Q1. Frobenius series solution

$$2xy'' + (x+1)y' + 3y = 0 \quad \text{near } x = 0.$$

$$P(x) = \frac{x+1}{2x} \qquad Q(x) = \frac{3}{2x}$$

$x = 0$ is a singular point. (Regular singular because!)

$$xP(x) = \frac{1}{2} + \frac{x}{2} \quad \text{analytic for } x > 0$$

$$x^2 Q(x) = \frac{3}{2}x \quad \text{analytic for } x > 0$$

$$p_0 = \frac{1}{2},\ p_1 = \frac{1}{2},\ p_2 = 0 \ \ldots$$

$$q_0 = 0,\ q_1 = \frac{3}{2},\ q_2 = 0 \ \ldots$$

Indicial Equation

$$f(m) = m(m-1) + mp_0 + q_0 = 0$$

$$= m(m-1) + m\frac{1}{2} = 0$$

$$m(m - 1 + 1/2) = 0$$

$$m = 0,\ m = 1/2$$

$$m_1 = 1/2,\ m_2 = 0 \quad (m_1 - m_2 = 1/2) \text{ not an integer}$$

Two frobenius series solution exists.

Let us take $m_1 = 1/2$, we assume the Frobenius series solution

$$y = x^{1/2} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

where a_n are given by the recursion formula:

$$a_n f(m+n) + a_0 (m p_n + q_n) + a_1 ((m+1) p_{n-1} + q_{n-1})$$
$$+ \cdots a_{n-1} ((m+n-1) p_1 + q_1) = 0. \quad — (*)$$

$n = 1.$ $\quad f(m) = m(m - 1/2)$, $\boxed{m = m_1 = 1/2}$

$$a_1 f(1/2 + 1) + a_0 \left(\frac{1}{2} p_1 + q_1\right) = 0$$

$$a_1 f\left(\frac{3}{2}\right) = -a_0 \left(\frac{1}{4} + \frac{3}{2}\right)$$

$$a_1 = -\frac{7}{6} a_0$$

$n = 2$

$$a_2 f\left(\frac{1}{2} + 2\right) + a_0 \left(\frac{1}{2} p_2 + q_2\right) + a_1 \left(\frac{3}{2} p_1 + q_1\right) = 0$$

$$a_2 f\left(\frac{5}{2}\right) + 0 + a_1 \left(\frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2}\right) = 0$$

$$a_2 f\left(\frac{5}{2}\right) = -a_1 \frac{9}{4}$$

$$a_2 = \frac{21}{40} a_0$$

one of the solution

$$y_1(x) = a_0 x^{1/2}\left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \cdots\right)$$

the solution converges for $x > 0$.

The second Frobenius series solution

$$y_2(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \qquad a_0 \neq 0$$

a_n are given by the recursion formula ⊗

$n = 1$,

$$a_1 f(0+1) + a_0 (0 \cdot p_1 + q_1) = 0$$

$$\cancel{a_1 f(1) + a_0 (\tfrac{7}{6}) = 0}$$

$$a_1 \times \frac{1}{2} = -\frac{3}{2} a_0$$

$$a_1 = -3a_0$$

$n = 2$

$$a_2 = 2a_0$$

solution :-

$$y_2(x) = a_0\left(1 - 3x + 2x^2 - \cdots\right)$$

It converges for

$-\infty < x < \infty$.

Q2. $f(x) = (1-x^2)x^p \qquad x \in [0,1]$

the Bessel series is given by

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$$

where λ_n's are positive zeros of $J_p(x)$ and

$$a_n = \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x)\, dx$$

$$= \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^1 (1-x^2) x^{p+1} J_p(\lambda_n x)\, dx$$

let $\lambda_n x = t \qquad x = \frac{t}{\lambda_n}$ & $dx = \frac{dt}{\lambda_n}$

$$= \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^{\lambda_n} \left(1 - \frac{t^2}{\lambda_n^2}\right) \frac{t^{p+1}}{\lambda_n^{p+1}} J_p(t) \frac{dt}{\lambda_n}$$

$$= \frac{2}{J_{p+1}^2(\lambda_n)\, \lambda_n^{p+4}} \int_0^{\lambda_n} (\lambda_n^2 - t^2)\, t^{p+1} J_p(t)\, dt$$

$$I = \int_0^{\lambda_m} (\lambda_m^2 - t^2)\, t^{p+1} J_p(t)\, dt \quad \left(\because \frac{d}{dx}\left(x^p J_p(x)\right) = x^p J_{p-1}(x) \right)$$

$$= \int_0^{\lambda_m} (\lambda_m^2 - t^2) \frac{d}{dt}\left(t^{p+1} J_{p+1}(t)\right) dt$$

$$= (\lambda_m^2 - t^2)\, t^{p+1} J_{p+1}(t) \Big|_0^{\lambda_m} + 2 \int_0^{\lambda_m} t^{p+2} J_{p+1}(t)\, dt$$

$$= 2 \int_0^{\lambda_n} t^{p+2} J_{p+1}(t)\, dt = 2 \int_0^{\lambda_n} \frac{d}{dt}\left(t^{p+2} J_{p+2}(t)\right) dt$$

$$= 2\, t^{p+2} J_{p+2}(t) \Big|_0^{\lambda_n}$$

$$= 2 \lambda_m^{p+2} J_{p+2}(\lambda_n).$$

$$\Rightarrow \quad a_n = \frac{2}{J_{p+1}^2(\lambda_n)\, \lambda_n^{p+4}} \cdot I$$

$$= \frac{4 J_{p+2}(\lambda_n)}{J_{p+1}^2(\lambda_n)\, \lambda_n^2}$$

$$\Rightarrow \quad (1 - x^2)\, x^p = \sum_{n=1}^{\infty} \frac{4 J_{p+2}(\lambda_n)}{J_{p+1}^2(\lambda_n)\, \lambda_n^2} J_p(\lambda_n x)$$

$$x \in (0,1).$$

Q3. $(x-1)y'' + (3x-4)y' - (4x+5)y = 0, \; y(1) = 0.$

Let $x - 1 = t,$ $\quad \frac{dy}{dx} = \frac{dy}{dt}, \; \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$

$\Rightarrow x = t+1.$

$y(0) = 0$

the equation becomes:-

$$t\frac{d^2y}{dt^2} + (3t-1)\frac{dy}{dt} - (4t+9)y = 0, \; y(0) = 0.$$

Now taking Laplace transform w.r.t. t.

let $L[y(t)] = Y(p)$

$$\Rightarrow L\left[t\frac{d^2y}{dt^2}\right] + 3L\left[t\frac{dy}{dt}\right] - L\left[\frac{dy}{dt}\right] - 4L[ty] - 9L[y] = 0$$

$$\Rightarrow -\frac{d}{dp}\left[p^2Y - py(0) - y'(0)\right] - 3\frac{d}{dp}\left[pY - y(0)\right] - (pY - y(0)) + 4\frac{dY}{dp} - 9Y = 0$$

$$\Rightarrow (p^2 + 3p - 4)\frac{dY}{dp} = -(3p+12)Y$$

$$\Rightarrow \quad \frac{dY}{dp} = \frac{-3(p+4)}{(p^2+3p-4)} Y = \frac{-3(p+4)}{(p+4)(p-1)} Y$$

$$\frac{dY}{dp} = \frac{-3}{(p-1)} Y \Rightarrow Y(p) = \frac{c}{(p-1)^3}$$

$$\boxed{\ln Y = -3 \ln(p-1) + \ln c}$$

$$Y = \frac{c}{(p-1)^3}$$

taking inverse transform :-

$$y(t) = L^{-1}\left(\frac{c}{(p-1)^3}\right) = c t^2 e^t$$

$$\Rightarrow \boxed{y(x) = c(x-1)^2 e^{(x-1)} \qquad c \in \mathbb{R}}$$

Q4. Prove $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

we know by generating function :-

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Now, differentiating both sides w.r.t t,

we get

$$\frac{(x-t)}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

Comparing the coefficient of t^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow \boxed{(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)}$$

Use it to show that $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Differentiating the recursion formula w.r.to x

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1}(x)$$

now, we know

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

$$\Rightarrow \quad xP'_n(x) = nP_n(x) + P'_{n-1}(x)$$

Sub. in the above formula for $xP'_n(x)$

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)\left(nP_n(x) + P'_{n-1}(x)\right) - nP'_{n-1}(x)$$

$$\Rightarrow \quad (n+1)P'_{n+1}(x) = (2n+1)(n+1)P_n(x) + (n+1)P'_{n-1}(x)$$

$$\Rightarrow \quad \boxed{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$$

Q5. Fourier Series of

$$f(x) = \cos\left(\frac{x}{2}\right) \quad -\pi \leq x \leq \pi.$$

The Fourier series is given by :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) dx$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} \cos\left(\frac{x}{2}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left[\frac{\cos\left(nx - \frac{x}{2}\right) + \cos\left(nx + \frac{x}{2}\right)}{2}\right]dx$$

$$= \frac{1}{\pi}\left[\frac{2}{2n-1}\sin\frac{(2n-1)x}{2} + \frac{2}{2n+1}\sin\frac{(2n+1)x}{2}\right]_0^{\pi}$$

$$= \frac{4\cos(n\pi)}{1-4n^2}$$

$b_n = 0 \quad \because \cos\left(\frac{x}{2}\right)$ is an even function.

Hence, the Fourier series is :-

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{(-1)^n}{(1-4n^2)} \cos nx .$$

$$x \in [-\pi, \pi].$$

Now to ~~show~~ evaluate the series

$$\sum_{n=1}^{\infty} \frac{1}{1-4n^2}$$

we use $x = \pi$, in the Fourier series,

$$f(\pi) = 0$$

$$0 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left(\frac{4}{\pi}\right) \frac{(-1)^n (-1)^n}{1-4n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{1-4n^2} = \frac{-1}{2}}$$