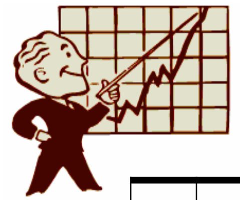

Discrete Fourier Transform

	TOPIC
1	Introduction to DTFT and DFT
2	Relation between DFT and DTFT
3	Properties of DFT
4	DFT computation using DFT properties
5	Linear and Circular Convolution using DFT

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Chapter-2 : Discrete Fourier Transform

Objective : To explore the properties of DFT in mathematical problem solving

Outcome :

At the end of module, students will be able to ,

- **Derive** DFT from DTFT
- **Covert** signal from time domain to frequency domain
- **Justify** the need of DFT
- **Evaluate** DFT and IDFT equations,
- **Apply** DFT properties in problem solving
- **Perform** Linear and Circular Convolution using DFT.

Discrete Time Fourier Transform (DTFT)

(1) DTFT of DT signal x[n] is defined as ,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n).e^{-jn\omega}$$

(2) Inverse DTFT of X(w) is is defined as ,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Properties of DTFT

Periodicity: $X(\mathrm{w}+2\pi) = X(\mathrm{w})$

Linearity: $a x_1[n] + b x_2[n] \longleftrightarrow aX_1(\mathrm{w}) + bX_2(\mathrm{w})$

Time Shifting: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(\mathrm{w})$

Frequency Shifting: $e^{j\omega_0 n} x[n] \longleftrightarrow X(\omega-\omega_0)$

Time Reversal: $x[-n] \longleftrightarrow X(-\omega)$

Symmetry: $x[n]$ real $\Rightarrow X(\omega) = X^*(-\omega)$

Limitations of DTFT

DTFT is

- Not practical for (real-time) computation on a digital computer

- **Solution**: Limit the extent of the summation to *N* points and evaluate the continuous function of frequency at *N* equi-spaced points.

Relation between DFT and DTFT.

DFT is frequency sampling of DTFT

$$X[k] = X(w)\Big|_{w=\frac{2\pi k}{N}}$$

Frequency spacing $w = \dfrac{2\pi}{N}$

- **The DFT is simply a sampling of the DTFT at equi spaced points along the frequency axis.**

Derivation of DFT equation

By DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n).e^{-jn\omega}$$

Put $w = \dfrac{2\pi k}{N}$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n).e^{-jn\left(\frac{2\pi k}{N}\right)}$$

$$X(k) = \sum_{n=-\infty}^{\infty} x(n).e^{\left(\frac{-j2p}{N}\right)nk}$$

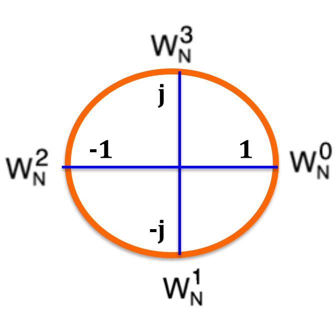
Put $W_N^1 = e^{\frac{-2p}{N}}$

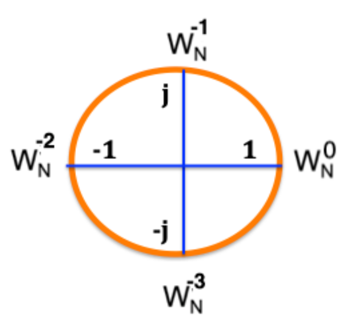
$$X[k] = \sum_{n=0}^{N-1} x[n]\, w_N^{nk}$$

Cyclic Property of Twiddle factor W_N

Twiddle factor W_N is periodic with period = N

(1) Twiddle factor W_N for N = 4 :





(2) Twiddle factor W_N^k for N = 8

W_N^0 = **1**

W_N^1 = **0.707- j 0.707**

W_N^2 = -j

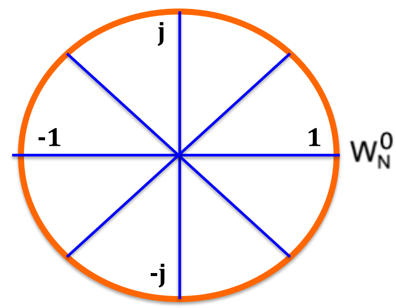
W_N^3 = **-0.707 – j 0.707**

W_N^4 = -1

W_N^5 = **-0.707 + j 0.707**

W_N^6 = j

W_N^7 = **0.707 + j 0.707**



(3) Twiddle factor W_N^{-k} for N = 8

W_N^0 = **1**

W_N^{-1} = **0.707 + j 0.707**

W_N^{-2} = j

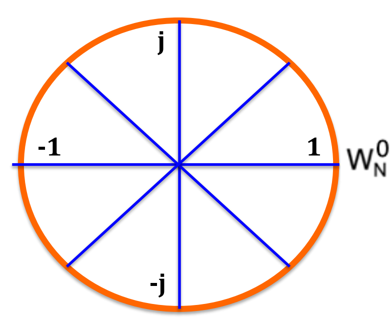
W_N^{-3} = **-0.707 + j 0.707**

W_N^{-4} = -1

W_N^{-5} = **-0.707 – j 0.707**

W_N^{-6} = -j

W_N^{-7} = **0.707 – j 0.707**



Ex-1. Let $x[n] = \{ 1, 2, 3, 4 \}$ **Find X[k]**

Solution : To Find X[k]

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

$$where\ N = 4\ and\ W_N^1 = e^{-j\frac{2\pi}{N}}$$

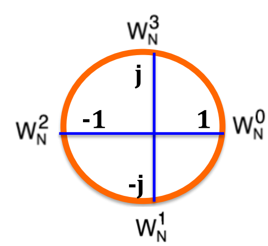
$$X[k] = \sum_{n=0}^{3} x[n] w_N^{nk}$$

$X[k] = X[0] + X[1] W_N^{k} + X[2] W_N^{2k} + X[3] W_N^{3k}$

$X[k] = 1 + 2 W_N^{k} + 3 W_N^{2k} + 4 W_N^{3k}$

(i) $X[0] = 1 + 2 + 3 + 4$
$= 10$

(ii) $X[1] = 1 + 2 W_N^{1} + 3 W_N^{2} + 4 W_N^{3}$
$= 1 + 2(-j) + 3(-1) + 4(j)$
$X[1] = -2 + 2j$



(iii) $X[2] = 1 + 2 W_N^{2} + 3 W_N^{4} + 4 W_N^{6}$
$= 1 + 2(-1) + 3(1) + 4(-1)$
$X[2] = -2$

X[k] =

(iv) $X[3] = 1 + 2 W_N^{3} + 3 W_N^{6} + 4 W_N^{9}$
$= 1 + 2(j) + 3(-1) + 4(-j)$
$X[3] = -2-2j$

Matrix Representation of DFT and Inverse DFT

Let $x[n] = \{ 1, 2, 3, 4 \}$

By DFT, $X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$

$$X[k] = x[0]\, W_N^{0} + x[1]\, W_N^{k} + x[2]\, W_N^{2k} + x[3]\, W_N^{3k}$$

In Matrix Form :

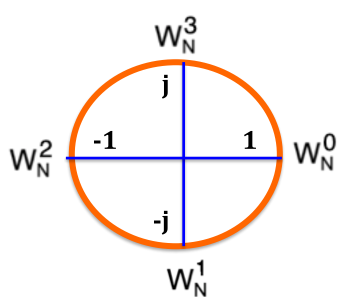
$$\begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}}_{X[k]} == \underbrace{\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix}}_{\text{DFT Matrix}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}}_{x[n]}$$

By Substituting :

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} (1) + (2) + (3) + (4) \\ (1) + (-2j) + (-3) + (4j) \\ (1) + (-2) + (3) + (-4j) \\ (1) + (-2j) + (-3) + (4j) \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \end{bmatrix}$$



Ex-1. **Let x[n] = { 1, 2, 3, 2 }** **Find X[k]**

Solution : To Find X[k]

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

$$where\ N = 4\ and\ W_N^1 = e^{-j\frac{2\pi}{N}}$$

In Matrix Form :

$$X[k] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

By Substituting :

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} (1) + (-2j) + (3) + (2j) \\ (1) + (-2j) + (-3) + (2j) \\ (1) + (-2) + (3) + (-2) \\ (1) + (2j) + (-3) + (-2j) \end{bmatrix}$$

Ans :

$$X[k] = \begin{bmatrix} 8 & k=0 \\ -2 & \\ -2 & \\ -2 & \end{bmatrix}$$

Ex-2. Given $X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$ Find x[n].

Solution : To Find x[n]

By IDFT,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, w_N^{-nk}$$

$$where\ N = 4\ and\ W_N^1 = e^{-j\frac{2\pi}{N}}$$

In Matrix Form :

$$x[n] = \frac{1}{N} \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & W_N^{-3} \\ W_N^0 & W_N^{-2} & W_N^{-4} & W_N^{-6} \\ W_N^0 & W_N^{-3} & W_N^{-6} & W_N^{-9} \end{bmatrix} \begin{bmatrix} X[O] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

By Substituting :

$$x[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

x[n] = { 1, 2, 3, 4 }

Ex-2. **Let $x[n] = \{ 1, 0, 2, 0, 3, 0, 4, 0 \}$ Find X[k]**

Solution : To Find X[k]

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

$$\text{where } N = 8 \text{ and } W_N^1 = e^{-j\frac{2\pi}{N}}$$

$$X[k] = \sum_{n=0}^{7} x[n] w_N^{nk}$$

$X[k] = x[0] + x[1]\ W_N^{k} + x[2]\ W_N^{2k} + x[3]\ W_N^{3k} +$
$x[4]\ W_N^{4k} + x[5]\ W_N^{5k} + x[6]\ W_N^{6k} + x[7]\ W_N^{7k}$

$X[k] = 1 + 2\ W_N^{2k} + 3\ W_N^{4k} + 4\ W_N^{6k}$

$$x[n] = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

(i) $X[0] = 1 + 2 + 3 + 4$
$X[0] = 10$

(ii) $X[1] = 1 + 2\ W_N^2 + 3\ W_N^4 + 4\ W_N^6$
$= 1 + 2(-j) + 3(-1) + 4(j)$

$X[1] =$

(iii) $X[2] = 1 + 2\ W_N^4 + 3\ W_N^8 + 4\ W_N^{12}$
$= 1 + 2(-1) + 3(1) + 4(-1)$

$X[2] =$

$$X[k] = 1 + 2\,W_N^{2k} + 3\,W_N^{4k} + 4\,W_N^{6k}$$

(iv) $X[3] = 1 + 2\,W_N^{6} + 3\,W_N^{12} + 4\,W_N^{18}$
$= 1 + 2(j) + 3(-1) + 4(-j)$
$X[3] =$

(v) $X[4] = 1 + 2\,W_N^{8} + 3\,W_N^{16} + 4\,W_N^{24}$
$= 1 + 2(1) + 3(1) + 4(1)$
$X[4] =$

(vi) $X[5] = 1 + 2\,W_N^{10} + 3\,W_N^{20} + 4\,W_N^{30}$
$= 1 + 2(-j) + 3(-1) + 4(j)$
$X[5] =$

vii) $X[6] = 1 + 2\,W_N^{12} + 3\,W_N^{24} + 4\,W_N^{36}$
$= 1 + 2(-1) + 3(1) + 4(-1)$
$X[6] =$

$$x[n] = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$X[k] = 1 + 2\,W_N^{2k} + 3\,W_N^{4k} + 4\,W_N^{6k}$$

(viii) $X[7] = 1 + 2\,W_N^{14} + 3\,W_N^{28} + 4\,W_N^{42}$
$= 1 + 2(j) + 3(-1) + 4(-j)$
$X[7] = -2 - 2j$

$$\text{ANS}\quad X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \\ 10 & \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

Ex-2 : Find DFT of the following Sequences :

(a) x[n] = { 1, 1, 1, 1 }   (b) x[n] = { 1, 0, 0, 0 }

Solution :

(a) To Find X[k]

$$X[k] = \begin{array}{l} \\ \\ \\ \\ \end{array}\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$$

(b) To Find X[k]

$$X[k] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$$

Note :

1. What is the DFT of δ[n] ?

- **Ans :** **DFT { δ[n] } = 1**

2. What is the DFT of N pt signal u[n]

- **Ans :** **DFT {u[n] } = N δ[k]**

Where

$$\delta[k] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{array}{l} k=0 \\ \\ \\ \\ \end{array}$$

Ex-3 Find DFT x[n] where x(n) = { 1, 2, 3, 4 }

Solution :

Step-1 : Find X(w) i.e. DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n).e^{-jn\omega}$$

$X(w) = x[-1]\, e^{jw} + x[o] + x[1]\, e^{-jw} + x[2]\, e^{-j2w}$

$X(w) = e^{jw} + 2 + 3\, e^{-jw} + 4\, e^{-j2w}$

$X(w) = e^{jw} + 2 + 3\, e^{-jw} + 4\, e^{-j2w}$

$X(w) = \text{Cos}(w) + j\,\text{Sin}(w) + 2$
$+ 3\,\text{Cos}(w) - 3j\,\text{Sin}(w)\, e^{-jw}$
$+ 4\,\text{Cos}(2w) - 4j\,\text{Sin}(2w)$

$X(w) = [\ 2 + 4\,\text{Cos}(w) + 4\,\text{Cos}(2w)\ ]$
$-j\ [\ 2\,\text{Sin}(w) + 4\,\text{Sin}(2w)\ ]$

DTFT of x[n]

Step-2 : Find X[k] by Sampling X(w)

Now X(w) = [2 + 4 Cos(w) + 4 Cos(2w)]
-j [2 Sin(w) + 4 Sin (2w)]

$$X[k] = X(w)\Big|_{w=\frac{2\pi k}{N}}$$

Put $w = \frac{2\pi k}{N} = \frac{2\pi k}{4} = \frac{\pi k}{2}$

$$X[k] = \left[2 + 4\cos\left(\frac{\pi k}{2}\right) + 4\cos(\pi k)\right] - j\left[2\sin\left(\frac{\pi k}{2}\right) + 4\sin(\pi k)\right]$$

$$X[k] = \left[2 + 4\cos\left(\frac{\pi k}{2}\right) + 4\cos(\pi k)\right] - j\left[2\sin\left(\frac{\pi k}{2}\right) + 4\sin(\pi k)\right]$$

By evaluating X[k] for k = 0,1, 2, 3 We get,

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2-2j & \\ 2 & \\ -2+2j & \end{bmatrix}$$

Properties of DFT

[1] Scaling and Linearity Property

If $x_1[n] \rightarrow X_1[k]$

$x_2[n] \rightarrow X_2[k]$

Then

DFT { a $x_1[n]$ + b $x_2[n]$ } =

Ex. Let x[n] = { 1, 2, 3, 4 }

(a) Find X[k]

Solution : (a) To Find X[k]

(i) **Formula**

(ii) **Matrix Representation**

(iii) **Matrix Substitution**

(iv) **Matrix Multiplication**

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(b) Let $p[n] = 2\,\delta[n] + x[n]$ Find $P[k]$ using $X[k]$

Solution (b) : To find $P[k]$ using $X[k]$

Given $p[n] = 2\,\delta[n] + x[n]$

By Linearity Property of DFT,

$P[k] = 2\,\text{DFT}\{\,\delta[n]\,\} + \text{DFT}\{x[n]\}$

$P[k] =$

$P[k] = 2 + X[k]$ where $X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$

==========================

$k=0$, $P[0] = 2 + X[0]$ ==

$k=1$, $P[1] = 2 + X[1]$ ==

$k=2$, $P[2] = 2 + X[2]$ ==

$k=3$, $P[3] = 2 + X[3]$ ==

Ans : $P[k] = \begin{bmatrix} & k=0 \\ & \\ & \\ & \end{bmatrix}$

(C) Let q[n] = 2 + x[n] Find Q[k] using X[k]

Solution (c): **To find Q[k] using X[k]**

Now, q[n] = 2 + x[n]

q[n] = 2 {1} + x[n]

Let, q[n] = 2 {u[n] } + x[n]

By Linearity Property of DFT,

Q[k] = 2 DFT {u[n]} + DFT{x[n]}

Q[k] = 2 DFT {u[n]} + DFT{x[n]}

Q[k] = 2 { 4 δ[k] } + X[k]

$$Q[k] = 8 \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \right\}$$

$$Q[k] = \begin{bmatrix} & k=0 \\ & k=1 \\ & k=2 \\ & k=3 \end{bmatrix}$$

HW-1. Let X[k] be 4 point DFT of x[n] with
X[k] ={ 1, 2, 3, 4 }.
Find 4 point DFT of p[n] such that
p[n] = 2 + 3 δ[n] + 4 x[n]

HW-2. Let x[n] = { 1, 2, 3, 4 } and x[n] ←→ X[k].
Find inverse DFT of the following without using DFT/iDFT equations.
(a) P[k] = 8 X[K] (b) Q[k] = 8 + X[k]

[2] Periodicity Property

If x[n] → X[k]

Then

(i) **x[n] = x[n+N]** **i.e. x[n] is periodic**

= **x[n Mod N]**

= $x[((n))]_N$

(ii) **X[k] = X[k+N]** **i.e. X[k] is periodic**

= **X[k Mod N]**

= $X[((k))]_N$

NOTE :
Both DFT and IDFT equations produce periodic results with period = N.

[3] Time Shift Property

If x[n] → X[k]

Then

DFT { x[n - m] } =

[4] Frequency Shift Property

If x[n] → X[k]

Then

DFT { W_N^{-mn} x[n] } =

Ex. Let x[n] = { 1, 2, 3, 4 }

(a) Find X[k]

Solution : (a) To Find X[k]

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(b) Let p[n] = { 4, 1, 2, 3 }. Find P[k] using X[k].

Solution :

(b) To find P[k]

Now, x[n] = { 1, 2, 3, 4 }

Given p[n] = { 4, 1, 2, 3 }

By comparing x[n] and p[n] we get,

p[n] =

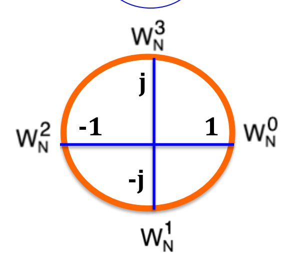
Now p[n] = x[n – 1]

Time Shift Property of DFT
DFT {x[n-m]} = $W_N^{mk} X[k]$

By Time Shift Property of DFT,

$$P[k] = W_N^k X[k]$$

$$P[k] = \begin{bmatrix} W_N^0 \\ W_N^1 \\ W_N^2 \\ W_N^3 \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$



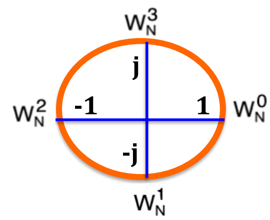
$$P[k] = \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \rightarrow P[k] = \begin{bmatrix} 10 & k=0 \\ 2+2j & \\ 2 & \\ 2-2j & \end{bmatrix}$$

(c) Let p[n] = (– 1)^n x[n] Find P[k] using X[k].

Solution (b): To find P[k]

Given p[n] = (– 1)^n x[n]

For N = 4, $W_N^2 = -1$



By Substituting,

p[n] = W_N^{2n} x[n]

By Frequency Shift Property of DFT,

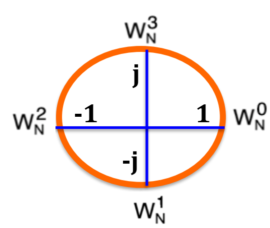
P[k] =

Frequency Shift Property of DFT
DFT {W_N^{-mn} x[n]} ==X[k-m]

Solution (b): To find P[k]

Given **p[n] = (– 1)^n x[n]**

For N = 4, $W_N^2 = -1$



By Substituting,

p[n] = W_N^{2n} x[n]

By Frequency Shift Property of DFT,

P[k] = X[k+2]

P[k] = $\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \end{bmatrix}$$

EX-2. **Let $x[n] = \{ 1, 2, 3, 4 \}$**
Find the Inverse DFT of the following sequences without using using DFT/iDFT equation. Given (a) $X[k-2]$ (b) $X[k+2]$

Solution (a): To find Inverse DFT { X[k-2] }

Let $P[k] = X[k-2]$

By Frequency Shift Property of IDFT,

$p[n] = W_N^{-2n}\ x[n]$

$p[n] = (-1)^n\ x[n]$

ANS : p[n] = { 1,

Frequency Shift Property of DFT
DFT $\{W_N^{-mn}\, x[n]\}$ $== X[k-m]$

Solution (a): To find Inverse DFT { X[k+2] }

Let **$P[k] = X[k+2]$**

By Frequency Shift Property of IDFT,

$p[n] = W_N^{-2n}\ x[n]$

$p[n] = (-1)^n\ x[n]$

ANS : p[n] = {

Frequency Shift Property of DFT
DFT $\{W_N^{-mn}\, x[n]\}$ $== X[k-m]$

Here,
$x[n] = \{1, 2, 3, 4\}$

HW-1. Find the DFT of the following sequences :

(a) $x[n] = \cos(0.5 \pi n)$

(b) $x[n] = \sin(0.25 \pi n)$

Hint :

1. Calculate one Period of Periodic x[n]

2. Calculate X[k] by DFT

HW-2. Find the DFT of the following sequences :

(a) $x[n] = \cos(0.5 \pi n)\ u[n]$

(b) $x[n] = \sin(0.25 \pi n)\ u[n]$

Hint :

1. Let $$x[n] = \left(\frac{e^{j0.5pn} + e^{-j0.5pn}}{2} \right) u[n]_{4pt}$$

2. Calculate X[k] by Frequency Shift Property of DFT

[5] Time Reversal Property

If $x[n] \rightarrow X[k]$

Then

DFT { x[- n] } =

Ex. Let x[n] = { 10, 20, 30, 40 }

(a) Find X[k]

(b) Let p[n] = { 1, 4, 3, 2 } Find P[k] using X[k]

Solution (a) To Find X[k] :

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

Solution (b): To find P[k]

Given **p[n] = { 1, 4, 3, 2 }**

By comparing p[n] and x[n] we get,

p[n] = x[-n]

By Time Reversal Property of DFT,

P[k] =

P[k] = $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

x[n] = { 1, 2, 3, 4 }

p[n] = { 1, 4, 3, 2 }

Time Reversal Property of DFT

DFT {x[-n]} = X[-k]

X[k] = $\begin{bmatrix} 10 & k=0 \\ -2+2j & k=1 \\ -2 & k=2 \\ -2-2j & k=3 \end{bmatrix}$

Ex-2 Let **X[k] = {1, 2, 3, 4}**.
Find the DFT of the following sequences using X[k] and not otherwise
(a) x[-n] (b) x[–n+1]

Solution (a): To find DFT { x[-n]}

Let **p[n] = x[-n]**

By Time Reversal Property of DFT,

P[k] = X[-k]

P[k] = $\begin{bmatrix} 1 & k=0 \\ 4 & \\ 3 & \\ 2 & \end{bmatrix}$

X[k] = $\begin{bmatrix} 1 & k=0 \\ 2 & k=1 \\ 3 & k=2 \\ 4 & k=3 \end{bmatrix}$

Solution (b): To find DFT { x[-n+1] }

Let **p[n] = x[-n]**

Replace (n) by (n-1)

p[n-1] = x[-(n-1)]

p[n-1] = x[-n+1]

By DFT,

DFT (p[n-1]) = DFT(x[-n+1])

Time Shift Property of DFT
DFT { x[n-m]} = W_N^{mk} X[k]

To find DFT { x[-n+1] }...

DFT (p[n-1]) = DFT(x[-n+1])

DFT (x[-n+1]) = DFT(p[n-1])

By Time Shift Property of DFT,

DFT (x[-n+1]) = W_N^k P[k]

DFT (x[-n+1]) = W_N^k X[-k]

$$\text{DFT (x[-n+1])} = \begin{bmatrix} 1 \\ -j \\ -1 \\ -j \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4j \\ -3 \\ -2j \end{bmatrix}$$

Time Shift Property of DFT
DFT { x[n-m]} = W_N^{mk} X[k]

[6] Symmetry Property

If x[n] is **Real valued** sequence

Then

X[k] =

=

=

Ex-1 The first five points of the eight point DFT of a real valued sequence are

X[k] = { 25, **0.12 – j 0.30**, 6.4, **0.20 + j 0.18**, 10 }.

Determine the remaining three points.

Solution :

Here x[n] is real valued N=8 point DT Signal.

By symmetry property of DFT,

If DFT { x[n] } = X[k]

Then

X[k] =

X[k] =

By symmetry property of DFT,

If DFT { x[n] } = X[k]

Then

$$X[k] = X^*[-k]$$
$$= X^*[N-k]$$
$$X[k] = X^*[8-k]$$

=================

k=5, $X[5] = X^*[3]$

= 0.20 - j 0.18

k=6, $X[6] = X^*[2]$

= 6.4

k=7, $X[7] = X^*[1]$

= 0.12 + j 0.30

Answer :

X[k] =		
	25	k=0
	0.12 – j 0.30	k=1
	6.4	k=2
	0.20+ j 0.18	k=3
	10	k=4
	0.20 - j 0.18	k=5
	6.4	k=6
	0.12 + j 0.30	k=7

NOTE:

X[k] =		
	25	k=0
	0.12 – j 0.30	
	6.4	
	0.25+ j 0.18	
	10	
	0.25 - j 0.18	
	6.4	
	0.12 + j 0.30	

NOTE:

If x[n] is Real valued sequence,
Then
Real { X[k] } is Symmetric @ k = N/2
And
Imaginary {X[k]} is Anti-Symmetric @ k = N/2

Ex-2 **The Find the unknown values of x[n] and X[k]**

(a) $x[n] = \{\ ?\ , 3, -4, 0, 2\ \}$

$X[k] = \{\ 5,\ ?,\ -1.28+4.39j\ ,\ ?,\ 8.78–1.4j\ \}$

(b) $x[n] = \{\ 2\ , 3, -4, 2, 0, 1\}$

$X[k] = \{\ 4,\ ?,\ 4–5.2j,\ -8\ ,\ ?,\ 4+1.73\ j\ \}$

•

[7] Even Signal Property

If $x[n] =$

Then $X[k] =$

[8] Odd Signal Property

If $x[n] =$

Then $X[k] =$

Ex-1 : **Let x[n]= { 1, 2, 3, 4 }**

(a) Find X[k].

(b) Find DFT of $x_e[n]$ and $x_o[n]$ using X[k] and not otherwise

Solution (a) To Find X[k] :

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

Solution : To find DFT of $x_e[n]$ using X[k]

$$x_e[n] = \tfrac{1}{2}\,(x[n] + x[-n])$$

$x[n] = x_e[n] + x_0[n]$

$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$

$x_0[n] = \tfrac{1}{2}(x[n] - x[-n])$

By Linearity Property of DFT,

$$X_e[k] = \tfrac{1}{2}\,(X[k] + X[-k])$$

$$X_e[k] = \tfrac{1}{2}\left(\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} + \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix} \right)$$

$$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

$$X_e[k] = \begin{bmatrix} 10 & k=0 \\ -2 & \\ -2 & \\ -2 & \end{bmatrix}$$

Xe[k] = Real part of X[k]

This is valid only for real valued sequence x[n]

Solution : **To find DFT of $x_o[n]$ using X[k]**

$x_0[n] = ½ \; (x[n] - x[-n])$

By Linearity Property of DFT,

$X_o[k] = ½ \; (X[k] - X[-k])$

$X_o[k] = ½ \left(\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} - \begin{bmatrix} 10 \\ -2-2j \\ -2 \\ -2+2j \end{bmatrix} \right)$

$x[n] = x_e[n]+x_0[-n]$
$x_e[n]=0.5(x[n] + x[-n])$
$x_0[n]=0.5(x[n] - x[-n])$

$X[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$

$X_o[k] = \begin{bmatrix} 0 & k=0 \\ -2j & \\ 0 & \\ -2j & \end{bmatrix}$

Xo[k] = Imaginary part of X[k]
This is valid only for real valued sequence x[n]

[9] Complex Conjugate Sequence Property

If $x[n] \rightarrow X[k]$

Then

DFT { x^*[n] } =

Ex : Let $x[n] = \begin{bmatrix} 1+j & n=0 \\ 2+2j & \\ 3+3j & \\ 4+2j & \end{bmatrix}$

(a) Find X[k].
(b) Let p[n] = { 1,2,3,4} and q[n] = { 1,2,3,2}
Find P[k] and Q[k] using X[k]

Solution (a) To Find X[k] :

By DFT, $$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

$$where\ N = 4\ and\ W_N^1 = e^{-j\frac{2\pi}{N}}$$

In Matrix Form :

$$X[k] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

By Substituting :

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+1j \\ 2+2j \\ 3+3j \\ 4+2j \end{bmatrix}$$

$$X[k] = \begin{bmatrix} (1+j) + (2+2j) + (3+3j) + (4+2j) \\ (1+j) + (-2j+2) + (-3-3j) + (4j-2) \\ (1+j) + (-2-2j) + (3+3j) + (-4-2j) \\ (1+j) + (2j-2) + (-3-3j) + (-4j+2) \end{bmatrix}$$

$$X[k] = \begin{bmatrix} 10+8j & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(b) Let p[n] = { 1, 2, 3, 4} and q[n] = { 1, 2, 3, 2}
Find P[k] and Q[k] using X[k]

Solution (b) To Find P[k] and Q[k]

$$x[n] = \begin{bmatrix} 1 + 1j \\ 2 + 2j \\ 3 + 3j \\ 4 + 2j \end{bmatrix}$$

Let x[n] = p[n] + q[n] j

To Find P[k] using X[k]

Now, x[n] = p[n] + j q[n] ….. (I)

By Complex Conjugate on both sides :

x*[n] = p[n] – j q[n] ….. (II)

====================================

Adding (I) and (II) we get;

x[n] + x*[n] = 2 p[n]

So, p[n] = ½ (x[n] + x*[n])

To Find P[k] using X[k].. .. ..

Now, $p[n] = \frac{1}{2} (x[n] + x^*[n])$

By Linearity & Complex Conjugate Property of DFT,

$$P[k] = \frac{1}{2} (X[k] + X^*[-k])$$

$$P[k] = \frac{1}{2} \left(\begin{bmatrix} 10+8j \\ -2 \\ -2 \\ -2-4j \end{bmatrix} + \begin{bmatrix} 10-8j \\ -2 \\ -2 \\ -2-4j \end{bmatrix} \right)$$

$$P[k] = \begin{bmatrix} 10 & k=0 \\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

To Find Q[k] using X[k]

Now, $x[n] = p[n] + j\ q[n]$ ….. (I)

By Complex Conjugate on both sides :

$x^*[n] = p[n] – j\ q[n]$ ….. (II)

====================================

By (I) - (II) we get;

$x[n] - x^*[n] = 2j\ q[n]$

So, $q[n] = \frac{1}{2j} (x[n] - x^*[n])$

To Find Q[k] using X[k].. .. ..

Now, $q[n] = {}^{1}/_{2j} (x[n] - x^*[n])$

By Linearity & Complex Conjugate Property of DFT,

$$Q[k] = {}^{1}/_{2j} (X[k] - X^*[-k])$$

$$Q[k] = {}^{1}/_{2j}\left(\begin{bmatrix} 10+8j \\ -2 \\ -2 \\ -2-4j \end{bmatrix} - \begin{bmatrix} 10-8j \\ -2+4j \\ -2 \\ -2 \end{bmatrix} \right)$$

$$Q[k] = \begin{bmatrix} 8 & k=0 \\ -2 & \\ 0 & \\ -2 & \end{bmatrix}$$

[10] Circular Convolution Property

If $x[n] \rightarrow X[k]$

And $h[n] \rightarrow H[k]$

Then

$$\underbrace{DFT\{ x[n] \otimes h[n]\}}_{\text{Circular Convolution in Time Domain}} = \underbrace{X[k]\ H[k]}_{\text{Multiplication in Freq. Domain}}$$

Algorithm to find CC using DFT

INPUT : **L point x[n] and M point h[n]**

ALGORITHM:

I. Select N

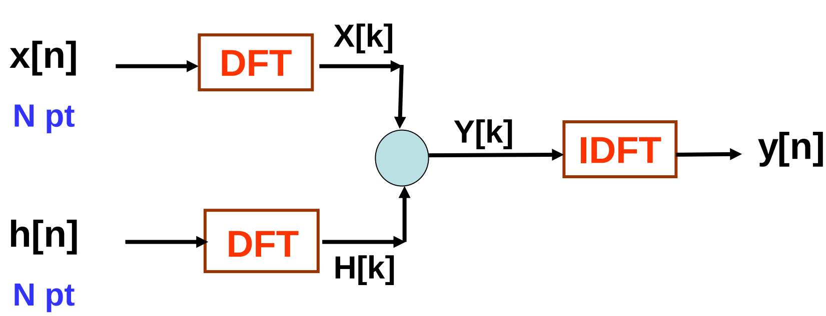
$$\boxed{N = MAX(L, M)}$$

II. Zero Padding

Append x[n] by (N-L) zeros and
Append h[n] by (N-M) zeros

III. **Find y[n] = x[n] ⊗ h[n] using DFT**

III : Find y[n] = x[n] ⊗ h[n] using DFT



Ex-1 **Let x[n] = [1, 2, 3, 4 } and h[n] = { 5, 6, 7 }**
Find Circular Convolution using DFT

Solution :

Here x[n] is L= 4 point and h[n] is M = 3 point

I. Select N

N = Max (L,M)

N = Max (4,3) == **4**

II. **Zero Padding**

x[n] = [1, 2, 3, 4 }

h[n] = { 5, 6, 7, 0 }

III Find $y[n] = x[n] \otimes h[n]$ using DFT

Now, $Y[k] = X[k] \; H[k]$

By Circular Convolution Property of IDFT ,

$$y[n] = x[n] \otimes h[n]$$

(1) Find X[k]

By DFT,

$$X[k]=\sum_{n=0}^{N-1} x[n]\,w_N^{nk}$$

$$X[k]=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$X[k]=\begin{bmatrix}10 & k=0\\ -2+2j & \\ -2 & \\ -2-2j & \end{bmatrix}$$

(2) Find H[k]

By DFT,

$$H[k]=\sum_{n=0}^{N-1} h[n]\,w_N^{nk}$$

$$H[k]=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}5\\6\\7\\0\end{bmatrix}$$

$$H[k]=\begin{bmatrix}18 & k=0\\ -2-6j & \\ 6 & \\ -2+6j & \end{bmatrix}$$

(3) Find Y[k]

$$Y[k] = X[k]\ H[k]$$

$$Y[k]=\begin{bmatrix}10\\ -2-2j\\ -2\\ -2+2j\end{bmatrix}\begin{bmatrix}18\\ -2-6j\\ 6\\ -2+6j\end{bmatrix}$$

$$Y[k]=\begin{bmatrix}180 & k=0\\ 16+8j & \\ -12 & \\ 16-8j & \end{bmatrix}$$

(4) Find y[n]

$$y[n]=\frac{1}{N}\sum_{k=0}^{N-1} Y[k]\ w_N^{-nk}$$

$$y[n]=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}\begin{bmatrix}180\\ 16+8j\\ -12\\ 16-8j\end{bmatrix}$$

$$y[n]=\begin{bmatrix}50 & n=0\\ 44 & \\ 34 & \\ 52 & \end{bmatrix}$$

Algorithm to find LC using DFT

INPUT : L point x[n] and M point h[n]

ALGORITHM:

I. Select N

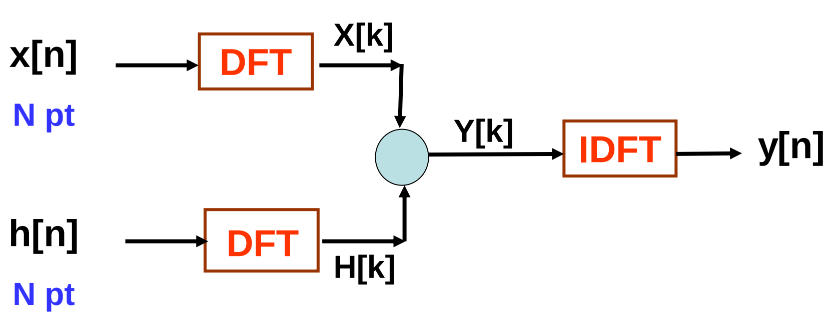
$$N \geq L + M - 1$$

II. Zero Padding

Append x[n] by (N-L) zeros and
Append h[n] by (N-M) zeros

III. **Find y[n] = x[n] ⊗ h[n] using FFT**

III. **Find y[n] = x[n] ⊗ h[n] using DFT**



Here, y[n] is LC of x[n] & h[n] i.e. y[n] = x[n] * h[n]

Ex-2 Let x[n] = [1, 2, 3 } and h[n] = { 5, 6 }
Find Circular Convolution using DFT

Solution :

Here x[n] is L= 3 point and h[n] is M = 2 point

I. Select N

$N \geq L+M-1$

$N \geq 3+2-1 == 4$

II. Zero Padding

x[n] = [1, 2, 3, 0 }
h[n] = { 5, 6, 0, 0 }

III Find $y[n] = x[n] \otimes h[n]$ using DFT

Now, $Y[k] = X[k]\ H[k]$

By Circular Convolution Property of IDFT ,

$y[n] = x[n] \otimes h[n]$

(1) Find X[k]

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

$$X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$X[k] = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(2) Find H[k]

By DFT,

$$H[k] = \sum_{n=0}^{N-1} h[n] w_N^{nk}$$

$$H[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

$$H[k] = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(3) Find Y[k]

$$Y[k] = X[k]\ H[k]$$

$$Y[k] = \begin{bmatrix} 6 \\ -2-2j \\ 2 \\ -2+2j \end{bmatrix} \begin{bmatrix} 11 \\ 5-6j \\ -1 \\ 5+6j \end{bmatrix}$$

$$Y[k] = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

(4) Find y[n]

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k]\ w_N^{-nk}$$

$$y[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 66 \\ -22+2j \\ -2 \\ -22-2j \end{bmatrix}$$

$$y[n] = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

[11] Parseval's Energy Theorem

Energy in Time Domain	==	Energy in Frequency Domain

(i) Energy in Time Domain :

$$E = \sum_{n=0}^{N-1} |x[n]|^2$$

(ii) Energy in Frequency Domain :

$$E = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Ex. Let x[n] = { 1, 2, 3, 2 }

(i) Find Energy in Time Domain :

$$E = \sum_{n=0}^{N-1} |x[n]|^2$$

$E = (1)^2 + (2)^2 + (3)^3 + (2)^2$

$E = 18$

(ii) Find Energy in Frequency Domain :

$$E = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

To Find X[k]

$$X[k] = \begin{bmatrix} 8 & k=0 \\ -2 & \\ 0 & \\ -2 & \end{bmatrix}$$

To Find Energy :

$$E = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

$$E = \tfrac{1}{4}\{ (8)^2 + (-2)^2 + (0)^3 + (-2)^2 \}$$

E =

* **Find Complex Multiplications and Complex additions in DFT**

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

Total Complex Multi = N^2

Total Complex Additions = $N^2 - N$

* Find Real Multiplications and Real additions in DFT

By DFT,

$$X[k] = \sum_{n=0}^{N-1} x[n] w_N^{nk}$$

Total Complex Multi = N^2

Total Complex Additions = $N^2 - N$

Let $P = a + j b$ and $Q = c + j d$

(1) $P \times Q = (a + j b)\ (c + j d)$
$= (ac - bd) + j\,(bc + ad)$

For 1 Complex Multi :
4 Real Multi
2 Real Additions

For 1 Complex Multiplication we require,
4 Real Multiplications and
2 Real Additions

(2) $P + Q = (a + j b) + (c + j d)$
$= (a + c) + j\,(b + d)$

For 1 Complex Additions :
2 Real Additions

For 1 Complex Addition we require 2 Real Additions

- **In DFT,**
- Total Complex Multi = N^2
- For N^2 Complex Multiplications we require
- $4\,N^2$ Real Multiplications
- $2\,N^2$ Real Additions
- Total Complex Additions = $N^2 - N$
- For $N^2 - N$ Complex Additions we require $2\,(N^2-N) = 2N^2 - 2N$ Real Additions
- **i.e.** $2N^2 + 2N^2 - 2N == 4N^2 - 2N$

For 1 Complex Multi :
4 Real Multi
2 Real Additions

For 1 Complex Additions :
2 Real Additions

By DFT :-

Total Real Multiplications $= 4\,N^2$

Total Real Additions $= 4\,N^2 - 2\,N$

By FFT :-

(i) Total Real Multiplications $= 2\ N \log_2 N$

(ii) Total Real Additions $= 3\ N \log_2 N$

Stay Connected……..

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- He is a recipient of P.R. Bapat IEEE Bombay Section Outstanding Volunteer Award in 2019.