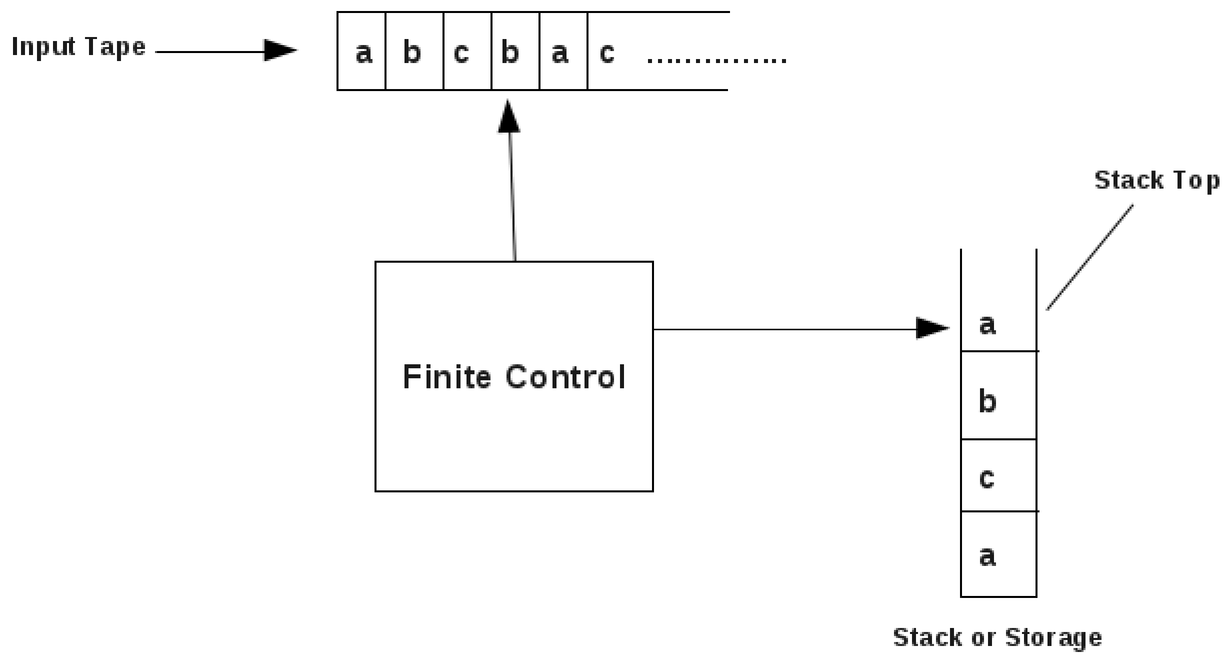
# UNIT III-PUSH DOWN AUTOMATA

## Pushdown Automata (PDA)

As we learned, context free languages are generated using context free grammars. Context free languages are recognized using pushdown automata.
Following diagram shows a pushdown automation.




The PDA has three components: An input tape,A control mechanism, and A stack.
From the input tape, the finite control reads the input, and also the finite control reads a symbol from the stack top.

### 6 Definition of PDA

A pushdown automation, P is a system,

$$P = (Q, \sum, \Gamma, \delta, q_0, F)$$

where

$Q$ is a set of states,

$\sum$ is a set of input symbols,

$\Gamma$ is a set of stack symbols (pushdown symbols),

$q_0$ is the start state,

$F$ is a set of final states,

$\delta$ is a transition function which maps,

$$(Q \times \sum^* \times \Gamma^*) \longrightarrow (Q \times \Gamma^*)$$

The meaning of $\delta$ is explained below:

Consider a transition function, $\delta$ in a PDA defined by,

$$\delta = (p, a, \beta) \longrightarrow (q, \gamma)$$

This means that

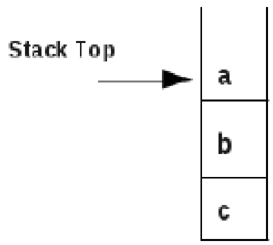
read $a$ from the input tape,

pop the string $\beta$ from stack top,

move from state $p$ to state $q$,

push string $\gamma$ on to the stack top.

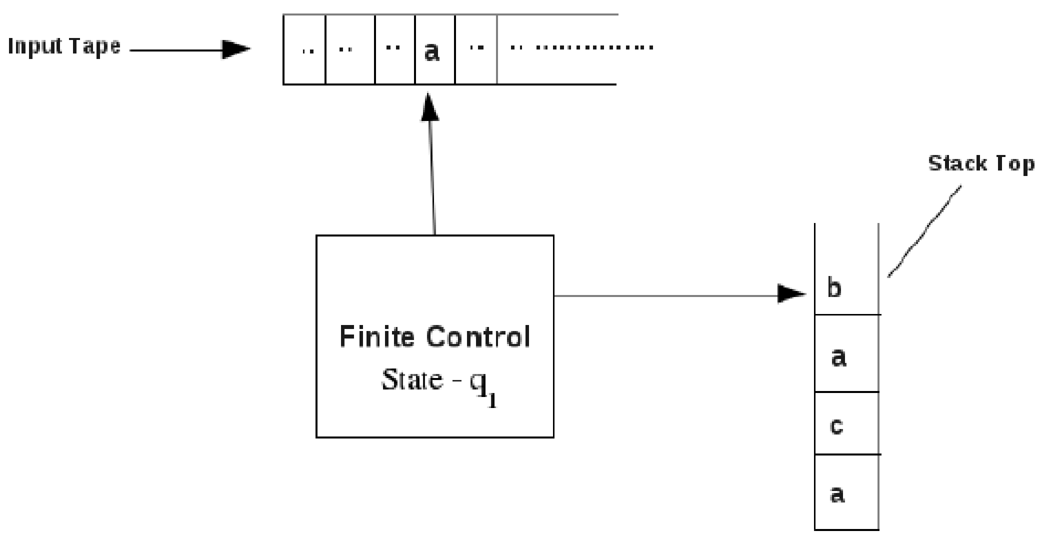
Pushing the string $abc$ on to the stack means,




Stack or Storage

Example:

Consider the following PDA,




Stack or Storage

Suppose PDA is currently in state $q_1$ and the finite control points to state $a$ and the stack contents are as shown above.

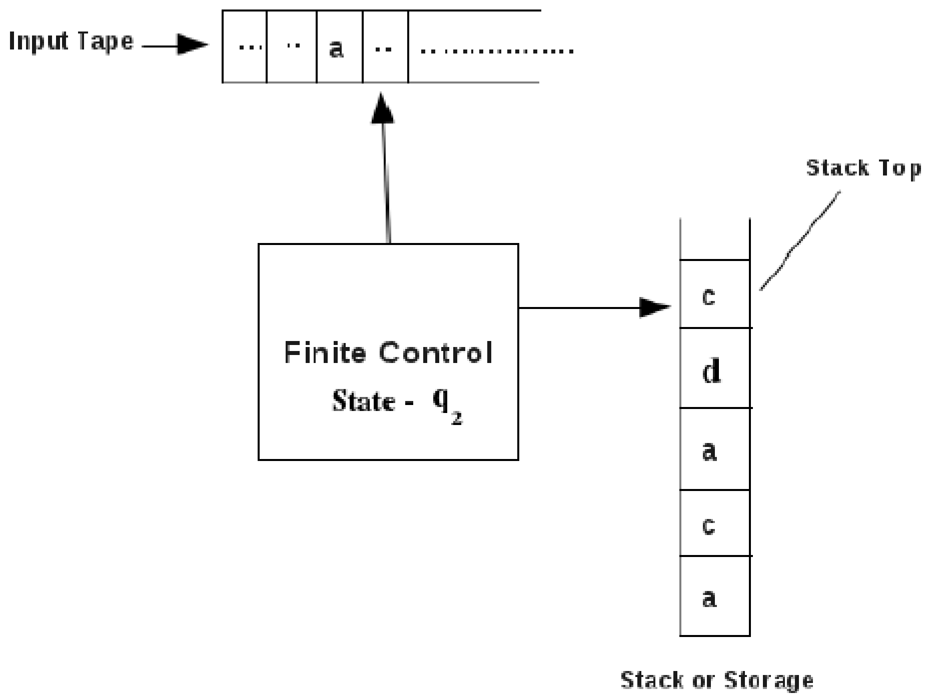
Let a transition is given as,

$$\delta = (q_1, a, b) \longrightarrow (q_2, cd)$$

This means PDA currently in state $q_1$, on reading the input symbol, $a$ and popping $b$ from the stack top changes to state

$q_2$ and pushes $cd$ on to the stack top.

The new PDA is shown below:




## 7 Language Acceptability by PDA

Example 1:

Consider a PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{s, q, f\}$

$\sum = \{a, b\}$

$\Gamma = \{a, b, c\}$

$q_0 = \{s\}$

$F = \{f\}$

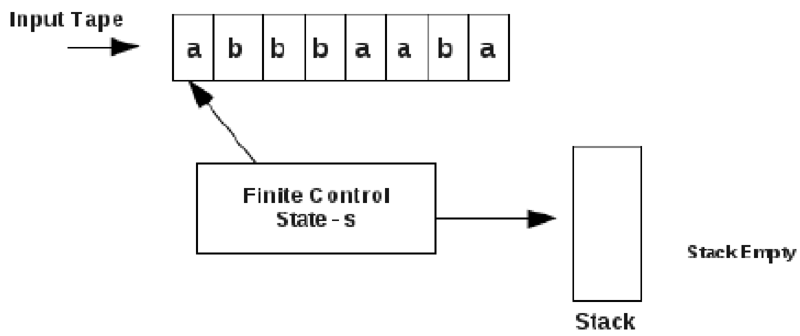
$\delta$ is given as follows:

1. $(s, a, \varepsilon) \longrightarrow (q, a)$
2. $(s, b, \varepsilon) \longrightarrow (q, b)$
3. $(q, a, a) \longrightarrow (q, aa)$
4. $(q, b, b) \longrightarrow (q, bb)$
5. $(q, a, b) \longrightarrow (q, \varepsilon)$
6. $(q, b, a) \longrightarrow (q, \varepsilon)$
7. $(q, b, \varepsilon) \longrightarrow (q, b)$
8. $(q, \varepsilon, \varepsilon) \longrightarrow (f, \varepsilon)$



Check whether the string abbbaaba is accepted by the above pushdown automation.

From the above, stack is initially empty, the start state of the PDA is $s$ and PDA points to symbol, $a$ as shown below:



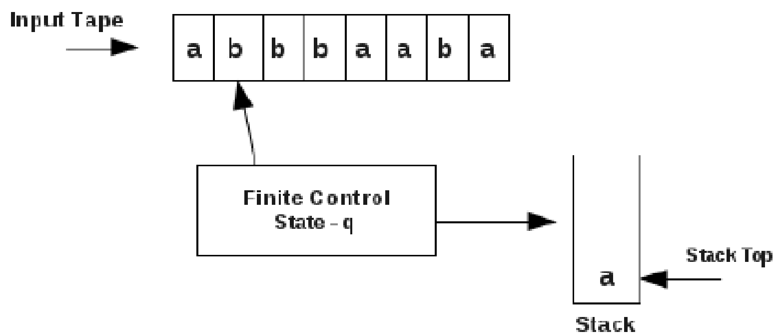

1.

The PDA is in state $s$, stack top contains symbol $\varepsilon$. Consider the transition,

1. $(s, a, \varepsilon) \longrightarrow (q, a)$

This means PDA in state $s$, reads $a$ from the tape, pops nothing from stack top, moves to state $q$ and pushes $a$ onto the stack.

PDA now is,



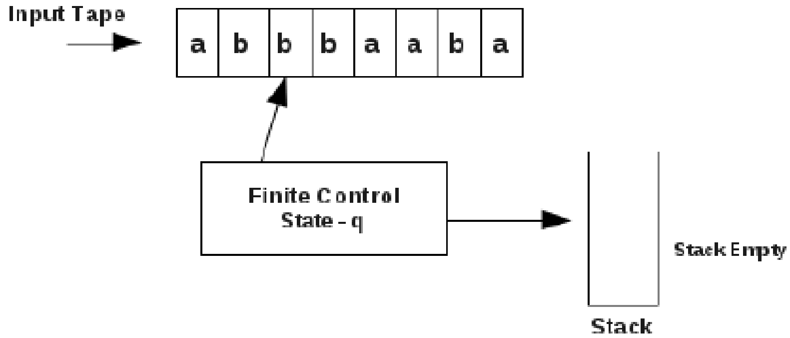

2.

The PDA is in state $q$, finite control points to symbol $b$ in the input tape, stack top contains symbol $a$. Consider the transition,

6. $(q, b, a) \longrightarrow (q, \varepsilon)$

This means PDA is in state $q$, reads $b$ from the input tape, pops $a$ from stack top, moves to state $q$ and pushes nothing onto the stack.

PDA now is,



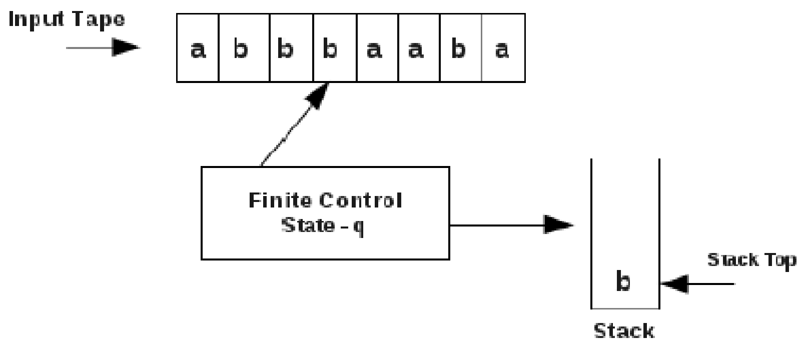

3.

The PDA is in state $q$, finite control points to symbol $b$ in the input tape, stack top contains symbol $\varepsilon$. Consider th transition,

7. $(q, b, \varepsilon) \longrightarrow (q, b)$

This means PDA is in state $q$, reads $b$ from the input tape, pops $\varepsilon$ from stack top, moves to state $q$ and pushes $b$ ont the stack.

PDA now is,



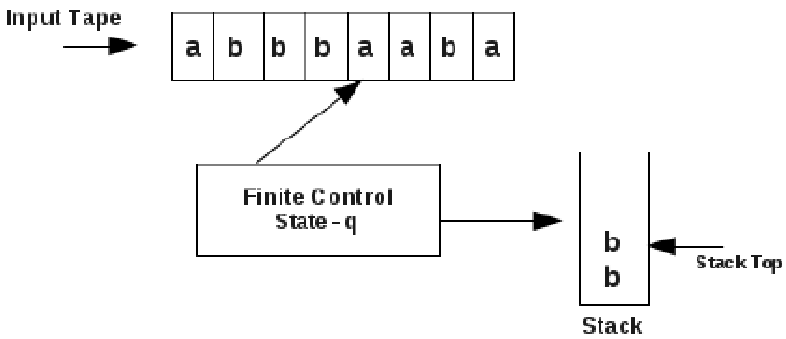

4.

The PDA is in state $q$, finite control points to symbol $b$ in the input tape, stack top contains symbol $b$. Consider th transition,

4. $(q, b, b) \longrightarrow (q, bb)$

This means PDA is in state $q$, reads $b$ from the input tape, pops $b$ from stack top, moves to state $q$ and pushes $bb$ ont the stack.

PDA now is,



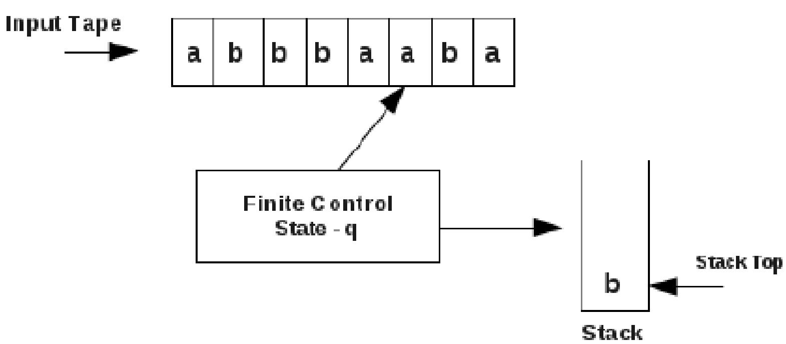

5.

The PDA is in state $q$, finite control points to symbol $a$ in the input tape, stack top contains symbol $b$. Consider th transition,

5. $(q, a, b) \longrightarrow (q, \varepsilon)$

This means PDA is in state $q$, reads $a$ from the input tape, pops $b$ from stack top, moves to state $q$ and pushes $\varepsilon$ ont the stack.

PDA now is,



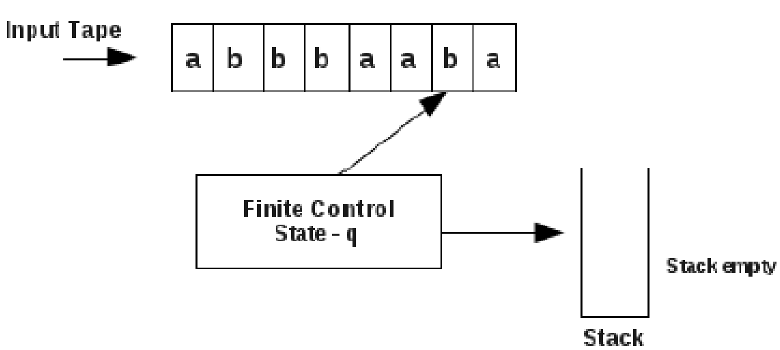

6.

The PDA is in state $q$, finite control points to symbol $a$ in the input tape, stack top contains symbol $b$. Consider th transition,

5. $(q, a, b) \longrightarrow (q, \varepsilon)$

This means PDA is in state $q$, reads $a$ from the input tape, pops $b$ from stack top, moves to state $q$ and pushes nothin onto the stack.

PDA now is,



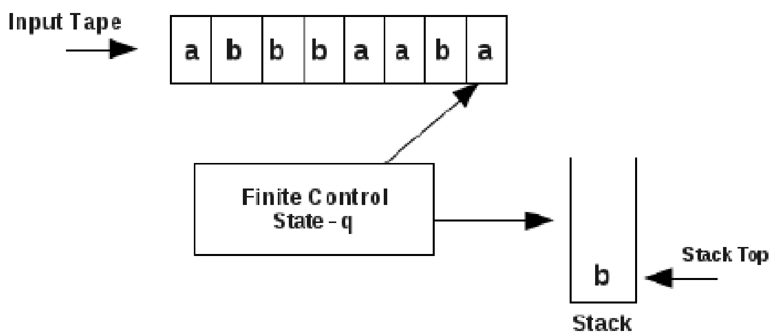
7.

The PDA is in state $q$, finite control points to symbol $b$ in the input tape, stack top contains symbol $\varepsilon$. Consider the transition,

7. $(q, b, \varepsilon) \longrightarrow (q, b)$

This means PDA is in state $q$, reads $b$ from the input tape, pops $\varepsilon$ from stack top, moves to state $q$ and pushes $b$ onto the stack.

PDA now is,



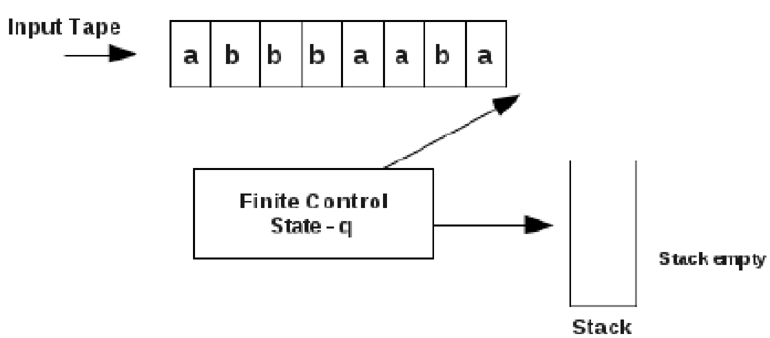

8.

The PDA is in state $q$, finite control points to symbol $a$ in the input tape, stack top contains symbol $b$. Consider the transition,

5. $(q, a, b) \longrightarrow (q, \varepsilon)$

This means PDA is in state $q$, reads $a$ from the input tape, pops $b$ from stack top, moves to state $q$ and pushes $\varepsilon$ onto the stack.

PDA now is,



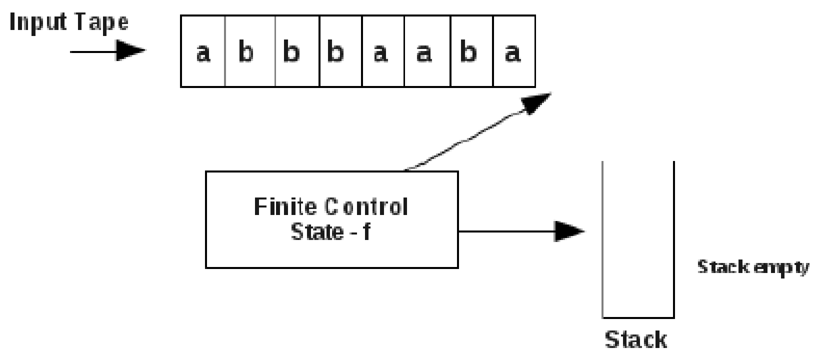

8.

The PDA is in state $q$, finite control points to symbol $\varepsilon$ in the input tape, stack top contains symbol $\varepsilon$. Consider the transition,

8. $(q, \varepsilon, \varepsilon) \longrightarrow (f, \varepsilon)$

This means PDA is in state $q$, reads $\varepsilon$ from the input tape, pops $\varepsilon$ from stack top, moves to state $f$ and pushes nothing onto the stack.

PDA now is,




Now PDA is in final state, $f$, and stack is empty. There are no more symbols in the input tape.

So the string $abbbaaba$ is accepted by the above pushdown automation.

Example 2:

1.

Consider a PDA,

$$P = (Q, \sum, \Gamma, \delta, q_0, F)$$

where

$Q = \{s, f\}$

$\sum = \{a, b, c\}$

$\Gamma = \{a, b\}$

$q_0 = \{s\}$

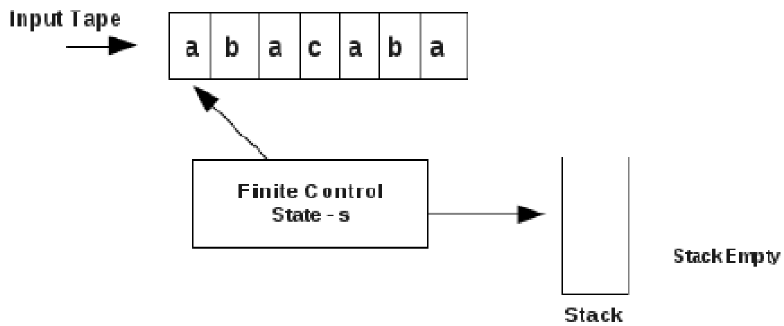
$F = \{f\}$

$\delta$ is given as follpws:

1. $(s, a, \varepsilon) \longrightarrow (s, a)$
2. $(s, b, \varepsilon) \longrightarrow (s, b)$
3. $(s, c, \varepsilon) \longrightarrow (f, \varepsilon)$
4. $(f, a, a) \longrightarrow (f, \varepsilon)$
5. $(f, b, b) \longrightarrow (f, \varepsilon)$

Check whether the string $abacaba$ is accpeted by the above pushdown automation.

From the above, stack initially is empty, the start state of the PDA is $s$ as shown below:



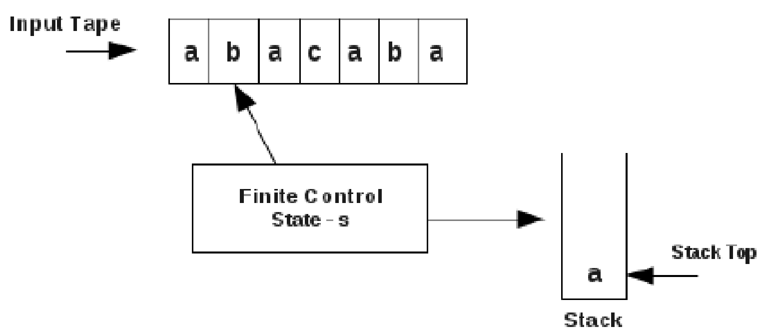

1.

The PDA is in state $s$, finite control points to symbol $a$ in the input tape, stack top contains symbol $\varepsilon$. Consider the transition,

1. $(s, a, \varepsilon) \longrightarrow (s, a)$

This means PDA is in state $s$, reads $a$ from the input tape, pops nothing from stack top, moves to state $s$ and pushes $a$ onto the stack.

PDA now is,



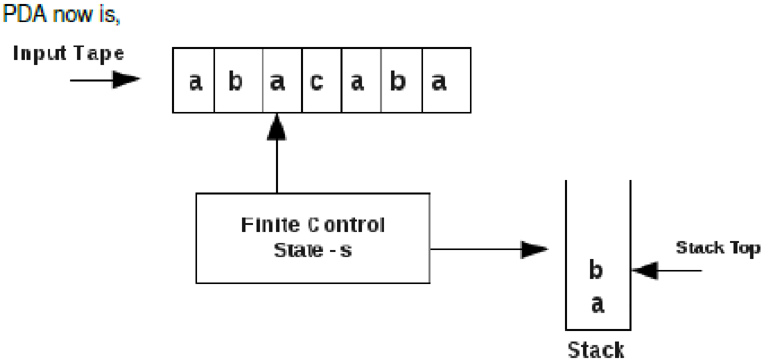

2.

The PDA is in state $s$, finite control points to symbol $b$ in the input tape, stack top contains symbol $a$. Consider the transition,

2. $(s, b, \varepsilon) \longrightarrow (s, b)$

This means PDA is in state $s$, reads $b$ from the input tape, pops nothing from stack top, moves to state $s$ and pushes $b$ onto the stack.

PDA now is,



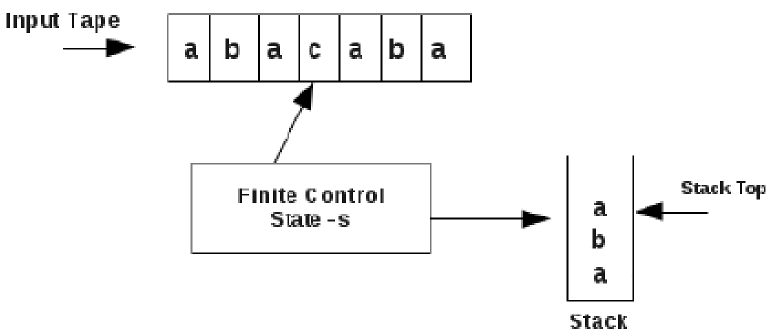
3.

The PDA is in state $s$, finite control points to symbol $a$ in the input tape, stack top contains symbol $b$. Consider the transition,

1. $(s, a, \varepsilon) \longrightarrow (s, a)$

This means PDA is in state $s$, reads $a$ from the input tape, pops nothing from stack top, moves to state $s$ and pushes $a$ onto the stack.

PDA now is,



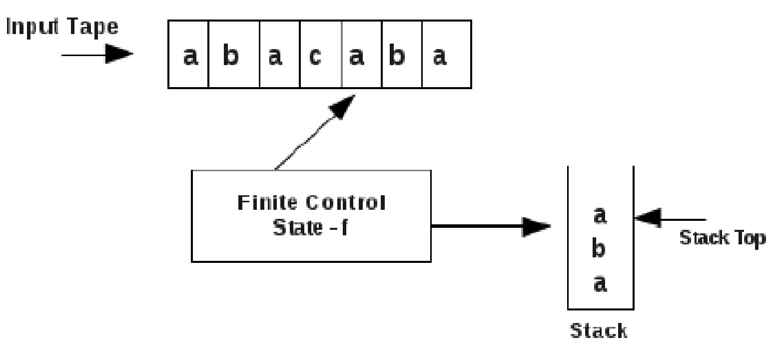

4.

The PDA is in state $s$, finite control points to symbol $c$ in the input tape, stack top contains symbol $a$. Consider the transition,

3. $(s, c, \varepsilon) \longrightarrow (f, \varepsilon)$

This means PDA is in state $s$, reads $c$ from the input tape, pops nothing from stack top, moves to state $f$ and pushes nothing onto the stack.

PDA now is,



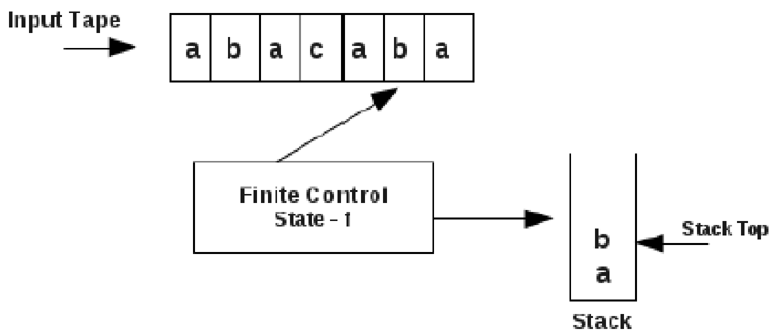

5.

The PDA is in state $f$, finite control points to symbol $a$ in the input tape, stack top contains symbol $a$. Consider the transition,

4. $(f, a, a) \longrightarrow (f, \varepsilon)$

This means PDA is in state $f$, reads $a$ from the input tape, pops $a$ from stack top, moves to state $f$ and pushes nothing onto the stack.

PDA now is,



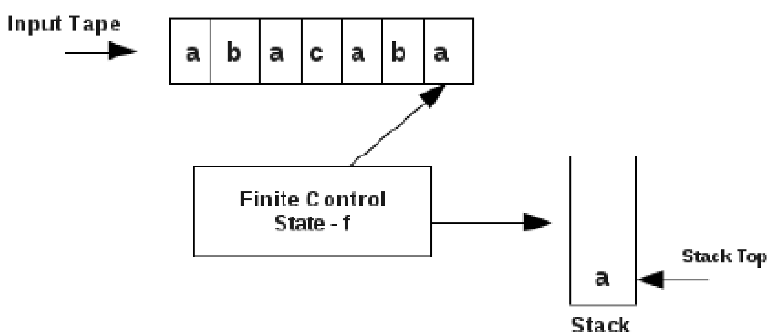

6.

The PDA is in state $f$, finite control points to symbol $b$ in the input tape, stack top contains symbol $b$. Consider the transition,

5. $(f, b, b) \longrightarrow (f, \varepsilon)$

This means PDA is in state $f$, reads $b$ from the input tape, pops $b$ from stack top, moves to state $f$ and pushes nothing onto the stack.

PDA now is,



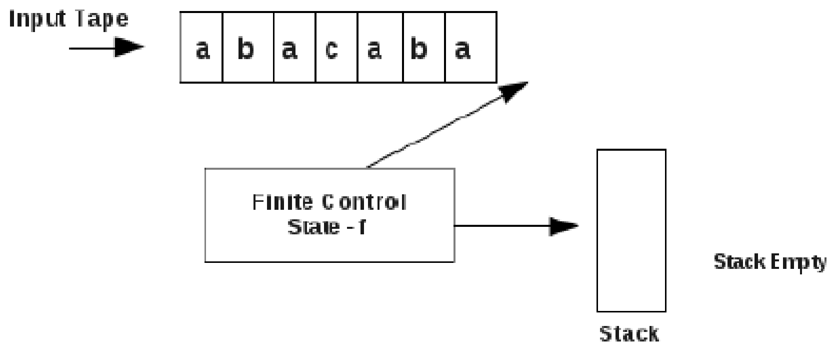

7.

The PDA is in state $f$, finite control points to symbol $a$ in the input tape, stack top contains symbol $a$. Consider the transition,

4. $(f, a, a) \longrightarrow (f, \varepsilon)$

This means PDA is in state $f$, reads $a$ from the input tape, pops $a$ from stack top, moves to state $f$ and pushes nothing onto the stack.

PDA now is,




Now there are no more symbols in the input string, stack is empty. PDA is in final state, f.

So the string $abacaba$ is accepted by the above pushdown automation.

**Exercises:**

1.

Consider a PDA,

$$P = (Q, \sum, \Gamma, \delta, q_0, F)$$

where

$Q = \{q_0, q_1, q_2, q_3\}$

$\sum = \{0, 1\}$

$\Gamma = \{0, 1\}$

$q_0 = \{q_0\}$

$F = \{q_3\}$

$\delta$ is given as follpws:

1. $(q_0, 0, \varepsilon) \longrightarrow (q_1, 0)$
2. $(q_1, 0, 0) \longrightarrow (q_1, 00)$
3. $(q_1, 1, 0) \longrightarrow (q_2, \varepsilon)$
4. $(q_2, 1, 0) \longrightarrow (q_2, \varepsilon)$
5. $(q_2, \varepsilon, \varepsilon) \longrightarrow (q_3, \varepsilon)$

Check whether the string $000111$ is accpeted by the above pushdown automation.

2.

Consider a PDA,

$$P = (Q, \sum, \Gamma, \delta, q_0, F)$$

where

$Q = \{q_0, q_1, q_2, q_3\}$

$\sum = \{a, b\}$

$\Gamma = \{a, b\}$

$q_0 = \{q_0\}$

2. $(s, b, \varepsilon) \longrightarrow (s, b)$

3. $(s, \varepsilon, \varepsilon) \longrightarrow (q, \varepsilon)$

4. $(q, a, a) \longrightarrow (q, \varepsilon)$

4. $(q, b, b) \longrightarrow (q, \varepsilon)$

5. $(q, \varepsilon, \varepsilon) \longrightarrow (f, \varepsilon)$

Check whether the string $aabbaa$ is accepted by the above pushdown automation.

## 8 Deterministic Pushdown Automata (DPDA)

A PDA is said to be deterministic, if

1. $\delta(q, a, b)$ contains at most one element, and
2. if $\delta(q, \varepsilon, b)$ is not empty, then

$\delta(q, c, b)$ must be empty for every input symbol, c.

For example, consider the following PDA,

Example 1:

Consider the PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{s, f\}$

$\sum = \{a, b, c\}$

$\Gamma = \{a, b\}$

$q_0 = \{s\}$

$F = \{f\}$

$\delta$ is given as follpws:

1. $(s, a, \varepsilon) \longrightarrow (s, a)$
2. $(s, b, \varepsilon) \longrightarrow (s, b)$
3. $(s, c, \varepsilon) \longrightarrow (f, \varepsilon)$
4. $(f, a, a) \longrightarrow (f, \varepsilon)$
5. $(f, b, b) \longrightarrow (f, \varepsilon)$

Above is a deterministic pushdown automata (DPDA) because it satisfies both conditions of DPDA.

Example 2:

Consider the PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{s, f\}$

$\sum = \{a, b, c\}$

$\Gamma = \{a, b\}$

$q_0 = \{s\}$

$F = \{f\}$

$\delta$ is given as follpws:

1. $(s, a, \varepsilon) \longrightarrow (s, a)$
2. $(s, b, \varepsilon) \longrightarrow (s, b)$
3. $(s, c, \varepsilon) \longrightarrow (f, \varepsilon)$
4. $(f, a, a) \longrightarrow (f, \varepsilon)$
5. $(f, b, b) \longrightarrow (f, \varepsilon)$

Above is a deterministic pushdown automata (DPDA) because it satisfies both conditions of DPDA.

## 9 Non-Deterministic Pushdown Automata (NPDA)

A PDA is said to be non- deterministic, if

1. $\delta(q, a, b)$ may contain multiple elements, or
2. if $\delta(q, \varepsilon, b)$ is not empty, then

   $\delta(q, c, b)$ is not empty for some input symbol, c.

Example 1:
Consider the following PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{q_0, q_1, q_2, q_3\}$

$\sum = \{a, b\}$

$\Gamma = \{0, 1\}$

$q_0 = \{q_0\}$

$F = \{q_3\}$

$\delta$ is given as follpws:

1. $(q_0, a, 0) \longrightarrow (q_1, 10), (q_3, \varepsilon)$
2. $(q_0, \varepsilon, 0) \longrightarrow (q_3, \varepsilon)$
3. $(q_1, a, 1) \longrightarrow (q_1, 11)$
4. $(q_1, b, 1) \longrightarrow (q_2, \varepsilon)$
5. $(q_2, b, 1) \longrightarrow (q_2, \varepsilon)$
5. $(q_2, \varepsilon, 0) \longrightarrow (q_3, \varepsilon)$

Above is a non deterministic pushdown automata (NPDA)

Consider the transition 1, ie.

1. $(q_0, a, 0) \longrightarrow (q_1, 10), (q_3, \varepsilon)$

Here $(q_0, a, 0)$ can go to $q_1$ or $q_3$. That this transition is not deterministic.

Example 2:

Consider the following PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{q_0, q_f\}$

$\sum = \{a, b\}$

$\Gamma = \{0, 1\}$

$q_0 = \{q_0\}$

$F = \{q_f\}$

$\delta$ is given as follpws:

1. $(q_0, \varepsilon, \varepsilon) \longrightarrow (q_f, \varepsilon)$
2. $(q_0, a, \varepsilon) \longrightarrow (q_0, 0)$
3. $(q_0, b, \varepsilon) \longrightarrow (q_0, 1)$
4. $(q_0, a, 0) \longrightarrow (q_0, 00)$
5. $(q_0, b, 0) \longrightarrow (q_0, \varepsilon)$
6. $(q_0, a, 1) \longrightarrow (q_0, \varepsilon)$
7. $(q_0, b, 1) \longrightarrow (q_0, 11)$

Above is a non deterministic pushdown automata (NPDA).

Consider the transitions, 1, 2 and 3.

1. $(q_0, \varepsilon, \varepsilon) \longrightarrow (q_f, \varepsilon)$
2. $(q_0, a, \varepsilon) \longrightarrow (q_0, 0)$
3. $(q_0, b, \varepsilon) \longrightarrow (q_0, 1)$

Here $(q_0, \varepsilon, \varepsilon)$ is not empty, also,

$(q_0, a, \varepsilon)$ and $(q_0, b, \varepsilon)$ are not empty.

Example 3:

Consider the following PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{s, p, q\}$

$\sum = \{a, b\}$

$\Gamma = \{a, b\}$

$q_0 = \{s\}$

$F = \{q\}$

$\delta$ is given as follpws:

1. $(s, a, \varepsilon) \longrightarrow (p, a)$
2. $(p, a, a) \longrightarrow (p, aa)$
3. $(p, a, \varepsilon) \longrightarrow (p, a)$
4. $(p, b, a) \longrightarrow (p, \varepsilon),\ (q, \varepsilon)$

Above is a non deterministic pushdown automata (NPDA).

This is due to the transition,

4. $(p, b, a) \longrightarrow (p, \varepsilon),\ (q, \varepsilon)$

Example 4:

Consider the following PDA,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{q_0, q_1, q_2, q_3, f\}$

$\sum = \{a, b\}$

$\Gamma = \{a, b\}$

$q_0 = \{q_0\}$

$F = \{f\}$

$\delta$ is given as follpws:

1. $(q_0, a, \varepsilon) \longrightarrow (q_1, a)$
2. $(q_1, a, a) \longrightarrow (q_1, a)$
3. $(q_1, b, a) \longrightarrow (q_2, \varepsilon)$
4. $(q_2, b, a) \longrightarrow (q_3, \varepsilon)$
5. $(q_3, b, a) \longrightarrow (q_2, \varepsilon)$
6. $(q_2, \varepsilon, a) \longrightarrow (q_2, \varepsilon)$
7. $(q_2, \varepsilon, \varepsilon) \longrightarrow (f, \varepsilon)$
8. $(q_1, \varepsilon, a) \longrightarrow (q_1, \varepsilon)$
9. $(q_1, \varepsilon, \varepsilon) \longrightarrow (f, \varepsilon)$

Above is a non deterministic pushdown automata (NPDA).

This is due to the transitions, 6 and 4, ie,

6. $(q_2, \varepsilon, a) \longrightarrow (q_2, \varepsilon)$
4. $(q_2, b, a) \longrightarrow (q_3, \varepsilon)$

Also due to the transitions, 8 and 2 and 3, ie,



8. $(q_1, \varepsilon, a) \longrightarrow (q_1, \varepsilon)$
2. $(q_1, a, a) \longrightarrow (q_1, a)$

3. $(q_1, b, a) \longrightarrow (q_2, \varepsilon)$

## 10 Design of Pushdown Automata

For every CFG, there exists a pushdown automation that accepts it.

To design a pushdown automation corresponding to a CFG, following are the steps:

Step 1:

Let the start symbol of the CFG is $S$. Then a transition of PDA is,

$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$

Step 2:

For a production of the form, $P \longrightarrow AaB$, a transition of PDA is,

$\delta(q, \varepsilon, P) \longrightarrow (q, AaB)$

For a production of the form, $P \longrightarrow a$, a transition of PDA is,

$\delta(q, \varepsilon, P) \longrightarrow (q, a)$

For a production of the form, $P \longrightarrow \varepsilon$, a transition of PDA is,

$\delta(q, \varepsilon, P) \longrightarrow (q, \varepsilon)$

Step 3:

For every terminal symbol, $a$ in CFG, a transition of PDA is,

$\delta(q, a, a) \longrightarrow (q, \varepsilon)$

Thus the PDA is,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{p, q\}$

$\sum =$ set of terminal symbols in the CFG

$\Gamma =$set of terminals and non-terminals in the CFG

$q_0 = p$

$F = q$

$\delta$ is according to the above rules.

Example 1:

Consider the following CFG,

$S \longrightarrow aA$

$A \longrightarrow aABC|bB|a$

$B \longrightarrow b$

$C \longrightarrow c$

where S is the start symbol.

Design a pushdown automata corresponding to the above grammar.

Step 1:

Here start symbol of CFG is S. A transition is,

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

Step 3:

Consider the production, $S \longrightarrow aA$.

A transition is

$$\delta(q, \varepsilon, S) \longrightarrow (q, aA)$$

Consider the production, $A \longrightarrow aABC|bB|a$.

Transitions are

$$\delta(q, \varepsilon, A) \longrightarrow (q, aABC),$$

$$\delta(q, \varepsilon, A) \longrightarrow (q, bB),$$

$$\delta(q, \varepsilon, A) \longrightarrow (q, a).$$

Consider the production, $B \longrightarrow b$.

Corresponding transition is

$$\delta(q, \varepsilon, B) \longrightarrow (q, b)$$

Consider the production, $C \longrightarrow c$

Corresponding transition is

$$\delta(q, \varepsilon, C) \longrightarrow (q, c)$$

Step 3:

The terminal symbols in the CFG are, a, b, c.

Then the transitions are,

$$\delta(q, a, a) \longrightarrow (q, \varepsilon)$$

$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

$$\delta(q, c, c) \longrightarrow (q, \varepsilon)$$

Thus the PDA is,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{p, q\}$

$\sum = \{a, b, c\}$

$\Gamma = \{S, A, B, C, a, b, c\}$

$q_0 = p$

$F = q$

$\delta$ is given as follows:

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

$$\delta(q, \varepsilon, S) \longrightarrow (q, aA)$$

$$\delta(q, \varepsilon, A) \longrightarrow (q, aABC),$$

$$\delta(q, \varepsilon, A) \longrightarrow (q, bB),$$

$$\delta(q,\varepsilon,A) \longrightarrow (q,a).$$

$$\delta(q,\varepsilon,B) \longrightarrow (q,b)$$

$$\delta(q,\varepsilon,C) \longrightarrow (q,c)$$

$$\delta(q,a,a) \longrightarrow (q,\varepsilon)$$

$$\delta(q,b,b) \longrightarrow (q,\varepsilon)$$

$$\delta(q,c,c) \longrightarrow (q,\varepsilon)$$

Example 2:

Consider the CFG,

$S \longrightarrow AB|BC$

$A \longrightarrow aB|bA|a$

$B \longrightarrow bB|cC|b$

$C \longrightarrow c$

where S is the start symbol.

Step 1:

Here start symbol of CFG is S. A transition is,

$$\delta(p,\varepsilon,\varepsilon) \longrightarrow (q,S)$$

Step 3:

Consider the production, $S \longrightarrow AB|BC$

Transitions are

$$\delta(q,\varepsilon,S) \longrightarrow (q,AB)$$

$$\delta(q,\varepsilon,S) \longrightarrow (q,BC$$

Consider the production, $A \longrightarrow aB|bA|a$

Transitions are

$$\delta(q,\varepsilon,A) \longrightarrow (q,aB),$$

$$\delta(q,\varepsilon,A) \longrightarrow (q,bA),$$

$$\delta(q,\varepsilon,A) \longrightarrow (q,a).$$

Consider the production, $B \longrightarrow bB|cC|b$

Transitions are

$$\delta(q,\varepsilon,B) \longrightarrow (q,bB)$$

$$\delta(q,\varepsilon,B) \longrightarrow (q,cC),$$

$$\delta(q,\varepsilon,B) \longrightarrow (q,b)$$

Consider the production, $C \longrightarrow c$

Transitions are

$$\delta(q,\varepsilon,C) \longrightarrow (q,c)$$



Step 3:

The terminal symbols in the CFG are, a, b, c.

Then the transitions are,

$$\delta(q,a,a) \longrightarrow (q,\varepsilon)$$
$$\delta(q,b,b) \longrightarrow (q,\varepsilon)$$
$$\delta(q,c,c) \longrightarrow (q,\varepsilon)$$

Thus the PDA is,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{p, q\}$

$\sum = \{a, b, c\}$

$\Gamma = \{S, A, B, C, a, b, c\}$

$q_0 = p$

$F = q$

$\delta$ is given as follows:

$$\delta(p,\varepsilon,\varepsilon) \longrightarrow (q,S)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,AB)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,BC)$$
$$\delta(q,\varepsilon,A) \longrightarrow (q,aB)$$
$$\delta(q,\varepsilon,A) \longrightarrow (q,bA)$$
$$\delta(q,\varepsilon,A) \longrightarrow (q,a)$$
$$\delta(q,\varepsilon,B) \longrightarrow (q,bB)$$
$$\delta(q,\varepsilon,B) \longrightarrow (q,cC)$$
$$\delta(q,\varepsilon,B) \longrightarrow (q,b)$$
$$\delta(q,\varepsilon,C) \longrightarrow (q,c)$$
$$\delta(q,a,a) \longrightarrow (q,\varepsilon)$$
$$\delta(q,b,b) \longrightarrow (q,\varepsilon)$$
$$\delta(q,c,c) \longrightarrow (q,\varepsilon)$$

Example 3:

Design a pushdown automata that accepts the language, $L = \{wcw^R | w \in (a,b)^*\}$

We learned earlier that the CFG corresponding to this language is,

$S \longrightarrow aSa$

$S \longrightarrow bSb$

$S \longrightarrow c$

where S is the start symbol.



Next we need to find the PDA corresponding to the above CFG.

Step 1:

Here start symbol of CFG is S. A transition is,

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

Step 3:

Consider the production, $S \longrightarrow aSa$

Transitions are

$$\delta(q, \varepsilon, S) \longrightarrow (q, aSa)$$

Consider the production, $S \longrightarrow bSb$

Transitions are

$$\delta(q, \varepsilon, S) \longrightarrow (q, bSb),$$

Consider the production, $S \longrightarrow c$

Transitions are

$$\delta(q, \varepsilon, S) \longrightarrow (q, c)$$

Step 3:

The terminal symbols in the CFG are, a, b, c.

Then the transitions are,

$$\delta(q, a, a) \longrightarrow (q, \varepsilon)$$

$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

$$\delta(q, c, c) \longrightarrow (q, \varepsilon)$$

Thus the PDA is,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{p, q\}$

$\sum = \{a, b, c\}$

$\Gamma = \{S, a, b, c\}$

$q_0 = p$

$F = q$

$\delta$ is given as follows:

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

$$\delta(q, \varepsilon, S) \longrightarrow (q, aSa)$$

$$\delta(q, \varepsilon, S) \longrightarrow (q, bSb)$$

$$\delta(q, \varepsilon, S) \longrightarrow (q, c)$$

$$\delta(q, a, a) \longrightarrow (q, \varepsilon)$$

$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

$$\delta(q, c, c) \longrightarrow (q, \varepsilon)$$



Example 4:

Design a pushdown automata that accepts the language, $L = \{w|w$ contains equal number of a's and b's $\}$ from the input alphabet {a,b}.

First, we need to find the CFG corresponding to this language. We learned in a previous section, the CFG corresponding to this is,

$S \longrightarrow aSbS|bSaS|\varepsilon$

where S is the start symbol.

Next we need to find the PDA corresponding to the above CFG.

Step 1:

Here start symbol of CFG is S. A transition is,

$$\delta(p,\varepsilon,\varepsilon) \longrightarrow (q,S)$$

Step 3:

Consider the production, $S \longrightarrow aSbS|bSaS|\varepsilon$

Transitions are

$$\delta(q,\varepsilon,S) \longrightarrow (q,aSbS)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,bSaS)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,\varepsilon)$$

Step 3:

The terminal symbols in the CFG are, a, b.

Then the transitions are,

$$\delta(q,a,a) \longrightarrow (q,\varepsilon)$$
$$\delta(q,b,b) \longrightarrow (q,\varepsilon)$$

Thus the PDA is,

$P = (Q,\sum,\Gamma,\delta,q_0,F)$

where

$Q = \{p,q\}$

$\sum = \{a,b\}$

$\Gamma = \{S,a,b\}$

$q_0 = p$

$F = q$

$\delta$ is given as follows:

$$\delta(p,\varepsilon,\varepsilon) \longrightarrow (q,S)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,aSbS)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,bSaS)$$
$$\delta(q,\varepsilon,S) \longrightarrow (q,\varepsilon)$$
$$\delta(q,a,a) \longrightarrow (q,\varepsilon)$$

$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

Example 5:

Design a pushdown automata that accepts the language corresponding to the regular expression, $(a|b)^*$.

First, we need to find the CFG corresponding to this language. We learned in a previous section, the CFG corresponding to this is,

$S \longrightarrow aS|bS|a|b|\varepsilon$

where S is the start symbol.

Next we need to find the PDA corresponding to the above CFG.

Step 1:

Here start symbol of CFG is S. A transition is,

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

Step 3:

Consider the production, $S \longrightarrow aS|bS|a|b|\varepsilon$

Transitions are

$$\delta(q, \varepsilon, S) \longrightarrow (q, aS)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, bS)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, a)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, b)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, \varepsilon)$$

Step 3:

The terminal symbols in the CFG are, a, b.

Then the transitions are,

$$\delta(q, a, a) \longrightarrow (q, \varepsilon)$$
$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

Thus the PDA is,

$P = (Q, \sum, \Gamma, \delta, q_0, F)$

where

$Q = \{p, q\}$

$\sum = \{a, b\}$

$\Gamma = \{S, a, b\}$

$q_0 = p$

$F = q$

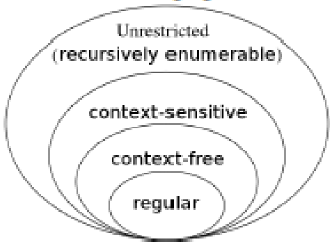
$\delta$ is given as follows:

$$\delta(p, \varepsilon, \varepsilon) \longrightarrow (q, S)$$

$$\delta(q, \varepsilon, S) \longrightarrow (q, aS)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, bS)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, a)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, b)$$
$$\delta(q, \varepsilon, S) \longrightarrow (q, \varepsilon)$$
$$\delta(q, a, a) \longrightarrow (q, \varepsilon)$$
$$\delta(q, b, b) \longrightarrow (q, \varepsilon)$$

# Part V. Pumping Lemma for Context Free Languages (CFLs)

Consider the following figure:




From the diagram, we can say that not all languages are context free languages. All context free languages are context sensitive and unrestricted. But all context sensitive and unrestricted languages are not context free from the above diagram.

We have a mechanism to show that some languages are not context free. The pumping lemma for CFLs allow us to show that some languages are not context free.

## 12 Pumping lemma for CFLs

Let $G$ be a CFG. Then there exists a constant, $n$ such that if $W$ is in $L(G)$ and $|W| \geq n$, then we may write $W = uvxyz$ such that,

1. $|vy| \geq 1$ that is either v or y is non-empty,
2. $|vxy| \leq n$ then for all $i \geq 0$,

   $uv^ixy^iz$ is in $L(G)$.

This is known as pumping lemma for context free languages.

Pumping lemma for CFLs can be used to prove that a language, L is not context free.

We assume L is context free. Then we apply pumping lemma to get a contradiction.

Following are the steps:

Step 1:

Assume $L$ is context free. Let n be the natural number obtained by using the pumping lemma.

Step 2:

Choose $W \in L$ so that $|W| \geq n$. Write $W = uvxyz$ using the pumping lemma.

Step 3:

Find a suitable $k$ so that $uv^kxy^kz \notin L$. This is a contradiction, and so L is not context free.

### Examples:

Example 1:

Show that $L = \{a^nb^nc^n | n \geq 1\}$ is not context free.

Step 1:

Assume $L$ is context free. Let n be the natural number obtained by using the pumping lemma.

Step 2:

Choose $W \in L$ so that $|W| \geq n$. Write $W = uvxyz$ using the pumping lemma.

Let $W = a^n b^n c^n$. Then $|W| = 3n > n$.

Write $W = uvxyz$, where $|vy| \geq 1$, that is, at least one of v or x is not $\varepsilon$.

Step 3:

Find a suitable $k$ so that $uv^k xy^k z \notin L$. This is a contradiction, and so L is not context free.

$uvxyz = a^n b^n c^n$. As $1 \leq |vy| \leq n$, v or x cannot contain all the three symbols a, b, c.

i) So v or y is of the form $a^i b^j$ (or $b^j c^j$) for some i,j such that $i + j \leq n$. or

ii) v or y is a string formed by the repetition of only one symbol among a, b, c.

When v or y is of the form $a^i b^j$, $v^2 = a^i b^j a^i b^j$ (or $y^2 = a^i b^j a^i b^j$). As $v^2$ is a substring of the form $uv^2 xy^2 z$, 'e cannot have $uv^2 xy^2 z$ of the form $a^m b^m c^m$. So $uv^2 xy^2 z \notin L$.

When both v and y are formed by the repetition of a single symbol (example: $u = a^i$ and $v = b^j$ for some i nd j, $i \leq n$, $j \leq n$), the string $uxz$ will contain the remaining symbol, say $a_1$. Also $a_1^n$ will be a substring of $uxz$ as $a_1$ does ot occur in v or y. The number of occurrences of one of the other two symbols in $uxz$ is less than n (recall $uvxyz = a^n b^n c^n$), nd n is the number of occurrences of $a_1$. So $uv^0 xy^0 z = uxz \notin L$.

Thus for any choice of v or y , we get a contradiction. Therefore, L is not context free.

Example 2:

Show that $L = \{a^p | p$ is a prime number$\}$ is not a context free language.

This means, if $w \in L$, number of characters in w is a prime number.

Step 1:

Assume $L$ is context free. Let n be the natural number obtained by using the pumping lemma.

Step 2:

Choose $W \in L$ so that $|W| \geq n$. Write $W = uvxyz$ using the pumping lemma.

Let p be a prime number greater than n. Then $w = a^p \in L$.

We write $W = uvxyz$

Step 3:

Find a suitable $k$ so that $uv^k xy^k z \notin L$. This is a contradiction, and so L is not context free.

By pumping lemma, $uv^0 xy^0 z = uxz \in L$.

So $|uxz|$ is a prime number, say q.

Let $|vy| = r$.

Then $|uv^q xy^q z| = q + qr$.

Since, $q + qr$ is not prime, $uv^q xy^q z \notin L$. This is a contradiction. Therefore, L is not context free.

Example 3:

Show that the language $L = \{a^n b^n c^n | n \geq 0\}$ is not context free.

Step 1:

Assume $L$ is context free. Let n be the natural number obtained by using the pumping lemma.

Step 2:

Choose $W \in L$ so that $|W| \geq n$. Write $W = uvxyz$ using the pumping lemma.

Let $W = a^n b^n c^n$.

We write $W = uvxyz$ where $|vy| \geq 1$ and $|vxy| \leq n$, then

pumping lemma says that $uv^i xy^i z \in L$ for all $i \geq 0$.

Observations:

a. But this is not possible; because if v or y contains two symbols from {a,b,c} then $uv^2xy^2z$ contains a 'b' before an 'a' or a 'c' before 'b', then $uv^2xy^2z$ will not be in L.

b. In other case, if v and y each contains only a's, b's or only c's, then $uv^ixy^iz$ cannot contain equal number of a's, b's and c's. So $uv^ixy^iz$ will not be in L.

c. If both v and y contains all three symbols a, b and c, then $uv^2xy^2z$ will not be in L by the same logic as in observation (a).

So it is a contradiction to our statement. So L is not context free.

Step 3:

Find a suitable $k$ so that $uv^kxy^kz \notin L$. This is a contradiction, and so L is not context free.

By pumping lemma, $uv^0xy^0z = uxz \in L$.

So $|uxz|$ is a prime number, say q.

Let $|vy| = r$.

Then $|uv^qxy^qz| = q + qr$.

Since, $q + qr$ is not prime, $uv^qxy^qz \notin L$. This is a contradiction. Therefore, L is not context free.