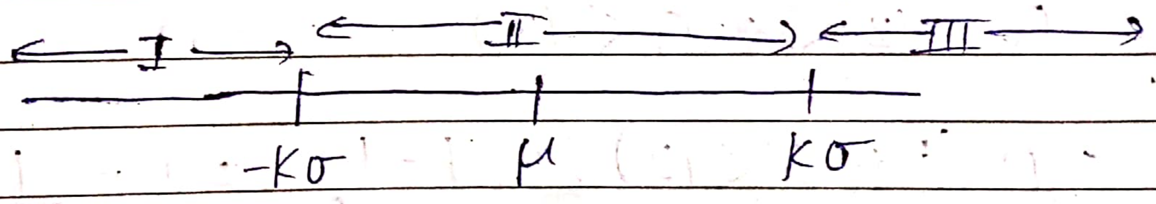
Chebyshev's Inequality

In ~~probable~~ prob. theory, Chebyshev's inequality guarantees that; for a wide class of prob. dist., no more than certain fraction of values can be more than a certain distance from the mean.

Specifically, no more than $\frac{1}{k^2}$ of the dist. values can be more than k std. deviations away from the mean (or equivalently, at least $1 - \frac{1}{k^2}$ of the dist. values are within k std. deviations of the mean)



$X \rightarrow$ r.v. Variance $= \sigma^2$

$\mu =$ mean $\qquad k =$ any real no., $k > 0$

$$\boxed{P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}}$$

OR

$$\boxed{P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}}$$

The rule is called Chebyshev's ~~inequality~~ thm., about the range of S.D. around the mean in statistics.

The inequality has great utility, $\because$ it can be applied to any prob. dist. in which mean & variance are defined.

e.g. 1) A r.v. X has mean 8 & variance = 9 & an unknown prob. dist. Find $P(-4 < X < 20)$ & $P(|X-8| \geq 6)$

Soln: $\mu = 8 \quad \sigma^2 = 9 \quad \sigma = 3$

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(-4 < X < 20)$$
$$= P(-4-8 < X-8 < 20-8)$$
$$= P(-12 < X-8 < 12)$$

$$= P(|X-8| < 12) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{16} = \frac{15}{16}$$

$\therefore$ Comp.

$$k\sigma = 12$$
$$k = \frac{12}{3} = 4 \qquad \boxed{\therefore k = 4}$$

Reqd. Prob. =

$$P(|X-8| \geq 6) \leq \frac{1}{k^2} \leq \frac{1}{4}$$

$k\sigma = 6$

$\therefore k = 2$

(2) Suppose that it is ~~unkc unknown~~ known that the number of items produced in a factory during a week is a r.v. with mean 50. If the variance of a week's production is known to equal to 25, then what can be said about the prod. that would be betn 40 & 60?

$\mu = 50$

$\sigma^2 = 25$

$\sigma = 5$

Reqd. Prob.

$= P(40 < X < 60)$

$= P(40-50 < X-50 < 60-50)$

$= P(-10 < X-50 < 10)$

$= P(|X-50| < 10) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$k\sigma = 10$

$k = 2$

(3) A r.v. X has the prob. dist.

x	0	1	2	4
$p(x)$	1/2	1/4	1/8	1/8

(a) Find an upper bound for $P(|X-1| > 2)$ by Chebyshev's inequality

(b) Find $P(|X-1|>2)$ by direct computation

~~Sol~~ Soln :

Mean = $Exp(x) = 1$

Variance = $E(X^2) - (E(X))^2 = \frac{7}{4}$

$P(|X-1|>2) \leq \frac{1}{k^2}$

$k\sigma = 2$

$k\frac{\sqrt{7}}{2} = 2$

$k = \frac{4}{\sqrt{7}}$

$$\boxed{P(|X-1|>2) \leq \frac{7}{16}}$$

2nd part:

The only x which satisfy $P(|X-1|>2)$ is

is $x = 4$

$$\boxed{\therefore P(X=4) = \frac{1}{8}}$$

Exact value which is less than $7/16$, an upper bound computed by Chebyshev's ineq.

(4) Does there exist a variate X for which

$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.6$?

$\Rightarrow P(-2\sigma \leq x - \mu \leq 2\sigma)$

$\Rightarrow P(|x-\mu| \leq 2\sigma) \geq 1 - \frac{1}{2^2} \geq 0.75$

$\because 0.6 \ngeq 0.75$, $\therefore$ there does not exist any such variate x