Relation

What is relation

Relation or Binary relation R from set A to B is a subset of AxB which can be defined as

aRb ↔ (a,b) € R ↔ R(a,b).

A Binary relation R on a single set A is defined as a subset of AxA. For two distinct set, A and B with cardinalities m and n, the maximum cardinality of the relation R from A to B is mn.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then −

- The **domain** of R, Dom(R), is the set $\{x \mid (x, y) \in R \; for \; some \; y \; in \; B\}$
- The **range** of R, Ran(R), is the set $\{y \mid (x, y) \in R \; for \; some \; x \; in \; A\}$

Let, $A = \{1, 2, 3, 9\}$ and $B = \{1, 3, 7\}$

- Case 1 – If relation R is 'equal to' then $R = \{(1, 1), (3, 3)\}$

  Dom(R) = $\{1, 3\}, Ran(R) = \{1, 3\}$

- Case 2 – If relation R is 'less than' then $R = \{(1, 3), (1, 7), (2, 3), (2, 7)\}$

  Dom(R) = $\{1, 2\}, Ran(R) = \{3, 7\}$

- Case 3 – If relation R is 'greater than' then $R = \{(2, 1), (9, 1), (9, 3), (9, 7)\}$

  Dom(R) = $\{2, 9\}, Ran(R) = \{1, 3, 7\}$

Representation of Relations

- Matrices
- Directed Graphs

Matrix Representation

$$A = \{a_1, a_2, a_3 \ldots a_m\}$$

$$B = \{b_1, b_2, \ldots b_n\}$$

M_R : $m \times n$ matrix

$$m_{ij} = 1 \quad ; \quad \text{if } (a_i, b_j) \in R$$

$$= 0 \quad ; \quad \text{if } (a_i, b_j) \notin R$$

Matrix representation of the relations R_1 and R_2 is given by ,

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \quad M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. If matrix representation of R_1 is $[a_{ij}]$ and matrix representation of R_2 is $[b_{ij}]$. Then the matrix representation of $R_1 \cup R_2$ is $[max\{a_{ij}, b_{ij}\}]$.

Hence for the given two relation matrix representation of $R_1 \cup R_2$ of given two relation is ,

$$M_{R_1 \cup R_2} = \begin{bmatrix} \max\{0,0\} & \max\{1,1\} & \max\{0,0\} \\ \max\{1,0\} & \max\{1,1\} & \max\{1,1\} \\ \max\{1,1\} & \max\{0,1\} & \max\{0,1\} \end{bmatrix}$$

$$\Rightarrow M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence the required matrix for the relation $R_1 \cup R_2$ is ,

Answer : $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Representing Relations Using Matrices

- **Example 2**: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ Which ordered pairs are in the relation R represented by the matrix:

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

  - **Solution:**

  $R = \{(a_1,b_2), (a_2,b_1),(a_2,b_3), (a_2,b_4),(a_3,b_1), (a_3,b_3), (a_3,b_5)\}$

Composition of Relations -

Let A, B and C be sets.

Let R is the relation from A to B i.e $R \subseteq A \times B$

S is the relation from B to C i.e $S \subseteq B \times C$

The composition of R and S, denoted by RoS, where

$$RoS = \{(a,c) \in A \times C : \text{for some } b \in B, (a,b) \in R \text{ and } (b,c) \in S\}$$

Eg:- let $A=\{1,2,3\}$, $B=\{p,q,r\}$, $C=\{x,y,z\}$

$R=\{(1,p),(1,r),(2,p),(2,q)\}$

$S=\{(p,y),(q,x),(q,y),(r,z)\}$

compute RoS.

Given $R=\{(1,p),(1,r),(2,p),(2,q)\}$

$S=\{(p,y),(q,x),(q,y),(r,z)\}$

$RoS=\{(1,y),(1,z),(2,y),(2,x)\}$

Sol^n: Given $R=\{(1,2),(3,4),(2,2)\}$

$S=\{(4,2),(2,5),(3,1),(1,3)\}$

(i) $R \circ S=\{(1,5),(3,2),(2,5)\}$

(ii) $S \circ R=\{(4,2),(3,2),(1,4)\}$

(iii) $R=\{(1,2),(3,4),(2,2)\}$

$S \circ R=\{(4,2),(3,2),(1,4)\}$

$R \circ (S \circ R)=\{(3,2)\}$

$R=\{(1,2),(3,4),(2,2)\}$

$R=\{(1,2),(3,4),(2,2)\}$

(iv) $R \circ R=R^2$

$=\{(1,2),(2,2)\}$

(v) $S \circ S=S^2$

$S=\{(4,2),(2,5),(3,1),(1,3)\}$

$S=\{(4,2),(2,5),(3,1),(1,3)\}$

$S \circ S=\{(4,5),(3,3),(1,1)\}$

matrix representation

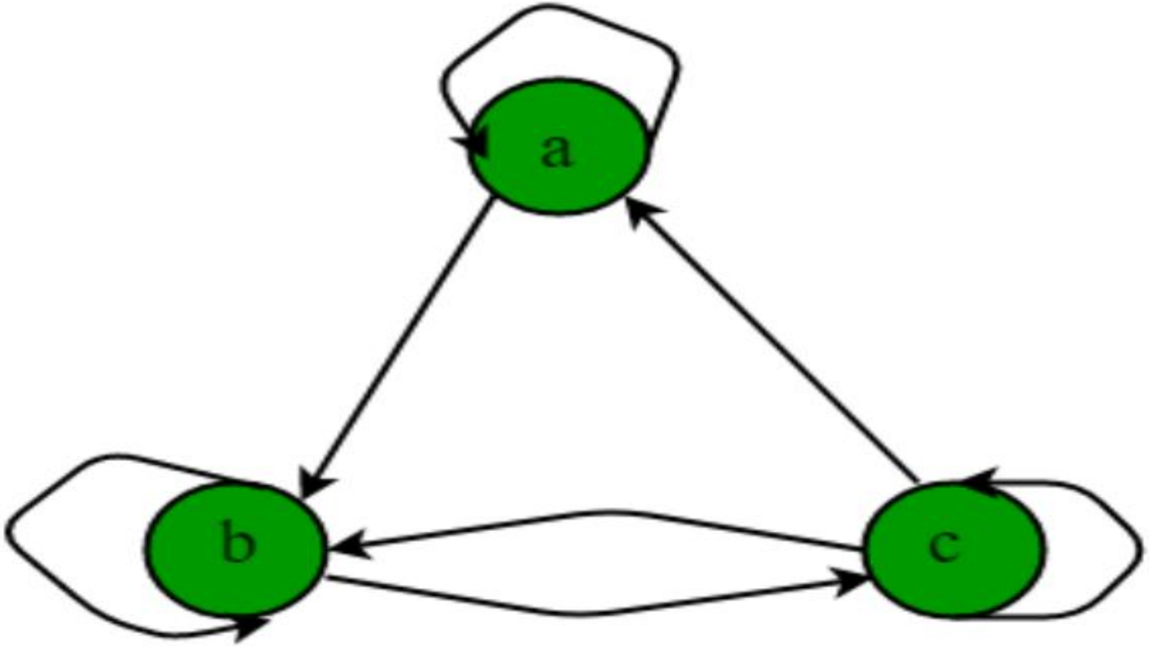
- A relation can be represented by an $m \times n$ zero-one matrix.
- Ex 7.17. Consider $\Re_1=\{(1, x), (2, x), (3, y), (3, z)\}$ and $\Re_2=\{(w, 5), (x, 6)\}$. What is $\Re_1 \circ \Re_2$?

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M(\mathcal{R}_1 \circ \mathcal{R}_2).$$

Matrix as Directed graph

The directed graph of relation R = {(a,a),(a,b),(b,b),(b,c),(c,c),(c,b),(c,a)} is represented as :



Relations

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Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$, $R_4 = \{(a,b) \mid a = b\}$,
$R_2 = \{(a,b) \mid a > b\}$, $R_5 = \{(a,b) \mid a = b + 1\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$, $R_6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs

$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Note that these relations are on an infinite set and each of these relations is an infinite set. Checking the conditions that define each relation, we see that

$(1,1)$ is in R_1, R_3, R_4 , and R_6:
$(1,2)$ is in R_1 and R_6:
$(2,1)$ is in R_2, R_5, and R_6:
$(1, -1)$ is in R_2, R_3, and R_6 :
$(2,2)$ is in R_1, R_3, and R_4.

Properties of relations

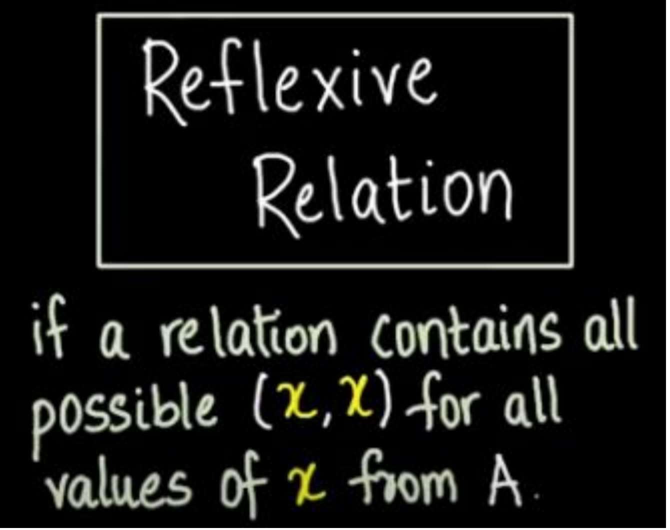
Let R be a relation on E, and let $x,y,z \in E$.

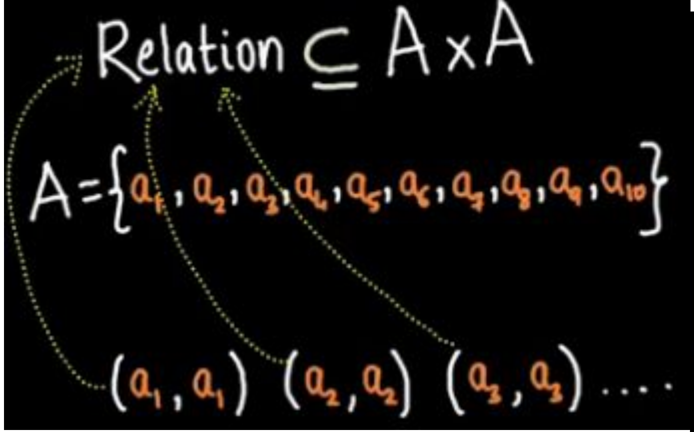
A relation R is ...	if ...		A relation R is ...	if ...
reflexive	xRx		***irreflexive***	xRy implies $x \neq y$
symmetric	xRy implies yRx		***antisymmetric***	xRy and yRx implies $x=y$
transitive	xRy and yRz implies xRz			

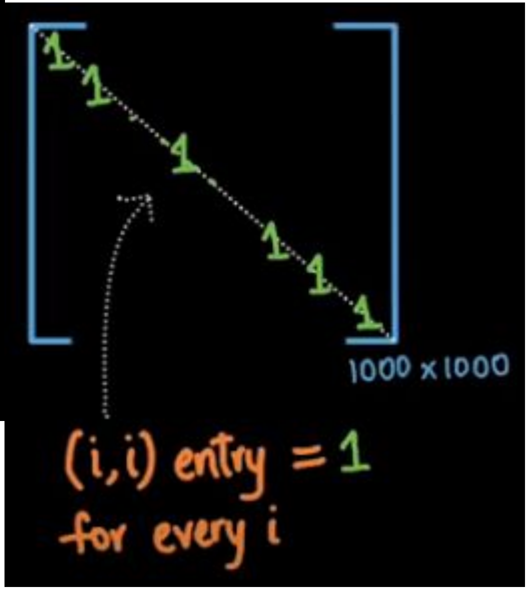
Examples using $=$, $<$, and $\leq$ on integers:

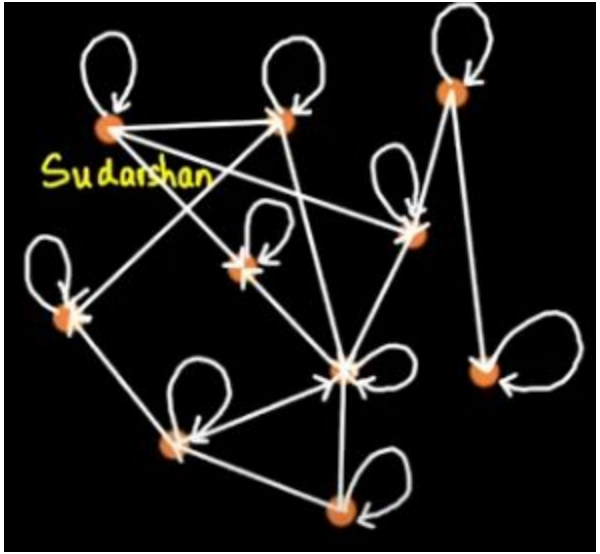
- $=$ is reflexive ($2=2$)
- $=$ is symmetric ($x=2$ implies $2=x$)
- $<$ is transitive ($2<3$ and $3<5$ implies $2<5$)
- $<$ is irreflexive ($2<3$ implies $2 \neq 3$)
- $\leq$ is antisymmetric ($x \leq y$ and $y \leq x$ implies $x=y$)

Reflexive Relation

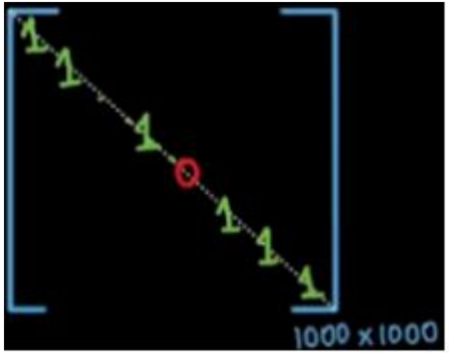








Non diagonal entries may or may not be 1 but all diagonal entries have to be 1



Not a reflexive

Reflexive Relation

Q.1: A relation R is on set A (set of all integers) is defined by "x R y if and only if 2x + 3y is divisible by 5", for all x, y ∈ A. Check if R is a reflexive relation on A.

Solution: Let us consider $x \in A$.

Now $2x + 3x = 5x$, which is divisible by 5.

Therefore, xRx holds for all 'x' in A

Hence, R is reflexive.

Reflexive Relation

Q.2: A relation R is defined on the set of all real numbers N by ‘a R b’ if and only if |a-b| ≤ b, for a, b ∈ N. Show that the R is not reflexive relation.

Solution: The relation is not reflexive if a = -2 ∈ R

But |a – a| = 0 which is not less than -2(= a).

Therefore, the relation R is not reflexive.

Reflexive Relation

Q.3: A relation R on the set A by “x R y if x – y is divisible by 5” for x, y $\in$ A. Check if R is a reflexive relation on set A.

Solution: Let us consider, x $\in$ A.

Then x – x is divisible by 5.

Since x R x holds for all x in A

Therefore, R is reflexive.

Reflexive Relation

Q.4: Consider the set A in which a relation R is defined by ‘x R y if and only if x + 3y is divisible by 4, for x, y ∈ A. Show that R is a reflexive relation on set A.

Solution: Let us consider x ∈ A.

So, x + 3x = 4x, is divisible by 4.

Since x R x holds for all x in A.

Therefore, R is reflexive.

Empty relation is reflexive?

For a relation to be reflexive: For all elements in A, they should be related to themselves. $(x \; R \; x)$. Now in this case there are no elements in the Relation and as A is non-empty no element is related to itself hence the empty relation is not reflexive.

Reflexive Relation

1\. $A = \{1, 2, 3, 4, 5\}$ $R \subseteq A \times A$

$R = \{\boxed{(1,1)}, (1,4), \boxed{(2,2)}, (2,3), \boxed{(3,3)}, (3,1), \boxed{(4,4)}, \boxed{(5,5)}, (5,1)\}$

Is R a reflexive relation?

Ans: R is reflexive

$(a, a) \in$

Reflexive Relation

2. $R = \{(a,b) \mid a, b \in \mathbb{N},\ b = a^2\}$

Is R reflexive?

$R = \{(1,1), (2,4), (3,9), (4,16), (5,25), \ldots\ldots\}$

$(1,1) \in R$

$(2,2) \notin R$

R is not reflexive.

Example: **The following relations on the integers are reflexive:**

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Reflexive Relation

$$S = \{a_1, a_2, \ldots\ldots\ldots, a_n\}$$

What are the total number of reflexive relations on S?

$$R \subseteq S \times S$$

Not all subsets are valid reflexive relations.

$$\underbrace{\{(a_1, a_1), (a_2, a_2), \ldots\ldots\ldots\ldots, (a_n, a_n)\}}_{n \text{ elements}}$$

Reflexive Relation

$A = \{1, 2, 3\}$.

Maximum size of a R.

$R \subseteq A \times A$.

$|A \times A| = 9$.

$R = \{\underbrace{(1,1), (2,2), (3,3)}_{3}, \underbrace{\ldots}_{\text{6 potential Candidates to be Included}}\}$.

Reflexive

6 potential Candidates to be Included → Present in R / Not Pres in R.

The total number of reflexive relations on a finite set having n elements is

__________.

Consider a set A with n elements

Say A ={1, 2, ... n^{-1}, n }

out of n^2 elements n elements are compulsory for relation to be reflexive.

i.e (1, 1) (2, 2) (3, 3) .... (n, n)

and for remaining $n^2 - n$ elements, we have choice of filling i.e either they are present or absent.

Hence, Total number of reflexive relation are 2^{n^2-n}.

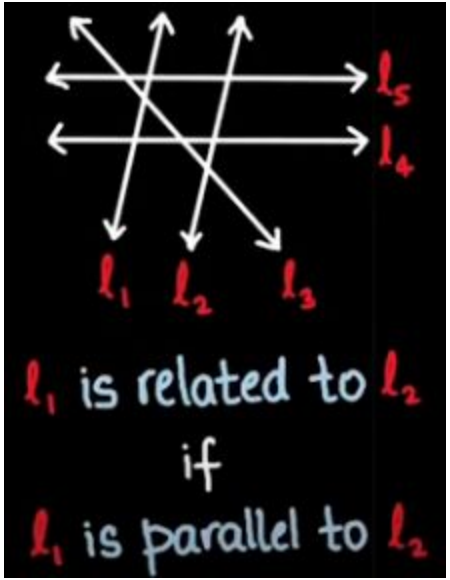
A relation R is reflexive if the matrix diagonal elements are 1.

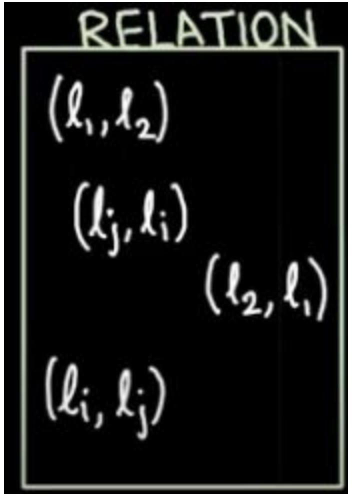
$$\left\{\begin{matrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{matrix}\right\}$$

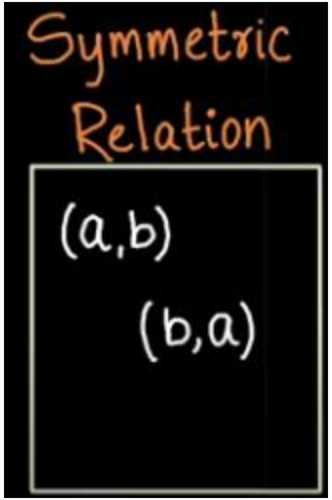
If we take a closer look the matrix, we can notice that the size of matrix is n^2. The n diagonal entries are fixed. For remaining $n^2 - n$ entries, we have choice to either fill 0 or 1. So there are total $\mathbf{2^{n(n-1)}}$ *ways of filling the matrix.*

Symmetric Relation

Symmetric Relation







Symmetric Relation

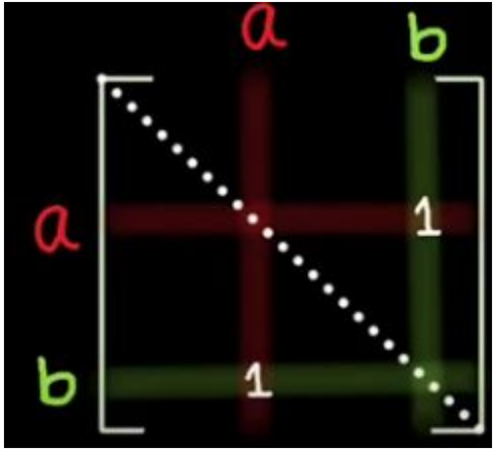
3. Consider a relation natural numbers.

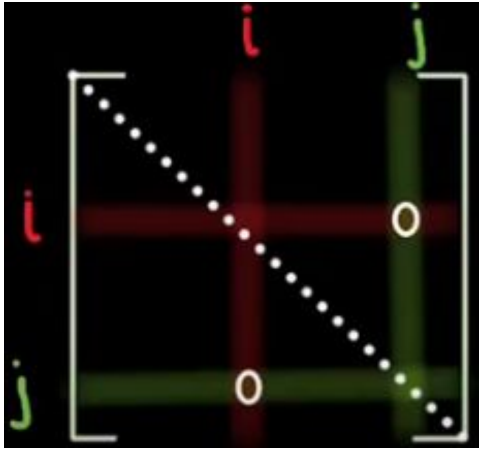
$R = \{(a,b) \mid a, b \in \mathbb{N},\ a \cdot b = 14\}$. Is R symmetric?

$R = \{\underline{(1,14)}, \underline{(2,7)}, \underline{(7,2)}, \underline{(14,1)}\}$

$\therefore R$ is symmetric.

If a relation is symmetric, then it is symmetric across the diagonal





Definition: A relation R on a set A is **symmetric** if whenever aRb then bRa, i.e., if whenever $(a, b) \in R$ then $(b, a) \in R$.

Thus R is **not symmetric** if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Example: Consider the following relations on the set $A = \{1, 2, 3\}$:

$$R_1 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3)\}$$
$$R_2 = \{(1, 1), (1, 2), (2, 2)\}$$

Determine which relation is symmetric.

Example 1: Suppose R is a relation on a set A where A = {1, 2, 3} and R = {(1,1), (1,2), (1,3), (2,3), (3,1)}. Check if R is a symmetric relation.

Solution: As we can see $(1, 2) \in R$. For R to be symmetric (2, 1) should be in R but $(2, 1) \notin R$.

Hence, R is not a symmetric relation.

Answer: R = {(1,1), (1,2), (1,3), (2,3), (3,1)} is not a symmetric relation.

Example 2: Suppose R is a relation on a set A where A = {a, b, c} and R = {(a, a), (a, b), (a, c), (b, c), (c, a)}. Determine the elements which should be in R to make R a symmetric relation.

Solution: To make R a symmetric relation, we will check for each element in R.

$(a, a) \in R \Rightarrow (a, a) \in R$

$(a, b) \in R \Rightarrow (b, a) \in R$, but $(b, a) \notin R$

$(a, c) \in R \Rightarrow (c, a) \in R$

$(b, c) \in R \Rightarrow (c, b) \in R$, but $(c, b) \notin R$

Hence, (b, a) and (c, b) should belong to R to make R a symmetric relation.

Answer: (b, a) and (c, b) should belong to R to make R a symmetric relation.

Q.1 A relation R is defined on the set of integers as:

$(x, y) \in R$ if and only if x is a multiple of y.

Is R a symmetric relation?

- ○ Yes
- ◉ No ✔

Example 2: Let Z be the set of two female kids in a family and R be a relation defined on the set Z as;

R = "is the sister of".

Verify whether R is symmetric or not.

Given a number n, find out the number of Symmetric Relations on a set of first n natural numbers {1, 2, ..n}.

Examples:

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Input  : n = 2
Output : 8

Given set is {1, 2}. Below are all symmetric relation.
{}
{(1, 1)},
{(2, 2)},
{(1, 1), (2, 2)},
{(1, 2), (2, 1)}
{(1, 1), (1, 2), (2, 1)},
{(2, 2), (1, 2), (2, 1)},
{(1, 1), (2, 2), (2, 1), (1, 2)}
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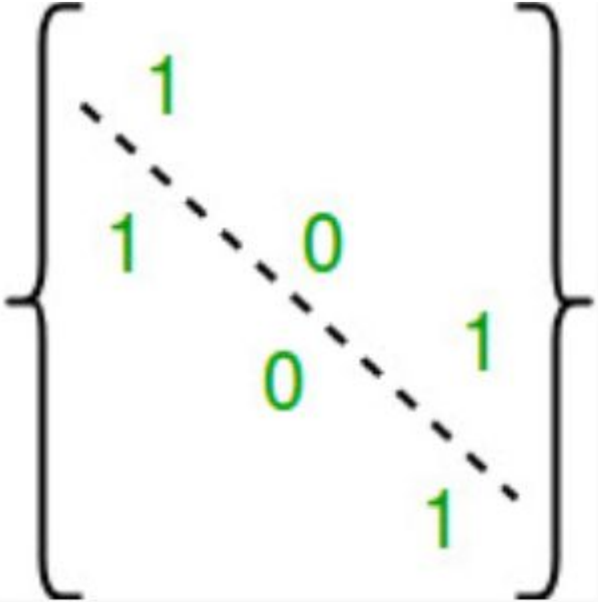


There are n diagonal values, total possible combination of diagonal values = 2^n

There are $n^2 - n$ non-diagonal values. We can only choose different value for half of them, because when we choose a value for cell (i, j), cell (j, i) gets same value.

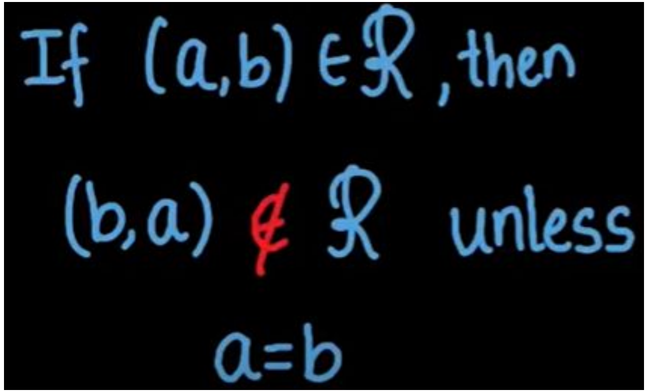
So combination of non-diagonal values = $2^{(n2 - n)/2}$

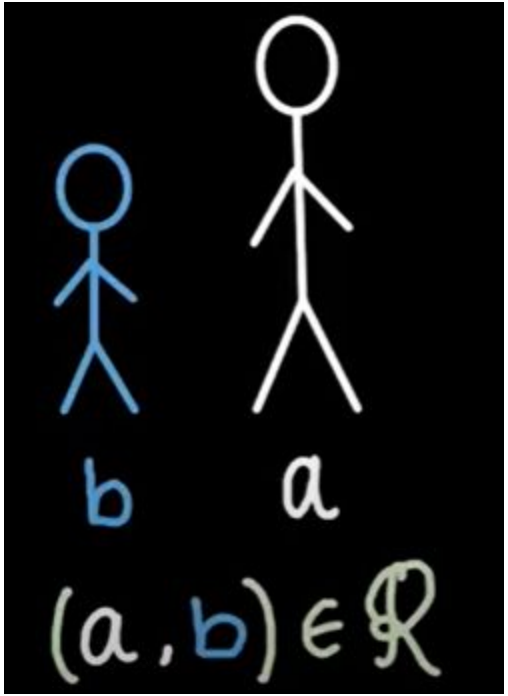
Overall combination = $2^n * 2^{(n2 - n)/2}$ = $\mathbf{2^{\underline{n(n+1)/2}}}$

Total number of symmetric relations is $\mathbf{2^{n(n+1)/2}}$.

Anti Symmetric Relation

Height of any 2 students in a class is an example of anti symmetric relation





Consider a relation R on $\mathbb{N}$.

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}$$

$$R = \{(1,2), (2,3), (3,4), (4,5), \ldots\ldots\ldots,\}$$

$(1,2) \in R \qquad (2,1) \notin R$

$n = 2 \qquad n+1 = 3$

If $(3,4) \in R \quad (4,3) \notin R$

If $(n, n+1) \in R, \quad (n+1, n) \notin R$

$\therefore$ R is anti-symmetric

Antisymmetric is not same as not symmetric

$A = \{1, 2, 3, 4, 5\}$ $R = \{(1,2), (2,1), (3,4)\}$

R is not symmetric because $(3,4) \in R$

but $(4,3) \notin R$

R is not antisymmetric. $(1,2) \in R$ but $(2,1) \in R$

Antisymmetric Relation

In set theory, the relation R is said to be antisymmetric on a set A, if xRy and yRx hold when $x = y$. Or it can be defined as, relation R is antisymmetric if either $(x,y) \notin R$ or $(y,x) \notin R$ whenever $x \neq y$.

A relation R is not antisymmetric if there exist $x,y \in A$ such that $(x,y) \in R$ and $(y,x) \in R$ but $x \neq y$.

Note: If a relation is not symmetric that does not mean it is antisymmetric.

Q.1: Which of these are antisymmetric?

(i) R = {(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

(ii) R = {(1,1),(1,3),(3,1)}

(iii) R = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

Solution:

(i) R is not antisymmetric here because of $(1,2) \in R$ and $(2,1) \in R$, but $1 \neq 2$.

(ii) R is not antisymmetric here because of $(1,3) \in R$ and $(3,1) \in R$, but $1 \neq 3$.

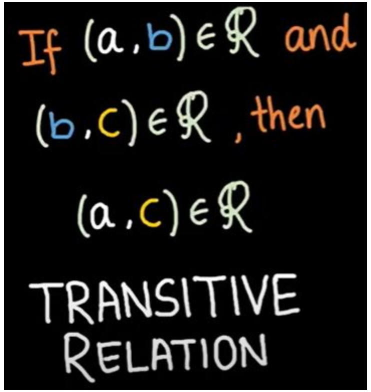
(iii) R is not antisymmetric here because of $(1,2) \in R$ and $(2,1) \in R$, but $1 \neq 2$ and also $(1,4) \in R$ and $(4,1) \in R$ but $1 \neq 4$.

Q.2: If A = {1,2,3,4} and R is the relation on set A, then find the antisymmetric relation on set A.

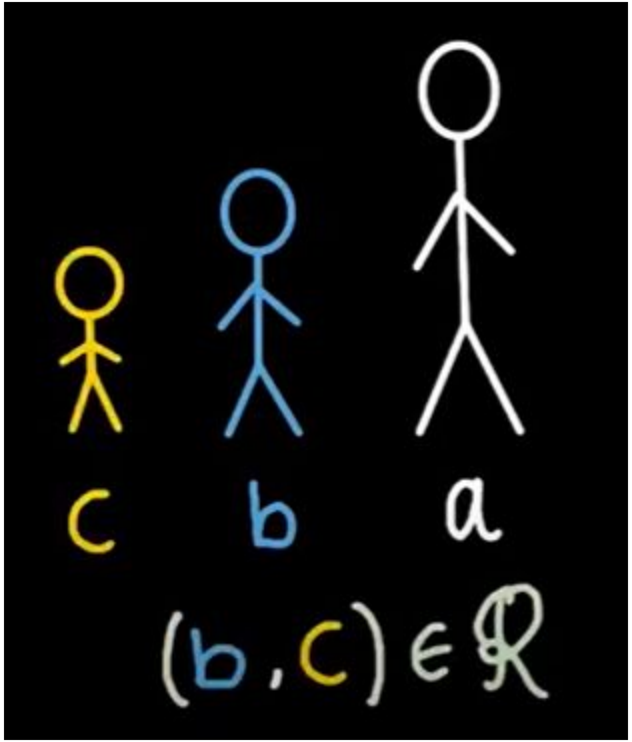
Solution: The antisymmetric relation on set A = {1,2,3,4} will be;

R = {(1,1), (2,2),(3,3),(4,4)}

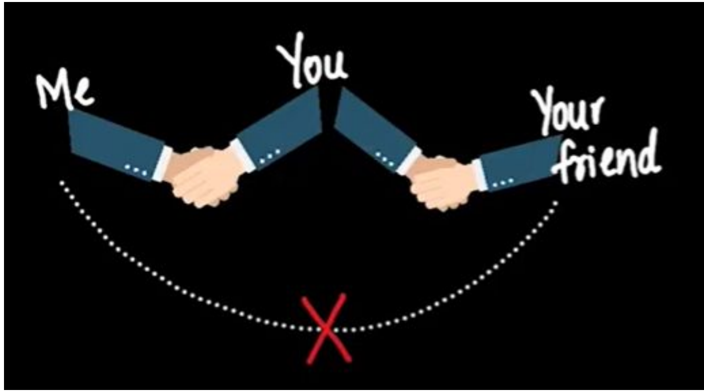
Transitive Relation



Transitive Relation



Non Transitive Relation



1. Consider a relation on the set of integers as

$$R = \{(a,b) \mid a+b=0\}.$$

$$R = \{(0,0),(1,-1),(-1,1),(2,-2),(-2,2),\ldots\}$$

Is R transitive?

Ans: R is symmetric.

$(1,-1) \in R$ $(-1,1) \in R$ $(1,1) \in R$? NO

$(1,1) \notin R$

R is not transitive.

2. $R = \{(a,b) \mid \sin a = \sin b\}$

$\sin 0 = 0 \quad \sin \pi = 0$

$(0,\pi) \in R \quad (0,2\pi) \in R \quad (0,n\pi) \in R$

$\sin \pi/2 = 1, \quad \sin 3\pi/2 = -1 \quad (\pi/2, 3\pi/2) \notin R$

$(a,b) \in R \quad (b,c) \in R$

$\sin a = \sin b \quad \sin b = \sin c$

$\sin a = \sin c \quad (a,c) \in R$

$\therefore R$ is transitive.

Equivalence Relation

A relation R on a set A is an equivalence relation iff R is reflexive, symmetric, and transitive.

$A = \{0, 1, 2, 3\}$

$R_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$

Is R_1 an equivalence relation?

Yes.

$R_2 = \{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

Is R_2 an equivalence relation?

Is R_2 reflexive?

No. Because $(1, 1)$ is not a member of R_2.

$R_3 = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

Is R_3 an equivalence relation?

Ask yourself: Is R_3 reflexive? Yes

Is R_3 symmetric? Yes

Is R_3 transitive? Yes

$R_4 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Is R_4 an equivalence relation? Ask yourself: Is R_4 reflexive? Yes

Is R_4 symmetric? No

Therefore, R_4 is not an equivalence relation.

$R_5 = \varnothing$ Is R_5 an equivalence relation?

Ask yourself: Is R_5 reflexive? No

Is R_5 symmetric?

Therefore, R_5 is not an equivalence relation.

$R_6 = AXA$ Is R_6 an equivalence relation?

Ask yourself: Is R_6 reflexive? Yes

Is R_6 symmetric? Yes

Is R_6 transitive? Yes

Therefore, R_6 is an equivalence relation.

Closure of a Relation

Definition: Reflexive closure of a binary relation R on a set A is the smallest reflexive relation of the set A that contains R.
Reflexive closure of R is usually denoted by R_r^+.

$$R_r^+ = R \cup \{(a, a) \mid a \in A\}$$

Let say we have a binary relation R.
$R = \{(1, 1), (2, 2), (2, 3)\}$ defined on a set $A = \{1, 2, 3\}$.

The relation R is not reflexive.
Smallest reflexive relation that contains R must include the ordered pair (3, 3).

$$R_{New} = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$$

Problem: Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the reflexive closure of R.

Solution: $R = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$

$A = \{0, 1, 2, 3\}$

Reflexive closure of R

$R_r^+ = R \cup \{(a, a) \mid a \in A\}$

$R_r^+ = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (0, 0), (3, 3)\}$

Transitive closure

Find the transitive closure of R defined in Example 3.
Solution:

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we have $n = 4$.

To find W_1, $k = 1$. We can see that W_0 has 1's in column 1 at location 2, and in row 1 at location 2. Thus W_1 has a new 1 at position (2, 2).

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_2, $k = 2$. W_1 has 1's in column 2 at locations 1 and 2, and in row 2 at locations 1, 2 and 3. So the new 1's would go to positions (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (2, 3) (if not already there).

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_2, $k = 2$. W_1 has 1's in column 2 at locations 1 and 2, and in row 2 at locations 1, 2 and 3. So the new 1's would go to positions (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (2, 3) (if not already there).

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_3, $k = 3$. W_2 has 1's in column 3 at locations 1 and 2, and in row 3 at location 4. So the new 1's would come at positions (1, 4) and (2, 4) (if not already there).

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_4, $k = 4$. W_3 has 1's in column 4 at locations 1, 2 and 3 but no 1's in row 4. So no new 1's are added. Hence $W_4 = W_3$. This gives us the matrix representation of

Use Warshall's algorithm to find the transitive closure of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \; W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \; W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \; W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \; W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \; W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \; W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Partially Ordered Set - POSET

A binary relation R on a set S is called a **partial ordering**, or partial order if and only if it is:

- Reflexive
- Antisymmetric
- Transitive

“divisibility” is a partial order relation on A

$$R = \{(a,b) \in A \times A \mid a \mid b\} \quad A \in \mathbb{Z}$$

Reflexive: if for all $a \in A$, $(a,a) \in R$	$a \mid a \quad and \quad a \in A$
Antisymmetric: $if\ (a,b) \in R\ and\ (b,a) \in R\ then\ a = b$	$a \mid b \quad then \quad b \mid a \quad only\ if \quad a = b$
Transitive: $if\ (a,b) \in R\ and\ (b,c) \in R\ then\ (a,c) \in R$	$a \mid b \quad and \quad b \mid c \quad then \quad a \mid c$

Partial Order Relations

Now, the three fundamental partial order relations are:

1. Less Than Or Equal To
2. Subset
3. Divisibility

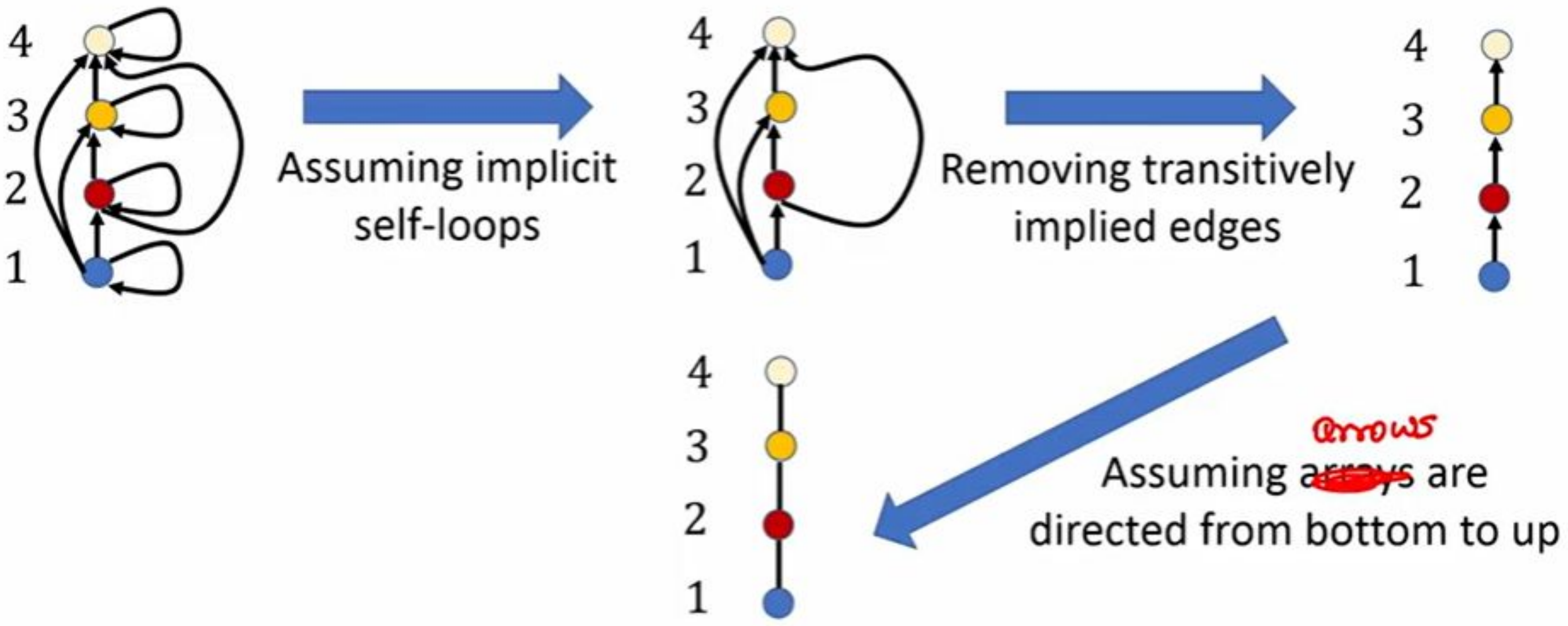
If (S, R) is an arbitrary poset, then it is represented as $(S, \leq)$

- $a \leq b$: represents that $(a, b) \in R$
  - Ex: In the poset$(\mathbb{Z}^+, |)$, we have $2 \leq 4$, but $2 \nleq 3$

- $a < b$: represents that $(a, b) \in R$ and a, b are distinct $(a \neq b)$
  - Ex: In the poset$(\mathbb{Z}^+, |)$, we have $2 < 4$, but $2 \nless 2$

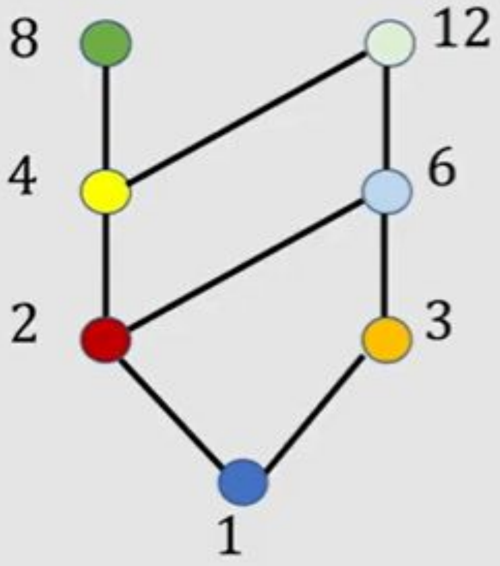
Hasse Diagrams for Representing Posets

- Consider $(\{1, 2, 3, 4\}, \ \leq)$, where $\leq$ is the "less-than or equal-to" relation



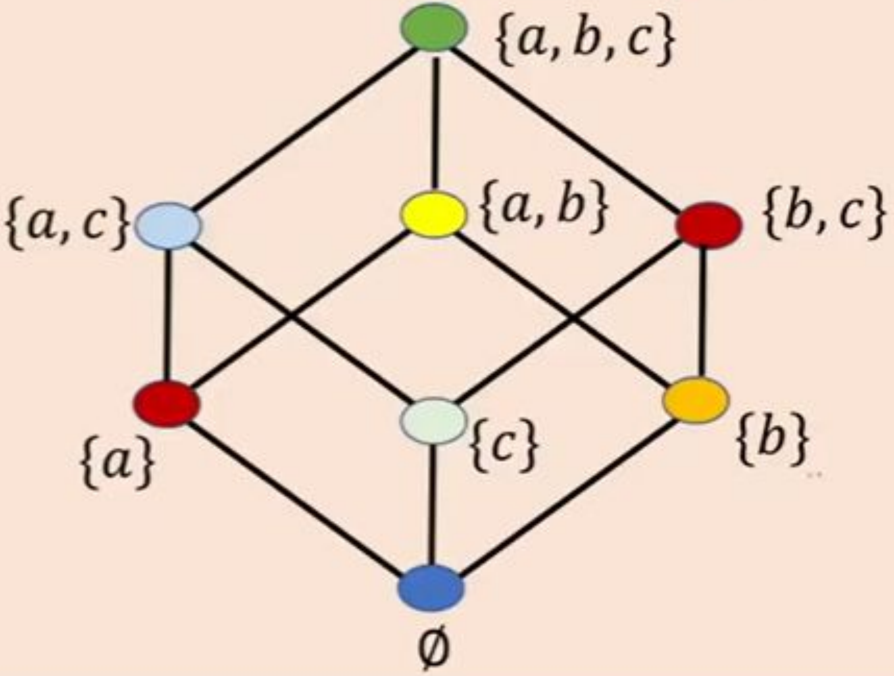
Hasse Diagram: Examples

- ($\{1, 2, 3, 4, 6, 8, 12\}$, $\leq$), where $\leq$ is the "divide" relation



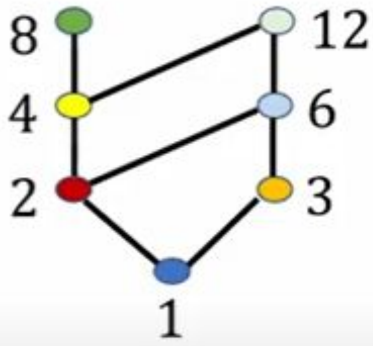
Hasse Diagram: Examples

- $(P(S), \leq)$, where $\leq$ is the "subset" relation and $S = \{a, b, c\}$ $|P(S)| = 8$



Maximal and Minimal Element

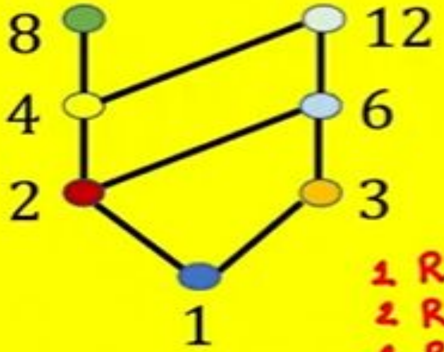
- Let $(S, \leq)$ be an arbitrary poset and $a \in S$
  - a is called a maximal element, if it has no cover
    - There is no $b \in S$, with $a < b$
    - Ex: $8, 12$ are the maximal elements
  - a is called a minimal element, if it covers no element
    - There is no $b \in S$, with $b < a$



- Every poset has at least one maximal and one minimal element
- An element of a poset can be both maximal as well as a minimal element
  - Ex: $(\mathbb{Z}, \leq)$, where $\leq$ is the "equal-to" relationship

Greatest and Least Element

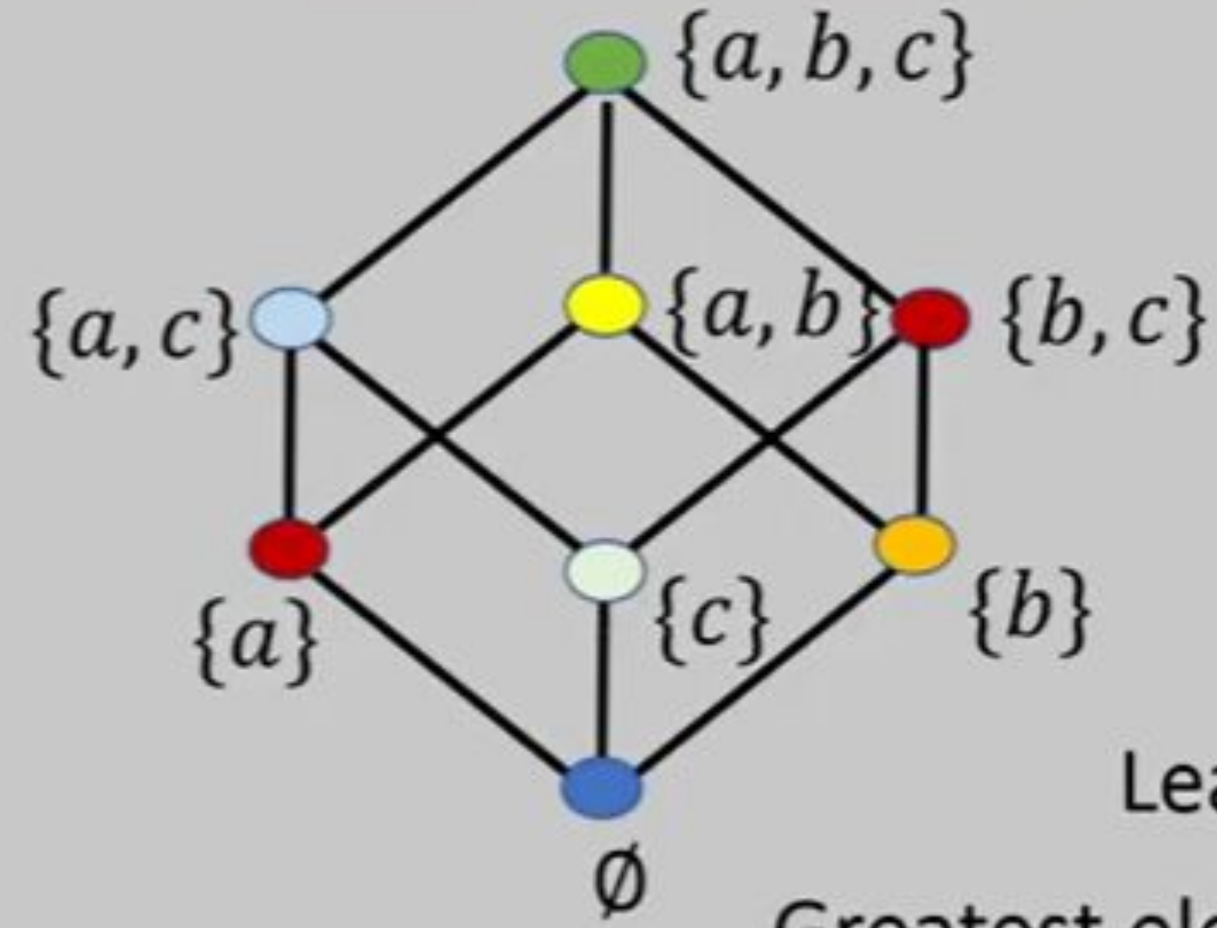
- Let $(S, \leq)$ be an arbitrary poset and $a \in S$
  - a is called the greatest element if $b \leq a$, for every $b \in S$
  - a is called the least element if $a \leq b$, for every $b \in S$



1 R 2
1 R 3
1 R 4
1 R 8
1 R 6
1 R 12

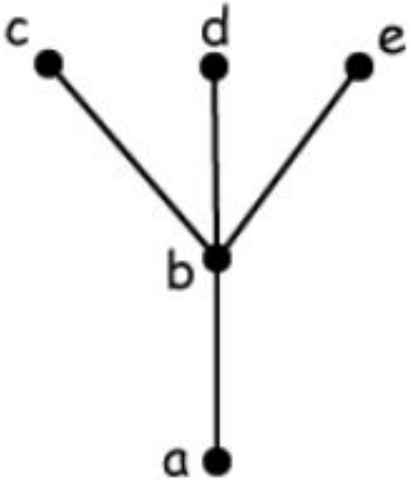
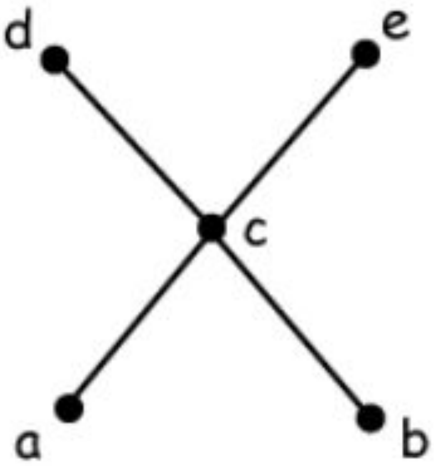
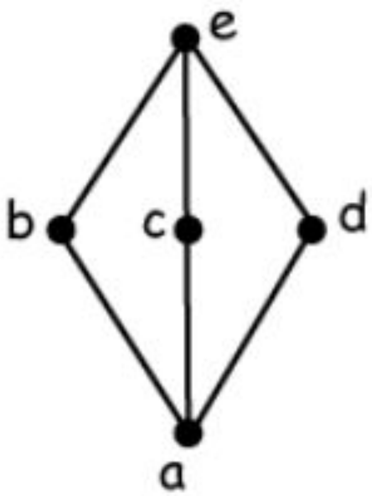
Least element: 1

No greatest element



Least element: $\emptyset$

Greatest element: $\{a, b, c\}$

		
Maximal: *c , d , e*	Maximal: *d , e*	Maximal: *e*
Greatest: *none*	Greatest: *none*	Greatest: *e*
Minimal: *a*	Minimal: *a , b*	Minimal: *a*
Least: *a*	Least: *none*	Least: *a*

Greatest Lower Bound and Least Upper Bound of a subset of a given poset

A poset $\langle\{1, 2, 3, 4, 6, 24, 36, 72\}, |\rangle$, with $|$ being the divides relation find the least upper bound and greatest lower bound of the subset $\{4, 6\}$?

Example 1: Find the greatest lower bound and the least upper bound of {b, d, g} if they exist in the poset with the Hasse diagram shown below:

Solution: Upper bounds of {b, d, g} are g and h because b is related to g, d is related to g and g is related to g.
Also, b is related to h, d is related to h, and g is related to h.

Out of g and h, minimum element is g. Therefore, g is the least upper bound of {b, d, g}.

Solution: Lower bounds of {b, d, g} are a and b.

Out of a and b, greatest element is b. Therefore, b is the greatest lower bound of {b, d, g}.

Example 2: Consider the following Hasse diagram

Find the least upper bound of {a, b, c} and the greatest lower bound of {f, g, h}, if they exist.

Solution:

Upper bounds of {a, b, c} are k, l, and m.

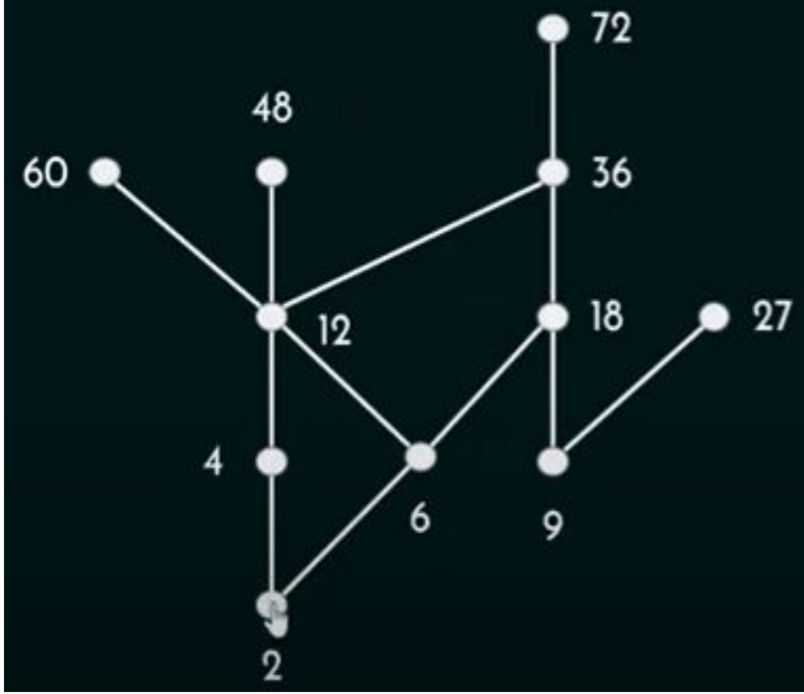
Least upper bound of {a, b, c} is k.

Lower bound of {f, g, h} is Ø.

Greatest lower bound of {f, g, h} is Ø.

Example 3: Find the least upper bound of $\{2, 9\}$ and the greatest lower bound of $\{60, 72\}$ for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$

Solution:



Upper bounds of $\{2, 9\}$ are 18, 36, 72.
Least upper bound of $\{2, 9\}$ is 18.

Lower bounds of $\{60, 72\}$ are 2, 4, 6, 12.
Greatest lower bound of $\{60, 72\}$ is 12.