SEComp / SEIT ESE : Linear Algebra

Day/Date : Sat, 07 May 2022 **Time : 11:00–13:15**

ATTEMPT ANY FOUR QUESTIONS
(Whereever necessary Justify your answers)

Question 1

(a) Let A be the 4×4 matrix A whose (i,j)-th entry is $(-1)^{i+j-1}$ and let

$$\mathrm{E}\ :\ AX = \begin{pmatrix} 3\\ -3\\ 3\\ -3 \end{pmatrix}$$

be the linear system of 4 equations in 4 variables.

Find the solution space $\mathrm{L}(\mathrm{E}_0)$ of the *homogeneous* linear system E_0 of equations corresponding to the linear system E. [5 Marks]

(b) Using one of the following suitable vector as a particular solution of the linear system E in the part (b) above, write down the set of general solution set of the linear system E.

$$(1)\ \begin{pmatrix} -2\\ 0\\ -2\\ -1 \end{pmatrix},\qquad (2)\ \begin{pmatrix} -5\\ 1\\ 5\\ -1 \end{pmatrix},\qquad (3)\ \begin{pmatrix} 2\\ 0\\ 2\\ 1 \end{pmatrix}.$$

[5 Marks]

(c) Describe all functions $f(x) = ax^2 + bx + c$ such that $f(1) = 2$. [5 Marks]

Question 2

(a) For what values of $a \in \mathbb{R}$, the homogeneous linear system (of 3 equations in 3 variables)

$$\mathrm{E}_0\ :\ \begin{pmatrix} a & 1 & 1\\ -1 & -1 & -1\\ 1 & 1 & a \end{pmatrix}\cdot\begin{pmatrix} X_1\\ X_2\\ X_3 \end{pmatrix} = \mathbf{0}$$

has infinitely many solutions? [5 Marks]

(b) Find the reduced echelon form of the matrix

$$A := \begin{pmatrix} 2 & 1 & 1 & 3\\ 6 & 4 & 1 & 2\\ 1 & 5 & 1 & 5 \end{pmatrix} \in \mathbb{M}_{3,4}(\mathbb{R}).$$

[5 Marks]

(c) Three truck drivers went into a roadside cafe. One truck driver purchased 4 sandwiches, a cup of coffee, and 10 doughnuts for Rs. 820. Second truck driver purchased 3 sandwiches, a cup of coffee, and 7 seven doughnuts for Rs. 590. What did the third truck driver pay for a sandwich, a cup of coffee, and a doughnut? [5 Marks]

Question 3

(a) For the 3×3 matrix

$$A := \begin{pmatrix} 0 & 0 & 5\\ 0 & -2 & 4\\ 2 & 3 & -2 \end{pmatrix} \in \mathbb{M}_3(\mathbb{R}),$$

by using the Gauss-Jordan elimination method, find the rank and the inverse, if it exists. [5 Marks]

(b) Let A and B be two $n\times n$ matrices over real numbers. If the sum matrix $A+B$ is invertible, then does it mean that each of the A and B is also invertible? Justify your answer. [5 Marks]

(c) Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ be a $m\times n$ matrices over real numbers. An $n\times m$ matrix $X \in \mathbb{M}_{n,m}(\mathbb{R})$ is called a left-inverse of A if $XA = \mathrm{I}_n$ =(the $n\times n$ identity matrix). Similarly, an $m\times n$ matrix $Y \in \mathbb{M}_{m,n}(\mathbb{R})$ is called a right-inverse of A if $AY = \mathrm{I}_m$ =(the $m\times m$ identity matrix).

Show that the 3×2 matrix

$$A := \begin{pmatrix} 1 & 0\\ 0 & 1\\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{3,2}(\mathbb{R})$$

has infinitely many left-inverses, bur it has no right-inverse. [5 Marks]

Question 4

(a) Show that the set of vectors

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 + x_2 - x_3 + x_4 = 0 \right\} \subseteq \mathbb{R}^4$$

in $\mathbb{R}^4$, is a vector space under the operations inherited from the vector space $\mathbb{R}^4$. [5 Marks]

(b) Show that the set of 2×2 matrices

$$\left\{ \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \subseteq \mathbb{M}_2(\mathbb{R})$$

in $\mathbb{M}_2(\mathbb{R})$, is NOT a vector space under the operations inherited from the vector space $\mathbb{M}_2(\mathbb{R})$. [5 Marks]

(c) Prove or disprove that the subset

$$W := \{ f : \mathbb{R} \to \mathbb{R} \mid f(7) = 0 \}$$

(of the real-valued functions f of one real variable) is a vector space under (the natural operations) the addition and scalar multiplication defined by :

$$(f+g)(t) := f(t) + g(t) \text{ and } (af) := a f(t) \text{ for all } t \in \mathbb{R},\ a \in \mathbb{R}.$$ [5 Marks]

Question 5

(a) Determine all the real numbers $a, b \in \mathbb{R}$ such that the vectors

$$\mathbf{v}_1 := \begin{pmatrix} 2 \\ a-b \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_2 := \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

are linearly independent over $\mathbb{R}$. [5 Marks]

(b) Find a generating system of the following subspace (of $\mathbb{M}_2(\mathbb{R})$)

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) \;\middle|\; a + b + d = 0 \right\} \subseteq \mathbb{M}_2(\mathbb{R}).$$ [5 Marks]

(c) Let

$$\mathrm{P}_3(X) := \{ a_0 + a_1 X + a_2 X^2 + a_3 X^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

be the vector space of polynomials (in one variable X with real coefficients) of degrees ≤ 3. Show that the polynomials

$$3,\ X,\ 1 + X + X^2,\ 5X^3$$

form a basis of the vector space $\mathrm{P}_3(X)$. [5 Marks]
