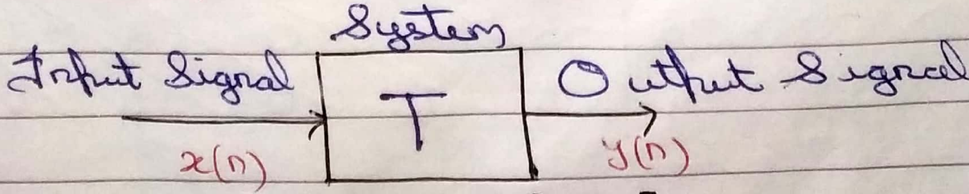
★★★ → Convolution ♡ Heart of Syllabus ♡

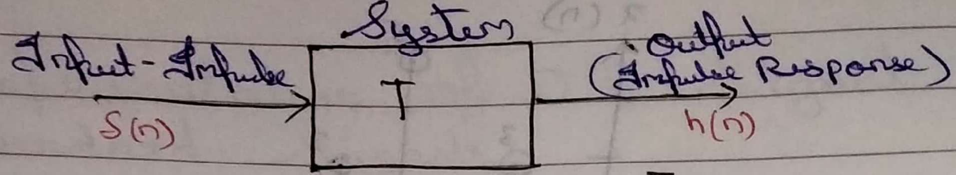
→ What is Convolution

If we know how a system responds to the Impulse Input. Then we can find response / output of the system to any D.T. signal $x(n)$.

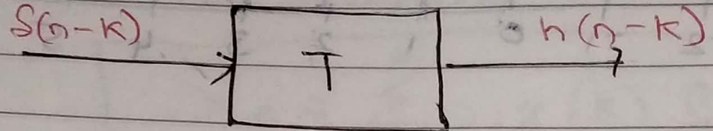
Explaination



$$y(n) = T[x(n)]$$



$$h(n) = T[\delta(n)]$$



$$h(n-k) = T[\delta(n+k)]$$

We ~~know~~

$$y(n) = T[x(n)]$$

$$= T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, T[\delta(n-k)]$$

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)}$$ ⇒ Convolution Formula

$$\boxed{y(n) = x(n) * h(n)}$$ ⟹ Convolution Notation

(Shift, Multiply, Add)

EQ Perspective on Convolution

Convolution

- Linear Convolution
  - Non-periodic Signal
  - $L = L_1 + L_2 - 1$
  - Time Domain $x(n)$
    - Graphical Method
      If asked
      10 Marks
    - Tabular Method
      5M ★★★
  - Frequency Domain $x[k]$
    - DFT
      If asked
      10 M
    - Fast (FFT)
      If asked
      10 M
- (Linear using Circular)
- Circular Convolution
  - Periodic Signal
  - $L = L_1 = L_2$
  - Circular Matrix ★★★
  - DFT/FFT

Method I :- Tabular Method

EX 5M Obtain Convolution of

$x(n) = \{\underset{\uparrow}{2}, 3, 1, 4\}$

$h(n) = \{\underset{\uparrow}{1}, 2, 1\}$

Solution :- We know $y(n) = x(n) * h(n)$

	2	3	1	4
1	2	3	1	4
2	4	6	2	8
1	2	3	1	4

$L = L_1 + L_2 - 1$
$= 4 + 3 - 1$
$= 6$

$\therefore y(n) = \{\underset{\uparrow}{2}, 7, 9, 9, 9, 4\}$

Eg - Obtain Convolution of 5M

$x(n) = \{1, 1, \underset{\uparrow}{0}, 1, 1\}$

$L = 5$
$M = 4$

$h(n) = \{1, -2, -3, \underset{\uparrow}{4}\}$

Solution :- We know $y(n) = x(n) * h(n)$

	1	1	0	1	1
1	1	1	0	1	1
-2	-2	-2	0	-2	-2
-3	-3	-3	0	-3	+3
4	4	4	0	4	4

$\therefore y(n) = \{1, -1, -5, 2, 3, \underset{\uparrow}{-5}, 1, 4\}$

$L = 5 + 4 - 1 = 8$

$N = L + M - 1 =$

EQ
5M

Obtain Convolution of
$$x(n) = \{\underset{\uparrow}{1}, 2, 3\}$$
$$h(n) = \{\underset{\uparrow}{1}, \tfrac{1}{2}\}$$

Solution :- We know $y(n) = x(n) * h(n)$

	1	2	3
1	1	2	3
0.5	0.5	1	1.5

$$\therefore y(n) = \{\underset{\uparrow}{1}, 2.5, 4, 1.5\}$$

$$\therefore L = 4$$

Circular Convolution :- Matrix Representation

EX : Obtain Circular Convolution of
$$x_1(n) = \{1, 2, 3, 4\}$$
$$x_2(n) = \{1, 2\}$$
max length = 4.

Solution :- For Circular Convolution
$$L = L_1 = L_2$$

Hence we perform zero padding of $x_2(n)$

$$\therefore x_2(n) = \{1, 2, 0, 0\}$$

$$\therefore y(n) = x_1(n) \circledast x_2(n)$$

$$= \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+8=9 \\ 2+2=4 \\ 3+4=7 \\ 4+6=10 \end{bmatrix}$$

$\therefore y(n) = \{9, 4, 7, 10\}$

EQ
5M

Find Circular Convolution of given two sequences

$x_1(n) = \{1, -1, 2, -4\}$

$x_2(n) = \{1, 2\}$

Solution :- For Circular Convolution

$L = L_1 \neq L_2$

$\therefore$ We perform zero padding of $x_2(n)$

$\therefore x_2(n) = \{1, 2, 0, 0\}$

$\therefore y(n) = x_1(n) \circledast x_2(n)$

$$= \begin{bmatrix} 1 & -4 & 2 & -1 \\ -1 & 1 & -4 & 2 \\ 2 & -1 & 1 & -4 \\ -4 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + (-8) = -7 \\ (-1) + 2 = 1 \\ 2 + (-2) = 0 \\ (-4) + 4 = 0 \end{bmatrix}$$

$\therefore y(n) = \{-7, 1, 0, 0\}$

EQ
10M

Find Circular Convolution & Linear using Circular for

$x_1(n) = \{1, 2, 3, 4\}$

$x_2(n) = \{1, 2, 1, 2\}$.

Using Time Domain Formula Method

Solution :- For Circular Convolution

We obtain circular convolution for x_1 & x_2

$$\therefore y(n) = x_1(n) \circledast x_2(n)$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+8+3+4 = 16 \\ 2+2+4+6 = 14 \\ 3+4+1+8 \; 16 \\ 4+6+2+2 \; 14 \end{bmatrix}$$

$$\therefore y(n) = \{16, 14, 16, 14\}$$

Now we obtain Linear Convolution using Circular Convolution

We know, for Linear Convolution

$$L = L_1 + L_2 - 1$$
$$= 4 + 4 - 1 = 7$$

Hence we perform zero padding to make $L = L_1 = L_2 = 7$

$$\therefore x_1(n) = \{1, 2, 3, 4, 0, 0, 0\}$$
$$\therefore x_2(n) = \{1, 2, 1, 2, 0, 0, 0\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \\ 14 \\ 15 \\ 10 \\ 8 \end{bmatrix}$$

$$\therefore y(n) = \{1, 4, 8, 14, 15, 10, 8\}$$

Cross-Check

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4
2	2	4	6	8

$\therefore y(n) = \{1, 4, 8, 14, 15, 10, 8\}$

EQ
5M

Find Linear Convolution using Circular Convolution

$x(n) = \{1, 2, 3, 4\}$ $\quad \therefore L_1 = 4$

$h(n) = \{1, 7\}$ $\quad \therefore L_2 = 2$

$\therefore L = L_1 + L_2 - 1$

<u>Solution</u>:- For Circular Convolution $\quad = 4 + 2 - 1$

$L = L_1 * = L_2$ $\quad = 5$

$\therefore$ We perform zero padding of $x(n)$ & $h(n)$

$\therefore x(n) = \{1, 2, 3, 4, 0\}$

$\therefore h(n) = \{1, 7, 0, 0, 0\}$

$\therefore y(n) = x(n) \circledast h(n)$

$$= \begin{bmatrix} 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 17 \\ 25 \\ 28 \end{bmatrix}$$

$\therefore y(n) = \{1, 9, 17, 25, 28\}$

Cross-Check

	1	2	3	4
1	1	2	3	4
7	7	14	21	28

$\therefore y(n) = \{1, 9, 17, 25, 28\}$

→ **Properties of Convolution**

1. Length Property

Linear	$L = L_1 + L_2 - 1$
Circular	$L = L_1 = L_2$

2. Commutative Property

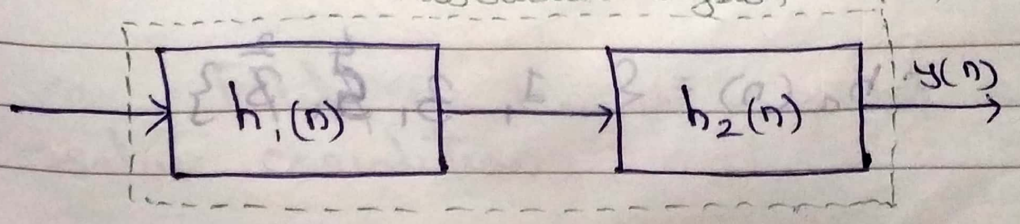
Convolution is Commutative

$$y(n) = x(n) * h(n)$$
$$= h(n) * x(n)$$

3. Associative Property

$$x(n) * h_1(n) * h_2(n)$$
$$= [x(n) * h_1(n)] * h_2(n)$$
$$= [x(n)] * [h_1(n) * h_2(n)]$$

Explaination Cascaded System $= h_1(n) * h_2(n)$



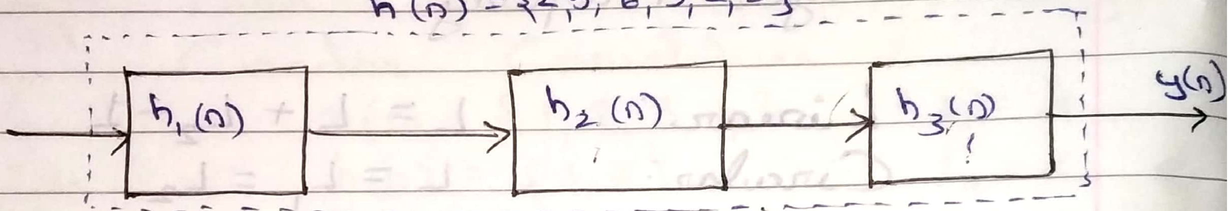
*** EQ
10 Marks

A causal FIR system has three cascaded block, first two of them have individual impulse responses

$$h_1(n) = \{1, 2, 2\}$$
$$h_2(n) = u(n) - u(n-2)$$

Find impulse response of third block $h_3(n)$, if an overall impulse response is $h(n) = \{2, 5, 6, 3, 2, 2\}$

<u>Solution</u>:- Consider 3-cascaded block



Overall $h(n) = h_1(n) * h_2(n) * h_3(n)$

We know $h_1(n) = \{1, 2, 2\}$

Also $h_2(n) = u(n) - u(n-2)$

$$= \{1, 1 \ldots\}$$

Now, we obtain $h_1(n) * h_2(n)$

Let $h_a(n) = h_1(n) * h_2(n)$

	$h_1(n)$: 1	2	2
$h_2(n)$: 1	1	2	2
1	1	2	2

$\therefore h_a(n) = \{1, 3, \overset{4}{\cancel{5}}, \overset{2}{\cancel{3}}\}$

Now, $h(n) = h_1(n) * h_2(n) * h_3(n)$

$L(6) = L_1(4) + L_2(?) - 1 \quad \therefore L_2 = 3$

$6 = 4 + x - 1$

$\therefore x = 3$

$h(n) = h_a(n) * h_3(n)$

$h_3(n)$

$h_a(n)$	x	y	z
1	x	y	z
3	$3x$	$3y$	$3z$
4	$4x$	$4y$	$4z$
2	$2x$	$2y$	$2z$

$\therefore h(n) = \{x, 3x+y, 4x+3y+z, 2x+4y+3z, 2y+4z, 2z\}$

$\therefore h(n) = \{2, 5, 6, 3, 2, 2\}$

$2z = 2$

$z = \frac{2}{2} = 1$

$3x + y = 5$

$6 + y = 5$

$y = 5 - 6$

$y = -1$

$\therefore h_3(n) = \{2, -1, 1\}$

→ Auto & Cross Correlation

*** ~~Always Asked - 5 Marks~~ Ques 1

- Correlation is obtained to find similarity between signals
- Measure of similarity

Correlation

- Auto-Correlation (Similarity with itself)
- Cross-Correlation (Between 2-different signals)

How to Solve Correlation Problem

Flip or Reverse second signal and solve convolution.

Auto Correlation (Even Signal)

$$\gamma_{xx}(n) = x(n) * x(-n)$$

Cross Correlation

x - reference $\gamma_{xy}(n) = x(n) * y(-n)$
y - reference $\gamma_{yx}(n) = y(n) * x(-n)$

EX:- Find Auto-Correlation of
$x(n) = \{2, 3, 1\}$

Solution:- Auto Correlation is given as

$$\gamma_{xx}(n) = x(n) * x(-n)$$

$x(-n)$ \ $x(n)$	2	3	1
1	2	3	1
3	6	9	3
2	4	6	2

$$\gamma_{xx}(n) = \{2, 9, \underset{\uparrow}{14}, 9, 2\}$$

DEC 15 1(d) 5M :- Find the cross correlation of the sequences $x(n) = \{1, 2, 3, 4\}$ & $h(n) = \{2, 4, 6\}$

Solution:- We first obtain Cross Correlation $\gamma_{xy}(n)$

$$\therefore \gamma_{xy}(n) = x(n) * y(-n)$$

$x(n)$

		$\boxed{1}$	2	3	4
	6	6	12	18	24
$y(-n)$	4	4	8	12	16
	$\boxed{2}$	2	4	6	8

$$\therefore \gamma_{xy}(n) = \{6, 16, \underset{\uparrow}{28}, 40, 22, 8\}$$

Now we find $\gamma_{yx}(n) = y(n) * x(-n)$

	$\boxed{2}$	4	6
4	8	16	24
3	6	12	18
2	4	8	12
$\boxed{1}$	2	4	6

$$\therefore \gamma_{yx}(n) = \{8, 22, 40, \underset{\uparrow}{28}, 16, 6\}$$

MAY 16
1(a) 5M

For the given causal sequence $x(n) = \{8, 9, 2, 3\}$ & $h(n) = \{4, 3, 6\}$ find the cross correlation

Solution :- We first obtain Cross Correlation $\gamma_{xy}(n)$

$$\therefore \gamma_{xy}(n) = X(n) * y(-n)$$

	$\boxed{8}$	9	2	3
6	48	54	12	18
3	24	27	6	9
$\rightarrow \boxed{4}$	32	36	8	12

$$\therefore \gamma_{xy}(n) = \{48, 78, \underset{\uparrow}{71}, 60, 17, 12\}$$

$\therefore$ Now we find $\gamma_{yx}(n) = y(n) * x(-n)$

	3	2	9	8
4	12	8	36	32
3	9	6	27	24
6	18	12	54	48

$\therefore \gamma_{yx}(n) = \{12, 17, 60, \underset{\uparrow}{71}, 78, 48\}$