Sardar Patel Institute of Technology

Bhavan's Campus, Munshi Nagar , Andheri (West),Mumbai-400058, India

(Autonomous College Affiliated to University of Mumbai)

End Semester Examination January 2022

Max. Marks:- 60 **Duration: 2 Hours**
Class: SE (ETRX and EXTC) **Semester:- III**
Course Code:- MA201 **Branch:- ETRX and EXTC**
Name of the Course: Linear Algebra

Instructions:
1) **Attempt any 3 out of the 4 questions.**
2) **Assume suitable data if necessary.**

Q No.		Max. Marks	CO
Q.1 (a)	Apply Gauss-Jordan elimination to the following linear system of equations : $$E: \begin{aligned} x + 2z + 3w &= 2 \\ x + 3y + 2z &= 5 \\ 2x + 4z + 9w &= 10 \end{aligned}$$ to find a particular solution, solutions space of the homogeneous system E_0 correspondingto the system E and hence find the solution set of the system E.	5	2
Q.1 (b)	Let $A \in M_n(R)$ be a square matrix of order n. If A is an upper triangular matrix when exactly is the reduced echelon form of the matrix A is the identity matrix I_n. Justify your answer.	5	1
Q.1 (c)	Let $A \in M_{m,n}(R)$, $m, n \in N^+$ and let E : $A.x = b$ be the linear system of m equations in n variables. Further, let r be the natural number obtained by performing Gauss - elimination process (note that $n - r$ is the number of free variables) on the system E. Then what are the relations between the natural numbers r, m and n if the system E has:		
	(i) no solution for some b.	1	2
	(ii) infinitely many solutions for every b.	1	2
	(iii) exactly one solution for some b and no solution for other b.	1	2
	(iv) exactly one solution for every b.	2	2
Q.1 (d)	Let $E : A.x = b$ be a system of m linear equations in n variables and let $E_0 : A.x = 0$ be the homogeneous linear system corresponding to the system E.		

	Suppose that the vectors x_1 and $x_2 \in R^n$ are two solutions of the system E, then: (i) Find two solutions of the system E_0. (ii) Find another solution (other than the solutions in (i)) to the system E_0 and another solution other than x_1 and x_2 to the system E.	2 3	1 1
Q.2 (a)	(i) Solve the system of linear equations: $$\text{E:} \quad \begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$ For what values of $a \in$ R does the system E fail to have solutions, and for what values of $a \in$ R are there infinitely many solutions? (ii) Describe all functions $f : R \rightarrow R$ defined by: $$f(x) := ax^2 + bx + c,$$ where $a, b, c \in R$ (are fixed) such that $f(1) = 2$ and $f(-1) = 6$.	3 2	1 4
Q.2 (b)	Let $A = \begin{pmatrix} 2 & 1 & 1 & 3 \\ 6 & 4 & 1 & 2 \\ 1 & 5 & 1 & 5 \end{pmatrix} \epsilon\ M_{3,4}(R)$. (i) Find two distinct echelon forms of the matrix A. (ii) Find the reduced echelon form of A.	2 3	1 1
Q.2 (c)	Show that the matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon\ M_2(R)$ is invertible if and only if $ad - bc \neq 0$. Moreover, in this case find the inverse A^{-1} of the matrix A.	5	1
Q.2 (d)	Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \in R^3$ be two vectors in R^3. (i) Are v_1 and v_2 linearly independent over R ? (ii) Find the dimension of the subspace W of R^3 generated by v_1 and v_2. (iii) Describe all vectors $v_3 \in R^3$ such that v_1, v_2, v_3 form a basis of R^3.	1 2 2	4 4 4
Q.3 (a)	Let $f : M_2(R) \rightarrow M_2(R)$ be the linear map defined by $$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$ (i) Find the matrix of f with respect to the basis $E_{11}, E_{12}, E_{21}, E_{22}$. Where $E_{ij} \in M_2(R)$ is the matrix with (i,j)-th entry 1 and all other entries are 0. (ii) Is the map f an isomorphism ? If yes, find the inverse map f^{-1} and its matrix with respect to the basis $E_{11}, E_{12}, E_{21}, E_{22}$. (iii) Find all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ such that	1 2 2	4 4 4

	$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.		
Q.3 (b)	Let P_4 be the vector space of polynomials with coefficients in R of degree ≤ 4 and let $$f = \frac{d}{dX} : P_4 \to P_4, \qquad F \longmapsto \frac{d}{dX}(F)$$ Then:		
	(i) Find the matrix of f with respect to the basis $1, X, X^2, X^3, X^4$ of P_4 and the basis $1, 2X, 3X^2, 4X^3, X^4$ of P_4 respectively.	1.5	4
	(ii) Find the matrix of f with respect to the basis $X^4, X^3, X^2, X, 1$ of P_4 and the basis $4X^3, 3X^2, 2X, 1, X^4$ of P_4 respectively.	1.5	4
	(iii) Is the map f surjective ? Justify your answer.	2	4
Q.3 (c)	Let f : $R^3 \to P_2$ be the linear map defined by the matrix $$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$ with respect to the standard basis e_1, e_2, e_3 of R^3 and the basis $1, 1+X^2, X$ of P_2 respectively. Find the matrix of the map f with respect to the basis $-e_1, 2e_2, 3e_3$ of R^3 and the basis $1, -X^2, 2X$ of P_2 respectively.	5	4
Q.3 (d)	(i) Is the linear map $$f : M_2(R) \to R_4 \text{ defined by } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ 0 \end{pmatrix}$$ isomorphism? Justify your answer.	1	4
	(ii) Find two isomorphisms between the vector spaces R^{16} *and* $M_4(R)$.	1	4
	(iii) For what $k \in N$, the vector spaces $M_{m,n}(R)$, and R^k are isomorphic ?	1	4
	(iv) Let P_5 be the vector space of polynomials with coefficients in R of degree ≤ 5. Is the linear map $$f : P_5 \to P_5, \text{ defined by } F(X) \longmapsto F(X-1).$$ an isomorphism of vector spaces ? If yes, then what is the inverse map f^{-1} of f ?	2	4

Q.4 (a)	Let V and W be two vector spaces with bases $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$, respectively and let $f: V \to W$ be a linear map defined by $f(v_1) = w_2$, $f(v_2) = f(v_3) = w_1 + 2w_3$. Then:		
	(i) Find the matrix A of f with respect to the bases $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$.	1	4
	(ii) Find a basis of the null-space $\mathrm{Ker}\, f = \{v \in V \mid f(v) = 0\}$. What is its dimension?	2	4
	(iii) Find all solutions $v \in V$ of the equation: $f(v) = w_2$.	2	4
Q.4 (b)	Let $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(R)$ be a square matrix.		
	Show that:		
	(i) If either $a = 0$, or $d = 0$, or $f = 0$, then the column vectors of A are linearly dependent over R.	3	4
	(ii) If all a, d and f are non-zero, then the column vectors of A are linearly independent over R.	2	4
Q.4 (c)	Let $f : M_2(R) \to M_2(R)$ be the linear map defined by $$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$		
	Then:		
	(i) Show that the map f is injective. (**Hint:** Use Question 2 (c) above.)	2	4
	(ii) Use (i) and Rank-Nullity Theorem to deduce that the map f is surjective.	3	4
Q.4 (d)	Let f : M_2(R) $\longrightarrow$ M_2(R) be the linear map defined $$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$		
	Then:		
	(i) Find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ with $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ (=the zero matrix).	1.5	1
	(ii) Describe all matrices $f\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.	1.5	1
	(iii) Describe all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ with	2	1

	$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ (= the zero matrix).		

*********** All the Best **********