Chaos and Machine Learning: A Case study of the Duffing Equation*

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Chaotic systems are characterised by the property that a small change in their initial conditions result in large unprecedented changes to the future trajectory. We implement methods from machine learning to classify and predict the Lyapunov exponent of a Duffing Oscillator for a physical range of input parameters and hence classify the oscillator as being chaotic or not. Our results show that ML algorithms are successful in predicting whether a Duffing Oscillator is chaotic or not. We also use novel advances in machine learning, particularly known as Echo state networks to predict the future trajectory of a given Duffing oscillator and hence render it model-free. We analyze and get a Poincare plot similar to that of a chaotic attractor using solely the trained echo state network, without any information about the system given.

I. INTRODUCTION

Chaotic phenomena is seen in most places in the sciences, they can be found in nearly all branches of nonlinear modeling. In mechanics for instance where at least two degrees of freedom play a part, or alternatively, if we have a nonlinear oscillator with external or parametric forcing. In these problems non-chaotic, regular behaviour is exceptional. Consequently this holds also in modeling based on theoretical mechanics as can be found in celestial mechanics, meteorology and other fields.

Chaotic systems are characterized majorly by the sensitive dependence of the final solutions of the system to the initial conditions. Moreover, in general we do not have a closed form solution for chaotic equations of motion and certain chaotic systems contain an infinite number of unstable periodic solutions.

There are two main types of dynamical systems: differential equations and iterated maps (also known as difference equations). Differential equations describe the evolution of systems in continuous time, whereas iterated maps arise in problems where time is discrete. Differential equations are used much more widely in science and engineering, and we shall therefore concentrate on them. Duffing equation is one such differential equation which has chaotic properties at some parameter range.

II. MOTIVATION

The Duffing Oscillator is an instance of a system which yields chaotic behaviour at certain parameter ranges of its governing differential equation. Physically, duffing oscillator is an example of a forced oscillator with a period and a nonlinear elasticity. The Duffing equation reads :

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma cos\omega t \tag{1}$$

The above equation may or may not be chaotic depending upon the values of the given parameters. Interestingly, there have also been works to relate the Duffing equation to market fluctuations. The Poincare section or the phase trajectory of a duffing oscillator is shown in figure 1.

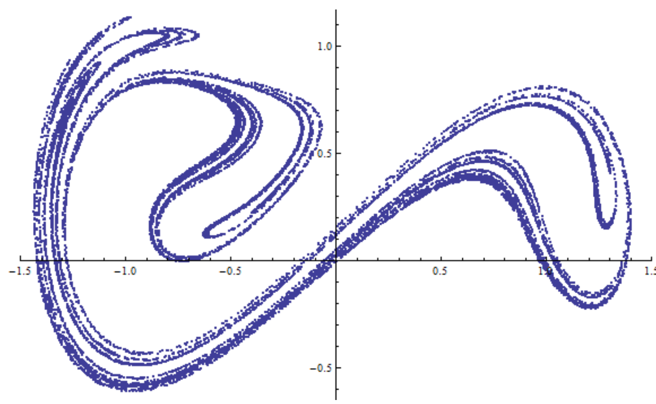

FIG. 1. Poincare section of a Duffing Oscillator with $\delta = 0.15$ $\gamma = 0.3$ and all the other parameters as unity

Physically the Duffing equation corresponds to a forced oscillator with a nonlinear spring or a model of a periodically forced steel beam which is deflected toward the two magnets. As the physical picture would not be of much interest for predicting the chaotic parameters and future evolution of the already chaotic system, we would focus on the solution generation for the first part of our work.

III. RELATED WORKS

Making use of recent advances in machine learning and neural networks, namely echo torch networks, predicting chaotic parameters and future data generation of already chaotic systems have become computationally less intensive than conventionally used methods.We

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take inspiration from these and apply various machine learning algorithms to predict the Lyapunov exponent of the Duffing oscillator for a variety of parameter ranges. Pathak et. al. demonstrated the effectiveness of using machine learning for model-free prediction of spatiotemporally chaotic systems of arbitrarily large spatial extent and attractor dimension purely from observations of the system's past evolution. We extend our work and attempt to predict the future data of a chaotic duffing oscillator and compare it with the already generated time series plots.

IV. EXPERIMENTS

A. Data Generation

The presence of chaos in any Duffing Oscillator system depends on the value of its parameters - $\alpha, \beta, \gamma, \delta$ and ω. For example, when these parameters take the values 1, 5, 8, 0.02, 0.5 respectively, the system is chaotic, while when the parameters have values (ADD VALUES) respectively, the system is not chaotic.

For a given set of parameters, the Duffing Oscillator system was solved using Runge-Kutta method for 2nd order Differential Equations (RK2) method. The solution is obtained in the form of time-series data. This time series data is then used as an input for Rosenstein et al's and Eckmann et al's algorithms which, given a time series data, finds the maximum Lyapunov exponent. After obtaining the Lyapunov exponent, the given set of parameters is labelled as 1 if the max Lyapunov exponent is positive (chaotic) and 0 otherwise (non-chaotic).

A total of 2048 set of parameters were considered. The range of values for each parameter is given below.

Parameter	Start Value	End Value	Count
α	1	10	4
β	1	50	4
γ	0.02	5	4
δ	1	25	4
ω	0.5	12	4

TABLE I. Parameter ranges

B. Classification

For the task of classification, logistic regression, k-Nearest Neighbors and Neural Networks were used. The data was split in a 80:20 ratio as training and testing data. All three classification algorithms were very successful in predicting whether a given system is chaotic or not. For k-nearest neighbors with k=8, the confusion matrix looks like

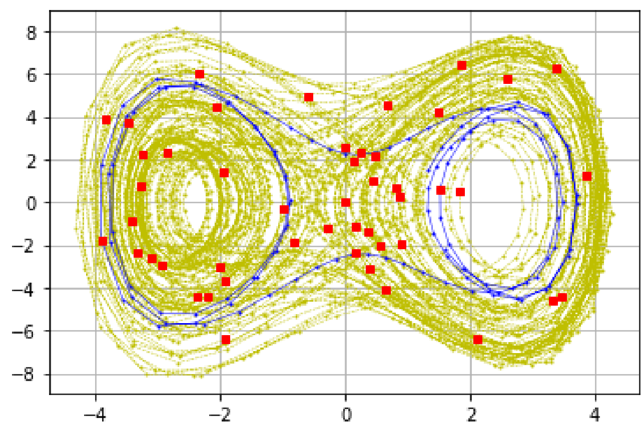

FIG. 2. Phase plot of the solutions generated of the Duffing equation using 2nd order Runge-Kutta

	Precision	Recall	f1-score	Support
0	0.87	0.99	0.93	318
1	0.96	0.47	0.63	92
Accuracy			0.88	410
Macro Avg	0.91	0.73	0.78	410
Micro Avg	0.89	0.88	0.86	410

TABLE II. Confusion matrix for kNN with k=8

For logistic regression with the default sklearn hyperparameters, the results observed were better when compared to kNN.

	Precision	Recall	f1-score	Support
0	0.97	0.98	0.98	318
1	0.92	0.91	0.92	92
Accuracy			0.96	410
Macro Avg	0.95	0.95	0.95	410
Micro Avg	0.96	0.96	0.96	410

TABLE III. Confusion matrix for logistic regression

V. ECHO STATE NETWORKS

A. Introduction

Reservoir computation is a computational structure with the recurrent neural network theory in its core. This

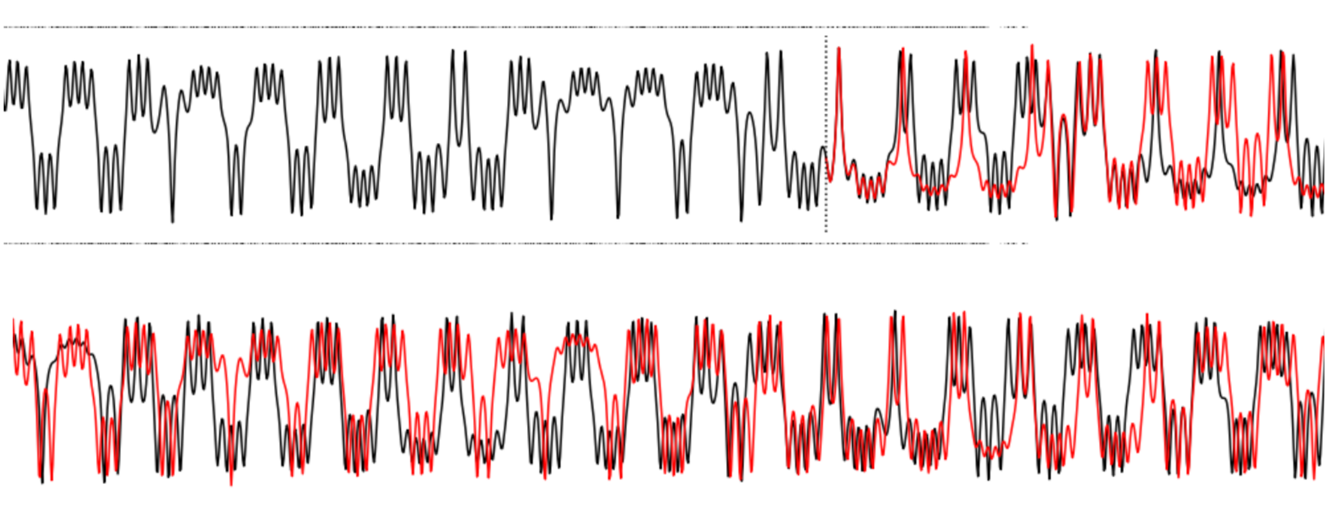

FIG. 3. ESN Prediction of time series data for n-resorvoir=3000 and spectral radius=1

framework maps the input signals we feed from our solution generation into higher dimensional computational spaces with a fixed non-linear system monitoring its dynamics, this is called its reservoir.The reservoir is treated as a black box and after the input feed is completed, we can train a readout mechanism to read the state of the reservoir, hence mapping it to the desired output. This method is advantageous because the computational power of naturally available systems can be put to great use in order to reduce the final computational cost.

This is a perfect candidate to mimic our chaotic oscillator because it will have a governing non-linear computational model as its core and would be considerably good in extrapolation of chaotic data in future timestamps.

B. Results

The main idea is to drive a random, large, fixed recurrent neural network with the input signal, thereby inducing in each neuron within this "reservoir" network a nonlinear response signal, and combine the desired output signal by a trainable linear combination of all of these response signals.

- Step 1: We provide a random RNN.
- Step 2: We harvest the reservoir states.
- Step 3: We finally compute output weights.

We used a library, namely pyESN, for our approach. With spectral radius as 1 and number of reservoirs to be 3000, we get the prediction result as illustrated in fig 3. Time series data obtained by using RK2 to solve the differential equation was given as input to ESN. The split in training and testing was done as 95:5. The red graph is the output predicted by the ESN, and the black graph is the original time series data obtained from RK2 method. The MSE error obtained for the predicted time series data is 2.31.

VI. CONCLUSION

It was observed that chaos in a Duffing Oscillator can be predicted by using machine learning algorithms. It was also observed that the Lyapunov coefficient of a duffing oscillator system, given the input parameters of the differential equation associated with that system can also be predicted with a reasonable mean square error (MSE).

We achieve an appreciable prediction/ data generation of a chaotic duffing oscillator with a relatively lower mean squared error. We plan to tune this further and extend it to other chaotic systems in general.

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