MATH1200 Practice Exam #1 Differentiation; Integration and Applications of integrals

1. Find the derivatives of the following functions. **Do not simplify your answer.** **[2 marks each]**

(a) $f(x) = \ln\left(\sqrt[3]{4x+5}\right)$ **Ans:** $f'(x) =$ ______________________________
(b) $i(t) = \sin^2(2\pi t^2)$ **Ans:** $i'(t) =$ ______________________________
(c) $y = \dfrac{-3x^2 - 10}{\cos(5x)}$ **Ans:** $y' =$ ______________________________
(d) $f(x) = \sqrt{x}\ e^{-\tan(x^2)}$ **Ans:** $y' =$ ______________________________

2. Given $f(x, y) = \ln\left(\dfrac{2}{x}\right) - 4ye^{-3x^2} + \sin(2x^2) - 3\sqrt{x}$, find the partial derivative $\dfrac{\partial f}{\partial x}$. **[3 marks]**

3. Integrate the following expressions. **<u>Simplify as shown in class.</u>** **[3 marks each]**

(a) $y = \int \left(\frac{4}{\sqrt[3]{x}} - \frac{5}{3x^{10}} + 3x^4 + \pi \right) dx$

(b) $\int \left(-4x - 12x^3\right)\left(4x^2 + 6x^4 + 3\right)^5 \, dx =$

(c) $\int \left(\frac{5x^2 - 3}{\sqrt[4]{x}} + \pi \right) dx$

(d) $\int \frac{x}{\sqrt[3]{4x^4 + 4x^2 + 1}} dx$

(e) Evaluate to 3 significant digits: $\int_0^2 2x^2 \sqrt{4x^3 + 4} \, dx =$

4. Find the equation of the curve whose slope is $2x^3 - 5$ and that passes through the point $(-1,\ 2)$.

[3 marks]

5. Use the Trapezoidal rule with $n = 4$ to approximate the value of the integral: $\int_1^3 \frac{1}{2x-1}\,dx.$

Give your final answer to 3 significant digits. **[3 marks]**

6. An object is accelerating according to $a(t) = 1 - 3\sqrt{t} \quad (m/s^2)$. Determine the velocity of the object after 4 seconds, given that the object has an initial velocity of $25 \ (m/s)$. **[3 marks]**

7. A current given by $i(t)=\sqrt{t+1}\,(A)$, where time t is measured in seconds, flows through a circuit containing a $10mF$ capacitor. Assume the capacitor is initially uncharged.

(a) Determine a general expression for the voltage $V_C(t)$ (in Volts) across the capacitor.

(b) Determine the voltage across the capacitor after $124\ ms$. Round answer to 3 significant digits.

[4 marks]

8. Given the current passing through a 15 H inductor is given by $i=12e^{-100t}\,(mA)$, find the voltage across the inductor $V_L(t)$ at $t=2\ ms$. Round answer to 3 significant digits. **[3 marks]**

Detailed Solutions

MATH1200 Practice Exam #1 **Differentiation; Integration and Applications of integrals**

1. Find the derivatives of the following functions. **Do not simplify your answer.** **[2 marks each]**

(a) $f(x)=\ln\left(\sqrt[3]{4x+5}\right)$ use properties of logarithms $=$

$= \ln(4x+5)^{1/3}$

$\boxed{f(x) = \frac{1}{3}\cdot \ln(4x+5)}$ $\Rightarrow$ **Ans:** $f'(x)= \frac{1}{3}\cdot\frac{4}{(4x+5)} = \boxed{\frac{4}{3(4x+5)}}$

(b) $i(t)=\sin^2(2\pi t^2)$ $= [\sin(2\pi t^2)]^2$ $\Rightarrow$ chain Rule $i'(t) = 2\sin(2\pi t^2)\cdot\cos(2\pi t^2)\cdot 4\pi t$

Ans: $i'(t) = \boxed{8\pi t\,\sin(2\pi t^2)\cdot\cos(2\pi t^2)}$

(c) $y=\dfrac{-3x^2-10}{\cos(5x)}$ Quotient Rule $\Rightarrow$

$$y'(x) = \frac{-6x\cdot\cos(5x) - (-3x^2-10)\cdot(-\sin(5x)\cdot 5)}{[\cos(5x)]^2}$$

Ans: $y' = \boxed{\dfrac{-6x\cos(5x) - 5(3x^2+10)\sin(5x)}{\cos^2(5x)}}$

(d) $f(x)=\sqrt{x}\;e^{-\tan(x^2)}$ $= x^{1/2}\cdot e^{-\tan(x^2)}$

Product Rule $\Rightarrow$

Ans: $y' = \frac{1}{2}\cdot x^{-\frac{1}{2}}\cdot e^{-\tan(x^2)} + x^{1/2}\cdot e^{-\tan(x^2)}\cdot(-\sec^2(x^2))(2x)$

OR $\boxed{y'(x) = \dfrac{e^{-\tan(x^2)}}{2\sqrt{x}} - 2x^{3/2}\sec^2(x^2)\,e^{-\tan(x^2)}}$

2. Given $f(x,y)=\ln\left(\dfrac{2}{x}\right)-4ye^{-3x^2}+\sin(2x^2)-3\sqrt{x}$, find the partial derivative $\dfrac{\partial f}{\partial x}$. **[3 marks]**

$\ln\left(\dfrac{2}{x}\right) = \ln 2 - \ln x$, $\quad \sqrt{x} = x^{1/2}$

$\Downarrow$

$$\frac{\partial f}{\partial x} = 0 - \frac{1}{x} - 4y\cdot e^{-3x^2}(-6x) + \cos(2x^2)\cdot(4x) - 3\left(\frac{1}{2}\right)\cdot x^{-1/2}$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial x} = -\frac{1}{x} + 24xy\,e^{-3x^2} + 4x\cos(2x^2) - \frac{3}{2\sqrt{x}}}$$

3. Integrate the following expressions. **Simplify as shown in class.** **[3 marks each]**

(a) $y=\int\left(\frac{4}{\sqrt[3]{x}}-\frac{5}{3x^{10}}+3x^4+\pi\right)dx$ $=\int\left(4x^{-1/3}-\frac{5}{3}\cdot x^{-10}+3x^4+\pi\right)dx$

$=4\cdot\frac{x^{2/3}}{(2/3)}-\frac{5}{3}\cdot\frac{x^{-9}}{-9}+\frac{3x^5}{5}+\pi x+C$ $=\boxed{6x^{2/3}+\frac{5}{27x^9}+\frac{3x^5}{5}+\pi x+C}$

(b) $\int\left(-4x-12x^3\right)\left(4x^2+6x^4+3\right)^5\,dx=$ $-\frac{1}{2}\int(4x^2+6x^4+3)^5\underbrace{(4x+12x^3)\,dx\,(2)}_{du}$

$u=4x^2+6x^4+3$
$du=(8x+24x^3)\,dx$

$=-\frac{1}{2}\int u^5\cdot du=-\frac{1}{2}\cdot\frac{u^6}{6}+C=-\frac{u^6}{12}+C$

$\Rightarrow$ $\boxed{\text{Int}=-\frac{1}{12}(4x^2+6x^4+3)^6+C}$

(c) $\int\left(\frac{5x^2-3}{\sqrt[4]{x}}+\pi\right)dx$

$=\int\left(\frac{5x^2}{\sqrt[4]{x}}-\frac{3}{\sqrt[4]{x}}+\pi\right)dx=\int\left(5\cdot x^{2-\frac{1}{4}}-3x^{-\frac{1}{4}}+\pi\right)dx$

$=\int\left(5x^{7/4}-3x^{-1/4}+\pi\right)dx=5\cdot x^{11/4}\cdot\frac{4}{11}-\frac{3x^{3/4}}{3/4}+\pi x+C$

$=\boxed{\frac{20}{11}x^{11/4}-4x^{3/4}+\pi x+C}$

(d) $\int\frac{x}{\sqrt[3]{4x^4+4x^2+1}}dx$ $=\int(4x^4+4x^2+1)^{-1/3}\cdot x\,dx$

$u=2x^2+1$
$du=4x\,dx$

$=\frac{1}{4}\int(2x^2+1)^{-2/3}(4x\,dx)=\frac{1}{4}\cdot\int u^{-2/3}du$

$=\frac{1}{4}\cdot\frac{u^{1/3}}{1/3}+C=\boxed{\frac{3}{4}(2x^2+1)^{1/3}+C}$

(e) Evaluate to 3 significant digits: $\int_0^2 2x^2\sqrt{4x^3+4}\,dx=\frac{1}{6}\int_0^2(4x^3+4)^{1/2}\underbrace{(2x^2dx)(6)}_{du}$

$u=4x^3+4$
$du=12x^2dx$
$x=0\Rightarrow u=4$
$x=2\Rightarrow u=36$

$=\frac{1}{6}\cdot\int_4^{36}u^{1/2}\cdot du$

$=\frac{1}{6}\cdot\left.\frac{u^{3/2}}{(3/2)}\right]_{u=4}^{u=36}=\frac{1}{9}\cdot\left.u^{3/2}\right]_4^{36}=\frac{1}{9}\left(36^{3/2}-4^{3/2}\right)$

$=\boxed{\frac{208}{9}}$ OR $\boxed{23.1}$

Find $y = f(x)$ s.t. slope $= \frac{dy}{dx} = 2x^3 - 5$

4. Find the equation of the curve whose slope is $2x^3 - 5$ and that passes through the point $(-1,\ 2)$.

[3 marks]

$$\frac{dy}{dx} = 2x^3 - 5 \Rightarrow y = \int (2x^3 - 5)\,dx = \cancel{2}\cdot\frac{x^4}{\cancel{4}_2} - 5x + C$$

$$\Rightarrow y(x) = \frac{1}{2}\cdot x^4 - 5x + C$$

Find C: use $\boxed{\begin{matrix} x = -1 \\ y = 2 \end{matrix}}$ $\Rightarrow 2 = \frac{1}{2}(-1)^4 - 5(-1) + C$

$$2 = \frac{1}{2} + 5 + C \Rightarrow C = -\frac{7}{2}$$

$$\Rightarrow \boxed{y(x) = \frac{1}{2}\cdot x^4 - 5x - \frac{7}{2}}$$

5. Use the Trapezoidal rule with $n = 4$ to approximate the value of the integral: $\int_1^3 \frac{1}{2x-1}\,dx$.

Give your final answer to 3 significant digits. **[3 marks]**

$a = 1$, $b = 3$, $n = 4$ $\Rightarrow h = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$

x	$y = \frac{1}{2x-1}$
1	$1 = y_0$
1.5	$\frac{1}{2(1.5)-1} = \frac{1}{2} = y_1$
2	$\frac{1}{2(2)-1} = \frac{1}{3} = y_2$
2.5	$\frac{1}{2(2.5)-1} = \frac{1}{4} = y_3$
3	$\frac{1}{2(3)-1} = \frac{1}{5} = y_4$

$$\int_1^3 \frac{1}{2x-1}\,dx \simeq A_T = \frac{h}{2}\cdot\left(y_0 + 2\cdot y_1 + 2\cdot y_2 + 2\cdot y_3 + y_4\right)$$

$$= \frac{0.5}{2}\cdot\left(1 + 2\left(\frac{1}{2}\right) + 2\cdot\left(\frac{1}{3}\right) + 2\cdot\left(\frac{1}{4}\right) + \frac{1}{5}\right)$$

$$\Rightarrow \boxed{A_T = 0.842}$$

6. An object is accelerating according to $a(t) = 1 - 3\sqrt{t}\quad (m/s^2)$. Determine the velocity of the object after 4 seconds, given that the object has an initial velocity of $25\ (m/s)$. **[3 marks]**

i.e when $t = 0$, $v = 25$

$a(t) = 1 - 3\sqrt{t} = 1 - 3\cdot t^{1/2}$ $\Rightarrow$ velocity $v(t) = \int a(t)\,dt$

$$v(t) = \int (1 - 3t^{1/2})\,dt$$

$$v(t) = t - \frac{3(t^{3/2})}{3/2} + C$$

$$(*)\ \boxed{v(t) = t - 2\cdot t^{3/2} + C}\ \left(\frac{m}{sec}\right)$$

subst. $C = 25$ into $(*)$

Find C: use $t = 0$, $v = 25$ $\Rightarrow 25 = 0 - 2(0^{3/2}) + C \Rightarrow \boxed{C = 25\ m/sec}$

$$\Rightarrow \boxed{v(t) = t - 2\cdot t^{3/2} + 25}$$

$\Rightarrow$ velocity after 4 sec is $\Rightarrow v(4) = 4 - 2(4^{3/2}) + 25$

$$\boxed{v(4) = 13\ m/sec}$$

ICE $\xrightarrow{diff}$ $i(t) = C \cdot \frac{dV_c}{dt}$

At $t=0 \Rightarrow V_c = 0$ V

$C = 10$ mF

7. A current given by $i(t)=\sqrt{t+1}\,(A)$, where time t is measured in seconds, flows through a circuit containing a $10mF$ capacitor. Assume the capacitor is initially uncharged.

(a) Determine a general expression for the voltage $V_C(t)$ (in Volts) across the capacitor.

(b) Determine the voltage across the capacitor after 124 ms. Round answer to 3 significant digits.

[4 marks]

a) $V_c(t) = \frac{1}{C} \cdot \int i(t)\,dt = \frac{1}{\underbrace{10 \cdot 10^{-3}}_{C}} \cdot \int \sqrt{t+1} \cdot dt = 100 \cdot \int (t+1)^{1/2} dt$

$$\boxed{\begin{aligned} u &= t+1 \\ du &= 1 \cdot dt \end{aligned}}$$

$\Rightarrow V_c(t) = 100 \cdot \int u^{1/2} du$

$\Rightarrow V_c(t) = 100 \cdot \frac{u^{3/2}}{3/2} + K$

$$\boxed{V_c(t) = \frac{200}{3} \cdot (t+1)^{3/2} + K}$$

$0 = \frac{200}{3} \cdot (1)^{3/2} + K$

$K = -\frac{200}{3}$

(a) $\boxed{V_c(t) = \frac{200}{3} \cdot (t+1)^{3/2} - \frac{200}{3}}$ (Volts)

(b) $t = 124$ ms $= 0.124$ s.

$V_c(0.124) = \frac{200}{3} \cdot (0.124+1)^{3/2} - \frac{200}{3}$

$= 12.8$ V

8. Given the current passing through a 15 H inductor is given by $\boxed{i = 12e^{-100t}\,(mA)}$, find the voltage across the inductor $V_L(t)$ at $t = 2\ ms$. Round answer to 3 significant digits. **[3 marks]**

Given $L = 15$H
$i = 12e^{-100t}$ (mA) $\Rightarrow V_L(t) = ?$ at $t = 2$ ms.

$V_L = L \cdot \frac{di}{dt} = 15 \cdot \left(12e^{-100t}\right)'$ mV

15

$= 15 \cdot 12 \cdot e^{-100t} \cdot (-100)$

$V_L(t) = -18000\,e^{-100t}$

$V_L(0.002) = -18000\,e^{-100(0.002)} = -14737 \times 10^{-3}$ mV

sec

V

$= -14.7$ V