Transfer Function of D.T system

D.T system

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{Impulse}\\ \text{Responce}\\ h(n)\end{matrix}} \longrightarrow \text{o/p responce } y(n)$$

Response of D.T signal can be obtain by convolution

$$x(n) * h(n) = y(n)$$

In frequency Domain we get

$$X[k] \cdot H[k] = Y[k]$$

$$\text{Transfer Function} = \frac{Y[k]}{X[k]} = \frac{\text{o/P}}{\text{I/P}}$$

Hence Transfer Function = DFT of Impulse Response

Q. Obtain Transfer function of the system in Frequency Domain if its.

Sol^n Transfer function = DFT of h(n)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2+1+2 = 6 \\ 1-2j-1+2j = 0 \\ 1-2+1-2 = -2 \\ 1+2j-1-2j = 0 \end{bmatrix}$$

$x[k] = \{6, 0, -2, 0\}$

19/16

4.1 Effective Computation of DFT :

- We have studied how to obtain DFT of a sequence by using direct computation.
- Basically, the direct computation of DFT requires large number of computations.
- So more processing time is required.
- For the computation of N-point DFT, N^2 complex multiplications and $N^2 - N$ complex additions are required.
- If the value of N is large then the number of computations will go into lakhs.
- This proves inefficiency of direct DFT computation.
- In 1965, Cooley and Tukey developed very efficient algorithm to implement the DFT.
- This algorithm is called as Fast Fourier Transform (FFT).
- These FFT algorithms are very efficient in terms of computations.
- By using these algorithms, number of arithmetic operations involved in the computation of DFT are greatly reduced.
- Different FFT algorithms are available : out of which Radix-2 FFT algorithm is most important FFT algorithm.

4.2.1 Properties of Twiddle Factor :

- The twiddle factor W_N is given by,

$$W_N = e^{-\frac{j 2\pi}{N}} \qquad \ldots(4.2.5)$$

1. $W_N^k = W_N^{k+N}$

- Using Equation (4.2.5) we can write,

$$W_N^{k+N} = \left[e^{-\frac{j 2\pi}{N}}\right]^{k+N} = e^{-\frac{j2\pi k}{N}} \cdot e^{-j\,2\pi}$$

But $e^{-j2\pi} = \cos 2\pi - j \sin 2\pi = 1 - j0 = 1$

$$\therefore\ W_N^{k+N} = e^{-j\frac{2\pi k}{N}}$$

$$\therefore\ W_N^{k+N} = \left(e^{-j\frac{2\pi}{N}}\right)^k \qquad \ldots(4.2.6)$$

- In Equation (4.2.6), the bracket term is W_N

$$\boxed{\therefore\ W_N^{k+N} = W_N^k} \qquad \ldots(4.2.7)$$

- Equation (4.2.7) indicates that **twiddle factor is periodic.**

2. $W_N^{k+N/2} = -W_N^k$

- Using Equation (4.2.6) we can write,

$$W_N^{k+N/2} = \left[e^{-\frac{j2\pi}{N}}\right]^{(k+N/2)} = e^{-\frac{j2\pi k}{N}} \cdot e^{-\frac{j2\pi}{N}\cdot\frac{N}{2}}$$

$$\therefore W_N^{k+N/2} = e^{-\frac{j2\pi k}{N}} \cdot e^{-j\pi} \qquad \ldots(4.2.8)$$

$$\text{But } e^{-j\pi} = \cos\pi - j\sin\pi = -1 - 0 = -1$$

$$\therefore W_N^{k+N/2} = -e^{-j\frac{2\pi k}{N}}$$

$$\therefore W_N^{k+N/2} = -\left(e^{-\frac{j2\pi}{N}}\right)^k \qquad \ldots(4.2.9)$$

- In Equation (4.2.9), the bracket term is, W_N

$$\boxed{\therefore W_N^{k+N/2} = -W_N^k} \qquad \ldots(4.2.10)$$

- Equation (4.2.10) indicates that **twiddle factor is symmetric.**

3. $W_N^2 = W_{N/2}$

- From Equation (4.2.5) we can write,

$$W_{N/2} = e^{-\frac{j2\pi}{N/2}} = e^{-\frac{j2\pi}{N}\cdot 2}; \qquad \therefore W_{N/2} = \left[e^{-\frac{j2\pi}{N}}\right]^2$$

$$\boxed{\therefore W_{N/2} = W_N^2}$$

4.3 Radix-2 Decimation In Time (DIT) Algorithm (DIT FFT) :

- To decimate means to break into parts.
- Thus DIT indicates dividing (splitting) the sequence in time domain.
- The different stages of decimation are as follows :

First stage of decimation :

- Let x (n) be the given input sequence containing 'N' samples.
- Now for decimation in time we will divide x (n) into even and odd sequences.

$$\therefore x(n) = f_1(m) + f_2(m) \qquad \ldots(4.3.1)$$

- Here $f_1(m)$ is even sequence and $f_2(m)$ is odd sequence

$$\therefore f_1(m) = x(2n), \qquad m = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(4.3.2)$$

and

$$f_2(m) = x(2n+1), \qquad m = 0, 1, \ldots, \frac{N}{2} - 1 \qquad \ldots(4.3.3)$$

- Input sequence x (n) has 'N' samples. So after decimation; $f_1(m)$ and $f_2(m)$ will contain $\frac{N}{2}$ samples.

  Now according to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \ldots(4.3.4)$$

- Since we have divided x (n) into two parts; we can write separate summation for even and odd sequences as follows :

$$X(k) = \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn} \quad \ldots(4.3.5)$$

- The first summation represents even sequence. So we will put n = 2m in first summation.
- While the second summation represents odd sequence, so we will put n = (2m + 1) in second summation.
- Since even and odd sequences contain $\frac{N}{2}$ samples each; the limits of summation will be from $m = 0$ to $\frac{N}{2} - 1$.

$$\therefore X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{k(2m+1)} \quad \ldots(4.3.6)$$

- But x (2m) is even sequence, so it is $f_1(m)$ and x (2m + 1) is odd sequence, so it is $f_2(m)$.

$$\therefore X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} f_2(m) W_N^{2km} \cdot W_N^k$$

$$\therefore X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) \left(W_N^2\right)^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m) \left(W_N^2\right)^{km} \quad \ldots(4.3.7)$$

- Now we have $W_N^2 = W_{N/2}$

$$\therefore X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m) W_{N/2}^{km} \quad \ldots(4.3.8)$$

- Comparing each summation with the definition of DFT,

$$X(k) = F_1(k) + W_N^k F_2(k), \; k = 0, 1, \ldots, N-1 \quad \ldots(4.3.9)$$

- We will consider an example of 8 point DFT. That means N = 8.

- Here $F_1(k)$ is $\frac{N}{2}$ point DFT of $f_1(m)$ and $F_2(k)$ is $\frac{N}{2}$ point DFT of $f_2(m)$. That means $F_1(k)$ and $F_2(k)$ are 4–point DFTs.
- Equation (4.3.9) indicates that $F_2(k)$ is multiplied by W_N^k and it is added with $F_1(k)$, to obtain (4 + 4) i.e. 8–point DFT.
- Graphically Equation (4.3.9) is represented as shown in Fig. 4.3.1.
- Remember that in Equation (4.3.9), k varies from 0 to N – 1 (i.e. 0 to 7 for 8 point DFT).

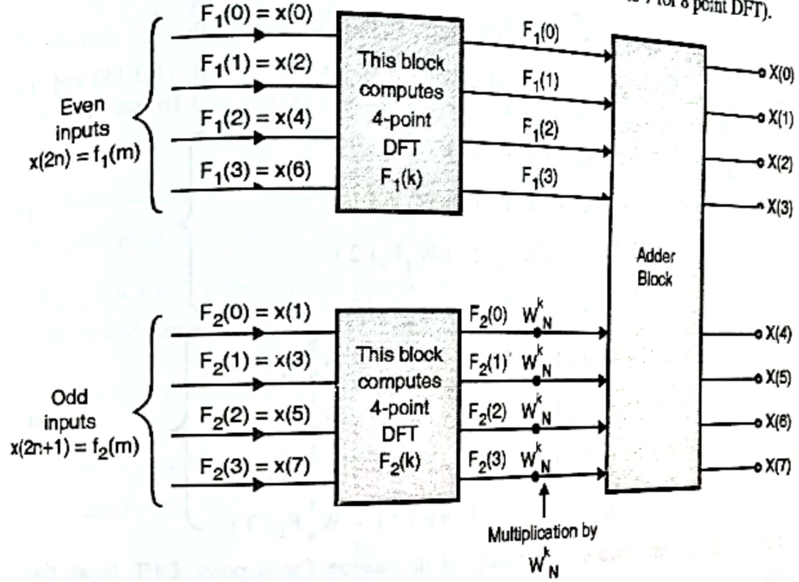


Fig. 4.3.1 : Graphical representation of $X(k) = F_1(k) + W_N^k F_2(k)$

- Now $F_1(k)$ and $F_2(k)$ are 4–point $\left(\frac{N}{2}\right)$ DFTs. They are periodic with period $\frac{N}{2}$.
- Using periodicity property of DFT we can write, ...(4.3.10)

$$F_1\left(k+\frac{N}{2}\right) = F_1(k) \quad \ldots(4.3.11)$$

$$\text{and } F_2\left(k+\frac{N}{2}\right) = F_2(k)$$

- Replacing k by $k + \frac{N}{2}$ in Equation (4.3.9) we get,

$$X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) + W_N^{k+N/2} F_2\left(k+\frac{N}{2}\right) \quad \ldots(4.3.12)$$

- Now we have,

$$W_N^{k+N/2} = -W_N^k \quad \ldots(4.3.13)$$

$$\therefore X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) - W_N^k F_2\left(k+\frac{N}{2}\right)$$

- Using Equations (4.3.10) and (4.3.11) we get,

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k\, F_2(k) \qquad \text{...(4.3.14)}$$

- Here X (k) is 'N' point DFT. We can take $k = 0$ to $\frac{N}{2} - 1$ then, by using Equations (4.3.9) and (4.3.14) we can obtain combined N–point DFT.

$$\therefore X(k) = F_1(k) + W_N^k\, F_2(k),\ k = 0, 1 \ldots \frac{N}{2} - 1 \qquad \text{...(4.3.15)}$$

$$\text{and } X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k\, F_2(k),\ k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \text{...(4.3.16)}$$

- We are considering an example of 8 point DFT (N = 8). So in Equations (4.3.15) and (4.3.16), k varies from 0 to 3. Now putting k = 0 to 3 in Equations (4.3.15) and (4.3.16) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_N^0 F_2(0) \\ X(1) &= F_1(1) + W_N^1 F_2(1) \\ X(2) &= F_1(2) + W_N^2 F_2(2) \\ X(3) &= F_1(3) + W_N^3 F_2(3) \end{aligned}\right\} \qquad \text{...(4.3.17)}$$

and,

$$\left.\begin{aligned} X(0+4) &= X(4) = F_1(0) - W_N^0 F_2(0) \\ X(1+4) &= X(5) = F_1(1) - W_N^1 F_2(1) \\ X(2+4) &= X(6) = F_1(2) - W_N^2 F_2(2) \\ X(3+4) &= X(7) = F_1(3) - W_N^3 F_2(3) \end{aligned}\right\} \qquad \text{...(4.3.18)}$$

- The graphical representation of first stage of decimation for 8 point DFT is as shown in Fig. 4.3.2.

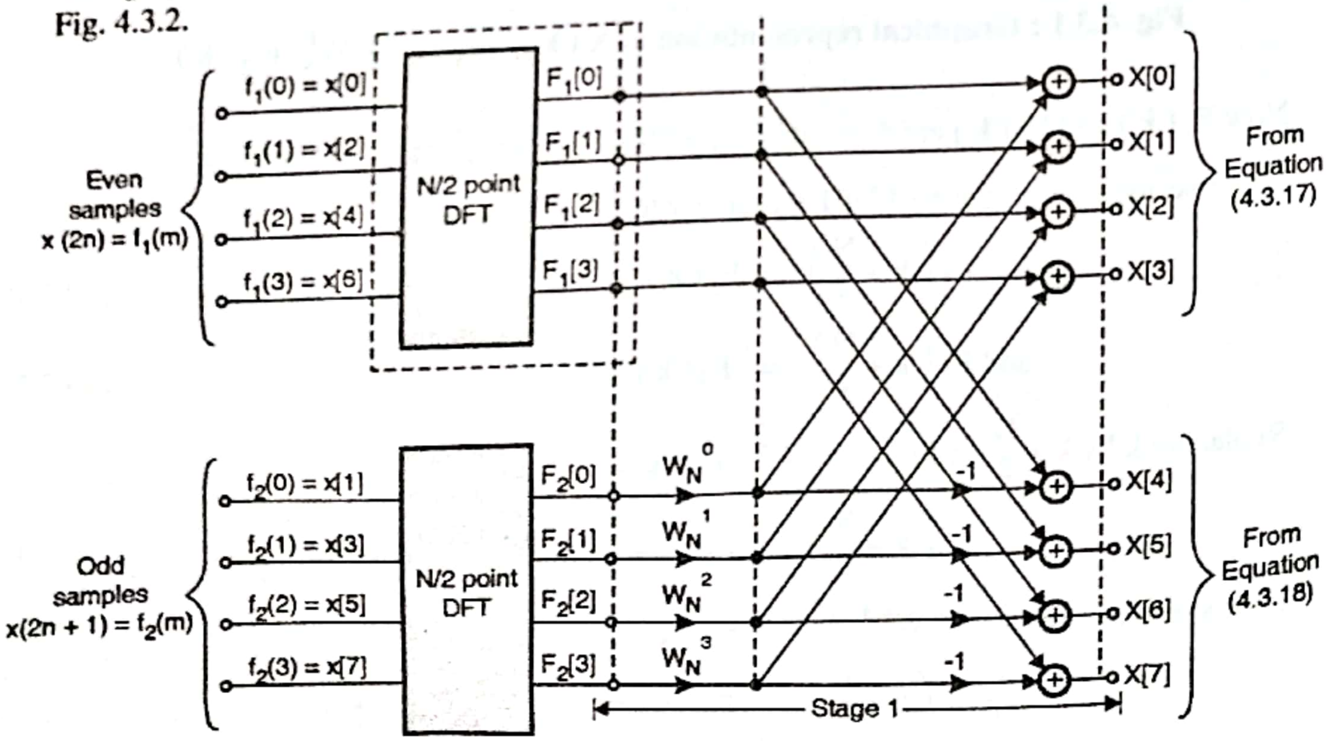


Fig. 4.3.2 : First stage of decimation

EQ - Numericals — 15- ma

Fourier Transform

- Direct computation (DFT)
  N-Point DFT ⇒ N×N matrix
  Not Suitable for large N
- FFT (Divide & conq
  Divide the seque
  and get the solⁿ
  - Decimation In Time (DIT)
    - Radix - 2
      Base - 2
    - Radix - 4
      Base 4
  - Decimation In Frequency (DIF

Butterfly signal flow graph

case-I

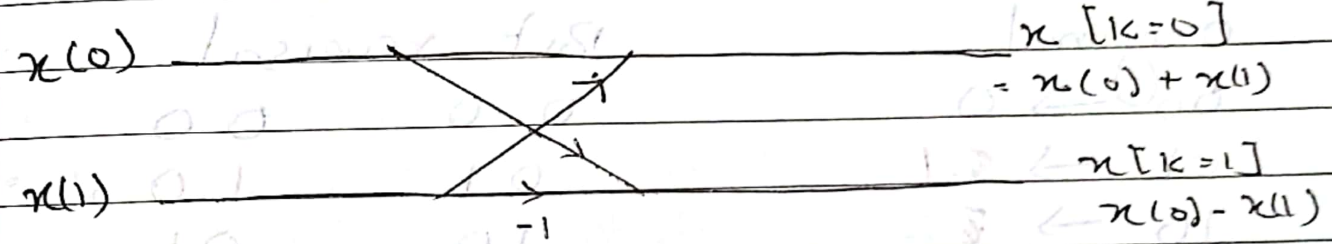
For $N=2$

$$X[K] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \end{bmatrix}$$

$x[K=0] = x(0) + x(1)$

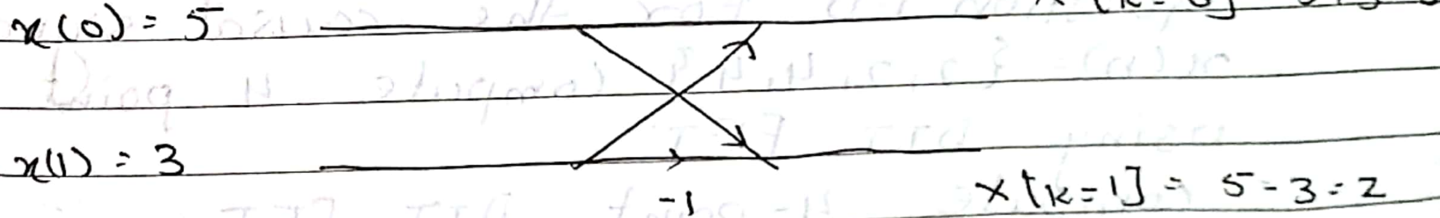
$x[K=1] = x(0) - x(1)$

Butterfly Diagram | Signal Flow Graph.



// Example obtain DIT - FFT of $x(n) = \{5, 3\}$

Sol^n:- Consider signal flow graph for $N=2$



$X[K] = \{8, 2\}$

$n = 0$

$$X(k) = G(k) + W_N^k H(k)$$

$$G(k) = \text{DFT}\{x(2n)\}$$

$$H(k) = \text{DFT}\{x(2n+1)\}$$

$\boxed{\text{FFT } N = 2}$

$$X(k) = \sum_{n=0}^{1} x(n)\, W_2^{nk}$$

$$X(0) = \sum_{n=0}^{1} x(n)\, W_2^{n \cdot 0} = x(0)\, W_2^0 + x(1)\, W_2^0$$

$$X(1) = \sum_{n=0}^{1} x(n)\, W_2^{n \cdot 1} = x(0)\, W_2^0 + x(1)\, W_2^1$$

$$W_2^l = e^{\frac{-j2\pi l}{2}}$$

$$W_2^l = e^{-j\pi l}$$

$l = 0$ $\qquad W_2^0 = e^0 = 1$

$l = 1$ $\qquad W_2^1 = e^{-j\pi} = -1$

$$\therefore X(0) = x(0) + x(1)$$

$$X(1) = x(0) - x(1)$$

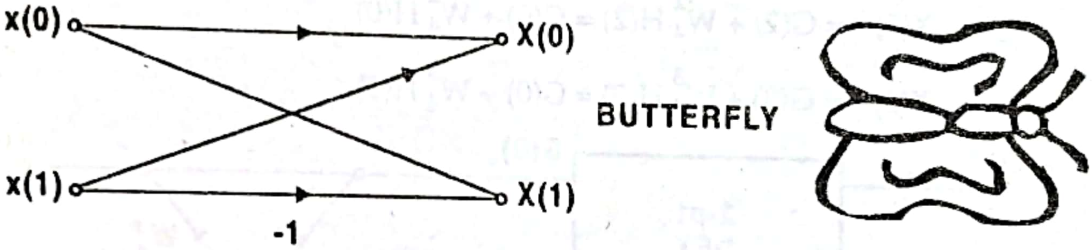


Fig. 4.23

$\boxed{N = 4}$

$$X(k) = G(k) + W_4^k\, H(k)$$

$$G(k) = \text{DFT}\,\{x(2n)\} = \text{DFT}\,\{x(0), x(2)\}$$

$$H(k) = \text{DFT}\,\{x(2n+1)\} = \text{DFT}\,\{x(1), x(3)\}$$

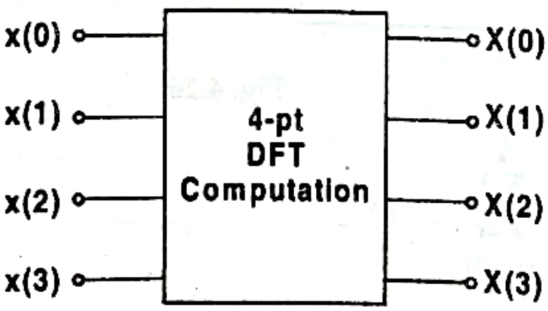


Fig. 4.24

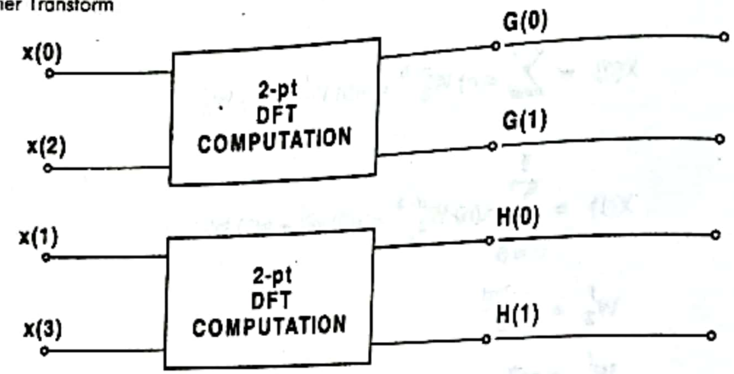


Fig. 4.25

$$X(k) = G(k) + W_4^k H(k)$$

$$X(0) = G(0) + W_4^0 H(0)$$

$$X(1) = G(1) + W_4^k H(1)$$

$$X(2) = G(2) + W_4^2 H(2) = G(0) + W_4^2 H(0)$$

$$X(3) = G(3) + W_4^3 H(3) = G(0) + W_4^3 H(1)$$

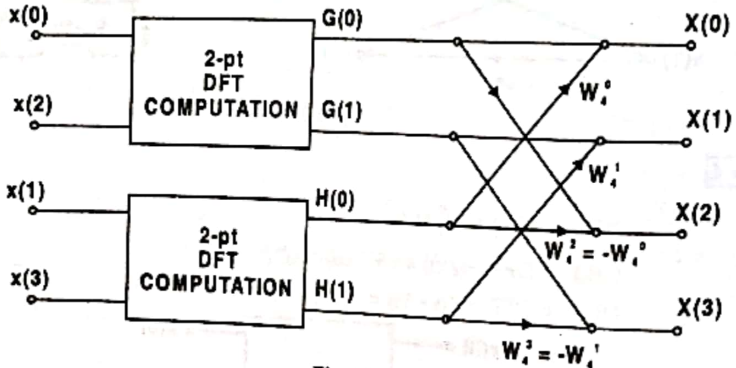


Fig. 4.26

$$G(k) = \sum_{n=0}^{1} x(2n)\, W_2^{nk}$$

$$G(0) = \sum_{n=0}^{1} x(2n)\, W_2^{n0} = x(0)\, W_2^0 + x(2)\, W_2^0$$

$$G(1) = \sum_{n=0}^{1} x(2n)\, W_2^{n \cdot 1} = x(0)\, W_2^0 + x(2)\, W_2^1$$

$$H(k) = \sum_{n=0}^{1} x(2n+1)\, W_2^{nk}$$

$$H(0) = \sum_{n=0}^{1} x(2n+1)\, W_2^{n \cdot 0} = x(1)\, W_2^0 + x(3)\, W_2^0$$

$$H(1) = \sum_{n=0}^{1} x(2n+1)\, W_2^{n \cdot 1} = x(1)\, W_2^0 + x(3)\, W_2^1$$

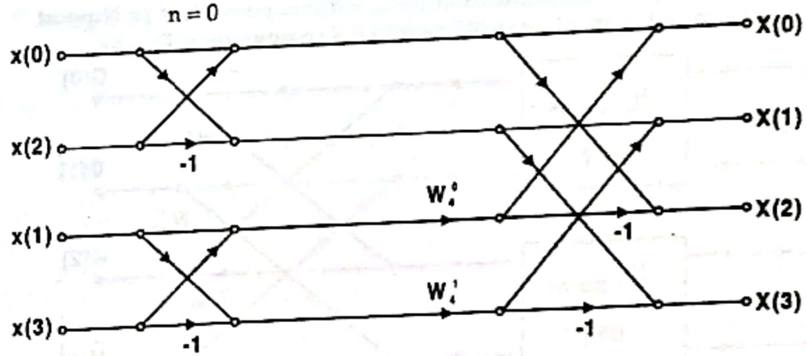


Fig. 4.27

$\boxed{N = 8}$

$$X(k) = G(k) + W_8^k H(k)$$

$$G(k) = L(k) + W_8^{2k} M(k)$$

$$H(k) = E(k) + W_8^{2k} F(k)$$

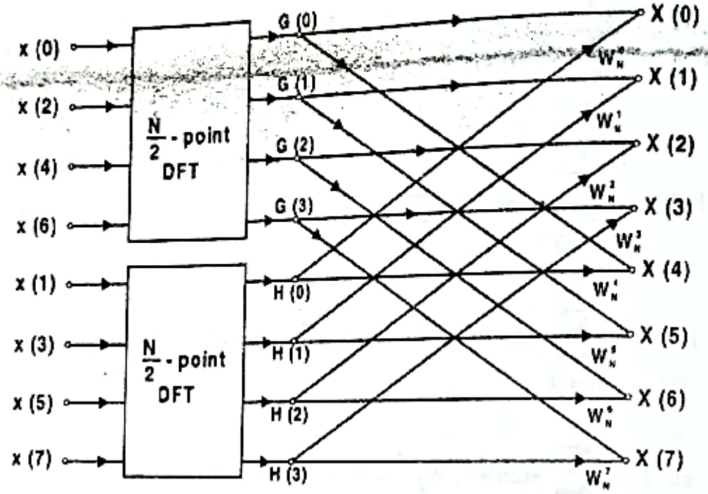


Fig. 4.28 Flow graph of the decimation-in-time decomposition of an N-point DFT computation into two N/2-point DFT computations (N = 8)

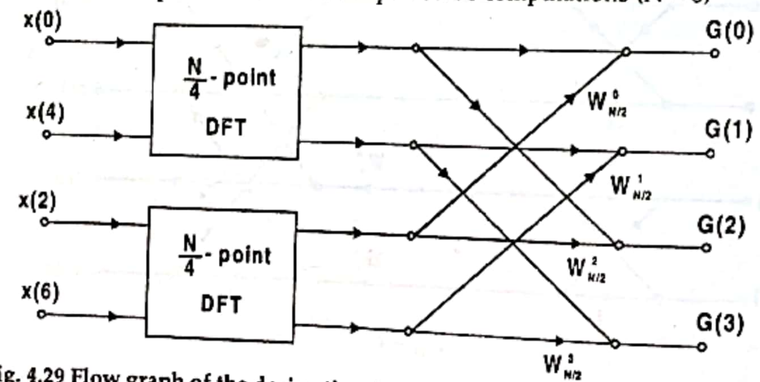


Fig. 4.29 Flow graph of the decimation-in-time decomposition of an N/2-point DFT computations into two N/4-point DFT computations (N = 8)

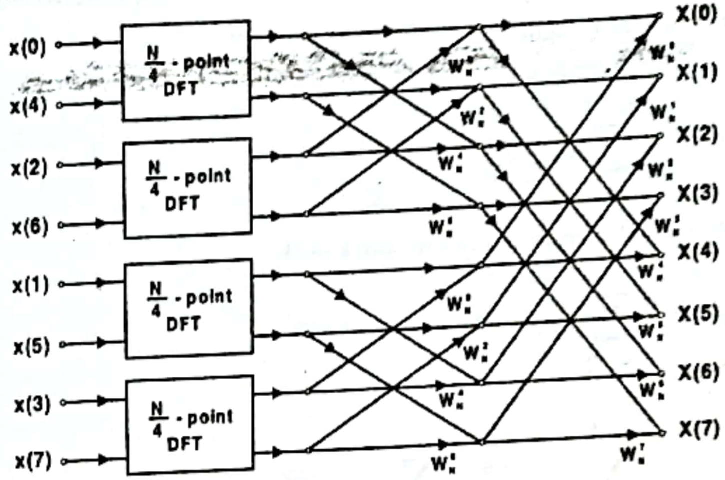


Fig. 4.30 Result of substituting Fig. 4.28 and Fig. 4.29

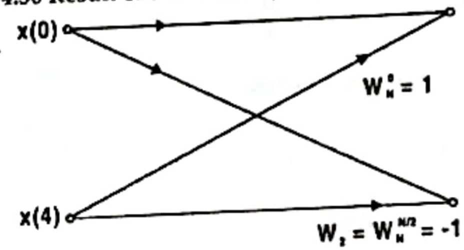


Fig. 4.31 Flow graph of a two-point DFT

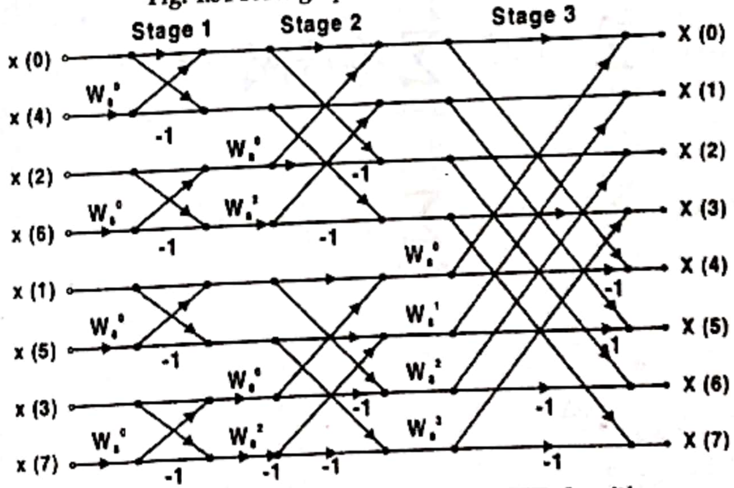


Fig. 4.32 Eight-point decimation in-time FFT algorithm.

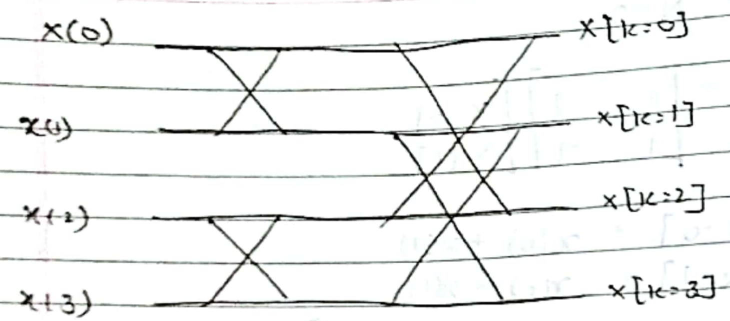
case-2

$x[k] = \{8, 2\}$

for $N = 4$

$\{0, 1, 2, 3\}$

$\{0, 2\}$ $\{1, 3\}$



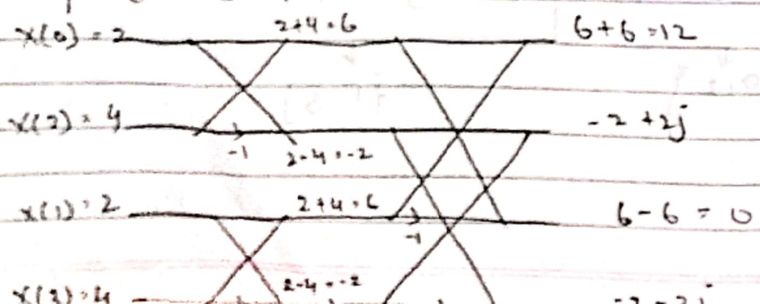
Bit - Reversal Technique

original		Bit reversal		
00 → 0		0 0	0 0	0
01 → 1		0 1	1 0	2
10 → 2		1 0	0 1	1
11 → 3		1 1	1 1	3

(Q) EQ. DIT - FFT ***

May 16
Question 1.D for the causal signal $x(n) = \{2, 2, 4, 4\}$ compute 4 point DFT using DIT FFT
compute 4-point DIT- FFT

Sol: Consider following butterfly Diagram for 4-point DIT-FTT



$X[k] = \{12, -2+2j, 0, -2-2j\}$

Dec-15

(Radian)

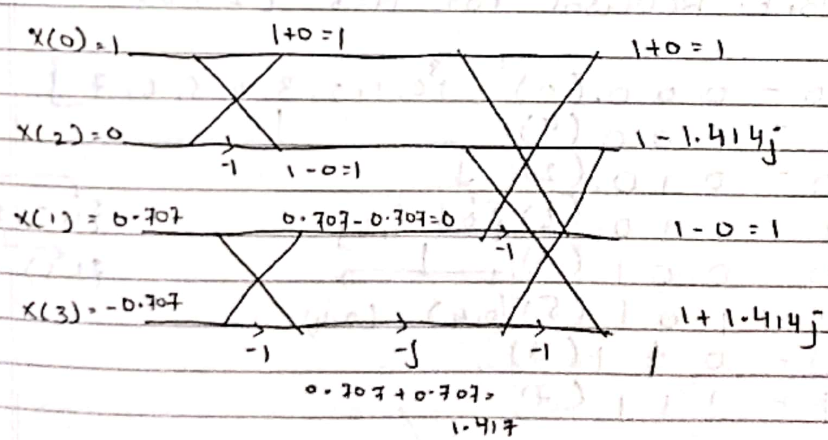
Find the value of $x(n) = \cos(0.25\pi n)$ for $n = 0, 1, 2, 3$, compute the DFT of $x(n)$ using FFT flow diagram.

$x(n) = \cos(0.25\pi n)$
but $n = 0, 1, 2, 3$

$n=0 \quad x(n) = \cos(0.25\pi \cdot 0) = \cos 0 = 1$
$n=1 \quad x(n) = \cos(0.25\pi\ 1) = \cos\pi/4 = 0.707$
$n=2 \quad x(n) = \cos(0.25\pi\ 2) = \cos 90 = 0$
$n=3 \quad x(n) = \cos(0.25\pi\ 3) = -\cos\pi/4 = -0.707$

By Butterfly diagram

$\{1, 0.707, 0, -0.707\}$
↓ ↓ ↓ ↓
0 1 2 3



$X[k] = \{1, 1-1.414j, 1, 1+1.414j\}$

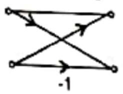
For N = 4

Natural Order	Binary Equivalent	Bit Reversed	Decimal Equivalent
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

For N = 8

Natural Order	Binary Equivalent	Bit Reversed	Decimal Equivalent
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Student should also note that the flow graph comprises of butterflies



which compute 2-point DFT.

For drawing DIT-FFT:

First draw small butterflies then bigger and bigger butterflies from left to right.

In decimation in frequency flow graphs small butterflies are there on the right and bigger and bigger butterflies on the left.

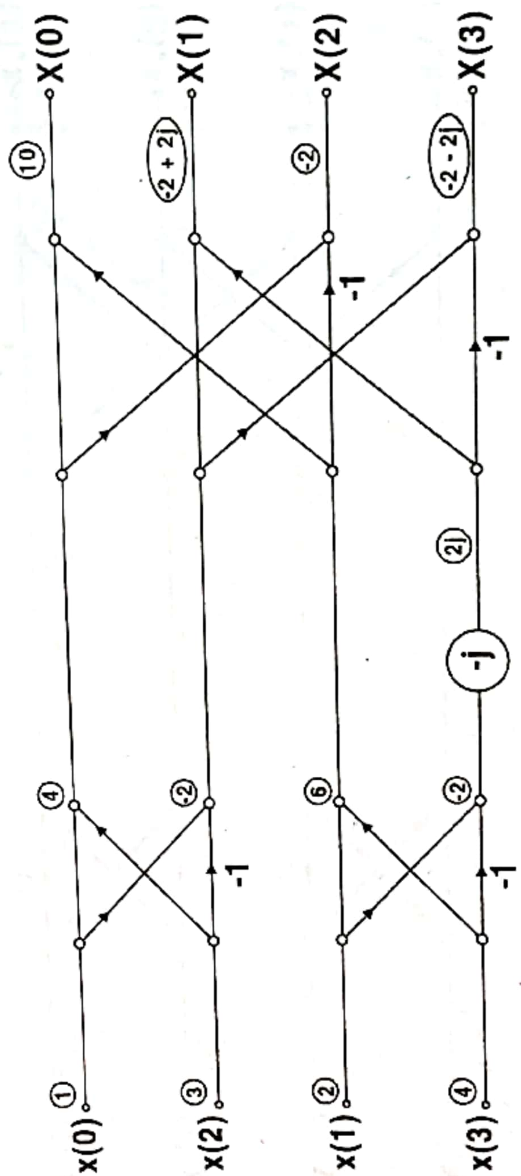
Another point worth noting is that the decimation in frequency flow graph can be obtained by folding or taking mirror image of the decimation in time flow graph and shuffling the output instead of the input.

Example 4.18:

Find the DFT of the following sequences using DIT-FFT

$$x(n) = \{1, 2, 3, 4\}$$

Solution:

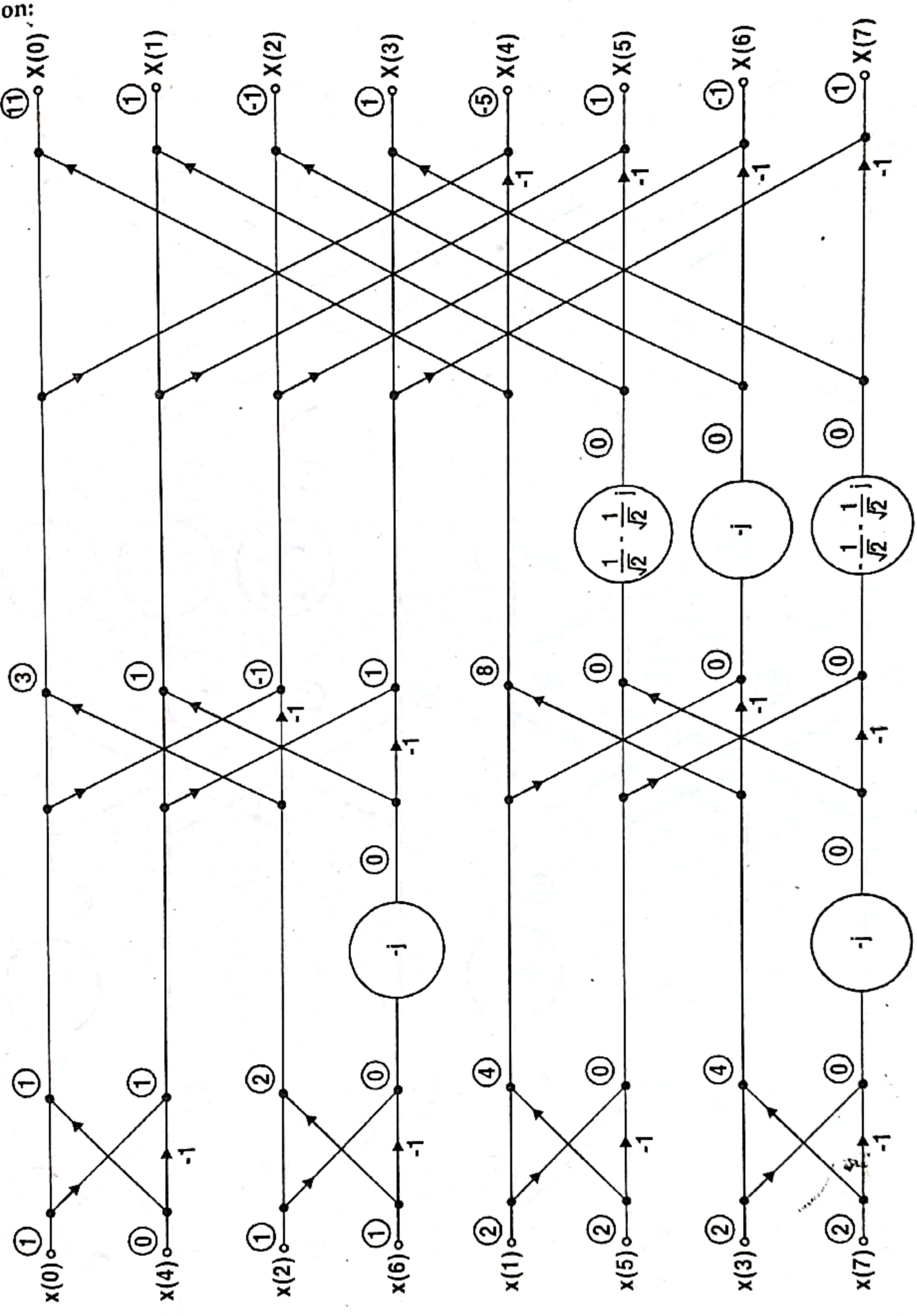


Example 4.22:

Find DFT of the following using DIT-FFT.

$$x(n) = \{1, 2, 1, 2, 0, 2, 1, 2\}$$

Solution:



case:- For N=8

Radix-2 DIT-FFT Flowgraph for N

Bit Reversal For N=8 ($2^3=8$)

0 0 0 - 0 0 0 (0)	{0, 1, 2, 3, 4, 5, 6, 7}	
0 0 1 - 1 0 0 (4)		
0 1 0 - 0 1 0 (2)		
0 1 1 - 1 1 0 (6)	{0, 2, 4, 6}	{1, 3, 5, 7}
1 0 0 - 0 0 1 (1)		{1, 5} (3, ~~7~~)
1 0 1 - 1 0 1 (5)	(0, 4) (2, 4)	
1 1 0 - 0 1 1 (3)		
1 1 1 - 1 1 1 (7)		