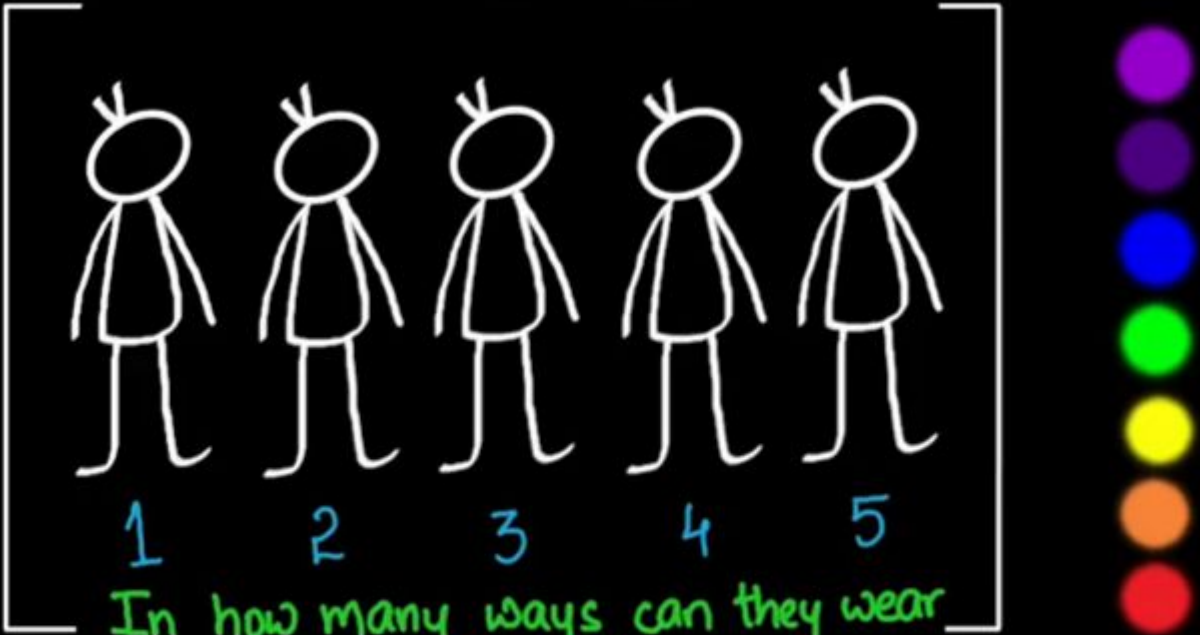


# Chromatic Polynomial

Graph Connectivity

Assume that student can not change their position



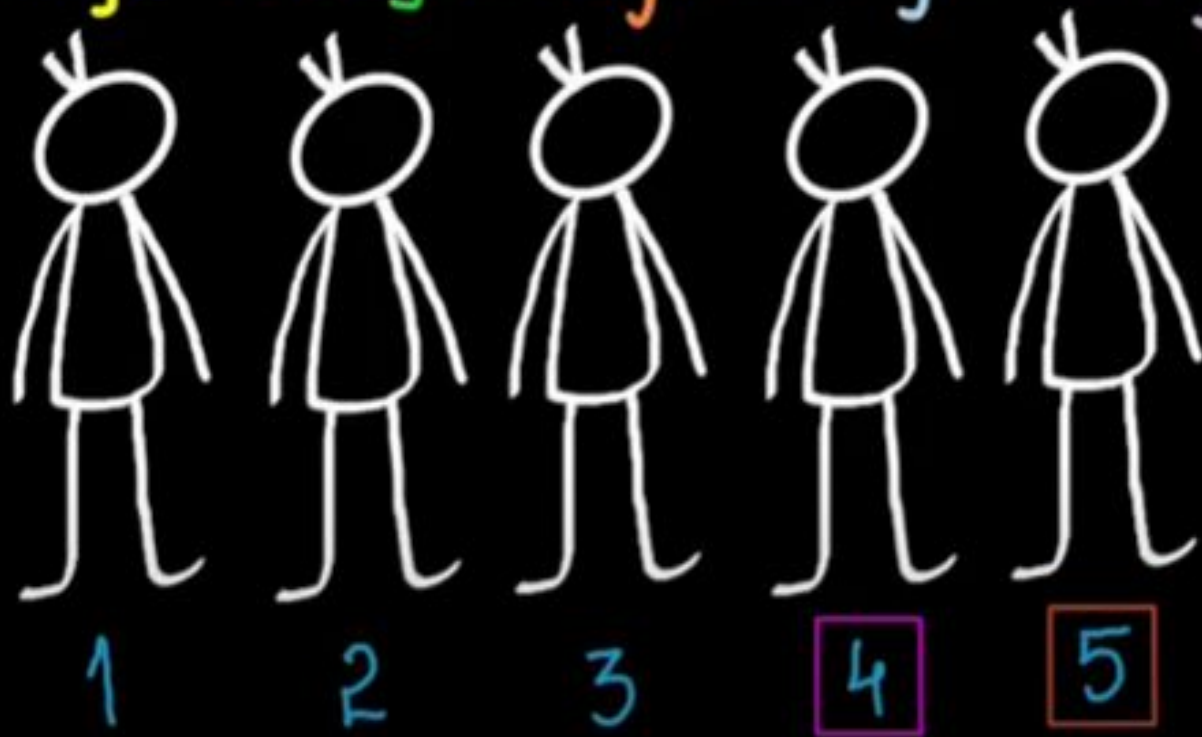
The diagram shows five stick figures standing in a row, enclosed in large square brackets. Below each figure is a number from 1 to 5. To the right of the figures is a vertical column of six colored dots: purple, dark purple, blue, green, yellow, and red. Below the figures, the text "In how many ways can they wear" is written in green, with "so that" written below it. At the bottom, the text "No two adjacent people are wearing the same colored shirt?" is written in green.

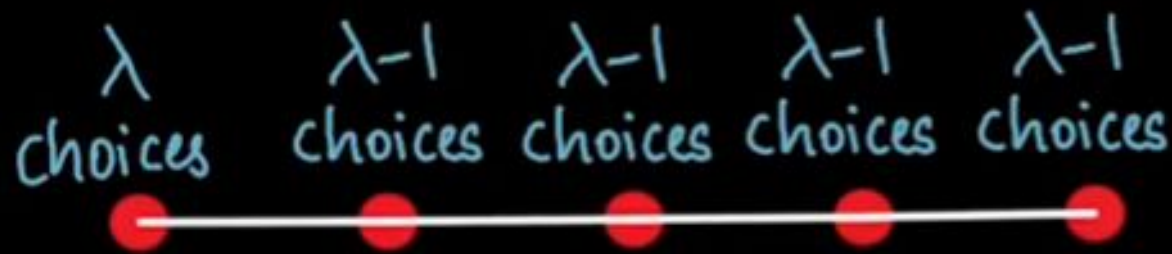
1 2 3 4 5

In how many ways can they wear  
so that

No two adjacent people are wearing the same colored shirt?

7 ways  $\times$  6 ways  $\times$  6 ways  $\times$  6 ways  $\times$  6 ways = 9072 ways





In how many ways can you color this graph with  $\lambda$  colors so that no two adjacent have the same color?

$$\lambda \times (\lambda-1)^4 \quad \text{for } P_5$$

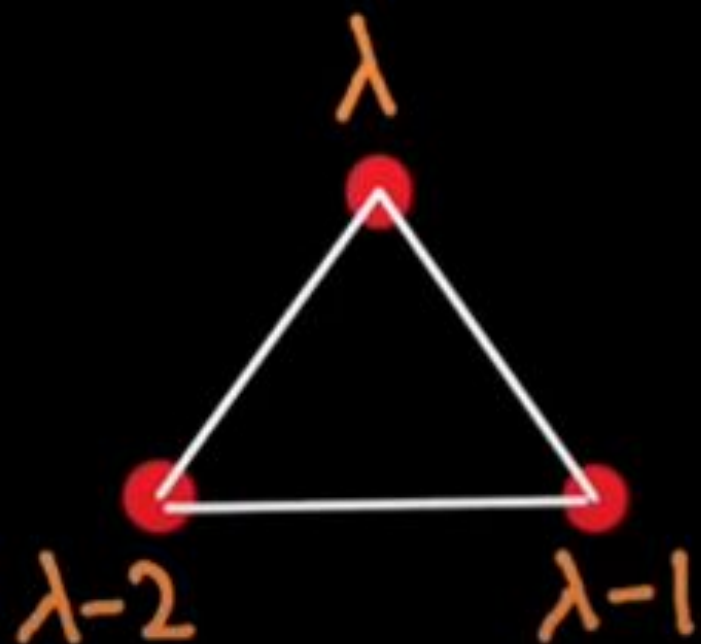
# G

In how many ways can you color this graph with  $\lambda$  colors so that no two adjacent have the same color?

Chromatic polynomial

$$C(P_5) = \lambda(\lambda-1)^4$$

# Chromatic polynomial of complete graphs



$$C(K_3) = \lambda(\lambda-1)(\lambda-2)$$



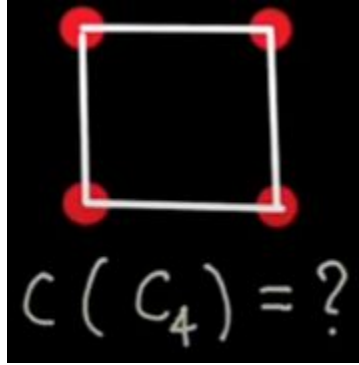
$$C(K_4) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$



$$C(K_5) = \lambda(\lambda-1)(\lambda-2) \\ (\lambda-3)(\lambda-4)$$

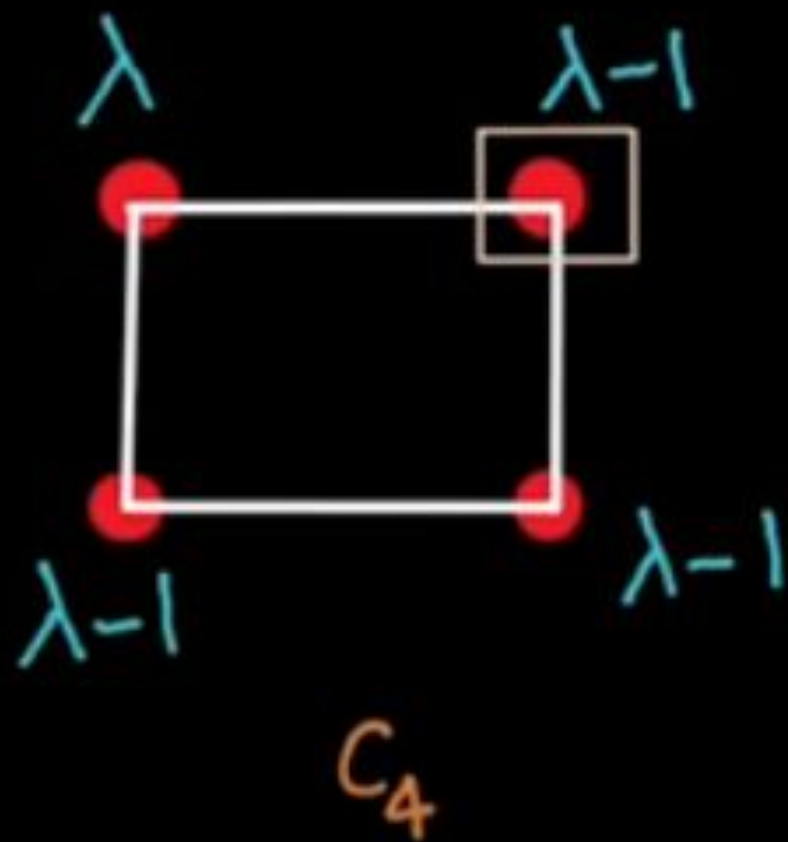
$K_n$

$$C(K_n) = \lambda(\lambda-1)(\lambda-2) \dots (\lambda-n+1)$$

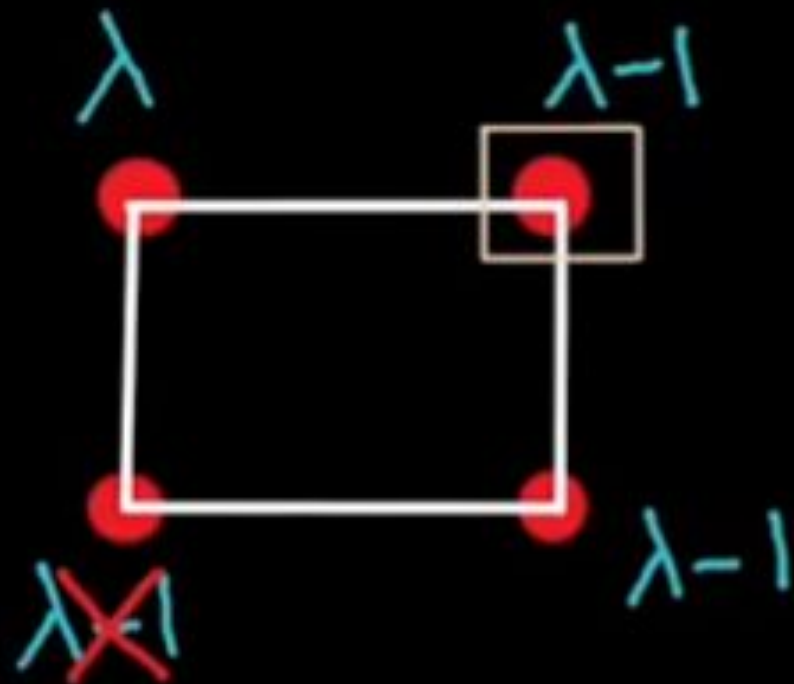


$$C(C_4) = ?$$



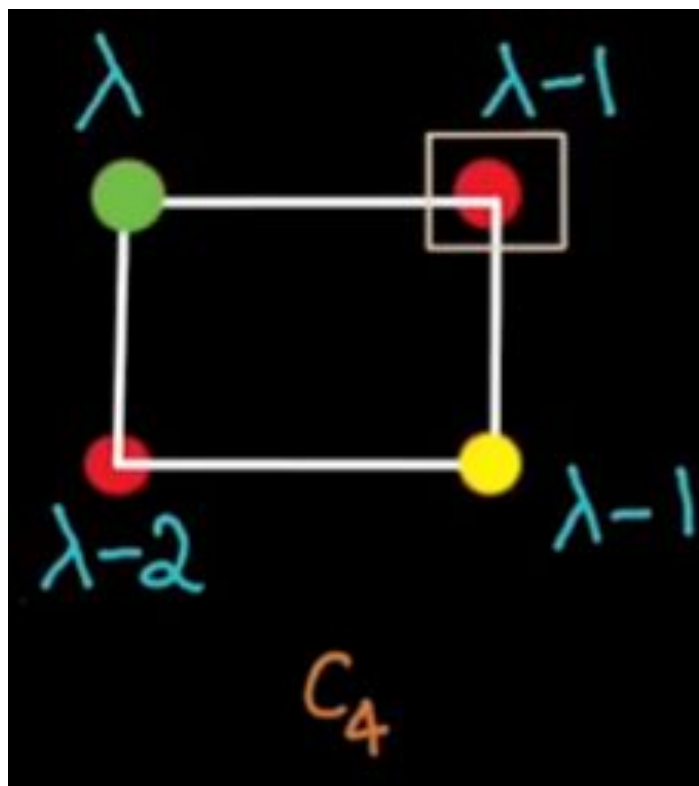


$$C(C_4) = ?$$

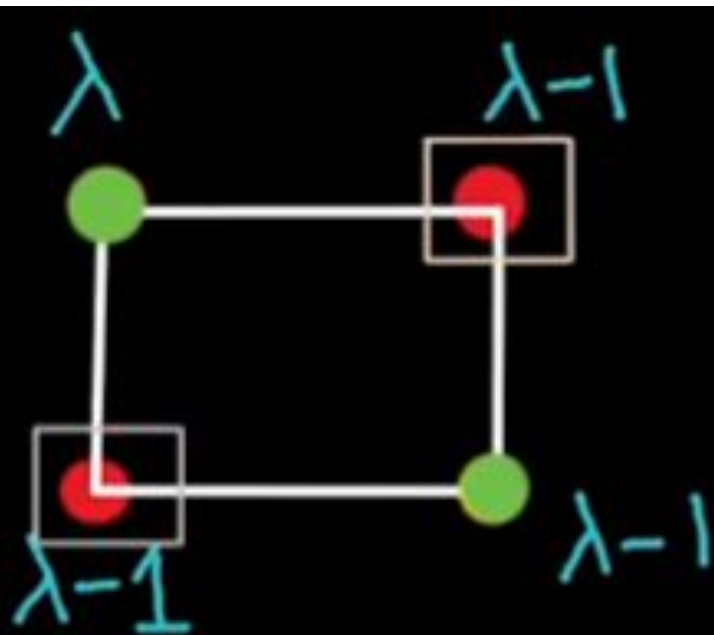


$C_4$

$$C(C_4) = ?$$

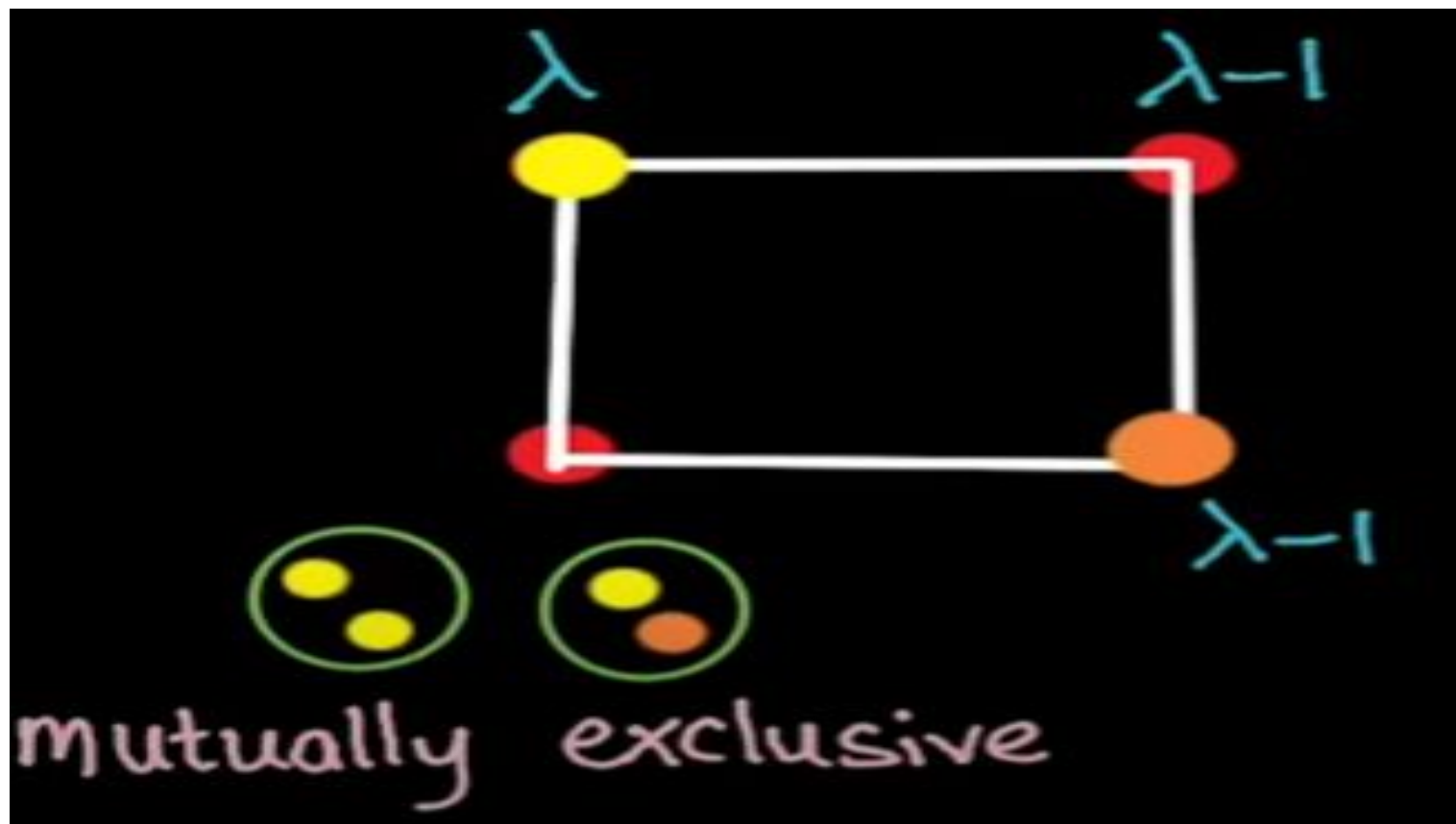


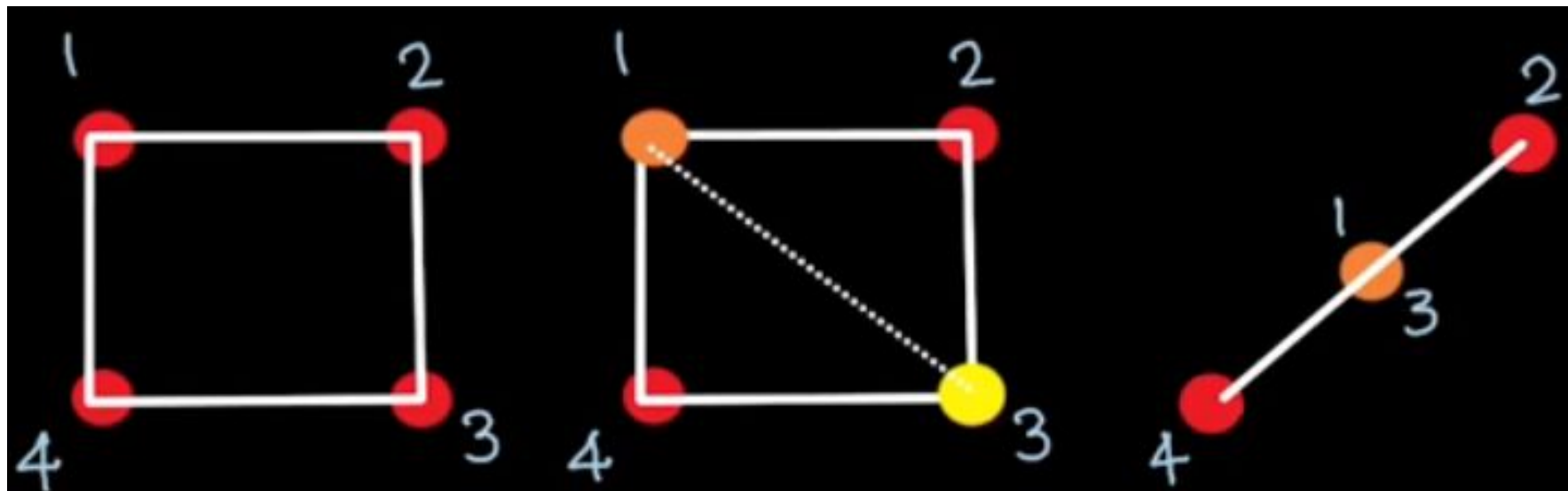
$$C(C_4) = ?$$



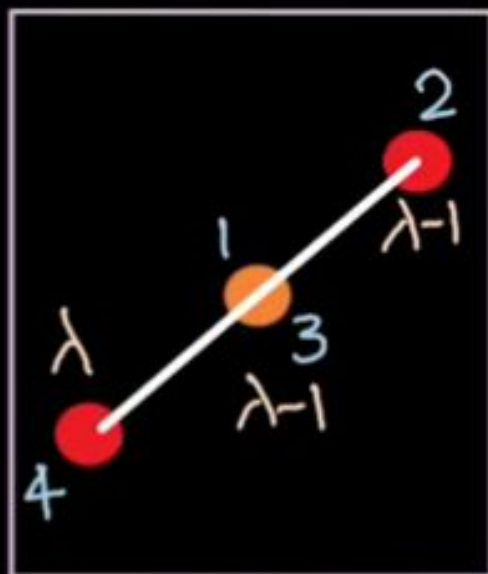
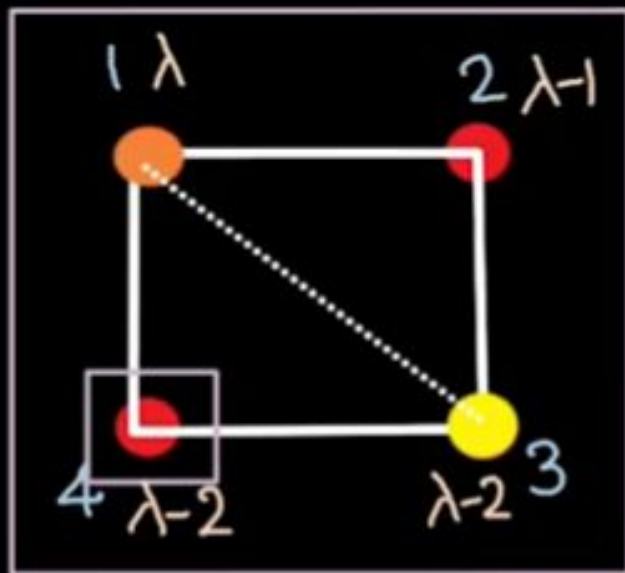
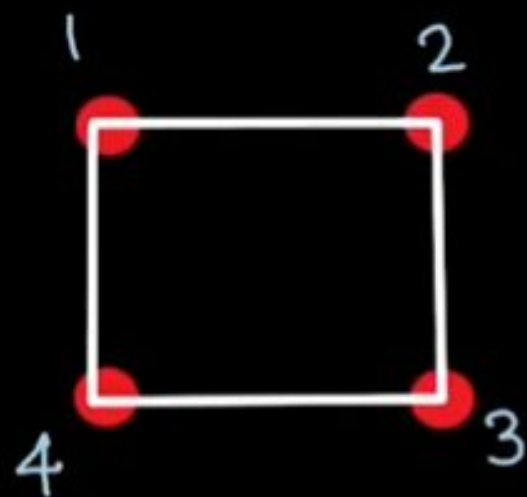
$C_4$

$$C(C_4) = ?$$





$$c(\square, \lambda) = c(\square \diagdown, \lambda) + c(\diagup, \lambda)$$



$$\begin{aligned}
 C(\square, \lambda) &= \lambda(\lambda-1)(\lambda-2)(\lambda-2) + \lambda(\lambda-1)(\lambda-1) \\
 &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda + \lambda^3 + \lambda - 2\lambda^2 \\
 &= \boxed{\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda}
 \end{aligned}$$

$$c(\square, \lambda) = c(\square \diagup, \lambda) + c(\diagdown, \lambda)$$

$$\Rightarrow c(\square \diagup_3, \lambda) = c(\square_3, \lambda) - c(\diagdown^{13}, \lambda)$$

$$C(G, \lambda) = \frac{C(G - e, \lambda)}{\frac{C(G \cdot e, \lambda)}}{}$$

Edge  $e$  removed from  $G$

Edge  $e$  coalesced in  $G$ .

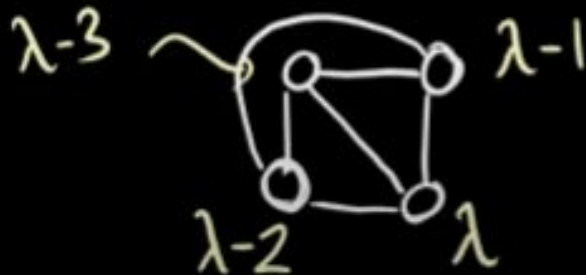


# Examples

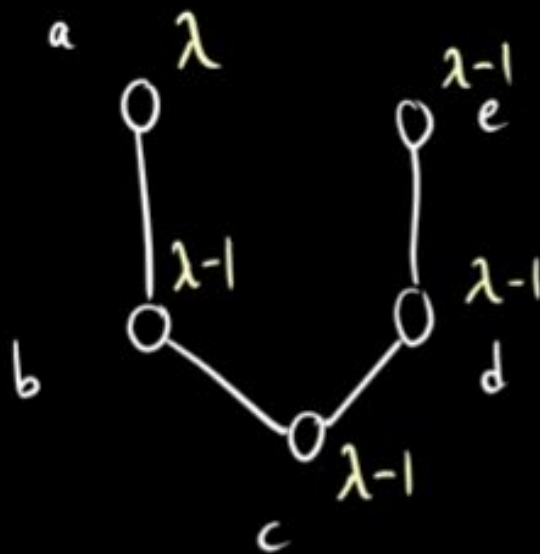
①  $|V|=n$  and  $E=\emptyset$ , then  $P(G, \lambda) = \lambda^n$

$$\begin{array}{ccc} \lambda \circ & \lambda \circ & \lambda \\ & & \circ \end{array} \quad \begin{array}{l} v=5 \\ \lambda^5 \end{array}$$
$$\lambda \circ \quad \lambda \circ$$

②  $G = K_n$ , then  $P(G, \lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1) = \frac{\lambda!}{(\lambda-n)!}$



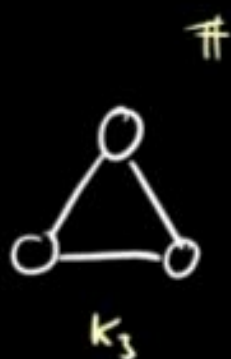
③ For paths, consider the number of choices from vertex to vertex.



$$\begin{aligned} n\text{-vertex path} \\ = \lambda(\lambda-1)^{n-1} \end{aligned}$$

$$P(G, \lambda) = \lambda(\lambda-1)^4$$

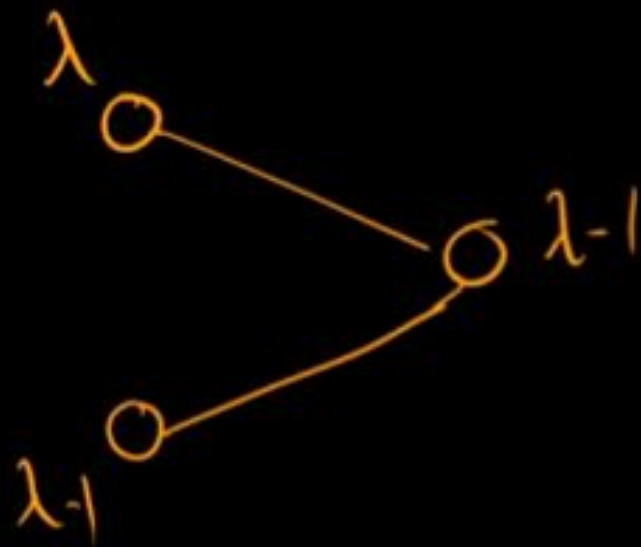
④ If  $G$  has components, multiply using the product rule.



$G$

$$P(G, \lambda) = \left( \lambda(\lambda-1)^2 \right) \left( \frac{\lambda!}{(\lambda-3)!} \right)$$

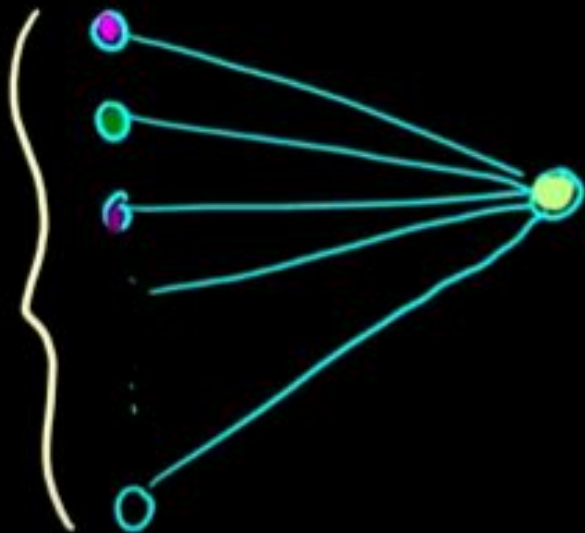
Find  $P(G, \lambda)$  for  $K_{2,1}$

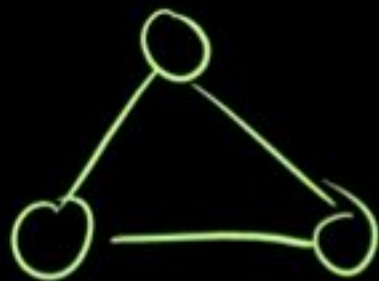


$$\lambda(\lambda-1)^2$$

What is  $\chi(k_{m,1})$ ?

$$\chi(k_{m,1}) = 2$$





$$K_3 = \lambda(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 2$$

$$\lambda = 3$$

$$3(3-1)(3-2)$$

$$= 3(2)(1)$$

$$= 6$$