MATH1200 – Exam #1 – Differentiation and Integration

Topics and Sections that will be tested in Exam #1 with pages in the textbook:

1. Review of Algebraic derivatives
2. Review of Transcendental derivatives
3. Review of Applications of derivatives: SVA, Electrical Applications
4. Partial derivatives
5. Anti-derivatives
6. Basic integration of algebraic functions
7. Integration with change of variables (u-substitution)
8. Solve for the constant of integration
9. Applications of indefinite integrals: Given acceleration or velocity find displacement or velocity
10. Applications of indefinite integrals: Electrical formulae
11. Definite integrals (algebraic functions)
12. Areas by integration
13. **Numerical integration and Area: Trapezoid Rule (Bonus Question)**

Exam #1: 60 minutes – in class; do Practice Exams # 1 (all versions posted in Moodle)

Math1200 Practice Exam 1

Simplify all fractions, use appropriate notation. All numerical answers must be expressed as specified in each question

1. Find the derivatives of the following functions. **[3 marks each]**

a. $y = \dfrac{e^{-3\cos(2x)}}{4x^2}$ $y' =$ ____________________

b. $y = \dfrac{2x-5}{\tan\left(x^2\right)}$ $y' =$ ____________________

c. $y = 2\cos^3\left(5t - \dfrac{\pi}{6}\right)$ $y' =$ ____________________

2. Find the equation of the **tangent line** to the curve of $y = \ln(2x^2 - 1)$ at the point where $x = -1$. Write the equation of the line in the slope-intercept form $y = mx + b$. **[3 marks]**

3. Find the following integrals. **Simplify** and **express your answers with positive exponents only**.

[3 marks each]

a) $\int\left(\frac{2x^2+5}{\sqrt{x}}+e\right)dx=$

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b) $\int\frac{6-2x^3}{\left(x^4-12x+1\right)^2}\,dx$

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c) Evaluate the definite integral $\int_0^2 3x\sqrt{2x^2+1}\,dx$. Express this value as an **exact** quantity.

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4. Given $f(x,y)=3y\cos(2x^2y)-y^2e^{-3x^3}+\ln\left(\frac{2y}{x^4}\right)$, find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

[3 marks]

3. An object is accelerating according to $a(t)=4\sqrt{t}-1 \quad (m/s^2)$. **Determine the displacement, of the object after 2 seconds**, given that when t = 0 seconds, the object travels at $v(0)=15 \ (m/s)$ and when t = 0 sec, the displacement is 3 m. **[3 marks]**

4. Given the current passing through a 15 H inductor is given by $i=4e^{\sin(2t)}\ (mA)$, find the voltage across the inductor at $t=123\ ms$. Round your answer to 3 significant digits using Engineering Notation and corresponding SI prefix. **[3 marks]**

5. Find the equation of the curve whose slope is $\sqrt[3]{x}-\frac{1}{2x^2}$ and that passes through the point $(-1,\ 2)$.

[3 marks]

6. Determine the area bounded by the following curves $\begin{array}{l} y=\sqrt{4-x} \\ y=2+x \\ y=0 \end{array}$ **using Calculus**. Use a definite integral to find the **exact area**. Sketch the two curves, find the point(s) of intersection between the two curves and shade the area bounded by the two curves. Show all your work. **<u>Round your answer to 3 significant digits (if needed).</u>** **[4 marks]**

7. Use the Trapezoidal rule with $n=4$ to approximate the value of the integral: $\int_{2}^{6} \frac{x}{3x-2}\,dx.$

Give your final answer to 3 significant digits. **[3 marks]**

MATH1200 – Exam #1 – Differentiation and Integration

Topics and Sections that will be tested in Exam #1 with pages in the textbook:

1. Review Algebraic derivatives (Chapter 23 – Sections 23.4, 23.5, 23.6, 23.7)
2. Transcendental derivatives (Chapter 27 – Sections 27.1, 27.2, 27.5, 27.6)
3. Partial derivatives (Section 29.3) Pg. 889: #3 - 17 odds
4. Anti-derivatives (Section 25.1) Pg. 737: # 9 - 31
5. Basic integration of algebraic functions (Section 25.2) Pg. 741: #5 - 22
   Worksheet in Moodle
6. Integration with change of variables (u-substitution) (Section 25.2) Pg. 742: #31 - 35
   Worksheet in Moodle
7. Solve for the constant of integration (Section 25.2) Pg. 742: #37, 39, 47 - 59
   Worksheet in Moodle
8. Area under a curve using approximations (backward/forward summations) of area of rectangles (Section 25.3) Pg. 747: #5, 7, 9, 15, 17, 21, 23
9. Definite integrals (Section 25.4) Pg. 750: #3 - 33
   Worksheet in Moodle
10. Area - Trapezoidal Rule (Section 25.5) Pg. 753: #3 - 13
11. Areas by integration (Section 26.2) Pg. 769: #3 – 27
    Worksheet in Moodle
12. Applications: distance, velocity, acceleration (Section 26.1) Pg. 764, #3 - 16
    Worksheet in Moodle

Detailed Solutions

Math1200 Practice Exam 1

Simplify all fractions, use appropriate notation. All numerical answers must be expressed as specified in each question

1. Find the derivatives of the following functions. **[3 marks each]**

a. $y = \dfrac{e^{-3\cos(2x)}}{4x^2}$ QR $\Rightarrow$ $y' = \dfrac{-3\left[-\sin(2x)\,2\right]\cdot e^{-3\cos(2x)}\cdot(4x^2) - e^{-3\cos(2x)}\cdot 8x}{(4x^2)^2}$ $y'(x) = \dfrac{\overset{3}{\cancel{24}}\,x^{\cancel{2}}\sin(2x)\,e^{-3\cos(2x)} - \cancel{8x}\,e^{-3\cos(2x)}}{\underset{2}{\cancel{16}}\,x^{\cancel{4}\,3}} \Rightarrow \boxed{y'(x) = \dfrac{e^{-3\cos(2x)}\cdot\left[3x\sin(2x) - 8\right]}{2x^3}}$
b. $y = \dfrac{2x-5}{\tan\left(x^2\right)}$ Q.R $\Rightarrow$ $y' = \dfrac{2\cdot\tan(x^2) - (2x-5)\cdot\sec^2(x^2)(2x)}{\left[\tan(x^2)\right]^2}$ $\Rightarrow \boxed{y'(x) = \dfrac{2\tan(x^2) - 2x(2x-5)\sec^2(x^2)}{\tan^2(x^2)}}$
c. $y = 2\cos^3\left(5t - \dfrac{\pi}{6}\right)$ chain Rule $\Rightarrow$ $y' = 2(3)\left[\cos\left(5t - \frac{\pi}{6}\right)\right]^2\cdot\left(-\sin\left(5t-\frac{\pi}{6}\right)\right)\cdot 5$ $y = 2\cdot\left[\cos\left(5t - \frac{\pi}{6}\right)\right]^3$ $\Downarrow$ $\boxed{y'(t) = -30\cos^2\left(5t - \frac{\pi}{6}\right)\cdot\sin\left(5t - \frac{\pi}{6}\right)}$

2. Find the equation of the **tangent line** to the curve of $y = \ln(2x^2 - 1)$ at the point where $x = -1$. Write the equation of the line in the slope-intercept form $y = mx + b$. **[3 marks]**

$y = \ln(2x^2-1) \Rightarrow y'(x) = \dfrac{4x}{2x^2-1} \Rightarrow$ Slope of tangent line:

$m_{tan} = y'(-1) = \dfrac{4(-1)}{2(-1)^2 - 1} = \dfrac{-4}{1} = -4$

Equation of tangent line at $x = -1$

$y = \ln\left(2(-1)^2 - 1\right) = \ln(1) = 0$ $\boxed{\underset{(x_1,\,y_1)}{(-1,0)}}$ & $\boxed{m_{tan} = -4}$

is given by: $T: y - \overset{0}{y_1} = \overset{-4}{m}\cdot(x - \underset{-1}{x_1}) \Rightarrow y - 0 = -4(x-(-1)) \Rightarrow \boxed{y = -4x - 4}$

$x^2 \cdot x^{-1/2} = x^{3/2}$

3. Find the following integrals. **Simplify** and **express your answers with positive exponents only**.

[3 marks each]

a) $\int \left(\frac{2x^2+5}{\sqrt{x}} + e \right) dx =$ $\int \left(\frac{2x^2}{\sqrt{x}} + \frac{5}{\sqrt{x}} + e \right) dx = \int \left(2 \cdot x^{\frac{3}{2}} + 5x^{-1/2} + e \right) dx$

$= 2 \cdot \frac{x^{5/2}}{5/2} + 5 \cdot \frac{x^{1/2}}{1/2} + e \cdot x + C$

$\Rightarrow \boxed{\text{Int} = \frac{4}{5} \cdot x^{5/2} + 10 \cdot x^{1/2} + ex + C}$

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b) $\int \frac{6-2x^3}{\left(x^4-12x+1\right)^2} dx$

$= \int (x^4-12x+1)^{-2} (6-2x^3) dx = \frac{-2}{4} \int (x^4-12x+1)^{-2} \underbrace{(x^3-3)\, dx\, (4)}_{=du}$

$6-2x^3 = -2(x^3-3)$

factor out (-2)

u-substitution

$\boxed{\begin{aligned} u &= x^4-12x+1 \\ du &= (4x^3-12)\,dx \end{aligned}}$

$= -\frac{1}{2} \int u^{-2} \cdot du = -\frac{1}{2} \cdot \frac{u^{-1}}{-1} + C$

$= \frac{1}{2u} + C \Rightarrow \boxed{\text{Ans} = \frac{1}{2(x^4-12x+1)} + C}$

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c) Evaluate the definite integral $\int_0^2 3x\sqrt{2x^2+1}\ dx$. Express this value as an **exact** quantity.

$I = 3 \cdot \int_0^2 (2x^2+1)^{1/2} x\, dx = \frac{3}{4} \int_0^2 (2x^2+1)^{1/2} \underbrace{(4x\, dx)}_{du}$

$\boxed{\begin{aligned} u &= 2x^2+1 \\ du &= 4x\, dx \\ x=0 &\Rightarrow u=1 \\ x=2 &\Rightarrow u=9 \end{aligned}}$

$= \frac{3}{4} \int_1^9 u^{1/2} du = \frac{3}{4} \cdot \left. \frac{u^{3/2}}{3/2} \right]_{u=1}^{u=9} = \left. \frac{1}{2} \sqrt{u}^{\,3} \right]_{u=1}^{u=9} = \frac{1}{2} \left[(\sqrt{9})^3 - (\sqrt{1})^3 \right]$

$= \frac{1}{2}(26) = \boxed{13}$

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4. Given $f(x,y) = 3y\cos(2x^2y) - y^2e^{-3x^3} + \ln\left(\frac{2y}{x^4}\right)$, find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$\ln\left(\frac{2y}{x^4}\right) = \ln(2y) - 4\ln x$

[3 marks]

$f(x,y) = 3y\cos(2x^2 \cdot y) - y^2 e^{-3x^3} + \ln(2y) - 4\ln(x)$

$\Rightarrow \frac{\partial f}{\partial x} = 3y \cdot [-\sin(2x^2y) \cdot 4xy] - y^2 \cdot e^{-3x^3} \cdot (-9x^2) - 4 \cdot \frac{1}{x} \Rightarrow \boxed{\frac{\partial f}{\partial x} = -12xy^2\sin(2x^2y) + 9x^2y^2e^{-3x^3} - \frac{4}{x}}$

$f(x,y) = 3y\cos(2x^2y) - y^2 \cdot e^{-3x^3} + \ln(2y) - 4\ln(x)$

$\Rightarrow \frac{\partial f}{\partial y} = 3(1)\cos(2x^2y) + 3y[-\sin(2x^2y) \cdot 2x^2] - 2ye^{-3x^3} + \frac{\cancel{2}}{\cancel{2}y} \Rightarrow \boxed{\frac{\partial f}{\partial y} = 3\cos(2x^2y) - 6x^2y\sin(2x^2y) - 2ye^{-3x^3} + \frac{1}{y}}$

$v = \int a \cdot dt$ & $s = \int v \cdot dt$

5. An object is accelerating according to $a(t) = 4\sqrt{t} - 1 \quad (m/s^2)$. **Determine the displacement, of the object after 2 seconds**, given that when t = 0 seconds, the object travels at $v(0) = 15 \ (m/s)$ and when t = 0 sec, the displacement is 3 m. **[3 marks]**

$a(t) = 4\sqrt{t} - 1 \Rightarrow v(t) = \int (4 \cdot t^{1/2} - 1)\, dt$

$\boxed{v(t) = \frac{8t^{3/2}}{3} - t + 15} \left(\frac{m}{s}\right)$

t=0

V=15 m/s

15=C

$s(t) = \int v\,dt$

$\Rightarrow s(t) = \int \left(\frac{8}{3} t^{3/2} - t + 15\right) dt$

$s(t) = \frac{8}{3} \cdot \frac{t^{5/2}}{5/2} - \frac{t^2}{2} + 15t + D$

$\boxed{s(t) = \frac{16}{15} t^{5/2} - \frac{t^2}{2} + 15t + 3}$

t=0

s=3 (m)

$3 = 0 + D \Rightarrow D = 3 \ (m)$

Ans $s(2) = \frac{16}{15}(2^{5/2}) - \frac{2^2}{2} + 15(2) + 3 = \boxed{37.0\,m}$

6. Given the current passing through a 15 H inductor is given by $i = 4e^{\sin(2t)} \ (mA)$, find the voltage across the inductor at $t = 123\ ms$. Round your answer to 3 significant digits using Engineering Notation and corresponding SI prefix. **[3 marks]**

$i(t) = 4e^{\sin(2t)} \ (mA)$ $\rightarrow V_L(123\ ms) = ?$

$L = 15H$

L

$\vec{ELi}$

$\boxed{V_L = L \cdot i'(t)}$

$V_L(t) = 15 \cdot 4 \cdot e^{\sin(2t)} \cdot 2\cos(2t) = 120\cos(2t) \cdot e^{\sin(2t)}$ (mV)

$\uparrow$ L

$V_L(0.123\ \text{sec}) = 120 \cdot \cos(2 \cdot 0.123)\, e^{\sin(2 \cdot 0.123)} \approx 148\,mV$

7. Find the equation of the curve whose slope is $\sqrt[3]{x} - \frac{1}{2x^2}$ and that passes through the point $(-1,\ 2)$. **[3 marks]**

Find $y = y(x)$ s.t.

$\frac{dy}{dx} = \sqrt[3]{x} - \frac{1}{2x^2}$

$\Rightarrow y = \int \left(\sqrt[3]{x} - \frac{1}{2x^2}\right) dx = \int \left(x^{1/3} - \frac{1}{2} x^{-2}\right) dx$

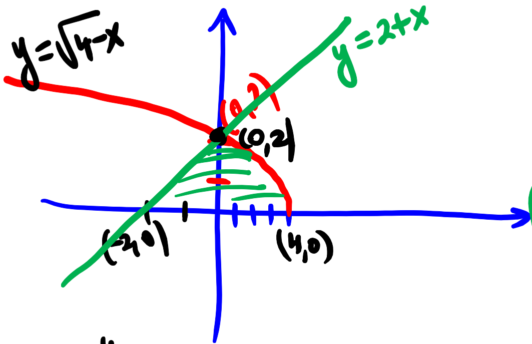
$y = \frac{x^{4/3}}{4/3} - \frac{1}{2} \cdot \frac{x^{-1}}{-1} + C$

$\Rightarrow y = \frac{3}{4} \cdot x^{4/3} + \frac{1}{2x} + C \rightarrow$ Subst $x = -1$, $y = 2 \Rightarrow$ $2 = \frac{3}{4}\sqrt[3]{(-1)^4} + \frac{1}{2(-1)} + C \rightarrow \boxed{C = \frac{7}{4}}$

Find constant of integration C

Ans: $\boxed{y = \frac{3}{4} \cdot x^{4/3} + \frac{1}{2x} + \frac{7}{4}}$

8. Determine the area bounded by the following curves $y=\sqrt{4-x}$, $y=2+x$, $y=0$ **using Calculus**. Use a definite integral to find the **exact area**. Sketch the two curves, find the point(s) of intersection between the two curves and shade the area bounded by the two curves. Show all your work. **<u>Round your answer to 3 significant digits (if needed).</u>** **[4 marks]**



Points of intersection: $\begin{cases} y=\sqrt{4-x} \\ y=2+x \end{cases} \Rightarrow \left(\sqrt{4-x}=2+x\right)^2$

$\Downarrow$

$\cancel{4}-x=\cancel{4}+4x+x^2$

$x^2+5x=0$

$x(x+5)=0$

$\begin{cases} x=0 \\ y=2 \end{cases}$ $\boxed{(0,2)}$ $\quad$ $x=-5$, $y=3$ Not possible

x_{Right}: $y=\sqrt{4-x} \Rightarrow \boxed{x=4-y^2}$ & x_{left}: $y=2+x$, $x=2-y$

$$A=\int_{y_B}^{y_T}\left(x_{Right}-x_{left}\right)\cdot dy=$$

$$A=\int_0^2\left[(4-y^2)-(2-y)\right]dy=\int_0^2\left(2-y^2+y\right)dy=\left(2y-\frac{y^3}{3}+\frac{y^2}{2}\right)\Big|_0^2$$

$$=2(2)-\frac{2^3}{3}+\frac{2^2}{2}=4-\frac{8}{3}+2=6-\frac{8}{3}=\frac{10}{3}\text{ units}^2$$

9. Use the Trapezoidal rule with $n=4$ to approximate the value of the integral: $\int_2^6 \frac{x}{3x-2}dx.$

Give your final answer to 3 significant digits. **[3 marks]**

$a=2$
$b=6$
$n=4 \Rightarrow h=\frac{b-a}{n}=\frac{6-2}{4}=1$

x	$y=\frac{x}{3x-2}$	
2	$\frac{2}{3(2)-2}=\frac{1}{2}$	y_0
3	$\frac{3}{3(3)-2}=\frac{3}{7}$	y_1
4	$\frac{4}{3(4)-2}=\frac{4}{10}=\frac{2}{5}$	y_2
5	$\frac{5}{3(5)-2}=\frac{5}{13}$	y_3
6	$\frac{6}{3(6)-2}=\frac{6}{16}=\frac{3}{8}$	y_4

+1

$$\int_2^6\frac{x}{3x-2}dx\approx\frac{h}{2}\cdot\left[y_0+2y_1+2y_2+2y_3+y_4\right]$$

$$=\frac{1}{2}\cdot\left[\frac{1}{2}+2\left(\frac{3}{7}\right)+2\left(\frac{2}{5}\right)+2\left(\frac{5}{13}\right)+\frac{3}{8}\right]$$

$$\approx\boxed{1.65}$$