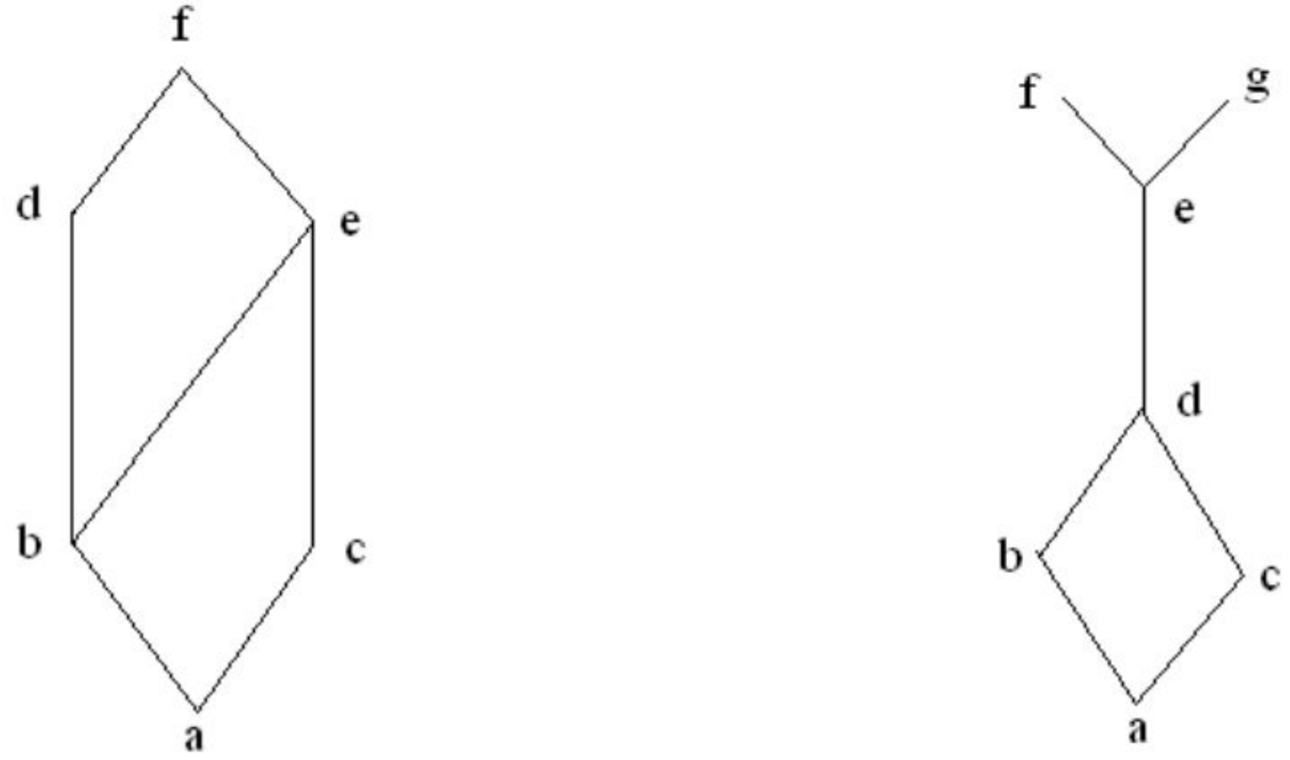
Lattice

Defination

A lattice is a poset in (L,≤) in which every subset {a,b} consisiting of two elements has a least upper bound and a greatest lower bound.
LUB({a,b}) is denoted by a v b and is called the join of a and b.
GLB({a,b}) is denoted by a Λ b and is called the meet of a and b.

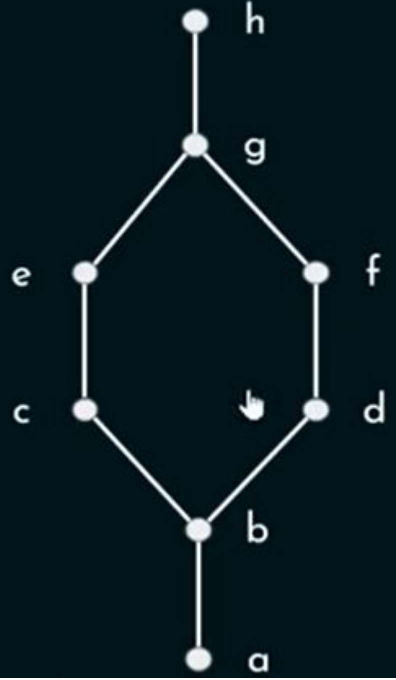
Definition: The poset (S, R) is called a lattice iff it is a meet semilattice and a join semilattice.



(a)
(b)

a) is a lattice.
b) Is not a lattice because f v g does not exist.

Example 1: Is the given Hasse diagram a lattice?



Solution:

We know that a Hasse diagram is called a lattice if it is both meet semilattice and join semilattice.

i.e.,

$\forall x, y \in S$, $GLB(x, y) \neq \emptyset$ and

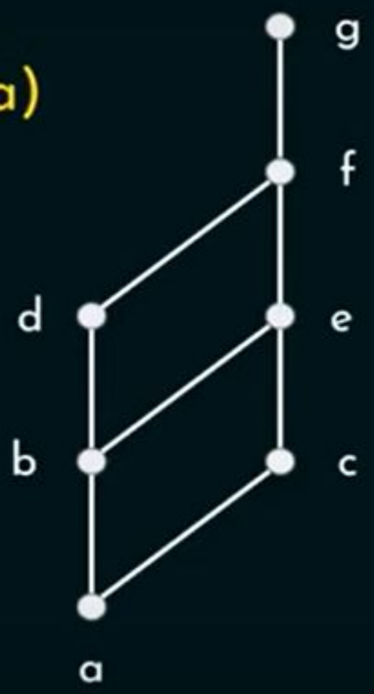
$\forall x, y \in S$, $LUB(x, y) \neq \emptyset$

Consider the incomparable pairs

$GLB(f, e) = b$	$GLB(e, d) = b$
$LUB(f, e) = g$	$LUB(e, d) = g$
$GLB(c, d) = b$	$GLB(c, f) = b$
$LUB(c, d) = g$	$LUB(c, f) = g$

The given Hasse diagram is a lattice.

(a)



Consider the pair (d, e)
GLB(d, e) = b
LUB(d, e) = f

Consider the pair (b, c)
GLB(b, c) = a
LUB(b, c) = e

For every pair of elements, the greatest lower bound and the least upper bound exists.

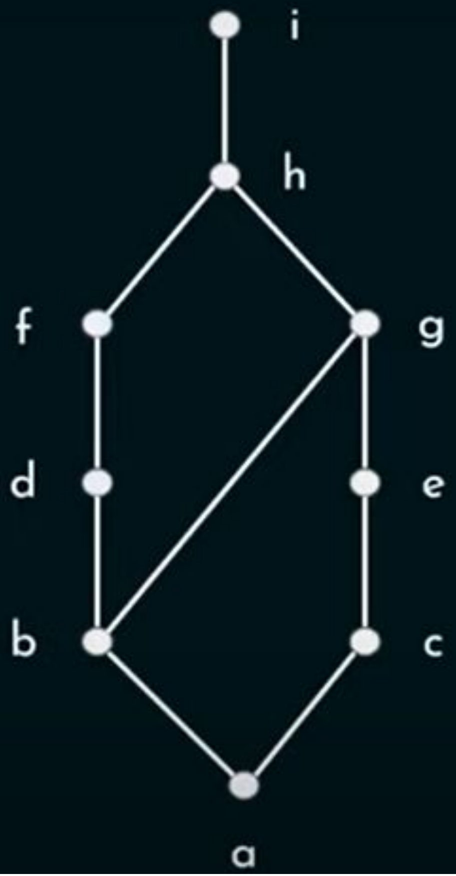
Therefore, the given Hasse diagram is a lattice.

Consider the pair (f, g)

GLB(f, g) = ∅

LUB(f, g) = h

The given Hasse diagram is not a meet semilattice and hence, it is not a lattice.



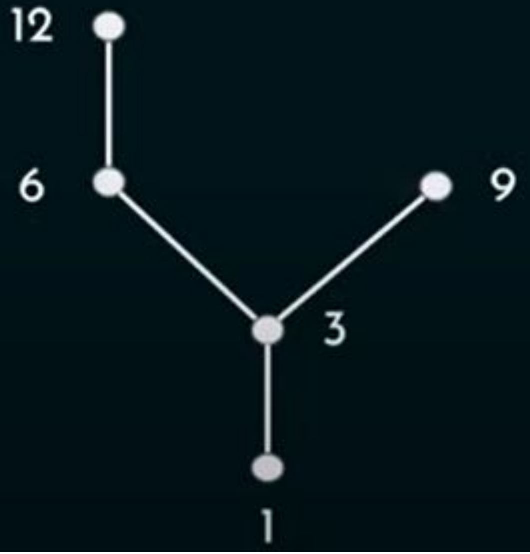
Consider the pair (f, g)
GLB(f, g) = b
LUB(f, g) = h

Consider the pair (d, e)
GLB(d, e) = a
LUB(d, e) = h

For every pair of elements, the greatest lower bound and the least upper bound exists.

Therefore, the given Hasse diagram is a lattice.

a) ({1, 3, 6, 9, 12}, |)



Consider the pair (9, 12)

GLB(9, 12) = 3

LUB(9, 12) = Ø

Therefore, the given Hasse diagram is not a lattice.

b) $(\{1, 5, 25, 125\}, |)$

It's a total order because every element is comparable. Hence, there is no need to check the least upper bound and the greatest lower bound of every pair.

Therefore, the given Hasse diagram is a lattice.

c) $(Z, \geq)$

It is a total order.

Therefore, the given Hasse diagram is a lattice (infinite lattice).

Lattice Using Operation Table

Q → Write the operation table for '$\vee$' & '$\wedge$' for $L = \{1, 2, 3, 5, 30\}$ under divisibility relation.

$a \vee b = l.u.b\{a, b\} = l.c.m\{a, b\}$

$a \wedge b = g.l.b$ of $\{a, b\} = g.c.d\{a, b\}$

$\vee$	1	2	3	5	30
1	1	2	3	5	30
2	2	2	6	10	30
3	3	6	3	15	30
5	5	10	15	5	30
30	30	30	30	30	30

$\wedge$	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

It is not a poset as all elements of LUB like 6 and 10 do not belong to L

Q → Write the operation table for '$\vee$' & '$\wedge$' for $L = \{\phi, \{1\}, \{2\}, \{1,2\}\}$ under '$\subseteq$' relation.

$a \vee b = l.u.b\{a, b\} = a \cup b$ $\quad$ $a \wedge b = g.l.b$ of $\{a, b\} = a \cap b$

$\vee$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$

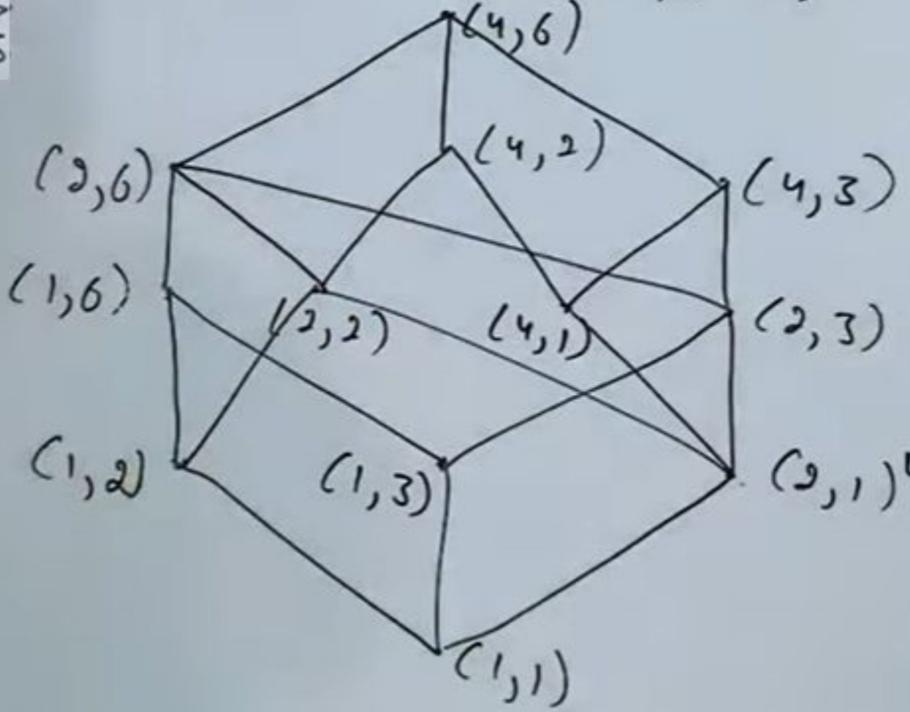
$\wedge$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	ϕ	$\{1\}$	ϕ	$\{1\}$
$\{2\}$	ϕ	ϕ	$\{2\}$	$\{2\}$
$\{1,2\}$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$

Q → Let D_4, D_6 be two lattice. Draw Hassie Diagram of $D_4 \times D_6$. Is it a lattice?

$D_4 = \{1, 2, 4\}$

$D_6 = \{1, 2, 3, 6\}$

$D_4 \times D_6 = \{(1,1), (1,2)\ (1,3)\ (1,6), (2,1)\ (2,2)\ (2,3)$
$(2,6), (4,1)\ (4,2)\ (4,3), (4,6)\}$



$(1,1) \le (1,2)$

$1 \le 1$

$1 \le 2$

Properties of Lattice

1. Idempotent Properties

   a) $a \vee a = a$
   b) $a \wedge a = a$

2. Commutative Properties

   a) $a \vee b = b \vee a$
   b) $a \wedge b = b \wedge a$

3. Associative Properties

   a) $a \vee (b \vee c) = (a \vee b) \vee c$
   b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties

   a) $a \vee (a \wedge b) = a$
   b) $a \wedge (a \vee b) = a$

Dual of a lattice

$$A \cup ((B^C \cup A) \cap B)^C = U$$

When we perform duality, then the union will be replaced by intersection, or intersection will be replaced by the union. The universal will also be replaced by null, or null will be replaced by universal.

$$A \cap ((B^C \cap A) \cup B)^C = \Phi$$

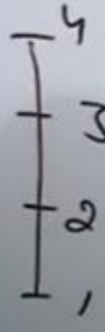
Dual of a lattice

$\leq$ $\quad\quad$ $\geq$

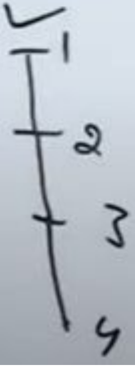
$a \vee b$ l.u.b $\longrightarrow$ g.l.b, $a \wedge b$

$a \wedge b$ g.l.b $\longrightarrow$ l.u.b, $a \vee b$

$\{1, 2, 3, 4\}$, usual $\leq$ $\quad\quad$ $\{1, 2, 3, 4\}$, $\geq$



$2 \wedge 3 = 2$



$2 \vee 3 = 2$

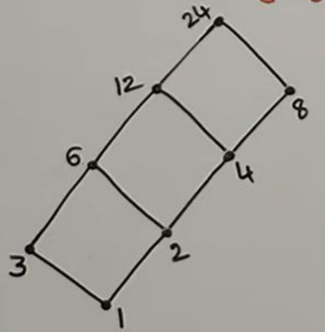
Let $(L, \vee, \wedge)$ be a Lattice and Let $S \subseteq L$ be a Subset of L. The algebraic System $(S, \vee, \wedge)$ is a Sublattice of $(L, \vee, \wedge)$ if and only if S is closed under both the operations $\vee$ and $\wedge$.

$\rightarrow$ From the above definition, it is obvious that the Sublattice itself is a Lattice with respect to $\vee$ and $\wedge$

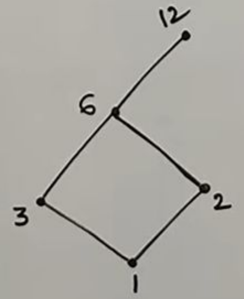
$\langle L, \vee, \wedge \rangle$ is a Lattice

S, $S \subseteq L$, $\langle S, \vee, \wedge \rangle$

Example-2:- consider the Lattice $(S, |)$ represented by Hasse diagram shown below, where $S = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Check whether the following sets $M_1 = \{1, 2, 3, 6, 12\}$, $M_2 = \{1, 2, 6, 12, 24\}$, $M_3 = \{1, 2, 4, 8, 24\}$ are Sublattices of S or not.

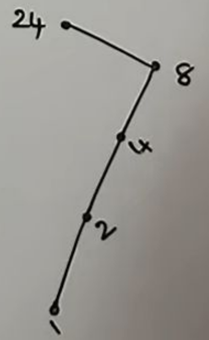


Hasse Diagram $\langle S_{24}, | \rangle$



$\langle M_1, | \rangle$ is a Sublattice of $\langle S_{24}, | \rangle$

$\langle M_2, | \rangle$ is a Sublattice of $\langle S_{24}, | \rangle$



$\langle M_3, | \rangle$ is a Sublattice of $\langle S_{24}, | \rangle$

Types of Lattice

1. Complete Lattice
2. Bounded Lattice
3. Isomorphic Lattice.
4. Distributive lattice
5. Complemented Lattice

Complete Lattice

Consider a poset (S, R).
Definition: A partially ordered set (S, R) is a complete lattice if every subset A of set S has both a greatest lower bound (or meet) and a least upper bound (or join) in (S, R).

Consider the following infinite lattice $(Z, \leq)$

Consider a subset of set Z

$B = \{x \mid x \geq 0\}$

$GLB(B) = 0$
$LUB(B) = \emptyset$

B is an infinite set and least upper bound of B does not exist.
Therefore, $(Z, \leq)$ is not a complete lattice.

Bounded Lattice

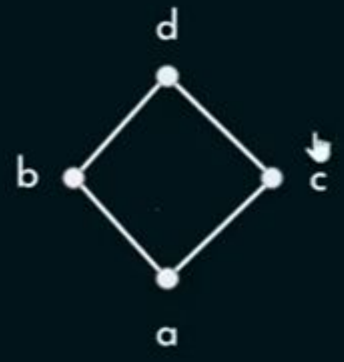
Consider a poset $(S, \leq)$.

Definition: A partially ordered set $(S, \leq)$ is called a bounded lattice if it has the greatest element (1) and the least element (0).

Greatest element: 1 is called the greatest element if $\forall x \in S$, $x \leq 1$.

Least element: 0 is called the least element if $\forall x \in S$, $0 \leq x$.

Example: Consider the following Hasse diagram.



a is the least element because a ≤ b, a ≤ c, and a ≤ d.
d is the greatest element because d ≥ b, d ≥ c, and d ≥ a.

Both least and greatest elements exist in the above lattice.
Therefore, the given lattice is a bounded lattice.

Properties of Bounded lattice

Consider the following Hasse diagram.

Least element is a.
Greatest element is f.
Therefore, the given lattice is a Bounded lattice.

We know that a is denoted by 0 and f is denoted by 1.

Consider some element x. Then, following properties must be satisfied for a bounded lattice.

$$x \vee 1 = 1$$
$$x \wedge 1 = x$$

$$x \vee 0 = x$$
$$x \wedge 0 = 0$$

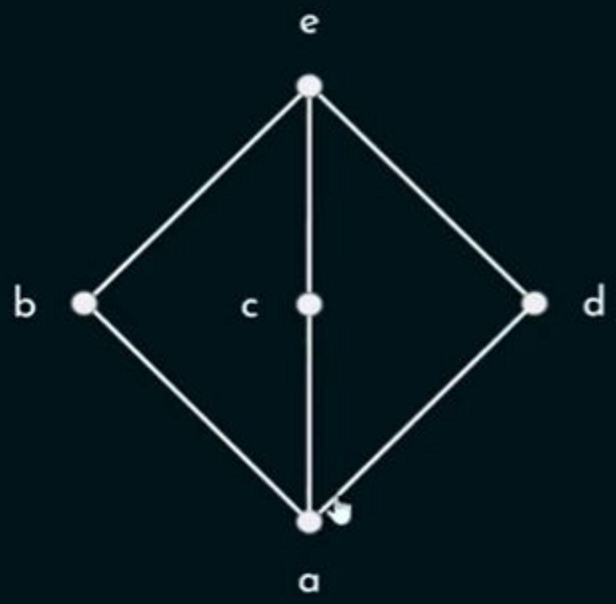
Distributive Lattice

Definition: A lattice L is said to be a distributive lattice if $\forall a,b,c \in L$

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Distributive Lattice



For a lattice to be distributive lattice, the following properties must be satisfied.

$\forall a,b,c \in L$

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

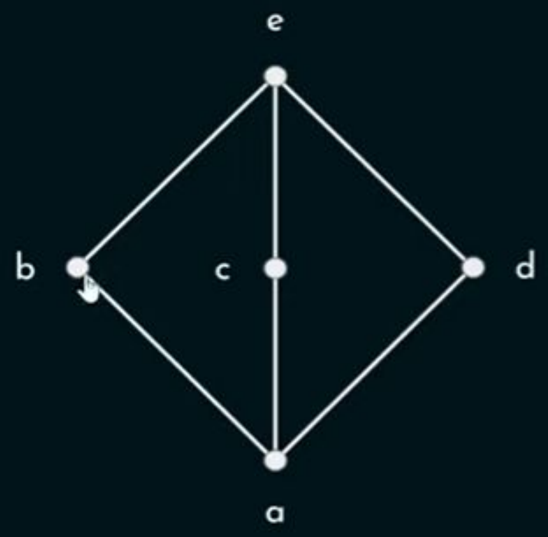
Let's consider the elements b, c, and d.

(i) $LHS = b \vee (c \wedge d)$
$= b \vee a$
$= b$

(ii) $RHS = (b \vee c) \wedge (b \vee d)$
$= e \wedge e$
$= e$

Therefore, the above lattice is **not a distributive lattice.**

Distributive Lattice using Complement



Least element: a
Greatest element: e

Complement of b

01 c is the complement of b because

$$LUB(c, b) = e$$
$$GLB(c, b) = a$$

02 d is the complement of b because

$$LUB(d, b) = e$$
$$GLB(d, b) = a$$

b has two complements. Therefore, the given lattice is not a distributive lattice.

Least element: a
Greatest element: f

Complement of d

01 e is not the complement of d because

$$LUB(e, d) = f$$
$$GLB(e, d) = c \neq a$$

02 c is not the complement of d because

$$LUB(c, d) = d \neq f$$
$$GLB(c, d) = c \neq a$$

Therefore, complement of d does not exist.

There is only one complement of e.

Least element: a
Greatest element: f

Complement of e

01 d is not the complement of e because

LUB(e, d) = f
GLB(e, d) = c ≠ a

02 b is the complement of e because

LUB(b, e) = f
GLB(b, e) = a

Similarly, complement of c does not exist.
Complement of a is f.
Complement of f is a.
And complement of b is e.

Therefore, every element in the given lattice has atmost one complement.

Hence, the given lattice is a distributive lattice.

Isomorphic Lattice

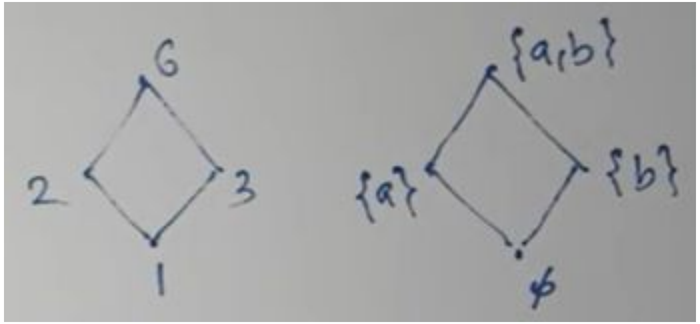
Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from L_1 to L_2 i,e $f: L_1 \to L_2$ such that

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and}$$

$$f(a \vee b) = f(a) \vee f(b)$$

for every element $a, b \in L_1$.

Eg:- Let $L = \{1, 2, 3, 6\}$ and $A = \{a, b\}$. Then prove that the lattices $(L, /)$ and $(P(A), \subseteq)$ are isomorphic.



Sol^n: consider the mapping $f: L \rightarrow P(A)$ where $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

and the mapping $f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a,b\}$

Also

$f(a \wedge b) = f(a) \wedge f(b)$ and

$f(a \vee b) = f(a) \vee f(b)$ holds.

$\boxed{a = 1, \ b = 2}$

$f(1 \wedge 2) = f(1) \wedge f(2)$

$\Rightarrow f(1) = \phi \wedge \{a\}$

$\Rightarrow \phi = \phi$

Isomorphic lattice will have same diagram