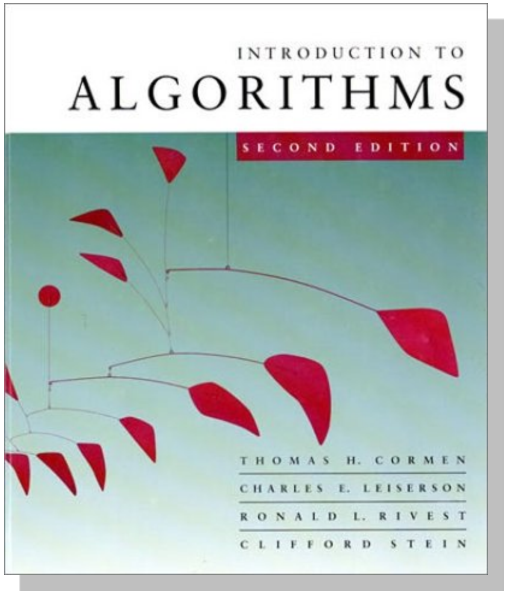
Introduction to Algorithms

6.046J/18.401J

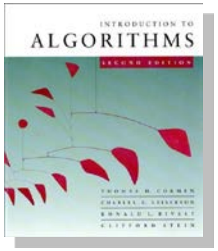


LECTURE 3

Divide and Conquer

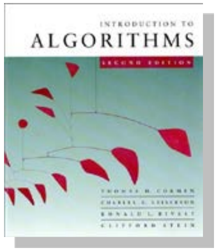
- Binary search
- Powering a number
- Fibonacci numbers
- Matrix multiplication
- Strassen's algorithm
- VLSI tree layout

Prof. Erik D. Demaine



The divide-and-conquer design paradigm

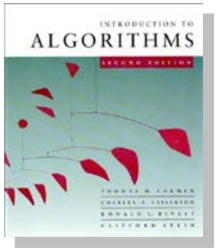
1. ***Divide*** the problem (instance) into subproblems.
2. ***Conquer*** the subproblems by solving them recursively.
3. ***Combine*** subproblem solutions.



Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.



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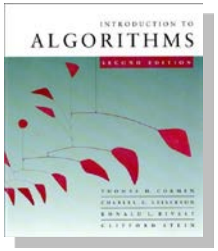
Example: Find 9

3	5	7	8	9	12	15



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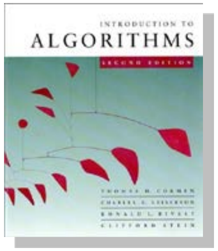

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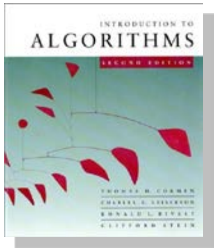

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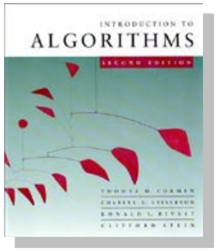

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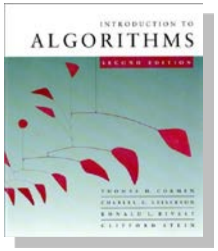

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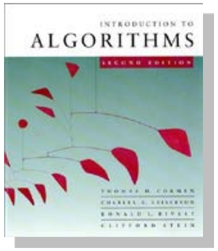


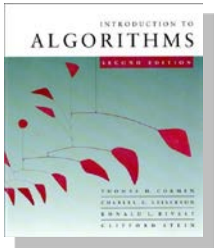
Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining



Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

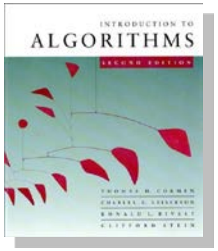
$$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2 } (k = 0)$$

$$\Rightarrow T(n) = \Theta(\lg n)\,.$$

Powering a number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.



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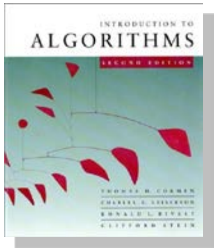
Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$



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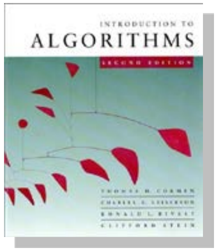
$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

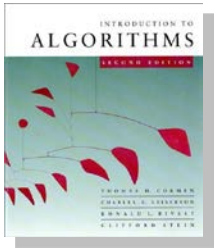


Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \cdots$$



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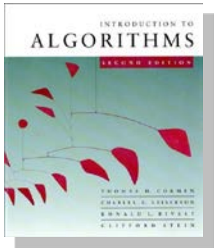
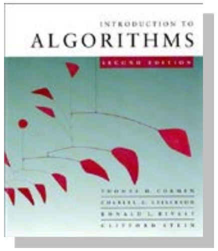
Naive recursive algorithm: $\Omega(\phi^n)$ (exponential time), where $\phi = (1 + \sqrt{5})/2$ is the ***golden ratio***.



Computing Fibonacci numbers

Bottom-up:

- Compute $F_0, F_1, F_2, \ldots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.



Computing Fibonacci numbers

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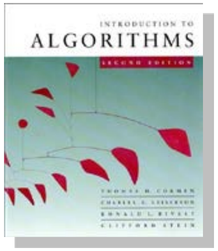
- Running time: $\Theta(n)$.

Naive recursive squaring:

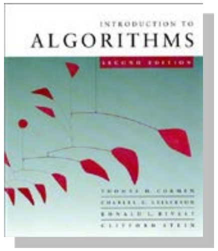
$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.



Recursive squaring

Theorem: $$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n.$$



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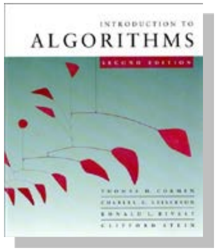
Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.



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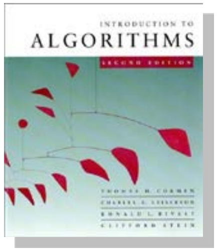
Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.

Proof of theorem. (Induction on n.)

Base ($n = 1$): $$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 .$$



Recursive squaring

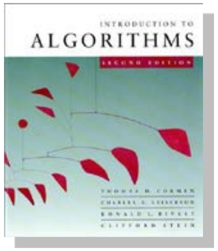
Inductive step ($n \geq 2$):

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n}$$

■



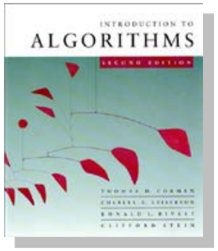
Matrix multiplication

Input: $A = [a_{ij}],\ B = [b_{ij}].$
Output: $C = [c_{ij}] = A \cdot B.$

$\left.\right\}\ i, j = 1, 2, \ldots, n.$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



Standard algorithm

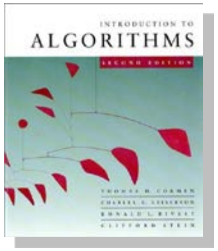
for $i \leftarrow 1$ **to** n

  do for $j \leftarrow 1$ **to** n

    do $c_{ij} \leftarrow 0$

      for $k \leftarrow 1$ **to** n

        do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$



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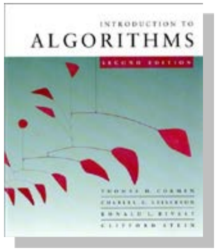
Running time $= \Theta(n^3)$



Divide-and-conquer algorithm

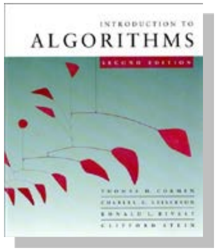
IDEA:

$n\times n$ matrix = 2×2 matrix of $(n/2)\times(n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}\right\} \begin{array}{l} 8 \text{ mults of } (n/2)\times(n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2)\times(n/2) \text{ submatrices} \end{array}$$



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$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dh \\ u &= cf + dg \end{aligned}\right\}$$

recursive

8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices

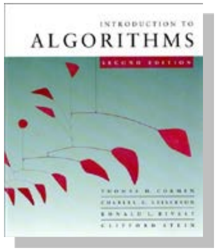
Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices

submatrix size

work adding submatrices



Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices

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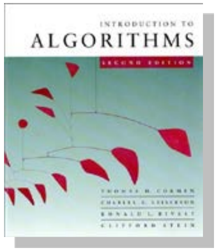
$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \text{CASE 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$



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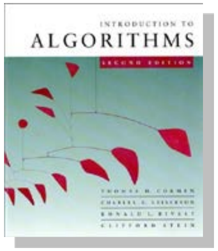
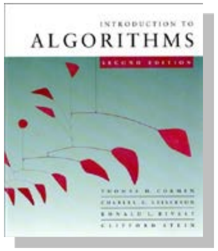
$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \textrm{CASE 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.



Strassen’s idea

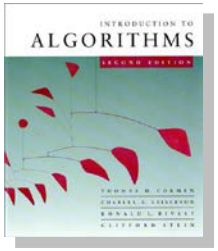
- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$
\begin{aligned}
P_1 &= a \cdot (f - h) \\
P_2 &= (a + b) \cdot h \\
P_3 &= (c + d) \cdot e \\
P_4 &= d \cdot (g - e) \\
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$



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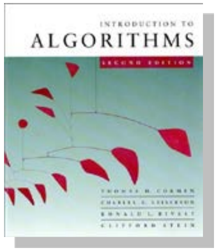
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$



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s &= P_1 + P_2 \\
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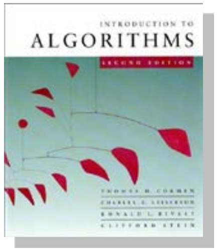
\end{aligned}
$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!



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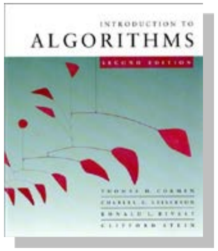
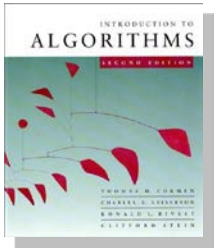
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$



Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.



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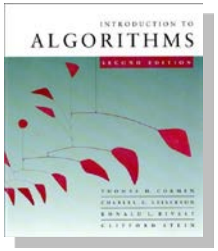
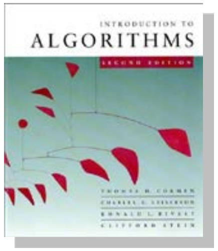
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$



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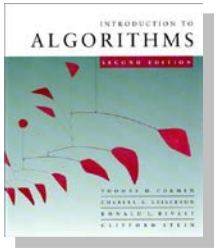
$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \;\Rightarrow\; \text{Case 1} \;\Rightarrow\; T(n) = \Theta(n^{\lg 7}).$$



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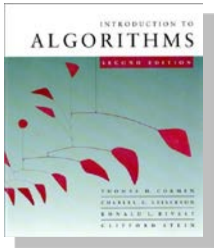
The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.



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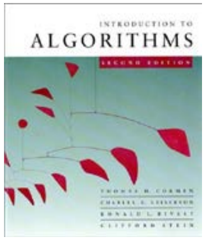
Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.



Merge sort

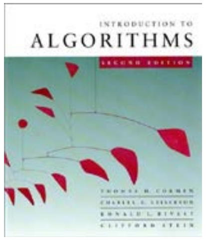
MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merging two sorted arrays

20	12
13	11
7	9
2	1



Merging two sorted arrays

20	12
13	11
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1

Merging two sorted arrays

20	12	‖	20	12
13	11	‖	13	11
7	9	‖	7	9
2	1	‖	2	

1

Merging two sorted arrays

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7	9	‖	7	9
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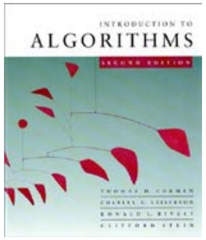
1 2



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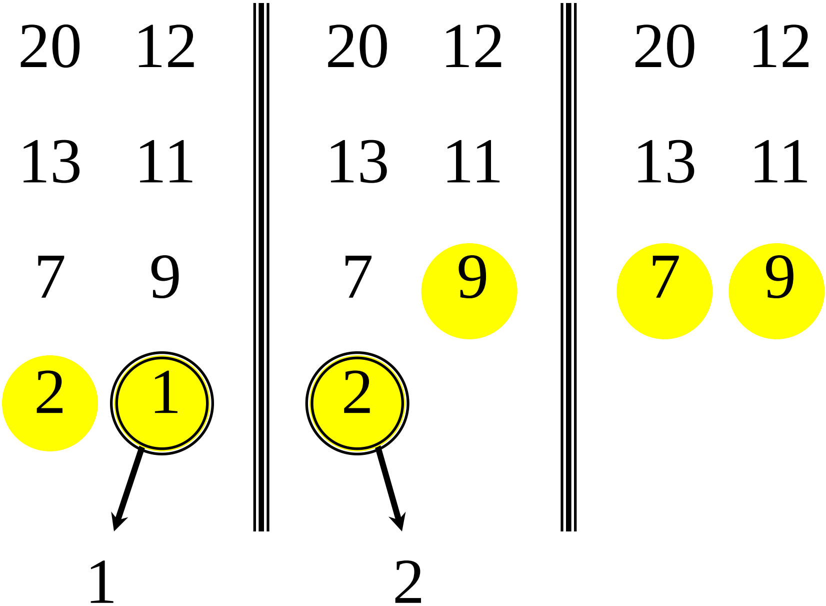
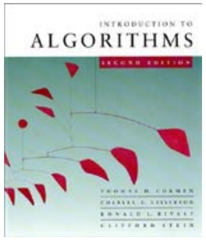
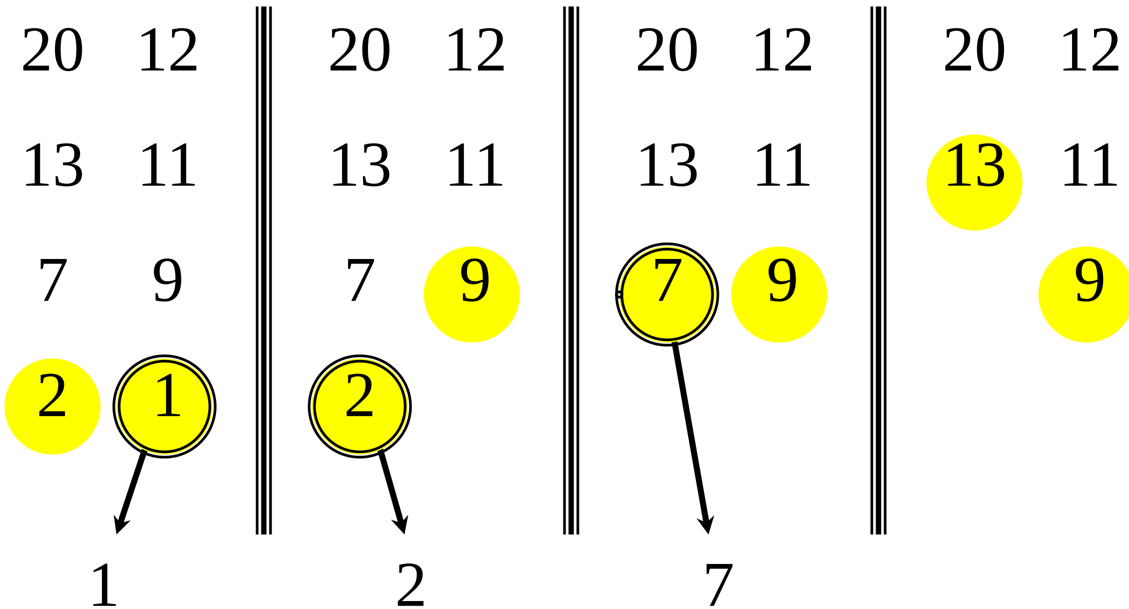


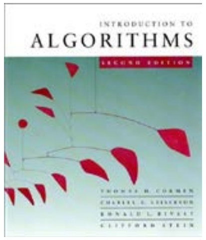


Merging two sorted arrays

20	12	20	12	20	12
13	11	13	11	13	11
7	9	7	9	7	9
2	1	2			
1		2		7	

Merging two sorted arrays





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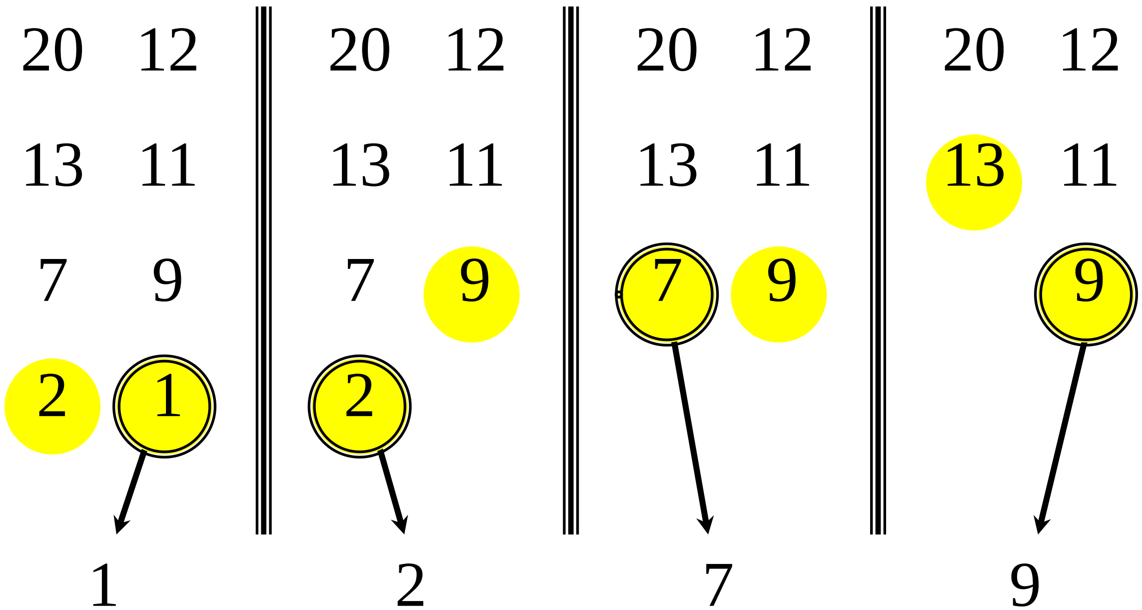

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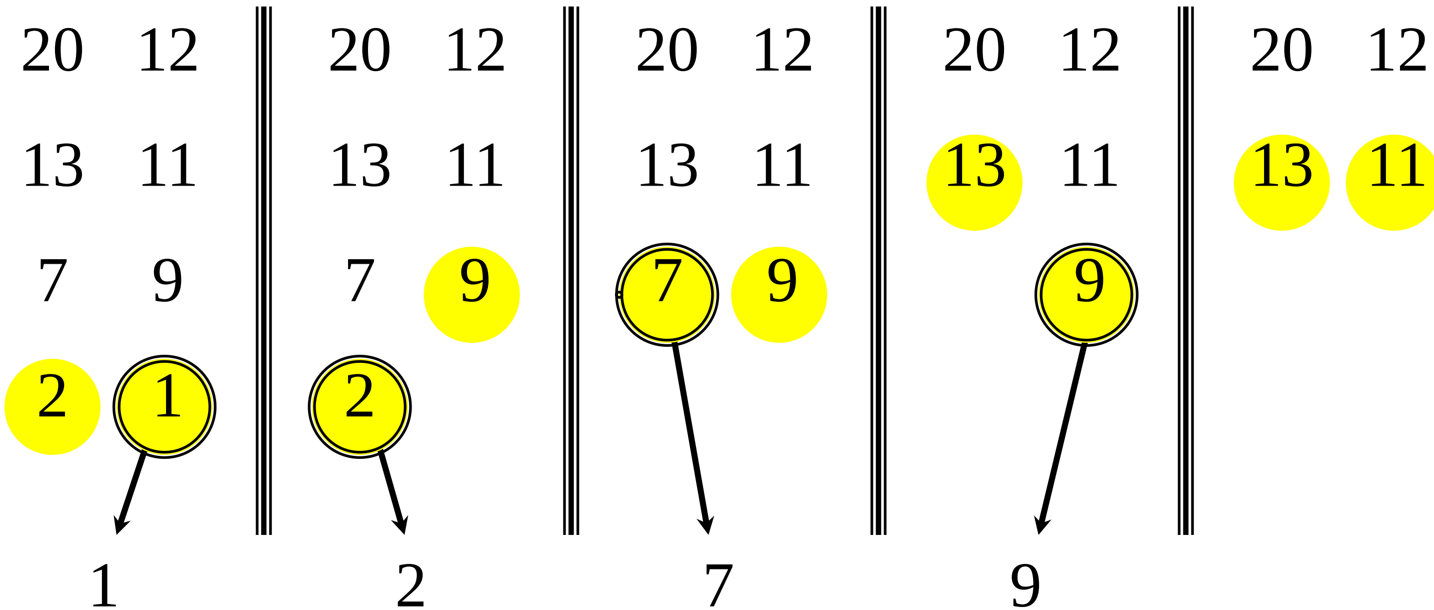

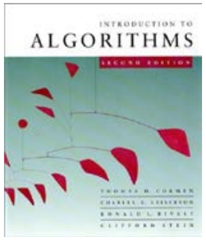

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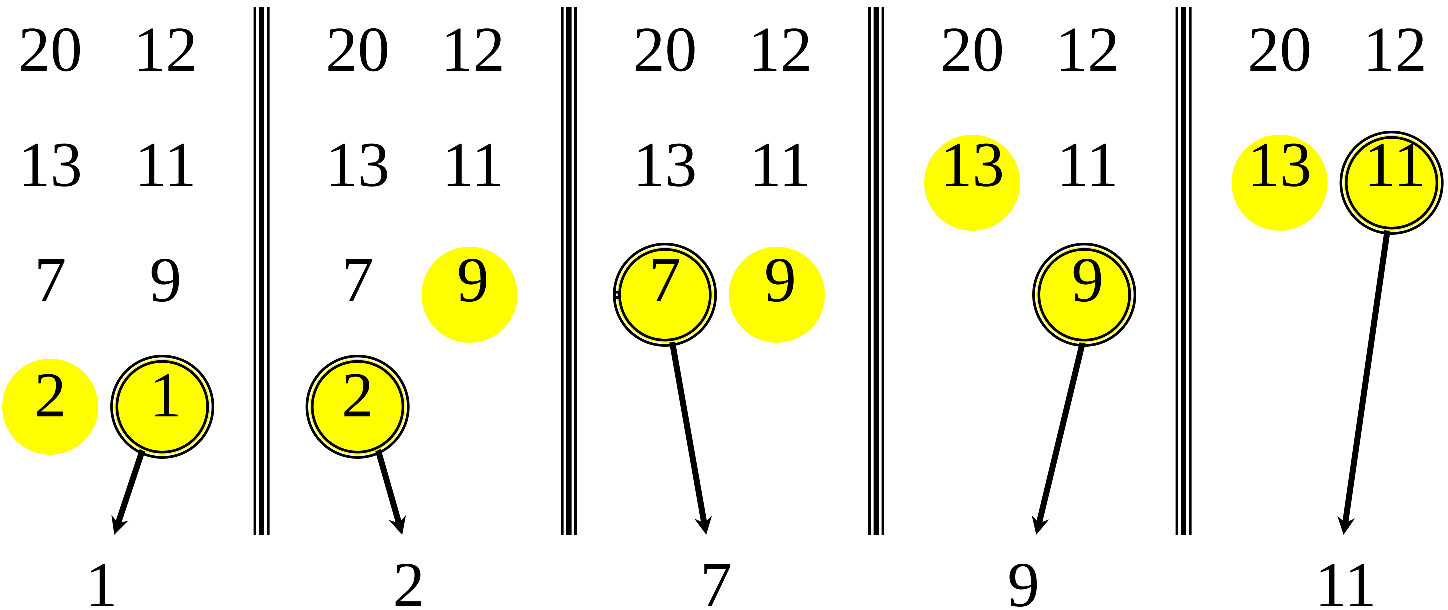

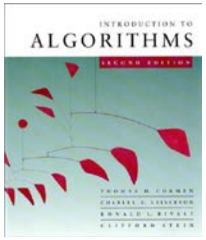

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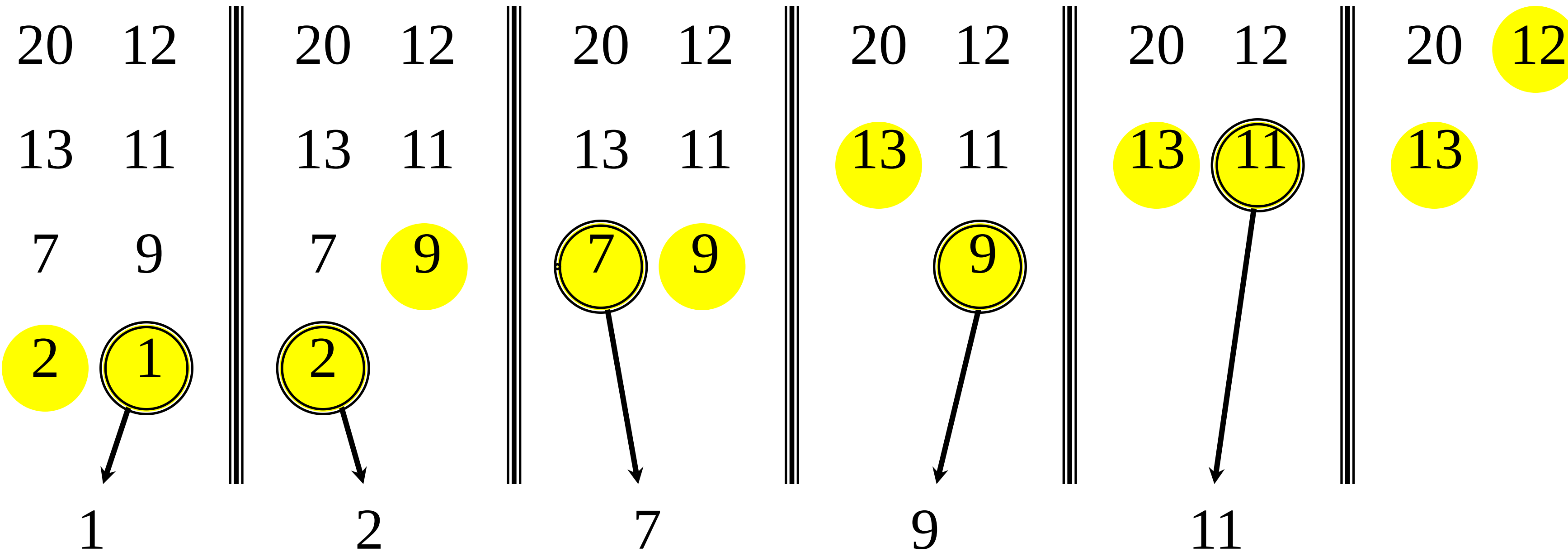

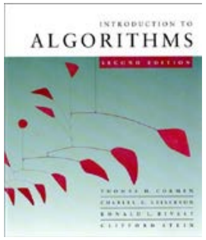

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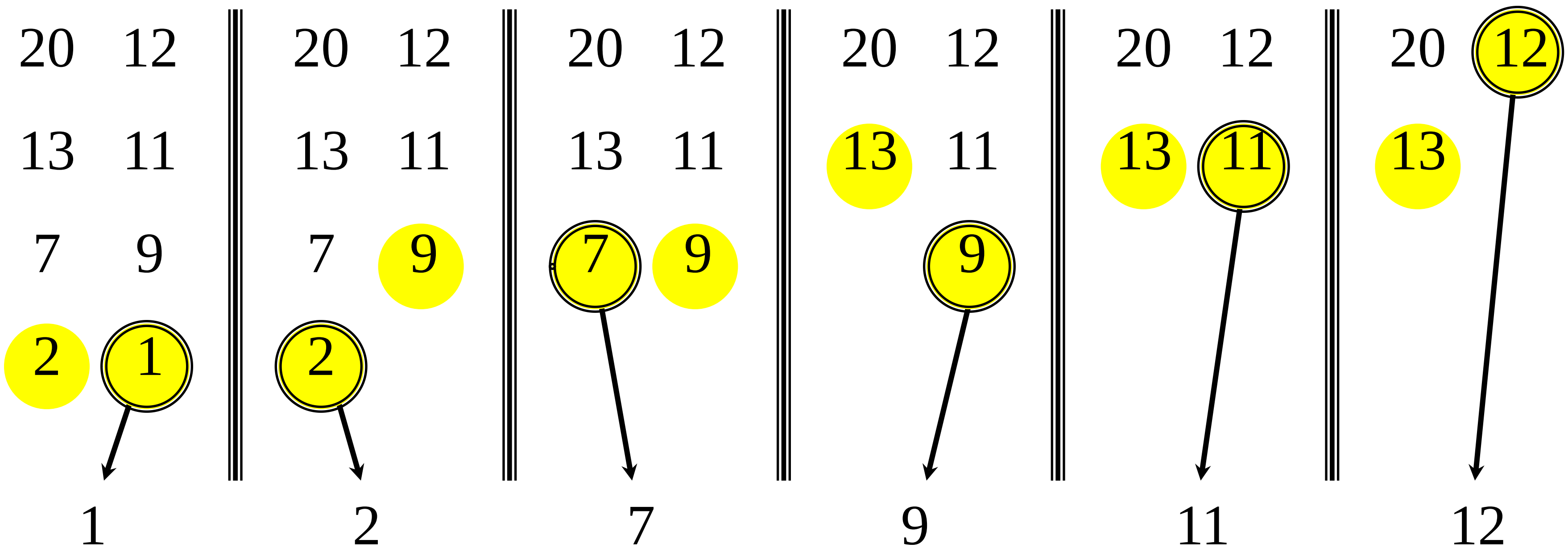

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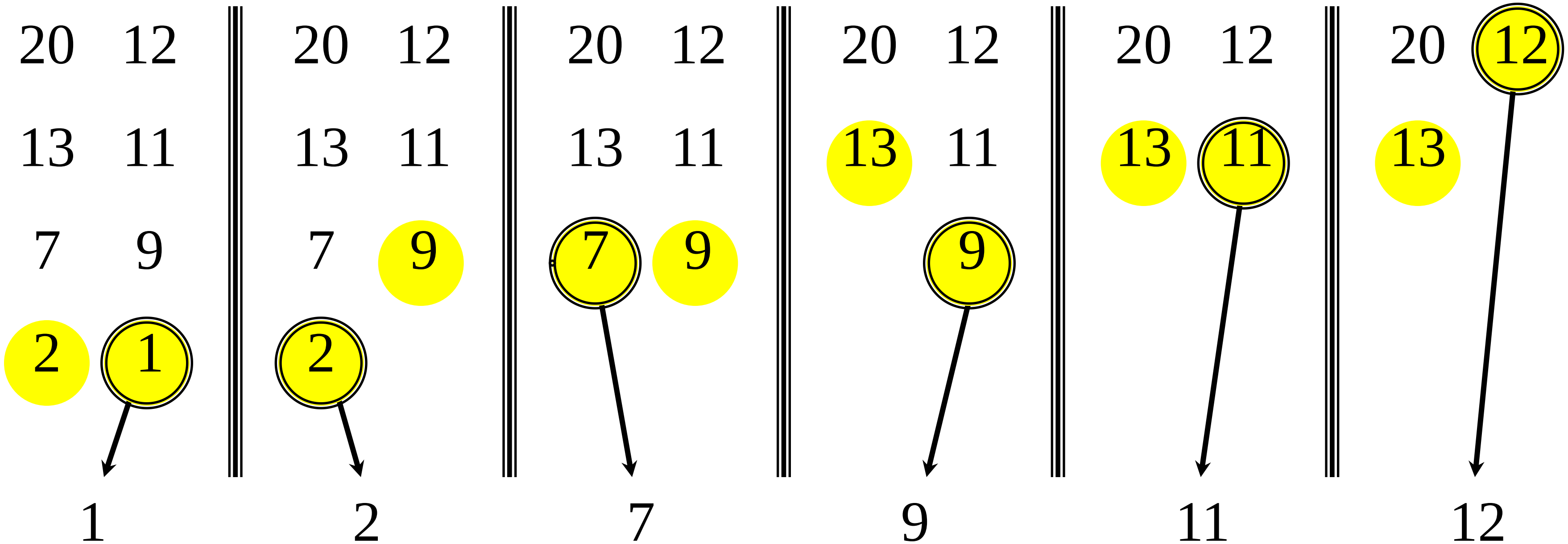
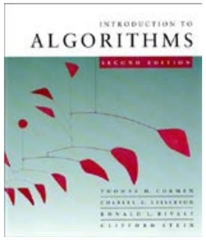


Time = $\Theta(n)$ to merge a total of n elements (linear time).

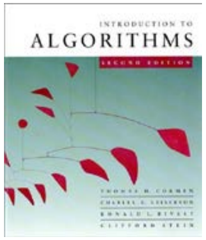


Analyzing merge sort

$T(n)$	MERGE-SORT $A[1 \,.\,.\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
$\Theta(n)$	3. ***"Merge"*** the 2 sorted lists

Abuse

Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.



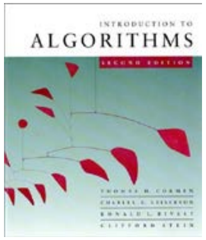
Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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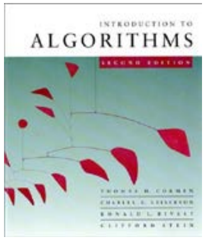
$$T(n)$$



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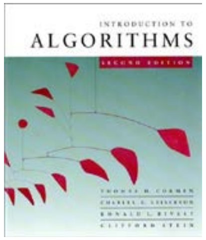
$$cn$$

$$T(n/2) \qquad\qquad T(n/2)$$



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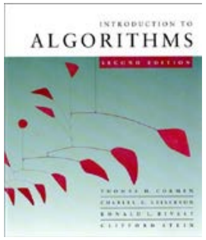
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$cn$$

$$cn/2 \qquad\qquad cn/2$$

$$T(n/4) \qquad T(n/4) \qquad T(n/4) \qquad T(n/4)$$



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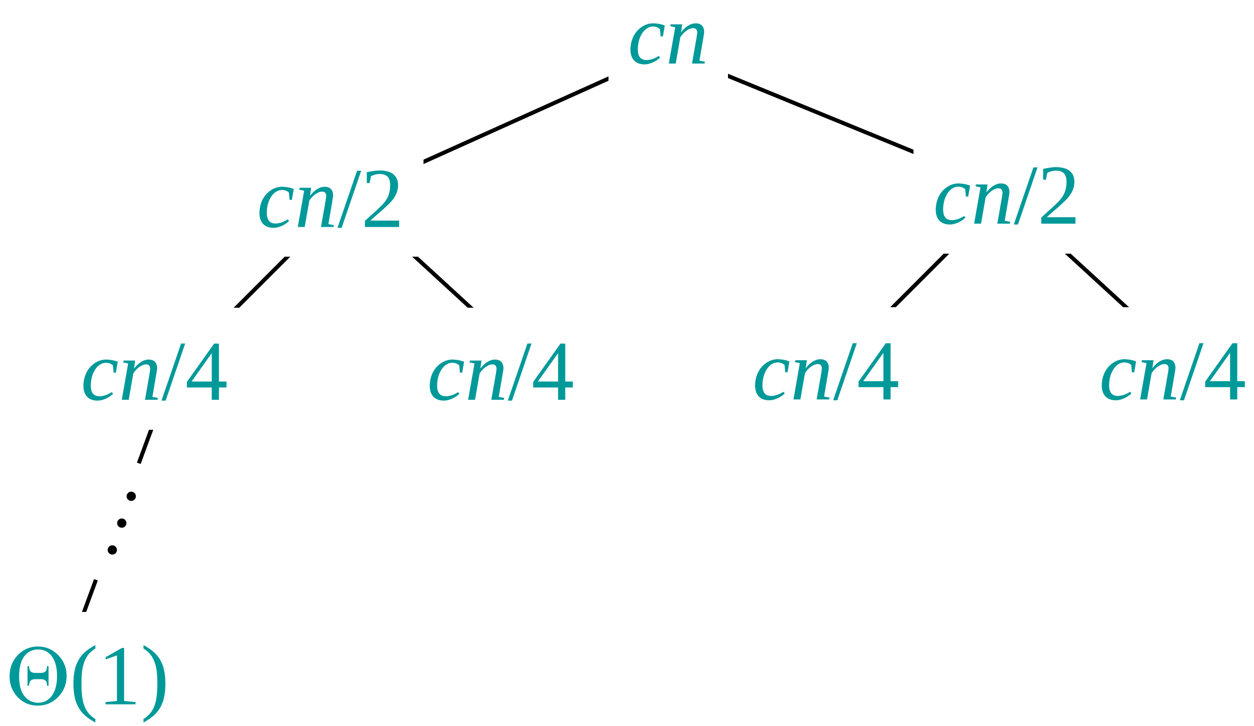

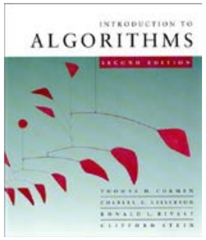

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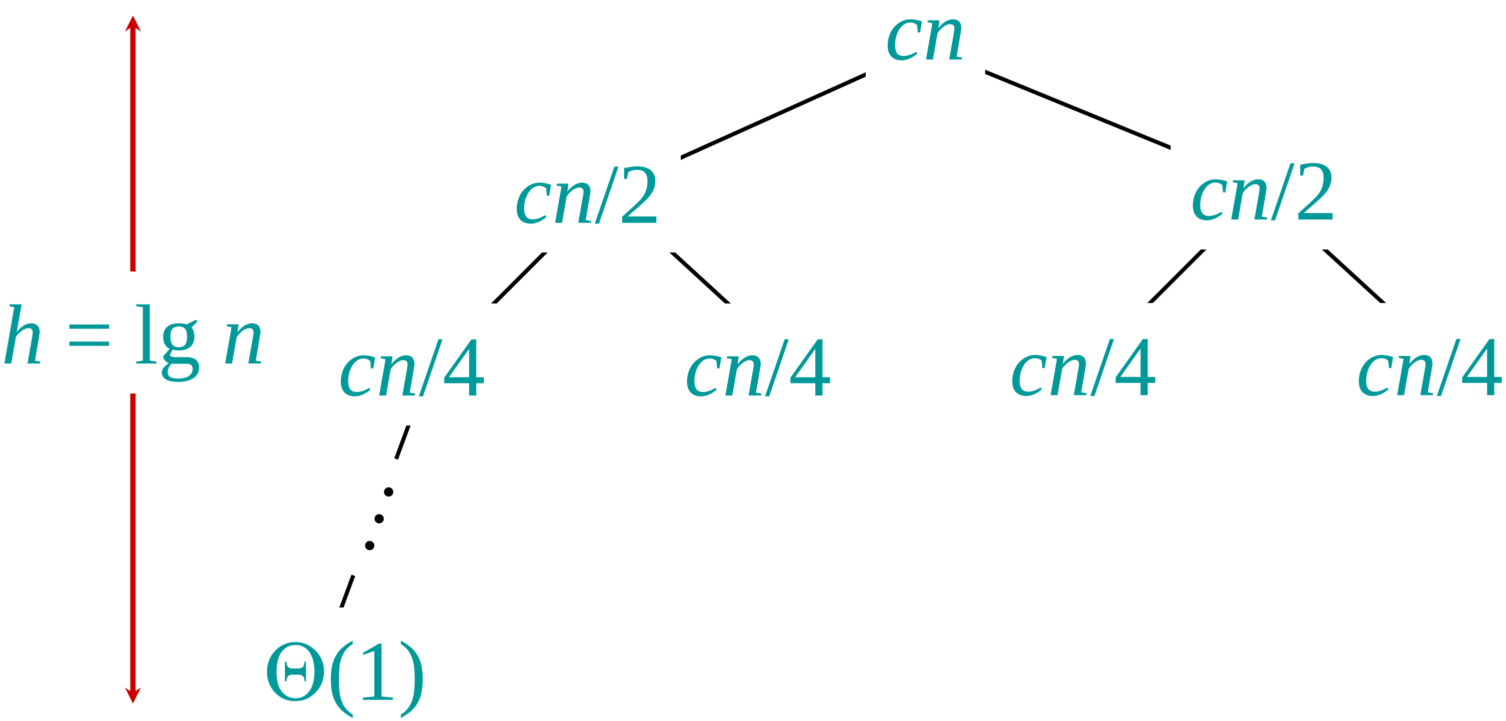

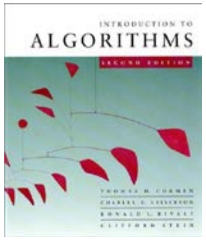

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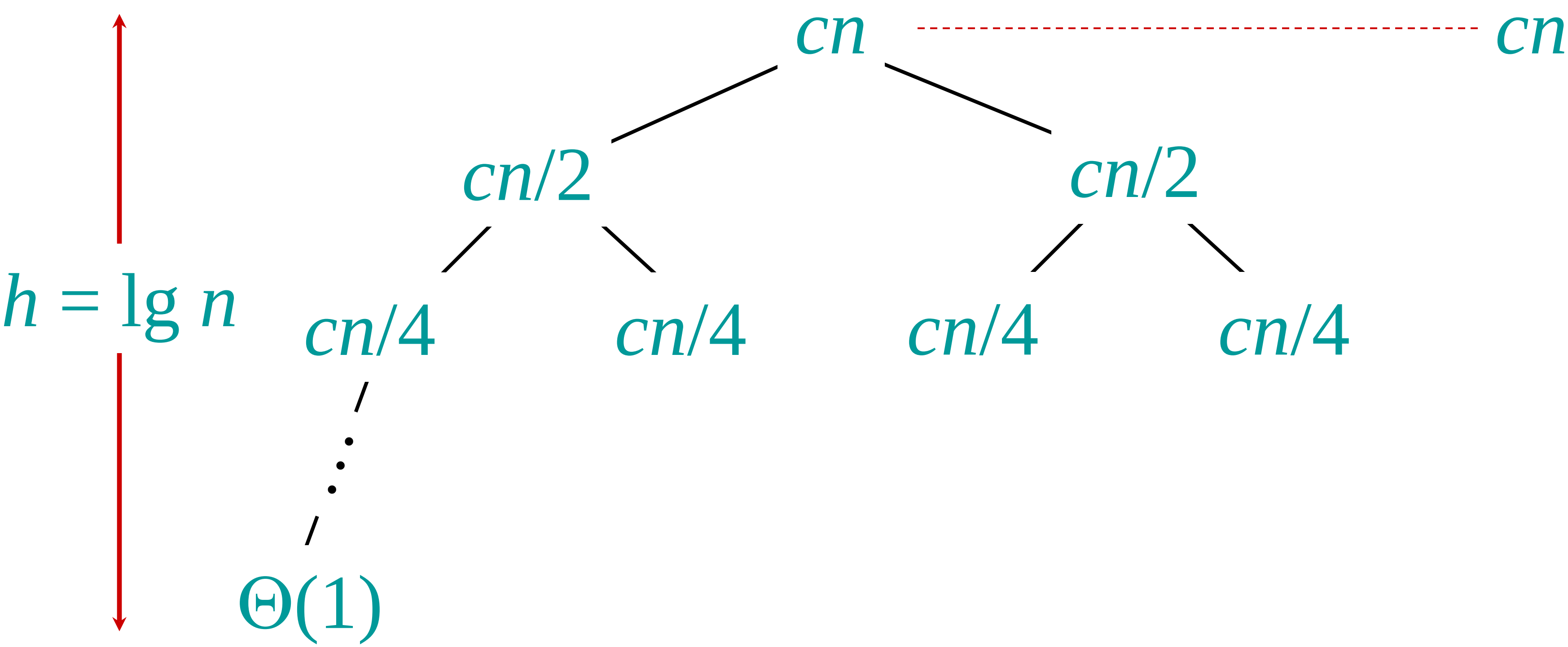

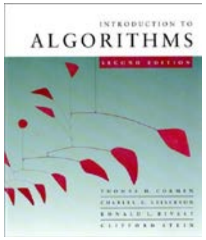

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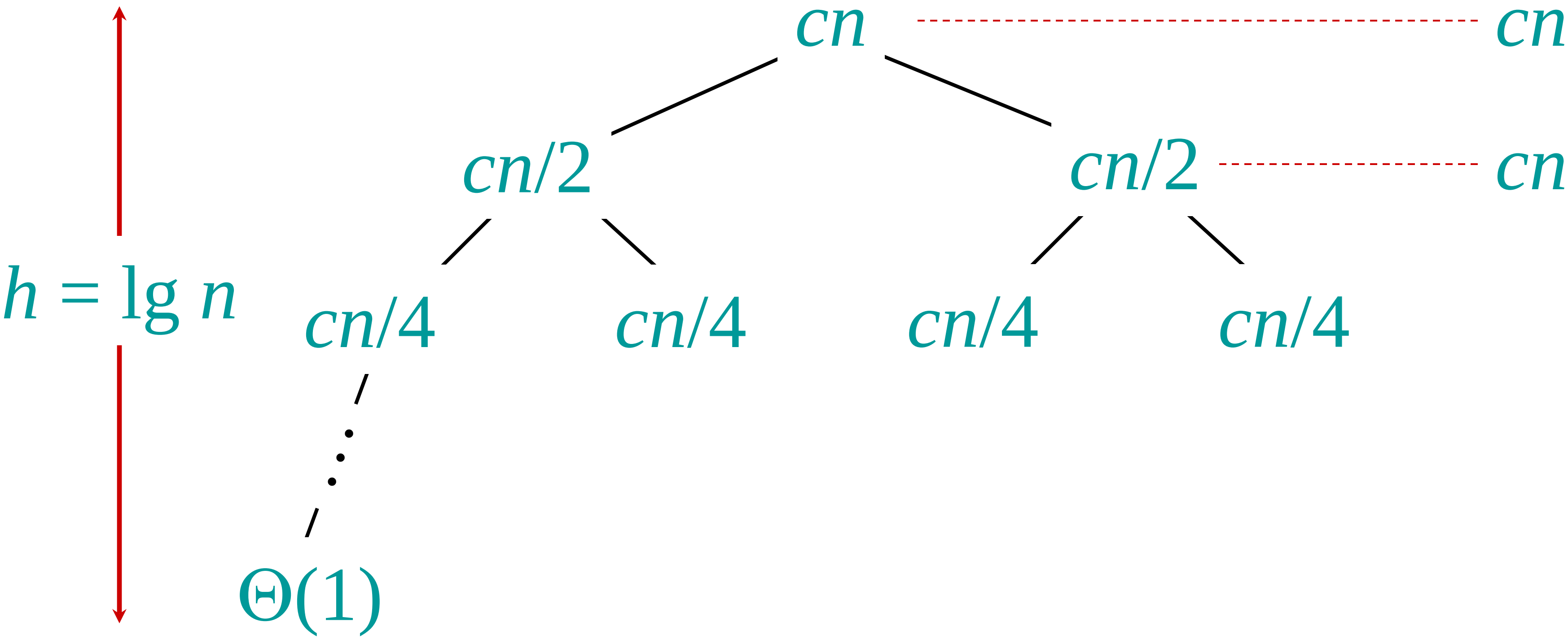

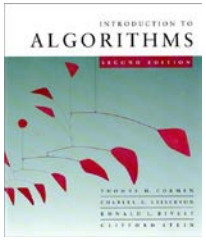

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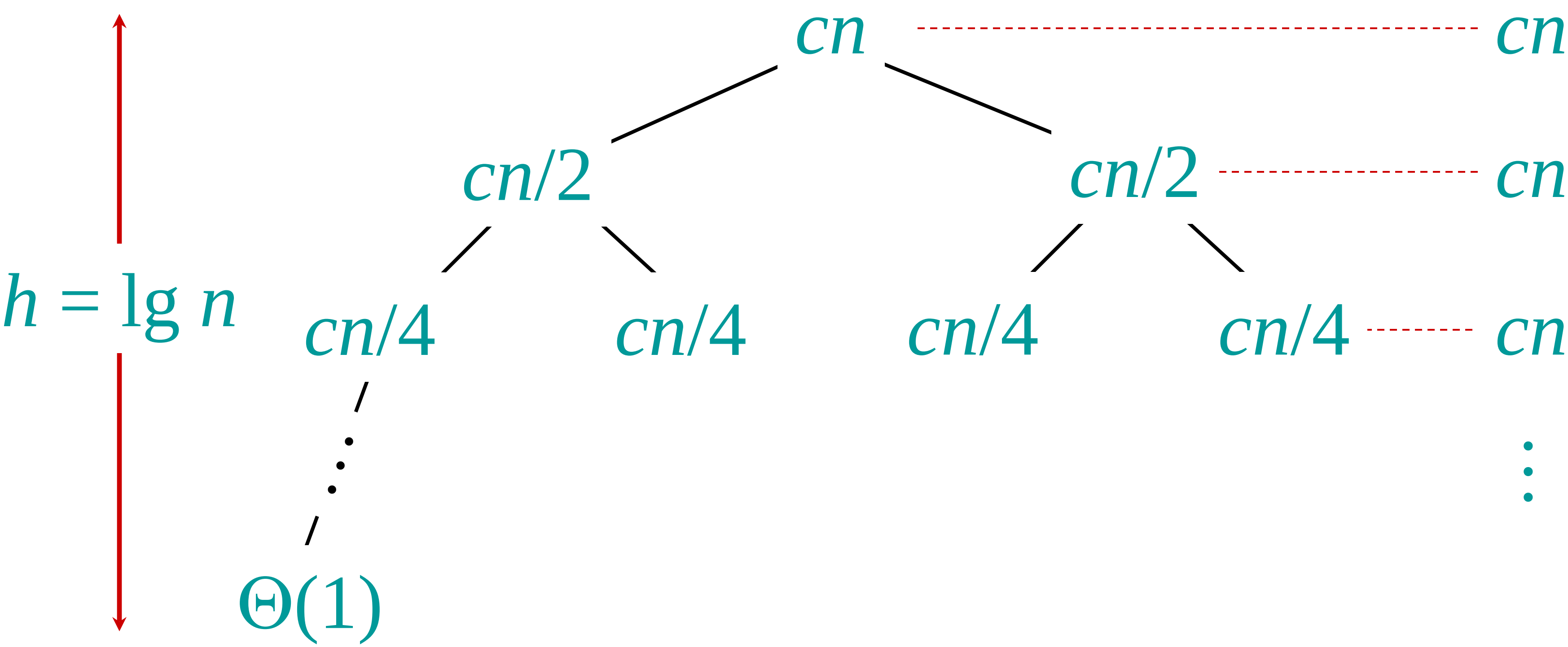

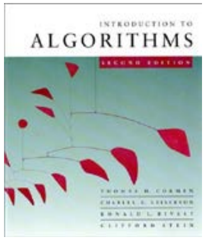

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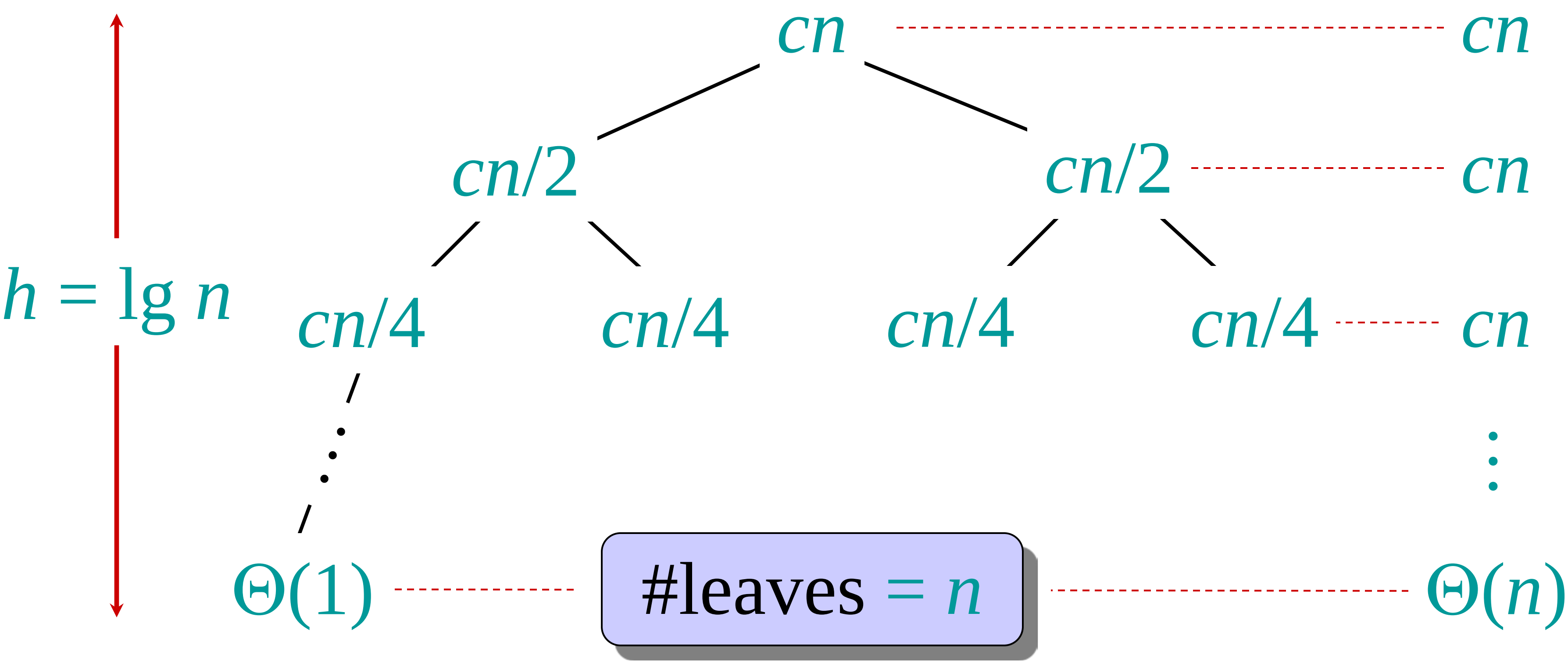
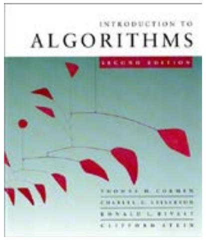




Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

cn .... cn

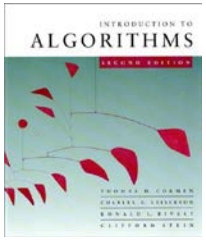
$cn/2$ $cn/2$ .... cn

$h = \lg n$ $cn/4$ $cn/4$ $cn/4$ $cn/4$ .... cn

$\vdots$

$\Theta(1)$ .... #leaves $= n$ .... $\Theta(n)$

Total $= \Theta(n \lg n)$

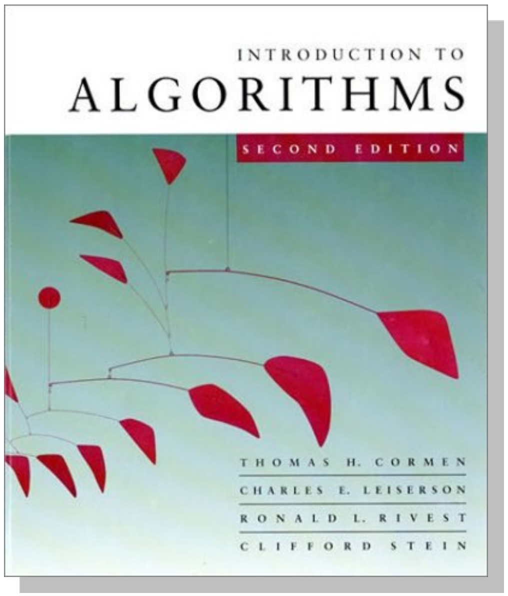


Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!

Introduction to Algorithms

6.046J/18.401J

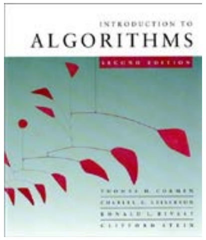


LECTURE 4

Quicksort

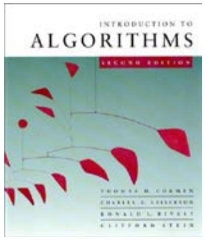
- Divide and conquer
- Partitioning
- Worst-case analysis
- Intuition
- Randomized quicksort
- Analysis

Prof. Charles E. Leiserson



Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).



Divide and conquer

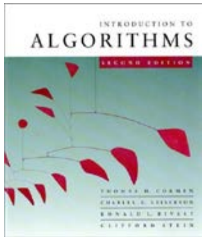
Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

$\leq x$	x	$\geq x$

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

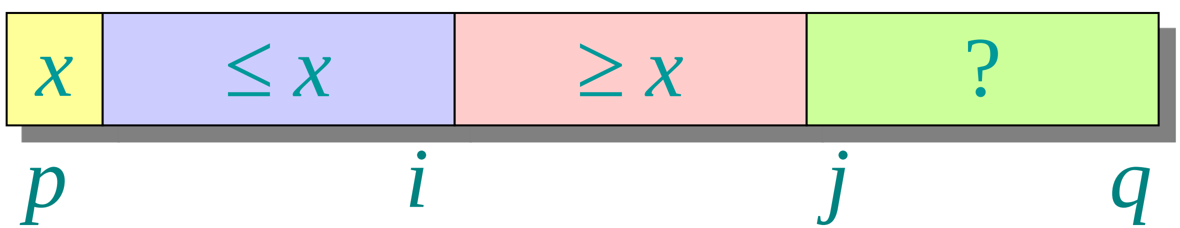


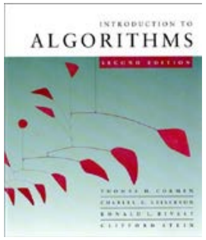
Partitioning subroutine

PARTITION(A, p, q) ⊳ $A[p\,.\,.\,q]$
    $x \leftarrow A[p]$ ⊳ pivot $= A[p]$
    $i \leftarrow p$
    for $j \leftarrow p + 1$ **to** q
        do if $A[j] \le x$
            then $i \leftarrow i + 1$
                exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    return i

Running time $= O(n)$ for n elements.

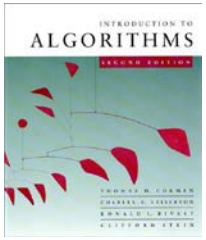
Invariant:





Example of partitioning

6	10	13	5	8	3	2	11
i	*j*						



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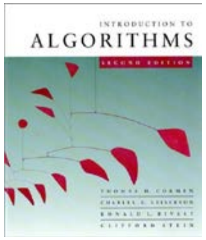

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i $\longrightarrow j$

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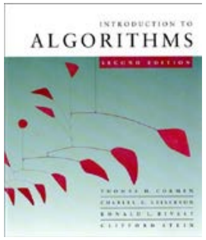
i $\qquad\qquad$ j



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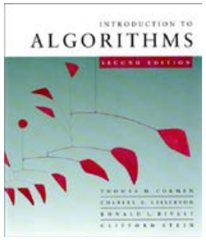

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	i				j		

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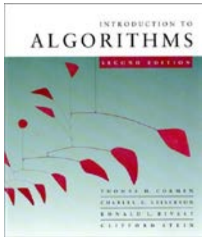
6	5	3	10	8	13	2	11

i j



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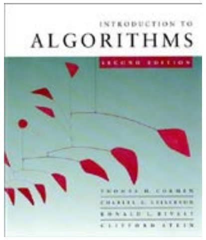

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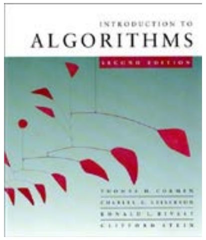
6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

$\longrightarrow i$ $\qquad\qquad j$



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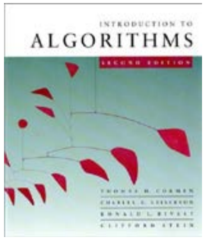
6	5	3	2	8	13	10	11
			i				*j*



Example of partitioning

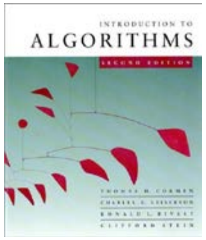
6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

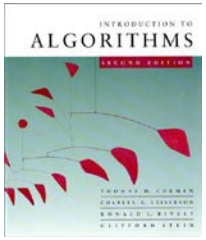
6	5	3	2	8	13	10	11

i $\quad$ j



Example of partitioning

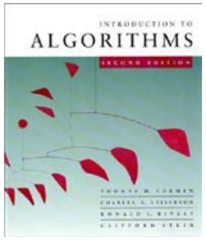
6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11
			i				



Pseudocode for quicksort

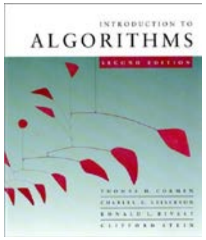
QUICKSORT(A, p, r)
    if $p < r$
        then $q \leftarrow$ PARTITION(A, p, r)
            QUICKSORT$(A, p, q–1)$
            QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$



Analysis of quicksort

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ = worst-case running time on an array of n elements.



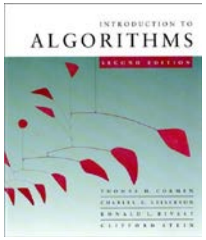
Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textbf{\textit{(arithmetic series)}}
\end{aligned}
$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



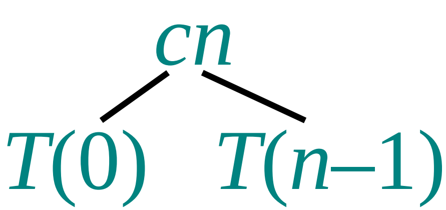
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$$T(n)$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

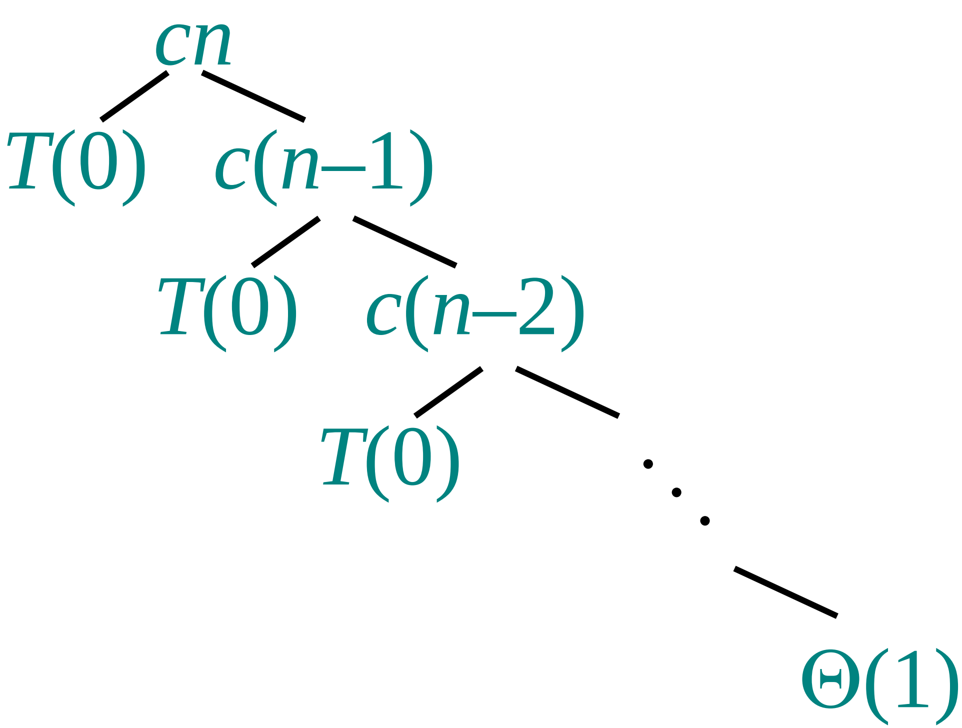
$$cn$$

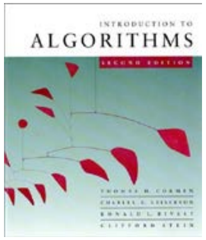
$$T(0) \qquad c(n-1)$$

$$T(0) \qquad T(n-2)$$

Worst-case recursion tree

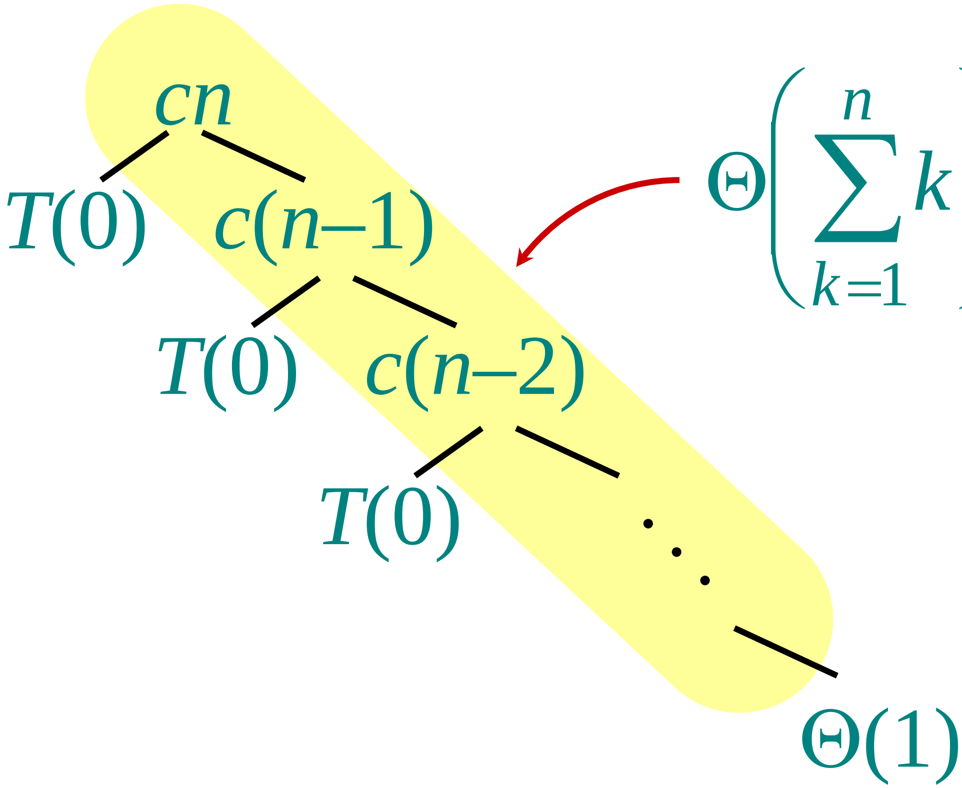
$$T(n) = T(0) + T(n-1) + cn$$





Worst-case recursion tree

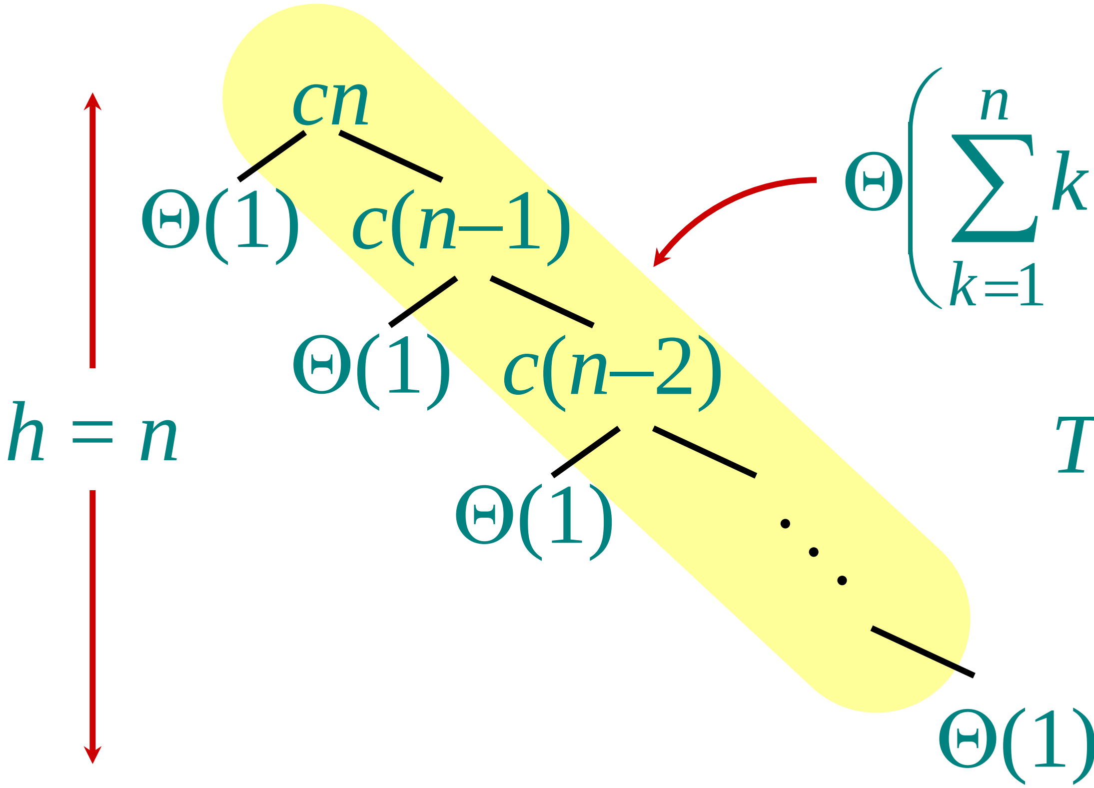
$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

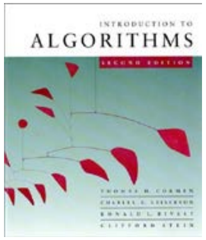
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta\left(n^2\right)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$



Best-case analysis

(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

What is the solution to this recurrence?

Analysis of "almost-best" case

$$T(n)$$

Analysis of "almost-best" case

$$cn$$

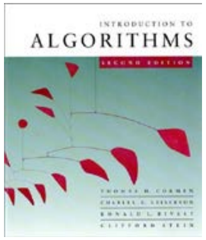
$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

Analysis of "almost-best" case

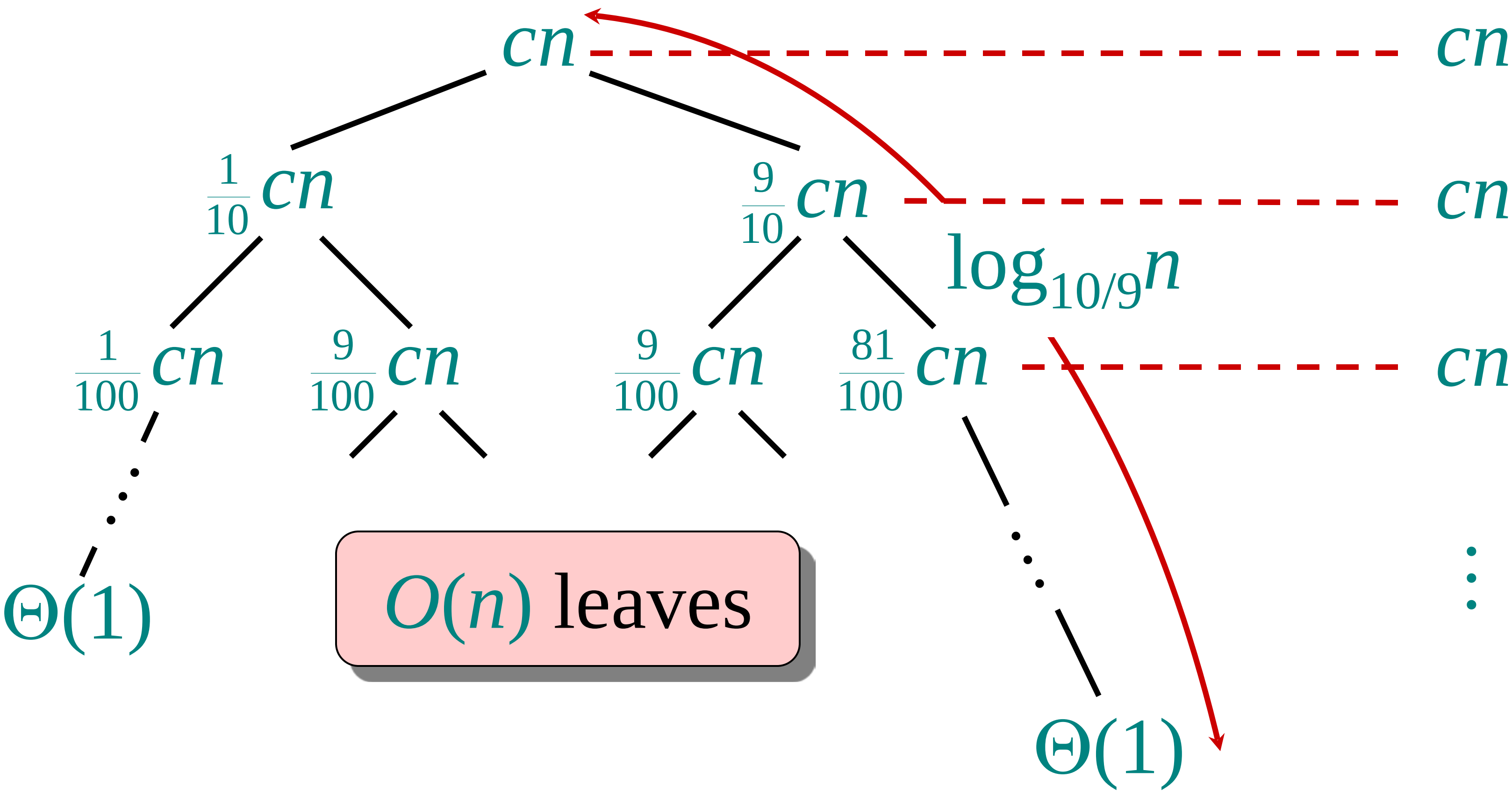
$$cn$$

$$\tfrac{1}{10}cn \qquad\qquad \tfrac{9}{10}cn$$

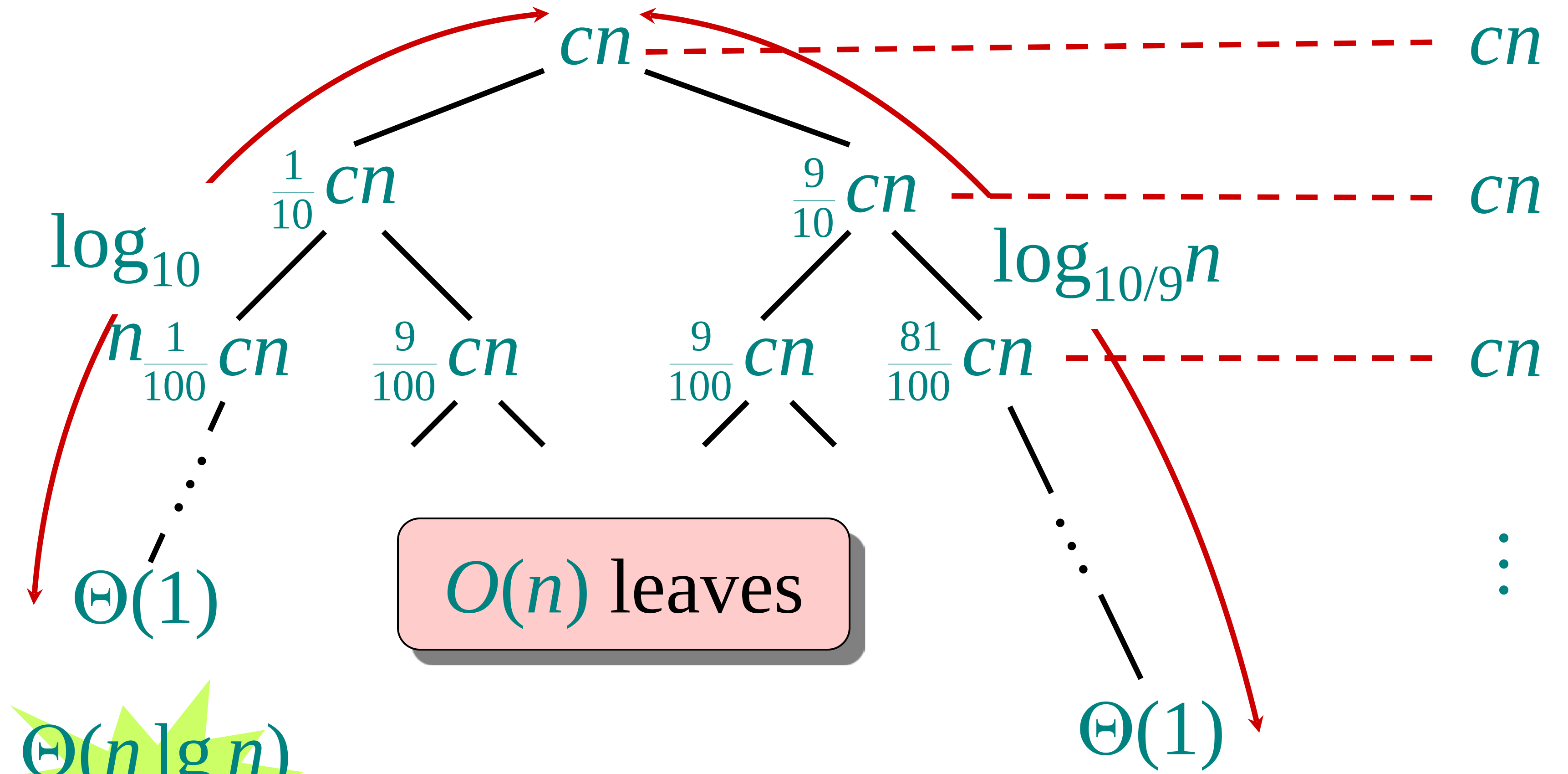
$$T\left(\tfrac{1}{100}n\right) \; T\left(\tfrac{9}{100}n\right) \qquad T\left(\tfrac{9}{100}n\right) \; T\left(\tfrac{81}{100}n\right)$$



Analysis of “almost-best” case



Analysis of “almost-best” case



$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$$L(n) = 2U(n/2) + \Theta(n) \quad \textit{lucky}$$

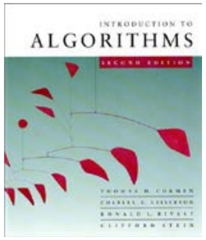
$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

Lucky!

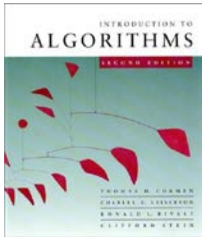
How can we make sure we are usually lucky?



Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.



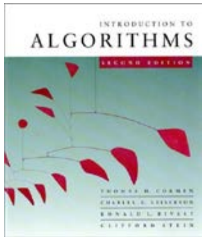
Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

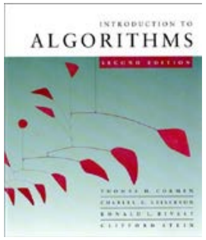
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

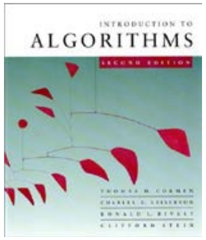
$$= \sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n) \big)$$



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

Take expectations of both sides.

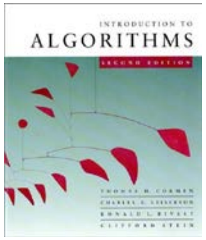


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.



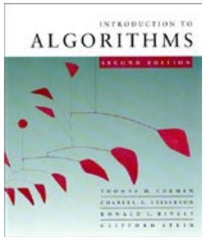
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choices.



Calculating expectation

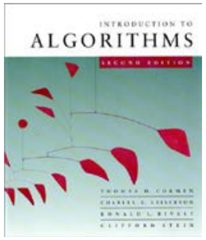
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

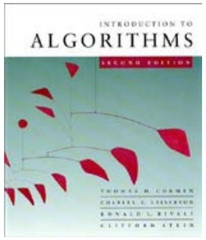
$$= \sum_{k=0}^{n-1} E\left[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.



Hairy recurrence

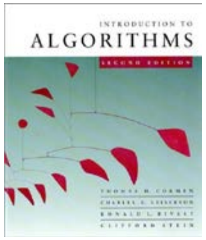
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a\,n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a\,n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

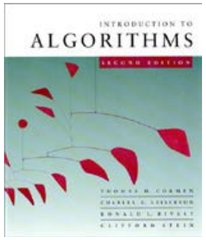
Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



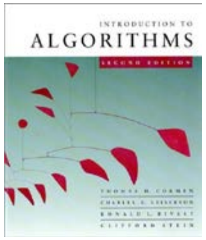
Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired* – *residual***.



Substitution method

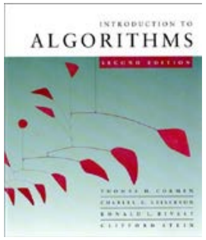
$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an\lg n,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.



Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.

