Types of Matrices

1) Row matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

2) Column matrix

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

3) Scalar matrix

$$\begin{bmatrix} k\cdot 1 & k\cdot 2 & k\cdot 3 \\ k\cdot 4 & k\cdot 5 & k\cdot 6 \\ k\cdot 7 & k\cdot 8 & k\cdot 9 \end{bmatrix}$$

k being any scalar

4) Rectangular matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}_{2\times 3}$$

5) Upper triangular matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

6) Lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$$

7) Symmetric matrix

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

$a_{ij} = a_{ji} \quad \forall\ i, j$

8) Skew symmetric matrix

$$\begin{bmatrix} 0 & 4 & -5 \\ -4 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$$

9) Hermitian matrix

Complex	conjugate
$7i$	$-7i$
$5+i$	$5-i$

$a_{ij} = \overline{a_{ji}}$

Diagonal elements always real no.s.

$$\begin{bmatrix} 1 & 1+i & 4-i \\ 1-i & 2 & 6i \\ 4+i & -6i & 3 \end{bmatrix}$$

10) Skew hermitian matrix

$a_{ij} = -\overline{a_{ji}}$

Leading diagonal elements are either purely imaginary or zero.

$$\begin{bmatrix} 6i & 1-2i & 3-4i \\ -1-2i & 7i & 2i \\ -3-4i & 2i & 8i \end{bmatrix}$$

* Transposed conjugate of a matrix

$(\bar{A})' = A^\theta$

$$A = \begin{bmatrix} 0 & 6i & 8i \\ 7+i & 0 & 1-2i \\ 8+i & 9i & 0 \end{bmatrix}$$

It is conjugate of transpose of a matrix.

$$\bar{A} = \begin{bmatrix} 0 & -6i & -8i \\ 7-i & 0 & 1+2i \\ 8-i & -9i & 0 \end{bmatrix}$$

$$(\bar{A})' = A^\theta = \begin{bmatrix} 0 & 7-i & 8-i \\ -6i & 0 & -9i \\ -8i & 1+2i & 0 \end{bmatrix}$$

Orthogonal Matrices

$$AA' = I$$
$$AA' = A'A = I$$
$$AA' = I$$
$$A^{-1}AA' = A^{-1}I$$
$$\boxed{A' = A^{-1}}$$

Unitary Matrices

$$AA^{\theta} = I$$
$$A^{\theta} \cdot A = AA^{\theta} = I$$
$$\boxed{A^{\theta} = A^{-1}}$$

Involuntary matrix

For a matrix A, $\boxed{A^2 = I}$

Idempotent matrix

For a matrix A, $\boxed{A^2 = A}$

Q. If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$

Find a, b, c if A is orthogonal

$\rightarrow$

$$AA' = I$$

$$\begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{3/9 + a/3}$ $\quad 6/9 + 2b/3$

$\cancel{3/9 + b/3 \quad c^2}$

$$1/9 + 4/9 + a^2$$
$$4/9 + 1/9 + b^2$$
$$4/9 + 4/9 + c^2$$

$5/9 + a^2 = 1$	$5/9 + b^2 = 1$	$8/9 + c^2 = 1$
$a^2 = 4/9$	$b^2 = 4/9$	$c^2 = 1/9$
$a = \pm 2/3$	$b = \pm 2/3$	$c = \pm 1/3$

Q. Show that the matrix

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is unitary & hence find A^{-1}

$$A^{-1} = A^{\theta} = (\bar{A})'$$

$$\bar{A} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ -i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = A^{-1}$$

$$A \cdot A^{\theta} =$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$A^{-1} = A^{\theta}$

$\therefore \cancel{A^{-1} = A^{\theta}}$

✱ Row Echelon Form (REF)

A matrix $m \times n$ is said to be in row Echelon form if it satisfies following points.

1) The non zero entry in every non zero row may or may not be unity. This entry is also called leading entry or pivot entry.
2) The zero rows (if any) are always placed below the non-zero rows.
3) No. of zeros before the pivot are less than no. of such zeroes in the succeding rows.

eg. Row echelon form ✓

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

✓

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

✓

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 3 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

✗

$$\begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ✓

* Reduced row echelon form.
1) It should be row echelon form.
2) Pivot entries necessarily be unity (1).
3) All the entries in pivot column has to be zero other than 1.

Pivot elements or entries

eg. $$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ Reduced (REF) ✓

Pivot column

no increased zero.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ REF ✗

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ RREF ✓

* Converting a matrix into REF & RREF.

eg. - $$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{bmatrix}$$

To convert into REF

1) Follow only Row Elementary Transformation.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\underline{REF.}}$$

$\frac{1}{2} R_1$

$\frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_3$
$R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\underline{RREF}}$$

Q Convert to REF & RREF

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 5 \end{bmatrix}$$

$\to$ $R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & 7 \\ 0 & -6 & -12 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & 7 \\ 0 & 0 & 0 & -2 \end{bmatrix} \Rightarrow \underline{\underline{REF}}$$

~~$R_1 \to R_1 + \frac{2}{3} R_2$~~

~~1~~
~~0~~
~~0 0 0 -2~~

$R_2 \to -R_2/3$
$R_3 \to -R_3/2$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -7/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-1 - 2\left(\frac{-7}{3}\right)$

$-1 + \frac{14}{3}$

$-11/3$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -11/3 \\ 0 & 1 & 2 & -7/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + 11/3\, R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -7/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 7/3\, R_3$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \text{RREF}$$

Q. $$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_1 + 2R_3$ $$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{REF}$$

$R_1 \to \cancel{R_2}\ R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RREF}$$

~~R~~. No. of non-zero rows in a matrix after converting into RREF is called Rank of a matrix.

Q. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & -8 & 0 \end{bmatrix}$

$\longrightarrow$

$R_3 \to R_3 - R_1$
$R_4 \to R_4 + 2R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

$R_4 \to R_4/2$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2/4$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 3/2 & -3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to 4R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{REF}$$

$R_1 \to R_1 - \frac{1}{3} R_3$

$R_2 \to R_2 + \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RREF}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

Q. $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix}$

$R_3 \to \frac{1}{5} R_3$, $R_4 \to \frac{1}{10} R_4$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 5 \\ 2 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 3 & -2 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{2}{3} R_2$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 19/3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\text{REF}}$$

$5 + \frac{4}{3}$

$\frac{19}{3}$

$R_1 + \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 3 & -2 \\ 0 & 0 & 19/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - \frac{1}{19} R_3$

$R_2 = R_2 + \frac{6}{19} R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 19/3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\text{RREF}}$$

Rank = 3

Q. $$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 8 & 2 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\to$ $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2} R_2$

$$\begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \Rightarrow \underline{\text{REF}}$$

$\cancel{R_2 \to \frac{R_2}{2}}$

$\cancel{R_3 \to 2R_3}$

$\cancel{1 \quad 4 \quad 2 \quad 3}$

0

$R_1 = R_1 + R_2$

$$\begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$R_1 = R_1 - 4R_3$
$R_2 = R_2 + 2R_3$

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$R_3 \to 2R_3$
$R_2 \to -\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{RREF}$$

Rank = 3

Assignment.

* ~~Addressing modes~~

Q1
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

Q2.
$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$$

Q3.
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Q1 ⇒
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

$R_2 = R_2 - 3R_1$
$R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{REF}$$

R

$R_1 \to R_1 + R_2$

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_2 \to -R_2/3$

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \underline{\underline{\text{RREF}}}$$

Q2 →
$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -5 \\ 0 & -9 & -9 \\ 0 & -3 & -3 \end{bmatrix}$$

$\cancel{R_3 \to R_3 - \frac{9}{2} R_2}$

$$\cancel{\begin{matrix} 1 & 2 & 4 \\ 0 & -2 & -5 \\ 0 & 0 & 27/2 \end{matrix}}$$

$\cancel{-9 + \frac{45}{2}}$

$\cancel{-9 + \frac{15}{2}}$

$R_3 \to R_3 - 3R_4$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & -3 & -3 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{2} R_2$

$-3 + \frac{15}{2}$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -2 & -5 \\ 0 & 0 & 7/2 \\ 0 & 0 & 0 \end{pmatrix} \to \text{REF}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & -5 \\ 0 & 0 & 7/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{10}{7} R_3$

$R_1 \to R_1 + \frac{2}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to \frac{2}{7} R_3 \quad R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\text{RREF}}$$

Q7 →

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

→ $R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - \frac{6}{7} R_2$
$R_4 \to R_4 - \frac{3}{7} R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$$

$5 - \frac{30}{7}$

$\frac{5}{7}$

$-4 + \frac{48}{7}$

$\frac{-28+48}{7}$

$\frac{-15}{7} + 2$

$+\frac{24}{7} - 4$

$R_4 \to R_4 + \frac{1}{5} R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\text{REF}}$$

$R_1 \to R_1 + \frac{2}{7} R_2$

$$\begin{bmatrix} 1 & 0 & 3/7 & 5/7 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{10}{7} - 1$

$\frac{3}{7}$

$\frac{-16}{7} + 3$

$\frac{5}{7}$

$R_2 \to R_2 - 7 R_3$
$R_1 \to R_1 - \frac{3}{5} R_3$

$-\frac{3}{5} \times \frac{20}{7}$

$-\frac{12}{7} + \frac{5}{7}$

$-\frac{7}{7} - 1$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -7 & 0 & -28 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\text{RREF}}$$

Topics

1) REF
2) RREF
3) Solution of systems.

Direct method — Indirect method. (Interactive methods)

4) Rank of matrix.
5) System of homo & non-homo linear eq^n

Solutions of system of equations.

I) Direct method.

a) Gauss elimination method.

b) Gauss Jordan method

Consider system of eq^ns.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$$

$$AX = B$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Augmented matrix

$$[A : B]$$

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$ Augmented matrix.

a) Gauss elimination method

3 zeros

x	y	z
x	x	x
0	x	x
0	0	x

Gauss Jordan Method.

x	0	0
0	x	0
0	0	x

only J values rest 0.

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

first you will get z then y then x.

you will get all three values.

Gauss elimination → Get values by REF.

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

Augmented matrix is $[A : B]$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{array}\right]$$

$R_3 \to 3R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{array}\right]$$

$$-32z = -64$$
$$\boxed{z = 2}$$

$$3y + z = 2$$
$$3y + 2 = 2$$
$$\boxed{y = 0}$$

$$x - y + 2z = 3$$
$$x - 0 + 4 = 3$$
$$\boxed{x = -1}$$

Q. $x + y + z = 3$
$2x + 3y + 7z = 0$
$x + 3y - 2z = 17$

Augmented matrix is $[A : B]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 2 & 3 & 7 & 0 \\ 1 & 3 & -2 & 17 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 2 & -3 & 14 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & -13 & 26 \end{array}\right]$$

$$-13z = 26$$
$$\boxed{z = -2}$$

$$y + 5z = -6$$
$$y - 10 = -6$$
$$\boxed{y = 4}$$

$$x + y + z = 3$$
$$x + 4 - 2 = 3$$
$$\boxed{x = 1}$$

Q.
$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$\cancel{R_1 \to 5R_1 - 2R_2}$
$\cancel{R_3 \to 5R_3 + 9R_2}$

$$\cancel{\left[\begin{array}{ccc:c} 1 & 0 & 9 & 35 \\ 0 & 5 & 3 & 5 \\ 0 & 0 & 22 & \end{array}\right]}$$

$R_1 \to R_1 + 2R_2$
$R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc:c} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

3P - 36

$R_1 \to 2R_1 - 9R_3$
$R_2 \to 2R_2 - 3R_3$

$$\left[\begin{array}{ccc:c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$2x = 2$ $\qquad$ $2y = -2$ $\qquad$ $2z = 4$

$\boxed{x = 1}$ $\qquad$ $\boxed{y = -1}$ $\qquad$ $\boxed{z = 2}$

Jordan

Q.
$$x + 2y - z = 3$$
$$2x + 5y + 2z = -3$$
$$4x - 2y + z = 12$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 2 & 5 & 2 & -3 \\ 4 & -2 & 1 & 12 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -9 \\ 0 & -10 & 5 & 0 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$
$R_3 \to R_3 + 10R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -9 & 21 \\ 0 & 1 & 4 & -9 \\ 0 & 0 & 45 & -90 \end{array}\right]$$

$R_1 \to 5R_1 + R_3$
$R_2 \to 45R_2 - 4R_3$

$105x - 90$

$$\left[\begin{array}{ccc:c} 5 & 0 & 0 & 15 \\ 0 & 45 & 0 & -45 \\ 0 & 0 & 45 & -90 \end{array}\right]$$

45
9
45
$-405 + 360$
45

$5x = 15$ $\quad$ $45y = -45$ $\quad$ $45z = -90$

$\boxed{x = 3}$ $\quad$ $\boxed{y = -1}$ $\quad$ $\boxed{z = -2}$

Indirect Methods.
(Interactive Methods).

a) Gauss - Jacobi
b) Gauss - Siedel

Gauss - Jacobi & Gauss Siedel.

Consider
$$a_{11}x + a_{12}y + a_{13}z = d_1$$
$$a_{21}x + a_{22}y + a_{23}z = d_2$$
$$a_{31}x + a_{32}y + a_{33}z = d_3.$$

Conditions check.

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Now represent

$$x = \frac{1}{a_{11}}(d_1 - a_{12}y - a_{13}z)$$

$$y = \frac{1}{a_{22}}(d_2 - a_{21}x - a_{23}z)$$

$$z = \frac{1}{a_{33}}(d_3 - a_{31}x - a_{32}y)$$

Gauss Jacobi

Q. $27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

$|27| > |6| + |-1|$

$|15| > |6| + |2|$

$|54| > |1| + |1|$

$x = \frac{1}{27}(85 - 6y + z)$

$y = \frac{1}{15}(72 - 6x - 2z)$

$z = \frac{1}{54}(110 - x - y)$

I^{st} iteration $x_0 = y_0 = z_0 = 0$

$x_1 = \frac{1}{27}(85 - 0 + 0) = 3.148$

$y_1 = \frac{1}{15}(72 - 0 - 0) = 4.8$

$z_1 = \frac{1}{54}(110 - 0 - 0) = 2.037$

$x_1 = 3.148, \quad y_1 = 4.8, \quad z_1 = 2.037$

$x_2 = 2.157$
$y_2 = 3.269$
$z_2 = 1.89$

Iterate till the ~~cor~~ successive iterations give more or less same value.

k	x_k	y_k	z_k
1			
2			
3 ~~2.497~~	2.492 ~~3.6~~	3.655	1.967
4	2.401	3.545	1.925
5	2.452	3.583	1.927
6	2.423	3.583	1.927
7	2.462	3.570	1.926

Soln

Gauss-Siedel

Q. $20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$

$|20| > |1| + |-2|$
$|20| > |3| + |-1|$
$|20| > |2| + |-3|$

$$x = \frac{1}{20}\left(17 - \cancel{20}y + 2z\right)$$

$$y = \frac{1}{20}\left(-18 - 3x + z\right)$$

$$z = \frac{1}{20}\left(25 - 2x + 3y\right)$$

$$y_0 = z_0 = 0$$

$$x_1 = \frac{1}{20}\left(17 - 0 + 0\right) = 0.85$$

$$y_1 = \frac{1}{20}\left(-18 - 3(0.85) + 0\right) = -1.0275$$

$$z_1 = \frac{1}{20}\left(25 - 2(0.85) + 3(-1.0275)\right) = 1.0109$$

$$y_1 = -1.0275 \qquad z_1 = 1.0109$$

$$x_2 = \frac{1}{20}\left(17 - (-1.0275) + 2(1.0109)\right)$$

$$= 1.0075$$

$$y_2 = \frac{1}{20}\left(-18 - 3(1.0075) + 1.0109\right)$$

$$= -0.9998$$

$$z_2 = \frac{1}{20}\left(25 - 2(1.0075) + 3(-0.9998)\right)$$

$$= 0.9998$$

$$x_3 = 1.0000$$
$$y_3 = -1.0000$$
$$z_3 = 1.0000$$

Q. Jacobi

$$6x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

$$|6| > |-2| + |3|$$
$$|9| > |-3| + |1|$$
$$|7| > |2| + |-1|$$

$$x_1 = \frac{1}{6}\left(-1 + 2x_2 - 3x_3\right)$$

$$x_2 \; \cancel{y_1} = \frac{1}{9}\left(2 + 3x_1 - x_3\right)$$

$$x_3 \; \cancel{z} = -\frac{1}{7}\left(3 - 2x_1 + x_2\right)$$

$$\cancel{x_0 = y_0 = z_0 = 0} \qquad x_{01} = x_{02} = x_{03} = 0$$

$$x_{11} = \frac{1}{6}(-1) = -0.16667$$

$$x_{21} = \frac{2}{9} = 0.2222$$

$$x_{31} = \frac{-3}{7} = -0.4286$$

$$x_{21} = \frac{1}{6}\left(-1 + 2(0.2222) - 3(-0.4286)\right) = 0.1217$$

$$x_{22} = \frac{1}{9}\left(2 + 3(-0.16667) + 0.4286\right) = 0.2143$$

$$x_{23} = -\frac{1}{7}\left(3 - 2(-0.16667) + 0.2222\right) = -0.5079$$

$x_{31} = 0.1587$
$x_{32} = 0.3192$
$x_{33} = -0.4244$

$x_{41} = 0.1519$
$x_{42} = 0.3222$
$x_{43} = -0.4288$

$x_{51} = 0.1551$
$x_{52} = 0.3205$
$x_{53} = -0.4312$

$$\left.\begin{aligned} x_{61} &= 0.15576 \\ x_{62} &= 0.32183 \\ x_{63} &= -0.43004 \end{aligned}\right\} \text{Sol}^n.$$

Gauss Siedel

Q

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

* Crout's Method.
  (LU Decomposition method).

Q.
$$\left.\begin{aligned} x + y + z &= 3 \\ 2x - y + 3z &= 16 \\ 3x + y - z &= -3 \end{aligned}\right\}$$

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$$

Let $A = LU$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

representation.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & (l_{31}u_{13} + l_{32}u_{23} + u_{33}) \end{bmatrix}$$

Comparing two matrices.

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$l_{21}u_{11} = 2$ $\qquad l_{21} = 2$

$l_{21}u_{12} + u_{22} = -1$ $\qquad 2 + u_{22} = -1$ $\qquad u_{22} = -3$

$l_{21}u_{13} + u_{23} = 3$ $\qquad 2 + u_{23} = 3$ $\qquad u_{23} = 1$

$l_{31}u_{11} = 3$ $\qquad l_{31} = 3$

$l_{31}u_{12} + l_{32}u_{22} = 1$ $\qquad 3 + l_{32}(-3) = 1$ $\qquad l_{32} = 2/3$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -1$ $\qquad 3 + 2/3 + u_{33} = -1$ $\qquad u_{33} = \frac{-14}{3}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -14/3 \end{bmatrix}$$

$A = LU$

$AX = B$

$(LU)X = B$

$L(UX) = B$

Let $UX = Y$

$LY = B$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$$

$y_1 = 3 \qquad y_2 = 10 \qquad y_3 = -56/3$

We have $UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -14/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -56/3 \end{bmatrix}$$

Given system

$AX = B$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Comparing equating we get matrix L & U

$AX = B$

& $A = LU$

We get $LUX = B$

$L(UX) = B$

Let $UX = Y$

Now, $LY = B$

get y_1, y_2, y_3 for $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

We have $UX = Y$

Comparing get values of x, y & z.

Q.
$$2x + 3y + z = -1$$
$$5x + y + z = 9$$
$$3x + 2y + 4z = 11$$

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$u_{11} = 2$, $\quad u_{12} = 3$, $\quad u_{13} = 1$

$l_{21}u_{11} = 5 \quad l_{21} = 5/2$

$l_{21}u_{12} + u_{22} = 1 \quad \frac{15}{2} + u_{22} = 1 \quad u_{22} = -13/2$

$l_{21}u_{13} + u_{23} = 1 \quad \frac{5}{2} + u_{23} = 1 \quad u_{23} = -3/2$

$l_{31}u_{11} = 3 \quad l_{31} = 3/2$

$l_{31}u_{12} + l_{32}u_{22} = 2 \quad \frac{9}{2} + l_{32}\left(\frac{-13}{2}\right) = 2$

$l_{32} = \frac{-5}{\cancel{2}} \times \frac{-\cancel{2}}{13} = 5/13$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4 \quad \frac{3}{2} - \frac{15}{26} + u_{33} = 4$

$u_{33} = 4 - \frac{12}{13}$

$u_{33} = 40/13$

$$LUX = B$$
$$L(UX) = B$$
$$UX = Y$$
$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5/2 & 1 & 0 \\ 3/2 & 5/13 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$y_1 = -1$

$-5/2 + y_2 = 9$

$y_2 = 23/2$

$-3/2 + \frac{115}{26} + y_3 = 11$

$\frac{-39 + 115}{26} + y_3 = 11$

$\frac{76}{26} + y_3 = 11$

$\frac{38}{13} + y_3 = 11$

$y_3 = 11 - \frac{38}{13}$

$y_3 \quad \frac{143 - 38}{13} = \frac{105}{13}$

$UX = Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -13/2 & -7/2 \\ 0 & 0 & 40/13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 23/2 \\ 105/13 \end{bmatrix}$$

$\frac{40}{13} z = \frac{105}{13}$

$z = \frac{105}{40} = 3.125$

$-\frac{13}{2} y \quad -\frac{3}{2} \times 3.125 = \frac{23}{2}$

$-\frac{13 y}{2} = \frac{23}{2} +$

$y = 10.778$

$\boxed{z = 21/8 = 2.625}$

$-\frac{13}{2} y - \frac{63}{16} = \frac{23}{2}$

$-\frac{13}{2} y = \frac{247}{16}$

$\boxed{y = -2.375}$

$2x = \frac{783}{16}$

$x = 24.46$

$2x + 9.75 = -1$

$2x = -10.75$

$2x - 4.5 = -1$

$2x = 3.5$

$\boxed{x = 1.75}$

* Solutions of non-homogenous & homogenous system of equations.

Linearly Dependent Vectors.

Consider the matrix equation

$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$

$0 \rightarrow$ zero vector with all components zero.

$k_1, k_2, k_3 \dots$ scalar not all zero.

$x_1, x_2, \dots x_r$ are vectors

Also linearly dependent vectors can be expressed as a linear combination of other vectors

$x_1 = \frac{-1}{k_1} (k_2 x_2 + \dots + k_r x_r)$

Linearly Independent Vectors

Consider the matrix eqn

$k_1 x_1 + \dots + k_r x_r = 0$

If $k_1 = k_2 = \dots = k_r = 0$ then $x_1, x_2 \dots x_r$ are called as linearly independent vectors.

6. ~~ST~~ Show that x_1, x_2, x_3 are LI (Linearly indep.)

& x_1, x_2, x_3, x_4 are LD (Li. De.)

where

$x_1 = (1,2,4)$, $x_2 = (2,-1,3)$, $x_3 = (0,1,2)$

$x_4 = (-3,7,2)$

Consider the matrix eqn

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

$$k_1(1,2,4) + k_2(2,-1,3) + k_3(0,1,2) = 0$$

$$k_1 + 2k_2 + 0k_3 = 0$$
$$2k_1 - k_2 + k_3 = 0$$
$$4k_1 + 3k_2 + 2k_3 = 0$$

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$k_3 = 0$, $k_2 = 0$, $k_1 = 0$

$\therefore x_1, x_2, x_3$ are linearly independent vectors.

Consider the matrix eqn

$$k_1 x_1 + \dots + k_4 x_4 = [0]$$

$$k_1 + 2k_2 + 0k_3 - 3k_4 = 0$$
$$2k_1 - k_2 + k_3 + 7k_4 = 0$$
$$4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$k_1 + 2k_2 + 0k_3 - 3k_4 = 0$$
$$-5k_2 + k_3 + 13k_4 = 0$$
$$k_3 + k_4 = 0$$

$K_3 = -K_4$

Let $K_4 = t$

$K_3 = -t$

$-5K_2 - t + 13t = 0$

$K_2 = -\frac{1}{5}(-12t) = +\frac{12}{5}t$

$K_1 + 2\left(+\frac{12}{5}t\right) - 3t = 0$

$K_1 = -\frac{24t}{5} + 3t$

$K_1 = \frac{-9t}{5}$

$K_1X_1 + K_2X_2 + K_3X_3 + K_4X_4 = 0$

$-\frac{9t}{5}X_1 + \frac{12t}{5}X_2 - tX_3 + tX_4 = 0$

$X_1 = \frac{-5}{9}\left(-X_4 + X_3 - \frac{12}{5}X_2\right)$

$\therefore$ X_1, X_2, X_3, X_4 are Linearly Dependent

* Non-Homogenous system of eq^n.

$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = d_1$

$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = d_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n = d_n$

$AX = B$

$$\begin{bmatrix} a_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \text{matrix of } A$$

$$[A : B] = \begin{bmatrix} a_{11} & \dots & a_{1n} & : d_1 \\ \vdots & & & \vdots \\ a_{n1} & \dots & a_{nn} & : d_n \end{bmatrix}$$

augmented matrix

For the system to be consistent

→ rank (A) < rank(A : B)
System is inconsistent & has no solution.

→ rank (A) = rank(A : B)
System is consistent

rank (A : B) = n (no. of unknowns)

(a) rank(A) = rank (A : B)
⇒ System has unique sol^n

(b) rank(A) = rank(A : B) < n (no. of unknowns)
System has infinitely ~~many~~ many sol^ns.

Non-homogenous sys. of eq^ns.

Steps:

- $AX = B$
- Reduce above form to row-echelon form & hence determine rank of A and rank of $(A:B)$

[Rank is no. of non-zero roots in the matrix after row echelon form]

- Consistency check.

Solve

$$x + y = 1$$
$$2x + 3y = 1$$
$$5x - y = 11$$

$$Ax = B$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & -1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 1 \\ 1 \\ 11 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$x + y = 1$$
$$y = -1$$
$$x = 2$$

$$\boxed{\rho_{(A)} = 2}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & : & 1 \\ 0 & 1 & : & -1 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$$\boxed{\rho_{(A:B)} = 2}$$

$\because \rho_A = \rho_{A:B}$

$\therefore$ system is consistent

$\rho_A = \rho_{A:B} = n$ (no. of unknowns) $= 2$

$\therefore$ system has unique sol^n

$y = -1, \quad x = 2$

2)

$5x_1 + 3x_2 + 7x_3 = 4$
$3x_1 + 26x_2 + 2x_3 = 9$
$7x_1 + 2x_2 + 10x_3 = 3$

$Ax = B.$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$5R_2 - 3R_1$
$5R_3 - 7R_1$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ -3 \end{bmatrix}$$

$R_3 + \frac{1}{11} R_2$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \\ 0 \end{bmatrix}$$

$\frac{1}{11} R_2$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$\rho(A) = \rho(A:B) = \overset{r}{2} < \overset{n}{3}$ (no. of unknowns)

Find this and infinitely many soln. $\Rightarrow$ $\therefore$ The eq^ns are consistent but have infinite solns.

$\therefore$ No of parameters $= n - r = 3 - 2$
$= 1$

$5x_1 + 3x_2 + 7x_3 = 4$
$11x_2 - x_3 = 3$

Put $x_3 = t$

$x_2 = \frac{3+t}{11}$

$5x_1 = 4 - \left(\frac{9+3t}{11}\right) - 7t$

$x_1 = \frac{44 - 9 - 3t - 77t}{55}$

$x_1 = \frac{35 - 80t}{55}$

$= \frac{7 - 16t}{11}$

* Show that if $\lambda \neq -5$ the system of eq^ns

$3x - y + 4z = 3$
$x + 2y - 3z = -2$
$6x + 5y + \lambda z = -3$

has a unique soln. If $\lambda = -5$ show that $\therefore$ eq^ns are consistent. Determine soln in each case.

$\rightarrow$

Rearrange

$x + 2y - 3z = -2$
$3x - y + 4z = 3$
$6x + 5y + \lambda z = -3$

$Ax = B$

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 4 \\ 6 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 13 \\ 0 & -7 & \lambda+18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 13 \\ 0 & 0 & \lambda+5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}$$

$\lambda \neq -5$

(I) $\lambda \neq -5$
$z = 0$

$-7y + 13z = 9$
$y = -9/7$

$x + 2y - 3z = -2$
$x = 4/7$

System has unique solⁿ.

(II) $\lambda = -5$

$\rho(A) = \rho(A : B) = 2 < 3$ (no. of unknowns)

No. of parameters $= 3 - 2 = 1$

Let $z = t$

$-7y + 13z = 9$

$-7y + 13t = 9$

$y = \frac{9 - 13t}{-7}$

$y = \frac{13t - 9}{7}$

$x + 2y - 3z = -2$

$x + \frac{26t - 18}{7} - 3t = -2$

$x = -2 + 3t - \left(\frac{26t - 18}{7}\right)$

$x = \frac{-14 + 21t - 26t - 18}{7}$

$x = \frac{-5t - 32}{7}$

$x = -\left(\frac{5t + 32}{7}\right)$

The eqⁿs are consistent but have infinitely many solⁿs.

* Investigate for what values of λ and μ the eqⁿs

$x + y + z = 6$
$x + 2y + 3z = 10$
$x + 2y + \lambda z = \mu$

have
1) No solⁿ
2) a Unique solⁿ
3) infinite solⁿ.

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu-10 \end{bmatrix}$$

1) if $\rho(A) < \rho(A:B)$

$\therefore$ if $\boxed{\lambda = 3}$

$\rho(A) = 2$

$\rho(A:B) = 3$ (if $\mu \neq 10$)

No. solⁿ

2) if $\lambda \neq 3$ & $\mu \neq 10$ &

then $\rho(A) = \rho(A:B) = 3 - 3 = 0$

unique solⁿ

3) if $\lambda = 3$ & $\mu = 10$

$\rho(A) = \rho(A:B) < n$ (i.e. no. of unknowns)

i.e. $2 < 3$

$\therefore$ infinite solⁿs.

Q. Check whether

$2x - 3y + 7z = 5$

$3x + y - 3z = 13$

$2x + 19y - 47z = 32$

is consistent or not.

→

$AX = B$

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

$R_2 \to 2R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & -7 & -27 \\ 0 & 22 & -54 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ \\ - \end{bmatrix}$$

$\rho(A) \neq \rho(A:B)$

$\therefore$ eq's are inconsistent.

Homogeneous system

$AX = 0$

Working rule

Write the system in the form

$AX = 0$

Reduce matrix A to Echelon form

If $\rho(A) = n$ (no. of unknowns) then zero soln (trivial)

If $\rho(A) < n$ then infinite soln.

Q. Solve.

$x_1 + 2x_2 + 3x_3 = 0$

$2x_1 + 3x_2 + x_3 = 0$

$4x_1 + 5x_2 + 4x_3 = 0$

$x_1 + x_2 - 2x_3 = 0$

$AX = 0$

$\boxed{n = r} \rightarrow$ zero soln

$r < n \rightarrow$ infinite soln.

Consider $AX = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_1 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

$R_4 \rightarrow R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \\ 0 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$R_4 \rightarrow R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

no. of unknowns $= 3$
r or $\rho(A) = 3$

$x_3 = x_2 = x_1 = 0$ zero or trivial solⁿ

Q. $x_1 + x_2 - x_3 + x_4 = 0$
$x_1 - x_2 + 2x_3 - x_4 = 0$
$3x_1 + x_2 + x_4 = 0$

$AX = 0$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n = 4$
$r = 2$
$r < n$

infinite solⁿ.

~~$n = 3$~~
~~$\rho(A) = 2$~~

$n = 4$, $r = 2$
no. of parameters $= n - r = 2$

$-2x_2 + 3x_3 - 2x_4 = 0$

$x_1 + x_2 - x_3 + x_4 = 0$

REF
RREF
Find rank
5 methods
Homo & non Homo
Eigen values, Eigen vectors & properties
Diagonalization

Eigen values, Eigen vectors

Consider a matrix A

I) Characteristic polynomial.

Let A be any matrix

λ any scalar

I - unit matrix

$|A - \lambda I|$ is called characteristic polynomial.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$ characteristic matrix.

Now,

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 11\lambda - 5$$

Characteristic polynomial.

$\therefore |A - \lambda I| = 0$

Characteristic eq^n

For A,

$$\boxed{\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0}$$

$\boxed{AX = B}$ Characteristic eq^n also can be written as $\boxed{AX = \lambda X}$

Solve characteristic eq^n

$\lambda = 1, 1, 5$

The values of λ are called as

Eigen values | Proper values | Characteristic roots | Latent roots | Spectral roots.

Properties of Eigen values

I) Eigen values of any Hermitian matrix are all real.

Proof

Consider any matrix A

For A to be Hermitian

$A^\theta = A$ $\qquad$ $A^\theta \Rightarrow$ complex conjugate transpose

Let λ be Eigen value of matrix A

(λ is any scalar).

let x be non-zero vector
We have
Taking complex conjugate transpose on both sides,

$$(Ax)^\theta = (\lambda x)^\theta$$

$$(AB)^T = (B^T A^T)$$

$$(AB)^\theta = B^\theta A^\theta$$

$$x^\theta \cdot A^\theta = \bar{\lambda} x^\theta$$

$$x^\theta \cdot A = \cancel{\bar{\lambda}\theta} \; \bar{\lambda} x^\theta$$

Post multiplying both sides by x.

$$\therefore x x^\theta A = x \bar{\lambda} x^\theta$$

$$\cancel{x\theta} \quad x x^\theta A = \bar{\lambda} x x^\theta$$

$$x^\theta A x = \bar{\lambda} x^\theta x$$

$$x^\theta \lambda x = \bar{\lambda} x^\theta x$$

$$\cancel{x x^\theta x = \bar{\lambda} x^\theta x}$$

$$\lambda x^\theta x = \bar{\lambda} x^\theta x$$

$$\lambda x^\theta x - \bar{\lambda} x^\theta x = 0$$

$$(\lambda - \bar{\lambda}) x^\theta x = 0$$

$$x^\theta x = 0$$

We have x non-zero vector

$$\cancel{x^\theta x \neq 0} \quad x^\theta x \neq 0$$

$$\therefore (\lambda - \bar{\lambda}) = 0$$

$$\lambda = \bar{\lambda}$$

II) Eigen values of unitary matrix are of unit modulus

$$AA^\theta = A^\theta A = I \quad \text{(unitary matrix)}$$

$$Ax = \lambda x \qquad x \neq 0 \qquad \lambda \text{ scalar}$$

$$Ax = \lambda x \quad \text{——} \; (1)$$

$$(Ax)^\theta = (\lambda x)^\theta$$

$$x^\theta A^\theta = \bar{\lambda} x^\theta \quad \text{——} \; (2)$$

$$x^\theta A^\theta A x = \bar{\lambda} x^\theta \lambda x$$

$$x^\theta x = \bar{\lambda} \lambda x^\theta x$$

$$\bar{\lambda}\lambda x^\theta x - x^\theta x = 0$$

$$(\bar{\lambda}\lambda - 1) x^\theta x = 0$$

$$x^\theta x \neq 0$$

$$\therefore \bar{\lambda}\lambda - 1 = 0$$

$$\therefore \boxed{|\lambda| = 1}$$

* Eigen values of A & $P^{-1}AP$ are same

Let λ be eigen value of X is
is non-zero eigen vector

Then $AX = \lambda X$
Premultiply by P^{-1}
$P^{-1}AX = P^{-1}\lambda X$

$P^{-1}AIX = \lambda P^{-1}X$
$P^{-1}APP^{-1}X = \lambda P^{-1}X$

$(P^{-1}AP)(P^{-1}X) = \lambda P^{-1}X$

$[P^{-1}AP - \lambda]P^{-1}X = 0$

$X \neq 0$

$\therefore \boxed{P^{-1}AP = \lambda}$

λ is eigen value of $P^{-1}AP$ &
$P^{-1}X$ is corresponding eigen vector of $P^{-1}AP$.

→ Eigen values of Hermitian matrix are all real
→ Eigen values of skew Hermitian matrix are either 0 or imaginary
→ Eigen values of orthogonal / unitary matrix are of unit modulus.
→ Eigen values of Idempotent matrix are 0 or 1.

→ Eigen values of Involuntary matrix are ± 1

~~(f) Adj $A \rightarrow \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \ldots \frac{|A|}{\lambda_n}$~~

→ Eigen values of A & A^T are same
→ Let $\lambda_1, \ldots \lambda_n$ be eigen values of A.
Then eigen values.

(a) $A^2 \rightarrow \lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$
(b) $A^3 \rightarrow \lambda_1^3, \lambda_2^3, \ldots \lambda_n^3$
(c) $A^n \rightarrow \lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$
(d) $K.A \rightarrow K\lambda_1, \ldots K\lambda_n$
(e) $A^{-1} \rightarrow \lambda_1^{-1}, \lambda_2^{-1} \ldots \lambda_n^{-1}$
(f) Adj $A \rightarrow \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \ldots \frac{|A|}{\lambda_n}$

* Nullity of matrix

Nullity = order of A − rank(A)
= no. of zero rows in echelon rows

rank(A) = no. of non-zero rows in echelon form.

eg. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - R_1$

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

nullity $(A) = 1$

To find eigen value & eigen vectors

① $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$

$[A - \lambda I]$

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & -8-\lambda & -4 \\ 2 & -4 & -12-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0 \rightarrow$ characteristic eqn.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & -8-\lambda & -4 \\ 2 & -4 & -12-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (8-8-12) + (\quad)\lambda - |A| = 0$$

$+ \quad - \quad + \quad -$

$(8-\lambda)[(-8-\lambda)(-12-\lambda) - 16] +$
$6[-6(-12-\lambda) + 8] + 2[24 - 2(-8-\lambda)]$

$(8-\lambda)[(96 + 8\lambda + 12\lambda + \lambda^2) - 16]$

$+ 6[72 + 6\lambda + 8] + 2[24 + 16 + 2\lambda]$

① $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$[A - \lambda I]$

$$A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

checking.

$\rightarrow \lambda^3 - (8+7+3)\lambda^2 + (\quad)\lambda - |A| = 0$

$+ \quad - \quad + \quad -$

$\lambda_1 + \lambda_2 + \lambda_3 = \frac{-b}{a}$

$\lambda_1 \lambda_2 \lambda_3 = \frac{-d}{a}$

$\therefore -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\therefore \lambda = 0, 3, 15$ distinct eigen values.

Note: Eigen vectors corresponding to distinct eigen values are always LI.

For $\lambda = 0$ $\quad [A - \lambda I]X = 0$ (consider)

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}}$$

(Cramer's rule)

$$\frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14}$$

$$\frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10}$$

$$x_1 = \frac{x_2}{2} = \frac{x_3}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 4 & -4 \\ -4 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -6 & -4 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 15$

$[A - 15I]X = [0]$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16}$$

$$\frac{x_1}{80} = \frac{-x_2}{80} = \frac{x_3}{40}$$

Every matrix with distinct eigen values is diagonalizable

$$\frac{x_1}{2} = \frac{-x_2}{2} = \frac{x_3}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

* Diagonalization of matrix.

Diagonal elements → eigen values.

& other elements = 0

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Transforming matrix P

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\boxed{P^{-1}AP = D}$$

* Algebraic Multiplicity.

Algebraic Multiplicity of an eigen value is the no. of times it appear as a root of characteristic eqⁿ.

eg. ch. root of any eqⁿ are

1, 1, 2

$AM(1) = 2$

$AM(2) = 1$

* Geometric Multiplicity

Geometric Multiplicity of an eigen value say (λ) is as below.

$$GM(\lambda) = \text{Nullity } (A - \lambda I)$$

we have

$$\text{Nullity } (A) = O(A) - \rho(A)$$

Note:

1) For distinct eigen values ~~are~~ we say that the matrix is diagonalizable.

2) For repeated eigen values, the matrix is diagonalizable if its

$$AM(\lambda) = GM(\lambda)$$

Check whether the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

is diagonalizable or not? If yes find the transforming matrix

→

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Consider,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\therefore \lambda = 1, 1, 5$$

are eigen values.

For $\lambda = 5$

$$[A - 5I][X] = 0$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_2 + 4x_3 = 0$$

$$x_2 = x_3 = k$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 + (-k) = 0$$

$$x_1 = k$$

$$x_1 = x_2 = x_3 = k$$

$\therefore$ Eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

i.e. $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For ~~λ~~ $\lambda = 1$

$$[A - \lambda][X] = [0]$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AM(1) = 2$

~~AM(2) = 1~~

$$GM(1) = n(A - I)$$
$$= O(A - I) - \rho(A - I)$$
$$= 3 - 1$$
$$= 2$$

$AM(1) =$ ~~GM(1)~~ $GM(1) = 2$

$\therefore$ A is diagonalizable

For $\lambda = 1$

(1)

$$x_1 + 2x_2 + x_3 = 0$$

$x_1 = -2k$

$x_2 = 0,\ x_3 = k$ $\qquad$ $x_2 = k,\ x_3 = 0$

$X_2 = 2k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad$ $X_3 = k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad$ $X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

OR

OR (2) Method

$n - r = 2$

2 parameters

Let $x_2 = s$ $\qquad$ $x_3 = t$

$$x_1 + 2s + t = 0$$
$$x_1 = -2s - t$$

$$\begin{bmatrix} -2s - t \\ s + 0 \\ 0 + t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

$$\overset{X_2}{s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} + \overset{X_3}{t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$\therefore$ Eigen vector $k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

ie. $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\qquad$ $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} A P = D$$

[$\because$ A is diagonalizable]

Q. $$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Consider

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$6-\lambda(9 - 6\lambda + \lambda^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda)$$

$$-6+\lambda \quad 54 - 36\lambda + 6\lambda^2 - 9\lambda + 6\lambda^2 - \lambda^3 - 12 + 4\lambda + 4$$

$$+4 - 12 + 4\lambda$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$\lambda = 2, 2, 8$ are eigen values

For $\lambda = 8$

$$[A - 8I][X] = [0]$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_2 - 3x_3 = 0$$

$$x_2 + x_3 = 0$$

* Check whether $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable

Find transforming matrix

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$-9-\lambda\left(21 - 10\lambda + \lambda^2 - 32\right) - 4\left(-56 + 8\lambda + 64\right)$$
$$+ 4\left(-64 + 48 - 16\lambda\right)$$
$$-189 + 90\lambda - 9\lambda^2 + 288 - 21\lambda + 10\lambda^2$$
$$-\lambda^3 + 32\lambda + 224 - 32\lambda - 256$$
$$- 256 + 192 - 64\lambda$$

$$-\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$$

$$\lambda = 3, -1, -1$$

For $\lambda = -1$ $\quad AM(-1) = 2$

$$[A + I]\,[x] = [0]$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to \frac{-1}{4} R_1$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$GM(-1) = 2$$

$\therefore$ A is diagonalizable.

$$2x_1 - x_2 - x_3 = 0$$

or

$x_1 = 0,\ x_2 = k$ $\qquad$ $x_1 = k,\ x_2 = 0$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ +2 \end{bmatrix}$$

or

$x_1 = s$ $\qquad x_2 = t$

$2s - t - x_3 = 0$

$x_3 = 2s - t$

$$X = \begin{bmatrix} s+0 \\ 0+t \\ 2s-t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

For $\lambda = 3$.

$$[A - 3I][x] = [0]$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -12 & 4 & 4 \\ +4 & -4 & 0 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -12 & 4 & 4 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to \frac{1}{4} R_1 \qquad R_2 \to \frac{1}{4} R_2$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{R_2 \to R_2 + \frac{1}{3} R_1}$

$R_2 \to 3R_2 + R_1$

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{x_3 =}$

$$-3x_1 + x_2 + x_3 = 0$$
$$-2x_2 + x_3 = 0$$

~~Parameters n - r = 3 - 1 = 2~~

$x_2 = s \quad , \quad x_3 = t$

$$-3x_1 + s + t = 0$$
$$-2s + t = 0$$
$$t = 2s$$

$$-3x_1 + s + 2s = 0$$
$$-3x_1 + 3s = 0$$
$$x_1 = s$$
$$x_2 = s$$
$$x_3 = t = 2s$$

Q. <u>H/W</u>

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$