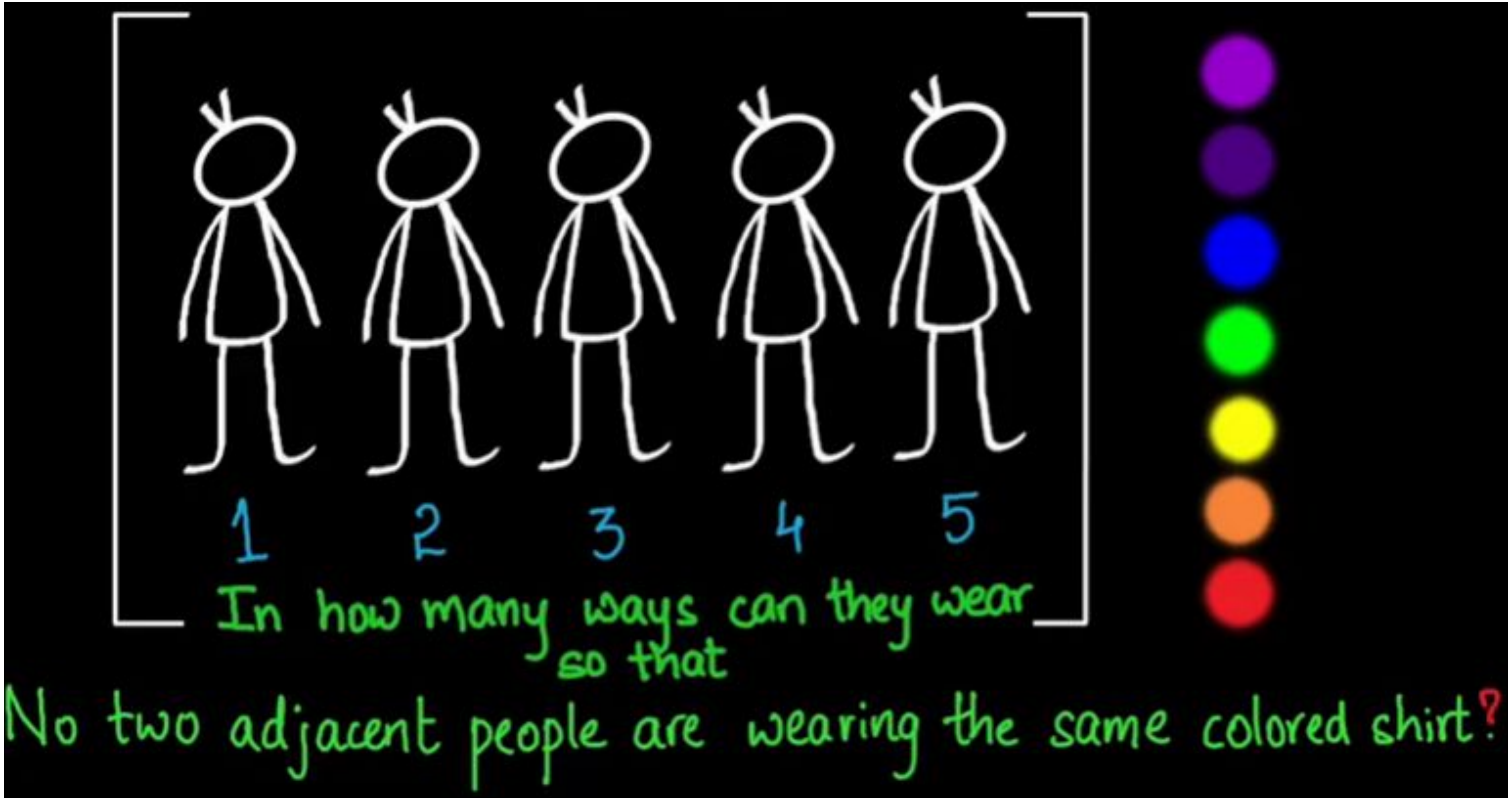
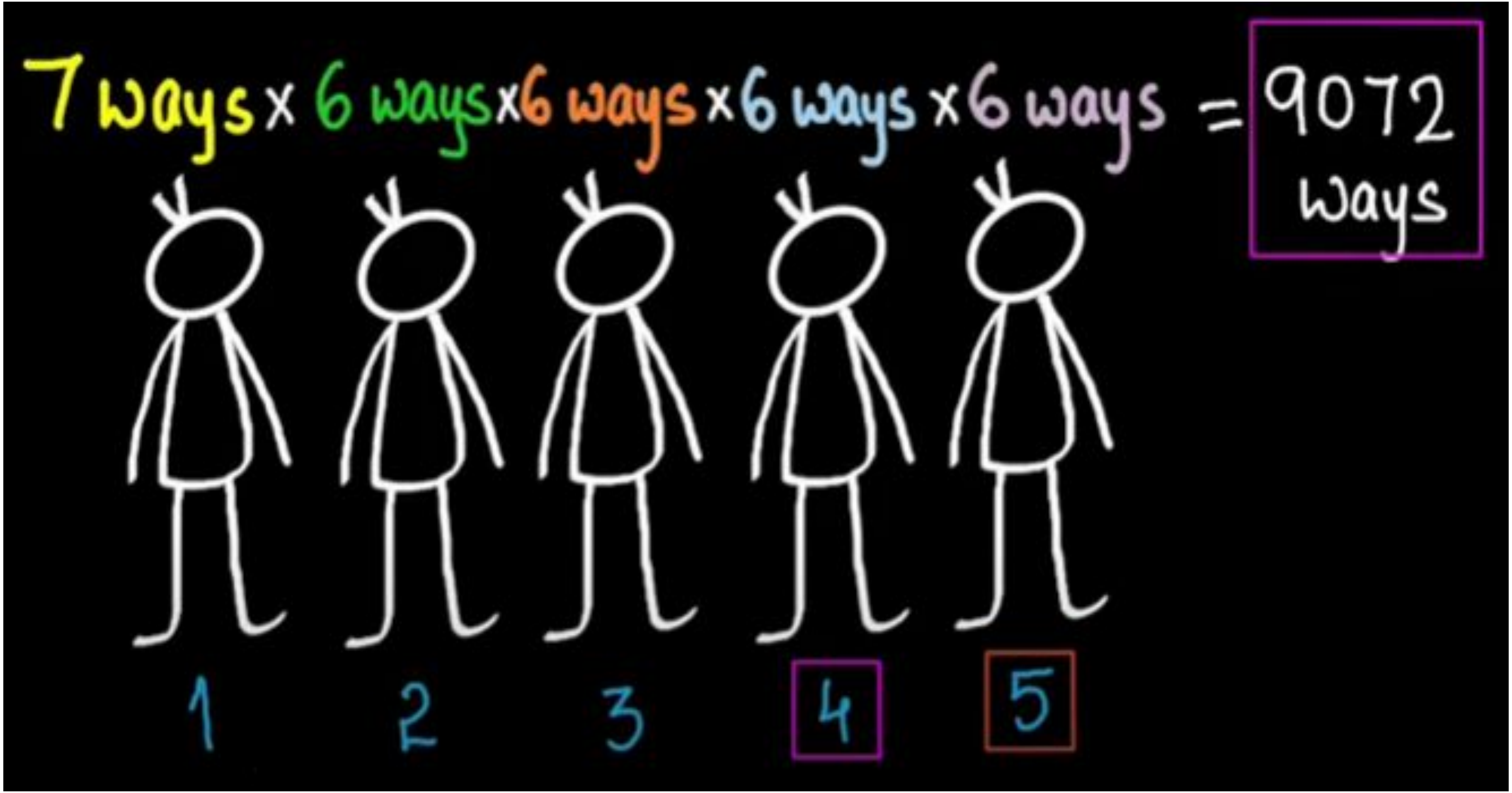
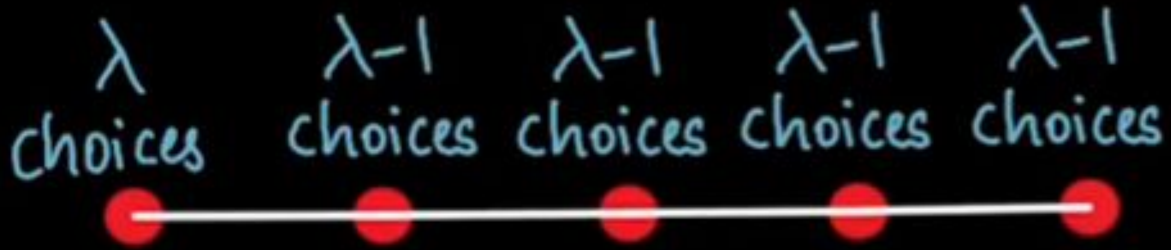
Chromatic Polynomial

Graph Connectivity

Assume that student can not change their position




7 ways x 6 ways x 6 ways x 6 ways x 6 ways = 9072 ways
1
2
3
4
5



In how many ways can you color this graph with λ colors so that no two adjacent have the same color?

$$\boxed{\lambda \times (\lambda-1)^4} \quad \text{for } P_5$$

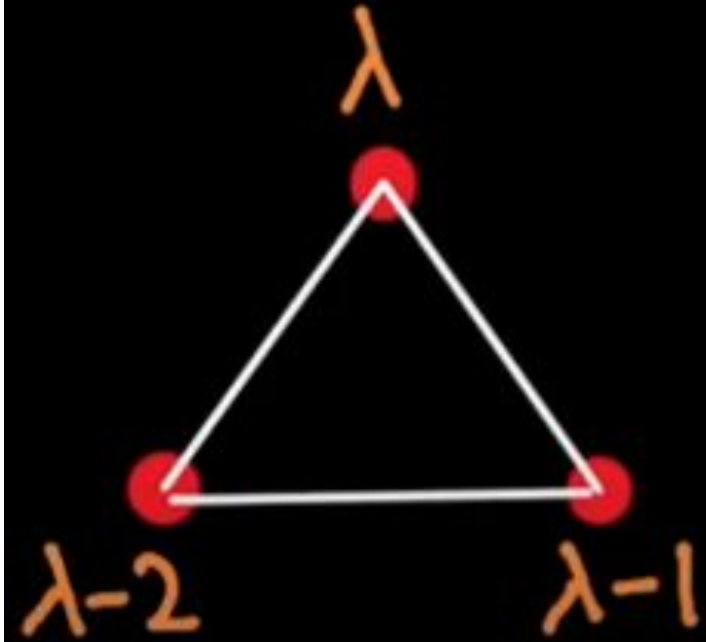
G

In how many ways can you color this graph with λ colors so that no two adjacent have the same color?

Chromatic polynomial

$$C(P_5) = \lambda(\lambda-1)^4$$

Chromatic polynomial of complete graphs

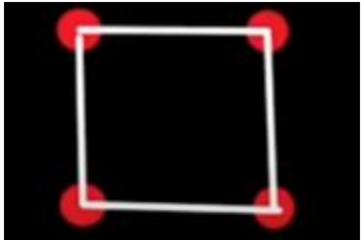


$$C(K_3) = \lambda(\lambda-1)(\lambda-2)$$

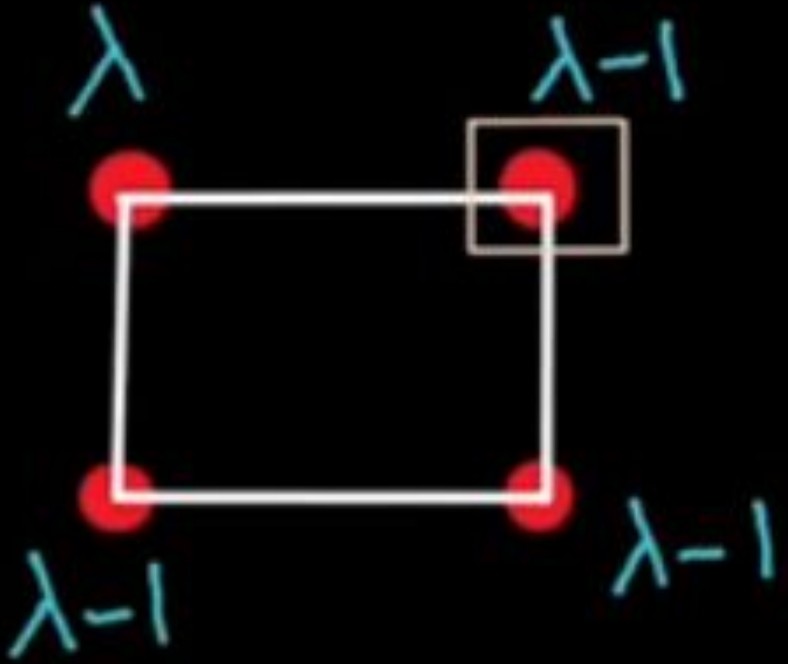
$$C(K_4) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

$$C(K_5) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

K_n $$C(K_n) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

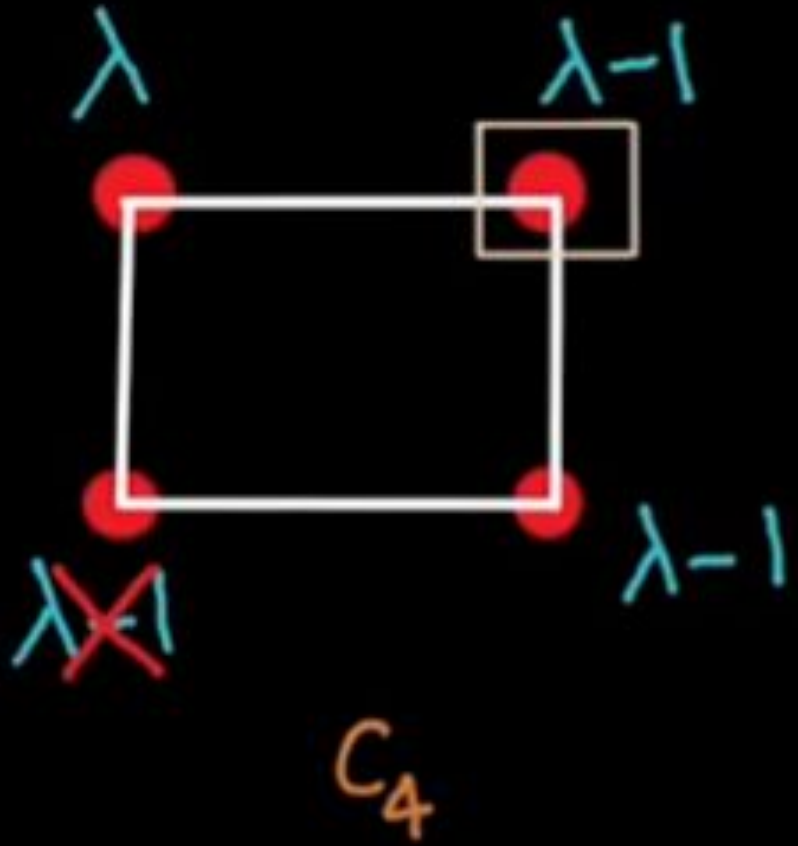


$c(C_4) = ?$

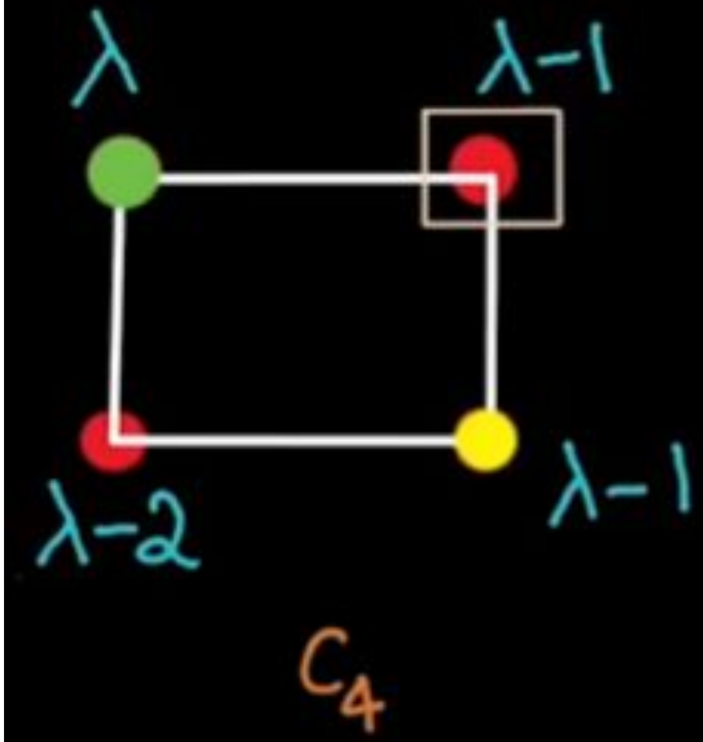


C_4

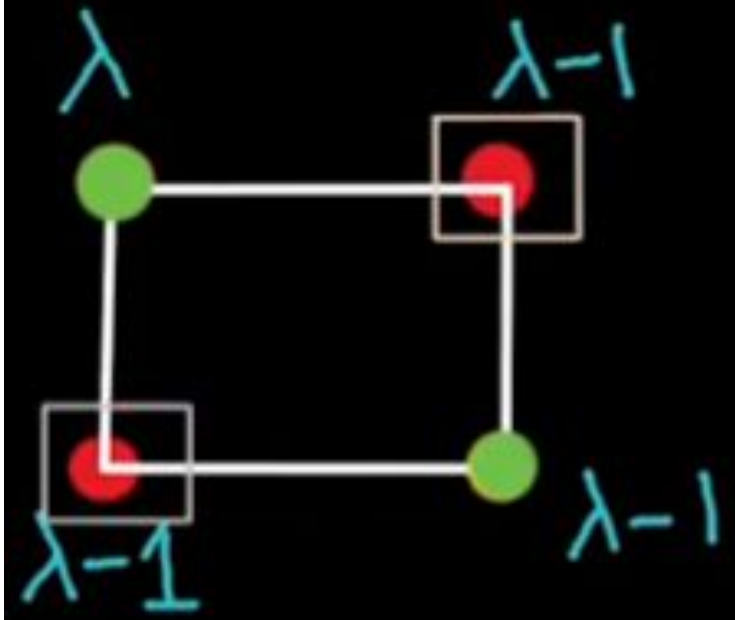
$$C(C_4) = ?$$



$$C(C_4) = ?$$

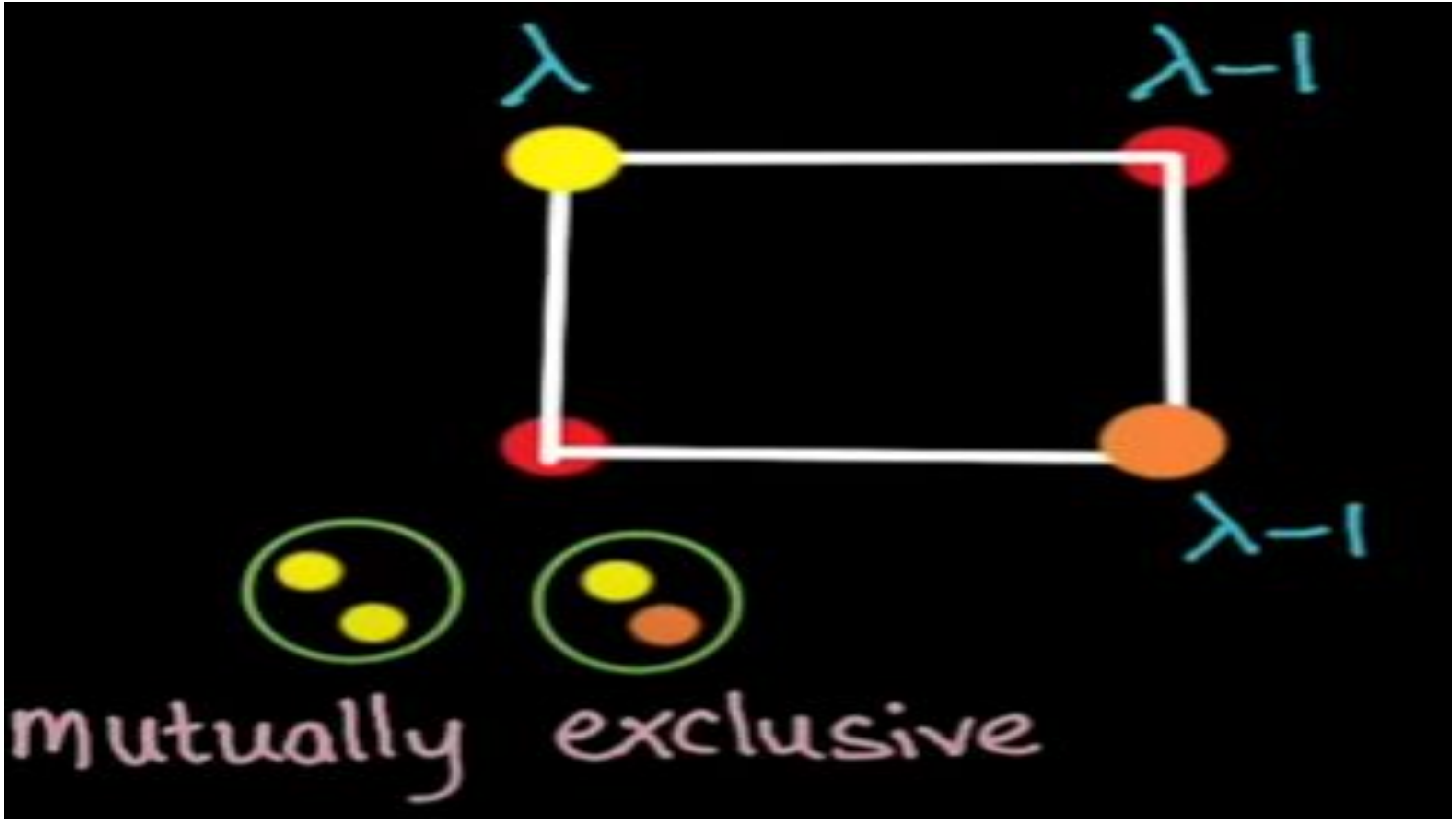


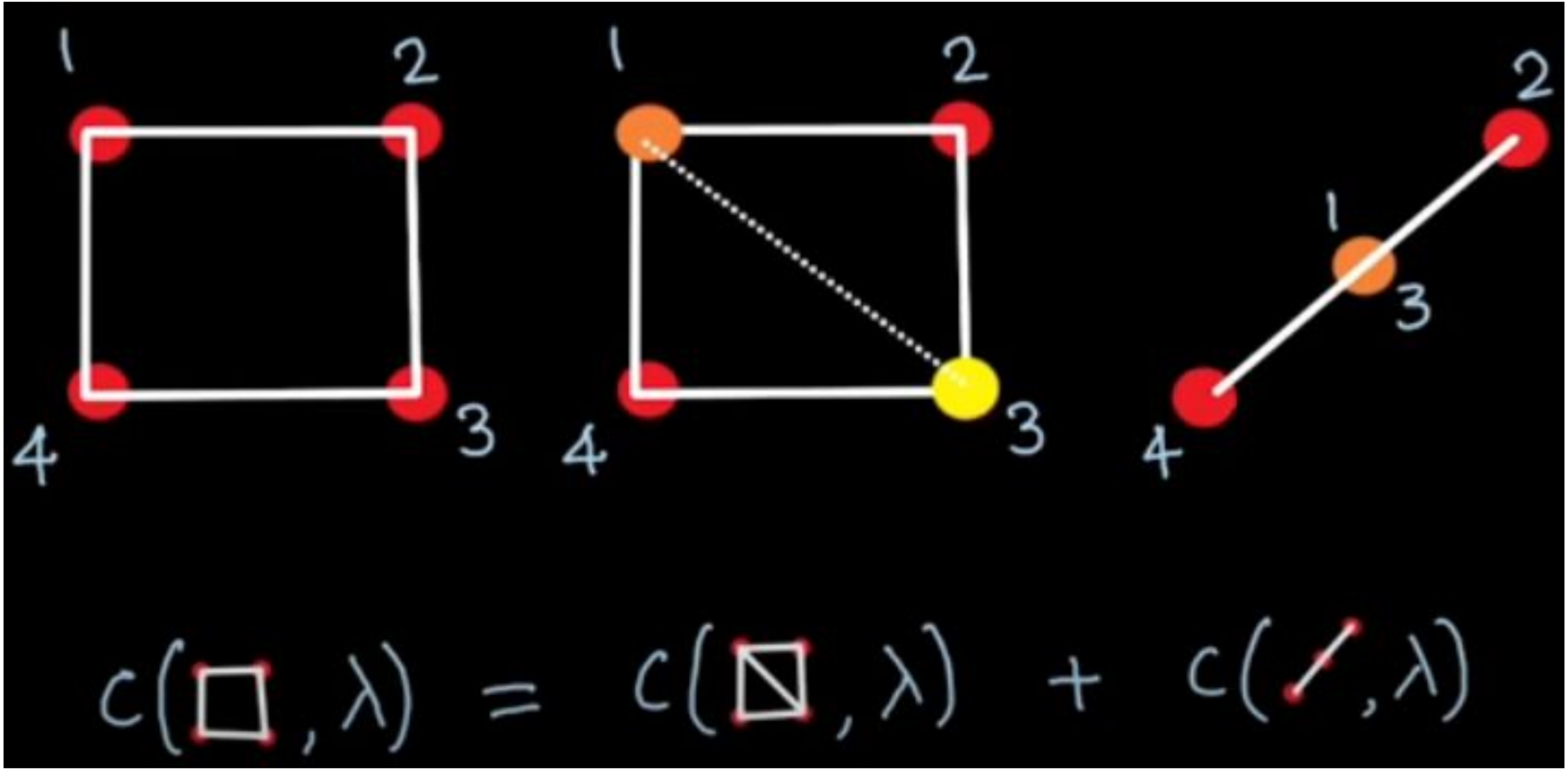
$$C(C_4) = ?$$

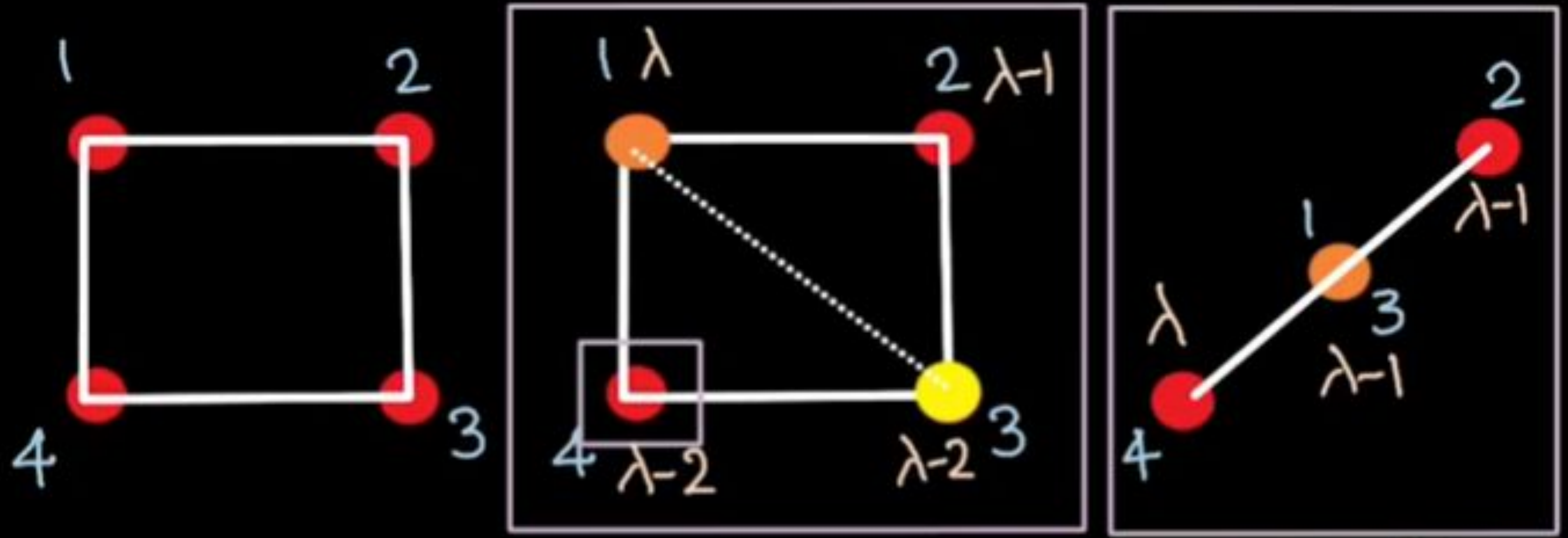


C_4

$C(C_4) = ?$


λ
λ-1
λ-1
mutually exclusive


1
2
3
4
1
2
3
4
1
2
3
4
c(, λ) = c(, λ) + c(, λ)



$$
\begin{aligned}
C(\square, \lambda) &= \lambda(\lambda-1)(\lambda-2)(\lambda-2) + \lambda(\lambda-1)(\lambda-1) \\
&= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda + \lambda^3 + \lambda - 2\lambda^2 \\
&= \boxed{\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda}
\end{aligned}
$$

$C($  $,\lambda) = C($  $,\lambda) + C($  $,\lambda)$

$\Rightarrow C($  $,\lambda) = C($  $,\lambda) - C($  $,\lambda)$

$$\boxed{C(G,\lambda) = \underline{C(G-e,\lambda)} - \underline{C(G\cdot e,\lambda)}}$$

Edge e removed from G

Edge e coalesced in G.

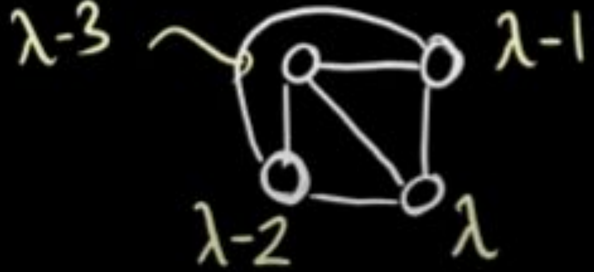
Examples

① $|V| = n$ and $E = \emptyset$, then $P(G,\lambda) = \lambda^n$

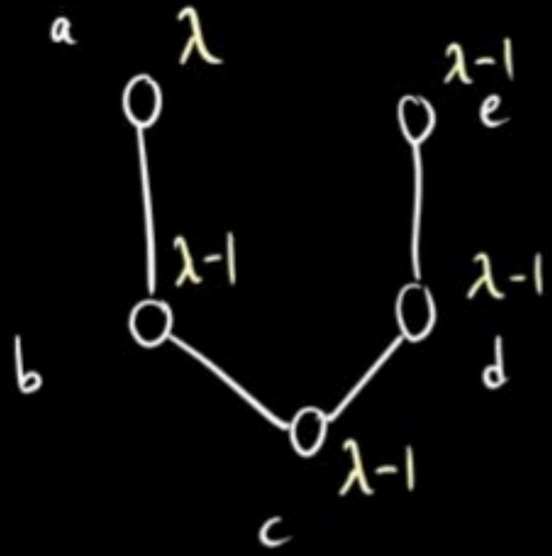
$v = 5$

λ^5

② $G = K_n$, then $P(G, \lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) = \dfrac{\lambda!}{(\lambda-n)!}$



③ For paths, consider the number of choices from vertex to vertex.

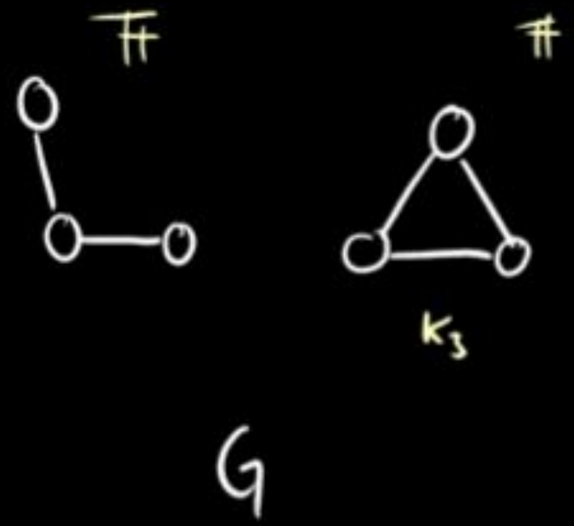


n-vertex path

$$= \lambda(\lambda-1)^{n-1}$$

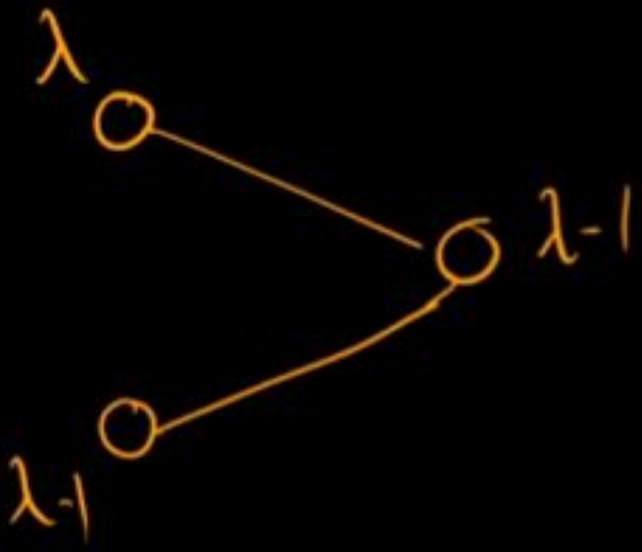
$$P(G,\lambda) = \lambda(\lambda-1)^4$$

④ If G has components, multiply using the product rule.



$$P(G,\lambda) = \left(\lambda(\lambda-1)^2\right)\left(\frac{\lambda!}{(\lambda-3)!}\right)$$

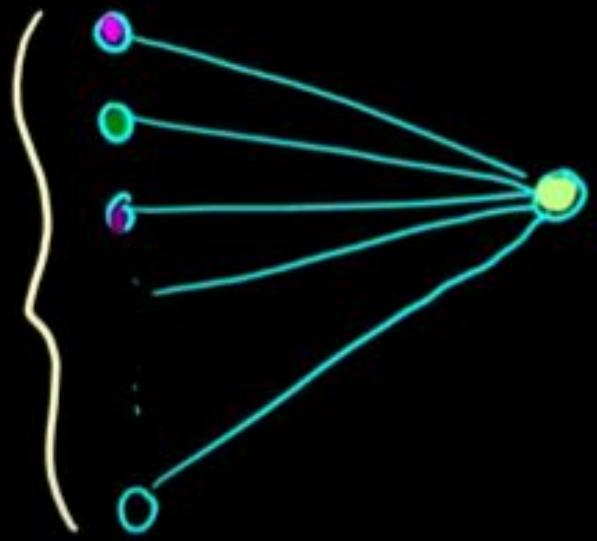
Find $P(G, \lambda)$ for $K_{2,1}$

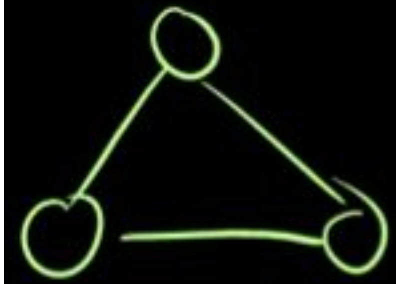


$$\lambda(\lambda-1)^2$$

What is $\chi(K_{m,1})$?

$$\chi(K_{m,1}) = 2$$





$K_3 = \lambda(\lambda-1)(\lambda-2) = 0$

$\lambda = 2$

$\lambda = 3$

$3(3-1)(3-2)$

$= 3(2)(1)$

$= 6$