for the 2-brane with a 3d worldvolume will be :

$$ds^2 = v^{-\frac{2}{3}} dy_a dy_a + v^{\frac{1}{3}} dx_i dx_i \tag{5.33}$$

and the metric for the 5-b4ane with a 6d worldvolume (p=5,q=6) will be:

$$ds^2 = v^{-\frac{1}{3}} dy_a dy_a + v^{\frac{2}{3}} dx_i dx_i \tag{5.34}$$

5.2 Dynamic and thermodynamic stability of Black branes

5.2.1 Stability of Black Holes and Black Branes

Black Holes are the perfect possibility of being end products of gravitational collapse in general relativity in 4-dimensional spacetimes. We see that:

- It is important to determine whether the Kerr black hole is stable.
- Black holes in higher dimensional space-time play very important roles in both general relativity and string theory.
- As there are a wide range of black hole solutions in higher dimensions, it is interesting and important to find their stability.

Now, we consider the stability of black brane solutions, which takes the metric form in (D+p) dimensions as:

$$d\tilde{s}^2_{D+p} = ds_D{}^2 \sum_{i=1}^{p} dz_i{}^2 \tag{5.35}$$

In the above equation, ${ds_D}^2$ is a black hole metric.

We can define a quantity as the canonical energy (ϵ) for a small perturbation $_{ab}$ of a black hole. The positivity of this ϵ should be sufficient for indicating linear stability to these perturbations (axisymmetric).

- for negative ϵ one may have mode stability (no exponentially growing perturbations)
- the γ_{ab} cannot reach a stationary approach at late times for negative ϵ values.

5.2.2 Stability of Black Holes and Black Branes contd.

There is a correspondence here of black holes and black branes with thermodynamic systems and thus we can use similar treatments to extract important and interesting results regarding the stability of such systems. As Black branes are homogeneous systems, the sufficient condition for showing instability of a black brane would be that the matrix:

$$H_A = \begin{pmatrix} \frac{\partial^2 A}{\partial M^2} & \frac{\partial^2 A}{\partial J_i \partial M} \\ \frac{\partial^2 A}{\partial M \partial J_i} & \frac{\partial^2 A}{\partial J_i \partial J_j} \end{pmatrix} \tag{5.36}$$

will have a positive eigevalue. First conjectured by Gubser and Mitra, this condition should be sufficient to indicate the instability of black branes.

5.2.3 Horizon Gauge conditions

We can consider stationary black holes with a positive surface gravity, which have a bifurcate event horizon, with the bifurcation on a surface. If we consider an arbitrary perturbation $\gamma = \delta g$ then the gauge condition would ensure that the location of the horizon does not change within the first order. The condition is given by:

$$\delta v|_B = 0 \tag{5.37}$$

5.2.4 Canonical Energy

We can define the canonical energy of a perturbation $\gamma = \delta g$ by:

$$\epsilon = W_\Sigma(g; \gamma, \mathcal{L}_t \gamma) \tag{5.38}$$

Here it should be noted that our fundamental identity will have a second variation which will yield:

$$\epsilon = \delta^2 M - \Sigma \Omega_i \delta^2 J_i - \frac{\kappa}{8\pi} \delta^2 A \tag{5.39}$$

5.2.5 Instability of black branes

Theorem : If we take a family of black holes which are parameterized by their mass and angular momentum such that at a particular (M_0, J_{0A}) there will exist a perturbation within the considered black hole family (for $\epsilon < 0$). Then for a corresponding black brane, we can find a sufficiently long wavelength perturbation for which the variation of the observables (like mass, temperature, pressure and area) is 0 and $\tilde{\epsilon} < 0$

The result, although not derived rigorously by us, is proven to be true by modifying the perturbations of the initial data for the black hole with $\epsilon < 0$ and then readjusting the modified data to fit the constraints.

But we are not over with the linear stability theory of black holes because in this work the formula for ϵ is very complicated and the initial data bust support all the linearized constraints, hence it is not easy to determine the sign of ϵ. Moreover only axisymmetric perturbations are studied in this work and there is a long way of developing the linear stability and instability from just the sign of ϵ.

5.2.6 Positivity of kinetic energy

We cab break up the canonical energy term into a kinetic energy and a potential energy term. Here, the kinetic energy will arise from the perturbation part which is odd under a reflection in t-ϕ, while the potential energy will arise from the perturbation part which is even.

We see that the kinetic energy will always be a positive quantity in any perturbative case of any black hole or a black brane. Moreover, the perturbation must grow exponentially if the potential energy is negative for a perturbation with the form $\mathcal{L}_t\gamma'_{ab}$.

5.2.7 Main Conclusion

We can draw a correspondence between the dynamical stability of a black hole and its thermodynamic stability with respect to the axisymmetric perturbations of the black hole. Hence this is a very interesting connection which relates the laws of black hole physics and the laws of thermodynamics, extending it to the dynamical stability.

We see that branes like p-branes and D1-D5 branes are interesting systems to study black holes and they incorporate various theories from string theory, whose full understanding will be a future objective of this thesis.

Chapter 6

Generating Solutions from String actions

6.1 String actions

In this thesis I do not delve deep into the detailed physics of string theory but rather use some insightful results which plays an important role in constructing the string theory description of black holes. Here I discuss only Type IIA and IIB supergravity theories. They posses supersymmetry in 10 dimensions and essentially have 32 real supercharges. The Type IIA theory is nonchiral (because spinors have opposite chirality) while the Type IIB theory has two Majorana-Weyl 16-component spinors with the same chirality. The effective Type IIA d=10 supergravity string action in the low energy regime is:

$$\begin{aligned} S_A =& \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x\sqrt{-G} \left\{ \frac{e^{-2\Phi}}{g_s^2} \left[R_G + 4(\partial\Phi)^2 - \frac{3}{4}\left(\partial B_2\right)^2 \right] + (\text{ fermions }) \right. \\ & \left. -\frac{1}{4}\left(2\partial C_1\right)^2 - \frac{3}{4}\left(\partial C_3 - 2\partial B_2 C_1\right)^2 \right\} + \frac{1}{64}\epsilon\partial C_3 \partial C_3 B_2 \end{aligned} \tag{6.1}$$

The dilaton field is shifted here so that the zero is at infinity. The antisymmetrisation looks like:

$$(\partial A)_{\mu\nu} \equiv \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{6.2}$$

the Hodge dual (8-n)-form potentials are not used in the above action to avoid the existence of both electric and magnetic potentials which would otherwise result in propagating ghosts. Additionally, supersymmetry requires the cross terms to be present.

There is no covariant action in type IIB string theory because the R-R (Ramond-Ramond) 5-form field strength $F_5^+ = \partial C_4$ is dual to itself. So we cannot derive the field equations from a covariant action. Now we define $\tilde{H}_3 \equiv \partial C_2, \ell \equiv C_0; H_3 \equiv \partial B_2$. We get the equation of motions for the metric.

$$\begin{aligned} R_{\mu\nu} =& 2\nabla_\mu \partial_\nu \Phi - \frac{9}{4} H^{\lambda\rho}_{(\mu} H_{\nu)\lambda\rho} - e^{2\Phi}\frac{1}{2}\left(\partial_\mu \ell \partial_\nu \ell - \frac{1}{2} G_{\mu\nu}(\partial \ell)^2\right) \\ &+ \frac{9}{4} e^{2\Phi}\left[(\bar{H} - \ell H)^{\lambda\rho}_{(\mu} H_{\nu)\lambda\rho} - \frac{1}{6} G_{\mu\nu}(\bar{H} - \ell H)^2\right] + \frac{25}{6} e^{2\Phi}\left(F_{\mu\lambda\rho\sigma\kappa} F_\nu^{\lambda\rho\sigma\kappa}\right) \end{aligned}$$

while for the scalars they are

$$\begin{aligned} \nabla^2 \Phi &= (\partial\Phi)^2 + \frac{1}{4} R_G + \frac{3}{16} H^2 \\ \nabla^2 \ell &= -\frac{3}{2} H^{\mu\lambda}(\bar{H} - \ell H)_{\mu\nu\lambda} \end{aligned}$$

and for the gauge fields

$$\begin{aligned} \nabla^\mu \left[\left(\ell^2 + e^{-2\Phi}\right) H - \ell \bar{H}\right]_{\mu\nu\rho} &= +\frac{10}{3} F_{\nu\rho\sigma\lambda\kappa} \bar{H}^{\sigma\lambda\kappa}, \\ \nabla^\mu [\bar{H} - \ell H]_{\mu\nu\rho} &= -\frac{10}{3} F_{\nu\rho\sigma\lambda\kappa} H^{\sigma\lambda\kappa} \\ F_5^+ &= {}^*F_5^+ \end{aligned}$$

The p-brane in 10 dimensions couple in two ways - C_{p+1} or C_{7-p}. This is analogous to when a charged particle in electromagnetism couples to its field strength F_2 electrically or its dual field strength $\star F_2$ magnetically.In the p-brane case, we find that 1-branes have F1 coupling while 5-branes have NS5 coupling to the NS-NS potential (B_2), while p-branes are Dp coupled to the R-R potentials (C_{p+1}). It must be noted that the branes here are on T^d or $\mathbb{R}^d$, here further intricate effects would not play a part.

6.2 Conserved Quantities

String theory has a lot of objects with conserved quantum numbers. For example, the conserved energy in rest frame becomes the mass M. In D dimensions, the skew matrix is also conserved which has eigenvalues as the independent angular momenta. The gauge charge Q, which couples to the long range R-R field is also conserved. The low energy action for a brane with supergravity as bulk is :

$$S = S_{SUGRA} + S_{brane} \tag{6.3}$$

This is a well defined action in string theory. In 10 dimensions, p-branes which have a codimension of less than three gives rise to non-asymptotically flat spacetimes which are not relevant in our case. Hence we consider p¡7. In the Einstein frame the action reads:

$$S = \int d^D x \left(\frac{\sqrt{-g} R[g]}{16\pi G_D} + \mathcal{L}_{\text{matter}} \right) \tag{6.4}$$

Here the einstein metric g is defined in terms of the string metric G as:

$$g_{\mu\nu} = \exp^{-4\Phi/(D-2)} G_{\mu\nu} \tag{6.5}$$

As a result, the field equation for this metric in D dimensions will be:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_D T_{\mu\nu}^{(matter)} \tag{6.6}$$

In the above equation, the Ricci tensor is denoted by $R_{\mu\nu}$ and the energy momentum tensor is denoted by $T_{\mu\nu}^{(matter)}$. As we see that the metric becomes asymptotically flat, we can linearize it about the minkowski metric and consider only the first order deviations, getting:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.7}$$

The deviation will transform as $\delta h_{\mu\nu} = -2\partial_{(\mu}\xi_{\nu)}$ under the coordinate transformation $\delta x^\mu = \xi^\mu$. The below condition fixes the symmetry but only partially, the harmonic gauge condition reads:

$$\partial_\nu \left(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h_\lambda^\lambda \right) = 0 \tag{6.8}$$

The field equation for the deviation can be calculated:

$$\left(\partial^i \partial_i\right) h_{\mu\nu} = 16\pi G_D \left[T_{\mu\nu}^{(\text{matter })} - \frac{1}{(D-2)} \eta_{\mu\nu} T_\lambda^{(\text{matter })\lambda} \right] \equiv -16\pi G_D \tilde{T}_{\mu\nu}. \tag{6.9}$$

The Laplace equation is the field equation for the harmonic gauge, the solution is given by:

$$h_{\mu\nu}(x) = \frac{16\pi G_D}{(D-3)\Omega_{D-2}} \int d^{D-1}\vec{y} \frac{\vec{T}_{\mu\nu}(|\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|^{D-3}}, \tag{6.10}$$

Here the prefactor terms are the D dimensional Green's function and the area of a n-sphere. Expanding the h term in moments, we get:

$$h_{\mu\nu}(x) = \frac{16\pi G_D}{(D-3)\Omega_{D-2}} \left\{ \frac{1}{r^{D-3}} \int d^{D-1}y\tilde{T}_{\mu\nu}(y) + \frac{x^j}{r^{D-1}} \int d^{D-1}yy^j\tilde{T}_{\mu\nu}(y) + \cdots \right\} \tag{6.11}$$

Now we see the form of the ADM linear and angular momenta:

$$P^{\mu} = \int d^{D-1}yT^{\mu 0} \quad J^{\mu\nu} = \int d^{D-1}y \left(y^{\mu}T^{\nu 0} - y^{\nu}T^{\mu 0} \right) \tag{6.12}$$

Let us evaluate the metric in rest frame for the sake of simplification. We get:

$$g_{tt} \longrightarrow -1 + \frac{16\pi G_D}{(D-2)\Omega_{D-2}} \frac{M}{r^{D-3}} + \cdots; \tag{6.13}$$

$$g_{ij} \longrightarrow 1 + \frac{16\pi G_D}{(D-2)(D-3)\Omega_{D-2}} \frac{M}{r^{D-3}} + \cdots \tag{6.14}$$

$$g_{ti} \longrightarrow \frac{16\pi G_D}{\Omega_{D-2}} \frac{x^j J^{ji}}{r^{D-1}} + \cdots \tag{6.15}$$

For Dp-branes, the action part which is relevant to analyse the conserved charges carried by the branes (other than the bulk part) is:

$$S_{\text{brane}} = -\frac{1}{(2\pi)^p \ell_s^{p+1}} \int C_{p+1} + \ldots \tag{6.16}$$

C_{p+1} refers to the R-R potential (or its Hodge dual). Ignoring the cross terms in the action (with supergravity), we get:

$$S_{\text{SUGRA}} = \frac{-1}{(2\pi)^7 \ell_s^8} \int d^{10}x\sqrt{-G} \frac{|(p+2)[\partial C]_{p+2}|^2}{2(p+2)!} + \ldots \tag{6.17}$$

We get the field equation:

$$d^* \left(dC_{p+1} \right) = (2\pi)^7 \ell_s^{8*} \left(J_{p+1} \right) \tag{6.18}$$

The conserved current (p+1) "J" can be derived as:

$$J_{p+1}(x) = -(2\pi)^p \ell_{\mathrm{s}}^{p+1} \int dX^0 \dots dX^p \delta^{10}(X - x) \tag{6.19}$$

We see that it is easier to visualize the physics in the static gauge given below:

$$X^{\mu_i}(\sigma) = \sigma^{\mu_i}, \quad i = 0 \dots p \tag{6.20}$$

In this form, the conserved Noether charge can be written as the integral of the current. We use the field equations derived to form:

$$Q_p = \int_{S^{8-p}}^{*} (dC)_{p+2} \tag{6.21}$$

The Bianchi identity is additionally present, other than the field equation C, this is:

$$d\left([dC]_{p+2}\right) = 0 \tag{6.22}$$

This identity can be used to show the presence of another charge - the topological charge:

$$P_{7-p} = \textstyle\int_{S^{p+2}} (dC)_{p+2} \tag{6.23}$$

These follow the Dirac quantisation condition - $Q_p P_{7-p} = 2\pi n, \quad n \in \mathbb{Z}$.

6.3 Some Important Results

The fundamental string tension is of the form:

$$\tau_{\mathrm{F1}} = \frac{1}{2\pi\alpha'} \equiv \frac{1}{2\pi\ell_{\mathrm{s}}^2} \tag{6.24}$$

In the same way, the tensions for the Dp-brane and NS5-brane are:

$$\tau_{\mathrm{D}p} = \frac{1}{g_{\mathrm{s}}(2\pi)^p \ell_{\mathrm{s}}^{p+1}} \tag{6.25}$$

$$\tau_{\text{NS5}} = \frac{1}{g_s^2 (2\pi)^5 \ell_s^6} \tag{6.26}$$

We can relate the Newton's constant to the gravitational coupling, in 10 dimensions, by:

$$16\pi G_{10} \equiv 2\kappa_{10}^2 = (2\pi)^7 g_s^2 \ell_s^8 \tag{6.27}$$

We can find any lower dimensional Newton's constant from the ten dimensional one by the formula:

$$G_d = \frac{G_{10}}{(2\pi)^{10-d} V_{10-d}} \tag{6.28}$$

We redefine the planck length in d dimensions:

$$16\pi G_d \equiv (2\pi)^{d-3} \ell_d^{d-2} \tag{6.29}$$

With these re-definitions, we can consistently say that there is a connection/translation of the Bekenstein-Hawking entropy earlier defined to the Bekenstein Hawking entropy in higher dimensions. If we take a p-brane wrapped on T^p (to make a 10-p dimensional black hole), we have a translational symmetry along the p-brane. The entropy of the given black hole is:

$$\begin{aligned} S_{\text{BH}} = \frac{A_{d+p}}{4G_{d+p}} &= \frac{A_d (2\pi)^p V_p}{4G_{d+p}} \\ &= \frac{A_d}{4G_d} \end{aligned} \tag{6.30}$$

We notice that this is identical to the black hole entropy derived in the earlier chapter. We can then compare the event horizon area in this formula with the einstein frame. The kinetic term for the metric is canonically normalized in this frame.

$$S_{\text{grav}} = \frac{1}{16\pi G_d} \int \sqrt{-g} R[g] \tag{6.31}$$

Now we give an example of Kaluza Klein reduction of fields which are in the string frame, we do this by reducing on a circle of radius R. Labelling the d dimensional system with hats, the (d-1) system without hats and splitting the indices, we get the following results where the vielbeins decompose as:

$$\left(E_{\mu}^{a}\right)=\begin{pmatrix}\hat{E}_{\mu}^{\hat{a}} & e^{\dot{X}}\hat{A}_{\hat{\mu}}\\ 0 & e^{\dot{x}}\end{pmatrix}\quad\Rightarrow\quad\left(G_{\mu\nu}\right)=\begin{pmatrix}\hat{G}_{\hat{\nu}\hat{\nu}}+e^{2\dot{\chi}}\hat{A}_{\hat{A}}\hat{A}_{\bar{\nu}} & e^{2\hat{\chi}}\hat{A}_{\hat{\mu}}\\ \hat{A}_{\hat{\nu}}e^{2\hat{\chi}} & e^{2\dot{\chi}}\end{pmatrix}\tag{6.32}$$

$$\Phi=\hat{\Phi}+\frac{1}{2}\hat{\chi}\frac{1}{16\pi G_{d}}\int d^{d}x\sqrt{-G}e^{-2\Phi}R_{G}=\tag{6.33}$$

$$\frac{1}{16\pi G_{d-1}}\int d^{d-1}x\sqrt{-\hat{G}}e^{-2\hat{\Phi}}\left[R_{G}+4(\partial\hat{\Phi})^{2}-(\partial\hat{\chi})^{2}-\frac{1}{4}e^{2\dot{\chi}}(2\partial\hat{A})^{2}\right]\tag{6.34}$$

This procedure can also be done in the Einstein frame. After using the metric:

$$ds^{2}=e^{2\alpha\dot{\chi}}d\hat{s}^{2}+e^{2\beta\hat{\chi}}\left(dz+\hat{A}_{\hat{\mu}}dx^{\hat{\mu}}\right)^{2}\tag{6.35}$$

where $\beta=(2-D)\alpha$ and $\alpha^{2}=1/[2(D-1)(D-2)]$, we get(Here F again is the field strength):

$$\sqrt{-g}R_{g}=\sqrt{-\hat{g}}\left(R_{\tilde{g}}-\frac{1}{2}(\partial\hat{\chi})^{2}-\frac{1}{4}e^{-2(D-1)\alpha\dot{\chi}}F^{2}\right),\tag{6.36}$$

Let's now analyze some dualities.

6.3.1 Type IIA - M-theory duality

We can compactify the 11th coordinate on a circle of radius $R_4=g_{\mathrm{s}}\ell_{\mathrm{s}}$, then the fields with supergravity has the following decomposition equations:

$$\begin{aligned}ds_{11}^{2}&=e^{-2\Phi/3}dS_{10}^{2}+e^{4\Phi/3}\left(dx^{h}+C_{1\mu}dx^{\mu}\right)^{2}\\(\partial A_{3})&=e^{4\Phi/3}\left(\partial C_{3}-2H_{3}C_{1}\right)+\frac{1}{2}e^{\Phi/3}\left(\partial B_{2}\right)dx_{\mathrm{b}}.\end{aligned}\tag{6.37}$$

Interestingly, M-theory objects can be turned into Type IIA objects (the vertical arrow shows what happens if we dont point it towards the 11th direction, while the diagonal arrow shows how it transforms if pointed in the 11th direction).

$$\begin{array}{cccccccc} & \mathrm{W} & & \mathrm{M2} & & \mathrm{M5} & & \mathrm{KK} \\ \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathrm{D0} & \mathrm{W} & \mathrm{F1} & \mathrm{D2} & \mathrm{D4} & \mathrm{NS5} & \mathrm{D6} & \mathrm{KK} \end{array} .$$

6.3.2 The S-duality of Type-IIB string theory

Type-IIB supergravity is the low energy limit of type-IIB string theory. It has a SL(2,R) symmetry which is SL(2,Z) in the full string theory picture. We can define:

$$\lambda \equiv C_0 + ie^{-\Phi} \quad \text{and} \quad H \equiv \begin{pmatrix} \partial B_2 \\ \partial C_2 \end{pmatrix} \tag{6.38}$$

The fields transform under the representation matrix:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \tag{6.39}$$

as

$$H \to UH, \quad \lambda \to \frac{a\lambda + b}{c\lambda + d} \tag{6.40}$$

The 10 dimensional Einstein metric and the self-dual 5-form field strength are invariant. We can check that the tensions of F1 and D1 transform into each other under operations like a Z_2 flip.

6.3.3 T-duality

T-duality on a circle switches the winding and momentum modes of fundamental strings. The radius is inverted in string units and the d-1 dimensional string coupling is left unchanged.

$$\frac{\tilde{R}}{\tilde{\ell}_{\mathrm{s}}} = \frac{\ell_{\mathrm{s}}}{R}, \quad \frac{\tilde{g}_{\mathrm{s}}}{\sqrt{\tilde{R}/\tilde{\ell}_{\mathrm{s}}}} = \frac{g_{\mathrm{s}}}{\sqrt{R/\ell_{\mathrm{s}}}} \quad , \quad \tilde{\ell}_{\mathrm{s}} = \ell_{\mathrm{s}}. \tag{6.41}$$

Unlike other cases, T-duality will not leave all branes invariant instead it demands a change in dimension of the D-brane subject to the transformation performed on a circle which is parallel or perpendicular to the given worldvolume. Moreover, performing a T-duality along the direction of isometry of the KK will give us an NS-5.Hence:

$$\mathrm{D}p \leftrightarrow \mathrm{D}p-1(\|) \text{ or } \mathrm{D}p+1(\perp), \quad \text{KK (isom) } \leftrightarrow \text{NS5} \tag{6.42}$$

If z is the isometry direction then the T-duality rules on an NS-NS field are as follows:

$$\begin{aligned} &e^{2\Phi} = e^{2\Phi}/G_{zz}, \quad \tilde{G}_{zz} = 1/G_{zz}, \quad \tilde{G}_{\mu z} = B_{\mu z}/G_{zz}, \quad \tilde{B}_{\mu z} = G_{\mu z}/G_{zz} \\ &\tilde{G}_{\mu\nu} = G_{\mu\nu} - \left(G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z}\right)/G_{zz} \\ &\tilde{B}_{\mu\nu} = B_{\mu\nu} - \left(B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z}\right)/G_{zz} \end{aligned} \tag{6.43}$$

It must be noted here that the T-duality formulae for fields with supergravity will be applied only if the direction along which the T-duality acts is the isometry direction.

6.4 Example of generating solutions

Due to the nonlinear nature of supergravity actions, it is generally very difficult to find new solutions of it. We have the no-hair theorem at our aid here, it says that if we can determine the conserved charges of the system then the spacetime geometry we consider is unique. It must be noted that this theorem fails in spacetimes which have naked singularities present. There are purely algebraic solution generating techniques which we use at our disposal.

Let us consider a d-1 dimensional neutral black hole, this is a generalised version of the Schwarzschild black hole:

$$d\hat{S}^2_{d-1} = -(1-K(\rho))dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega^2_{d-3} \tag{6.44}$$

$$K(\rho) \equiv \left(\frac{r_H}{\rho}\right)^{d-4} \tag{6.45}$$

We again compute the metric after analyzing the agreement of the harmonic gauge condition in this case, like we did earlier. Hence from:

$$g_{tt} \sim -1 + \frac{16\pi G_{d-1} M_{d-1}}{(d-3)\Omega_{d-3}\rho^{d-4}}, \tag{6.46}$$

We get:

$$M_{d-1} = \frac{(d-3)\Omega_{d-3} r_H^{d-4}}{16\pi G_{d-1}}. \tag{6.47}$$

Now we explain a procedure termed as lift. As the black hole is a solution of the d-1 dimensional Einstein equations, if we take a direct product of it with the real line, then it will satisfy the d dimensional Einstein equations. Lifting results in a configuration in d dimensions which has a translational invariance in the direction it is lifted.Here:

$$\begin{aligned} dS_d^2 &= dz^2 - (1-K(\rho))dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \\ &= (-dt^2 + dz^2) + K(\rho)dt^2 + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2. \end{aligned} \tag{6.48}$$

We do the given boost to the system:

$$\begin{pmatrix} dt \\ dz \end{pmatrix} \to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} dt \\ dz \end{pmatrix}. \tag{6.49}$$

This boosting takes solutions to solutions and is a general procedure used to construct new solutions. The transformed metric is:

$$\begin{aligned} dS_d^2\prime &= \left(-dt^2 + dz^2\right) + K(\rho)(\cosh\gamma dt + \sinh\gamma dz)^2 \\ &\quad + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \\ &= -\, dt^2\left(1 - K(\rho)\cosh^2\gamma\right) + dz^2\left(1 + K(\rho)\sinh^2\gamma\right) \\ &\quad + 2dtdz\cosh\gamma\sinh\gamma K(\rho) + (1-K(\rho))^{-1}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \end{aligned} \tag{6.50}$$

It must be noted here that the horizon ($G^{\rho\rho} \to 0$) occurs when the value of K(ρ) goes to unity, in other terms when $\rho = r_H$. It is not at $G_{tt} = 0$. We see that adding longitudinal momentum can make the compactified dimension larger at the horizon. Now, if we KK down the d-1 dimensional black hole, we get:

$$dS_d^2 = d\hat{S}_{d-1}^2 + e^{2\hat{\chi}}\left(dz + \hat{A}_\mu dz^\mu\right)^2 \tag{6.51}$$

$$e^{\Phi} = e^{\hat{\Phi}+\frac{1}{2}\hat{x}} \tag{6.52}$$

As an example, we can have:

$$\hat{G}_{tt} = G_{tt} - G_{tz}^2/G_{zz} = -1 + K\cosh^2\gamma - \frac{(K\cosh\gamma\sinh\gamma)^2}{\left(1 + K\sinh^2\gamma\right)} \tag{6.53}$$

and we obtain here:

$$d\hat{S}_{d-1}^{2\prime} = \frac{-(1-K(\rho))}{\left(1+K(\rho)\sinh^2\gamma\right)}dt^2 + \frac{1}{(1-K(\rho))}d\rho^2 + \rho^2 d\Omega_{d-3}^2 \tag{6.54}$$

$$\begin{aligned} \hat{A}_t &= \frac{K(\rho)\cosh\gamma\sinh\gamma}{\left(1+K(\rho)\sinh^2\gamma\right)} \\ e^{\hat{\Phi}} &= e^{-\frac{1}{2}\hat{x}} = \left(1+K(\rho)\sinh^2\gamma\right)^{-\frac{1}{4}} \\ M' &= \frac{\Omega_{d-3}r_H^{d-4}}{16\pi G_{d-1}}\left[(d-3)+(d-4)\sinh^2\gamma\right] \\ Q' &= R\frac{\Omega_{d-3}r_H^{d-4}}{16\pi G_{d-1}}\left[\frac{1}{2}\sinh(2\gamma)\right] \end{aligned} \tag{6.55}$$

The last two equations show the conserved quantum numbers of the new spacetime. If we want to regain the original geometry, then we can simply take $\gamma \to 0$

We then take the opposite limit, which is $\gamma \to \infty$ in the procedure given below:

$$\frac{1}{2}r_H^{d-4}e^{2\gamma} \equiv k = \text{ fixed} \quad , \quad \text{so } K(\rho) = \frac{k}{\rho^{d-4}} \tag{6.56}$$

In the light cone coordinates at the higher dimension, we get:

$$dS_d^2 = -2dz^+\left[dz^- - \frac{k}{\rho^{d-4}}dz^+\right] + \left(d\rho^2 + \rho^2 d\Omega_{d-3}^2\right) \tag{6.57}$$

This denotes the gravitational wave with no ADM mass in d dimensions. For the d-1 dimensional black hole, the same limit yields an extremal black hole with zero Hawking temperature, these two are connected via:

$$M_d^2 = 0 = M_{d-1}^2 - \frac{Q^2}{R^2} \tag{6.58}$$

This gravitational wave is a purely gravitational BPS object in string theory, so is the KK monopole. The metric reads:

$$\begin{aligned} &ds^2 = -dt^2 + dy_{1\cdots5}^2 + H^{-1}(x)\left(dz + A_i dx^i\right)^2 + H(x)dx_{1\cdots3}^2, \\ &2\partial_{[i}A_{j]}(x) = \epsilon_{ijk}\partial_k H(x) \end{aligned} \tag{6.59}$$

The azimuthal angle must have a periodicity of 4π in order to avoid conical singularities.

In the Boyer-Lindqvist-type coordinates, if we set G_d to unity temporarily and have one angular momentum a, we have the higher dimensional metric for a Kerr-type black hole as:

$$\begin{aligned} ds_d^2 = &- \frac{\left(\rho^2 + a^2\cos^2\theta - 2m\rho^{5-d}\right)}{\left(\rho^2 + a^2\cos^2\theta\right)}dt^2 + 2dtd\varphi\frac{2m\rho^{5-d}a\sin^2\theta}{\left(\rho^2 + a^2\sin^2\theta\right)} \\ &+ \frac{\sin^2\theta}{\left(\rho^2 + a^2\cos^2\theta\right)}\left[\left(\rho^2 + a^2\right)\left(\rho^2 + a^2\cos^2\theta\right) + 2ma^2\sin^2\theta\rho^{5-d}\right]d\varphi^2 \\ &+ \frac{\left(\rho^2 + a^2\cos^2\theta\right)}{\rho^2 + a^2 - 2m\rho^{5-d}}d\rho^2 + \left(\rho^2 + a^2\cos^2\theta\right)d\theta^2 + \rho^2\cos^2\theta d\Omega_{d-4}^2. \end{aligned} \tag{6.60}$$

The horizon for the above metric is at:

$$\rho^2 + a^2 - 2m\rho^{5-d} = 0 \tag{6.61}$$

The similar equations and analysis for this metric is far more complicated with more angular momentum parameters. We can have these type of objects as the starting point for generating rotating string and brane solutions using boosts.

Chapter 7

p-branes

The common objects seen in string theory spacetimes with conserved quantum numbers are black p-branes, rather than black holes. They have a translation symmetry in p spatial directions because of which their horizon has a topology of $\mathbb{R} \times S^{d-1}$. Here d is the number of transverse spacetime dimensions. Now we present a brief review of the BPS M-brane and the D-brane solutions.

7.1 BPS M-brane and D-brane

The BPS M2-brane spacetime has a worlvolume symmetry group of SO(1,2) and a transverse symmetry group of SO(8). If we define coordinates that are parallel and perpendicular to the brane, then using these symmetry groups and the no hair theorem, we get:

$$ds_{11}^2 = H_2^{-2/3} dx_{\parallel}^2 + H_2^{1/3} dx_{\perp}^2, \quad A_{012} = -H_2^{-1} \tag{7.1}$$

It is of no surprise that the same function appears in the gauge field as well as the metric, this is a consequence of invoking supersymmetry. There is no string frame in 11 dimensions and hence the string is already in the Einstein frame. We should use supergravity equations of motion here to make the function H satisfy the equation. As H_2 satisfies the laplace equation in the perpendicular coordinate, it must be harmonic. Hence we get the solution (l_{11} is the Planck length in 11 dimensions):

$$H_2 = 1 + \frac{r_2^6}{r^6}, \quad \text{where } r_2^6 = 32\pi^2 N_2 \ell_{11}^6 \tag{7.2}$$

Now, we analyze the BPS M5-brane which belongs to the symmetry group SO(1,5) × SO(5). The metric reads:

$$ds_{11}^2 = H_2^{-1/3} dx_{\parallel}^2 + H_2^{2/3} dx_{\perp}^2 \tag{7.3}$$

Similar to the above treatment, the harmonic function is:

$$H_5 = 1 + \frac{r_2^3}{r^3}, \qquad \text{where } r_5^3 = \pi N_5 \ell_{11}^3 \tag{7.4}$$

It must be noted that the gauge field here is magnetically coupled and the field strength (F_4) is proportional to the volume element on the 4-sphere which is transverse to the brane. In this case, the origin of the coordinates at r=0 has a singularity, hence it has a delta function source. This is due to the electric coupling of the brane. Contrasting to this, the magnetically coupled brane will be solitonic and devoid of any singularity. Additionally, the spacetime here will have a maximal analytic extension. The near horizon geometry of the M2 spacetime is $AdS_4 \times S_7$ while for the M5 brane it is $AdS_7 \times S_4$. Similar to the case of the Reissner-Nordstorm black hole, here we have an interpolation between two highly supersymmetric vacua.

In 10 dimensions, the BPS D-branes have a symmetry group of SO(1,p) × SO(9-p). We have the solutions in the string frame as:

$$\begin{aligned} dS^2 &= H_p(r)^{-\frac{1}{2}} \left(-dt^2 + dx_{\parallel}^2\right) + H_p(r)^{+\frac{1}{2}} dx_{\perp}^2 \\ e^{\Phi} &= H_p(r)^{\frac{1}{4}(3-p)} \\ C_{01\cdots p} &= g_s^{-1} \left[1 - H_p(r)^{-1}\right] \end{aligned} \tag{7.5}$$

Again here the function $H_p(r)$ will be harmonic and satisfies the condition $\partial_{\perp}^2 H_p(r) = 0$. Hence,

$$H_p = 1 + \frac{c_p g_s N_p \ell_s^{7-p}}{r^{7-p}}, \quad c_p \equiv (2\sqrt{\pi})^{(5-p)} \Gamma\left[\frac{1}{2}(7-p)\right] \tag{7.6}$$

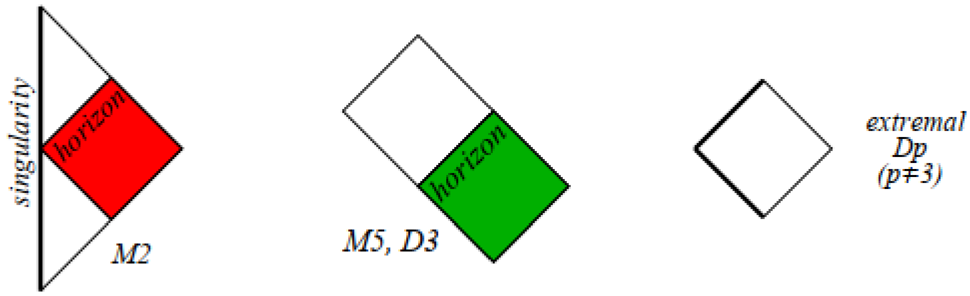


Figure 7.1: Penrose diagram for M2, M5, D3 and Dp branes

This geometry has an interesting property, it has a double horizon occuring at r=0 with a singularity present there (except for D3-branes). Hence for these branes, the singularity is null.

Now we have an interesting problem at our disposal. In this case, since the horizon and the true singularity coincides, one can think that the singularity is not properly censored behind the horizon, hence being a naked singularity. In this case, if we demand that a null geodesic coming from infinity should not reach the singularity in a set affine parameter then the D6-brane is the only brane which possesses a naked singularity.

Now if we analyze D3 branes, in this case the dilation will be constant and the spacetime does not have singularities, allowing a smooth analytic extension inside the horizon. The near horizon geometry in this case is $AdS_5 \times S_5$ and the penrose diagram is similar to that of the M5-brane. The penrose diagrams are given above..

7.2 Arraying the BPS branes

In this section we introduce another technique used to generate relevant solutions. If we consider BPS Dp-branes as described in the previous section, then we can see that multi-centered BPS solutions are allowed as a result of the form for the harmonic function.

$$H_{\bar{p}} = 1 + c_p g_{\mathrm{s}} N_p \ell_{\mathrm{s}}^{7-p} \sum_i \frac{1}{|x_\perp - x_{\perp i}|^{7-p}} \tag{7.7}$$

Physically, we can say that BPS branes which are of the same kind are in static equilibrium - this is because the repulsive gauge forces present cancels out the attractive gravitational and dilatonic forces.

If we construct an infinite array of Dp-branes along a x^{p+1} direction (with $2\pi R$ periodicity), and define:

$$r^2 \equiv \hat{r}^2 + \left(x^{p+1}\right)^2 \tag{7.8}$$

then we get the harmonic function to be:

$$H_{\bar{p}} = 1 + c_p g_{\mathrm{s}} N_p \ell_{\mathrm{s}}^{7-p} \sum_{n=-\infty}^{\infty} \frac{1}{\left[\hat{r}^2 + (x^{p+1} - 2\pi R n)^2\right]^{\frac{1}{2}(7-p)}} \tag{7.9}$$

Performing a change of variables to u:

$$x^{p+1} \equiv 2\pi R n - \hat{r} u \tag{7.10}$$

We get:

$$H_{\bar{p}} \simeq 1 + c_p g_{\mathrm{s}} N_p \ell_{\mathrm{s}}^{7-p} \frac{1}{2\pi R} \frac{1}{\hat{r}^{7-[p+1]}} \underbrace{\int du \frac{1}{\sqrt{1+u^2}(7-p)}}_{\equiv I_p.}, \tag{7.11}$$

We now evaluate the integral I_p as:

$$I_p = \sqrt{\pi}\Gamma\left[\frac{1}{2}(7 - \{p+1\})\right] / \Gamma\left[\frac{1}{2}(7-p)\right] \tag{7.12}$$

We invoke :

$$b_p = (2\sqrt{\pi})^{5-p}\Gamma\left[\frac{1}{2}(7-p)\right] \tag{7.13}$$

to obtain:

$$H_{\bar{p}} \simeq 1 + \left[\frac{N_p}{(R/\ell_s)}\right] g_s c_{p+1} \left(\frac{\ell_s}{\hat{r}}\right)^{7-[p+1]} \tag{7.14}$$

Now we make a linear density of the branes from the arrayed objects and match it with a "smeared" harmonic function getting:

$$N_{p+1} = \frac{N_p}{(R/\ell_s)} \tag{7.15}$$

Using T-duality rules mentioned before and setting $x^{p+1} = z$ as the isometry direction, we get:

$$\begin{aligned} d\tilde{S}^2 &= H_{\bar{p}}^{-\frac{1}{2}} \left(-dt^2 + dx_{1\ldots p}^2 + dz^2\right) + H_{\bar{p}}^{\frac{1}{2}} \left(d\hat{r}^2 + \hat{r}^2 d\Omega^2_{[8-(p+1)]}\right) \\ e^{\dot{\Phi}} &= H_{\bar{p}}^{\frac{1}{4}(3-p)} / H_{\bar{p}}^{\frac{1}{4}} = H_{\bar{p}}^{\frac{1}{4}}[3-(p+1)] \\ \tilde{C}_{01\ldots p+1} &= g_s^{-1}\left[1 - H_{\bar{p}}^{-1}\right] \end{aligned} \tag{7.16}$$

These are precisely the supergravity fields for our D(p+1)-brane. As mentioned earlier, this procedure of taking an array of infinite branes and taking a linear limit is known as smearing, this results in a larger brane to be formed as a result.

It is very insightful to note that using dualities and arraying we can indeed interconnect all M-branes and D-branes with NS-branes.

7.3 Non-extremal branes

We can write the metric and the dilation fields of the non extremal Dp-branes in the string frame with a Schwarzschild type radial coordinate as:

$$\begin{gathered} dS^2 = -\Delta_+(\rho)\Delta_-(\rho)^{-\frac{1}{2}} dt^2 + \Delta_-(\rho)^{+\frac{1}{2}} dx_\parallel^2 + \\ \Delta_+(\rho)^{-1}\Delta_-(\rho)^{\frac{1}{2}(p-3)/(7-p)-1} d\rho^2 + \rho^2 \Delta_-(\rho)^{\frac{1}{2}(p-3)/(7-p)} d\Omega^2, \\ e^{\Phi} = \Delta_-(\rho)^{\frac{1}{4}(p-3)}, \end{gathered} \tag{7.17}$$

Here we define :

$$\Delta_{\pm}(\rho) \equiv 1 - \left(\frac{r_{\pm}}{\rho}\right)^{7-p} \tag{7.18}$$

Also defining :

$$r_{+}^{7-p} = r_{H}^{7-p} \cosh^2 \beta, \quad r_{-}^{7-p} = r_{H}^{7-p} \sinh^2 \beta \tag{7.19}$$

In order to simplify the metric we make a change of coordinates, $r^{7-p} = \rho^{7-p} - r_{-}^{7-p}$, hence we get:

$$dS^2 = D_p(r)^{-\frac{1}{2}} \left(-K(r)dt^2 + dx_{\|}^2\right) + D_p(r)^{\frac{1}{2}} \left(dr^2/K(r) + r^2 d\Omega_{8-p}^2\right) \tag{7.20}$$

We see here that:

$$D_p(r) = 1 + \left(r_H/r\right)^{7-p} \sinh^2 \beta, , \quad K(r) = 1 - \left(r_H/r\right)^{7-p} \tag{7.21}$$

while the other corresponding fields become:

$$\begin{gathered} e^{\Phi} = D_p(r)^{(3-p)/4} \\ C_{01\ldots p} = (\coth \beta) g_{\mathrm{s}}^{-1} \left[1 - D_p(r)^{-1}\right] \end{gathered} \tag{7.22}$$

We have this expression for the boost parameter β :

$$\sinh^2 \beta = -\frac{1}{2} + \sqrt{\frac{1}{4} + \left[c_p g_{\mathrm{s}} N \left(\ell_{\mathrm{s}}/r_H\right)^{7-p}\right]^2} \tag{7.23}$$

It must be noted here that isometry directions allow us to produce new solutions using boost. For example - in a 4 dimensional case, an angular boost can be done but is a bit tricky.

In the extremal limit we have $r_h \to 0, \beta \to \infty$. Here non-extremality refers to how much a quantity is deviating from unity, hence we can also invoke the change of the harmonic function due to this non-extremality with a parameter : $\zeta = \tanh \beta$

$$D_p(r) = 1 + \zeta c_p g_s N \left(\ell_s/r\right)^{7-p}, \quad \zeta = \sqrt{1 + \left[\frac{r_H^{7-p}}{2c_p g_s N \ell_s^{7-p}}\right]^2} - \left[\frac{r_H^{7-p}}{2c_p g_s N \ell_s^{7-p}}\right] \quad (7.24)$$

Now, the gauge field can be expressed by :

$$C_{01\ldots p} = \zeta^{-1} g_s^{-1}\left[1 - D_p(r)^{-1}\right] \quad (7.25)$$

Consequently, we have the ADM mass per unit p-volume and the charge:

$$\begin{aligned} \frac{M_p}{(2\pi)^P V_p} &= \frac{(r_H/\ell_s)^{7-p}}{c_p g_s^2 (2\pi)^p \ell_s^{p+1}}\left[\cosh^2\beta + \frac{1}{(7-p)}\right] \\ N_p &= \frac{1}{c_p g_s}\left(\frac{\sqrt{r_+ r_-}}{\ell_s}\right)^{7-p}. \end{aligned} \quad (7.26)$$

Finally, we get the relations for the Hawking temperature and the Bekenstein-Hawking entropy:

$$\begin{aligned} T_H &= \frac{(7-p)}{4\pi r_H \cosh\beta} \\ S_{BH} &= \frac{\Omega_{8-p}}{4G_{10-p}} r_H^{8-p}\cosh\beta \end{aligned} \quad (7.27)$$

We can introduce a factor ϵ which accounts for the energy density above the said extremality and construct a relation in the near extremal limit (i.e when $\zeta \to 1$) :

$$r_H^{7-p} = \varepsilon G_{10} 8\pi^{\frac{1}{2}(p-7)}\Gamma\left[\frac{1}{2}(7-p)\right] \quad (7.28)$$

ϵ can be connected to the thermodynamic temperature and entropy and hence in the near extremal limit we get these corresponding relations, it must be noted that they are much smaller than the original Dp-brane case :

$$T_H \sim \varepsilon^{\frac{1}{2}(5-p)/(7-p)}, \quad \text{and} \quad S_{BH} \sim \varepsilon^{\frac{1}{2}(9-p)/(7-p)} \quad (7.29)$$

It again follows that we can obtain non-extremal NS-branes from these Dp-branes via duality transformations.

7.4 The Gregory-Laflamme instability

Gregory-Laflamme instability is an important instability of nonextremal p-branes which puts forward a simple question. We know that we can construct a d dimensional neutral black hole in two ways - one is from a compactified black string while the other is from an array of evenly spaced d-dimensional black holes. The question arises that if we have such a black hole at hand, which system would the black hole settle towards and be stable in. This instability says that whichever configuration has the highest entropy will qualify. Let us see how this goes:

$$M_{array} = M_{string} \tag{7.30}$$

From the properties shown in the previous sections, we have :

$$M \sim \frac{{r_H}^{d-3}}{G_d}, \quad S \sim \frac{{r_H}^{d-2}}{G_d}. \tag{7.31}$$

We can derive a scaling relation for the mass per unit length of the array:

$$\frac{M_{\text{array}}}{R} \sim \frac{{r_H}^{d-3}}{G_d}, \quad \frac{M_{\text{string}}}{R} \sim \frac{\hat{r}_H^{d-4}}{G_{d-1}}, \tag{7.32}$$

By our equal mass assumption, which we mentioned in the beginning, we have:

$$r_H^{d-3} \sim \hat{r}^{d-4} R. \tag{7.33}$$

Let's find out which configuration has the higher entropy :

$$\frac{S_{\text{array}}}{S_{\text{string}}} \sim \frac{r_H^{d-2}}{G_d} \frac{G_{d-1}}{\hat{r}_H^{d-3}} \sim \left(\frac{R}{\hat{r}_H}\right)^{1/(d-2)} \sim \left(\frac{R}{r_H}\right)^{1/(d-3)}. \tag{7.34}$$

Hence, we see that the array configuration is dominating for small radii of the horizon, while the black string dominates in the case of a larger horizon radii. If we take an asymptotic limit of the radius then we see that the uncompactified neutral black string will remain unstable. Although, this instability was originally shown by calculating presence of tachyonic modes after doing perturbation theory.

It is most important to note that the Gregory-Laflamme instability is not the same as the instability due to Hawking radiation. Moreover, this also violates the cosmic censorship hypothesis, because a neutral black string can fall into an array of black holes.

Chapter 8

Constructing brane black holes

In this chapter we concentrate upon constructing various black holes with branes as the building blocks. Here we are primarily focusing on BPS systems because of simplicity.

8.1 Branes : The Lego model

There are various reasons for choosing BPS branes. Two clumps of BPS branes will be in static equilibrium with each other and additionally even BPS p-branes and q-branes can be in equilibrium with each other under certain specific conditions.

We again invoke the properties of T-duality and S-duality to obtain the following relations. Intersections are denoted by parallel or perpendicular symbols depending on how they are intersecting correspondingly :

$$\begin{aligned} &\mathrm{D}m \| \mathrm{D}m+4(m), m=0,1,2 \quad \rightarrow \mathrm{D}p \perp \mathrm{D}q(m), \quad p+q=4+2m \\ &\mathrm{F1} \parallel \mathrm{NS5}\ , \quad \mathrm{NS5} \ \perp\ \mathrm{NS5}(3), \quad Dp \perp \mathrm{NS}\,5(p-1) \end{aligned} \tag{8.1}$$

Now, the rules in 11 dimensions will be:

$$\begin{aligned} &\mathrm{M2} \perp \mathrm{M2}(0), \quad \mathrm{M2} \perp \mathrm{M5}(1), \quad \mathrm{M5} \perp \mathrm{M5}(1) \text{ or } \mathrm{M5} \perp \mathrm{M5}(3) \\ &\mathrm{W} \| \mathrm{M2}, \quad \mathrm{W} \| \mathrm{M5}, \quad \mathrm{M2} \| \mathrm{KK} \text{ or } \mathrm{M2} \perp \mathrm{KK}(0) \\ &\mathrm{M5} \| \mathrm{KK} \text{ or } \mathrm{M5} \perp \mathrm{KK}(1) \text{ or } \mathrm{M5} \perp \mathrm{KK}(3) \\ &\mathrm{W} \| \mathrm{KK}, \mathrm{KK} \perp \mathrm{KK}(4,2) \end{aligned} \tag{8.2}$$

It must be noted that when the D0-brane crosses the D8-brane, a fundamental string is created in order to preserve the force cancellation, interestingly, a dual to this is in a particular setup. In this case, a D5-brane crosses a NS5-brane to create a D3-brane (Commonly known as the Hanany-Witten setup).

8.2 Supergravity with intersecting branes

We study the 11 dimensional supergravity system, the action for this theory gives rise to the M-brane intersection rules. We have the action for the bosonic gauge potential as :

$$S\left[A_3\right] = \frac{1}{16\pi G_{11}} \int \left\{ - \left[d^{11}x\sqrt{-g}\frac{\left|F_4\right|^2}{2(4!)} \right] + \left[\# F_4 \wedge F_4 \wedge A_3\right] \right\} \tag{8.3}$$

We then define the field strength as:

$$F_4 = dA_3 \tag{8.4}$$

This obeys the Bianchi identity : $dF_4 = 0$ and we get the conserved charge to be an integral over a transverse 4-sphere :

$$Q_5 = \int_{S^4} F_4 \tag{8.5}$$

It must be noted that the M5-brane cannot end on anything else due to the Bianchi identity. Now, with some convenient normalisation of the constant we get the differential forms :

$$\begin{aligned} d^{*}F_4 &= -F_4 \wedge F_4 \\ &= -\left(dA_3\right) \wedge F_4 = -d\left(A_3 \wedge F_4\right), \end{aligned} \tag{8.6}$$

Hence, the M2-brane conserved charge is :

$$\hat{Q}_2 = \int_{S^7} \left[{}^{*}F_4 + A_3 \wedge F_4\right] \tag{8.7}$$

If the M2-brane ends on something, then by some intermediate steps involving deformation, it can be shown that the charge factorizes into

$$\hat{Q}_2 = \underbrace{\int_{S^3} dV_2}_{\text{string charge}} \underbrace{\int_{S^4} F_4}_{Q_5} \tag{8.8}$$

Here the first factor is the magnetic charge of the string. It is the boundary of the M2-brane in the M5-brane worldvolume.

8.3 BPS black holes from the harmonic function rule

Typically BPS black holes have zero horizon area, therefore they do not have any macroscopic entropy. There exists a systematic ansatz which is known as the "harmonic function rule" which makes the metric factorize into a product structure, superposing the harmonic functions. This process results in only smeared and intersecting brane solutions while working for both parallel and perpendicular intersections.

Let us consider two examples. We denote "-" as the brane is extended in a given direction while " " as the dependence on the coordinates smeared away although not extended. In the first one, we consider a D5-brane is with a smeared D1 brane, while in the second one we consider two perpendicular smeared D2-branes :

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{D1} & - & - & \sim & \sim & \sim & \sim & . & . & . & . \\ \text{D5} & - & - & - & - & - & - & . & . & . & . \end{array} \tag{8.9}$$

$$\begin{array}{lcccccccccc} \text{D2} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{D2}\,' & - & - & - & \sim & \sim & . & . & . & . & . \\ & - & \sim & \sim & - & - & . & . & . & . & . \end{array}$$

If we define an overall transverse coordinate $r^2 = x_\perp^2 \equiv \sum_{i=1}^{4} \left(x^i\right)^2$ then the metric in the string frame becomes (we use the harmonic function rule here) :

$$\begin{aligned} dS_{10}^2 = {} & H_1(r)^{-\frac{1}{2}} H_5(r)^{-\frac{1}{2}} \left(-dt^2 + dx_1^2\right) + H_1(r)^{+\frac{1}{2}} H_5(r)^{-\frac{1}{2}} dx_{2\cdots 5}^2 \\ & + H_1(r)^{+\frac{1}{2}} H_5(r)^{+\frac{1}{2}} \left(dr^2 + r^2 d\Omega_3^2\right) \end{aligned} \tag{8.10}$$

The dilation term becomes :

$$e^{\Phi} = H_1(r)^{+\frac{1}{2}} H_5(r)^{-\frac{1}{2}} \tag{8.11}$$

The gauge fields remain unchanged :

$$C_{01} = g_{\rm s}^{-1} \left[1 - H_1(r)^{-1}\right], \quad C_{01\ldots 5} = g_{\rm s}^{-1} \left[1 - H_5(r)^{-1}\right] \tag{8.12}$$

We see that the independent harmonic functions are proportional to r^{-2}, this is a naturally explicable consequence :

$$H_5(r) = 1 + \frac{\#'}{r^2}, \quad H_1(r) = 1 + \frac{\#}{r^2} \tag{8.13}$$

If the other four coordinates are wrapped on a 4-torus (to make a 6 dimensional black hole), the volume of the T^4 will be finite at r=0 (which is the event horizon).

$$\frac{\mathrm{Vol}\,(T^4)}{(2\pi)^4 V_4} = \sqrt{G_{22}\cdots G_{55}} = \left(\frac{H_1}{H_5}\right)^{\frac{1}{4}4} \to \left(\frac{\#}{\#'}\right). \tag{8.14}$$

If the first compactifying direction along the string is taken on a circle then consequently, the radius becomes 0 at the horizon and hence the volume also goes to 0, no matter how large the radius is at infinity.

$$\frac{\mathrm{Vol}\,(S^1)}{(2\pi)R} = \sqrt{G_{11}} = (H_1 H_5)^{-\frac{1}{2}} \to r/\sqrt{\#\#'} \to 0 \tag{8.15}$$

It must be noted here that generally supergravity solutions formed by intersecting branes are not smeared or delocalised. But there is an interesting delocalisation phenomenon in the case of a Dp-brane and a Dp+4-brane where the transverse seperation of the brane the Dp-brane is parallel to tends to zero.

8.4 Some examples

8.4.1 The 3-charge d=5 black hole

As previously seen, the black hole made up of D1, D5-brane charges does not have a finite horizon area and hence no macroscopic entropy. In this case we use the solution generating techniques at our disposal, specifically, we perform a boost in the longitudinal direction to make the horizon size macroscopic. Hence, we can build a black hole like this with the previous ingredients after adding a gravitational wave W. The arrow below indicates the direction in which the gravitational wave moves.

$$
\begin{array}{lcccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{D1} & - & - & \sim & \sim & \sim & \sim & . & . & . & . \\
\text{D5} & - & - & - & - & - & - & . & . & . & . \\
\text{W} & - & \to & \sim & \sim & \sim & \sim & . & . & . & .
\end{array}
\tag{8.16}
$$

We can thus obtain the BPS metric for this system from an ordinary D1-D5 system metric by performing a boost and then taking its extremal limit (as discussed previously). Now in order to make a 5 dimensional black hole, we have to get rid of the extra dimensions by compactifying the D5-brane on a 4-torus of volume $(2\pi)^4 V$ and the D1 brane with the remaining extended dimension on a 1-sphere of radius R. Hence the Einstein metric in five dimension is :

$$
\begin{aligned}
ds_5^2 = & -\left(H_1(r)H_5(r)(1+K(r))\right)^{-2/3} dt^2 \\
& + \left(H_1(r)H_5(r)(1+K(r))\right)^{1/3}\left[dr^2 + r^2 d\Omega_3^2\right]
\end{aligned}
\tag{8.17}
$$

The harmonic functions mentioned above are of the form :

$$
H_1(r) = 1 + \frac{r_1^2}{r^2}, \quad H_5(r) = 1 + \frac{r_5^2}{r^2}, \quad K(r) = \frac{r_m^2}{r^2},
\tag{8.18}
$$

performing the same arraying procedure for H_1 and K, we get :

$$
r_1^2 = \frac{g_\mathrm{s} N_1 \ell_\mathrm{s}^6}{V}, \quad r_5^2 = g_\mathrm{s} N_5 \ell_\mathrm{s}^2 \quad r_m^2 = \frac{g_\mathrm{s}^2 N_m \ell_\mathrm{s}^8}{R^2 V}
\tag{8.19}
$$

Now, we compute the thermodynamic quantities of this spacetime. We know that the BPS black hole is extremal with a zero Hawking temperature. But we can expand the Bekenstein Hawking entropy in terms of the harmonic functions as:

$$
\begin{aligned}
S_\mathrm{BH} = & \frac{A}{4G_5} = \frac{1}{4G_5}\pi^2 r^3 \left[H_1(r)H_5(r)(1+K(r))\right]^{3/6} \text{ at } r=0 \\
= & \frac{\pi^2}{4\left[\frac{1}{8}\pi/8 g_\mathrm{s}^2 \ell_\mathrm{s}^8/(VR)\right]}\left(r_1 r_5 r_m\right)^{\frac{1}{2}} = \frac{2\pi V R}{g_\mathrm{s}^2 \ell_\mathrm{s}^8}\left(\frac{g_\mathrm{s} N_1 \ell_\mathrm{s}^6}{V} g_\mathrm{s} N_5 \ell_\mathrm{s}^2 \frac{g_\mathrm{s}^2 N_m \ell_\mathrm{s}^8}{R^2 V}\right)^{\frac{1}{2}} \\
= & 2\pi\sqrt{N_1 N_5 N_m}
\end{aligned}
\tag{8.20}
$$

As seen, this entropy is macroscopically large and independent of R and V, while the ADM mass given below has an explicit dependence on the radius and volume terms.

$$M = \frac{N_m}{R} + \frac{N_1 R}{g_\mathrm{s}\ell_\mathrm{s}^2} + \frac{N_5 RV}{g_\mathrm{s}\ell_\mathrm{s}^6} \tag{8.21}$$

For a general black hole solution, we can have a maximal supergravity arising by compactifying the Type II theory on T^5, the entropy for such a solution will be:

$$\begin{gathered} S_\mathrm{BH} = 2\pi\sqrt{\frac{\Delta}{48}}, \\ \Delta = 2\sum_{i=1}^{4} \lambda_i^3 \end{gathered} \tag{8.22}$$

here Δ is the cube invariant of the $E_{6,6}$ duality group while λ_i are the eigenvalues of the central charge matrix Z.

8.4.2 The 4-charge d=4 black hole

We can now embed the extremal Reissner-Nordstrom black hole (discussed before) in string theory using these ingredients known as D-branes. We had earlier seen that there is an H^2(r) term appearing in the RN metric while a generic p-brane metric has $H^{\frac{1}{2}}$ term. Hence one can speculate that we might require 4 ingredients to construct a RN black hole.

Hence, the following table illustrates the ingredients required to make a four dimensional Reissner-Nordstorm black hole.

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathrm{D2} & - & - & - & \sim & \sim & \sim & \sim & . & . & . \\ \mathrm{D6} & - & - & - & - & - & - & - & & . & \\ \mathrm{NS5} & - & - & - & - & - & - & \sim & . & . & . \\ \mathrm{W} & - & \rightarrow & \sim & \sim & \sim & \sim & \sim & . & . & \end{array} \tag{8.23}$$

In 10 dimensions, we can construct BPS solutions using the same harmonic function rule. As we know that the Einstein metric is invariant under S-duality, we can use this fact and the D5 metric form to get :

$$\begin{aligned}dS_{10}^2 =&H_2(r)^{-\frac{1}{2}}H_6(r)^{-\frac{1}{2}}\left[-dt^2+dx_1^2+K(r)\left(dt+dx_1\right)^2\right]\\&+H_5(r)H_2(r)^{-\frac{1}{2}}H_6(r)^{-\frac{1}{2}}\left(dx_2^2\right)\\&+H_2(r)^{+\frac{1}{2}}H_6(r)^{-\frac{1}{2}}H_5(r)\left(dx_{3\cdots 6}^2\right)\\&+H_5(r)H_2(r)^{+\frac{1}{2}}H_6(r)^{+\frac{1}{2}}\left(dr^2+r^2d\Omega_2^2\right)\end{aligned} \tag{8.24}$$

$$e^{\Phi}=H_5^{+\frac{1}{2}}H_2^{+\frac{1}{4}}H_6^{-\frac{1}{4}(3)}.$$

After arraying, we can find the Newton's constant using the general formula :

$$G_4=\frac{G_{10}}{(2\pi)^6\left(VR_aR_b\right)}=\frac{g_s^2\ell_s^8}{8VR_aR_b} \tag{8.25}$$

We find the 4 gravitational radii to be :

$$r_2=\frac{g_sN_2\ell_s^5}{2V},r_6=\frac{g_sN_6\ell_s}{2},r_5=\frac{N_5\ell_s^2}{2R_b},r_m=\frac{g_s^2N_m\ell_s^8}{2VR_a^2R_b} \tag{8.26}$$

We use the formula mentioned in one of the previous sections (Kaluza Klein reduction) to reduce the object to a 5 dimensional black string.

$$\begin{aligned}dS_5^2=&H_2(r)^{-\frac{1}{2}}H_6(r)^{-\frac{1}{2}}\left[-dt^2+dx_1^2+K(r)\left(dt+dx_1\right)^2\right]\\&+H_5(r)H_2(r)^{+\frac{1}{2}}H_6(r)^{+\frac{1}{2}}\left(dr^2+r^2d\Omega_2^2\right)\end{aligned} \tag{8.27}$$

The dilaton will get some factors of the harmonic function :

$$e^{2\Phi_5}=e^{2\Phi_{10}}\frac{1}{\sqrt{G_{44}\cdots G_{88}}}=H_5^{+\frac{1}{2}}H_2^{-\frac{1}{4}}H_6^{-\frac{1}{4}} \tag{8.28}$$

Again using the KK formula which reads:

$$\hat{G}_{00}=G_{00}-G_{01}^2/G_{11} \tag{8.29}$$

We get the metric upon reducing (KK reduction) on the last direction:

$$\begin{aligned}dS_4^2=&-H_2(r)^{-\frac{1}{2}}H_6(r)^{-\frac{1}{2}}(1+K(r))^{-1}dt^2\\&+H_2(r)^{+\frac{1}{2}}H_6(r)^{+\frac{1}{2}}H_5(r)\left(dr^2+r^2d\Omega_2^2\right)\end{aligned} \tag{8.30}$$

Because of this, the dilaton again gets changed into L

$$e^{2\Phi_4} = \frac{H_5^{+\frac{1}{2}} H_2^{-\frac{1}{4}} H_6^{-\frac{1}{4}}}{\sqrt{(1+K(r))H_2(r)^{-\frac{1}{2}} H_6(r)^{-\frac{1}{2}}}} = \frac{H_5^{\frac{1}{2}}}{1+K(r)} \tag{8.31}$$

Hence we get the 4- dimensional Einstein metric as :

$$\begin{aligned} ds^2 &= -dt^2 \left[\sqrt{(1+K(r))H_2(r)H_6(r)H_5(r)}\right]^{-1} \\ &+ \left(dr^2 + r^2 d\Omega_2^2\right) \left[\sqrt{(1+K(r))H_2(r)H_6(r)H_5(r)}\right]. \end{aligned} \tag{8.32}$$

Here we can easily find the Bekenstein Hawking entropy to be :

$$\begin{gathered} S_{\mathrm{BH}} = 2\pi\sqrt{N_2 N_6 N_5 N_m} \\ \diamond = \sum_{i=1}^{4} |\lambda_i|^2 - 2\sum_{i<j}^{4} |\lambda_i|^2 |\lambda_j|^2 + 4\left(\overline{\lambda_1\lambda_2\lambda_3\lambda_4} + \lambda_1\lambda_2\lambda_3\lambda_4\right) \end{gathered} \tag{8.33}$$

Similar to the previous section, $\diamond$ is the quartic invariant of the $E_{7,7}$ duality group while λ_i are complex eigenvalyes of Z. We can obtain the connection of the Reissner-Nordstorm black hole with the brane counterpart by setting all of the gravitational radii in the brane sector to be identical to each other.

Chapter 9

Entropy Calculation

Here we calculate the entropy of the BPS systems via the D-brane picture and show that the computed entropy agrees with the entropy in different pictures.

9.1 The Strominger-Vafa entropy matching

Let us review our setup of branes from the previous section for a D1-D5-W system. It is given below.

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \text{D1} & - & - & \sim & \sim & \sim & \sim & . & . & . & . \\ \text{D5} & - & - & - & - & - & - & . & . & . & . \\ \text{W} & - & \rightarrow & \sim & \sim & \sim & \sim & . & . & . & . \end{array} \quad . \tag{9.1}$$

Such a system has 4 real and preserved supercharges, seen from the constituent brane SUSY solutions. Each constituent breaks half of SUSY. Supersymmetry demands the branes to orient in a relatively supersymmetric way or else the D-brane would represent a black hole without supersymmetry.

Our first step is to identify the degeneracy of the system, a simplified result chooses the four volume to be small by comparison to the radius of the circle : $V^{\frac{1}{4}} << R$. The resulting theory on D-branes is a 2 dimensional theory with (4,4) supersymmetry. We then divide the system into right and left movers. The right movers is put in their ground state while the left movers are highly excited. This partly conserves SUSY. As these are excited BPS states in 2 dimensions, the relation : E=N_m/R must hold. Hence we find the partition function of the system for equal number of bosons and fermions ($4N_1N_5$).

$$Z=\left[\prod_{N_m=1}^{\infty}\frac{1+w^{N_m}}{1-w^{N_m}}\right]^{4N_1N_5}\equiv\sum\Omega\left(N_m\right)w^{N_m} \tag{9.2}$$

Here Ω is the degeneracy of states. For large N_m we use the Cardy formula to get :

$$\Omega\left(N_m\right)\sim\exp\sqrt{\tfrac{\pi cE(2\pi R)}{3}}=\exp\left(2\pi\sqrt{\tfrac{c}{6}ER}\right) \tag{9.3}$$

The above formula assumes that the lowest eigenvalue of the energy operator is zero (being in our system). Finally, the microscopic D-brane statistical entropy yields :

$$S_{\text{micro}}=\log\left(\Omega\left(N_m\right)\right)=2\pi\sqrt{N_1N_5N_m} \tag{9.4}$$

It can be seen that this agrees with the other results for the entropy of a black hole.

9.2 Extension in d=4

The canonical set of ingredients required to build a 4 dimensional system with a black hole is :

$$\begin{array}{cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{D2} & - & - & - & \sim & \sim & \sim & \sim & . & . \\ \text{D6} & - & - & - & - & - & - & - & & . \\ \text{NS5} & - & - & - & - & - & - & \sim & . & . \\ \text{W} & - & \rightarrow & \sim & \sim & \sim & \sim & \sim & . & . \end{array} \tag{9.5}$$

The system now can have D2-branes ending up on NS5-branes and hence it wont cost any energy to break up a D2-brane. The massless degrees of freedom incorporated in the system will give rise to a N_{NS5} term in the degeneracy , the entropy yielding :

$$S_{\text{micro}}=2\pi\sqrt{N_2N_6N_{\text{NS5}}N_m} \tag{9.6}$$

We see that this too obeys exactly the Bekenstein-Hawking black hole entropy.

Chapter 10

Non-BPS systems

As mentioned before, BPS systems have a high theoretical control because of supersymmetry, which implies the presence of nonrenormalization theorems for entropy and other such quantities. The non-BPS systems are also of much importance because of various reasons.

10.1 Nonextremality

The 10 dimensional string frame metric gives rise to the nonextremal black hole metric for the D1-D5-W system. The metric in the string frame is :

$$\begin{aligned} dS_{10}^2 &= D_1(r)^{-\frac{1}{2}} D_5(r)^{-\frac{1}{2}} \left[-dt^2 + dz^2 + K(r) \left(\cosh \alpha_m dt + \sinh \alpha_m dz \right)^2 \right] \\ &+ D_1(r)^{+\frac{1}{2}} D_5(r)^{-\frac{1}{2}} dx_{\|}^2 + D_1(r)^{+\frac{1}{2}} D_5(r)^{+\frac{1}{2}} \left[\frac{dr^2}{(1-K(r))} + r^2 d\Omega_3^2 \right] \\ K(r) &= \frac{r_H^2}{r^2}, \quad f_{1,5}(r) = 1 + K(r) \sinh^2 \alpha_{1,5} \end{aligned} \tag{10.1}$$

Given that α is the boost parameter, we can find the conserved charges for this metric as :

$$N_1 = \frac{V r_H^2}{g_s \ell_s^6} \frac{\sinh(2\alpha_1)}{2}, \quad N_5 = \frac{r_H^2}{g_s \ell_s^2} \frac{\sinh(2\alpha_5)}{2}, \quad N_m = \frac{R^2 V r_H^2}{g_s^2 \ell_s^8} \frac{\sinh(2\alpha_m)}{2} \tag{10.2}$$

The thermodynamic quantities, including mass are :

$$\begin{aligned} M_{ADM} &= \frac{R V r_H^2}{g_s^2 \ell_s^8} \left[\frac{\cosh(2\alpha_1)}{2} + \frac{\cosh(2\alpha_5)}{2} + \frac{\cosh(2\alpha_m)}{2} \right] \\ S_{\mathrm{BH}} &= \frac{2\pi R V r_H^3}{g_s^2 \ell_s^8} \left[\cosh \alpha_1 \cosh \alpha_5 \cosh \alpha_m \right] \\ T_{\mathrm{H}} &= \frac{\ell_s}{2\pi r_H \left[\cosh \alpha_1 \cosh \alpha_5 \cosh \alpha_m \right]} \end{aligned} \tag{10.3}$$

In order to simplify the ADM mass and take the condition away from extremality, we take the limit,

$$r_H^2 \sinh^2 \alpha_{1,5} \equiv r_{1,5}^2 \gg r_m^2 \equiv r_H^2 \sinh^2 \alpha_m \gg \ell_s^2 \tag{10.4}$$

Hence, the energy above extremality is of the form :

$$\Delta E \equiv M - \left(\frac{N_5 RV}{g_s \ell_s^6} + \frac{N_1 R}{g_s \ell_s^2} \right) \simeq \frac{RV r_H^2}{g_s^2 \ell_s^8} \frac{\cosh(2\alpha_m)}{2} \tag{10.5}$$

We have the relation :

$$\frac{N_m}{R} = \frac{RV r_H^2}{g_s^2 \ell_s^8} \frac{\sinh(2\alpha_m)}{2} \tag{10.6}$$

Here we introduce another term in order to split the system :

$$N_{L,R} = \frac{R^2 V r_H^2}{4 g_s^2 \ell_s^8} e^{\pm 2\alpha_m} \tag{10.7}$$

We see that the system has split into two independent sector of gases - left movers and right movers because :

$$\Delta E = \frac{1}{2\ell_s} (N_R + N_L), \quad N_m = N_L - N_R \tag{10.8}$$

This regime is known as the dilute gas regime as the energies and momentum are be additive. Now we state the result of the Bekenstein Hawking entropy of such a system, α_m is the finite boost parameter in the dilute gas regime.

$$\begin{aligned} \cosh \alpha_m &= \frac{1}{2} \left(e^{\alpha_m} + e^{-\alpha_m} \right) = \frac{1}{2} \left(\sqrt{\frac{4 g_s^2 \ell_s^8 N_L}{R^2 V r_H^2}} + \sqrt{\frac{4 g_s^2 \ell_s^8 N_R}{R^2 V r_H^2}} \right) \\ S_{\mathrm{BH}} &= \frac{2\pi RV r_H^3}{g_s^2 \ell_s^8} \left(\frac{g_s N_1 \ell_s^6}{V r_H^2} \right)^{\frac{1}{2}} \left(\frac{g_s N_5 \ell_s^2}{r_H^2} \right)^{\frac{1}{2}} \left(\frac{g_s^2 \ell_s^8}{R^2 V r_H^2} \right)^{\frac{1}{2}} \left[\sqrt{N_L} + \sqrt{N_R} \right] \\ &= 2\pi \left(\sqrt{N_1 N_5 N_L} + \sqrt{N_1 N_5 N_R} \right) \end{aligned} \tag{10.9}$$

We state that the entropy here becomes additive in nature, this same result is also seen in the case of D-brane entropy counting and even after introducing rotation.

10.2 Connection of BTZ black hole with the D1-D5 system

The rule given below no longer applies for spacetime with three dimensions without cosmological constant due to logarithmic divergence.

$$"g_{tt} = -1 + \left(\frac{r_H}{r}\right)^{d-3}" \tag{10.10}$$

We have good black hole solutions for spaces with negative cosmological constant, these are the BTZ black holes and they are solutions to the action:

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left(R_g + \frac{2}{\ell^2}\right) \tag{10.11}$$

Here the cosmological constant is $\Lambda = -1/l^2$ and the metric is :

$$\begin{aligned} ds^2_{\mathrm{BTZ}} = &-\frac{\left(w^2 - w_+^2\right)\left(w^2 - w_-^2\right)}{\ell^2 w^2}dt^2 + \frac{\ell^2 w^2}{\left(w^2 - w_+^2\right)\left(w^2 - w_-^2\right)}dw^2 \\ &+w^2\left(d\varphi + \frac{w_+ w_-}{\ell w^2}dt\right)^2 \end{aligned} \tag{10.12}$$

The coordinate φ has a periodicity of 2π while the event horizons are at $w_\pm$. The mass and the angular momenta are :

$$M = \frac{\left(w_+^2 - w_-^2\right)}{8\ell^2 G_3}, \quad J = \frac{(w_+ w_-)}{4\ell G_3} \tag{10.13}$$

We get the corresponding thermodynamic quantities (entropy and temperature) to be :

$$S_{\mathrm{BH}} = \frac{2\pi w_+}{4G_3}, \quad T_{\mathrm{H}} = \frac{\left(w_+^2 - w_-^2\right)}{2\pi w_+ \ell^2} \tag{10.14}$$

We consider this object to have the following mass and angular momentum parameters.

$$J = 0, \quad M = -\frac{1}{8\ell^2 G_3} \tag{10.15}$$

This object is indeed not a black hole, it has a metric which is AdS_3 in global coordinates :

$$ds^2 = -\frac{(r^2+1)}{\ell^2}dt^2 + \frac{\ell^2}{(r^2+1)}dr^2 + r^2 d\varphi^2 \tag{10.16}$$

Due to the properties of 3 dimensional gravity, it can be shown that the BTZ spacetime will be AdS_3 everywhere locally, although φ is compact.

Now we derive the connection between the BTZ black hole mentioned here and the D1-D5-W system. If we wrap the four dimensions of the D5 which are not parallel to D1 on a 4-torus and reduce it on these directions, then we get a 6- dimensional string :

$$\begin{aligned} dS_6^2 = {} & D_1(r)^{-\frac{1}{2}} D_5(r)^{-\frac{1}{2}} \left[-dt^2 + dz^2 + K(r)\left(\cosh\alpha_m dt + \sinh\alpha_m dz\right)^2\right] \\ & + D_1(r)^{+\frac{1}{2}} D_5(r)^{+\frac{1}{2}} \left[\frac{dr^2}{(1-K(r))} + r^2 d\Omega_3^2\right] \end{aligned} \tag{10.17}$$

We take the near horizon limit as :

$$r^2 \ll r_{1,5}^2 \equiv r_H^2 \sinh^2\alpha_{1,5} \tag{10.18}$$

In this limit the volume of the internal four torus goes to a constant at the horizon.

$$\mathrm{Vol}\left(T^4\right) \to V_4 \left(\frac{r_1^2}{r_5^2}\right) \tag{10.19}$$

Hence the dilation becomes :

$$e^\Phi \to \left(\frac{r_1}{r_5}\right). \tag{10.20}$$

The near horizon string and the Einstein metric only differ by a constant because the dilation is constant near the horizon. We have the simplified angular part of the metric which becomes a 3-sphere of constant radius λ :

$$G_{\Omega\Omega} = r^2\sqrt{1+\frac{r_1^2}{r^2}}\sqrt{1+\frac{r_5^2}{r^2}} \longrightarrow r_1 r_5 \equiv \lambda^2; \tag{10.21}$$

The other part of the metric reads as :

$$ds^2_{t,z,r} \to \frac{r^2}{\lambda^2}\left[-dt^2 + dz^2 + K(r)\left(\cosh\alpha_m dt + \sinh\alpha_m dz\right)^2\right] + \frac{\lambda^2 dr^2}{r^2(1-K(r))}. \quad (10.22)$$

by defining $w_\pm$ as the following, we get a metric in terms of these variables as :

$$\begin{aligned} w_+^2 &\equiv r_H^2\cosh^2\alpha_m, \quad w_-^2 \equiv r_H^2\sinh^2\alpha_m \\ ds^2_{t,r,z} &= \frac{1}{\lambda^2}\left[-dt^2\left(r^2 - w_+^2\right) + dz^2\left(r^2 + w_-^2\right) + 2dtdzw_+w_-\right] \\ &\quad + \frac{\lambda^2 dr^2}{r^2\left(1 - \left(w_+^2 - w_-^2\right)/r^2\right)} \end{aligned} \quad (10.23)$$

If we again perform a change in coordinates : $w^2 \equiv r^2 + w_-^2$ the 6 dimensional metric can be rearranged as :

$$\begin{aligned} ds^2 = &- dt^2\frac{\left(w^2 - w_+^2\right)\left(w^2 - w_-^2\right)}{\lambda^2 w^2} + \frac{w^2\lambda^2 dw^2}{\left(w^2 - w_+^2\right)\left(w^2 - w_-^2\right)} \\ &+ \frac{w^2}{\lambda^2}\left(dz + \frac{w_+w_-}{w^2}dt\right)^2 + \lambda^2 d\Omega_3^2 \\ z &\to \frac{z}{R} \equiv \varphi, \quad w \to \frac{wR}{\lambda} \quad t \to \frac{t\lambda}{R} \end{aligned} \quad (10.24)$$

This is a direct product of a 3-sphere and a BTZ black hole. The final result is :
wrapped extremal black string $\longrightarrow$ extremal BTZ $\times$ S^3,
wrapped nonextremal black string $\longrightarrow$ nonextremal BTZ $\times S^3$. This is trye because only the momentum charge controls the extremality. Such a BTZ spacetime can also be constructed for rotating blackholes with a local identification as there is a global obstruction.

10.3 Black hole absorption

Now we would like to know the absorption properties for our constructed black holes. Some important facts to note here are that the absorption probability is not unity because of backscattering due to the curved geometry, the dominant mode at low energy is the s-wave and there are some universal results for absorption in the case of these black holes. Let us first take a d dimensional spherically symmetric black hole in the Einstein frame.

$$ds^2 = -f(\rho)dt^2 + g(\rho)\left[d\rho^2 + \rho^2 d\Omega_{d-2}^2\right]. \tag{10.25}$$

The wave equation for minimally coupled scalars reads :

$$\nabla^\mu \nabla_\mu \Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Psi = 0 \tag{10.26}$$

In our case of s-wave, we let $\Psi = \Psi_\omega(\rho)e^{-i\omega t}$ to get :

$$\partial_t\left(g^{tt}\partial_t\right)\Psi_\omega - \frac{1}{\sqrt{f(\rho)g(\rho)^{d-1}}\rho^{d-2}}\partial_\rho\left(\sqrt{f(\rho)g(\rho)^{d-1}}\rho^{d-2}g(\rho)^{-1}\partial_\rho\right)\Psi_\omega = 0. \tag{10.27}$$

We take the frequency of the wave to be smaller than any energy scale set by the black hole, in order to fit into the definition of low-energy in string theory. Now we define

$$\partial_\sigma \equiv \sqrt{f(\rho)g(\rho)^{d-3}}\rho^{d-2}\partial_\rho, \tag{10.28}$$

This leads to the modified wave equation :

$$\left(\partial_\sigma^2 + \left[\rho^2(\sigma)g(\rho(\sigma))^{d-2}\omega^2\right]\right)\Psi_\omega(\sigma) = 0 \tag{10.29}$$

If the horizon is at $\rho = r_h$ then the entropy is :

$$S_{\mathrm{BH}} = \frac{\Omega_{d-2}r_H^{d-2}g\left(r_H\right)^{\frac{1}{2}(d-2)}}{4G_d} \equiv \frac{\Omega_{d-2}}{4G_d}R_H^{d-2} \tag{10.30}$$

Now, near the horizon we term the area as the near zone. Here, the wave equation is:

$$\left[\partial_\sigma^2 + \omega^2 R_H^{2(d-2)}\right]\Psi_\omega^{\mathrm{near}}(\sigma) = 0 \tag{10.31}$$

Here the solution is purely ingoing at the horizon, yielding :

$$\Psi_\omega^{\mathrm{near}}(\sigma) = e^{-i\omega R_H^{d-2}\sigma} \tag{10.32}$$

There is a region of overlapping validity of the near-zone wavefunction with the far-zone wave function. At the edge of this region of validity, we have :

$$\Psi_\omega^{\text{near}}(\rho)\big|_{\text{edge}} \sim 1 - i\omega R_H^{d-2}\frac{\rho^{3-d}}{(3-d)} \tag{10.33}$$

We use ρ to evaluate the form of the far zone wavefunction now.

$$\left[\rho^{d-2}\partial_\rho\left(\rho^{d-2}\partial_\rho\right) + \omega^2\rho^{2(d-2)}\right]\Psi_\omega^{\text{far}} = 0 \tag{10.34}$$

In order to eliminate the linear derivative, we perform a change in variables :

$$\Psi_\omega^{\text{far}} \equiv \rho^{-\frac{1}{2}(d-2)}\chi_\omega \tag{10.35}$$

We also define :

$$z \equiv \omega\rho, \tag{10.36}$$

Finally getting :

$$\left[\partial_z^2 + 1 - \tfrac{(d-2)(d-4)}{4z^2}\right]\chi_\omega = 0 \tag{10.37}$$

We see that the solution here for this will be Bessel functions for $\chi_\omega(z)$,

$$\Psi_\omega^{\text{far}}(z) = z^{\frac{1}{2}(3-d)}\left[AJ_{\frac{1}{2}(d-3)}(z) + BJ_{-\frac{1}{2}(d-3)}(z)\right]. \tag{10.38}$$

Now we investigate the behaviour of this wavefunction on the edge of the subtle region of validity. We use small z-series expansion to

$$J_\nu(z) \to \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}, \tag{10.39}$$

this expansion results in :

$$\Psi^{\text{far}}(\rho)\big|_{\text{edge}} \sim \frac{2^{\frac{1}{2}(3-d)}}{\Gamma\left[\frac{1}{2}(d-1)\right]}A + \frac{2^{\frac{1}{2}(d-3)}}{\Gamma\left[\frac{1}{2}(5-d)\right](\omega\rho)^{d-3}}B \tag{10.40}$$

we match this on the near-zone wavefunction situated at the edge to get :

$$A = \Gamma\left[\tfrac{1}{2}(d-1)\right] 2^{\frac{1}{2}(d-3)} \quad B = i\frac{\Gamma\left[\frac{1}{2}(5-d)\right] 2^{\frac{1}{2}(3-d)}(\omega R_H)^{d-2}}{(3-d)} \tag{10.41}$$

While in the far zone we see that the behaviour is oscillatory (like a wave) by the z$\to\infty$ expansion of the Bessel functions.

$$J_\nu(z) \to \sqrt{\tfrac{2}{\pi z}}\left[\cos\left(z - \tfrac{\pi\nu}{2} - \tfrac{\pi}{4}\right)\right] \tag{10.42}$$

Then we get :

$$\begin{aligned}\Psi_\omega^{\text{far}}(\omega\rho) \to & \sqrt{\frac{2}{\pi(\omega\rho)^{d-2}}}\left(e^{+i\left(\omega\rho - \frac{1}{4}\pi\right)}\left[e^{-i\frac{1}{4}(d-3)}\frac{1}{2}A + e^{+i\frac{1}{4}(d-3)}\frac{1}{2}B\right]\right.\\ & \left. + e^{-i\left(\omega\rho - \frac{1}{4}\pi\right)}\left[e^{+i\frac{1}{4}(d-3)}\frac{1}{2}A + e^{-i\frac{1}{4}(d-3)}\frac{1}{2}B\right]\right).\end{aligned} \tag{10.43}$$

The absorption probability is :

$$\begin{aligned}\Gamma &= 1 - \mid \text{ Reflection coefficient } \mid^2 \\ &= 1 - \left|\frac{A + Be^{+i\frac{1}{2}(d-3)}}{A + Be^{-i\frac{1}{2}(d-3)}}\right|^2\end{aligned} \tag{10.44}$$

Finally, we normalize the flux (ingoing plane waves are more used than ingoing spherical waves) :

$$e^{ikz} \equiv N\frac{e^{-i\omega\rho}}{\rho^{\frac{1}{2}}(d-2)}\left(Y_{0\cdots 0} = \frac{1}{\sqrt{\Omega_{d-2}}}\right). \tag{10.45}$$

Ultimately, everything together yields :

$$\sigma_{\text{abs}} = \Gamma|N|^2 = \frac{2\sqrt{\pi}^{d-1}R_H^{d-2}}{\Gamma\left[\frac{1}{2}(d-1)\right]} \equiv A_{\text{H}}. \tag{10.46}$$

This is a universal result for spherically symmetric black holes in the case of low energy minimally coupled scalar s-waves.

10.4 D-brane emission

In this section we analyze the emission properties of D-branes. We use the same gas picture as discussed previously and divide into left and right movers.

$$T_{L,R} = \frac{1}{\pi R} \frac{\sqrt{N_{L,R}}}{\sqrt{N_1 N_5}} \tag{10.47}$$

We can relate these temperatures to the Hawking temperature :

$$T_L^{-1} + T_R^{-1} = 2T_H^{-1} \tag{10.48}$$

In the dilute gas approximation, the right movers are far less than the left movers, hence the T_L must be very larger than T_R. Hence, both can be approximated to a good extent. If we consider low energy quanta with frequencies $\omega_{L,R}$ then in a nontrivial scattering process, the dominant process at low energy will be the collision of two open strings which collide to form a closed string and moves off into the bulk :

$$\left(\omega_L = \frac{+n}{RN_1N_5}\right) + \left(\omega_R = \frac{-n}{RN_1N_5}\right) \quad \longrightarrow \quad \left(\omega_c = \frac{2n}{RN_1N_5}\right). \tag{10.49}$$

It is of no surprise that the emission from the brane is the D$_b raneanalogyeoftheHawkingradiation.W$

$$d\Gamma \sim \underbrace{\frac{d^4k}{\omega_c}}_{\perp \text{ phase space}} \underbrace{\frac{\ell_s^5}{RV\omega_L\omega_R}}_{\text{normalizations}} \underbrace{\delta\left(\omega_c - (\omega_L + \omega_R)\right)}_{\text{momentum conservation}} \underbrace{|\mathcal{A}|^2}, \tag{10.50}$$

We consider emission of a quantum corresponding to a minimally coupled bulk scalar (like an internal component of $G_{\mu\nu}$). Let us take a piece of the brane action :

$$S_{\text{DBI}} = -\frac{1}{(2\pi)g_s\ell_s^2} \int d^2\sigma e^{-\Phi} \sqrt{-\det\left(\mathbb{P}\left(G_{\alpha\beta}\right)\right)} + \dots \tag{10.51}$$

We pick up a static gauge and expand the metric :

$$G_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_{10} h_{\mu\nu}(X) \tag{10.52}$$

The bulk part of the action reads :

$$S_{\text{bulk}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}e^{-2\Phi}R[G] + \dots \tag{10.53}$$

We can canonically normalise the kinetic term for h here to get :

$$\mathcal{L}_{\text{bulk}} \sim \frac{1}{2}\left(\partial h_{ij}\right)\left(\partial h_{ij}\right) \tag{10.54}$$

Hence, the brane action yields :

$$\mathcal{L} \sim \left(\delta_{ij} + 2\kappa_{10}h_{ij}\right)\partial_\alpha X^i \partial^\alpha X^j \tag{10.55}$$

It must be noted that there are a few more intermediate steps in the above calculation where a factor from the string tension is soaked up in the X^i to get a canonically normalised kinetic energy. We have the interaction lagrangian

$$\mathcal{L}_{\text{int}} \sim \kappa h_{ij}\partial_{\hat{\alpha}} X^i \partial^{\hat{\alpha}} X^j \tag{10.56}$$

We use the relation $\kappa_{10} \sim g_{\text{s}}\ell_{\text{s}}^4$ and assume that the outgoing graviton momentum is perpendicular to the D-string. The amplitude then is:

$$\mathcal{A} \sim g_{\text{s}}\omega^2\ell_{\text{s}}^2 \tag{10.57}$$

Using this amplitude as the starting point for calculating the emission probability, we average over the initial states and sum over final states to get the occupation factors.

$$\rho_{L,R}(\omega) = \frac{1}{e^{\omega/\left(2T_{L,R}\right)} - 1} \tag{10.58}$$

In the dilute gas approximation we hence have :

$$\rho_L(\omega) \simeq \frac{2T_L}{\omega}, \quad \rho_R(\omega) \simeq \frac{1}{e^{\omega/T_{\text{H}}} - 1} \tag{10.59}$$

The emission rate is hence proportional to :

$$d\Gamma \propto \frac{d^4k\ell_{\text{s}}^7}{\omega^3 RV}\left(N_1 N_5 R\right) g_{\text{s}}^2\omega^4 \frac{2T_L}{\omega}\frac{1}{e^{\omega/T_{\text{H}}} - 1} \tag{10.60}$$

The exact coefficient results in the precise relation.

$$d\Gamma = A_{\mathrm{H}} \frac{1}{e^{\omega/T_{\mathrm{H}}} - 1} \frac{d^4 k \ell_{\mathrm{s}}^4}{(2\pi)^4} \tag{10.61}$$

Finally we use detailed balance and find the absorption cross section to be :

$$\sigma = A_{\mathrm{H}} \tag{10.62}$$

It must be noted here that this agreement is only valid for cases with special conditions assumed like this. The agreement also in essence depends upon the presence of greybody factors which leads to interesting physical underpinnings.

Chapter 11

Summary and Conclusions

As we see, black holes are of tremendous importance, both in the astrophysical scenario and in the theoretical framework of physics, in an attempt to get a full theory of quantum gravity. The Schwarzschild and the Reissner–Nordström metric have pretty interesting mathematical and geometrical properties linked to the singularities, whose close study, as mentioned before yield us a corresponding set of laws for black hole mechanics, which is analogous to thermodynamical laws.

In the later part we see that how this framework breaks down for the case of quantum black holes, where problems of Information loss and non-unitary evolution seeps in. As we know, quantum mechanics cannot involve the breakdown of unitarity, so for a complete and correct picture of the phenomenon, some other theory should be developed.

Here we consider developments from string theory, being a viable framework for a complete theory of quantum gravity, and we analyze different string constructions of black holes in higher dimensions. Particularly, branes like D1-D5 branes and p-branes are the most interesting candidates as they implicitly invoke supersymmetry. Interestingly, the black holes occuring in such systems have a peculiar near horizon geometry and it can be studied using the AdS/CFT correspondence, as the near horizon limit will consist of an $AdS_3 \times S_3 \times T_4$ for the case of D1-D5 branes and $AdS_3 \times S_3 \times K_3$ for the case of p-branes. Hence the study of two dimensional CFT is beneficial and crucial for both of these cases.

Although complex, these black holes do have a lot of open areas associated with it. Certain concluding remarks for the report are:

- Certain higher dimensional black strings and branes are unstable (Gregory Laflamme instability) and it is not clear how they reach an equillibrium state and what is a more stable counterpart for such objects.
- Certain Black strings form naked singularities and hence show an apparent violation of the Weak cosmic censorship hypothesis.
- A rigorous proof of the AdS/CFT correspondence is not known from a background other than string theory.
- The information problem is not completely solved and clear for the cases of holographic systems or black holes in string theory.
- We can generate solutions from given string actions in an analogous way as we generate solutions in classical general relativity.
- We can construct various kinds of black holes using different branes as the base ingredients.
- These black holes obey the Bekenstein Hawking entropy of their counterpart and there is a neat microscopic derivation for the entropy of these black holes.
- There are interesting absorption and emission properties of these black holes which can be calculated and are still areas of active study.

Bibliography

References

1. Peet A. TASI Lectures on Black Holes in String Theory. Strings, Branes and Gravity. 2001.

2. Kraus P. Lectures on Black Holes and the AdS3/CFT2 Correspondence. Lecture Notes in Physics.:1-55.

3. Banks T, Susskind L, Peskin M. Difficulties for the evolution of pure states into mixed states. Nuclear Physics B. 1984;244(1):125-134.

4. Hotta M, Schützhold R, Unruh W. Partner particles for moving mirror radiation and black hole evaporation. Physical Review D. 2015;91(12).

5. Almheiri A, Marolf D, Polchinski J, Sully J. Black holes: complementarity or firewalls?. Journal of High Energy Physics. 2013;2013(2).

6. Mathur S. The information paradox: a pedagogical introduction. Class Quantum Gravity. 2009;26(22):224001.

7. Mathur S. The fuzzball proposal for black holes: an elementary review. Fortschritte der Physik. 2005;53(7-8):793-827.

Appendix A

(A brief summary of CFT in two dimensions)

A.1 The Conformal Group

A conformal transformation of coordinates is an invertible mapping $x \to x'$, which leaves the metric tensor invariant, upto a scale:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$$

A conformal transformation is locally equivalent to a (pseudo) rotation and a dilation. The set of conformal transformations forms a group (Poincare group as subgroup) $\to \Lambda(x) = 1$

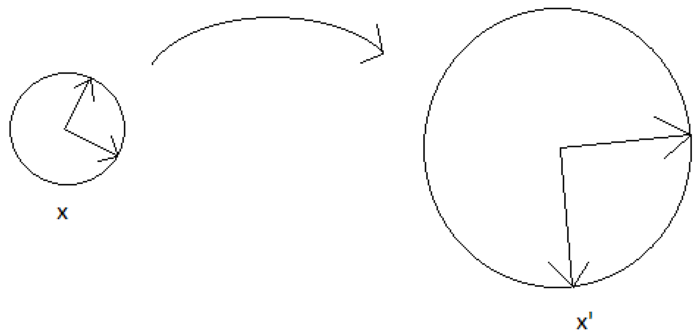
Conformal$\to$Angle does not change between two arbitrary curves crossing each other (despite local dilation)

$$x^\mu \to x'^\mu + \epsilon^\mu(x) g_{\nu\mu} \to x'^\mu + \epsilon^\mu(x)$$

Conformal field theory is a quantum field theory whose correlation functions are invariant under the conformal group. In this report we only consider CFTs in Euclidean $\mathbb{R}^d$ so there is no distinction between upper and lower indices.

A one to one conformal transformation can be shown as :

$$x \to x' = f(x)$$



Locally, these are dilation and rotation. Hence we get the expression:

$$J_{\mu\nu} = \frac{\partial f^{\mu}}{\partial x_{\nu}} = \Omega(x) R_{\mu\nu}(x) \leftarrow O(d) \quad [RR^T = 1]$$

Here fields $\phi_i(x)$ [i=1,2,3.. and $\phi_1 = 1$ by convention] are taken as operators labelled by two properties :

- Scaling dimension $\Delta_i \in \mathbb{R}_{>0}$
- ρ_i irreducible representations of SO(d)

[Here we take only bosonic theory for simplicity.] We consider the correlation function:

$$< \phi_1(x_1)...\phi_n(x_n) >= G(x_1...x_n)$$

Scale invariance would imply that : $G(\lambda x_1, ...\lambda x_n) = \lambda^{-\Delta_1 - \Delta_2 ... - \Delta_n} G(x_1, ...x_n)$ where Δ is the scaling dimension.

For ϕ_i scalars $x \to x'$:

$$G(x'_1, ...x'_n) = \Omega(x_1)^{-\Delta_1} ...\Omega(x_n)^{-\Delta_n} G(x_1, ...x_n)$$

ϕ_i transforms as: $\tilde{\phi}_i(x') = \Omega(x)^{-\Delta_i} \phi_i(x)$

Now we construct the conditions for ϕ_i under irreducible representation ρ_i. We know that invariance under Poincaire groups would mean : x'=Rx+b. Hence,

$$\tilde{\phi}_i(x') = \rho(R)\phi(x)$$

$$\tilde{\phi}_\mu(x') = R_{\mu\nu}\phi_\nu(x) \text{ for } \phi \text{ a vector.}$$

$\rho(R)$ is a finite dimensional representation of R and acts on indices of ϕ

For conformal transformations:-

$$\tilde{\phi}(x') = \Omega(x)^{-\Delta_\phi}\rho(R(x))\phi(x)$$

In short, Conformal transformation gives R(x), based on this construct $\rho(R(x))$ act on the field, giving the transformation.

A.2 Conformal transformations

Conformal transformations leave the metric invariant up to an x dependent factor.

$$(dx')^2 = \Omega(x)^2(dx)^2$$

According to Louville Theorem: Conformal transformations in $\mathbb{R}^{\mathrm{d}}$ are generated in (d>2) by:

- Translations
- Rotations
- Dilatations $x' = \lambda x$
- Inversions $x'^\mu = \frac{x^\mu}{x^2}$

These all transformations belong to a finite dimensional Lie group and has some topology. But, Inversions belong to a Lie group which is disconnected from 1

A.3 Special Conformal Transformation

SCT(a)=I.T(-a).I$\in$ connected components

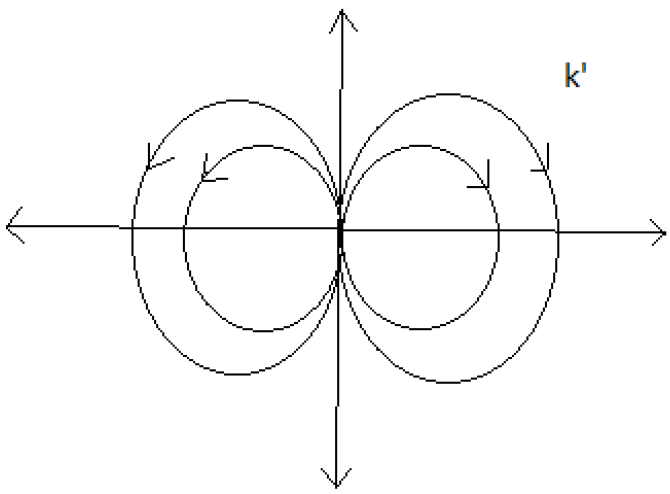
SCT(a):$x_\mu \to \frac{x_\mu - a_\mu x^2}{1-2ax+a^2x^2} \to x = \frac{a}{a^2} = \infty$ (There exists a point)

K_μ have x^2 behavior at $\infty . If \mathrm{K}_\mu$ is exponentiated and the corresponding differential equation ($y' = y^2$) is solved, it blows up at finite time.
Now we consider an infinitesimal transformation:

$$x' = x + \epsilon(x)$$

We know that $P^\mu, M_{\mu\nu}, D, K_\mu$ are the finite generators of translations, rotations, dilatations and special conformal translations. Likewise with some algebra we can construct the infinitesimal generators of the transformations:

$$p^\mu = \partial_\mu$$
$$\mathrm{m}_\mu \nu = x^\nu \partial_\mu - x^\mu \partial_\nu$$
$$\mathrm{d{=}x}^\lambda \partial_\lambda$$
$$\mathrm{k}_\mu = 2x_\mu (x.\partial) - x^2 \partial_\mu$$



Lines of flow for kμ (k')

The conformal killing equation satisfying all of the above vector fields is:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \epsilon)\delta_{\mu\nu}$$

The commutation relation for the generators are given below (worked out in rough) :-

$$
\begin{aligned}
[M, M] &\sim M \\
[M, P] &\sim P \\
[M, K] &\sim K \\
[D, P_\mu] &\sim P_\mu \\
[D, K_\mu] &\sim K_\mu \\
[K_\mu, P_\nu] &\sim 2\delta_{\mu\nu} D - 2M_{\mu\nu}
\end{aligned}
$$

In the above relations, P is a vector under rotations and K is a vector under Special Conformal transformations. Rotation generator measure the dimension of the generator (counting the number of x's and the derivatives).
Now we ask the question that how many generators are there in total?

$$\frac{d(d-1)}{2} + 1 + d + d = \frac{(d+1)(d+2)}{2}$$

This is same as for SO(d+2), more precisely SO(d+1,1).

- We find a linear combination of generators, such a way that it satisfies the commutation relations of SO(d+1,1)
- This acts on $\mathbb{R}^{\mathrm{d+1,1}}$ (Seen clearly in embedding formalism)

No subset of generators are left invariant by the lie brackets. This is true because the K_μ generator is added (simplifies).

A.4 Transformation rules of operators

The connected components of CFT are invariant under SO(d+1,1). The inversion invariance may or may not hold true. Inversion is conjugate to parity by SO(d+1,1).

CFTs are invariant under full conformal group or SO(d+1,1) if they are invariant under parity.

$$I = g^{-1}.P.g$$

$$g \in SO(d+1,1)$$

Conformal transformations act on $\mathrm{I\!R}^{\mathrm{d}} \cup \infty$. This is not a proper Riemannian manifold but a "Riemann sphere". Now we define Weyl Rescaling as:

$$(ds^2)_{new} = W(x)(ds^2)_{old}$$

If W(x) is chosen appropriately, $x = \infty$ can be set a finite distance away. Eg. $W(x) \sim \frac{1}{x^2}$

Since conformal transformations are defined as transformations which preserve the metric up-to an x dependent factor, it means that if we re-scale the metric by another x dependent factor then at least locally it does not change the set of conformal transformations.

Now we show why the rule $\tilde{\phi}(x') = \Omega(x)^{-\Delta_\phi}\rho(R(x))\phi(x)$ is important.

- First we imagine $\phi(x)$ (taking values in V_ρ which is a vector space of the irreducible representation ρ) to be just some function on $\mathrm{I\!R}^{\mathrm{d}}$.

  $$\pi_f : \phi \to \tilde{\phi}$$

  $\pi_{\Delta,\rho}$ is a representation of conformal group. (If we commute two conformal transformations then the Jacobi matrices also commute by chain rule and so the factors and rotation matrices will compose properly).

- Considering the correlation function :

$$G(x_1, ...x_n) =< \phi_1(x_1), ...\phi_n(x_n) >$$

> Correlation function of conformal field theory is an invariant tensor inside tensor product representation.

$$G(x_1, ...x_n) \in (\pi_{\Delta_1,\rho_1} \otimes \pi_{\Delta_2,\rho_2} \otimes .. \pi_{\Delta_n,\rho_n})^{SO(d+1,1)}$$

(The step 1 transformations will only apply to primaries)

In the above steps, the special aspects are:

- Only Σ and R are included but not their derivatives
- $\tilde{\phi}$ depends only on ϕ

It must be noted that derivatives of primaries of any order = descendants ($\neq$ primaries). So now we face a problem regarding fields which are neither primaries or descendants of primaries. The following fact must be noted :

> In CFT, any field is either primary or descendant of primary (or a linear combination thereof)

(The hidden assumptions in the above statement are stated later on.)

Now we try to introduce "locality" in our theory. The requirement that our theory is local helps in understanding why many theories have conformal invariance.

A.4.1 Virasoro algebra

Commutation relations[1] between phase space functions in classical field theory are promoted to that between operators acting on the Hilbert space in quantum theory

[1] which with certain conditions induce a corresponding Poisson bracket relation

using the Dirac quantisation procedure.

$$[t_a, t_b] = f^c_{ab} t_c \xrightarrow[\text{Quantisation}]{} [T_a, T_b] = \hbar f^c_{ab} T_c + \mathcal{O}(\hbar)^2 \tag{A.1}$$

The $O(\hbar)^2$ terms are constrained by the Jacobi identity. Under re-scaling and taking the $\mathcal{O}(\hbar)^2$ to be c-numbers, we obtain what is called the central extension of the algebra. Thus, the central extension of the Witt algebra, under the above procedure gives rise to the following commutation relations[2],

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)c \quad [L_n, c] = 0 \quad , \forall n, m \in \mathbb{Z} \tag{A.2}$$

where c is called the central charge. The algebra so obtained is called the Virasoro algebra

A.5 Representations of the Virasoro algebra

In 2-dimensional conformal field theory, the spectrum can be broken down into representations of the generic symmetry algebra - which is the two copies of Virasoro algebra in this case. The knowledge of either the left (L_m) or the right operator is sufficient because they are repetitive.

L_0 can be termed as the generator of the Cartan subalgebra[3] of the Virasoro algebra, physically L_o will be identified with the Hamiltonian of the system. It is very similar to what we do in creation and annihilation operators. Here there are two modes. The positive modes are the raising operators whereas the negative modes are the lowering operators.

[2]Note that the generators of anti-holomorphic transformations $\overline{L}_n$'s also obey the same commutation relations.

[3]a subalgebra of a Lie algebra that is self-normalising

A.5.1 Highest weight representations

The Highest weight representations are constructed when we start from a state being annihilated by all the raising operators. We can generate the representation by acting on this highest weight state, the lowering operators.

In quantum field theory, the operator product expansion (OPE) defines product of fields as a sum over the same fields. It axiomatically offers a non-perturbative approach to quantum field theory. In 2D Euclidean field theory, the operator product expansion is a Laurent series expansion associated to two operators.

If we have a primary field of a left-moving dimension, we can write the operator product expansion with the stress energy tensor (T(z)) and we can get the relation :

$$|L_n, \Phi(w)| = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\Phi(w) = \Delta(n+1)w^n \Phi(w) + w^{n+1}\partial\Phi(w) \tag{A.3}$$

By the State-Operator correspondence for conformal field theories there will be a state associated with the above operator, it is natural to note that the state will be :

$$|\Delta\rangle \equiv \Phi(0)|0\rangle$$

Since $|L_n, \Phi(0)| = 0, n > 0$, it follows that

$$L_{m>0}|\Delta\rangle = L_{m>0}\Phi(0)|0\rangle = |L_m, \Phi(0)|\,|0\rangle + \Phi(0)L_{m>0}|0\rangle = 0 \tag{A.4}$$

It is important to note here that the primary fields [4] are in one to one correspondence with the Highest Weight states. Hence, each primary field will generate a representation of the Virasoro algebra. We can generalize these expression to particular in-states (defined same as above)

$$L_0|\Delta, \bar{\Delta}\rangle = \Delta|\Delta, \bar{\Delta}\rangle, \quad \bar{L}_0|\Delta, \bar{\Delta}\rangle = \bar{\Delta}|\Delta, \bar{\Delta}\rangle, \quad L_{n>0}, \bar{L}_{n>0}|\Delta, \bar{\Delta}\rangle = 0 \tag{A.5}$$

[4] A field $f(z)$ is primary if it transforms as $f(z) \to g(\omega) = \left(\frac{d\omega}{dz}\right)^{-h} f(z)$ under an infinitesimal conformal transformation $z \to \omega(z)$.

The representation of the rest of the states generated by $|\Delta\rangle$ is:

$$|\psi\rangle = L_{-n_1} L_{-n_2} \cdots L_{-n_k} |\Delta\rangle \tag{A.6}$$

It is important to note that here all $n_i > 0$ will be called descendants as they are L_0 eigenstates with eigenvalues of $\Delta + \Sigma_k n_k$. This representation is termed as the Verma module.

Now, we can argue that as the spectrum of L_0 should be bounded below, it will have a lowest eigenvalue (termed h) and a corresponding eigenvector (termed $|\Psi_0\rangle$). These must satisfy the relations:

$$\begin{aligned} T_n^a \, |\Psi_0\rangle &= 0, \quad n > 0 \\ L_n \, |\Psi_0\rangle &= 0, \quad n > 0 \end{aligned} \tag{A.7}$$

A.5.2 Unitary Representations of the Virasoro Algebra

After imposing unitarity conditions with the highest weight representations above, we get:

$$\begin{aligned} T_n^{a\dagger} &= T_{-n}^a \\ L_n^\dagger &= L_{-n} \end{aligned} \tag{A.8}$$

We are interested in these representations as they carry physical implications. Although not always irreducible, these representations can be decomposed into irreducible representations.

We take the trace over the whole representation while q is an element of the conformal group and find χ to be:

$$\chi_\Delta(q) \equiv \mathrm{Tr}\left[q^{L_0 - c/24}\right] \tag{A.9}$$

This is an interesting function of a conformal representation from a primary operator with dimension Δ

In order to get a representation without any null vectors, the above character will be:

$$\chi_{\Delta}(q) = \frac{q^{\Delta - c/24}}{\prod_{n=1}^{\infty}(1-q^n)} \tag{A.10}$$

We term this as the vacuum representation while the rest of the representation will be generated by the negative modes of the Virasoro algebra. (except L_{-1} as it annihilates the vacuum state). When we have c greater than 1, we get :

$$\chi_0(q) = \frac{q^{-c/24}}{\prod_{n=2}^{\infty}(1-q^n)} \tag{A.11}$$

Now, to find the character, we have to find the central charge.

- We know that in a unitary theory, the norms should be positive, so we get the norm of the state $L_{-n}|0\rangle$ to be

$$\begin{aligned} \|L_{-n}|0\rangle\|^2 &= \left\langle 0 \left| L_{-n}^{\dagger} L_{-n} \right| 0 \right\rangle = \left\langle 0 \left| \left[\frac{c}{12}\left(n^3 - n\right) + 2nL_0 \right] \right| 0 \right\rangle \\ &= \frac{c}{12}\left(n^3 - n\right) \end{aligned} \tag{A.12}$$

- we get the above expression after using the Virasoro commutation relations and the global conformal invariance of the vacuum. When we have $0 < c < 1$, we get c to be of the form :

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, \ldots \tag{A.13}$$

A.6 Conclusion

So far we have studied and analysed the method of finding the characters and representations of any general conformal field theory in 2 dimensions. This is indeed very helpful for calculations in specific examples of CFTs like the Critical Ising model, String theory, free Scalar Field theories with conformal symmetry and N-free Majorana fermions as some examples.

Appendix B

(Penrose Carter Diagrams)

B.1 Introduction

Curved spacetime manifolds can be often approximated by manifolds with high degrees of symmetry. It would be useful to be able to draw spacetimes diagrams that capture global properities and casual structure of sufficiently symmetric spacetimes. What is needed to be done is a conformal transformation which brings entire manifold onto a compact region such that we can fit the spacetime (ie. its infinities) on a finite 2-dimensional diagram, known as Penrose-Carter diagram

B.2 For Schwarzschild Black Holes

One particularly interesting and useful example of a coordinate system in flat spacetime is used to construct its Penrose diagram. Let us consider the following transformations in the line element of flat spacetime in spherical polar coordinates

$$u \equiv t - r \qquad v \equiv t + r$$

Hence, the line element transforms to

$$ds^2 = -dudv + \frac{1}{4}(u - v)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

We can therefore say that the (u,v) axes are rotated by 45 degrees from the (t,r) axes in a spacetime diagram. Now we make a further transformation as follows

$$u' \equiv \tan^{-1} u \equiv t' - r' \qquad v' \equiv \tan^{-1} v \equiv t' + r'$$

Now we focus on the range of each of the basis of these coordinate systems. It is trivial to note that the initial (t,r) system has a range of $-\infty < t < +\infty$ and $0 < r < +\infty$. But we can see that the ranges of (u',v') become finite because of $\tan^{-1}$. If we see in terms of the metrics involved, first we are considering the Minkowski space-time metric g_{ab} which we understand as a physical metric and introduce an unphysical metric term $\tilde{g}_{ab}$, which is related to the physical metric by a conformal factor.

$$\tilde{g}_{ab} = \Omega g_{ab}$$

we can calculate the line elements of the physical and unphysical metrics after the first coordinate transformations as follows:

$$ds^2 = \frac{1}{4} sec^2 u sec^2 v [4dudv - \sin^2(u-v)(d\theta^2 + \sin^2\theta d\phi^2)]$$

$$d\tilde{s}^2 = 4dudv - \sin^2(u-v)(d\theta^2 + \sin^2\theta d\phi^2)$$

Hence the conformal factor,

$$\Omega = \frac{1}{4} sec^2 u \ \ sec^2 v$$

Finally we express the unphysical line element in terms of t' and r' as follows:

$$ds^2 = dt'^2 - dr'^2 - \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2)$$

B.3 For Reissner–Nordström Black Holes

For a Reissner–Nordström black hole, the penrose diagram becomes more complex. The Penrose diagram reveals the meaning of the inner horizon $r = r_-$.If we consider some initial data specified on a spatial surface, such a surface is referred to as as Cauchy surface. We then evolve this initial data forward using the equations of

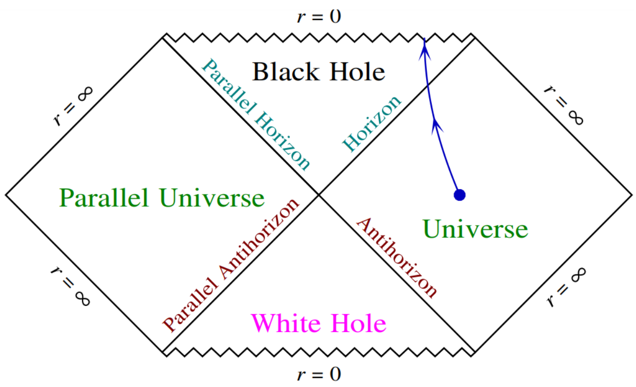


Figure B.1: Penrose Diagram of an analytically continued Schwarzschild black hole

motion. Sadly, once we encounter a timelike singularity, such evolution is no longer possible, because we need information about what the fields are doing at the singularity. We see that the data on the Cauchy surface can only be evolved as far as the inner horizon $r = r_-$. The null surface $r = r_-$ is called a Cauchy horizon.

The Cauchy horizon is believed to be unstable. To get some intuition for this, we can consider the two observers in the next figure. Observer A stays sensibly away from the black hole, sending signals with some constant frequency into the black hole for all eternity. Meanwhile, observer B ventures into the black hole where he receives the signals. But the signals get closer and closer together as he approaches $r = r_-$, an eternities worth of signals arriving a finite amount of time. These signals are therefore infinitely blue shifted, meaning that a small perturbation in the asymptotic region results in a divergent perturbation on the Cauchy horizon.

This instability means that much of the Penrose diagram of the Reissner-Nordstrom black hole, including the timelike singularity, is unphysical. It is unclear what the end point of the perturbation will be. One possibility is that the Cauchy horizon $r = r_-$ becomes a singularity.

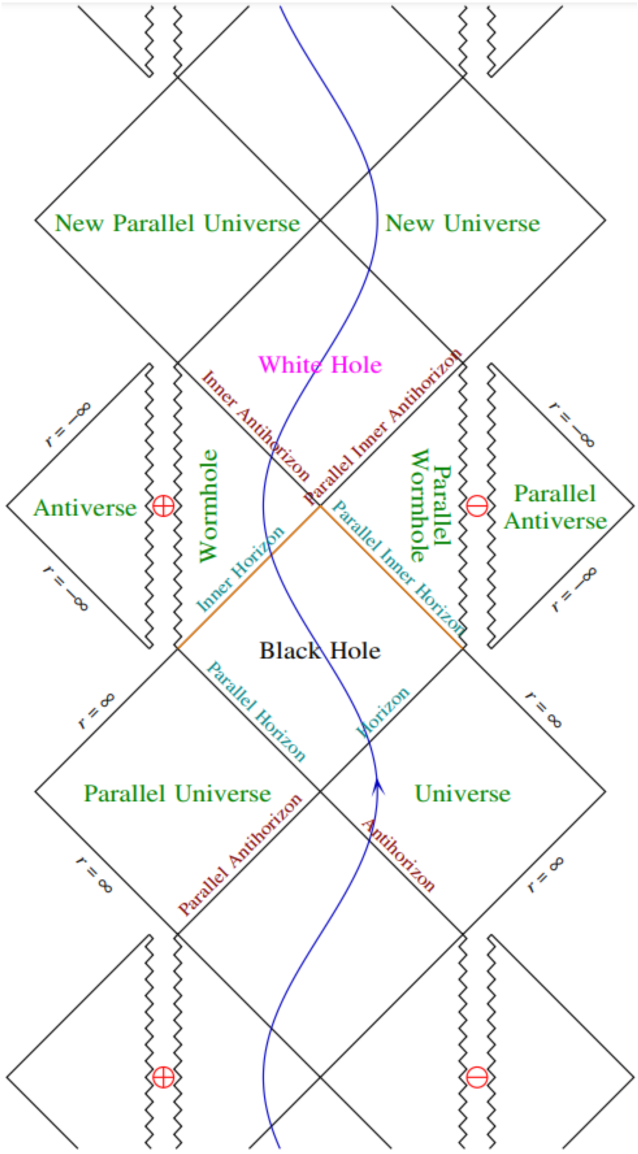


Figure B.2: Penrose Carter diagram of a Reissner–Nordström black hole

Appendix C

AdS$_3$/CFT$_2$

C.1 Introduction

In this chapter, we study an interesting correspondence resulted from string theory which relates an anti desitter space of D dimensions with a conformal field theory in D-1 dimensions. In our case we choose D=3 as for our systems of interest,the near horizon geometry has an AdS_3 structure.

C.1.1 AdS-CFT correspondence

The anti-de Sitter/conformal field theory correspondence is arguably the most successfully result from string theory, which provides a duality link between a space and a theory. The essential ingredients on the two sides of AdS-CFT correspondence are the near horizon Anti-de Sitter region of a black hole geometry, and the low energy Conformal Field Theory describing the underlying branes.

$$Z_{AdS} = Z_{CFT} \tag{C.1}$$

In the high energy limit, the left hand side of the above equation is dominated by black holes which asymptotically reach an AdS space, known as BTZ black hole. The general properties of conformal field theories imply that the asymptotic density of states will agree between the two sides.

Once we incorporate new contributions in the gravitational side beyond the single large black hole, the accuracy of the correspondence becomes clear.

We can define the AdS partition function as an Euclidean path integral. At finite temperature, the contributing Euclidean geometries should have a boundary as a two dimensional torus to match with the standard finite temp. description of the boundary CFT.The typical bulk geometry will be topologically a three-dimensional solid torus.

The Path Integral will include ideally an infinite series of higher derivative terms in the spacetime Lagrangian, also allowing particle, string and brane states to wind around the solid torus.

C.1.2 2-d CFT

In two dimensional CFT we have independent temperatures for left and right movers :

$$\tau \sim \frac{1}{T_L}(\bar{\tau} \sim \frac{1}{T_R}) \tag{C.2}$$

There is a spectrum of left and right moving conserved charges (turning on chemical potential - z_I and $\tilde{Z}_I$). While, Non zero chemical potentials help us study charged black holes

C.1.3 Black Hole entropy

To study the black hole entropy, we consider the high temperature behaviour of the partition function in CFT and we see its structure:

$$\ln Z = \frac{i\pi}{\tau}(\frac{c}{12} - 2C^{IJ}z_I z_J) - \frac{i\pi}{\bar{\tau}}(\frac{\tilde{c}}{12} - 2\tilde{C}^{IJ}\tilde{z}_I\tilde{z}_J) \tag{C.3}$$

+ exponentially suppressed terms c and $\tilde{c}$ are the left and right moving charges; $C^{IJ}$and$\tilde{C}^{IJ}$ are matrices appearing in the CFT current algebra. The two derivative approximation in the gravity side (neglecting the exponentially small terms) will reproduce the area law for the entropy of a general rotating, charged black hole.

We can go considerably further by computing the parameters (like c) by relating them with anomalies.

The higher derivative corrections to space time action is encoded in the parameter correction and in turn gives us a correction to the area law.We can express the degeneracy as a function of charges by a Laplace transform relation of the canonical ensemble to micro-canonical ensemble,thus deducing a series of 1/Q corrections to the degeneracy. The exponentially suppressed terms in the above equation arises from fluctuations in the Black hole geometry when we sum over in-equivalent black holes. Finally, we need to take Z as a supersymmetric partition function to get the computation for the exact form.

C.1.4 Stress Tensor

Now we can construct an AdS boundary stress tensor dual to the stress tensor of the CFT. In two derivative gravity, this stress tensor obeys Virasoro algebra with derivable central charges. It can be generalized to higher derivative gravity theories to find a generalised central charge by a simple extremization principle.

While computing the entropy, it is crucial to construct a BTZ as a quotient of AdS and its relation to a thermal AdS geometry via a modular transformation. Finally, we can establish the agreement between black hole and CFT entropies with agreeing central charges.

The Bulk Chern-Simons terms completely determine the currents and play a central role. Turning on flat connections for our gauge fields allow us to incorporate charged black holes.

C.1.5 String Theory constructions

In String theory, particularly the D1-D5 system, giving rise to five-dimensional black holes with near horizon geometry $AdS_3 \times S^3$ while wrapped M5-branes yields 4d black holes with near horizon geometry $AdS_3 \times S^2$. This will allow us to derive a class of corrections to the black hole area law.

C.2 Gravity in asymptotically AdS_3 spacetimes

C.2.1 Action and stress tensor

We start with the Einstein-Hilbert action supplemented by boundary terms:

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}(R - \frac{2}{l^2}) + I_{bndy}$$

Here, AdS_3 is a solution of the equation of motions for the action, it is a homogeneous space of constant negative curvature, having maximal symmetry with the isometry group $SL(2,\mathbb{C}) \sim \mathrm{SL}(2,\mathbb{R})_\mathrm{L} \times \mathrm{SL}(2,\mathbb{R})_\mathrm{R}$.

$$ds^2 = (1 + \frac{r^2}{l^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\phi^2 \tag{C.4}$$

A one parameter family of solutions of this is the non-rotating BTZ black hole:

$$ds^2 = \frac{(r^2 - r_+^2)}{l^2}dt^2 + \frac{l^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2 \tag{C.5}$$

This has a singularity at $r = r_+$ and after rotating to Lorentzian signature it can be described as a BH having the following Bekenstein Hawking Entropy :

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G}$$

- It should be noted from large r behaviour that the solution asymptotically approaches AdS_3
- Now we demand the existence of a well defined action and a variational principle.
- Action takes on a well defined meaning as giving the partition function of the CFT (in a semi-classical limit)

- Now we introduce Gaussian Normal coordinates for this construction:

$$ds^2 = d\eta^2 + g_{ij}dx^i dx^j$$

We can write the Einstein Hilbert action in the terms of Gaussian Normal Coordinates (and integrating it by parts we get):

$$I_{EH} = \frac{1}{16\pi G}\int d^3x d\eta\sqrt{g}(R^{(2)} + (TrK)^2 - TrK^2 - 2\Lambda) \tag{C.6}$$

$$-\frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{g}TrK \tag{C.7}$$

Here $R^{(2)}$ is the Ricci Scalar of the metric. K is the extrinsic curvature:

$$K_{ij} = \frac{1}{2}\partial_\eta g_{ij}$$

C.2.2 Introducing the GH term

As the boundary term variation will contain $\delta\partial_\eta g_{ij}$, it will spoil the total variation (with fixed induced metric on ∂M but not its normal derivative).We add a rectification term for this purpose to the action, known as the Gibbons-Hawking term: (How can we just add this?)

$$I_{GH} = \frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{g}TrK$$

The variation is composed of a Bulk piece (vanishes when EOM satisfied) and a boundary piece.

$$\delta(I_{EH} + I_{GH}) = -\frac{1}{16\pi G}\int_{\partial M} d^dx\sqrt{g}(K^{ij} - TrKg^{ij})\delta g_{ij}$$

The variation of the boundary term can be defined in terms of the stress tensor as:

$$\delta I = \frac{1}{2}\int_{\partial M} d^2x T^{ij}\delta g_{ij}$$

Hence:

$$T^{ij} = -\frac{1}{8\pi G}(K^{ij} - TrKg^{ij})$$

Once we find out the stress energy tensor from the Bulk side, we can also calculate the stress energy tensor from the CFT counterpart and equate them. We plan to do so in the upcoming continuation of the thesis and ultimately rigorously formulate the correspondence, which will help us in calculations related to the near horizon geometry of the D1-D5 brane system or the p-branes.