Fast Fourier Transform

	TOPIC
1	Radix-2 Cooley & Tuckey’s DIT-FFT Algorithm,
3	DIT-FFT Flowgraph for N=4 & 8,
3	Comparison of Complex and Real, Multiplication and Additions of DFT and FFT
4	Inverse FFT algorithm

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Chapter-2B : Fast Fourier Transform

Objective : To illustrate FFT calculations mathematically

Outcomes :

At the end of module, students will be able to ,

- **Develop** FFT flow-graph
- **Compare** DFT and FFT computationally
- **Perform** forward and Inverse FFT
- **Plot** signal spectrum in frequency domain

- In 1965, James W. Cooley and John W. Tukey (IEEE 1982 Medal of Honor recipient) published a paper describing the Fast Fourier Transform (FFT) algorithm, which led to an explosion in Digital Signal Processing.

- Their landmark research offered enormous improvements in processing speeds and played an essential role in the digital revolution.

James COOLEY

John TUKEY

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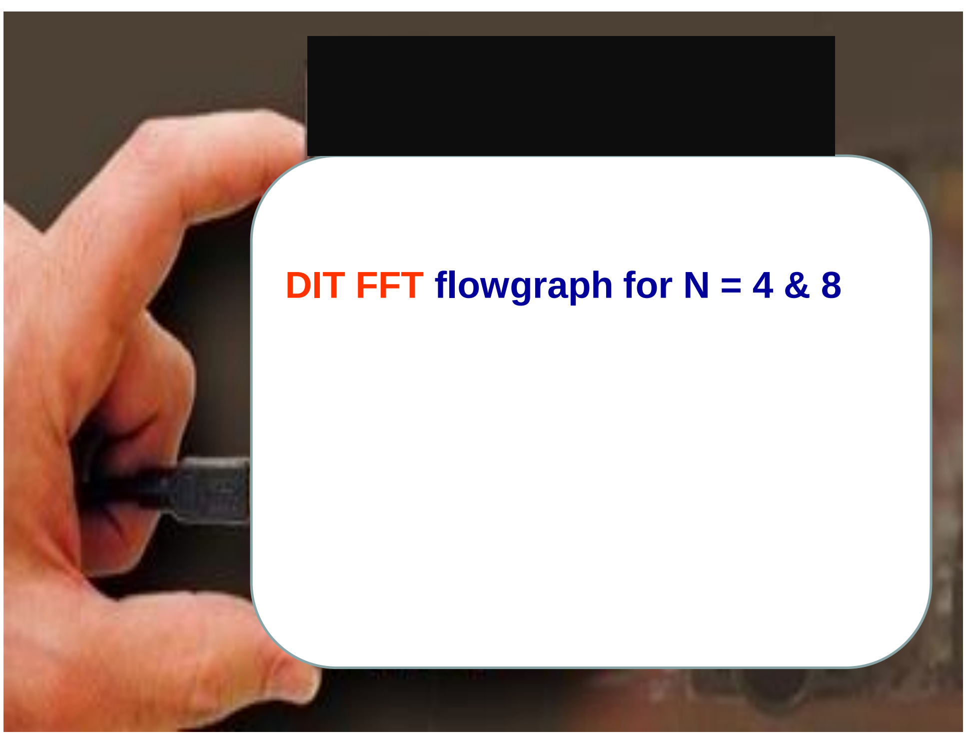


DIT FFT flowgraph for N = 4

Step-1 : Derive DIT-FFT equation

- By DFT, $X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$

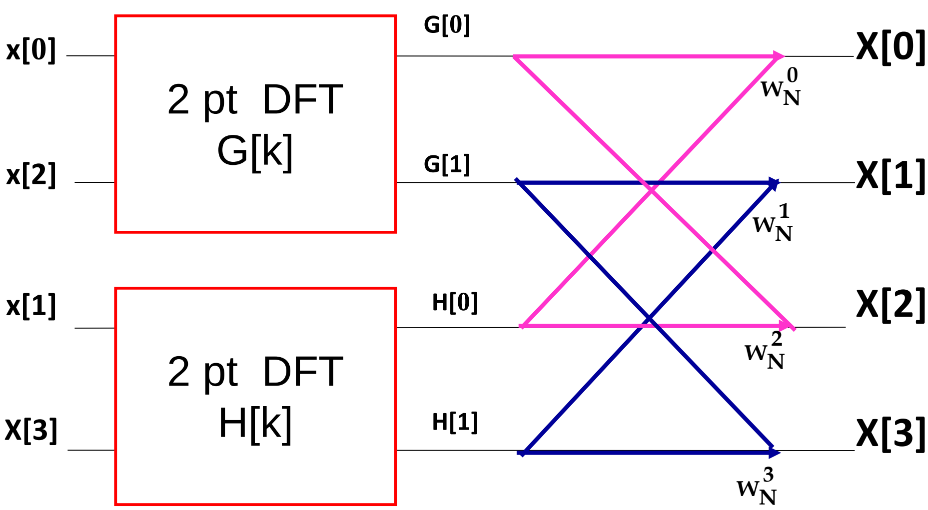
$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\; W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\; W_N^{(2r+1)k}$$

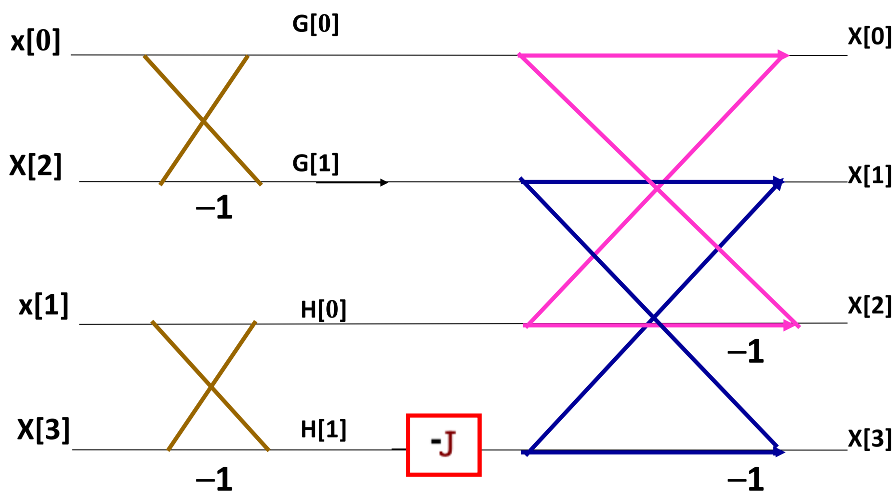
$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\; W_{\frac{N}{2}}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\; W_{\frac{N}{2}}^{rk}$$

$$\mathbf{X[k] = G[k] + W_N^k\; H[k]}$$

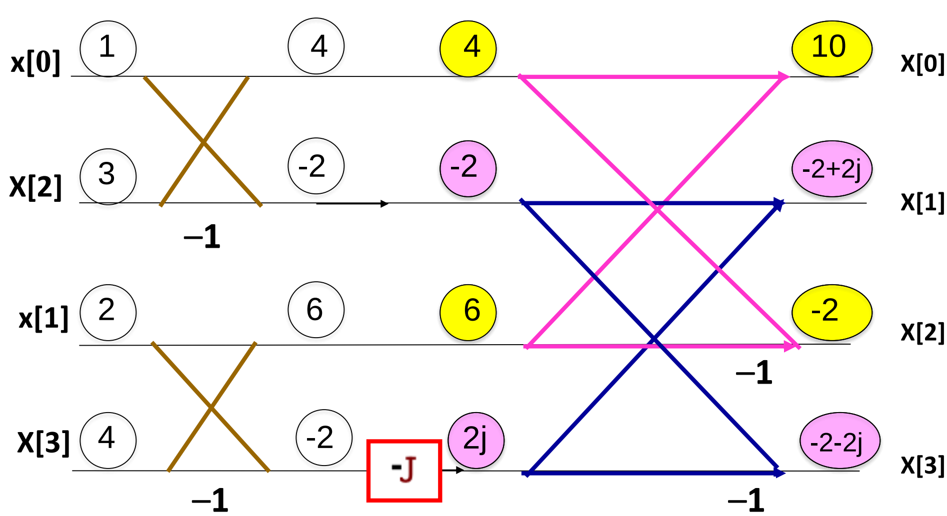
N pt $\quad \frac{N}{2}$ pt $\quad \frac{N}{2}$ pt

Step-2 : Derive DIT-FFT flowgraph



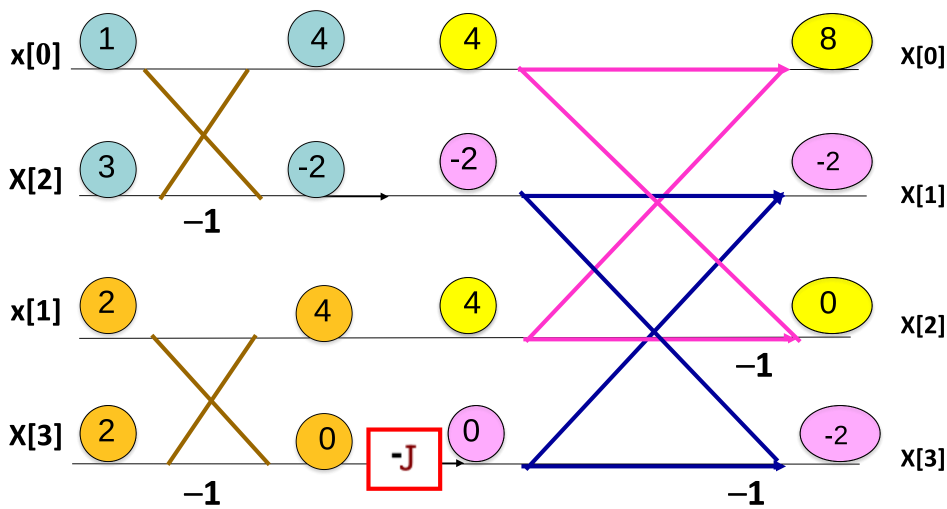


Ex-1 : Given x[n] = { 1, 2, 3, 4 }. Find X[k] using DITFFT



Ex-2 : Given x[n] = { 1, 2, 3, 4 }. Find X[k] using DITFFT

Solution : To Find X[k] using FFT



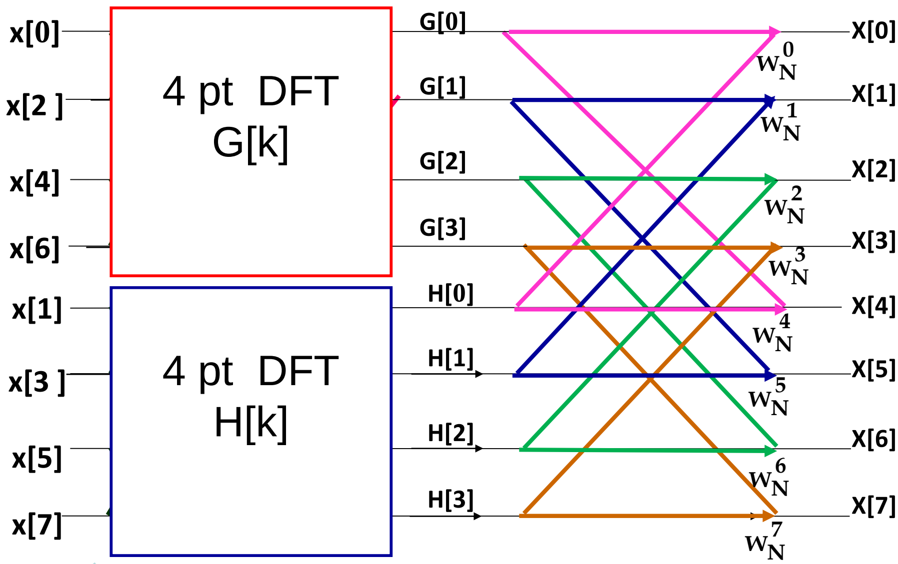
DIT FFT flowgraph for N = 8

Step-1 : Derive DIT-FFT equation

$$X[k] = G[k] + W_N^k \; H[k]$$

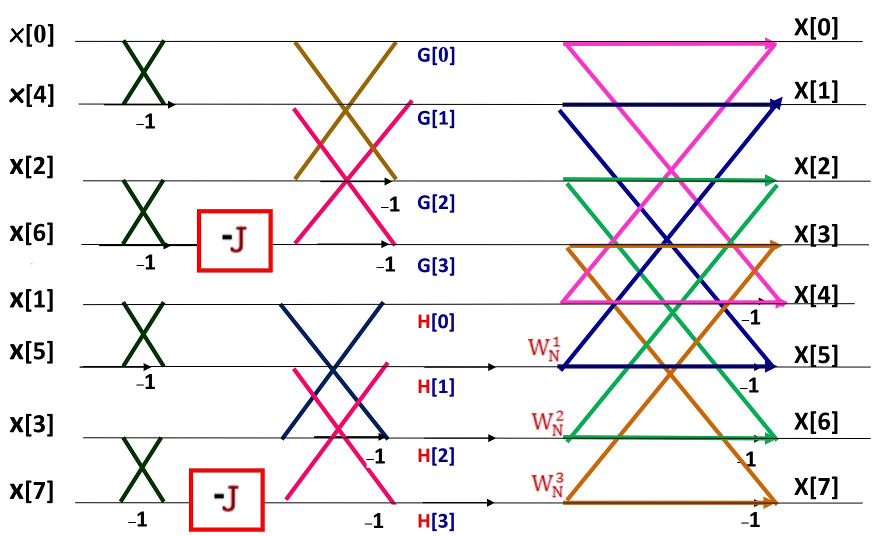
N pt $\quad \frac{N}{2}$ pt $\quad \frac{N}{2}$ pt

Step-2 : Derive DIT-FFT flowgraph



$$X[k] = G[k] + W_N^k \; H[k]$$

8 pt **4 pt** **4 pt**



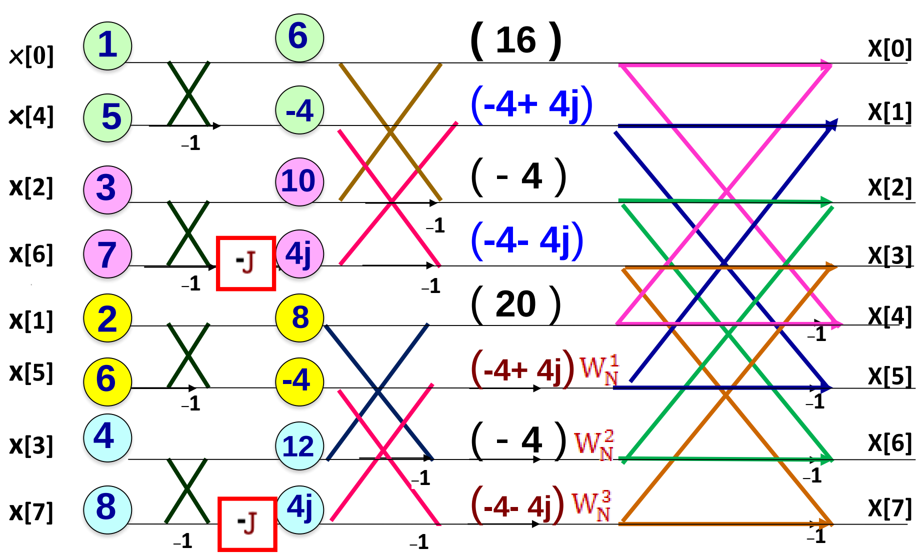
DIT-FFT Flowgraph for N = 8

Ex-1 : Given x[n] = { 1, 2, 3, 4, 5, 6, 7, 8 }.

Find X[k] using DIT-FFT.

Solution : To Find X[k] using DIT-FFT

Given x[n] = { 1, 2, 3, 4, 5, 6, 7, 8 }



Find X[k] :

(1) X[0] = (16) + (20)

X(0) = 36

(2) X[1] = (-4+4j) + (-4+4j) W_N^1

= (-4+4j) + (-4+4j) (0.707 –j 0.707)

X(1) = – 4 + j 9.656

(3) X[2] ≔ (-4) + (-4) W_N^2

= (-4) + (-4) (- j)

X(2) = - 4 + 4 j

(4) X[3] ≔ (-4-4j) + (-4-4j) W_N^3

= (-4-4j) + (-4-4j) (-0.707 –j 0.707)

X[3] = – 4 + j 1.656

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(5) X[4] = (16) - (20)

X[4] = - 4

(6) X[5] = (-4-4j) – (-4-4j) W_N^1

= (-4-4j) – (-4-4j) [0.707 –j 0.707]

X[5] = – 4 - j 1.656

(7) X[6] = (-4) - (-4) W_N^2

= (-4) - (-4) [-j]

X[6] = – 4 – 4 j

(8) X[7] = (-4-4j) – (-4-4j) W_N^3

X[7] = (-4-4j) – (-4-4j) [-0.707-j 0.707]

X[7] = – 4 – j 9.656

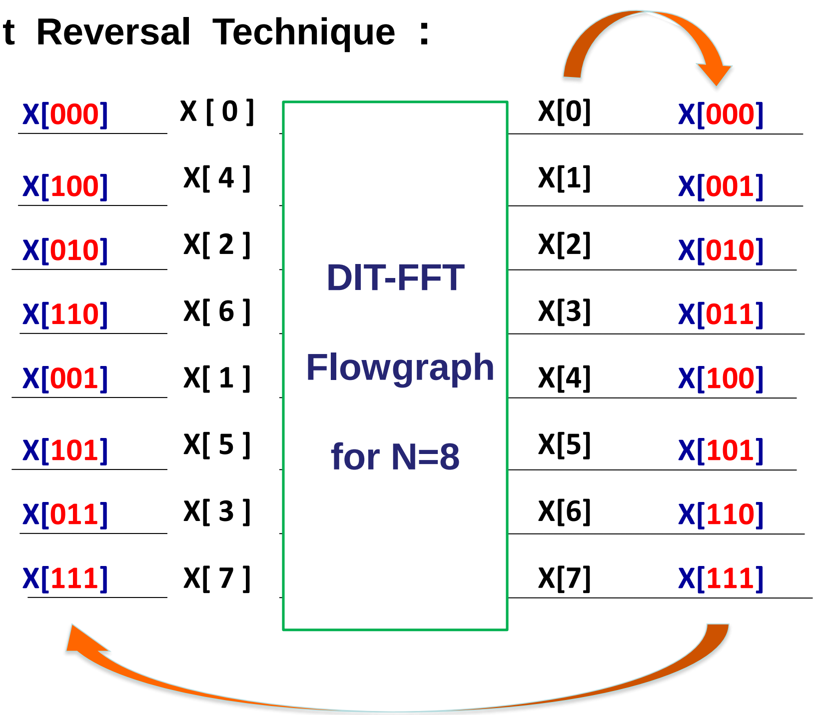
ANS :

$$X[k]=\begin{bmatrix} 36 & k=0 \\ -4+9.656j & \\ -4+4j & \\ -4+1.656j & \\ -4 & \\ -4-1.656j & \\ -4-4j & \\ -4-9.656j & \end{bmatrix}$$

Computational Efficiency of FFT

N	Complex Multiplications		Speed Improvement factor.
	By DFT	By FFT.	
4	16	4	4.0
8	64	12	5.3
16	256	32	8.0
32	1024	80	12.8
64	4096	192	21.3
128	16384	448	36.6
256	65536	1024	64.0

Bit Reversal Technique :



Bit Reversal Technique :

Input with Index in Binary	Input sequence		Output Sequence	Output with Index in Binary
x[0 0 0]	x[0]		X[0]	X[0 0 0]
x[1 0 0]	x[4]		X[1]	X[0 0 1]
x[0 1 0]	x[2]	**DIT-FFT**	X[2]	X[0 1 0]
x[1 1 0]	x[6]	Flowgraph	X[3]	X[0 1 1]
x[0 0 1]	x[1]	For	X[4]	X[1 0 0]
x[1 0 1]	x[5]	N = 8	X[5]	X[1 0 1]
x[0 1 1]	x[3]		X[6]	X[1 1 0]
X[1 1 1]	x[7]		X[7]	X[1 1 1]

Inverse FFT Algorithm

Using **Forward** FFT Flowgraph

Using **Inverse** FFT Flowgraph

Inverse FFT using Forward FFT Flowgraph

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$$

By Complex Conjugate on Both Sides,

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\, W_N^{nk}$$

$$x^*[n] = \frac{1}{N}\,\mathrm{FFT}\{X^*[k]\}$$

IFFT ALGORITHM

I. Find $X^*[k]$

II. Find FFT ($X^*[k]$)

III. Find x[n] using IFFT equation

By Complex Conjugate on Both Sides,

$$\boxed{x[n] = \frac{1}{N}\left(\mathrm{FFT}\{X^*[k]\}\right)^*}$$

This is an IFFT equation

Ex-1. Given $X[k] = \begin{bmatrix} 66 & k=0 \\ -22+2j & \\ -2 & \\ -22-2j & \end{bmatrix}$

Find x[n] using Forward FFT.

Solution : To find x[n]

By IFFT equation:

$$x[n] = \frac{1}{N}\left(FFT\{X^*[k]\}\right)^*$$

IFFT ALGORITHM

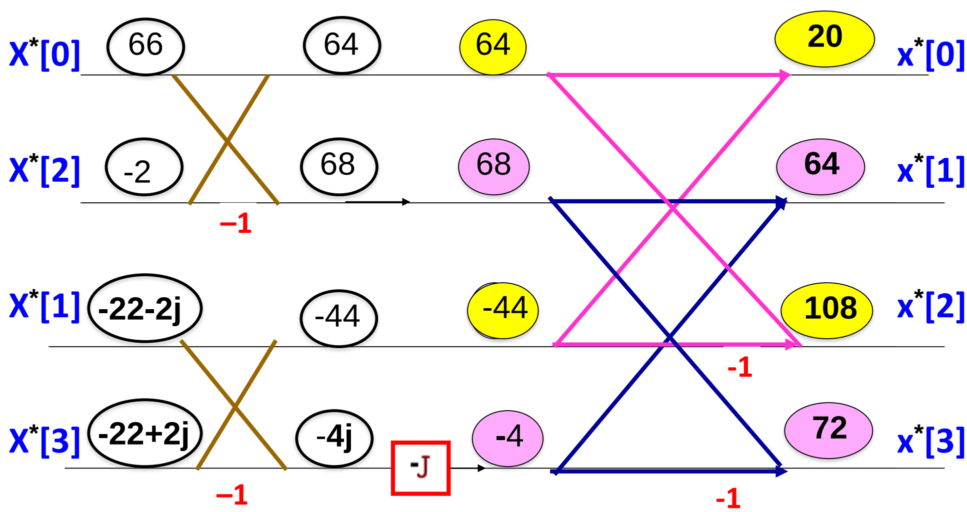
I. Find $X^*[k]$
II. Find FFT ($X^*[k]$)
III. Find x[n] using IFFT equation

I. Find $X^*[k]$

Now, $X[k] = \begin{bmatrix} 66 & k=0 \\ -22+2j & \\ -2 & \\ -22-2j & \end{bmatrix}$

$X^*[k] = \begin{bmatrix} 66 & k=0 \\ -22-2j & \\ -2 & \\ -22+2j & \end{bmatrix}$

II. Find FFT ($X^*[k]$)



III. Find x[n]

By IFFT : $x[n] = \frac{1}{N}\left(\text{FFT}\{X^*[k]\}\right)^*$

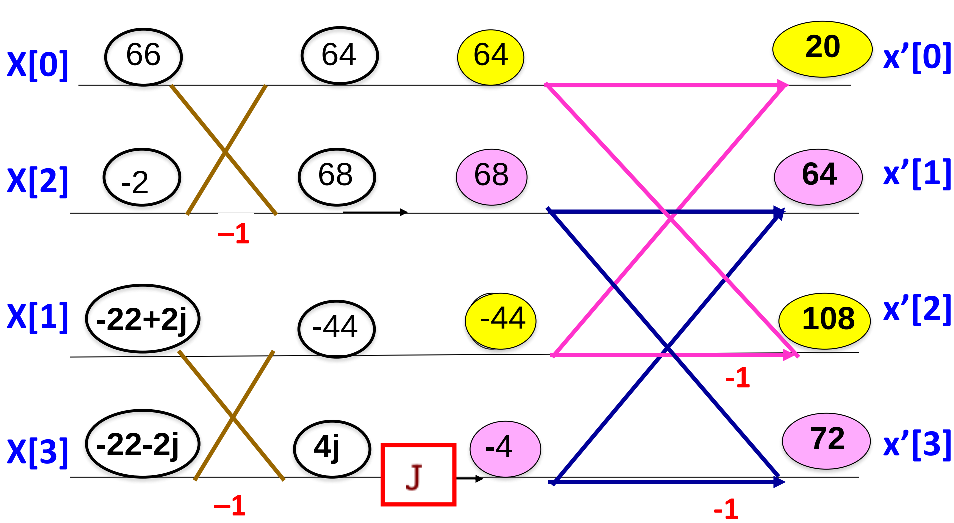
$$x[n] = \frac{1}{4}\left(\begin{array}{cc} \begin{bmatrix} 20 \\ 64 \\ 108 \\ 72 \end{bmatrix} & \begin{array}{l} n=0 \\ \\ \\ \\ \end{array} \end{array}\right)^*$$

$$x[n] = \begin{array}{cc} \begin{bmatrix} 5 \\ 16 \\ 27 \\ 18 \end{bmatrix} & \begin{array}{l} n=0 \\ \\ \\ \\ \end{array} \end{array}$$

Ex-2. Given $X[k] = \begin{array}{c} \left[\begin{array}{l} 66 \\ -22+2j \\ -2 \\ -22-2j \end{array}\right] \end{array} \begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$

Find x[n] using Inverse FFT.

To Find x[n] using Inverse FFT



III. Find x[n]

By IFFT : $x[n] = \frac{1}{N} x'[n]$

$$x[n] = \frac{1}{4} \left(\begin{bmatrix} 20 \\ 64 \\ 108 \\ 72 \end{bmatrix} \right) \quad n=0$$

$$x[n] = \begin{bmatrix} 5 \\ 16 \\ 27 \\ 18 \end{bmatrix} \quad n=0$$

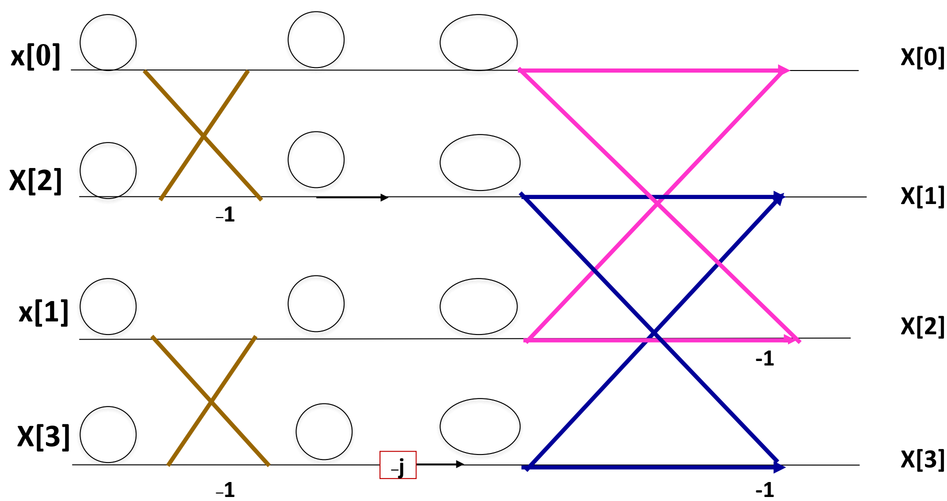
Ex : Let x[n] = {1, 2, 3, 4}

(a) Find X[k] using DIT-FFT.

(b) Let p[n] = { 1, 0, 2, 0, 3, 0, 4, 0 }.
Find P[k] using X[k].

Solution : (a) To finds X[k] using DITFFT

Given x[n] = { 1, 2, 3, 4 }.



(b) Let p[n] = { 1, 0, 2, 0, 3, 0, 4, 0 }.
Find P[k] using X[k].

- To find P[k]

Let $\underbrace{P[k]}_{\text{8 pt}} = \underbrace{G[k]}_{\text{4 pt}} + W_N^k \underbrace{H[k]}_{\text{4 pt}}$ ---Eqn (1)

Where G[k] = DFT{ p(2r) } and H[k] = DFT { p(2r+1) }

$G[k] = DFT\begin{bmatrix} p[0] \\ p[2] \\ p[4] \\ p[6] \end{bmatrix}$ $\quad H[k] = DFT\begin{bmatrix} p[1] \\ p[3] \\ p[5] \\ p[7] \end{bmatrix}$

$G[k] = DFT\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\quad H[k] = DFT\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

G[k] = X[k] H[k] = 0

By Substituting G[k] = X[k] and H[k] = 0 in Eqn (1) we get,

P[k] = X[k]

P[0] = X[0] = = 10

P[1] = X[1] = = -2+2J

P[2] = X[2] = = -2

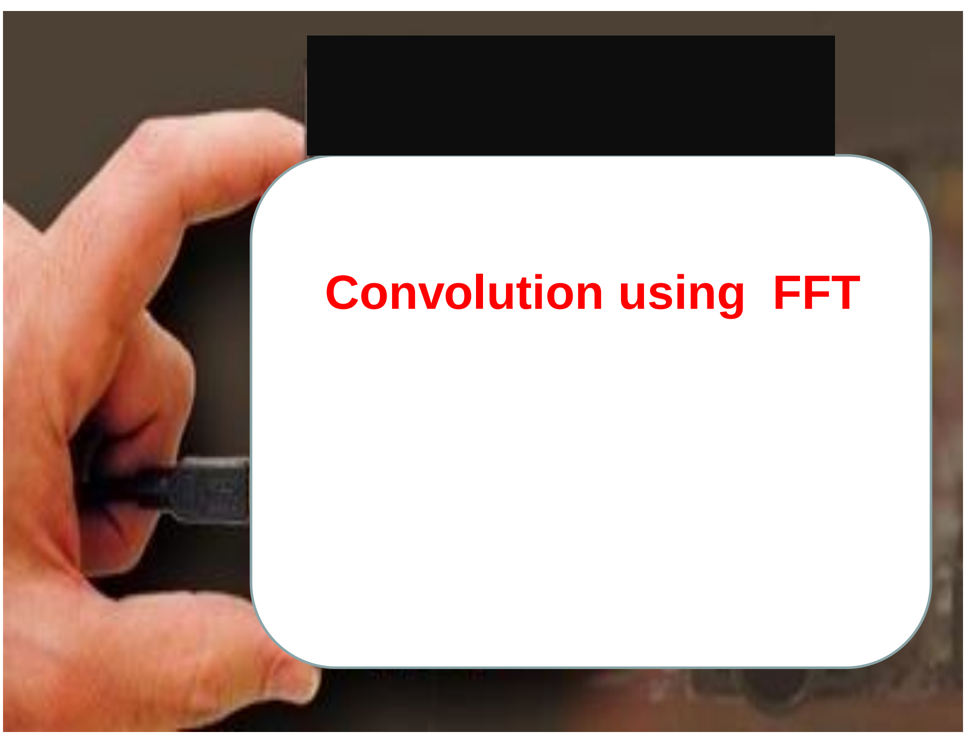
P[3] = X[3] = = -2-2J

P[4] = X[4] = = 10

P[5] = X[5] = = -2+2J

P[6] = X[6] = = -2

P[7] = X[7] = = -2-2J



Ex-1 **Let $x[n] = [\ 1, 2, 3, 4\ \}$ and $h[n] = \{\ 5, 6, 7\ \}$**

Find Circular Convolution using FFT

Solution :

Here x[n] is L=4 point and h[n] is M = 3 point

I. Select N

$$N = \text{Max}\ (L,M)$$

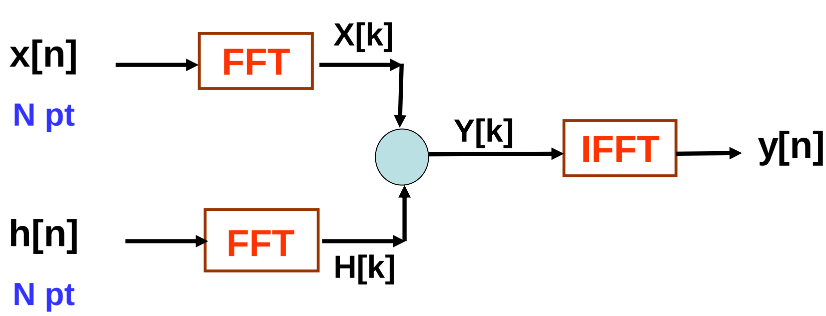
$$N = \text{Max}\ (4,3) == \mathbf{4}$$

II. Zero Padding

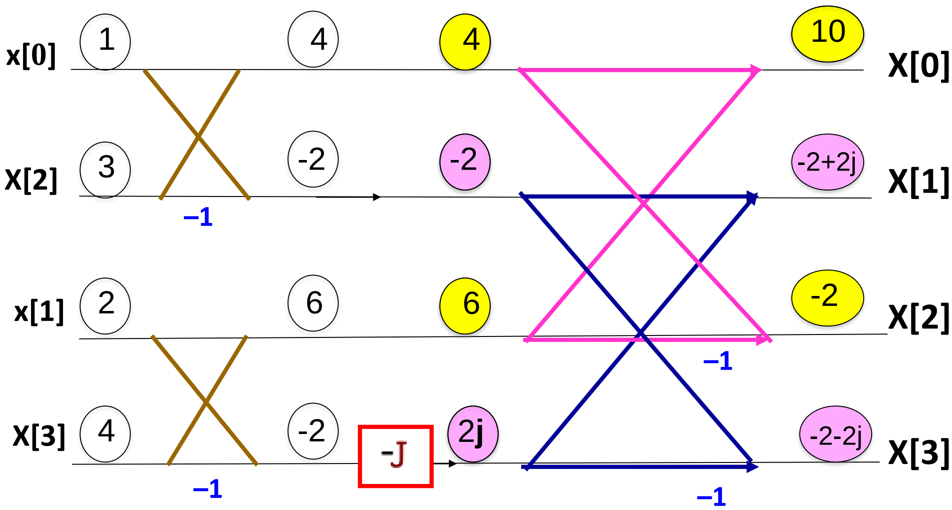
$x[n] = [\ 1, 2, 3, 4\ \}$

$h[n] = \{\ 5, 6, 7, 0\ \}$

III. Find $y[n] = x[n] \otimes h[n]$ using FFT



(1). **Find X[k] using DIT-FFT**



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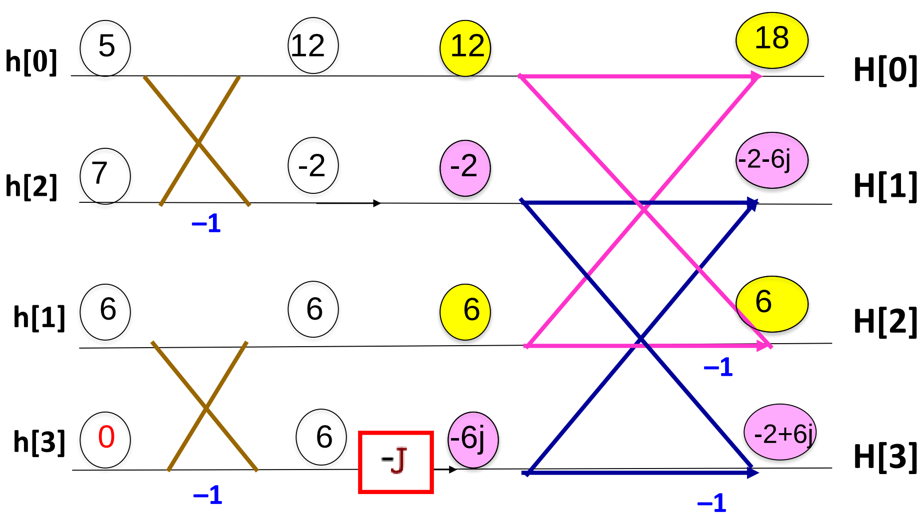
(2). **Find H[k] using DIT-FFT**



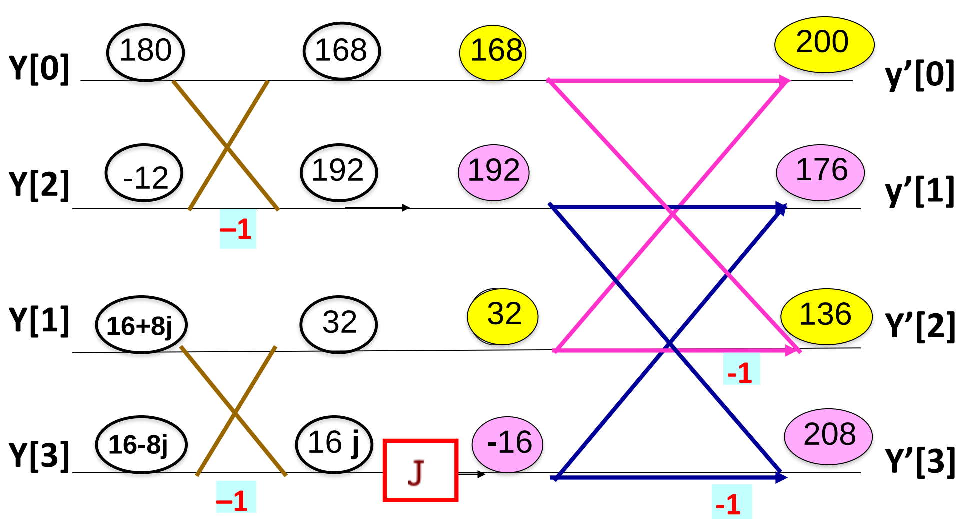
(3). Find Y[k]

$$Y[k] = X[k]\ H[k]$$

$$Y[k] = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array} \begin{bmatrix} 18 \\ -2-6j \\ 6 \\ -2+6j \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$$

$$Y[k] = \begin{bmatrix} 180 \\ 16+8j \\ -12 \\ 16-8j \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$$

III. Find y[n] By Inverse FFT



Now $y[n] = \frac{1}{N} y'[n]$

$$y[n] = \frac{1}{4}\begin{bmatrix} 200 & n=0 \\ 176 & \\ 136 & \\ 208 & \end{bmatrix}$$

$$y[n] = \begin{bmatrix} 50 & n=0 \\ 44 & \\ 34 & \\ 52 & \end{bmatrix}$$

To verify :

$$y[n] = \begin{bmatrix} 5 & 0 & 7 & 6 \\ 6 & 5 & 0 & 7 \\ 7 & 6 & 5 & 0 \\ 0 & 7 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$y[n] = \begin{bmatrix} 50 \\ 44 \\ 34 \\ 52 \end{bmatrix}$$

Ex-1 Let x[n] = [1, 2, 3 } and h[n] = { 5, 6 }
Find Linear Convolution using FFT

Solution :

Here x[n] is L= 3 point and h[n] is M = 2 point

I. Select N

$N \geq L+M-1$

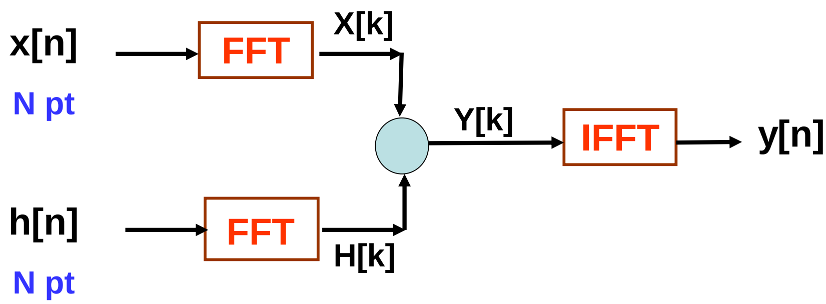
$N \geq 3+2-1 == 4$

II. Zero Padding

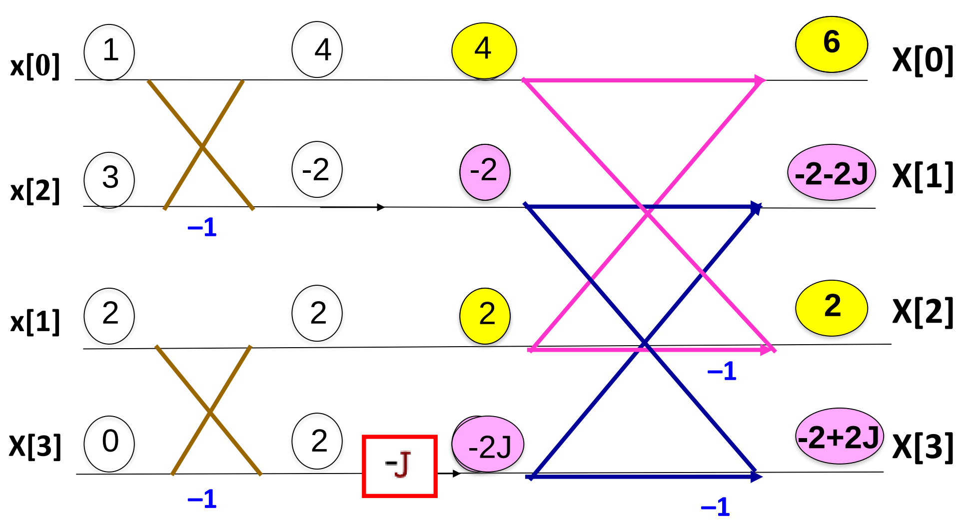
x[n] = [1, 2, 3, 0 }

h[n] = { 5, 6, 0, 0 }

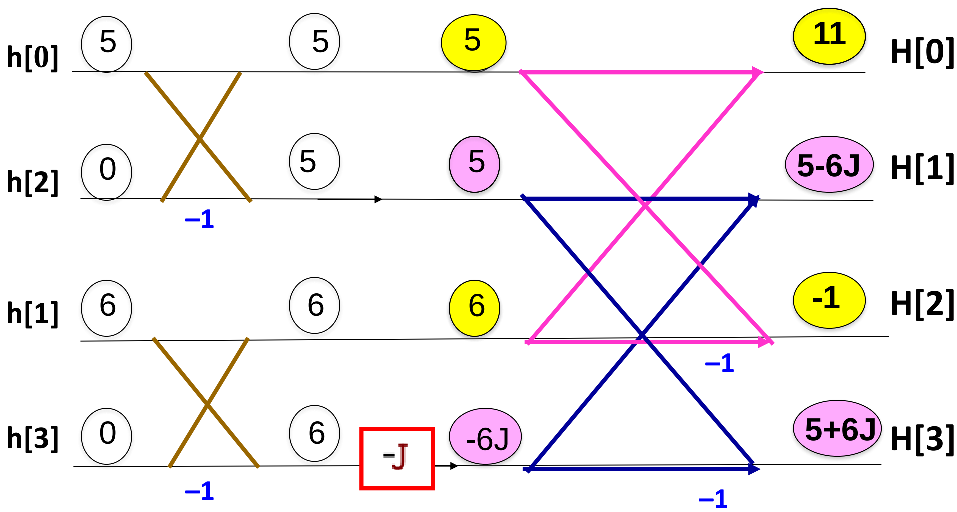
III. Find y[n] = x[n] ⊗ h[n] using FFT



(1). Find X[k] using DIT-FFT



(2). Find H[k] using DIT-FFT



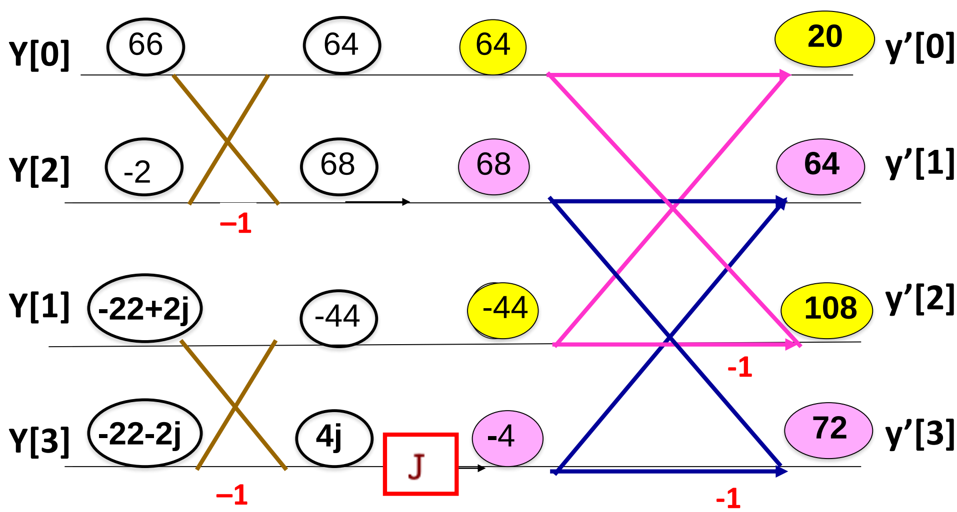
(3). Find Y[k]

$$Y[k] = X[k]\ H[k]$$

$$Y[k] = \begin{bmatrix} 6 & k=0 \\ -2-2j & \\ 2 & \\ -2+2j & \end{bmatrix} \begin{bmatrix} 11 & k=0 \\ 5-6j & \\ -1 & \\ 5+6j & \end{bmatrix}$$

$$Y[k] = \begin{bmatrix} 66 & k=0 \\ -22+2j & \\ -2 & \\ -22-2j & \end{bmatrix}$$

III. Find y[n] using Inverse FFT



Now $y[n] = \frac{1}{N} y'[n]$

$$y[n] = \frac{1}{4}\begin{bmatrix} 20 & n=0 \\ 64 & \\ 108 & \\ 72 & \end{bmatrix}$$

$$y[n] = \begin{bmatrix} 5 & n=0 \\ 16 & \\ 27 & \\ 18 & \end{bmatrix}$$

To verify :

x[n]		1	2	3	y[n]
h[-n]	6	5			5
		6	5		16
			6	5	27
				6	18

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