Conformal Field Theory in d>2 and bootstrap

Debaiudh Das

1. The Conformal Group

A conformal transformation of coordinates is an invertible mapping $x \to x'$, which leaves the metric tensor invariant, upto a scale:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x)$$

A conformal transformation is locally equivalent to a (pseudo) rotation and a dilation. The set of conformal transformations forms a group (Poincare group as subgroup) $\to \Lambda(x) = 1$

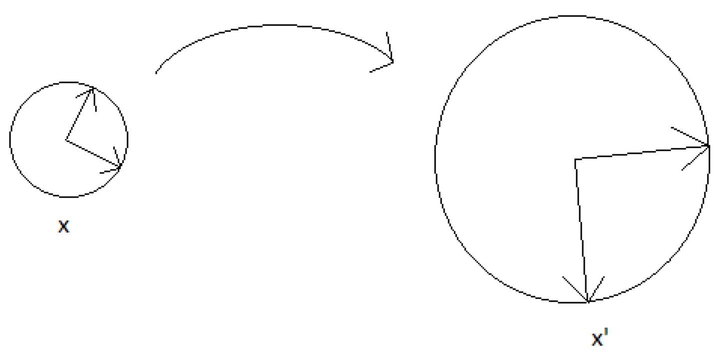
Conformal$\to$Angle does not change between two arbitrary curves crossing each other (despite local dilation)

$$x^\mu \to x'^\mu + \epsilon^\mu(x) g_{\nu\mu} \to x'^\mu + \epsilon^\mu(x)$$

Conformal field theory is a quantum field theory whose correlation functions are invariant under the conformal group. In this report we only consider CFTs in Euclidean $\mathbb{R}^d$ so there is no distinction between upper and lower indices.

A one to one conformal transformation can be shown as :

$$x \to x' = f(x)$$



Locally, these are dilation and rotation. Hence we get the expression:

$$J_{\mu\nu} = \frac{\partial f^\mu}{\partial x_\nu} = \Omega(x) R_{\mu\nu}(x) \leftarrow O(d) \quad [RR^T = 1]$$

Here fields $\phi_i(x)$ [i=1,2,3.. and $\phi_1 = \mathbb{1}$ by convention] are taken as operators labelled by two properties :

- Scaling dimension $\Delta_i \in \mathbb{R}_{>0}$
- ρ_i irreducible representations of SO(d)

[Here we take only bosonic theory for simplicity.] We consider the correlation function:

$$< \phi_1(x_1) ... \phi_n(x_n) >= G(x_1 ... x_n)$$

Scale invariance would imply that : $G(\lambda x_1,...\lambda x_n) = \lambda^{-\Delta_1-\Delta_2...-\Delta_n}G(x_1,...x_n)$ where Δ is the scaling dimension.
For ϕ_i scalars $x \to x'$:

$$G(x'_1,...x'_n) = \Omega(x_1)^{-\Delta_1}...\Omega(x_n)^{-\Delta_n}G(x_1,...x_n)$$

ϕ_i transforms as: $\tilde{\phi}_i(x') = \Omega(x)^{-\Delta_i}\phi_i(x)$
Now we construct the conditions for ϕ_i under irreducible representation ρ_i. We know that invariance under Poincaire groups would mean : x'=Rx+b. Hence,

$$\tilde{\phi}_i(x') = \rho(R)\phi(x)$$

$$\tilde{\phi}_\mu(x') = R_{\mu\nu}\phi_\nu(x) \text{ for } \phi \text{ a vector.}$$

$\rho(R)$ is a finite dimensional representation of R and acts on indices of ϕ
For conformal transformations:-

$$\tilde{\phi}(x') = \Omega(x)^{-\Delta_\phi}\rho(R(x))\phi(x)$$

In short, Conformal transformation gives R(x), based on this construct $\rho(R(x))$ act on the field, giving the transformation.

2. Conformal transformations

Conformal transformations leave the metric invariant up to an x dependent factor.

$$(dx')^2 = \Omega(x)^2(dx)^2$$

According to Louville Theorem: Conformal transformations in $\mathbb{R}^d$ are generated in (d>2) by:

- Translations
- Rotations
- Dilatations $x' = \lambda x$
- Inversions $x'^\mu = \frac{x^\mu}{x^2}$

These all transformations belong to a finite dimensional Lie group and has some topology. But, Inversions belong to a Lie group which is disconnected from $\mathbb{1}$

3. Special Conformal Transformation

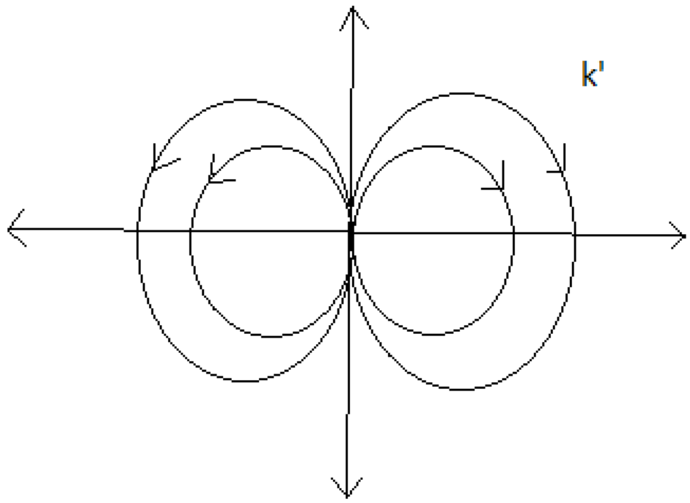
$$\text{SCT(a)=I.T(-a).I} \in \text{ connected components}$$
$$\text{SCT(a)}: x_\mu \to \frac{x_\mu - a_\mu x^2}{1-2ax+a^2x^2} \to x = \frac{a}{a^2} = \infty \text{ (There exists a point)}$$

K_μ have x^2 behavior at $\infty . If$K$_\mu$ is exponentiated and the corresponding differential equation ($y' = y^2$) is solved, it blows up at finite time.
Now we consider an infinitesimal transformation:

$$x' = x + \epsilon(x)$$

We know that $P^\mu, M_{\mu\nu}, D, K_\mu$ are the finite generators of translations, rotations, dilatations and special conformal translations. Likewise with some algebra we can construct the infinitesimal generators of the transformations:

$$p^\mu = \partial_\mu$$
$$\mathrm{m}_\mu \nu = x^\nu \partial_\mu - x^\mu \partial_\nu$$
$$\mathrm{d}=\mathrm{x}^\lambda \partial_\lambda$$
$$\mathrm{k}_\mu = 2x_\mu (x.\partial) - x^2 \partial_\mu$$



Lines of flow for kμ (k')

The conformal killing equation satisfying all of the above vector fields is:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial\epsilon)\delta_{\mu\nu}$$

The commutation relation for the generators are given below (worked out in rough) :-

$$[M, M] \sim M$$
$$[M, P] \sim P$$
$$[M, K] \sim K$$
$$[D, P_\mu] \sim P_\mu$$
$$[D, K_\mu] \sim K_\mu$$
$$[K_\mu, P_\nu] \sim 2\delta_{\mu\nu} D - 2M_{\mu\nu}$$

In the above relations, P is a vector under rotations and K is a vector under Special Conformal transformations. Rotation generator measure the dimension of the generator (counting the number of x's and the derivatives).
Now we ask the question that how many generators are there in total?

$$\frac{d(d-1)}{2} + 1 + d + d = \frac{(d+1)(d+2)}{2}$$

This is same as for SO(d+2), more precisely SO(d+1,1).

- We find a linear combination of generators, such a way that it satisfies the commutation relations of SO(d+1,1)
- This acts on $\mathbb{R}^{d+1,1}$ (Seen clearly in embedding formalism)

No subset of generators are left invariant by the lie brackets. This is true because the K_μ generator is added (simplifies).

4. Transformation rules of operators

The connected components of CFT are invariant under SO(d+1,1). The inversion invariance may or may not hold true. Inversion is conjugate to parity by SO(d+1,1). CFTs are invariant under full conformal group or SO(d+1,1) if they are invariant under parity.

$$I = g^{-1}.P.g$$

$$g \in SO(d+1,1)$$

Conformal transformations act on $\mathbb{R}^{\mathrm{d}} \cup \infty$. This is not a proper Riemannian manifold but a "Riemann sphere". Now we define Weyl Rescaling as:

$$(ds^2)_{new} = W(x)(ds^2)_{old}$$

If W(x) is chosen appropriately, $x = \infty$ can be set a finite distance away. Eg. $W(x) \sim \frac{1}{x^2}$

Since conformal transformations are defined as transformations which preserve the metric up-to an x dependent factor, it means that if we re-scale the metric by another x dependent factor then at least locally it does not change the set of conformal transformations.

Now we show why the rule $\tilde{\phi}(x') = \Omega(x)^{-\Delta_\phi}\rho(R(x))\phi(x)$ is important.

- First we imagine $\phi(x)$ (taking values in V_ρ which is a vector space of the irreducible representation ρ) to be just some function on $\mathbb{R}^{\mathrm{d}}$.

  $$\pi_f : \phi \rightarrow \tilde{\phi}$$

  $\pi_{\Delta,\rho}$ is a representation of conformal group. (If we commute two conformal transformations then the Jacobi matrices also commute by chain rule and so the factors and rotation matrices will compose properly).

- Considering the correlation function :

  $$G(x_1, ...x_n) =< \phi_1(x_1), ...\phi_n(x_n) >$$

  > Correlation function of conformal field theory is an invariant tensor inside tensor product representation.

  $$G(x_1, ...x_n) \in (\pi_{\Delta_1,\rho_1}\pi_{\Delta_2,\rho_2}...\pi_{\Delta_n,\rho_n})^{SO(d+1,1)}$$

  (The step 1 transformations will only apply to primaries)

In the above steps, the special aspects are:

- Only Σ and R are included but not their derivatives
- $\tilde{\phi}$ depends only on ϕ

It must be noted that derivatives of primaries of any order = descendants ($\neq$ primaries). So now we face a problem regarding fields which are neither primaries or descendants of primaries. The following fact must be noted :

In CFT, any field is either primary or descendant of primary (or a linear combination thereof)

(The hidden assumptions in the above statement are stated later on.)
Now we try to introduce "locality" in our theory. The requirement that our theory is local helps in understanding why many theories have conformal invariance.

5. QFT Background

UV complete QFTs are QFTs without cutoff (i.e) If a system is near some idealized mathematical object, that must be studied first, then the physical picture should be analyzed.

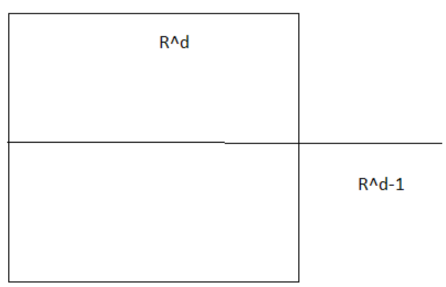
- Examples:RG fixed Points (a priori as scale invariant is also conformal invariant)
  - Free (Gaussian) massless scalar, massless fermions
  - Non-Gaussian - critical point of 3d ising model.
- Non scale invariant - UV complete QFTs (starting from RG fixed and perturbing it in some direction)

$$S_{FP} + \Sigma_i \int O_\Delta$$

QCDs, $(\phi^4)_{d=3}$,QED_3+fermions
Effective QFTs $(\Lambda_{UV}) \neq$ UV- complete QFTs
QED$_4$,$(\lambda\phi^4)_{4d}$ (Landau pole), Chiral lagrangians $\Lambda_{UV} \sim 1$GeV
It must be noted here that UV-complete QFTs are mathematically unaxiomatic.

5.1. Non-local QFTs

- Set of correlation coming from local d dimensional theory with $\mathbb{R}^{d-1}$ subspace.



- Scalar Field

$$\int d^d p \ \ |p|^s \phi_{-p}\phi_{-p} \ \ \sim \ \ \int d^d x \ \ d^d y \ \ \frac{\phi(x)\phi(y)}{|x-y|^{d+s}}$$

(Lattice models with Long range interactions)

A quantum field theory is local if it has a local stress tensor $T_{\mu\nu}$

- $\partial_\mu T_{\mu\nu}$=0 (Away from coincident points)
- $T_{\mu\nu} = T_{\nu\mu}$ (Symmetric)
- Generates translations

It should be stressed that we are not restricted to any local Lagrangian or a local path integral. Now we construct an operator.

Topological Surface Operator

$$Q_\nu(\Sigma) = \int_\Sigma dS^\mu \;\; T_{\mu\nu}(x)$$

The right hand integration is conserved and does not depend on the shape of the surface on Euclidean space.We can write :

$$< \phi_1(x_1)...Q_\nu(\Sigma) >=< \phi_1(x_1)...Q_\nu(\Sigma') >$$

where $\Sigma_R \rightarrow \infty$ (surrounding the surface with spheres)

$$0 = \Sigma_i < \phi_1(x_1)...(Q_\nu\phi_i)(x_i)... >$$

$$Q_\nu.\phi(x_i) = \lim_{\Sigma\rightarrow x} \int_\Sigma dS^\mu \;\; T^{\mu\nu}(x)\phi(x_i)$$

We see that $Q_\nu.\phi(x_i)$ is also a local operator which is equal to $\partial_\mu\phi(x_i)$. Here we assume the following:

- $Q_\nu = P_\nu$
- $Q_\nu.\phi(x) = \partial_\mu\phi(x)\forall\phi$

In framework of general Poincaire invariant QFT we see that scale invariance would imply conformal invariance. This is a miraculous fact.

- $T^{\mu]nu}(x)\phi(0) \sim$ (More singular terms corresponding to rotation, scaling etc.)...+$B^\phi_{\mu\nu\lambda}(x)\partial_\lambda\phi(0) + ...$
- $\int dS^\mu \;\; B^\phi_{\mu\nu\lambda}(x) = \delta_{\nu\lambda}$
- $B \sim \;\; \frac{1}{|x|^{d-1}}$
- $\Delta_T = d$

We can construct more Topological Surface Operators for some vector field $\epsilon(x)$:

$$Q_\epsilon(\Sigma) = \int_\Sigma dS^\mu \;\; T^{\mu\nu}(x)\epsilon_\nu(x)$$

Here ϵ is constant and $\epsilon_\mu = \omega_{\mu\lambda x^\lambda}$

$$\partial_\mu(T^{\mu\nu}\epsilon_\nu) = 0 = \delta_{\mu\nu} \; for \; \epsilon^D \implies TSO \tag{1}$$
$$\equiv T^{\mu\nu}\partial_\mu\epsilon_\nu = 0 \tag{2}$$
$$iff \;\; \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = 0 \tag{3}$$

For any TSO we have

$$\Sigma_i < \phi_1(x_1)...(Q_\epsilon\phi_i)(x_i)... >= 0$$

Correlators of stress tensors $\implies \frac{1}{dist^{2d}} \sim \frac{1}{l^{2d}}$

Q_ϵ has the same physical meaning $\implies$ spatial transformations corresponding to ϵ.

$$\boxed{Q_\epsilon T_{\mu\nu} = \epsilon_\rho \partial_\rho T_{\mu\nu} - \partial_\rho \epsilon_\mu T_{\rho\nu} + +\partial_\nu \epsilon_\rho T_{\rho\mu}}$$

This holds true for both constant ϵ and $\epsilon = \omega x$. In 2 dimensions we see that this takes the form of Virasoro generators but this is less straightforward in $d > 2$ CFT. After deforming the 2 surface to surround the insertion, we get:

$$[\epsilon_1, \epsilon_2] = (\epsilon_1 \partial)\epsilon_2 - (\epsilon_2 \partial)\epsilon_1$$

$$\boxed{[Q_{\epsilon_1}, Q_{\epsilon_2}] = Q_{-[\epsilon_1,\epsilon_2]}} \equiv Poincaire - Algebra$$

This simple formalism setting helps in generalizing scale invariance and conformal invariance equivalence. Moreover, we did not have to assume that rotations act correctly on stress tensor.

6. Scale Invariance- Conformal Invariance Equivalence

TSO $(\epsilon^D)_\mu = x_\mu$
$T_{\mu\mu} = 0$. I can get more TSOs $\forall$ conformal killing vector fields $\implies$ TSO $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \sim \delta_{\mu\nu} C(x)$
Working out the Poincaire algebra we see that:

$$Q_\epsilon T_{\mu\nu} = \epsilon_\rho \partial_\rho T_{\mu\nu} - \partial_\rho \epsilon_\mu T_{\rho\nu} + +\partial_\nu \epsilon_\rho T_{\rho\mu} + (\partial.\epsilon)T_{\mu\nu}$$

The last term is fixed by the condition that $\Delta_T = d$. The above expression is valid for any conformal killing ϵ.
The Conformal group is a group of transformations acting on space while the conformal group algebra is working on space of operators. Now we will derive the latter part of the statement and eventually show that:
Algebra working on local operators $\equiv$ Algebra working on space
There are counterexamples for this statement because the assumption that scale invariance requires $T_{\mu\mu} = 0$ is not generical. In general, Virial theorem states that $T_{\mu\mu} = \partial_{_\mu}$

Hence, $\boxed{V \neq \partial L \implies Scale\ without\ conformal}$
An example of such a case is the theory of elasticity, where the Virial current is not a total derivative.

$$\int d^d x^\alpha (\partial_/ muu_\nu)^2 + \beta(\partial_\nu u\mu)^2$$

But in interactive theories, we would not have candidates like this because the $T_{\mu\mu}$ and V_μ will always be conserved.

7. Euclidean Quantum Mechanics

For unknown states, the Hilbert space for QFT can be seen as space of states living on the surface. Hence our aim is the following:

$$[\phi(0)]^\dagger = \phi(0), H^\dagger = H \tag{4}$$
$$\phi(\tau) = e^{H\tau}\phi(0)e^{-H\tau} \tag{5}$$
$$[\phi(\tau)]^\dagger = \phi(-\tau) \tag{6}$$
$$Now: \ |\tau_1, ...\tau_n >= \phi(\tau_1)\phi(\tau_2)...\phi(\tau_n)|0> \ \ where \ 0 > \tau_1 > ... > \tau_n \tag{7}$$
$$Norm: < \tau_1...\tau_n|\tau_1...\tau_n >=< 0|\phi(-\tau_n)...\phi(-\tau_1)\phi(\tau_1)...\phi(\tau_n)|0> \geq 0 \tag{8}$$
$$\tag{9}$$

The last term is greater than equal to 0 because the reflection positivity is strictly weaker. Now we construct the Osterwalder Schroeder reflection positivity conditions:

$$|\Psi >= \Sigma_n \int d\tau^{(n)} f_n < \tau^{(n)}|\tau^{(n)} >$$

This is non zero for negative terms. Hence,

$$\Sigma_{n,m} \int d\tau^{(n)} d\tau^{(m)} f_n^*(\tau^{(n)}) f_m^*(\tau^{(m)}) < \text{H}\tau^{(n)}|\tau^{(m)} > \geq 0$$

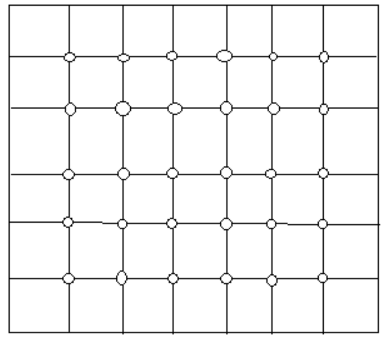
The above condition is known as Osterwalder-Schroeder Reflection Positivity where H inserts a bunch of operators at positive and negative times.
All I know is some bunch of correlation functions to construct a Hilbert space.
$< \phi(\tau_1)\phi(\tau_2)...\phi(\tau_n) >$ if reflection positivity holds.
Now we take a sequence of functions $|f >= (f_0 f_1 f_2 ...)$ and compute the inner product $< f|g >$. This is always positive as long as reflection positivity holds. Hence, we can say:

> If an Euclidean space is taken, foliated with lines of constant time, bunch of operators are inserted at negative time, then the Hilbert space can be constructed.

8. Poincaire invariant QFT

- If we take non RP Euclidean theory, it cannot be rotated to Minkowski. Hence, RP theories are considered for real time cases (particle theory) while statistical theories consider both RP and non-RP cases.
- RP theories have extra constraints, making it easier to study.
- Many well known theories are RP. if the theory is non-RP then the Hilbert space technology is lost. (Hilbert spaces are always positive definite)

If CFT arises at critical point on lattice theory, it is easier to check on lattice model because RP is easy to check. Eg. Cubic lattice model.



If the transfer matrix in spin variable is hermitian, then the lattice theory is reflection positive.
Lattice RP $\equiv$ hermitian transfer matrix.
$P_\mu, M_{\mu\nu}$ are the generators acting on the Hilbert space. Hence,

$$[P_\mu, \phi(x)] = \partial_\mu \phi \quad | \quad \phi(x+y) = e^{y^\mu P_\mu} \phi(x) e^{-y^\mu P_\mu} \tag{10}$$

$$[\phi(x)]^\dagger = \phi(\Theta x) \quad \textit{by construction} \tag{11}$$

The Hamiltonian is hermitian while the momentum generators are anti-hermitian.(We broke rotational invariance, so this is not surprising).

$$P_\mu^\dagger = -\Theta_{\mu\nu} P_\nu \qquad \Theta = diag(-1, 1...1)$$

For non scalars (with appropriate definition of complex conjugates) we have :

$$[O_\mu(x)]^\dagger = \Theta_{\mu\nu} O_\nu(\Theta x)$$

8.1. Comment on cutting Path integrals on arbitrary surfaces

Between two vector spaces, natural pairing is given by the correlation functions. If the surface is curved, it is sufficiently complicated because the vector space of states is not equal to the Hilbert space.

8.2. Radial Quantization

- We introduce a space of states living on a sphere
- The conjugate operator is implemented by inversion and hence:

  $$[\phi(x)]^\dagger = I(\phi(x)) = |x|^{2\Delta} \phi(Ix)$$

  For a non scalar,

  $$\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} = x^2 \frac{\partial I^\mu}{\partial x^\nu}$$

  By the definition of Unitary CFT we have:

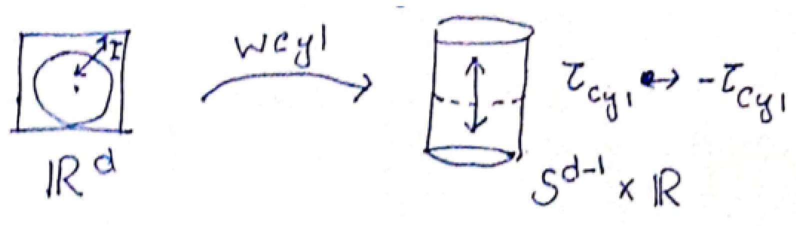
  $$< [\phi(x)]^\dagger ... \phi(x) > \ \geq 0$$

Why is this a reasonable assumption?

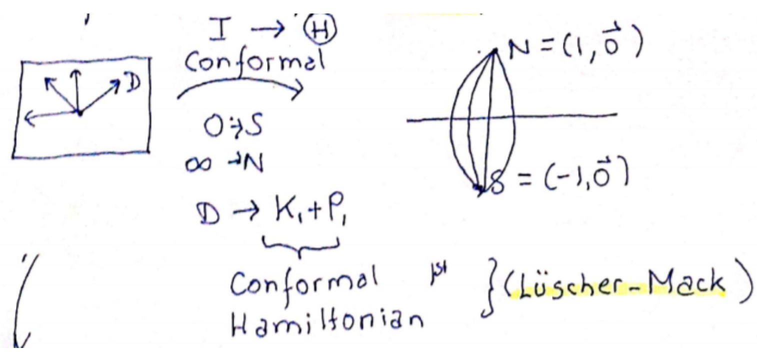
- If path integral theory is also a conformal theory conformal symmetry can be used along with radial quantization and also radial RP is true using the inversion.
- It must be known that radial RP is an extra assumption or a consequence of the property of the RP of Poincaire invariance.

There are two approaches to the problem:

- The well known way, in which we Weyl transform a sphere in $\mathbb{R}^d$ into $S^{d-1} \times \mathbb{R}$. In this way, the Schroeder sense reflection positive will remain also radial sense reflection positive.



- N-S Pole quantization:-



Here we apply conformal transformation, go back to radial quantization and see reflection positivity there in radial sense.

9. Conjugate rules for conformal algebra

$$Q_\epsilon(\Sigma) = \int_\Sigma dS^\mu [T_{\mu\nu}(x)]\epsilon_\nu(x)$$

Any generator can be given by an integral of the stress tensor. In D> $2, T_{\mu\nu}$ transforms as a primary under conformal transformations.

$$[Q_\epsilon(\Sigma)]^\dagger = \int_\Sigma dS^\mu [T_{\mu\nu}(x)]^\dagger \epsilon_\nu(x) \tag{12}$$

$$[T_{\mu\nu}(x)]^\dagger = I_{\mu\nu'} T_{\mu'\nu'}(I_x) I_{\nu'\nu} \tag{13}$$

$$[Q_\epsilon(\Sigma)]^\dagger = \int_\Sigma dS^\mu T_{\mu\nu} \tilde{\epsilon}_\nu^\dagger \tag{14}$$

$$(\tilde{(}\epsilon) = -I\epsilon I) \tag{15}$$

The last expression comes from change in orientation.Now we mention an important correspondence.

9.1. State-Operator Correspondence

We consider self adjoint non negative operators so that we can apply spectral theorem on them for rotation, dilataions. Under the discrete assumption : $D|\Delta_i >= \Delta_i|\Delta_i >$ we can say:-

States are local operators living at the origin.
$|\Delta_i >= \phi_\Delta(0)|0>$

$$[D, K_\mu] = -K_\mu$$

By acting with K_μ, I can lower the eigenvalue of D. By generality I know D is non negative in Reflection Postive theory. So this stops and hence a state can be annihilate by K_μ. We know that being primary $K_\mu|\phi >= 0$. Hence the vector field of K_μ vanishes to the quadratic order at origin. So special conformal transformations do nothing at origin.

$$K_\mu|\phi(0) >= 0$$

Hence, every state is a primary or a descendant of primary. Operator product expansion becomes a triviality here.

10. Embedding Formalism

Embedding formalism is done using the following steps:

- We introduce coordinates : $X^1, X^2...X^{d+1}, X^{d+2} \xrightarrow{timelike}$ in $\mathbb{R}^{d+1,1}$. Here:

$$X^{\pm} = X^{d+2d+1} \tag{16}$$

$$ds^2 = \Sigma_{r=1}^{d}\,(dX^\mu)^2 - dX^+dX^- \tag{17}$$

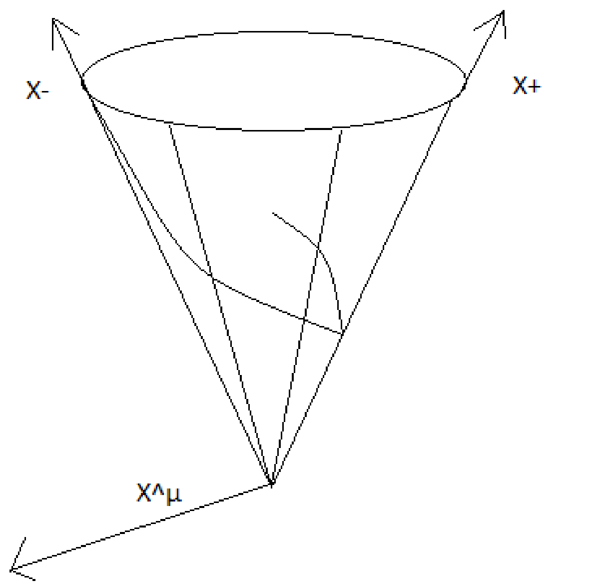
$$(X^\mu \rightarrow \Lambda_\mu^\mu X^\mu) \tag{18}$$

- Now we get rid of the extra 2 dimensions without spoiling the form of the linear transformation.
- We kill the two coordinates by:
  - Restricting to X^2=0 (Action restricted to light cone)
  - Making SO(d+1,1) act on the section, not on the light cone as a whole.
  - x^μ (defines the conformal transformation) $\rightarrow$ light ray $\xrightarrow[\Lambda_N^M]{}$light ray $\rightarrow x^{\mu'}$

  The natural metric is now the restriction of light cone on the section.

$$ds^2 = dx^2 - dX^+dX^- = g^{\mu\nu}(x)dx_\mu dx_\nu$$

  The above expression is at the limit $X^+ = f(X^\mu)$ (section equation) and $X^- = x^2/X^+$ (similar to inversion).



Here we see that $dX \xrightarrow{\Lambda} .dX$ is an isometry. Rescaling $\Omega(x)$ we get:

$$(d(\Omega(X)X))^2 = (\Omega.dX + X(\nabla\Omega.dX))^2$$

All the cross terms are dropped out because they are proportional to either X^2 or X.dX. The former is 0 because the points line on the light cone while the latter is 0 because both the initial and final points line on the light cone. Hence $(d(\Omega(x)X))^2 = \Omega^2 dX^2$.

$$\boxed{ds^2 = \Omega(X)^2 ds^2}$$

We know that $ds^2 = g_{\mu\nu}(x)dx_\mu dx_\nu$. For an Euclidean section we have the condition that $g^{\mu\nu} = \delta^{\mu\nu}$. Other sections are used for other contexts. For an Euclidean section we also know that f=1. So,

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu)$$

Now, considering infinitesimal generators, we have the expression:

$$-[J_{MN}, X^k] = \delta^k_M X_N - \delta^k_N X_M$$

Correspondingly, we get the following relations for the various conformal transformations:

$$J_{\mu\nu} = M_{\mu\nu} \tag{19}$$
$$J_{\mu+} = K_\mu \tag{20}$$
$$J_{\mu-} = P_\mu \tag{21}$$
$$J_{+-} = D \tag{22}$$

For Euclidean section, $X_- = X^2/X_+$, the infinitesimal transformation is:

$$X^\mu \to (1 - \epsilon^\mu x^\mu, x^2, x^\mu - \frac{1}{2}\epsilon^\mu x^2)$$

Going back to Euclidean section and rescaling, we get:

$$X^{\mu} \xrightarrow{rescaling} (1, *, x^{\mu} - \frac{1}{2}\epsilon^{\mu}x^{2} + x^{\mu}(\epsilon^{\nu}x^{\nu}))$$

The latter part is a special conformal transformation up to a certain constant (K^{μ} transformation).Hence with this we can generate conformal transformations not part of the connected conformal groups.
We realized all conformal transformations in a very nice way. Eventually we have to compute the correlation functions of fields which transform under conformal transformation in a simple way. In order to do this, we have to make SO(d+1,1) act on fields after making the fields inside the light cone.

$$\tilde{\Phi(X')} = \Phi(X)\ for\ scalar\ fields \tag{23}$$

$$Physical\ field\ \rightarrow d \tag{24}$$

$$light\ cone\ \rightarrow d+1 \tag{25}$$

$$\Phi(X)\ at\ section = \phi(x) \tag{26}$$

Hence our demand is that : $\boxed{\Phi(\lambda X) = \lambda^{-\Delta}\Phi(X)}$
We consider the condition that fields on the light cone are "homogeneous under rescaling". This by consistency is bound to represent some d dimensional conformal group acting on $\Phi(x)$. Now if conformal group acts on Φ:

$$\boxed{\tilde{\phi}(x') = \frac{1}{\Omega(x)^{\Delta}}\phi(x)}$$

10.1. Corollary

$$< \Phi_{\Delta_1}(X)\Phi_{\Delta_2}(Y)\Phi_{\Delta_3}(Z)... >$$

Suppose I'm able to construct such an object which depends upon $\Delta_1, \Delta_2, \Delta_3$ and X,Y,Z in such a way that it is consistent with all rescaling properties of Φ and moreover it depends on X,Y and Z in such a way that it is invariant under SO(d+1,1). This will guarantee that the projected object is going to be invariant under conformal group.

$$\rightarrow < \phi_{\Delta_1}(x)\phi_{\Delta_2}(y)... >$$

The reverse is also true, like a one-one correspondence. Here Φ has no physical meaning with no hidden dimensionality, it is like a crank used to extract the results.

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