Disco-Key

1

1. Need to use this : $4m \le n^2$ (T)

$4m = 20, \ n^2 = 16$ so $4m > n^2$

2. $\{d_i\}_{i=1}^{n} \rightarrow$ degrees. (T)

HL $\Rightarrow \quad d_1 + \cdots d_n = 2m$

$\varepsilon = \min(d_1, \ldots d_n)$; $d_i \ge \varepsilon \quad \forall i = 1, \ldots n$

$\Rightarrow d_1 + \cdots d_n \ge n\varepsilon$

Similarly, $\delta = \max(d_1, \ldots d_n) \Rightarrow d_1 + \cdots d_n \le n\delta$

$\Rightarrow \varepsilon \le \frac{2m}{n} \le \delta$

3. $G_1 = (V_1, E_1)$; $G_2 = (V_2, E_2)$; $G_1 \cong G_2$ (T)

$f : V_1 \rightarrow V_2$ is 1-1 & onto & preserve vertex adjacencies. $\Rightarrow$ for any $u, v \in V_1$

if $\{u, v\} \notin E_1 \Rightarrow \{f(u), f(v)\} \notin E_2 \Rightarrow f$ preserves adj for $\bar{G}_1$ & $\bar{G}_2$ $\Rightarrow$ $f : \bar{G}_1 \rightarrow \bar{G}_2$ is an isomorphism.

(F)

4.

	G_1		G_2
	x		$n-x$
Edges	$\frac{1}{2}x(x-1)$	+	$\frac{1}{2}(n-x)(n-x-1)$

$= x^2 - nx + \frac{1}{2}n(n-1) \equiv E$

$\downarrow$

$\frac{dE}{dx} = 0 \Rightarrow x = n/2 \Rightarrow \frac{d^2E}{dx^2} > 0$

$\text{Min } E = \frac{1}{4}n(n-2)$

5. $\underbrace{d_1, \ldots d_{n-2}}_{\text{non-pendant}}$; T has exactly $(n-1)$ edges (T)

$\underbrace{1 + 1}_{\text{pendant}} + d_1 + \cdots d_{n-2} = 2(n-1)$

$\Rightarrow d_1 + \cdots d_{n-2} = 2(n-2)$

$\downarrow$

each $\{d_i\}_{i=1}^{n-2}$ must be 2

(6) Use $n = mq+1 = \frac{mp-1}{m-1}$; use $m=2$ (F)

(7) $v \in V(G_1) \cap V(G_2)$; $a \in V(G_1)$, $b \in V(G_2)$ (T)

but $a, b \notin V(G_1) \cap V(G_2) \Rightarrow$ $a-v$ path P_1 in G_1

Let $P_1: a = x_0 x_1 \ldots x_k = v$. Let i be the smallest path $\ni x_i \in G_2$. Then $i \geq 1$ Let Q be the $x_i - b$ path in G_2. Then $x_0 x_1 \ldots x_{i-1}$ Q is an $a-b$ path in G (no x_j can occur in Q for $j < i$)

(8) $x = x^{-1}$ $\forall x \in G$ (F)

$\Rightarrow x^2 = e$

$x^n = e$, n even $\Rightarrow$ every integral power of x is e / x

$= ex$; n odd

no element $x \in G$ can be a generator

(9) $f : G \to H \times K$, $f(0) = (0,0)$, $f(1) = (1,1)$ (F)

$f(2) = (2,0)$; $f(3) = (0,1)$

$f(4) = (1,0)$; $f(5) = (2,1)$

$\Downarrow$

no two elements of G have the same image in $H \times K$ under $f \Rightarrow f$ is 1-1 . $|G| = |H \times K| \Rightarrow f$ is onto

$f(a+b) = f(a) + f(b)$ Easy check

$\longrightarrow f$ homom

11. k – # of spanning trees of K_n containing the edge e.

3 1 n^{n-2} spanning trees in total, each of which contains $(n-1)$ edges

3 2 $\frac{n(n-1)}{2}$ edges in K_n. Each of these edges is contained in K different spanning trees.

Consider the disjoint union of all spanning trees of K_n. We have 2 ways to count all the edges.

3

By 1: $(n-1)n^{n-2}$ $\Rightarrow$ $Kn/_2 = n^{n-2}$

By 2: $Kn(n-1)/2$ $\Rightarrow$ $K = 2n^{n-3}$

$\Downarrow$

$n^{n-2} - 2n^{n-3}$ spanning trees of K_n which

1

don't contain e ~~or~~

circles

(11)

Consider a person A and 9 others in the room.

(3) Case I: Spose A has at least 4 acquaintances, then if any of those acquaintances know each other, we have 3 mutual friends. If none of those acquaintances know each other, we have at least 4 mutual strangers, namely A's acquaintances.

(2) Case II: A has more than 5 strangers (less than 4 acquaintances). Consider B in the set of A's strangers. B has at least 3 acquaintances or 3 strangers.

(2) Suppose B has at least 3 acquaintances, then none of these acquaintances know each other, otherws we have 3 mutual acquaintances. But, since A doesn't know any of these people, B's 3 acquaintances and A form a group of 4 mutual strangers.

(3) OTOH, if B has at least 3 strangers, then if these strangers all know each other, we would have 3 mutual acquaintances. Thus, suppose there are at least 2 people of B's strangers that don't know each other. But then, we would have these 2 people who don't know each other, B, and A as a group of mutual strangers.

In all cases, we have 3 mutual acquaintances / 4 mutual strangers.