zero space.

Vector Space:-

Let V be a non empty set of elements for vectors K, m are scalars. If set V satisfies the following conditions axioms then it is called vector space.

Axioms of Vector Space:-

A] Closure Axiom:-

1] C_1:- If u & v are the elements of V then u+v are also in V
∴ V is closed under addition.

2] C_2: If K is any scalar then K.u is in V.
∴ V is closed under scalar multiplicate.

B] Addition Axiom:-

① A_1: $u+v = v+u$ Commutative property

② A_2: $(u+v)+w = u+(v+w)$ (Associativity)

③ A_3: There is an element called 0 in V such that $u+0 = u$ for all u in V (Existance of additive Identity) → zero

④ A_4: There is an element $-u$ in V called negative of u corresponding to every u in V such that $u+(-u)=0$ (Existance of additive inverse)

C] Scalar multiplicate Axiom:-

1] M_1: $K(u+v) = Ku+Kv$ (distributivity of scalar vectors)

2] M_2: $(K+l)u = Ku+lu$ (" of vectors)

3] M_3: $(Kl)u = K(lw)$

4] M_4:- There is an element 1 in V called

unity such that $u \cdot 1 = u$. That property is called as multiplicative Identity.

Q] If V is a vector space then S.T
i] Additive identity 0 is unique
ii] " inverse of vector u is unique

→] If possible let there be 2 distinct additive identities 0 & 0' —①
then $u+0=u$ —① & $u+0'=u$ —②
put $u=0'$ in ① $\therefore 0'+0=0'$ —③
Also put $u=0$ in ② $\therefore 0+0'=0$
$\therefore$ from ③ & ④ $0'=0$
i.e Additive identity 0 & 0' are unique

2] If possible let there be v_1 & v_2 be 2 additive inverses of v then by the defination of inverse.
$\therefore u+v_1=0$ —⑤
& $u+v_2=0$ —⑥
then add v_2 in ⑤ $\Rightarrow (u+v_1)+v_2 = v_2+0$
$\therefore (v_2+v)+v_1 = v_2$
$\therefore v_1 = u_2$ from ⑥

This shows that additive inverse of vector u is unique.

③] Check whether the set of all pairs of Real numbers of the form $(1, x)$ with the operates $(1,y)+(1,y'') = (1, y+y'')$
& $k(1,y) = (1, ky)$ is a vector space

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→ Here we have to show that V is a vector space
given operates
$(1,y)+(1,y') = (1, y+y')$ — ①
& $K(1,y) = (1, Ky)$ — ②

A] Closure Axiom
(1) if $u=(1,x)$ & $v=(1,y)$
then $u+v = (1,x)+(1,y') = (1+1, x+y)$
(∵ ①)

∴ V is closed under addition

(2) If K is any scalar then
$K(u) = K(1,x) = (1, Kx)$ — from ②
∴ V is closed under scalar multiplicate

B] Addition Axiom
1] A_1: $u+v = (1,x)+(1,y) = (1, x+y)$
$= (1, y+x)$ — from ①
$= (1,y)+(1,x)$
∴ $u+v = v+u$ (Commutativity)

2] A_2: $(u+v)+w = ((1,x)+(1,y))+(1,z)$
$= (1, x+y)+(1,z)$ — from ①
$= (1, x+y+z)$
~~$= (1, x+z+y)$~~
$= (1,x)+(1,y+z)$
$= (1,x)+((1,y)+(1,z))$
$= u+(v+w)$ (Associativity)

3] A_3: There is an element u in V such that
$u+0 = (1,x)+(1,0) = (1, x+0) = u$ (E of A Id)

4] A4: There is an element of $-u$ in V such that
$u+(-u) = (1,x)+(1,-x) = (1,0) = 0$
(∈ of A In)

c] Scalar Multiplicate Axioms:-
1] M_1: $K(u+v) = K[(1,x)+(1,y)]$
$= K(1, x+y)$ ____ from ①
$= [1, K(x+y)]$
$= (1, Kx+Ky)$
$= (1,Kx)+(1,Ky)$
$= K(1,x)+K(1,y)$
$= Ku+Kv$

2] M_2: $(K+l)u = (K+l)(1,x)$
$= [1,(K+l)x]$
$= (1, Kx+lx)$
$= (1,Kx)+(1,lx)$
$= K(1,x)+l(1,x)$
$= Ku+lu$ M_3, M_4 HW

Here all axioms are satisfied ∴ V is a vector space

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Linear Dependence & Independence of vectors:-

Linear combinate of vector:-

A vector w is called as linear combinate of vectors, $v_1, v_2 \dots v_n$ if it can be expressed in the form

$w = v_1k_1 + v_2k_2 + \dots + v_nk_n$

where k_1, k_2 upto k_n are scalars not all zero

Linear Dependence of vectors:-

In the theory of vector space, a set of vectors is said to be linearly dependent if atleast one of the vector in the set can be defined as linear combinate of other.

Linear Independence of vectors:-

In the theory of vector space, a set of vectors is said to be linearly independent if none of the vectors in the set can be defined as linear combinate of others.

Q] Examine whether the vectors $v_1 = [3, 1, 1]$, $v_2 = [2, 0, -1]$, $v_3 = [4, 2, 1]$ are linearly dependent or independent.

→ Consider the matrix eq^n

$k_1v_1 + k_2v_2 + k_3v_3 = 0$

$k_1[3, 1, 1] + k_2[2, 0, -1] + k_3[4, 2, 1] = [0, 0, 0]$

$$3K_1 + 2K_2 + 4K_3 = 0$$
$$K_1 + 0K_2 + 2K_3 = 0$$
$$K_1 - K_2 + K_3 = 0$$

which can be written as

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$$

By $R_2 - R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = 0$$

$4K_3 = 0 \quad K_3 = 0$

$K_2 = 0 \quad K_1 = 0$

∴ All K's are zero ∴ the given vectors are linearly independent.

Q) Show that $P_1 = 1 - 2x$, $P_2 = 2 - 3x^2$, $P_3 = 3 - 5x + x^2$ are linearly dependent.

-> we take $P_1 + P_2 - P_3$

$= [1 - 2x + (2 - 3x + x^2) - (3 - 5x + x^2)]$

$= [1 - 2 - 3, -2x - 3x + 5x, x^2 - x^2]$

$= (0, 0, 0) = 0$

$\therefore P_1 + P_2 - P_3 = 0$

$\therefore P_1 + P_2 = P_3$

$\therefore P_3$ is expressed as $P_3 = P_1 + P_2$

i.e P_3, P_2, P_1 are linearly dependence.

Q] Use suitable Trigo identities to determine whether the following set of functions are linearly dependent or not & find the relate between them.

-> i] $8, 4\sin^2 x, 2\cos^2 x$

ii] $\cos 2x, \cos^2 x, \sin^2 x$

i]

-> let $f_1 = 8, f_2 = 4\sin^2 x, f_3 = 2\cos^2 x$

$\therefore f_1 - 2f_2 - 4f_3$

$= 8 - 8\sin^2 x - 8\cos^2 x$

$= 8 - 8(\sin^2 x + \cos^2 x)$

$= 8 - 8 = 0$

$\therefore f_1 = 2f_2 + 4f_3$

ii] $f_1 = \cos 2x, f_2 = \cos^2 x, f_3 = \sin^2 x$

$f_1 - f_2 - f_3 = -2\sin^2 x$

linearly independent

$f_1 - f_2 + f_3 = 0$

linearly dependent

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∴ the given vectors are linearly dependent.

Q] Verify whether (1, 2) & (2, 3) are linearly dependent or not when placed with initial points at origin by analytic & geometric method.

Analytic method

→ Let the vectors be
$v_1 = (1, 2)$, $v_2 = (2, 3)$
Consider the matrix eq as

$$k_1[1, 2] + k_2[2, 3] = [0, 0]$$
$$\therefore k_1 + 2k_2 = 0$$
$$\& \; 2k_1 + 3k_2 = 0$$

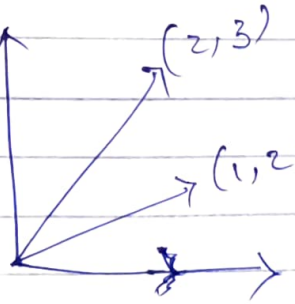
$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore k_2 = 0 \;\&\; k_1 = 0$$

∴ the points linearly independent

Now geometric method
when we place the vectors with initial point on origin there is no line containing both the vectors
∴ linearly independent.



Q] Verify whether (1, 2) & (3, 6) are linearly independent or not when placed with initial point.

Subspace:-

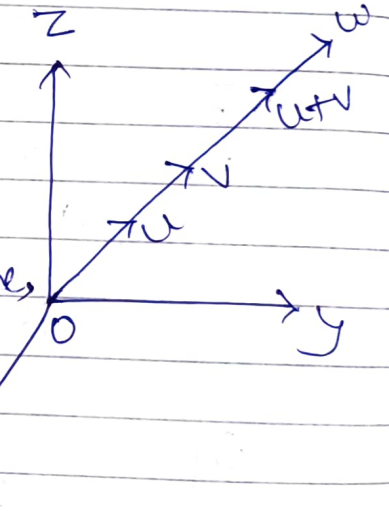
A subset W of vector space V is called a subspace of V if W itself is a vectors space under addition & scalar multiplicate defined on V.

Theorem:- If W is a non-empty ~~vector~~ subset ~~space~~ of a vector then W is a subspace of V

(1) for all u & v in W, u+v is also in W. (vectors)

(2) for any scalar k & a vector u of W then ku is in W.

(3) ST a line through the origin in R^3 (3D space) is a subspace of R^3



→ Consider a line through the origin in R^3. If u & v are any 2 vectors w.r.t. origin as the initial point on this line, u+v is also a vector on this line thus u+v is closed under addition.

Also if u is a vector on the line & k is any scalar then ku is also a vector on this line, thus W is closed under scalar multiplication.

∴ Both the conditions are satisfied.

∴ line through the origin is a subspace of R^3

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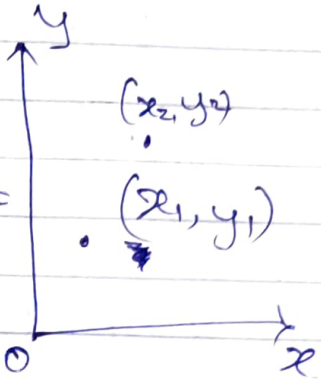
5] W is a set of all points of (x, y) in R^2 such that ~~x~~ ~~not~~ $x \geq 0$ & $y \geq 0$ then W is not a subspace of R^2

→ If (x_1, y_1) & (x_2, y_2) are any 2 points of W in the first quadrant then $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ is also a point of W in the first quadrant.



But $K(x, y) \neq (Kx, Ky)$ for all K for instance if $K = -2$

$\therefore -2(x, y) = (-2x, -2y)$ $(x \& y \geq 0)$

↓ not in first quadrant

$\therefore$ W is not a subspace of R^2

6] Let W be the set of all functions of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where n is a non-negative integer & $a_0, a_1, \ldots a_n$ are real nos. then W is a subspace of all real valued function, of V.

→ given,

$P(x) = a_0 + a_1 x + \ldots + a_n x^n$

& $q(x) = b_0 + b_1 x + \ldots + b_n x^n$ (assume)

then $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$

$\therefore$ W is closed under addition.

then $kP(x) = k(a_0 + a_1 x^1 + \ldots + a_n x^n)$

$= ka_0 + ka_1 x^n + \ldots + ka_n x^n$

Here $k \cdot p(x)$ is of the above form
$\therefore k \cdot p(x)$ is in W
Here both conditions are specified thus
W is a subspace of real valued functn.

Q] Show that any plane through the origin is a subspace of R^3.

Space spanned by a vector:-

Let W be a subspace of V consisting of all linear combinations of $v_1, v_2 \ldots v_n$ of S then W is called the space spanned by S. We also say that the vectors $v_1, v_2 \ldots v_n$ spans W.
To spans means to furnish with something that extends or streches over. Loosely speaking to span means to create or generate this is denoted by W = space of S (Span(S)) or $W = span(v_1, v_2 \ldots, v_n)$.
Span of set S is also denoted by L(S)

Q] ST the vectors $v_1 = (1, 0, 1)$, $v_2 = (2, 1, 4)$, $v_3 = (1, 1, 3)$ do not span the vector space R^3.

→ If v_1, v_2 & v_3 span the vector space R^3 then any vector in R^3 should be expressible in terms of v_1, v_2 & v_3
Let $w = (w_1, w_2, w_3)$ be a vector in R^3 then we try to find the scalars k_1, k_2 & k_3 such that
$w = k_1 v_1 + v_2 k_2 + k_3 v_3$ ——— ①

$\therefore (w_1, w_2, w_3) = k_1[1,0,1] + k_2[2,1,4] + k_3[1,1,3]$

$\therefore (w_1, w_2, w_3) = [(k_1 + 2k_2 + k_3), (k_2 + k_3), (k_1 + 4k_2 + 3k_3)]$

$\therefore w_1 = k_1 + 2k_2 + k_3$

$w_2 = k_2 + k_3$

$w_3 = k_1 + 4k_2 + 3k_3$

Note:- The system is consistent if

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} \neq 0$$

$\therefore 1(-1) - 2(-1) + 1(-1) = 0$

$\therefore$ The system is inconsistent

$\therefore$ There is no k_1, k_2 or k_3 such that the eqⁿ (1) can be satisfied

$\therefore$ No vector in R^3 can be expressed in terms of V_1, V_2 & V_3

$\therefore V_1$, V_2 & V_3 do not span the vector space R^3.

5) Determine whether the following vectors space P_2 (P_2 means vector space consisting of all polynomial of 2nd order. Where $P_1 = 1 - x + 2x^2$, $P_2 = 5 - x + 4x^2$ & $P_3 = -2 - 2x + 2x^2$)

Basis of a vector:-

Let V be a vector space. A minimal set of vectors in V that spans V is called the basis for V.
Equivalently a basis of V is a set of vectors, that are linearly independent & that spans V.
As a result to check, if a set of vectors form the basis of vector then one needs to check that it is L.I & it spans the vector space. If atleast 1 of the conditions fail to hold then it is not a basis.

Q] Is the set $S = \{(1,1), (1,-1)\}$ a basis of R^2?

→ Consider the set S as eqns

$$C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$\Rightarrow C_1 + C_2 = x$ — ① & $C_1 - C_2 = y$ — ②

This can be written as

$$\left[\begin{array}{cc|c}1 & 1 & x\\1 & -1 & y\end{array}\right].$$

By $R_2 \to R_2 - R_1$ $\left[\begin{array}{cc|c}1 & 1 & x\\0 & -2 & y-x\end{array}\right]$

Determinant $= -2 - 0 = -2 \neq 0$

∴ the system is consistent for every x & y

∴ S spans for R^2.

Now checking for L.I

∴ from the eqn we can write

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore c_1+c_2=0$ — (3) & $c_1-c_2=0$ — (4)

$$\left[\begin{array}{cc|c}1 & 1 & 0\\ 1 & -1 & 0\end{array}\right]$$

By $R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & -2 & 0\end{array}\right]$$

$\therefore c_1 = c_2 = 0$ Trivial sol^n

$\therefore S$ is L.I

$\therefore$ Set S is basis for R^2

6] find the basis for the vector space V. Spanned by the vector $w_1=(1,1,0)$, $w_2=(0,1,1)$, $w_3=(2,3,1)$, & $w_4=(1,1,1)$

→ We need to find the relation of the form $r_1w_1+r_2w_2+r_3w_3+r_4w_4=0$.

Which can be written in matrix form as

$$\begin{pmatrix}1 & 0 & 2 & 1\\ 1 & 1 & 3 & 1\\ 0 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}r_1\\ r_2\\ r_3\\ r_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

To solve the system of linear eqns for r_1, r_2, r_3, r_4 so we apply row reduction

$R_2 - R_1$

$$\begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore r_1 + 2r_3 + r_4 = 0$ — (1)

& $r_2 + r_3 = 0$ — (2)

$\Rightarrow r_2 = -r_3$

& $r_4 = 0$ — (3)

$\therefore$ eqn (1) becomes $r_1 + 2r_3 = 0$

$\therefore r_1 = -2r_3$

$\therefore (r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$

put particular soln $t = -1$

$\therefore (r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$

$\therefore$ We obtain $2w_1 + w_2 - w_3 = 0$

from particular soln

Hence any of the vectors w_1, w_2, w_3 can be dropped

Let us drop w_3

$\therefore V = \text{span}(w_1, w_2, w_4)$

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore w_1, w_2$ & w_4 are span & L.I

6] Show that the vectors $(1, 2, 1), (2, 1, 0), (1, -1, 2)$ form a basis for R^3.

Q] Construct an orthonormal basis of R^2 by applying GS process to the set $S=\{(3,1),(4,2)\}$

→ Let $u_1=(3,1)$ — (A) & $u_2=(4,2)$ — (B)

Step 1: Let $v_1=u_1=(3,1)$ — (C)

" 2: $v_2=u_2-\text{proj}_{u_2}$

$$=(4,2)-\frac{(u_2,v_1)}{\|v_1\|^2}\cdot v_1 \quad \text{—— (1)}$$

first find (u_2,v_1) & $\|v_1\|^2$

$\therefore (u_2,v_1)=(4,2)\cdot(3,1)$

$=12+2=14$ —— (D)

And $\|v_1\|^2=\left(\sqrt{3^2+1^2}\right)^2$

$=10$ —— (E)

Using eqn (A) to (E) in (1)

$$v_2=(4,2)-\frac{14}{10}\cdot(3,1)$$

$$=\left(4-\frac{7\cdot3}{5},\ 2-\frac{7}{5}\cdot1\right)$$

$$=\left(\frac{-1}{5},\frac{3}{5}\right)$$

Hence $v_1=(3,1)$ & $v_2=\left(\frac{-1}{5},\frac{3}{5}\right)$ form an orthogonal basis for R^2

Now the norms of these vectors are

$$\|v_1\|=\sqrt{3^2+1^2}=\sqrt{10}$$

$$\&\ \|v_2\|=\sqrt{(-1/5)^2+(3/5)^2}=\sqrt{\frac{2}{5}}$$

Hence the orthogonal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$\& \; q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-1}{5}\sqrt{\frac{5}{2}}, \frac{3}{5}\sqrt{\frac{5}{2}}\right)$$

$$= \left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

6) Construct an orthonormal basis for the subspace of R^3 by applying GS process for the set $S = \{(1, 2, 0), (0, 3, 1)\}$

→ Let $u_1 = (1, 2, 0)$ — (A)

$u_2 = (0, 3, 1)$ — (B)

Let $v_1 = u_2 = (1, 2, 0)$ — (C)

$v_2 = u_2 - \text{proj}_{u_2}$

$$\therefore v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} \cdot v_1 \quad \text{— (1)}$$

$(u_2, v_1) = (0, 3, 1) \cdot (1, 2, 0) = 0 + 6 + 0 = 6$ — (D)

$\|v_1\|^2 = 5$ — (E)

Using eqs (A) to (E) in (1)

$$\therefore v_2 = (0, 3, 1) - \frac{6}{5}(1, 2, 0)$$

$$= \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

$\|v_1\| = \sqrt{5}$

$$\|v_2\| = \sqrt{\left(\frac{-6}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + 1} = \sqrt{\frac{36 + 9 + 25}{25}} = \sqrt{\frac{14}{5}}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

~~$$q_2 = \frac{v_2}{\|v_2\|} = \left(0, \frac{8\sqrt{5}}{\sqrt{14}}, \sqrt{\frac{5}{14}}\right)$$~~

$$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \sqrt{\frac{5}{14}}\right)$$

Euclidean Inner Product:-

If $u = u_1, u_2, u_3 \ldots u_n$
& $v = v_1, v_2, v_3 \ldots v_n$
are any 2 vectors in R^n then the Euclidean Inner product of u, v is denoted by $u \cdot v$ & is defined by
$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

Q) Let R^3 have the Euclidean Inner product Use GS process to transform the basis u_1, u_2, u_3 into orthonormal basis where $u_1 = (1, 0, 0)$, $u_2 = (3, 7, -2)$, $u_3 = (0, 4, 1)$

→ Given $u_1 = (1, 0, 0)$ — (A)
$u_2 = (3, 7, -2)$ — (B)
$u_3 = (0, 4, 1)$ — (C)

Let $v_1 = u_1 = (1, 0, 0)$ — (D)

$$v_2 = u_2 - \text{proj}_{u_2} = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1 \quad \text{— (1)}$$

$(u_2, v_1) = 3 \quad \|v_1\|^2 = 1$

$v_2 = (3, 7, -2) - 3(1, 0, 0)$
$= (0, 7, -2)$

$$v_3 = u_3 - \text{proj}_{u_3}$$
$$= u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$$

$(u_3, v_1) = 0 \quad \|v_2\|^2 = 53 \quad (u_3, v_2) = 28 - 2 = 26$

$$v_3 = (0, 4, 1) - 0 - \frac{26}{53}(0, 7, -2)$$

$$= (0, 30/53, 105/53)$$

$$q_1 = \frac{v_1}{\|v_1\|} = (1, 0, 0) \qquad \|v_3\| = \frac{15}{\sqrt{53}}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(0, 7, -2)}{\sqrt{53}} = \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|^2} = \left(0, \frac{30}{53} \times \frac{\sqrt{53}}{15}, \frac{105}{53} \times \frac{\sqrt{53}}{15}\right)$$
$$= \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$$

Q) Let P^2 have the inner product defined by $(P, q) = \int_{-1}^{1} P(x) q(x) dx$

Use GS process to transform the basis $S = \{1, x, x^2\}$ into orthonormal basis.

* $u_1 = 1, u_2 = x, u_3 = x^2$

Let $v_1 = u_1$

$v_2 = u_2 - \text{proj} u_2$

$= u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$

$(u_2, v_1) = \int_{-1}^{1} x \cdot 1 \, dx = \left(\frac{x^2}{2}\right)_{-1}^{1}$

$= 0$

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$$\& \|v_1\|^2 = \int_{-1}^{1} [v_1(x)]^2 dx$$

$$= \int_{-1}^{1} 1^2 dx = 2$$

$$v_2 = x - \frac{0}{2} \cdot (2) = x$$

$$v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} \cdot v_2$$

$$(u_3, v_1) = \int_{-1}^{1} x^2 \cdot 2 \, dx = 2\left(\frac{x^3}{3}\right)_{-1}^{1}$$

$$=$$

Mod 4

Applicate of Matrices to Coding & Decoding

Cryptography:- Study of coding & Decoding a message.

Plain Text:- Message to be sent. Codes are Ciphers & code message are called as Cipher Codes

Encoding or Enciphering:-
The process of converting plain text by coding is called as encoding & the reverse process of getting plain text from cipher text is called as decoding deciphering.
The selected matrix is called as encoding matrix & its reverse is called as decoding matrix. Encoding matrix is called Key matrix.

The method is as follows:-
1] Replace words/letters by numbers.
2] To encode the message we premultiply the column matrix to the encoding matrix
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
3] To decode the message, In this step we
4] premultiply the column matrix by decoding matrix $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
4] To replace numbers by words/letter.

Note:- A ⇒ 1, B ⇒ 2 . . . . , Z ⇒ 26, space(*) ⇒ 27

Q] Using a suitable 2×2 matrix encode & decode a message WE CnO

→ 1] Replace words by numbers
Then we get the transformate
WE GO
23 5 7 15
2] Then we write this in a sequence of 2×1 matrix

$$\begin{bmatrix} 23 \\ 5 \end{bmatrix} \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

2] To encode the message multiply with the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
Premultiply each of the above column matrix by the encoding matrix
Thus we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 23 & 7 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 28 & 22 \\ 5 & 15 \end{bmatrix}$$

The columns of this matrix give the encoded message.
The above message is transmitted in the following linear form
28 5 22 15

3] To decode the message
The received message in column matrix as $\begin{bmatrix} 28 & 22 \\ 5 & 15 \end{bmatrix}$
Premultiply to the above matrix by the decoding matrix $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

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$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 28 & 22 \\ 5 & 15 \end{bmatrix} = \begin{bmatrix} 23 & 7 \\ 5 & 15 \end{bmatrix}$$

4] Replace numbers by words
we can write the matrix in linear form
as 23 5 7 15
W E G O

5] The message NOW*STUDY

⇓

NOW*STUDY*

14 15 23 27 19 20 21 4 25 27

→ We write this in a sequence of 2×1 matrix

$$\begin{bmatrix} 14 \\ 15 \end{bmatrix} \begin{bmatrix} 23 \\ 27 \end{bmatrix} \begin{bmatrix} 19 \\ 20 \end{bmatrix} \begin{bmatrix} 21 \\ 4 \end{bmatrix} \begin{bmatrix} 25 \\ 27 \end{bmatrix}$$

Premultiply by encoding matrix

$$\underset{2\times2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 14 & 23 & 19 & 21 & 25 \\ 15 & 27 & 20 & 4 & 27 \end{bmatrix}_{2\times5}$$

$$= \begin{bmatrix} 29 & 50 & 39 & 25 & 52 \\ 15 & 27 & 20 & 4 & 27 \end{bmatrix}_{2\times5}$$

Premultiply by decoding matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 29 & 50 & 39 & 25 & 52 \\ 15 & 27 & 20 & 4 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 23 & 19 & 21 & 25 \\ 15 & 27 & 20 & 4 & 27 \end{bmatrix}$$

6] Replace numbers by words
we can write the matrix in linear form

as 14 15 23 27 19 20 21 4 25 27
N O W * S T U D Y

Q] MOVE

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Let A be a m×n matrix the subspace of R^n stand by the row vector of A is called as a row space of A.

The subspace of R^m stand by the column vector of A is called a column space of A

Q) Find the basis for row space of matrix $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$

→ Reduced the matrix in to row echelon form

By $R_4 \to R_4 - R_1$, $R_5 \to R_5 - 2R_1$ $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & 3 \end{bmatrix}$

By $R_4 \to R_4 + 2R_2$, $R_5 \to R_5 + R_2$ $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

By $R_4 \to R_4 - R_3$, $R_5 \to R_5 + R_3$ $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

knowledge and resources and empowers students to pursue their passion and acheive their goals.

Thus $w_1 = [1, 1, 4, 1, 2]$
$w_2 = [0, 1, 2, 1, 1]$
$w_3 = [0, 0, 0, 1, 2]$

6) Find the basis for the column space of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$$

→ Reduce the given matrix to row

By $R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

→ Reduced the matrix form

By $R_4 \to R_4 - R_1$
$R_5 \to R_5 - 2R_1$

By $R_4 \to R_4 + 2R_2$
$R_5 \to R_5 + R_2$

By $R_4 \to R_4 - R_3$
$R_5 \to R_5 + R_3$

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$R_3 \to R_3 - R_2$
$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns (1's necessary for pivot columns)

~~Let the column space be L_2~~

L_2 = form a basis of column space

$$W_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

6) Column space

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 7 & 2 \\ -1 & -4 & -1 & 3 \\ 3 & 2 & -7 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + R_1$
$R_4 \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & -4 & -4 & -13 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$
$R_3 \to R_3/-2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & 1 & 1 & -7/2 \\ 0 & 0 & 0 & -27 \end{bmatrix}$$

R_4

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$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that 1st, 2nd & 4th columns are the pivot columns

$$\therefore w_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} \quad w_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

Q Find the basis for the row space of

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$$

Note: dim(row space(A)) = dim(column space(A)) = rank(A)

Least square method/Sol:-

Let A be a m×n matrix & b be a vector in $\mathbb{R}^n$. To find the LSS of $Ax = b$

1] Compute the matrix
$A^T \cdot A$ & $A^T \cdot b$

2] form the augmented matrix
$A^T \cdot A \mid A^T b$

3] Reduce the matrix $A^T A \mid A^T b$ into ref

4] The eqn is always consistent &

any sole $\hat{x}$ (x cap/vector x) is a least square sole

Q] Find LSS of $AX=b$ where $A=\begin{bmatrix}2 & -1\\ 1 & 2\\ 1 & 1\end{bmatrix}$ & $b=\begin{bmatrix}2\\1\\4\end{bmatrix}$

→ We have $A^T \cdot A=\begin{bmatrix}6 & 1\\ 1 & 6\end{bmatrix}$

Now
We form an augmented matrix & row reduced

$\therefore A^T \cdot A=\left[\begin{array}{cc|c}6 & 1 & 9\\ 1 & 6 & 4\end{array}\right]$

$R_1 \leftrightarrow R_2 \left[\begin{array}{cc|c}1 & 6 & 4\\ 6 & 1 & 9\end{array}\right]$

$R_2 \to R_2 - 6R_1 \left[\begin{array}{cc|c}1 & 6 & 4\\ 0 & -35 & -15\end{array}\right]$

$R_2 \to \frac{R_2}{-35} \left[\begin{array}{cc|c}1 & 6 & 4\\ 0 & 1 & 3/7\end{array}\right]$

By $R_1 \to R_1 - 6R_2 \left[\begin{array}{cc|c}1 & 0 & 10/7\\ 0 & 1 & 3/7\end{array}\right]$

$\therefore \hat{x}=\begin{pmatrix}10/7\\ 3/7\end{pmatrix}$ ∴ Unique LSS

Unique

~~∴ the LSS is $\hat{x}=10/$~~

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Q) Find LSS of matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{bmatrix}$ & $b = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

→ $A^T \cdot A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 5 & -7 \\ -3 & -7 & 11 \end{bmatrix}$

& $A^T b = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

Augmented form

$$\left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 3 & 5 & -7 & 0 \\ -3 & -7 & 11 & 6 \end{array}\right]$$

By $R_3 \to R_3 + R_1$ & $R_2 \to R_2 - R_1$

$$= \left[\begin{array}{ccc|c} 3 & 3 & -3 & 6 \\ 0 & 2 & -10 & -6 \\ 0 & & & 12 \end{array}\right]$$

Q) Find LSS of $AX = B$ where $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Application of Inverse of a Matrix to Coding Theory :-

Modular ~~Metho~~ Maths : If n is positive integer and a and b are any 2 Integers then we say that

a is Equivalent to b modulo n which is written as :

$$a = b \pmod{n}$$

If and only If $a-b$ is Integer multiple of n.

Residue under modulo n : The set $Z_n = \{0, 1, 2 \cdots n-1\}$ is called the Residue under modulo n.

Residue is denoted by r.

Theorem : for any int a and modulo n, Let r = remainder of $\frac{|a|}{n}$

then residue r of modulo n is given by :

R → remainder

$$r = \begin{cases} R & \text{If } a \geq 0 \\ n-R & \text{If } a < 0 \text{ and } R \neq 0 \\ 0 & \text{If } a < 0 \text{ and } R = 0 \end{cases}$$

Eg: find residue of 87, -38, -26, under modulo 26.

Soln:- Dividing $|87| = 87$ by 27 yields

$R = 9$

Here $a > 0$

$\therefore \quad r = R = 9$

$\boxed{r = 9}$ Hence $\therefore 87 \equiv 9 \pmod{26}$

$$\begin{array}{r} 3 \\ 26 \overline{)\,87} \\ \underline{78} \\ 9 \end{array} \longrightarrow R$$

Dividing $|-38| = 38$ by 26 yields

$R = 12$

Here $a < 0 \quad R \neq 0$

$\therefore \quad r = 26 - 12 = 14 \quad \therefore -38 \equiv 14 \pmod{26}$

Dividing $|-26| = 26$ by 26 yields $R = 0$

Here $a < 0$ and $R = 0$

$\therefore \boxed{r = 0} \quad \therefore \quad -26 \equiv 0 \pmod{26}$

Multiplicative Inverse of modulo n :-

If a is the no. in Z_n then $\bar{a}^1$ in Z_n is called reciprocal or multiplicative inverse of module n if and only if

$$a\bar{a}^1 = \bar{a}^1 a = 1 \pmod{n}$$

Eg. find reciprocal of 9 modulo 26.

Sol^n: The number 9 has a reciprocal modulo 26 because they have no prime factor.
But we know that

$$a\bar{a}^1 = 1 \pmod{n}$$

$$\Rightarrow a\bar{a}^1 = 1 \pmod{26}$$

$$\Rightarrow a\bar{a}^1 - 1 = \text{multiples of } 26$$

$$a\bar{a}^1 - 1 = 26n$$

$$\boxed{\bar{a}^{-1} = \frac{26n+1}{a}} \quad \text{---} (*)$$

for $a = 9$ in $(*)$

$$\therefore \bar{a}^1 = \frac{26n+1}{9}$$

for $n = 0$ $\quad \bar{a}^1 = 1/9$ ✗

$n = 1$ $\quad \bar{a}^1 = 3$ ✓

Note

1) If $a = 1$ in (*) $\bar{a}^1 = \frac{26n+1}{1}$

for $n = 0$ $\bar{a}^1 = 1$

2) If $a = 3$ in (*) $\bar{a}^1 = \frac{26n+1}{3}$

for $n = 0$

$\bar{a}^1 = 1/3$ ✗

$n = 1 \Rightarrow \bar{a}^1 = 9$

3) If $a = 5$ in (*)

$a^{-1} = \frac{26n+1}{5}$

for $n = 0$ $\bar{a}^1 = 1/5$

$n = 1$ $\bar{a}^1 = 27/5$

$n = 4$ $\bar{a}^1 = 21$

Q) Given a Hill 2 cipher Key $A = \begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}$

Compute A^{-1} modulo 27 and hence decode the following message XNUEYVCRSQEJ.

Soln :- Step 1 : To find inverse of matrix A.

$A^{-1} = \frac{adj(A)}{|A|}$ — (1) — Given

$adj(A) = \begin{bmatrix} 6 & -1 \\ -2 & 1 \end{bmatrix}$ $|A| = 4$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 6 & -1 \\ -2 & 1 \end{bmatrix}$$

Step 2:- $a\bar{a}^{-1} = 1 \pmod{n}$

Here $a = 4$ and $n = 27$

$$\therefore \quad 4\bar{a}^{-1} = 1 \pmod{27}$$
$$4\bar{a}^{-1} - 1 = \text{multiples of } 27$$
$$4\bar{a}^{-1} - 1 = 27n$$
$$\bar{a}^{-1} = \frac{27n+1}{4}$$

for $n = 1 : \bar{a}^{-1} = 7$

$$\therefore \quad A^{-1} = 7\begin{bmatrix} 6 & -1 \\ -2 & 1 \end{bmatrix}$$
$$A^{-1} = \begin{bmatrix} 42 & -7 \\ -14 & 7 \end{bmatrix} \quad \text{---} (*)$$

Here Consider various multiples of 27
i.e. -54, -27, 0, 27, 54, 81, 108

So now 42 lies b/w 27 and 54
. The no Smaller than 42 i.e. 27
Hence $42 - 27 = 15$

Now -7 lies b/w 0 and -27
Hence $-7 - (-27) = 20$

Now -14 lies b/w 0 and -27
Hence $-14 - (-27) = 13$

Now 7 lies b/w 0 and 27
$7 - 0 = 7$

Therefore

$$A^{-1} = \begin{bmatrix} 17 & 20 \\ 13 & 7 \end{bmatrix} \text{ mod } 27 \quad (I)$$

Step 3 : Now Consider Encoded message

XNUFYVCRSQEJ

24 14 21 6 25 22 3 18 13 17 5 10

Therefore we get matrix C as

$$C = \begin{bmatrix} 24 & 21 & 25 & 3 & 13 & 5 \\ 14 & 6 & 22 & 18 & 17 & 10 \end{bmatrix}$$

Step 4 :- Decoding can be performed as :

$$B = A^{-1}C \pmod{27}$$

$$= \begin{bmatrix} 15 & 20 \\ 13 & 7 \end{bmatrix} \begin{bmatrix} 24 & 21 & 25 & 3 & 13 & 5 \\ 14 & 6 & 22 & 18 & 17 & 10 \end{bmatrix} \pmod{27}$$

$$= \begin{bmatrix} 640 & 435 & 815 & 405 & 625 & 275 \\ 410 & 315 & 473 & 165 & 366 & 135 \end{bmatrix} \pmod{27}$$

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 10 & 20 & 3 & 4 & 0 \end{bmatrix}$$

Step 5 :- Hence the decoded message is :

19 5 3 18 5 20 0 3 15 4 5 0

S E C R E T C O D E

By removing last space we get decoded message as :

"SECRET CODE"

(null space part)

Rank Nullity Theorem :-

It states that

$$\text{Rank}(A) + \text{Nullity}(A) = \text{no. of Columns of } A$$

Eg: find the nullity of matrix $A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix}$

Using Rank nullity theorem

Sol^n.

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \end{bmatrix}$$

Convert matrix to ref

$R_3 \to R_3 + R_1$

$R_2 \to R_2 - 3R_1$

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & -7 & -1 & 12 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here : rank(A) = 2 and no. of Columns = 4

Hence by Rank nullity theorem

$\therefore$ Nullity (A) = 2