Linear Algebra

~~Design & Analysis of Algorithms~~

MODULE 1: Matrices

- Definition: A set of mn ~~et~~ numbers real or complex arranged in a rectangular array of m rows and n columns like:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

$i = 1, \ldots, m$

$j = 1, \ldots, n$

A is a matrix of order mxn. It is read as m by n. The elements $a_{11}, a_{12}, \ldots, a_{mn}$ constituting mxn matrix are called as elements/constituents. Matrix is usually denoted by capital letters. a_{ij} is the element in the ith row and jth column.

- Types of matrices:

(1) Square matrix: A matrix of order mxn is called as a square matrix if and only if m=n, i.e. the number of rows is same as columns.

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(2) Row matrix: A matrix which contains only one row.

(3) Column matrix: Only one column

(4) Diagonal matrix: It is a square matrix in which all non-diagonal entries are zero.

Eg: $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(5) Scalar matrix: It is a diagonal matrix in which all diagonal elements are same.

Eg: $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

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(6) Unit matrix: It is a scalar matrix ~~wh~~ with diagonal elements as unity.

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(7) Upper Triangular Matrix: It is a square matrix in which all the elements below the principle diagonals are zero.

Eg: $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(8) Lower Triangular Matrix: It is a square matrix in which all the elements above the principle diagonals are zero.

Eg: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

(9) Null matrix: All elements are 0.

Eg: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(10) Transpose of a matrix: A matrix which is obtained from the given matrix by changing its rows into columns and vice versa is called transpose of a matrix.

Eg: $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$ $A^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

(11) Idempotent Matrix: A square matrix such that $A^2 = A$ is called as Idempotent Matrix.

(12) Involuntary Matrix: A square matrix such that $A^2 = I$, where I is the unit matrix.

(13) Nilpotent Matrix: A square matrix is Nilpotent if there exists a positive integer m such that $A^m = 0$, m is called index.

(14) Hermitian Matrix: A square matrix is said to be Hermitian if and only if $A^\theta = A$ where θ is the complex conjugate.

(15) Skew-Hermitian Matrix: Skew ⇒ minus ∴ $A^{\theta} = -A$.

(16) Rectangular Matrix: m is not equal to n.

(17) Symmetric Matrix: A matrix is said to be symmetric if it's transpose coincides with itself, i.e, $A = A'$.

(18) Skew-symmetric Matrix: A matrix is skew-symmetric if and only if $A = -A'$ or $A' = -A$.

(19) Singular matrix: Matrix is singular if and only if $|A| = 0$.

(20) Orthogonal Matrix: Matrix is orthogonal if and only if $AA' = A'A = I$

- Minor of an element:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

~~Mino~~ Minor of $a_{11} = a_{22}a_{33} - a_{23}a_{32} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of an element of $|A|$ is a determinant obtained from $|A|$ ignoring the row and column containing that element.

- Cofactor of an element:

Cofactor of $a_{ij} = (-1)^{i+j} \times$ minor a_{ij}

If we multiply the minor of an element in the ith row and jth column of the determinant by (-1) raised to $i+j$, we get cofactor of an element.

- Adjoint of a matrix:

adj A = [matrix of cofactors]'

Let A be the square matrix of order m, adjoint of a matrix is denoted by Adj A and is defined by transpose matrix of its cofactors.

- Row Echelon form:

Eg.(1) Convert the matrix A into row echelon form.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

NOTE: Only apply row transformation for row echelon.

Soln: Given matrix A.

$R_1 \longleftrightarrow R_2$ Interchange row 1 and row 2 only to make the first non-zero entry in first column to be in row 1, i.e. a_{11} = non-zero

$$\therefore A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\therefore A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

← Lower triangle = 0

This is required Row Echelon Form.

(2) Reduce the given matrix to Row Echelon form and find its rank.

$$\rightarrow A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_3 \rightarrow} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_4]{R_2 \rightarrow R_2 - 3R_4} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = No of non-zero rows = 2 (After transformations)

(3) Reduce the following matrix to row echlon form and find its rank.

$$\rightarrow A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -1 & 1 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & -2 & -2 & 2 \end{bmatrix} \xrightarrow[R_4 - 2R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

(4) $\cancel{A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & -3 & -1 \\ -1 & 0 & 5 & 7 \end{bmatrix}}$ $\quad A \begin{bmatrix} 0 & 2 & -6 & -2 \\ 0 & 2 & 6 & 10 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$R_1 \leftrightarrow R_4 \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 2 & 6 & 10 \\ 3 & 1 & 0 & 2 \\ 0 & 2 & -6 & -2 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 2 & 6 & 10 \\ 0 & -2 & 6 & 2 \\ 0 & 2 & -6 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4 - R_2]{R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & -12 & -12 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

★ MODULE 2:

- Solution of System~~atic~~ of Linear Eqns:

> Gauss Elimination Method:

- In this method, we express the system of eqns as $AX = B$.
- Find the Augmented matrix for given matrix.
- Find the transform. of the augmented matrix and convert it into Row Echelon form.
- Find eqns corresponding to upper triangular matrix
- Using back substitution method, find solns for the given system of eqns.

- Egs:

(1) Apply Gauss elimination method to solve the following system of eqns:

$\longrightarrow$ $2x - 2y + 3z = 2$

$x + 2y - z = 3$

$3x - y + 2z = 1$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right]$$

Augmented matrix of abv eqns are: $[A|B]$ OR $[A:B]$

Then transforming augmented matrix to Row Echelon form.

$$\xrightarrow{R_2 \leftrightarrow R_1} [A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array}\right]$$

$\therefore (0) - (-y) + (0) = -4$

$\therefore -y = -4 \Rightarrow y = \boxed{4}$

$x + 2y - z = 3 \Rightarrow x - z = -5$

$-24 + 5z = -4 \Rightarrow 5z = 20 \Rightarrow z = \boxed{4}$

$x = \boxed{-1}$

This statement for ½ M $\rightarrow$ Therefore the required soln of given system is $x = -1$, $y = 4$ and $z = 4$.

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(2) Apply Gauss Elimination Method to solve the following system of eqns.

$x+2y+3z-t=10$

$2x+3y-3z-t=1$

$3x+2y-4z+3t=2$

$2x-y+2z+3t=7$

$$\cancel{[A|B]=\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 10 \\ 2 & 3 & -3 & -1 & 1 \\ 3 & 2 & -4 & 3 & 2 \\ 2 & -1 & 2 & 3 & 7 \end{array}\right]}$$

We first write the eqns in matrix form and then reduce the coeff matrix to upper triangular matrix by elementary row transformations.

write this 1M

$$\cancel{\xrightarrow[\substack{R_3-3R_1 \\ R_4-2R_1}]{R_2-2R_1}\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 10 \\ 0 & -1 & -9 & 1 & -19 \\ 0 & -4 & & & \end{array}\right]}$$

Dif method ↑.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & -3 & -1 \\ 3 & 2 & -4 & 3 \\ 2 & -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 2 \\ 7 \end{bmatrix} \xrightarrow[\substack{R_3-3R_1 \\ R_4-2R_1}]{R_2-2R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & -4 & -13 & 6 \\ 0 & -5 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ -28 \\ -13 \end{bmatrix} \xrightarrow[R_4-5R_2]{R_3-4R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -9 & 1 \\ 0 & 0 & 23 & 2 \\ 0 & 0 & 41 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 10 \\ -19 \\ 48 \\ 82 \end{bmatrix}$$

$\cancel{41t=82 \Rightarrow \boxed{t=2}}$ $\quad$ $41z=82 \Rightarrow \boxed{z=2}$

$\cancel{23z+4=48 \Rightarrow \boxed{z=2}}$ $\quad$ $46+2t=48 \Rightarrow \boxed{t=1}$

$\cancel{-y-18+2=-19-1 \Rightarrow \boxed{y=3}}$ $\quad$ $-y-18+1=-19 \Rightarrow \boxed{y=2}$

$\cancel{x+6+6-2=10}$ $\quad$ $x+\cancel{6}+\cancel{6}-1=\cancel{10} \Rightarrow \boxed{x=1}$

(3) Apply Gauss elimination method to solve the following system of eqns.

→ $x + 2y + 3z = 14$
$3x + 3y + 4z = 21$
$4x + 5y + 7z = 35$

We first write the eqn in matrix form and then reduce the coeff matrix to upper Δ matrix, by applying elementary row transformation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 35 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -21 \\ -21 \end{bmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -21 \\ 0 \end{bmatrix}$$

$x + 2y + 3z = 14$ — ①
$+3y + 5z = +21$ — ②

Put $z = t$ (In case you get all zeros in last row, put $z = t$)

~~$-5t = 21 + 3y$~~ $3y + 5t = 21 \Rightarrow \boxed{y = \frac{21 - 5t}{3}}$

$x + 2y + 3t = 14 \Rightarrow x = 14 - 3t - 2\left(\frac{21 - 5t}{3}\right)$

$\therefore x = \frac{\cancel{42} - 9t - \cancel{42} + 10t}{3} = \frac{t}{3} \Rightarrow \boxed{x = \frac{t}{3}}$

~~$\frac{t}{3} + \frac{42 - 10t}{3} + 3t = 14$~~

~~$t + 42 - 10t + 9t = 42$~~

(4) Apply Gauss elimination

$\rightarrow$ $x + 2y + 3z + 4w = 30$

$2x + 3y + 6z + 5w = 46$

$3x + 4y + 8z - 6w = 9$

$4x - y - z + 2w = 7$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 6 & 5 \\ 3 & 4 & 8 & -6 \\ 4 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 30 \\ 46 \\ 9 \\ 7 \end{bmatrix} \xrightarrow[\substack{R_3 - 3R_1 \\ R_4 - 4R_1}]{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & -3 \\ 0 & -2 & -1 & -18 \\ 0 & -9 & -13 & -14 \end{bmatrix} = \begin{bmatrix} 30 \\ -14 \\ -81 \\ -113 \end{bmatrix} \xrightarrow[R_4 - 9R_2]{R_3 - 2R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -12 \\ 0 & 0 & -13 & 12 \end{bmatrix} = \begin{bmatrix} 30 \\ -14 \\ -53 \\ 55 \end{bmatrix} \quad R_4 - 13R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -12 \\ 0 & 0 & 0 & 144 \end{bmatrix} = \begin{bmatrix} 30 \\ -14 \\ -53 \\ 744 \end{bmatrix}$$

$t = 5.16$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & -1 & -12 \\ 0 & 0 & -13 & 13 \end{bmatrix} = \begin{bmatrix} 30 \\ -14 \\ -53 \\ 13 \end{bmatrix}$$

> Gauss Jacobi Method:

Procedure: Consider the following system of equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{—} (1)$$

where a_1, b_2, c_3 are large as compared to remaining coeffs.

We rewrite the eqns from (1) as

$$\left.\begin{aligned} x &= \frac{1}{a_1}\left[d_1 - b_1y - c_1z\right] \\ y &= \frac{1}{b_2}\left[d_2 - a_2x - c_2z\right] \\ z &= \frac{1}{c_3}\left[d_3 - a_3x - b_3y\right] \end{aligned}\right\} \text{—} (2)$$

Now you start with the assumption that the roots of the eqn are $x = x_0$, $y = y_0$, $z = z_0$. Putting these values in eqn 2, the first approximation is

$$\therefore x_1 = \frac{1}{a_1}\left[d_1 - b_1y_0 - c_1z_0\right]$$

$$\therefore y_1 = \frac{1}{b_2}\left[d_2 - a_2x_0 - c_2z_0\right]$$

$$\therefore z_1 = \frac{1}{c_3}\left[d_3 - a_3x_0 - b_3y_0\right]$$

Also, we start with the assumption that the roots of the eqn are $x = x_1$, $y = y_1$, $z = z_1$. Putting these values in eqn 2, ~~the~~ we get a better approximation. Thus, second approximation is

$$\therefore x_2 = \frac{1}{a_1}\left[d_1 - b_1y_1 - c_1z_1\right]$$

$$y_2 = \frac{1}{b_2}\left[d_2 - a_2x_1 - c_2z_1\right]$$

$$z_2 = \frac{1}{c_3}\left[d_3 - a_3x_1 - b_3y_1\right]$$

Now, we iterate (repeat) the procedure as many times

as we want till we arrive at a desired accuracy

Eg:

(1) Applying Gauss Jacobi Method to solve the following system of equations:

$15x + 2y + z = 18$ upto sixth iterations

$2x + 20y - 3z = 19$

$3x - 6y + 25z = 22$

$$\left.\begin{aligned} x &= \frac{1}{15}[18 - 2y - z] \\ y &= \frac{1}{20}[19 - 2x + 3z] \\ z &= \frac{1}{25}[22 - 3x + 6y] \end{aligned}\right\} \text{—} (1)$$

$$\left.\begin{aligned} x_1 &= \frac{1}{15}[18 - 2(0) - (0)] = 18/15 = 1.2 \\ y_1 &= 1/20\,[19 - 0 + 0] = 19/20 = 0.95 \\ z_1 &= 22/25 = 0.88 \end{aligned}\right\} \text{First iteration}$$

Second iteration:

$\cancel{x_2 = 1/15\,[18 - 2(1.2) - 1.2] = 0.96}$

$\cancel{y_2 = 1/25\,[22 - 2(1.2) + 3(1.2)] =}$

$\cancel{x_2 = 1/15\,[18 - 2(0.95) - 0.88] = 1.0147}$

$\cancel{y_2 = 1/20\,[22 - 2(1.2) + 3(0.88)] = 0.964}$

$\cancel{z_2 = 1/}$

$x_2 = 1/15\,[18 - 2(0.95) - 0.88] = 1.0147$

$y_2 = 1/20\,[19 - 2(1.2) + 3(0.88)] = 0.962$

$z_2 = 1/25\,[22 - 3(1.2) + 6(0.95)] = 0.964$

Third iteration: $x_3 = 1.0075$, $y_3 = 0.9931$, $z_3 = 0.9891$

Fourth iteration: $x_4 = 1.0016$, $y_4 = 0.9976$, $z_4 = 0.9974$

Fifth iteration: $x_5 = 1.0005$, $y_5 = 0.994$, $z_5 = 0.9980$

Sixth iteration: $x_6 = 1.0002$, $y_6 = 0.996$, $z_6 = 0.9998$

→ is called dominant form

(2) Apply Gauss Jacobi Method to solve the following eqns:

→ $4x + y + 3z = 17$ upto 5 iterations

$x + 5y + z = 14$

$2x - y + 8z = 12$

$x = \frac{1}{4}\left[17 - y - 3z\right]$

$y = \frac{1}{5}\left[14 - x - z\right]$

$z = \frac{1}{8}\left[12 - 2x + y\right]$

First iteration: $x_1 = 17/4 = 4.25$

$y_1 = 14/5 = 2.8$

$z_1 = 12/8 = 1.5$

Second iteration: $x_2 = 2.425$

$y_2 = 1.65$

$z_2 = 0.7875$

Third iteration: $x_3 = 3.2468$, $y_3 = 2.1575$, $z_3 = 1.1$

4th iteration: $x_4 = 2.8856$, $y_4 = 1.9306$, $z_4 = 0.9579$

5th iteration: $x_5 = 3.0489$, $y_5 = 2.0313$, $z_5 = 1.0199$

(3) Apply Gauss Jacobi method to solve.

→ $20x + y - 2z = 17$ (here iterations are not mentioned, perform upto better approximation) (closest values)

$3x + 20y - z = -18$

$2x - 3y + 20z = 25$

$x = \frac{1}{20}\left[17 - y + 2z\right]$

$y = \frac{1}{20}\left[-18 - 3x + z\right]$

$z = \frac{1}{20}\left[25 - 2x + 3y\right]$

$x_1 = 0.85$ $y_1 = -0.9$ $z_1 = 1.25$

$x_2 = 1.02$ $y_2 = -0.965$ $z_2 = 1.03$

$x_3 = 1.00125$ $y_3 = -1.0015$ $z_3 = 1.013$

$x_4 = 1.0013$ $y_4 = -0.9995$ $z_4 = 0.99965$

REDUCED ROW ECHELON FORM:

Eg(1) Convert the following matrix to reduced row echelon.

$$\rightarrow A = \begin{bmatrix} 2 & 0 & 4 \\ 9 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - \frac{9}{2}R_1]{R_1 \to R_1/2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -15 \end{bmatrix} \xrightarrow{R_2 \to R_2/2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7.5 \end{bmatrix}$$

NOTE: We need matrix closest to ~~identity~~ unit matrix. For 2 rows, identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which we get as final result.

(2) Convert the following matrix to reduced row echelon form

$$\rightarrow A = \begin{bmatrix} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{bmatrix} \xrightarrow{R_1 \to R_1/2} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{bmatrix} \cancel{\xrightarrow{2R_2} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 5 \end{bmatrix}}$$

$$\cancel{\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & & & \end{bmatrix}}$$

$$\xrightarrow{R_3 = 4R_2} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{R_3}{2}} \begin{bmatrix} 1 & 1/2 & -1/2 & 4 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\cancel{\xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 1/2 & \end{bmatrix}}$$

Ans: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (Check calculations)

(3) $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -22 \\ 0 & 1 & 7 \end{bmatrix}$

★ GAUSS JORDAN METHOD:

In this method, we first express the system of equations as $AX = B$. Find augmented matrix for given system. Convert the augmented matrix into reduced row echelon form.

Eg(1) Apply Gauss Jordan method to solve the following eqns:

$\longrightarrow$ $x + 3y = 7$

$5x + 4y = 11$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right]$$

$$\text{By } R_2 \to \frac{R_2}{-5} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Thus, required soln is $x = 1$, $y = 2$.

(2) Apply Gauss Jordan method to solve following eqns:

$\longrightarrow$ $x + y + z = 5$

$2x + 3y + 5z = 8$

$4x + 5z = 2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right] \xrightarrow{R_3/13} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$x = 3$, $y = 4$, $z = -2$

(3) Apply Gauss Jordan method to solve the following eqns:

$\rightarrow$ $x + 2y = 4$

$x - 2y = 6$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -2 & 6 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2/-4} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & -1/2 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1/2 \end{array}\right]$$

$x = 5$, $y = -1/2$

(4) $x + 2y + 3z = 4$

$2x + 3y - 4z = 5$

$4x + 5y - 6z = 7$

$$\rightarrow [A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & -4 & 5 \\ 4 & 5 & -6 & 7 \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -10 & -3 \\ 0 & -3 & -18 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -10 & -3 \\ 0 & 0 & 12 & 0 \end{array}\right] \xrightarrow[R_2 \times (-1)]{R_3/12} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & +1 & +10 & +3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 10R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x = -2$, $y = 3$, $z = 0$

(5) $6x - 2y + 2z + 4w = 16$

$12x - 8y + 6z + 10w = 26$

$3x - 13y + 9z + 3w = -19$

$-6x + 4y + z - 18w = -34$

GAUSS SEIDEL METHOD: This is modification of Gauss Jacobi. $AX = B$.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{—(1)}$$

We write these eqns as

$$\left.\begin{aligned} \therefore x &= \frac{1}{a_1}\left[d_1 - b_1y - c_1z\right] \\ y &= \frac{1}{b_2}\left[d_2 - a_2x - c_2z\right] \\ z &= \frac{1}{c_3}\left[d_3 - a_3x - b_3y\right] \end{aligned}\right\} \text{—(2)}$$

Here we start with initial approximation x_0, y_0, z_0.
Then substituting $x = x_0, y = y_0, z = z_0$ in first eqn of (2)

$$x_1 = 1/a_1\left[d_1 - b_1y_0 - c_1z_0\right]$$

For getting y_1, substitute $x = x_1$ and $z = z_0$ in second eqn of (2), $y_1 = 1/b_2\left[d_2 - a_2x_1 - c_2z_0\right]$

For getting z_1, put $x = x_1$ and $y = y_1$

$$\therefore z_1 = 1/c_3\left[d_3 - a_3x_1 - b_3y_1\right]$$

As soon as the new approximation of an unknown is found, it is immediately used in the next step. This process of iteration is repeated till the values of x, y, z are obtained to desired degree of values.

Egs:

(1) Apply Gauss ~~Siedel~~ Seidel method to solve the following eqns:

$$\rightarrow \left.\begin{aligned} 5x + 2y + z &= 12 \\ x + 4y + 2z &= 15 \\ x + 2y + 5z &= 20 \end{aligned}\right\} \text{—(1)} \qquad a_1 = 5 \quad b_2 = 4 \quad c_3 = 5$$

First we check the leading diagonal elements

$$x = \frac{1}{5}\left[12 - 2y - z\right]$$

$$\left.\begin{aligned} y &= 1/4\,[15 - x - 2z] \\ z &= 1/5\,[20 - x - 2y] \end{aligned}\right\} \quad \text{— (2)}$$

First iteration: Put $y=0$, $z=0$ in (2)

$\therefore x_1 = 1/5\,[12 - 0 - 0] = 2.4$

$y_1 = 1/4\,[15 - 2.4 - 0] = 3.15$

$z_1 = 1/5\,[20 - 2.4 - 6.3] = 2.26$

Second iteration:

$\therefore x_2 = 1/5\,[12 - 2(3.15) - 2.26] = 0.688$

$y_2 = 1/4\,[15 - 0.688 - 2(2.26)] = 2.448$

$z_2 = 1/5\,[20 - 0.688 - 2(2.448)] = 2.8832$

Third iteration:

$\therefore x_3 = 1/5\,[12 - 2(2.448) - 2.8832] = 0.84416$

$y_3 = 1/4\,[15 - 0.84416 - 2(2.8832)] = 2.09736$

$z_3 = 1/5\,[20 - 0.84416 - 2(2.09736)] = 2.992224$

Fourth iteration:

$x_4 = 1/5\,[12 - 2(2.09736) - (2.992224)] = 0.9626$

$y_4 = 1/4\,[15 - 0.9626 - 2(2.992224)] = 2.0132$

$z_4 = 1/5\,[20 - 0.9626 - 2(2.0132)] = 3.0022$

Fifth iteration:

$x_5 = 1/5\,[12 - 2(2.0132) - 3.0022] = 0.99428$

$y_5 = 1/4\,[15 - 0.99428 - 2(3.0022)] = 2.00033$

$z_5 = 1/5\,[20 - 0.99428 - 2(2.00033)] = 3.001012$

Sixth iteration:

$x_6 = 1.00015 \quad y_6 = 1.9993 \quad z_6 = 3.0002$

$\therefore$ By sixth approximation, we get $x=1$, $y=2$, $z=3$.

(2) Solve in terms of x_1, x_2, x_3, x_4.

$\rightarrow$ $\overset{a}{13}x_1 + \overset{b}{5}x_2 - \overset{c}{3}x_3 + \overset{e}{x_4} = \overset{d}{18}$

$2x_1 + 12x_2 + x_3 - 4x_4 = 13$

$3x_1 - 4x_2 + 10x_3 + x_4 = 29$

$2x_1 + x_2 - 3x_3 + 9x_4 = 31$

$a_1 = 13$, $b_2 = 12$, $c_3 = 10$, $e_4 = 9$

First iteration:

$x_1 = 1/13\ [18 - 5(0) + 3(0) - (0)] = 1.3846$

$x_2 = 1/12\ [13 - 2(1.3846) - (0) + 4(0)] = 0.8525$

$x_3 = 1/10\ [29 - 3(1.3846) + 4(0.8525) - (0)] = 2.8256$

$x_4 = 1/9\ [31 - 2(1.3846) - (0.8525) + 3(2.8256)] = 3.9839$

Now consider the table:

Iterations	1	2	3	4	$x_1 = 1$
x_1	1.3846	1.4023	0.9994	1.0119	$x_2 = 2$
x_2	0.8525	1.9421	1.9684	1.9992	$x_3 = 3$
x_3	2.8256	2.8577	3.0005	2.9957	$x_4 = 4$
x_4	3.9839	3.8696	4.0038	3.996	upto sixth iteration
	5	6			

MODULE 5: Eigen value and Eigen vectors

- Characteristic matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

For the given matrix A, $A - \lambda I$ is called as a characteristic matrix, and I is the unit matrix.

- Characteristic Polynomial: $|A - \lambda I| = \lambda^3 - 7\lambda^2 + 11\lambda - 5$

The determinant $|A - \lambda I|$ when expanded will give a polynomial which we call as characteristic polynomial. After simplification we get the above eqn.

- Characteristic equation: The eqn $|A - \lambda I| = 0$ is called as a characteristic equation.

- Characteristic root or Eigen Value: The roots of the characteristic eqn $|A - \lambda I| = 0$ is called as characteristic root.

Eg: $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \Rightarrow (\lambda-1)(\lambda-1)(\lambda-5) = 0$

$\therefore \lambda = 1, 1, 5 \leftarrow$ Eigen values.

NOTE: If λ is an eigen value of matrix A and X is the corresponding eigen vector then $AX = \lambda X$

- Properties of eigen value and eigen vectors

(1) Show that the characteristic roots of a Hermitian matrix are all real.

→ Given that A is the Hermitian Matrix, i.e, $A^\theta = A$ — (*)

If λ is the eigen value and X is the corresponding eigen vector then $AX = \lambda X$ — (1). Then taking complex conjugate of eqn (1),

$(AX)^\theta = (\lambda X)^\theta$

$\Rightarrow A^\theta X^\theta = \bar{\lambda} X^\theta$ — (2)

But A is Hermitian,

$AX^\theta = \bar{\lambda} X^\theta$ {∵ (*)}

Now post multiply by X,

$\therefore X^\theta(AX) = \bar{\lambda}(X^\theta X)$

$\therefore X^\theta(\lambda X) = \bar{\lambda}(X^\theta X)$ {∵ (1)}

$\therefore \lambda(X^\theta X) - \bar{\lambda}(X^\theta X) = 0$

$\therefore (\lambda - \bar{\lambda})(X^\theta X) = 0.$

But, $X^\theta X$ cannot be 0 as it is a vector

$\therefore \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow$ characteristic roots are real. Hence, $\lambda = \bar{\lambda}$ shows that λ is real. Thus, the characteristic roots of a Hermitian matrix are all real.

(2) Show that the matrices A and $P^{-1}AP$ have the same characteristic roots.

→ We know that if λ is an eigen value of a matrix A and X is the corresponding eigen vector, then $AX = \lambda X$.

Premultiply by P^{-1} to both sides.

$\therefore P^{-1}AX = P^{-1}\lambda X$

$\therefore P^{-1}APP^{-1}X = \lambda(P^{-1}X)$

$\therefore (P^{-1}AP)P^{-1}X = \lambda(P^{-1}X)$

$\therefore [P^{-1}AP - \lambda] P^{-1}X = 0$

(write this statement) But $P^{-1}X$ is not a 0 vector

$\therefore \lambda = P^{-1}AP \Rightarrow \lambda$ is an eigen value of $P^{-1}AP$ and also A. Thus, they have same characteristic roots.

(3) Show that the eigen values of unitary matrix are of unit modulus (have absolute value 1)

→ Given that matrix A is unitary matrix, i.e, $A^{\theta}A = AA^{\theta} = I$ — ⊛

We know that λ is the eigen value of A and X is eigen vector.

$\therefore AX = \lambda X$ — ① Taking complex conjugate to both sides of the above eqn,

$\therefore A^{\theta}X^{\theta} = \bar{\lambda} X^{\theta}$ — ②

From ① and ②, by multiplication,

$\therefore X^{\theta}A^{\theta}AX = \bar{\lambda} X^{\theta} \lambda X$

$\therefore (A^{\theta}A)(X^{\theta}X) = (\lambda\bar{\lambda})(X^{\theta}X)$

$\therefore I(X^{\theta}X) = (\lambda\bar{\lambda})(X^{\theta}X)$ {from ⊛}

$\therefore (\lambda\bar{\lambda} - 1)(X^{\theta}X) = 0$

$\therefore \lambda\bar{\lambda} = 1$ Thus, we must say that eigen values of unitary matrix have unit modulus.

Egs How to find Eigen Value and Eigen Vectors

(1) $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

→ By using characteristic eqn i.e, $|A - \lambda I| = 0$

$\therefore \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

$\Rightarrow \begin{vmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{vmatrix} = 0$ — ①

$$(10-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ -5 & 5-\lambda \end{vmatrix} - 5\begin{vmatrix} -2 & 2-\lambda \\ -5 & 3 \end{vmatrix} = 0$$

$\therefore (10-\lambda)(10-2\lambda-5\lambda+\lambda^2-9) + 2(-10+2\lambda+15) - 5(-6+10-5\lambda) = 0$

$\therefore 10-\lambda-70\lambda+70\lambda^2+10\lambda^2-\lambda^3-20+4\lambda+30+30-50+25\lambda = 0$

$\therefore -\lambda^3+17\lambda^2-42\lambda = 0$

$\therefore \lambda^3-17\lambda^2+42\lambda = 0$

$\therefore \lambda(\lambda^2-17\lambda+42) = 0$

$\therefore \lambda(\lambda-14)(\lambda-3) = 0$

$\therefore \lambda = 0, 3, 14$ are the Eigen values/characteristic roots

Next to find the Eigen vector:

For the sake of Eigen vectors, let

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $[A-\lambda I]X = 0$

$\therefore$ Using (1) we get,

$$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{--- (2)}$$

Putting $\lambda = 0$ in (2)

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore 10x_1-2x_2-5x_3 = 0$

$-2x_1+2x_2+3x_3 = 0$

$-5x_1+3x_2+5x_3 = 0$

For finding Eigen vector, apply ~~Kramer's~~ Cramner's rule. For Cramner's rule, we must have 2 eqns

$$\frac{x_1}{\begin{vmatrix} -2 & -5 \\ 2 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 10 & -5 \\ -2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-6+10} = \frac{-x_2}{30-10} = \frac{x_3}{20-4}$$

$\therefore x_1/4 = -x_2/20 = x_3/16 \Rightarrow x_1 = +x_2/5 = x_3/4$

Thus, $\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$ is the first Eigen vector corresponding to $\lambda = 0$

Also put $\lambda = 3$ in ②,

$$\begin{bmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore \quad 7x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - x_2 + 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -2 & -5 \\ -1 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -5 \\ -2 & 3 \end{vmatrix}} = \frac{-x_3}{\begin{vmatrix} 7 & -2 \\ -2 & -1 \end{vmatrix}}$$

$$\therefore \frac{x_1}{-11} = \frac{+x_2}{-11} = \frac{x_3}{-11}$$

$\therefore \quad x_1/11 = x_2/11 = x_3/11$

$\therefore \quad x_1 = x_2 = x_3 = 1$

Putting $\lambda = 14$ in ②,

$$\begin{bmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore \quad -4x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - 12x_2 + 3x_3 = 0$

$$\therefore \frac{x_1}{\begin{vmatrix} -2 & -5 \\ -12 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -4 & -5 \\ -2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & -2 \\ -2 & -12 \end{vmatrix}}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$\therefore \quad x_1 = -3 \qquad x_2 = 1 \qquad x_3 = 2$

Eigen vectors are: $\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$

~~* DIAGONAL AND TRANSF~~

Q(2) Find Eigen values and Eigen vector of matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

* DIAGONAL & TRANSFORMING MATRIX

Egs

(1) Show that matrix A is diagonalisable. Also find the diagonal matrix and transforming matrix.

$$\rightarrow A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

If all Eigen values are distinct, matrix is diagonalisable. If 2 are equal, 1 is distinct matrix may or may not be diagonalisable.

By using characteristic eqn, i.e.,

$$\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{array}{r} +24 \\ -13 \\ \hline 11 \end{array}$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0 \quad \text{——} \; (1)$$

$$(8-\lambda)(-3+3\lambda-\lambda+\lambda^2) + 8(4-4\lambda+6) - 2(-16+9+3\lambda) = 0$$

$$\therefore -24+16\lambda+8\lambda^2+3\lambda-2\lambda^2-\lambda^3+80-24\lambda+14-6\lambda = 0$$

$$\therefore -\lambda^3+6\lambda^2-11\lambda+70 = 0$$

$\lambda = 1, 2, 3$ Thus, 1, 2, 3 are the Eigen values or characteristic roots. Next to find Eigen vectors, for the sake of Eigen vector,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{such that} \quad [A - \lambda I]X = 0$$

Using (1),

Put $\lambda = 1$,

$-12+8$ $16-8$

$M^{-1}AM = D$ diagonal

Transform

$$\therefore \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -4 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -8 \\ 4 & -4 \end{vmatrix}}$$

$$\therefore \frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4}$$

$\therefore x_1 = 4,\ x_2 = 3,\ x_3 = 2$

Put $\lambda = 2$,

$$\therefore \begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -8 & -2 \\ -5 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 6 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 6 & -8 \\ 4 & -5 \end{vmatrix}}$$

$$\therefore \frac{x_1}{6} = \frac{+x_2}{+4} = \frac{x_3}{2}$$

$x_1 = 3,\ x_2 = 2,\ x_3 = 1$

Put $\lambda = 3$,

$$\therefore \begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$\therefore \frac{x_1}{\begin{vmatrix} -8 & -2 \\ -6 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -8 \\ 4 & -6 \end{vmatrix}}$$

$\therefore$ Here, all Eigen values are distinct and algebraic multiplicity and geometric multiplicity are same. That means that given matrix A is diagonalisable.
Since, $M^{-1}AM = D$, the given matrix A will be diagonalisable to diagonal matrix D.

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by the transforming matrix $M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

CAYLEY - HAMILTON'S THEOREM (C-H Theorem):

Every square matrix which satisfies its own characteristic eqns is CH matrix.

Egs:

(1) ~~Verify~~ Verify the CH theorem for matrix A.

→ $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ By using $|A - \lambda I| = 0$,

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(-1 \cancel{-\lambda+\lambda} + \lambda^2 - 2) + 2(1 - 2 + 2\lambda) = 0$

$-1 + \lambda^2 - 2 + \lambda - \lambda^3 + 2\lambda - 2 + 4\lambda = 0$

$-\lambda^3 + \lambda^2 + 7\lambda - 5 = 0$

$\lambda^3 - \lambda^2 - 7\lambda + 5 = 0$

But as ~~per~~ C-H theorem states that every square matrix satisfies its own characteristic eqn, that means replace λ by A in the above eqn. Here ~~you~~ we have to show that

$A^3 - A^2 - 7A + 5 = 0$ — (*)

first find A^2, A^3

$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 0 \\ 6 & 3 & 2 \\ 1 & 0 & 7 \end{bmatrix}$

$A^3 = \begin{bmatrix} 7 & 2 & 14 \\ 13 & 5 & 16 \\ 15 & 7 & -5 \end{bmatrix}$

Now consider LHS of eqn (*),

$$\therefore \begin{bmatrix} 7 & 2 & 14 \\ 13 & 5 & 16 \\ 15 & 7 & -5 \end{bmatrix} - \begin{bmatrix} 5 & 2 & 0 \\ 6 & 3 & 2 \\ 1 & 0 & 7 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RHS}$$

Therefore A satisfies characteristic eqn, hence CH theorem is verified for given matrix. Now to find inverse, multiply be A^{-1} to both sides of (*)

$$A^{-1}(A^3 - A^2 - 7A + 5) = A^{-1}.0$$

$$\therefore A^2 - A - 7 + 5A^{-1} = 0$$

$$\therefore A^{-1} = -\frac{1}{5}\left[A^2 - A - 7I\right]$$

$$A^{-1} = -\frac{1}{5}\left[\begin{bmatrix} 5 & 2 & 0 \\ 6 & 3 & 2 \\ 1 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right]$$

$$A^{-1} = -\frac{1}{5}\begin{bmatrix} -3 & 2 & -2 \\ 5 & -5 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

(5) Find the characteristic eqn of the matrix A. And verify that it is satisfied by A, hence obtain A^{-1} and A^4.

$$\rightarrow A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

By using characteristic eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(\overset{3}{4} \overset{-4\lambda}{-2\lambda - 2\lambda} + \lambda^2 - 1) + 1(-2 + \lambda + 1) + 1(1 - 2 + \lambda) = 0$$

$$\therefore 6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 - 1 + \lambda - 1 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$22 - 36 + 18 - 4$

$\therefore A^3 - 6A^2 + 9A - 4 = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence proved.
CH theorem verified.

$A^2 - 6A + 9 - 4A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9\right]$$

$$= \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \right)$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

(Q) Find the inverse of the matrix A. Using CH theorem find A^{-1}.

$$\rightarrow A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (3-\lambda)\left[14 - 8\lambda + \lambda^2\right] + 1\left[\lambda - 2\right] + 1\left[-4 + \lambda\right] = 0$$

$$\begin{array}{r} 22 \\ 14 \\ \hline 36 \end{array}$$

$$\therefore 42 - 24\lambda - 14\lambda + 8\lambda^2 + 3\lambda^2 - \lambda^3 + \lambda - 2 - 4 + \lambda = 0$$

$$\therefore \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$\phi(A) = \alpha_1 A + \alpha_0 I$

$\phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$

$9+1+1$

$\therefore A^3 - 11A^2 + 36A - 36 = 0$

$$A^2 = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{bmatrix}$$

$A^2 - 11A + 36 - 36A^{-1} = 0$

$$\therefore A^{-1} = \frac{1}{36}\left[A^2 - 11A + 36\right]$$

$$= \frac{1}{36}\left(\begin{bmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{bmatrix} - \begin{bmatrix} 33 & -11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{bmatrix} + \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}\right)$$

$$= \frac{1}{36}\begin{bmatrix} 14 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 14 \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} 7 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

★ FUNCTION OF A SQUARE MATRIX:

- It is denoted by $\phi(A)$
- ~~Since~~ If the matrix is of order 2×2, then we take $\phi(A) = \alpha_1 A + \alpha_0 I$ where α_0 and α_1 are the arbitrary constants.
- If the matrix is of order 3×3, then we take $\phi(A) = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ where $\alpha_0, \alpha_1, \alpha_2$ are arbitrary constants.

Egs:

(1) $A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$ Then prove that $A^{50} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$

→ By using characteristic eqn i.e, $|A - \lambda I| = 0$,

$$\begin{vmatrix} 2-\lambda & 3 \\ -3 & -4-\lambda \end{vmatrix} = 0$$

$\therefore -8 - 2\lambda + 4\lambda + \lambda^2 + 9 = 0$

$\therefore \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1, -1$

Let $\phi(A) = A^{50} = \alpha_1 A + \alpha_0 I$ — (*) where α_0 and α_1 are arbitrary constants. We assume that the above equality is satisfied by the characteristic root of A, that means replace A by λ in (*).

$\therefore \lambda^{50} = \alpha_1 \lambda + \alpha_0 I$ — (I)

put $\lambda = -1$

$(-1)^{50} = \alpha_1 (-1) + \alpha_0 \Rightarrow 1 = -\alpha_1 + \alpha_0$ — (1)

As λ values are repeated, we can't directly substitute. So diff eqn (I) wrt λ

$\therefore 50 \lambda^{49} = \alpha_1$

$\therefore 50 (-1)^{49} = \alpha_1 \Rightarrow \alpha_1 = -50$

From eqn (1), $\alpha_0 = -49$

$A^{50} = \alpha_1 A + \alpha_0 I$

$$= -50 \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} + (-49) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -100 & -150 \\ +150 & +200 \end{bmatrix} - \begin{bmatrix} 49 & 0 \\ 0 & 49 \end{bmatrix} = \begin{bmatrix} -149 & -150 \\ 150 & 151 \end{bmatrix}$$

(2) If $A = \begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix}$. Find $\cos A$.

$\rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} \pi - \lambda & \pi/4 \\ 0 & \pi/2 - \lambda \end{vmatrix} = 0$$

$\therefore \pi^2/2 - \lambda\pi - \lambda\pi/2 + \lambda^2 = 0$

$\therefore \lambda^2 - \lambda\, 3\pi/2 + \pi^2/2 = 0$

$\therefore \lambda = \pi/2, \pi$

$\phi(A) = \cos A = \alpha_1 A + \alpha_0 I$ — (*)

$\therefore \cos(\lambda) = \alpha_1 (\lambda) + \alpha_0$

Put $\lambda = \pi/2$

$\therefore \cos(\pi/2) = \alpha_1 (\pi/2) + \alpha_0 = 0 \Rightarrow \alpha_0 = -\pi/2\, \alpha_1$

Put $\lambda = \pi$

$\therefore \cos(\pi) = \alpha_1 (\pi) + \alpha_0 = -1 \Rightarrow \alpha_0 = -1 - \pi\alpha_1$

$\therefore -\pi/2\, \alpha_1 = -1 - \pi\alpha_1 \Rightarrow \alpha_1 (-\pi/2) = 1 \Rightarrow \boxed{\alpha_1 = -2/\pi,\ \alpha_0 = 1}$

$$\therefore \cancel{\cos A = \frac{2}{3\pi}\begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\cancel{= \frac{\begin{bmatrix} 2 & 1/2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{3} = \begin{bmatrix} 1/3 & 1/6 \\ 0 & 0 \end{bmatrix}}$$

$$\therefore \cos A = \frac{-2}{\pi}\begin{bmatrix} \pi & \pi/4 \\ 0 & \pi/2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1/2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

(3) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ find A^{50}.

→ By using characteristic eqn,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(\lambda^2-1) = 0 \Rightarrow \lambda = 1, 1, -1$

$\phi(A) = A^{50} = \alpha_2 A^2 + \alpha_1 A + \alpha_0 I$ — (*)

$\therefore \lambda^{50} = \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0$ — (I)

Put $\lambda = 1$

$\therefore 1 = \alpha_2 + \alpha_1 + \alpha_0$ — (1)

Put $\lambda = -1$

$\therefore 1 = \alpha_2 - \alpha_1 + \alpha_0$ — (2)

Diff (I),

$\therefore 50 \cdot \lambda^{49} = 2\alpha_2 \lambda + \alpha_1$

Put $\lambda = -1$, $-50 = -2\alpha_2 + \alpha_1$ — (3). $\Rightarrow \alpha_1 = 2\alpha_2 - 50$

$\therefore 3\alpha_2 + \alpha_0 = 51$ from (1)

$-\alpha_2 + \alpha_0 = -49$

$4\alpha_2 = 100$

$\alpha_2 = 25 \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_0 = -24$

Ans = divisor (Q) + R

$\therefore A^{50} = 25A^2 - 24I$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

(8) Find the characteristic eqn of matrix A given below and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$\rightarrow A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Using characteristic eqn,

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\therefore (2-\lambda)[2 - 3\lambda + \lambda^2] - 1(0) + 1(-1+\lambda) = 0$

$\therefore 4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$

$\therefore -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ — (*)

This eqn is satisfied by A. Now dividing,

$$\frac{\lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + \lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1}{\lambda^3 - 5\lambda^2 + 7\lambda - 3}$$ } by long division

$Q = \lambda^5 - \lambda$, $R = \lambda^2 + \lambda + 1$

Ans = divisor (Q) + R

$= (A^3 - 5A^2 + 7A - 3)(A^5 - A) + (A^2 + A + 1)$ — (2)

But $(A^3 - 5A^2 + 7A - 3)$ is 0.

Ans = $A^2 + A + 1$

Substitute A and find answer.

★ HOW TO FIND EIGEN VALUE AND EIGEN VECTOR BY ELEMENTARY ROW TRANSFORMATION.

(Q) Find eigen values and bases for Eigen spaces.

→ $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ By using characteristic equation i.e, $|A - \lambda I| = 0$ — (1)

$$\therefore \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (6-\lambda)(9+\lambda^2-6\lambda+1) + 2(-6+2\lambda+2) + 2(-2-6+2\lambda) = 0$

$\therefore (6-\lambda)(10+\lambda^2-6\lambda) + 2(2\lambda-4) + 2(-8+2\lambda) = 0$

$\therefore 60 + 6\lambda^2 - 36\lambda - 10\lambda - \lambda^3 + 6\lambda^2 + 4\lambda - 6 - 16 + 4\lambda = 0$

$\therefore -\lambda^3 + 12\lambda^2 - 38\lambda + 38 = 0$

$\therefore \lambda^3 - 12\lambda^2 + 38\lambda - 38 = 0$

Wrong check calculations

$\Rightarrow \lambda = 2, 2, 8$

(i) for $\lambda_{1,2}$ in (1) $\therefore [A - \lambda I]X = 0$ gives

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & +1 & -1 \\ 2 & -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By $R_1 \leftrightarrow R_2$ $\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

By $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$ $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Thus, we will get 2 independently linear set of infinite solutions

$\therefore 2x_1 - x_2 + x_3 = 0$ — (2)

Here we see that, the rank of matrix is one and number of variables are 3. Hence there are $3-1 = 2$ linearly independent solutions. So, put x_2 and x_3 values

$x_2 = 2s$ and $x_3 = -2t$ 2 bcoz coeff of x_1 is 2.

+ve as x_2 has -ve sign

-ve as x_3 has +ve sign

$2x_1 - 2s - 2t = 0$

$\therefore x_1 = s + t$

$$\therefore X = \begin{bmatrix} s+t \\ +2s+0 \\ 0-2t \end{bmatrix} \Rightarrow X = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Thus, the vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ are linearly independent. Also the above Eigen vectors form a basis for the Eigen spaces corresponding to $\lambda = 2$. Thus, the dimension of the Eigen space is 2.

(ii) for $\lambda_2 = 8$ in (1) $\therefore [A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

By $R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Here we see that, the rank of the matrix is 2 and no of variables are 3. Hence there are $3-2=1$ linearly independent solutions.

$\therefore 2x_1 + 2x_2 - 2x_3 = 0 \Rightarrow x_1 + x_2 - x_3 = 0$ — (3)

$\therefore 3x_2 + 3x_3 = 0 \Rightarrow x_2 + x_3 = 0$ — (4)

Put $x_3 = t \Rightarrow x_2 = -t \Rightarrow x_1 = 2t$

$$X = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the basis for Eigen space corresponding to $\lambda = 8$

(Q) Find eigen value and basis for Eigen spaces for matrix

$\rightarrow A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ By $|A - \lambda I| = 0$, $\rightarrow$ (1)

$\therefore \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$ — (1)

$\therefore (2-\lambda)(6-3\lambda-2\lambda+\lambda^2-2) - 2(2-\lambda-1) + 1(2-3+\lambda) = 0$

$\therefore 12 - 10\lambda + 2\lambda^2 - 4 - 6\lambda + 6\lambda^2 - \lambda^3 - \cancel{2\lambda} - 2 + \cancel{2\lambda} + \lambda - 1 = 0$

$\therefore -\lambda^3 + 8\lambda^2 - 15\lambda + 5 = 0$

$\therefore \lambda^3 - 8\lambda^2 + 15\lambda - 5 = 0$

$\therefore \lambda = 1, 1, 5$

(i) for $\lambda_1 = 1$ in (1)

$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ By $R_2 - R_1$, $R_3 - R_1$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore x_1 + 2x_2 + x_3 = 0$ — (2)

Put $x_2 = -s$, $x_3 = -t$

$\therefore x_1 - 2s - t = 0 \Rightarrow x_1 = 2s + t$

$\therefore X = \begin{bmatrix} 2s+t \\ -s+0 \\ 0-t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore$ The vectors $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent and form basis for Eigen spaces corresponding to $\lambda = 1$.

Dimension of Eigen space is 2.

(ii) for $\lambda_2 = 5$ in (1)

$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

~~By $R_2 \rightarrow 3R_2 + R_1$, $R_3 \rightarrow 3R_3 + R_1$ $\begin{bmatrix} -3 & 2 & 1 \\ 0 & -4 & \end{bmatrix}$~~

$R_1 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_2 - R_1$, $R_3 + 3R_1$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$R_3 + 2R_2$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore x_1 + 2x_2 - 3x_3 = 0$

$\therefore 4x_2 = 4x_3 \Rightarrow x_2 = x_3 = t$

$\therefore x_1 = 3t - 2t = t$

$\therefore X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

★ MODULE 2: Condition of consistency and ~~increasing~~ inconsistency for Homogeneous and Non-Homogeneous eqn

$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$

Such eqns are called as non-homogeneous eqns. On the other hand, all $b_1, b_2, \ldots b_m$ are zeros, then eqns are homogenous eqns.

(1) If Rank A < Rank [A, B] then the equations are inconsistent.

(2) If Rank A = Rank [A, B] then equations are consistent, ie, they have a solution.

(a) If $r = n$, i.e, rank of A is equal to no of unknowns then system has unique soln. Also note that system has unique soln if and only if its coeff matrix is non-singular.

(b) If $r < n$, system has infinite solns. In this case $(n - r)$ unknowns

called parameters can be assigned arbitrary values. The remaining unknowns then can be explained in terms of these parameters.

Egs:

(1) Solve the eqns:

$\rightarrow$ $x+y+z=3$

$x+2y+3z=4$

$x+4y+9z=6$

We have $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$R_2 - R_1$, $R_3 - R_1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

$R_3 - 3R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ — (1)

Thus $A \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $[A,B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$

The rank of A is equal to the rank of $[A,B]$ which is 3. Thus, the eqns are consistent. Further, the rank of A is equal to no. of unknowns. Hence system has unique soln.

$\therefore$ $2z=0 \Rightarrow z=0$.

$y+2z=1 \Rightarrow y=1$

$x+y+z=3 \Rightarrow x=2$

$\therefore x=2, y=1, z=0$

(2) $2x-3y+7z=5$

$3x+y-3z=13$

$2x+19y-47z=32$

Check for consistency and if possible, solve the following eqns.

→ We have $\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$

$R_1 - R_2$ $\begin{bmatrix} -1 & -4 & 10 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \\ 32 \end{bmatrix}$

$R_1 \to -R_1$ $\begin{bmatrix} 1 & 4 & -10 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 32 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 1 & 4 & -10 \\ 0 & -11 & 27 \\ 0 & 11 & -27 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 16 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $\begin{bmatrix} 1 & 4 & -10 \\ 0 & -11 & 27 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 5 \end{bmatrix}$ — ①

Thus, $A = \begin{bmatrix} 1 & 4 & -10 \\ 0 & -11 & 27 \\ 0 & 0 & 0 \end{bmatrix}$ and $[A|B] = \left[\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right]$

Here, the rank of A is 2 and rank of [A|B] is 3. Since ranks are not equal, system of eqn is inconsistent. This is clear even otherwise from eqn ①, the last row gives $0x + 0y + 0z = 5$ which is absurd. Thus, the eqns are inconsistent.

(2) Investigate for what values of λ and μ, the eqns have (i) No soln (ii) Unique soln (iii) Infinite soln.

→ $x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

We have, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 10 \end{bmatrix}$ — ①

(i) System has linear soln if coeff matrix is non-singular.

$\therefore \lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$

If λ is not equal to 3, system has unique soln.

(ii) If $\lambda = 3$, coeff matrix and augmented matrix become

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu - 10 \end{bmatrix}$$

So rank of $A = 2$ and rank of $[A, B]$ is also 2 if $\mu = 10$.
Thus, if $\lambda = 3$ and $\mu = 10$, system is consistent. But the rank of $A = 2$ is less than no of unknowns, eqns will possess infinite solns.

(iii) If $\lambda = 3$ and $\mu \neq 10$. Rank of $A = 2$ and rank of $[A, B] = 3$
Hence, the eqns will be inconsistent and will not possess any solution.

> CROUT'S METHOD:

In linear algebra, crout matrix decomposition is an LU decomposition which decomposes a matrix into lower and upper triangular matrix. Steps:

(1) Get the matrix A and B from given set of eqns.

(2) First we take $LU = A$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$L \Rightarrow$ lowertriangular matrix with all diagonal elements $= 1$

(3) Then calculate the elements of lower ~~of~~ and upper triangular matrix.

(4) Find V by solving $LV = B$ by forward substitution, where $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$

(5) Then finally we take the original system of equation $UX = V$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find X by back substitution.

Egs:

(1) Apply Crout's method to solve the following eqns:

$3x+2y+7z=4$

$2x+3y+z=5$

$3x+4y+z=7$

We have $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

First we take $LU = A$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$\therefore \quad u_{11} = 3 \qquad u_{12} = 2 \qquad u_{13} = 7$

$l_{21} u_{11} = 2 \Rightarrow l_{21} = 2/3$

$l_{21} u_{12} + u_{22} = 3 \Rightarrow 4/3 + u_{22} = 3 \Rightarrow u_{22} = 3 - 4/3 = 5/3$

$l_{21} u_{13} + u_{23} = 1 \Rightarrow u_{23} = 1 - 14/3 = -11/3$

$l_{31} u_{11} = 3 \Rightarrow l_{31} = 1$

$l_{31} u_{12} + l_{32} u_{22} = 4 \Rightarrow 2 + 5 l_{32}/3 = 4 \Rightarrow l_{32} = 6/5$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} u_{33} = 1 \Rightarrow 7 - \frac{44}{15} + u_{33} = 1 \Rightarrow u_{33} = -8/5$

$\frac{44}{15} - 6 = \frac{44 - 90}{15} = \frac{-46}{15}$

$\frac{4 \times -11}{5 \quad 3} = \frac{44}{15}$

$\begin{array}{r} 45 \\ -28 \\ \hline 17 \end{array}$

Using $LV = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$\frac{4}{5} \times \frac{7}{3} = \frac{28}{15}$

$\therefore \quad v_1 = 4$

$2/3 \times 4 + v_2 = 5 \Rightarrow v_2 = 5 - 8/3 = 7/3$

$v_1 + 6v_2/5 + v_3 = 7 \Rightarrow \cancel{4 + 28/15 + v_3 = 7 \Rightarrow v_3 = 3 - 28/15 = 17/15}$

$\frac{28}{5} \times \frac{7}{3} = \frac{28 \times 7}{15}$

$\therefore V = \cancel{\begin{bmatrix} 4 \\ 7/3 \\ 17/15 \end{bmatrix}}$

$4 + 14/5 + v_3 = 7$

$\therefore v_3 = 3 - 14/5 = 1/5$

$\therefore V = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$

Using $UX = V$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$\therefore \quad -8/5\, z = 1/5 \Rightarrow \boxed{z = -1/8}$

$\cancel{5y/3 = 7/3 - 11/24 \Rightarrow \boxed{y = 31/20}}$

$5y/3 + \left(\frac{+11}{3} \times \frac{+1}{8}\right) = 7/3$

$5y/3 = 7/3 - \frac{11}{24} = 45/24 \Rightarrow \boxed{y = 9/8}$

$3x + 9/4 - 7/8 = 4 \Rightarrow \boxed{x = 7/8}$

$\cancel{3x + 31/15 - 7/8 = 4}$

$\frac{-23}{8} \times -7 = +\frac{161}{8}$

(2) Apply Crout's method to solve:

$\rightarrow$ $x_1 + x_2 + 4x_3 = 43$

$2x_1 + 10x_2 + x_3 = 63$

$25x_1 + 2x_2 + x_3 = 69$

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 10 & 1 \\ 25 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 43 \\ 63 \\ 69 \end{bmatrix}$$

$LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 10 & 1 \\ 25 & 2 & 1 \end{bmatrix}$$

$u_{11} = 1 \quad u_{12} = 1 \quad u_{13} = 4$

$l_{21} u_{11} = 2 \Rightarrow l_{21} = 2$

$l_{21} u_{12} + u_{22} = 10 \Rightarrow u_{22} = 8$

$l_{21} u_{13} + u_{23} = 1 \Rightarrow u_{23} = -7$

$l_{31} u_{11} = 25 \Rightarrow l_{31} = 25$

$l_{31} u_{12} + l_{32} u_{22} = 2 \Rightarrow 8 l_{32} = -23 \Rightarrow l_{32} = -23/8$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1 \Rightarrow 100 + 161/8 + u_{33} = 1 \Rightarrow u_{33} = -99 - \frac{161}{8} = -953/8$

$LV = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 25 & -23/8 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 43 \\ 63 \\ 69 \end{bmatrix}$$

$\therefore v_1 = 43$

$86 + v_2 = 63 \Rightarrow v_2 = -23$

$25(43) + \left(\frac{23 \times 23}{8}\right) + v_3 = 69$

$\therefore v_3 = 69 - 1075 - 529/8 = -8577/8$

$$V = \begin{bmatrix} 43 \\ -23 \\ -8577/8 \end{bmatrix}$$

$Ux = V$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & -7 \\ 0 & 0 & -953/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 43 \\ -23 \\ -8577/8 \end{bmatrix}$$

$\therefore \boxed{x_3 = 9}$

$8x_2 - 63 = -23 \Rightarrow 8x_2 = 40 \Rightarrow \boxed{x_2 = 5}$

$x_1 + 5 + 36 = 43 \Rightarrow \boxed{x_1 = 2}$

— NOTE: For 4 variables, L and U will be 4x4 matrices.

Derogatory and Non-derogatory Matrix:

- Monic Polynomial:

Eg: (i) $x^3 - 2x^2 + x - 1 = 0$ Monic

(ii) $2x^4 + x - 7 = 0$ Not monic

A polynomial in x in which the coeff of highest power of x is unity.

- Egs.

(1) Show that matrix A is a Non Derogatory Matrix

$\rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

By using characteristic matrix $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 3 & 4 & 5-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(15 - 8\lambda + \lambda^2 - 16) - 2(10 - 2\lambda - 12) + 3(8 - 9 + 3\lambda) = 0$

$\therefore (1-\lambda)(\lambda^2 - 8\lambda - 1) - 2(-2\lambda - 2) + 3(-1 + 3\lambda) = 0$

$\therefore \lambda^2 - 8\lambda - 1 - \lambda^3 + 8\lambda^2 + \lambda + 4\lambda + 4 - 3 + 9\lambda = 0$

$\therefore -\lambda^3 + 9\lambda^2 + 6\lambda = 0$

$\therefore \lambda(\lambda^2 - 9\lambda + 6) = 0$

$\therefore \lambda = 0, 9.62, -0.62$

Since all eigen values are distinct and characteristic eqn is satisfied by A, hence given matrix A is a non derogatory matrix.

(2) Show that A is Non Derogatory matrix.

$\rightarrow A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ using $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\therefore (2-\lambda)(-1-\cancel{\lambda}+\cancel{\lambda}+\lambda^2-3)+2(-1-\lambda-1)+3(3-1+\lambda)=0$

$\therefore (2-\lambda)(\lambda^2-4)+2(-\lambda-2)+3(2+\lambda)=0$

$\therefore 2\lambda^2-8-\lambda^3+4\lambda-2\lambda-4+6+3\lambda=0$

$\therefore -\lambda^3+2\lambda^2+5\lambda-6=0$

$\therefore \lambda^3-2\lambda^2-5\lambda+6=0$

$\lambda=1 \Rightarrow (\lambda-1)=0$

$$\therefore \begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & \downarrow & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2-\lambda-6)=0$

$\therefore (\lambda-1)(\lambda^2-3\lambda+2\lambda-6)=0$

$\therefore \lambda=1,-2,3$

Here, all eigen values are distinct and characteristic eqn is satisfied by A, Hence it is non-derogatory matrix

(3) Show that A is Derogatory Matrix.

$\rightarrow A=\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ By using $|A-\lambda I|=0$

$$\therefore \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix}=0$$

$\therefore (5-\lambda)(-16+\lambda^2+12)+6(4+\lambda-6)-6(6-12+3\lambda)=0$

$\therefore (5-\lambda)(\lambda^2-4)+6(\lambda-2)-6(3\lambda-6)=0$

$\therefore 5\lambda^2-20-\lambda^3+4\lambda+6\lambda-12-18\lambda+36=0$

$\therefore -\lambda^3+5\lambda^2-8\lambda+4=0$

$\therefore \lambda^3-5\lambda^2+8\lambda-4=0$

$\therefore \lambda=1 \Rightarrow (\lambda-1)=0$

$$\therefore \begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & \downarrow & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\therefore (\lambda-1)(\lambda^2-4\lambda+4)=0$

$(\lambda-1)(\lambda-2)^2=0 \Rightarrow \lambda=1,2,2$ ~~Hence not distinct~~ $\therefore$ ~~Derogatory~~

Here, 2 eigen values are repeated and 3rd value is different. So, first we find the minimal polynomial of A. But we know that each characteristic root of A is also a root of the minimal polynomial of A.

If $f(x)$ is minimal polynomial of A then $(x-1)(x-2)$ are factors of $f(x)$.

Let us see whether the polynomial is $(x-1)(x-2) = x^2-3x+2$ — (1)

Let us see whether the polynomial annihilates the matrix A or not.

Replace x by A in eqn (1), we get

$A^2 - 3A + 2I$

$$A^2 = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix}$$

Substituting we get,

$$\begin{bmatrix} 13 & -18 & -18 \\ -3 & 10 & 6 \\ 9 & -18 & -14 \end{bmatrix} - 3\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Thus, $f(x)$ annihilates the matrix A

Hence $f(x)$ is a minimal polynomial and monic polynomial. Thus, matrix is derogatory.

(4) Check whether following matrix A is derogatory or non-derogatory

$$\rightarrow A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix} = 0$$

$\therefore (7-\lambda)(28-11\lambda+\lambda^2-4) - 4(16-4\lambda-4) - 1(-16-28+4\lambda) = 0$

$\therefore (7-\lambda)(\lambda^2-11\lambda+24) - 4(12-4\lambda) - 1(4\lambda-44) = 0$

$\therefore 7\lambda^2 - 77\lambda + 168 - \lambda^3 + 11\lambda^2 - 24\lambda - 48 + 16\lambda - 4\lambda + 44 = 0$

$\therefore -\lambda^3 + 18\lambda^2 - 89\lambda + 164 = 0$

$\lambda = 12, 3, 3$

~~$f(x) = (x-1)(x-2) = x^2 - 3x + 2$~~ $f(x) = (x-3)(x-12)$

~~$A - 3A + 2$~~ $= x^2 - 15x + 36$

$$A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

★ SINGULAR VALUE DECOMPOSITION (SVD):

- ~~Theo~~ Theorem: A rectangular matrix A_{mn} can be decomposed into the product of 3 matrices, an orthogonal matrix U_{mm} and diagonal matrix D_{mn}, and the transpose of another orthoganal matrix V'_{nn}.

$\therefore A_{mn} = U_{mm} \times D_{mn} \times V'_{nn}$

Since U and V are orthogonal, we have $UU' = I$ and $VV' = I$. Further the columns of U are orthoganal eigen vectors of AA' and columns of V are orthonormal eigen vectors of $A'A$. The diagonal values of D are called as singular values and hence it is called Singular-Value-Decomposition.

The normalised eigen vectors of $A'A$ are the columns of V and the normalised eigen vectors of AA' are the columns of U.

Step 1: We first find eigen values of $A'A$.

2: Then we take the square roots of these eigen values obtained above in step 1, and write the diagonal matrix D whose diagonal elements are these square roots arranged in a descending order.

Let $D = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$ where $\sigma_1, \sigma_2, \sigma_3$ are the square roots of diagonal elements

3. Now we obtain eigen vectors of $A'A$. These eigen vectors when normalised are the columns of V say v_1, v_2, v_3.

4. The columns of U are given by

$$u_1 = \frac{1}{\sigma_1} A v_1 \quad , \quad u_2 = \frac{1}{\sigma_2} A v_2 \quad , \quad u_3 = \frac{1}{\sigma_3} A v_3$$

5. Thus, we get $A = UDV'$

(Q) Find the singular value decomposition of given matrix.

$\rightarrow A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ $\quad A' = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$

$\therefore A'A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$ First we find eigen values of $A'A$.

$|A'A - \lambda I| = 0$. Thus, characteristic eqn is

$$\therefore \begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix} = 0$$

$$\therefore (25-\lambda)^2 - 49 = 0$$

$$\therefore (25-\lambda) = \pm 7 \Rightarrow 25-\lambda = 7 \text{ OR } 25-\lambda = -7$$

$$\therefore \lambda = 18, 32$$

We now take the square roots of the above eigen values and arrange them in descending order to get D.

$\sigma_1 = \sqrt{32} = 4\sqrt{2}$, $\sigma_2 = \sqrt{18} = 3\sqrt{2}$

$$\therefore D = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$$

Now we obtain eigen vector of $A'A$

For $\lambda_1 = 32$ $\quad [A - \lambda I] x = 0$

$$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \xrightarrow[R_1 \to \frac{1}{7} R_1]{R_2 + R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\therefore x_2 = t \Rightarrow -x_1 = -t \Rightarrow x_1 = t$

$$\therefore X = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 18$ $\quad \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$v_2 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Since $v_1 . v_2 = (1 \quad 1) . (1 \quad -1) = 0 \Rightarrow v_1$ and v_2 are orthogonal. We ~~to~~ ~~m~~ now normalize ~~them~~ by dividing them by their norms. $\|v_2\|$

$\|v_1\| = \sqrt{x_1^2+x_2^2} = \sqrt{2}$

$\|v_2\| = \sqrt{x_1^2+x_2^2} = \sqrt{2}$

$\therefore$ The normalised vector $v_1 = \dfrac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$v_2 = \dfrac{v_1}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Thus, $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

For getting columns of U,

$u_1 = \dfrac{1}{4\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \cancel{\dfrac{1}{4\sqrt{2}}} \quad = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_2 = \dfrac{1}{3\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Thus, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Thus, we get $A = UDV'$

(Q) Find the ~~SDV~~ SVD of given matrix:

$\rightarrow A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \qquad A' = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

$A'A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

$(4-\lambda)(13-\lambda) - 36 = 0$

$\therefore 52 - 17\lambda + \lambda^2 - 36 = 0$

$\therefore \lambda^2 - 17\lambda + 16 = 0 \qquad \lambda^2 - 16\lambda - \lambda + 16 = 0 \Rightarrow \lambda = 16, 1$

$\sigma_1 = 4$, $\sigma_2 = 1$ $\qquad \lambda_1 = 16 \quad \lambda_2 = 1$

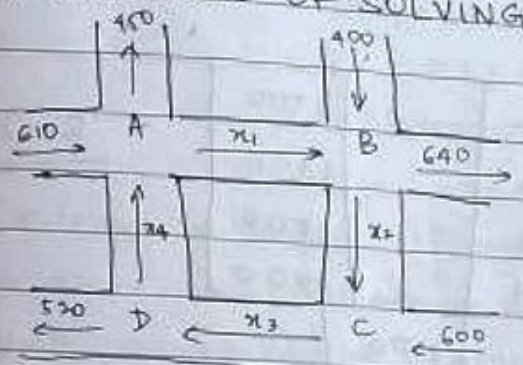
$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\cancel{\lambda = 16, \begin{bmatrix} -12 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \xrightarrow{2R_2 + R_1} \begin{bmatrix} -12 & 6 \\ 0 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0}$$

$$\lambda = 16 \quad \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

★ APPLICATIONS OF SOLVING SYSTEM OF EQNS IN TRAFFIC CONTROL

(Q)



Determine the amount of traffic between each of the 4 intersections

Soln: At each intersection, the incoming traffic has to match the outgoing traffic. Then we can write nodes A, B, C, D, ie

At node A: $610 + x_4 = 450 + x_1 \Rightarrow -x_1 + x_4 = -160$

B: $x_1 + 400 = 640 + x_2 \Rightarrow x_1 - x_2 = 240$

C: $x_2 + 600 = x_3 \Rightarrow x_2 - x_3 = 600$

D: $x_3 = x_4 + 520 \Rightarrow$ ~~$x_4 - x_3 = 520$~~ $x_3 - x_4 = 520$

On simplification we get,

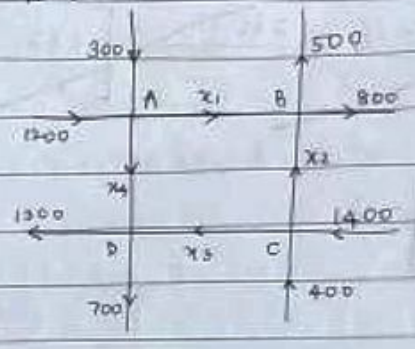
$x_2 = x_1 - 240$

$x_3 = x_1 + 360$

$x_4 = x_1 - 160$

We need to install only one sensor in terms of x_1 to count the no of cars passing through lane A. Plus, we can fully determine the traffic flow through these 4 branches if we can count the no of cars passing from A to B in a given time.

(Q) If x_1, x_2, x_3, x_4 are the no of vehicles travelling through each road per hour, find x_1, x_2, x_3, x_4 from the traffic diagram given below.



At A: $x_1 + x_4 = 1500$

B: $x_1 + x_2 = 1300$

C: $x_2 + x_3 = 1800$

D: $x_3 + x_4 = 2000$

$\therefore$ We get: $x_4 = 1500 - x_1$

$x_2 = 1300 - x_1$

$x_3 = 500 + x_1$

$\therefore$ In terms of matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1500 \\ 1300 \\ 1800 \\ 2000 \end{bmatrix} \xrightarrow{R_2 - R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1500 \\ -200 \\ 1800 \\ 2000 \end{bmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1500 \\ -200 \\ 2000 \\ 2000 \end{bmatrix} \xrightarrow{R_4 - R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1500 \\ -200 \\ 2000 \\ 0 \end{bmatrix}$$

$x_4 = t$

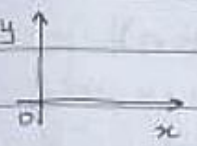
$x_1 = 1500 - t$

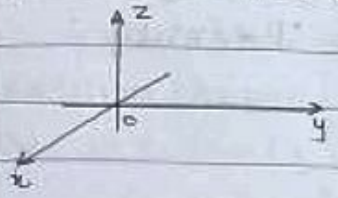
$x_2 = t - 200$

$x_3 = 2000 - t$

MODULE 3 : VECTOR SPACE

- One dimensional space:

Set of all points on a line is called as one dimensional space
- Two dimensional space:


The set of all ordered pairs (x,y) where $-\infty < x < \infty$, $-\infty < y < \infty$ is called two dimensional space.
- Three dimensional space:


The set of all ordered triplets (x,y,z) where $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$ is called three dimensional space.

- Ordered n tuple : If n is a positive integer, then the sequence $x_1, x_2, \ldots x_n$ of n real numbers is called as ordered n tuple, represents a vector or a point in n-dimensional space denoted by R^n.
- Zero space: A set of a single pt 0 is called zero space.
- Vector space: Let V be a non empty set of elements called vectors and k, m are scalars. If the same V satisfies the following axioms, then it is called as a vector space.
- Axioms of Vector Space:

(A) Closure Axiom:

(1) C_1 : If u and v are the elements in vector space V then $u+v$ is also in V. Thus, V is closed under addition.

(2) C_2: If k is any scalar then (ku) is in V. Thus, V is closed under scalar multiplication.

(B) Addition Axiom:

(1) A_1: $u+v = v+u$ Commutative

(2) A_2: $(u+v)+w = u+(v+w)$ Associative

(3) A_3: There is an element called zero in V such that $u+0=u$ for all u in V. Existence of Additive Identity.

(4) A4: There is an element $-u$ in V called negative of u corresponding to every u in V such that $u+(-u)=0$. Existence of Additive Inverse.

(C) Scalar Multiplication Axiom:

(1) M_1: $k(u+v) = ku + kv$ (distributivity of vectors)

(2) M_2: $(k+l)u = ku + lu$ "

(3) M_3: $(kl)u = k(lu)$ "

(4) M4: There is an element 1 in V called unity such that $u.1 = u$. Existence of Multiplicative Identity

Egs:

(1) If V is a vector space then show that:

(i) additive identity, 0 is unique

(ii) additive inverse of a vector u is unique

→ If possible, let their be 2 distinct additive identities 0 and $0'$.

Then $u+0=u$ — ① and $u+0'=u$ — ②

∴ put $u=0'$ in ①

∴ $0'+0 = 0'$ — ③

Put $u=0$ in ②

∴ $0+0'=0$ — ④

From 3 and 4 we get $0=0'$. Hence additive identity 0 is unique.

→ If possible, let their be u_1 and u_2 be two additive inverses of u, then by the definition of inverse,

$u+u_1=0$ — ⑤

$u+u_2=0$ — ⑥

Then add u_2 in ⑤

∴ $u_2+(u+u_1) = u_2+0$

∴ $(u_2+u)+u_1 = u_2$

∴ $0+u_1 = u_2$ (∵ ⑥)

∴ $u_1 = u_2$

This shows that the additive inverse of a vector u is unique.

(2) Check whether the set of all pairs of real numbers of the form $(1, x)$ with the operations $(1,y)+(1,y') = (1, y+y')$ ① and $k(1,y) = (1, ky)$ ②, is a vector space.

→ Here we have to show that V is a vector space.

(A) Closure axiom:

C1: $u = (1,x)$ and $v = (1,y)$

Then $u+v = (1,x)+(1,y) = (1, x+y)$ $(\because ①)$

$\therefore$ V is closed under addition.

C2: If k is any scalar then $ku = k(1,x) = (1, kx)$ $(\because ②)$

$\therefore$ V is closed under scalar multiplication.

(B) Addition Axiom.

A1: $u+v = (1,x)+(1,y) = (1, x+y)$ $(\because ①)$

$= (1, y+x) = (1,y)+(1,x)$

$\therefore u+v = v+u$ Commutative

A2: $(u+v)+w = ((1,x)+(1,y))+(1,z)$

$= (1,x+y)+(1,z) = (1, x+y+z)$ $(\because ①)$

$= (1, x+(y+z)) = (1,x)+(1,y+z) = (1,x)+((1,y)+(1,z))$

$= u+(v+w)$ Associative

A3: There is an element u in V such that $u = (1,x)$ and $0 = (1,0)$

$u+0 = (1,x)+(1,0) = (1, x+0) = (1,x) = u$ Additive Identity.

A4: There is an element $-u$ in V such that ~~u+(-u)~~

$u+(-u) = (1,x)+(1,-x) = (1, x-x) = (1,0) = 0$

$\therefore$ Additive Inverse.

(C) Scalar Multiplication:

~~M1. $k(u+v) = k((1,x)+(1,y)) = k((1, x+y))$~~

~~$= k(1,kx)+(1,ky)$~~

M1: $k(u+v) = k(1, x+y) = (1, k(x+y))$ $(\because ②)$

$= (1, kx+ky) = (1,kx)+(1,ky) = k(1,x)+k(1,y) = ku+kv$

M2: $(k+l)u = (k+l)(1,x) = (1, kx+lx)$ $(\because ②)$

$= (1,kx)+(1,lx) = k(1,x)+l(1,x) = ku+lu.$

M3: $(kl)u = (kl)(1,x) = (1, klx)$ $(\because ②)$

$= k(1, lx) = k(lu)$

As all axioms are satisfied, therefore V is a vector space.

☆ LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

- Linear combination of vectors: A vector w is called as a linear combination of vectors $v_1, v_2, \ldots v_n$ if it can be expressed in the form $w = v_1k_1 + v_2k_2 + \ldots + v_nk_n$ where $k_1, k_2, \ldots k_n$ are scalars, not all zero.
- Linear dependence of vectors: In the theory of vector space, a set of vectors is said to be linearly dependent if atleast one of the vector in the set can be defined as linear combination of others.
- Linear independence of vectors: In the theory of vector space, a set of vectors is said to be linearly independent if none of the vectors in the set can be defined as linear combination of others.

Egs:

(1) Examine whether the vectors $v_1 = [3, 1, 1]$, $v_2 = [2, 0, -1]$ and $v_3 = [4, 2, 1]$ are linearly dependent or independent.

→ Consider the matrix eqn $k_1v_1 + k_2v_2 + k_3v_3 = 0$

$\therefore k_1[3, 1, 1] + k_2[2, 0, -1] + k_3[4, 2, 1] = [0, 0, 0]$

$\therefore 3k_1 + 2k_2 + 4k_3 = 0$

$\therefore k_1 + 2k_3 = 0$

$k_1 - k_2 + k_3 = 0$

which can be written as

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \qquad \text{By } R_1 \leftrightarrow R_3$$

$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \qquad \begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \qquad R_3 - R_2$$

$-4k_3 = 0$

$0 \quad 0 \quad -4$

$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

$\therefore k_1 - k_2 + k_3 = 0$

$k_2 + k_3 = 0$

$4k_2 = 0 \Rightarrow k_2 = 0,\ k_3 = 0,\ k_1 = 0$

$\therefore$ Linearly indepent vectors

(2) Show that: $P_1 = 1-2x$, $P_2 = 2-3x+x^2$, $P_3 = 3-5x+x^2$ vectors are linearly dependent.

→ ~~P~~ We take

$P_1 + P_2 - P_3 = [1-2x + (2-3x+x^2) - (3-5x+x^2)]$

$= (1+2-3,\ -2x-3x+5x,\ x^2-x^2)$

$= (0,0,0) = 0$

$\therefore P_1 + P_2 - P_3 = 0 \Rightarrow P_1 + P_2 = P_3$

Thus, P_3 is expressable as P_1 and P_2. Hence, P_1, P_2, P_3 are linearly dependent.

(3) Use suitable trigonometric identities to determine whether the following set of functions are linearly dependent or not, and find the relation between them.

(i) $\overset{f_1}{8}$, $\overset{f_2}{4\sin^2 x}$, $\overset{f_3}{2\cos^2 x}$

(ii) $\cos 2x, \cos^2 x, \sin^2 x$

→ $f_1 - 2f_2 - 4f_3 = 8 - 2(4\sin^2 x) - 4(2\cos^2 x)$

$= 8 - 8\sin^2 x - 8\cos^2 x$

$= 8 - 8(1) = 0$

$\therefore f_1 = 2f_2 + 4f_3$ Hence, the functions are linearly dependent.

→ We take

$f_1 - f_2 - f_3 = \cos 2x - \cos^2 x - \sin^2 x = \cancel{\cos^2 x} - \sin^2 x - \cancel{\cos^2 x} - \sin^2 x$

$= -2\sin^2 x$

Now we take $f_1 - f_2 + f_3 = \cos 2x - \cos^2 x + \sin^2 x = 0$

$\therefore f_1 + f_3 = f_2$. Given functions are linearly dependent.

(4) Verify whether vectors (1,2) and (2,3) are linearly independent or dependent when placed with initial pts ~~as~~ at origin by analytic matrix and geometric method.

→ Let the vectors be $v_1 = (1,2)$ and $v_2 = (2,3)$ and consider the matrix eqn as $k_1 v_1 + k_2 v_2 = 0$

$\therefore k_1 [1,2] + k_2 [2,3] = [0,0]$

$\Rightarrow k_1 + 2k_2 = 0$

$2k_1 + 3k_2 = 0$

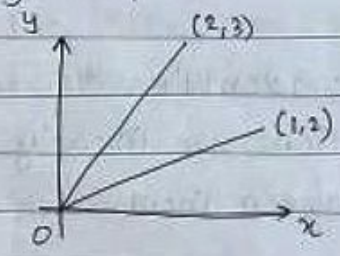
$$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \qquad R_2 - 2R_1$$

$$\therefore \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

$\therefore -k_2 = 0 \Rightarrow k_2 = 0 \Rightarrow k_1 = 0$

$\therefore$ Linearly independent vectors by analytical method.

When we place the vectors with initial pt on origin, there is a line containing both vectors. Therefore, given vectors here are linearly independent $\Rightarrow$ 2 lines.



HW: Verify whether (1,2) and (3,6) are linearly independent or not when placed with initial pts at origin.

☆ SUBSPACE:

A subset W of a vector space V is called a subspace of V if W is itself a vector space, under addition and scalar multiplication, defined on V.

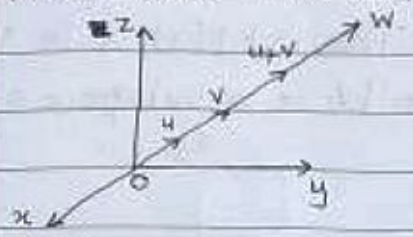
Theorem: If W is a non-empty subset of a vector space V then W is a subspace of V:

(i) for all u and v in W, u+v is in W

(ii) for any scalar k and vector u of W, then ku is in W.

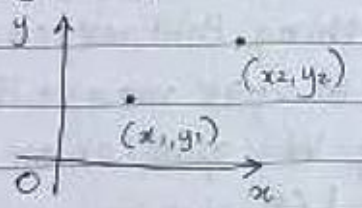
Egs:

(1) Show that line from origin in R^3 is a subspace of R^3.

→ 

If u and v are any 2 vectors wrt origin as the initial point on this line, u+v is also a vector on this line. Thus, u+v is closed under addition. Also if u is a vector on the line and k is any scalar then ku is also vector on this line. Thus, W is closed under scalar multiplication. Here both conditions are satisfied. Hence, a line through origin is a subspace of R^3.

(2) If W is a set of all points (x,y) in R^2 such that $x, y \geq 0$, then show that W is not a subspace of R^2.

→ 

If (x_1, y_1) and (x_2, y_2) are any two points of W in the first quadrant then,

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

But $k(x,y) \neq (kx, ky)$ for all k.

for instance if $k = -2$

$-2(x,y) = (-2x, -2y)$

and this point is not in the first quad. Hence, W is not closed under scalar multiplication. Thus, W is not a subspace of R^2.

(3) Let W be the set of all functions of the form $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$ where n is a non-negative integer and $a_0, a_1, a_2, \dots$ upto a_n are real numbers, then W is a subspace of all real valued functions, i.e, V

→ $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ & $q(x) = b_0 + b_1 x + \dots + b_n x^n$

Then (i) $p(x)+q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_n+b_n)x^n$
Here $p(x)+q(x)$ is of the above form, therefore $p(x)+q(x)$ is in W.

(ii) $kp(x) = k(a_0+a_1x+\dots+a_nx^n)$
$= ka_0 + ka_1x + ka_2x^2 + \dots + ka_nx^n$

Here $kp(x)$ is of the above form. Thus, $kp(x)$ is in W. Here both conditions are satisfied, thus W is a subspace of real valued function.

HW(4) Show that any plane through the origin is a subspace of R^3.

★ Space Spanned by a vector:
Let W be a subspace of V consisting of all linear combinations of $v_1, v_2, \dots, v_n$ of S, then W is called as a space spanned by S. We can also say that vectors $v_1, \dots, v_n$ spans W. To span means to furnish with something that extends or stretches over. Loosely speaking, to span means to create or to generate. This is denoted by W = span (S). OR $W = \text{span}(v_1 \dots v_n)$
Span of set S is also denoted by L(S)

Egs:

(1) Show that the vectors $v_1 = (1,0,1)$, $v_2 = (2,1,4)$, $v_3 = (1,1,3)$ do not span the vector space R^3.

→ If v_1, v_2, v_3 span the vector space R^3, then any vector in R^3 should be accesible in terms of v_1, v_2, v_3.
Let $W = (w_1, w_2, w_3)$ be a vector in R^3. Then we try to find the scalars k_1, k_2 and k_3 such that $W = k_1v_1 + k_2v_2 + k_3v_3$

$\therefore w = (w_1, w_2, w_3) = k_1[1,0,1] + k_2[2,1,4] + k_3[1,1,3]$
$= (k_1 + 2k_2 + k_3) + (k_2 + k_3) + (k_1 + 4k_2 + 3k_3)$

$\therefore w_1 = k_1 + 2k_2 + k_3$ $\quad w_3 = k_1 + 4k_2 + 3k_3$
$w_2 = k_2 + k_3$

NOTE: The system is consistent if $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} \neq 0$

But after solving we get $\Delta = (3-4) - 2(-1) + 1(-1) = 0$

This means that no k_1, k_2, k_3 satisfy eqn ① which means no vector in R^3 can be expressed as linear combination of v_1, v_2, v_3. Hence v_1, v_2, v_3 do not span the vectorspace in R^3.

HW (2) Determine whether the following vectors span P_2 (P_2 means vector space consisting of all polynomials of second order), where $P_1 = 1 - x + 2x^2$, $P_2 = 5 - x + 4x^2$, $P_3 = -2 - 2x + 2x^2$

★ BASIS OF A VECTOR:

- Let V be a vector space. A minimal set of vectors in V that spans V is called ~~the~~ a basis for V. Equivalently, a basis for V is a set of vectors i.e, linearly independent and spans V. As a result to check, if a set of vectors form a basis for the vector space, one needs to check that it is linearly independent and spans the vector space. If atleast one of these conditions fail to hold then it is not a basis.

Egs:

(1) Is the set $S = \{(1,1), (1,-1)\}$ a basis of R^2?

→ Consider the set S as equations

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\therefore c_1 + c_2 = x$ — ①

$\therefore c_1 - c_2 = y$ — ②

$$\therefore \left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y-x \end{array}\right] \quad \Delta = -2 \neq 0$$

Here, the system is consistent for every x and y. $\therefore$ S spans in R^2. Now checking for linearly independent.

From given, we can write,

$$\therefore c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $c_1 + c_2 = 0$ — (3)

$c_1 - c_2 = 0$ — (4)

$$\therefore \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

$\therefore$ $c_1 = c_2 = 0$ Trivial soln.

Given system has a unique soln i.e, $c_1 = c_2 = 0$

The set S is linearly independent and a basis for R^2

(2) Find the basis for vector space V spanned by a vector $w_1 = (1,1,0)$, $w_2 = (0,1,1)$, $w_3 = (2,3,1)$, $w_4 = (1,1,1)$

$\rightarrow$ We need to find the relation of the form $r_1 w_1 + r_2 w_2 + r_3 w_3 + r_4 w_4 = 0$ which can be written in matrix form as:

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To solve the system of linear eqns for r_1, r_2, r_3, r_4, we apply row transformations.

$$\xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = 0$$

$$\xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = 0$$

i.e, $r_1 + 2r_3 + r_4 = 0$ — (1)

$r_2 + r_3 = 0$ — (2) $\Rightarrow r_2 = -r_3$

$r_4 = 0$ — (3)

Using $x_4 = 0$ in ①,

$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$

$\therefore (x_1, x_2, x_3, x_4) = (-2t, -t, t, 0)$

$\therefore$ Particular soln at $t = -1$

$\therefore (x_1, x_2, x_3, x_4) = (2, 1, -1, 0)$

Thus, we obtain $2w_1 + w_2 - w_3 = 0$ from particular soln.
Hence any of vectors w_1, w_2, w_3 can be dropped. For instance $V = \text{span}(w_1, w_2, w_4)$

Check whether w_1, w_2, w_4 are LI (linearly independent)

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$$ Thus, w_1, w_2, w_4 are LI.

Therefore $V = R^3$ and w_1, w_2, w_4 is a basis for vector space V.

★ Gram Schmidts Orthogonalisation Process:

Gram Schmidts Process ~~use~~ gives us a method of finding orthonormal vectors. The following steps lead to an orthogonal basis v_1, v_2, v_3 for vector space V.

Step I: Let $v_1 = u_1$

Step II: We obtain vector v_2 that is orthogonal to v_1. Therefore

$$v_2 = u_2 - \text{proj}\, u_2$$

$$v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} \cdot v_1$$

Step III: To find v_3 that is orthogonal to v_1 and v_2. We find u_3 orthogonal to space w_2 spanned by the vector v_1, v_2 which is given by

$$v_3 = u_3 - \text{proj}\, u_3$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} \cdot v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} \cdot v_2$$

Thus we get the orthonormal basis as,

$$q_1 = \frac{v_1}{\|v_1\|}, \quad q_2 = \frac{v_2}{\|v_2\|}, \quad q_3 = \frac{v_3}{\|v_3\|}$$

Egs:

(1) Construct an orthonormal basis of R^2 by applying Gram Schmidt Process to set S.

→ $S = \{(3,1), (4,2)\}$

Let $u_1 = (3,1)$ — (A) and $u_2 = (4,2)$ — (B) Let $v_1 = u_1 = (3,1)$ — (C)

$v_2 = u_2 - \text{proj}\, u_2$

$= u_2 - \dfrac{(u_2, v_1)}{\|v_1\|^2} v_1$ — (1)

First find (u_2, v_1) and $\|v_1\|^2$.

$\therefore (u_2, v_1) = (4,2) \cdot (3,1) = 12 + 2 = 14$ — (D)

$\|v_1\|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = \left(\sqrt{3^2 + 1^2}\right)^2 = 10$ — (E)

Using eqn A, B, C, D and E in (1),

$\therefore v_2 = (4,2) - \dfrac{\cancel{14}^{7} \times (3,1)}{\cancel{10}\,5}$

$= \left(4 - \dfrac{7 \times 3}{5}, 2 - \dfrac{7}{5}\right) = \left(\dfrac{-1}{5}, \dfrac{3}{5}\right)$

Hence, $v_1 = (3,1)$ and $v_2 = (-1/5, 3/5)$ form an orthogonal basis for R^2. Now the norms of these vectors are:

$\|v_1\| = \sqrt{10}$

$\|v_2\| = \sqrt{1 + 9/25} = \sqrt{2/5}$

Hence, the orthonormal basis are:

$q_1 = \dfrac{v_1}{\|v_1\|} = \left(\dfrac{3}{\sqrt{10}}, \dfrac{1}{\sqrt{10}}\right)$

$q_2 = \dfrac{v_2}{\|v_2\|} = \left(\dfrac{-1}{5}\sqrt{\dfrac{5}{2}}, \dfrac{3}{5}\sqrt{\dfrac{5}{2}}\right) = \left(\dfrac{-1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\right)$

(2) Construct an orthonormal basis for subspace of R^3 by applying Gram Schmidts Process.

→ $S = \{(1,2,0), (0,3,1)\}$

Let $u_1 = (1,2,0)$ and $u_2 = (0,3,1)$

Let $v_1 = u_1 = (1,2,0)$.

$v_2 = u_2 - \text{proj}\, u_2$

$(u_2, v_1) = (0,3,1) \cdot (1,2,0) = 0 + 6 + 0 = 6$

$\|v_1\|^2 = \left(\sqrt{1^2 + 2^2 + 0^2}\right)^2 = 5$

$\therefore v_2 = (0,3,1) - \dfrac{6 \times (1,2,0)}{5}$

$= \left(\dfrac{-6}{5}, \dfrac{3}{5}, 1\right)$

$\|v_2\| = \sqrt{\dfrac{36+9}{25} + 1} = \sqrt{\dfrac{14}{5}}$

$q_1 = \dfrac{v_1}{\|v_1\|} = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}, 0\right)$

$q_2 = \dfrac{v_2}{\|v_2\|} = \left(\dfrac{-6}{\sqrt{70}}, \dfrac{3}{\sqrt{70}}, \sqrt{\dfrac{14}{5}}\right)$

Euclidean Inner Product:

If $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2 \ldots v_n)$ ~~then the~~ are any 2 vectors in R^n then the Euclidean Inner Product of u, v is denoted by $u.v$ and is defined as

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + \ldots + u_n v_n$

~~Let R^3 have the~~

Egs:

(1) Let R^3 have the Euclidean Inner Product. Use the Gram ~~Smid~~ Schmidt Process to transform the basis (u_1, u_2, u_3) into orthonormal basis, where $u_1 = (1,0,0)$, $u_2 = (3,7,-2)$ and $u_3 = (0,4,1)$.

→ Let $v_1 = u_1 = (1,0,0)$

$v_2 = u_2 - \text{proj}\, u_2$

$(u_2, v_1) = (3,7,-2) \cdot (1,0,0) = 3$

$\|v_1\|^2 = 1$

$\therefore v_2 = (3,7,-2) - 3(1,0,0) = (0,7,-2)$

$\|v_2\|^2 = 7^2 + (-2)^2 = 49 + 4 = 53$

$v_3 = u_3 - \dfrac{(u_3, v_1) \cdot v_1}{\|v_1\|^2} - \dfrac{(u_3, v_2)}{\|v_2\|^2} \cdot v_2$

$(u_3, v_1) = 0$

$(u_3, v_2) = (0,4,1)\cdot(0,7,-2) = 28 - 2 = 26$

$\therefore v_3 = (0,4,1) - 0 - \dfrac{26(0,7,-2)}{53}$

$\therefore v_3 = \left(0,\ 4 - \dfrac{182}{53},\ 1 + \dfrac{52}{53}\right) = \left(0,\ \dfrac{30}{53},\ \dfrac{105}{53}\right)$

$\|v_3\| = \sqrt{\dfrac{900 + 11025}{2809}} = \sqrt{\dfrac{11925}{2809}} = \dfrac{\sqrt{11925}}{53} = \dfrac{15}{\sqrt{53}}$

$q_1 = (1,0,0)$

$q_2 = \left(0, \dfrac{7}{\sqrt{53}}, \dfrac{-2}{\sqrt{53}}\right)$

$q_3 = \left(0, \dfrac{2}{\sqrt{53}}, \dfrac{7}{\sqrt{53}}\right)$

(e) Let the vector space P^2 have the inner product defined by $(p,q) = \int_{-1}^{1} p(x)q(x)\,dx$. Use Gram Schmidts Process to transform the standard basis ~~as equal to~~ $S = \{1, x, x^2\}$ into an orthonormal basis.

→ Given that

$u_1 = 1,\ u_2 = x,\ u_3 = x^2$

Let $v_1 = u_1 = 1$

$\therefore v_2 = u_2 - \text{proj}_{u_2}$

$(u_2, v_1) = \int_{-1}^{1} x(1)\,dx = 0$ ($\because$ odd function)

$\|v_1\|^2 = \int_{-1}^{1} [v_1(x)]^2\,dx$

$= \int_{-1}^{1} 1^2\,dx = 2\int_0^1 dx = 2$ ($\because$ even function)

$\therefore v_2 = x - 0 = x$

for v_3,

$(u_3, v_1) = \int_{-1}^{1} x^2(1)\,dx = 2\int_0^1 x^2\,dx = 2/3$

$(u_3, v_2) = \int_{-1}^{1} x^2(x)\,dx = 0$

$\therefore v_3 = x^2 - \dfrac{(2/3)\cdot 1}{2} - 0 = x^2 - \dfrac{1}{3}$

$\|v_2\| = \int_{-1}^{1} v_2(x)\,dx = \int_{-1}^{1} x\,dx = 0$

$\|v_3\| = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right) dx$

★ ROW AND COLUMN SPACE:

Let A be a $m \times n$ matrix. The subspace of R^n spanned by the row vector of A is called as a row space of A. The subspace of R^m spanned by the column vector of A is called as column space of A. ~~Find the basis for the row~~

Eg:

(1) Find the basis for the row space of matrix A

$$\rightarrow A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

Reduce the given matrix to row echelon form.

By $R_4 \to R_4 - R_1$, $R_5 \to R_5 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

By $R_4 \to R_4 + 2R_2$, $R_5 \to R_5 + R_2$

$$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

By $R_4 \to R_4 - R_3$, $R_5 \to R_5 + R_3$

$$= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $W_1 = [1, 1, 4, 1, 2]$

$W_2 = [0, 1, 2, 1, 1]$

$W_3 = [0, 0, 0, 1, 2]$

form a basis for the row space of the given matrix A.

(2) Find the basis for the column space of the following matrix.

$$\rightarrow A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$$

Reduce the given matrix to Row Echelon form

By $R_3 - R_1$, $R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow[R_4 - 3R_2]{R_3 - R_2} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is very clear that column 1 and column 2 are the pivot columns (according to unit matrix). Thus, 1st and 2nd column of matrix A form a basis for the column space.

$$\therefore W_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

(3) Find the basis for column space of given matrix.

$$\rightarrow A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 7 & 2 \\ -1 & -4 & -1 & 3 \\ 3 & 2 & -7 & -1 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 - 3R_1}]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & -4 & -4 & -13 \end{bmatrix}$$

So after solving completely and reducing to Row Echelon form, we get:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So here, colum 1, 2 and 4 are pivot ~~ster~~ columns. Pivot ~~et~~ column 1 shd have '1' ~~at 1~~ in 1st row, 2nd pivot column shd have '1' in 2nd row and 3rd pivot column shd have '1' in 3rd row. Thus, 4th column is pivot column.

$$W_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

form the basis for column space of A.

<u>Note</u>: dim (rowspace (A)) = dim (column space (A)) = Rank A.

★ LEAST SQUARE METHOD:

Let A be a mxn matrix and 'b' be a vector in R^n. To find least square solution of $AX = b$:

(1) Compute the matrix A^TA and A^Tb.

(2) Form the augmented matrix of $[A^TA \mid A^Tb]$

(3) Reduce the matrix $[A^TA \mid A^Tb]$ into Reduced Row Echelon form.

(4) This eqn is always consistent and any soln $\hat{x}$ is a least square soln.

<u>Egs:</u>

(1) Find the least square soln of $AX = b$ where:

$\longrightarrow$ $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Augmented matrix $= \left[\begin{array}{cc|c} 6 & 1 & 9 \\ 1 & 6 & 4 \end{array}\right]$

$R_1 \leftrightarrow R_2 = \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 6 & 1 & 9 \end{array}\right]$

$R_2 - 6R_1 = \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -35 & -15 \end{array}\right]$

$R_2 \to \frac{R_2}{-35} = \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & 3/7 \end{array}\right]$

$R_1 - 6R_2 = \left[\begin{array}{cc|c} 1 & 0 & 10/7 \\ 0 & 1 & 3/7 \end{array}\right]$ which is RREF.

$\hat{x} = \begin{pmatrix} 10/7 \\ 3/7 \end{pmatrix}$ Unique

$\therefore$ The least square soln is $(10/7, 3/7)$

(2) Find the least square soln of matrix Ax.

$\longrightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 5 & -7 \\ -3 & -7 & 11 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

Augmented matrix $= \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 3 & 5 & -7 & 0 \\ -3 & -7 & 11 & 6 \end{array}\right]$

Converting into reduced row echelon form.

$R_1 \to R_1/3$
$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$

$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 2 & -4 & -6 \\ 0 & -4 & 8 & 12 \end{array}\right] \xrightarrow[\text{row transformations}]{\text{after}}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving the system:

$x_1 + x_3 = 5$

$x_2 - 2x_3 = -3$

Assume $x_3 = 0$ OR $x_3 = 1$

$x_2 = -3$ OR $x_2 = -1$

$x_1 = 5$ OR $x_1 = 4$

} Any random value

$\therefore \hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ Infinite least square soln.

HW: Find least square solution of $Ax = b$ where A:

$\rightarrow A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$