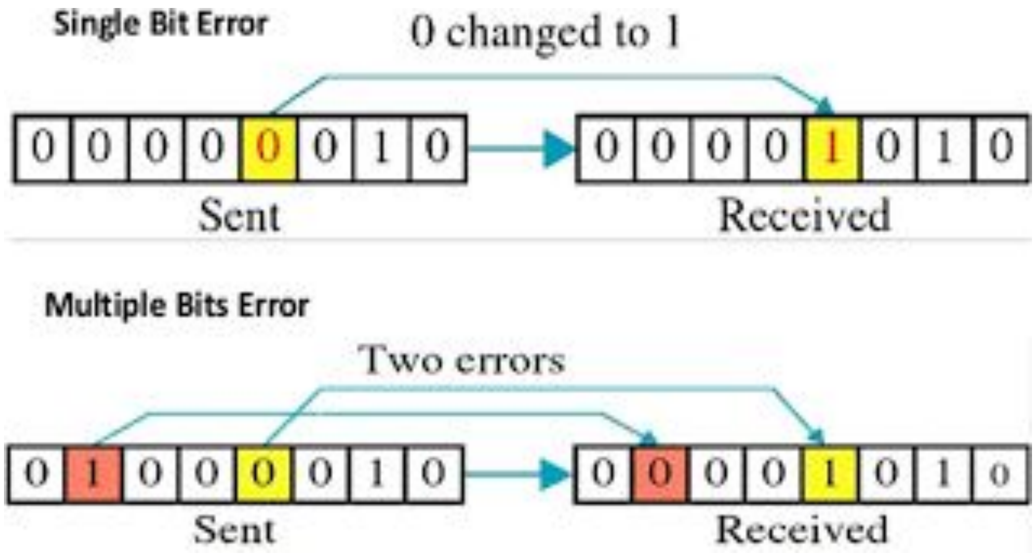
Coding Theory

Hamming Code

We can’t avoid the interference of noise. But, we can get back the original data first by detecting whether any error present and then correcting those errors.



we can use the following codes.

- Error detection codes
- Error correction codes

Error-correcting codes (ECC) are a sequence of numbers generated by specific algorithms for detecting and removing errors in data that has been transmitted over noisy channels.

Parity Code

Even Parity Code

Odd Parity Code

Even Parity Code

Binary Code	Even Parity bit	Even Parity Code
000	0	0000
001	1	0011
010	1	0101
011	0	0110
100	1	1001
101	0	1010
110	0	1100
111	1	1111

Odd Parity Code

Binary Code	Odd Parity bit	Odd Parity Code
000	1	0001
001	0	0010
010	0	0100
011	1	0111
100	0	1000
101	1	1011
110	1	1101
111	0	1110

Hamming Code

Hamming code is a block code that is capable of detecting up to two simultaneous bit errors and correcting single-bit errors. It was developed by R.W. Hamming for error correction.

In this coding method, the source encodes the message by inserting redundant bits within the message. These redundant bits are extra bits that are generated and inserted at specific positions in the message itself to enable error detection and correction. When the destination receives this message, it performs recalculations to detect errors and find the bit position that has error.

Example

Encode a binary word **11001** into the even parity hamming code.
Given, number of data bits, n =5.

$$2^k \geq n + k + 1$$

$$2^4 \geq 5 + 4 + 1$$

The equation is satisfied and so 4 redundant bits are selected.

So, total code bit = n+P = 9

The redundant bits are placed at bit positions 1, 2, 4 and 8.

Bit Location	9	8	7	6	5	4	3	2	1
Bit designation	D9	**P8**	D7	D6	D5	**P4**	D3	**P2**	**P1**
Information bits	1		1	0	0		1		
Parity bits		1				1		0	1

For P1: Bit locations 3, 5, 7 and 9 have three 1s. To have even parity, P1 must be 1.
For P2: Bit locations 3, 6, 7 have two 1s. To have even parity, P2 must be 0.
For P4: Bit locations 5, 6, 7 have one 1s. To have even parity, P4 must be 1.
For P8: Bit locations 8, 9 have one 1s. To have even parity, P8 must be 1.
Thus the encoded 9-bit hamming code is 111001101.

How to detect and correct the error in the hamming code?

After receiving the encoded message, each parity bit along with its corresponding group of bits are checked for proper parity. While checking, the correct result of individual parity is marked as 0 and the wrong result is marked as 1.

Let us assume the even parity hamming code from the above example (111001101) is transmitted and the received code is (110001101). Now from the received code, let us detect and correct the error.

Bit Location	9	8	7	6	5	4	3	2	1
Bit designation	D_9	**P_8**	D_7	D_6	D_5	**P_4**	D_3	**P2**	**P1**
Received code	1	1	0	0	0	1	1	0	1

For P1 : Check the locations 1, 3, 5, 7, 9. There is three 1s in this group, which is **wrong** for even parity. Hence the bit value for P1 is **1**.

For P2 : Check the locations 2, 3, 6, 7. There is one 1 in this group, which is **wrong** for even parity. Hence the bit value for P2 is **1**.

For P4 : Check the locations 4, 5, 6, 7. There is one 1 in this group, which is **wrong** for even parity. Hence the bit value for P4 is **1**.

For P8 : Check the locations 8, 9. There are two 1s in this group, which is **correct** for even parity. Hence the bit value for P8 is **0**.

The resultant binary word is 0111.
It corresponds to the bit location 7 in the above table.
The error is detected in the data bit D7. The error is 0 and it should be changed to 1. **Thus the corrected code is 111001101**.

Hamming Code

Hamming code is useful for both detection and correction of error present in the received data. This code uses multiple parity bits and we have to place these parity bits in the positions of powers of 2.

The **minimum value of 'k'** for which the following relation is correct $valid$ is nothing but the required number of parity bits.

$$2^k \geq n + k + 1$$

Where,

'n' is the number of bits in the binary code $information$

'k' is the number of parity bits

Therefore, the number of bits in the Hamming code is equal to n + k.

Let the **Hamming code** is $b_{n+k} b_{n+k-1} \ldots\ldots b_3 b_2 b_1$ & parity bits $p_k, p_{k-1}, \ldots. p_1$. We can place the 'k' parity bits in powers of 2 positions only.

Let us find the Hamming code for binary code, $d_4d_3d_2d_1$ = 1000. Consider even parity bits.

The number of bits in the given binary code is n=4.

We can find the required number of parity bits by using the following mathematical relation.

$$2^k \geq n + k + 1$$

Substitute, n=4 in the above mathematical relation.

$$\Rightarrow 2^k \geq 4 + k + 1$$

$$\Rightarrow 2^k \geq 5 + k$$

The minimum value of k that satisfied the above relation is 3. Hence, we require 3 parity bits p_1, p_2, and p_3. Therefore, the number of bits in Hamming code will be 7, since there are 4 bits in binary code and 3 parity bits. We have to place the parity bits and bits of binary code in the Hamming code as shown below.

The **7-bit Hamming code** is $b_7b_6b_5b_4b_3b_2b_1 = d_4d_3d_2p_3d_1p_2bp_1$

By substituting the bits of binary code, the Hamming code will be $b_7 b_6 b_5 b_4 b_3 b_2 b_1 = 100p_3 O p_2 p_1$. Now, let us find the parity bits.

$$p_1 = b_7 \oplus b_5 \oplus b_3 = 1 \oplus 0 \oplus 0 = 1$$

$$p_2 = b_7 \oplus b_6 \oplus b_3 = 1 \oplus 0 \oplus 0 = 1$$

$$p_3 = b_7 \oplus b_6 \oplus b_5 = 1 \oplus 0 \oplus 0 = 1$$

By substituting these parity bits, the **Hamming code** will be $b_7 b_6 b_5 b_4 b_3 b_2 b_1 = 1001011$.

For P1, check 1 bit and skip 1 bit

7	6	5	4	3	2	1
D_7	D_6	D_5	P_4	D_3	P_2	P_1

$$P_1 = D_3 \oplus D_5 \oplus D_7$$

For P2, check 2 bits and skip 2 bits

7	6	5	4	3	2	1
D_7	D_6	D_5	P_4	D_3	P_2	P_1

$$P_2 = D_3 \oplus D_6 \oplus D_7$$

For P4, check 4 bits and skip 4 bits

7	6	5	4	3	2	1
D_7	D_6	D_5	P_4	D_3	P_2	P_1

$$P_4 = D_7 \oplus D_6 \oplus D_5$$

In the above example, we got the Hamming code as $b_7 b_6 b_5 b_4 b_3 b_2 b_1 = 1001011$. Now, let us find the error position when the code received is $b_7 b_6 b_5 b_4 b_3 b_2 b_1 = 1001111$.

Now, let us find the check bits.

$$c_1 = b_7 \oplus b_5 \oplus b_3 \oplus b_1 = 1 \oplus 0 \oplus 1 \oplus 1 = 1$$

$$c_2 = b_7 \oplus b_6 \oplus b_3 \oplus b_2 = 1 \oplus 0 \oplus 1 \oplus 1 = 1$$

$$c_3 = b_7 \oplus b_6 \oplus b_5 \oplus b_4 = 1 \oplus 0 \oplus 0 \oplus 1 = 0$$

The decimal value of check bits gives the position of error in received Hamming code.

$$c_3 c_2 c_1 = (011)_2 = (3)_{10}$$

Therefore, the error present in third bit (b_3) of Hamming code. Just complement the value present in that bit and remove parity bits in order to get the original binary code.

Example

1001 message

$$2^k \geq n + k + 1$$

1	0	0		1		
D_7	D_6	D_5	P_4	D_3	P_2	P_1

$P_1 = D_3, D_5, D_7$ $\quad P_1 = 0$

0 1 0 1

$P_2 = D_3, D_6, D_7$

0 1 0 1

$P_4 = D_5, D_6, D_7$

1 0 0 1

$1001 \longrightarrow 1001100$ ✓

message 1 0 0 1

after hamming Code

1 0 0 1 1 0 0

after change bit

1 1 0 1 1 0 0

1	1	0	1	1	0	0
D_7	D_6	D_5	P_4	D_3	P_2	P_1

$P_1 = D_3, D_5, D_7$ $\quad P_1 = 0$

0 1 0 1 ✓

$P_2 = D_3, D_6, D_7$ × $\quad P_2 = 1$

0 1 1 1

$P_4 = D_5, D_6, D_7$ ×

1 0 1 1 $\quad P_4 = 1$

$P_4 P_2 P_1 = 110 = 6$

1 1 0 1 1 0 0

$\uparrow$ 6th