Recurrence Relation

Definition

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, … , a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation. The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Example - Fibonacci Series

Fibonacci Sequence

$$\langle 1, 1, 2, 3, 5, 8, --- \rangle$$

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2$$

$$a_3 = a_2 + a_1$$

$$a_2 = a_1 + a_0$$

Initially we take $a_0 = 1$ & $a_1 = 1$

Geometric Progression

$\rightarrow \langle 3, 9, 27, 81, \ldots \rangle$

$$a_n = 3 \cdot a_{n-1}$$

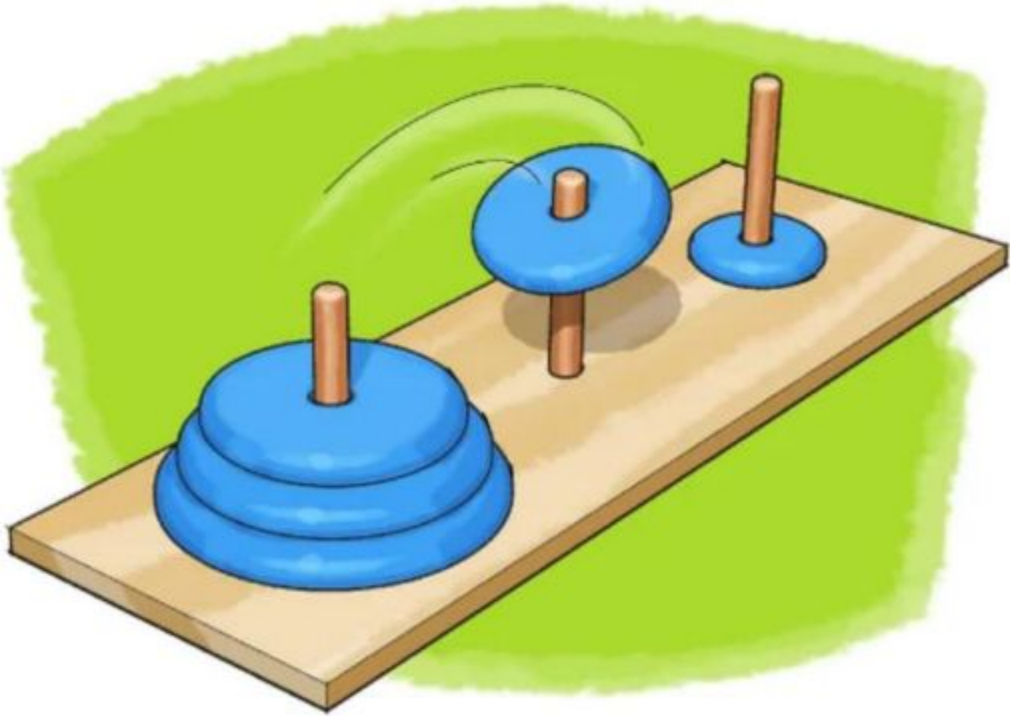
$$a_0 = 3$$

$$a_{227} = 3 \cdot a_{226}$$

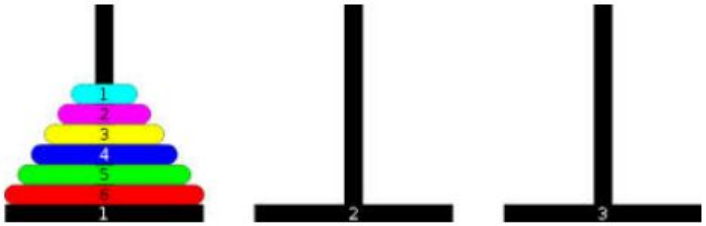
$$= 3 \cdot (3 \cdot a_{225})$$

$$a_n = 3 \cdot (3^n)$$ Closed Form

Example - The Tower of Hanoi



Towers of Hanoi



- Given 3 pegs where first peg contains n disks
- Goal: Move all the disks to a different peg (e.g., second one)
- Rule 1: Larger disks cannot rest on top of smaller disks
- Rule 2: Can only move the top-most disk at a time
- Question: How many steps does it take to move all n disks?

A Recursive Solution

- Solve recursively – T_n is number of steps to move n disks
- Base case: $n = 1$, move disk from first peg to second: $T_1 = 1$
- Induction: Suppose we can move $n - 1$ disks in T_{n-1} steps; how many steps does it take to move T_n disks?
- Idea: First move the topmost $n - 1$ disks to peg 3; can be done in T_{n-1} steps
- Now, move bottom-most disk to peg 2 – takes just 1 step
- Finally, recursively move $n - 1$ disks in peg 3 to peg 2 – can be done in T_{n-1} steps

T (n) = 2 T (n -1) + 1.

Find a recurrence relation and initial conditions for 1,5,17,53,161,485……

So $a_n = 3a_{n-1} + 2$ is our recurrence relation and the initial condition is $a_0 = 1$.

Solve the recurrence relation $a_n = a_{n-1} + n$ with initial term $a_0 = 4$.

To get a feel for the recurrence relation, write out the first few terms of the sequence: $4, 5, 7, 10, 14, 19, \ldots$. Look at the difference between terms. $a_1 - a_0 = 1$ and $a_2 - a_1 = 2$ and so on. The key thing here is that the difference between terms is n. We can write this explicitly: $a_n - a_{n-1} = n$. Now use this equation over and over again, changing n each time:

$$\begin{aligned} a_1 - a_0 &= 1 \\ a_2 - a_1 &= 2 \\ a_3 - a_2 &= 3 \\ \vdots \quad & \quad \vdots \\ a_n - a_{n-1} &= n. \end{aligned}$$

Add all these equations together. On the right-hand side, we get the sum $1 + 2 + 3 + \cdots + n$. We already know this can be simplified to $\frac{n(n+1)}{2}$. What happens on the left-hand side? We get

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}).$$

$-a_0 + a_n = \frac{n(n+1)}{2}$ or $a_n = \frac{n(n+1)}{2} + a_0$. But we know that $a_0 = 4$. So the solution to the recurrence relation, subject to the initial condition is

$$a_n = \frac{n(n+1)}{2} + 4.$$

Linear Recurrence relation with constant coefficients

A Recurrence Relations is called linear if its degree is one.

The general form of linear recurrence relation with constant coefficient is

$$C_0\, y_{n+r} + C_1\, y_{n+r-1} + C_2\, y_{n+r-2} + \cdots + C_r\, y_n = R\,(n)$$

Where $C_0, C_1, C_2 \ldots\ldots C_n$ are constant and R (n) is some function. Maximum degree of y is 1 hence it is linear.

The equation is said to be linear homogeneous difference equation if and only if $R(n) = 0$

The equation is said to be linear non-homogeneous difference equation if $R(n) \neq 0$.

Order of the Recurrence Relation:

The order of the recurrence relation or difference equation is defined to be the difference between the highest and lowest subscripts of f(x) or $a_r = y_k$.

Example1: The equation $13a_r + 20a_{r-1} = 0$ is a first order recurrence relation.

Example1: The equation $y^3_{k+3} + 2y^2_{k+2} + 2y_{k+1} = 0$ has the degree 3, as the highest power of y_k is 3.

Example2: The equation $a^4_r + 3a^3_{r-1} + 6a^2_{r-2} + 4a_{r-3} = 0$ has the degree 4, as the highest power of a_r is 4.

Example3: The equation $y_{k+3} + 2y_{k+2} + 4y_{k+1} + 2y_k = k(x)$ has the degree 1, because the highest power of y_k is 1 and its order is 3.

Example1: The equation $a_{r+3}+6a_{r+2}+12a_{r+1}+8a_r=0$ is a linear non-homogeneous equation of order 3.

Example2: The equation $a_{r+2}-4a_{r+1}+4a_r= 3r + 2^r$ is a linear non-homogeneous equation of order 2.

A linear homogeneous difference equation with constant coefficients is given by

$$C_0\, y_n+C_1\, y_{n-1}+C_2\, y_{n-2}+\cdots\ldots\ldots+C_r\, y_{n-r}=0$$

Determine which of these are linear homogeneous recurrence relations with constant coefficients.

a $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

Yes. Degree 3.

b $a_n = 2na_{n-1} + a_{n-2}$

No. $2n$ is not a constant coefficient.

c $a_n = a_{n-1} + a_{n-4}$

Yes. Degree 4.

d $a_n = a_{n-1} + 2$

No. This is nonhomogeneous because of the 2.

Determine which of these are linear homogeneous recurrence relations with constant coefficients.

e $a_n = a_{n-1}^2 + a_{n-2}$

No. This is not linear because of a_{n-1}^2.

f $a_n = a_{n-2}$

Yes. Degree 2.

g $a_n = a_{n-1} + n$

No. This is nonhomogeneous because of the n.

Solving Recurrence relation

1) Iteration
2) Characteristic Roots
3) Generating Functions

Use iteration to solve the recurrence relation $a_n = a_{n-1} + n$ with $a_0 = 4$.

Again, start by writing down the recurrence relation when $n = 1$. This time, don't subtract the a_{n-1} terms to the other side:

$$a_1 = a_0 + 1.$$

Now $a_2 = a_1 + 2$, but we know what a_1 is. By substitution, we get

$$a_2 = (a_0 + 1) + 2.$$

Now go to $a_3 = a_2 + 3$, using our known value of a_2:

$$a_3 = ((a_0 + 1) + 2) + 3.$$

We notice a pattern. Each time, we take the previous term and add the current index. So

$$a_n = ((((a_0 + 1) + 2) + 3) + \cdots + n - 1) + n.$$

Regrouping terms, we notice that a_n is just a_0 plus the sum of the integers from 1 to n. So, since $a_0 = 4$,

$$a_n = 4 + \frac{n(n+1)}{2}.$$

- Let $\{a_i\}$ be the sequence given by:

$$a_k = a_{k-1} + k \qquad \text{with } a_0 = 0.$$

- Solve this recurrence relation and find a_{100}.
- Now, $a_1 = a_0 + 1 = 1 + 0$

$$a_2 = a_1 + 2 = 2 + 1 + 0$$

$$a_3 = a_2 + 3 = 3 + 2 + 1 + 0$$

$$a_4 = a_3 + 4 = 4 + 3 + 2 + 1 + 0$$

- Thus $a_n = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 + 0$

  so $a_n = n(n+1)/2.$
- Plugging in $n = 100$: $a_{100} = 100(101)/2 = 5050.$

Characteristic Roots

Solve $a_n - 6a_{n-1} + 8a_{n-2} = 0$

$a_0 = 4,\ a_1 = 10$

Put $a_n = r^n$

$r^n - 6r^{n-1} + 8r^{n-2} = 0$

Divide b.s. by r^{n-2}

$r^2 - 6r + 8 = 0$

$r^2 - 4r - 2r + 8 = 0$

$r(r-4) - 2(r-4) = 0$

$(r-2)(r-4) = 0$

$r = 2, 4$

$a_n = C_1 \cdot 2^n + C_2 \cdot 4^n$

$a_0 = C_1 \cdot 2^0 + C_2 \cdot 4^0$

$\boxed{4 = C_1 + C_2}$ — (2)

Put $n = 1$ in (1)

$a_1 = C_1 \cdot 2^1 + C_2 \cdot 4^1$

$10 = 2C_1 + 4 \cdot C_2$

$\boxed{5 = C_1 + 2C_2}$ — (3)

(3) − (2)

$\boxed{1 = C_2}$ Put in (2)

$4 = C_1 + 1$

$\boxed{C_1 = 3}$

$\boxed{a_n = 3 \cdot 2^n + 1 \cdot 4^n}$

Characteristic Roots

$a_n = 2a_{n-1}$ for $n \geq 1, a_0 = 3$

Characteristic equation: $r - 2 = 0$
Characteristic root: $r = 2$

use the initial condition, $a_0 = 3$, to find it.
$3 = \alpha 2^0$
$3 = \alpha 1$
$3 = \alpha$

So our solution to the recurrence relation is $a_n = 3 \cdot 2^n$.

Characteristic Roots

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