Part II - Graph Theory

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Contents

0 Introduction

07 Oct 2022, Lecture 1

Notation. We write $[n]$ for $\{1, 2, \dots, n\}$. For a set X and $k \in \mathbb{N}$, define $X^{(k)} = \{S \subset X \mid |S| = k\}$, i.e. the set of all subsets of size k.

1 Fundamentals

Definition 1.1. A **graph** is an object $G = (V, E)$ where V is a set and $E \subseteq V^{(2)}$.

V is the set of vertices, and E is the set of edges.

$V(G)$ will denote V, $E(G)$ will denote E, and we define $|G| = |V(G)|$ (sometimes called the order) and $e(G) = |E(G)|$ (sometimes called the size).

Example 1.1. The **complete graph** on n vertices is denoted K_n. This is the graph where $V(K_n) = [n]$ and $E(K_n) = [n]^{(2)}$.

Remark. We assume the following:

- We don't order edges;
- We don't allow loops (an edge joining a vertex to itself);
- We don't allow multiple edges;
- Most of the time, $V(G)$ will be finite (we will explicitly say when it's not).

Example 1.2. The **empty graph** on n vertices, denoted $\overline{K_n}$, has $V(\overline{K_n}) = [n]$ and $E(\overline{K_n}) = \emptyset$.

Example 1.3. The path of length n, denoted P_n, is a path: it has $V(P_n) = [n+1]$ and $E(P_n) = \{\{i, i+1\} \mid 1 \leq i \leq n\}$.

Example 1.4. The cycle of length n, denoted C_n, has $V(C_n) = [n]$ and $E(C_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.

Let G be a graph and $x \in V(G)$. The **neighborhood** of x is $N(x) = \{y \mid xy \in E(G)\}$, i.e. all the vertices connected to x. If $y \in N(x)$, we write $x \sim y$ and say y is a **neighbor** of x or that y is **adjacent** to x.

The **degree** of x is $\deg(x) = |N(x)|$.

Just as a formality, we define graph isomorphism: let G, H be graphs. A graph isomorphism is a bijection $\phi : V(G) \to V(H)$ such that it maps edges to edges, i.e. $uv \in E(G) \iff \phi(u)\phi(v) \in E(H)$.

Definition 1.2 (Subgraph)**.** We say H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Two subgraph types that are important enough to have their own notation:

- If G is a graph, and $xy \in E(G)$, define $G - xy$ to be the graph $(V(G), E(G) \setminus \{xy\})$.
- For $x, y \in V(G)$, define $G + xy$ to be the graph $(V(G), E(G) \cup \{xy\})$.

Definition 1.3 (Path)**.** Let G be a graph, $x, y \in V(G)$. A $x - y$ **path in** G is a sequence $x_1, \ldots, x_k$ where $x_1 = x,\ x_k = y$ and $x_i x_{i+1} \in E(G)\ \forall 1 \leq i \leq k - 1$ and all the x_i are distinct.

Definition 1.4. A graph is **connected** if $\forall x \neq y \in V(G)$, there exists an $x - y$ path in G.

Remark. A little annoyingly, if P is a $x - y$ path and P' is a $y - z$ path, then the concatenation PP' may not be a path (since the vertices of the new path might not be unique).

So let an $x - y$ **walk** in a graph G be a sequence $x_1, \ldots, x_k$ where $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E(G)\ \forall 1 \leq i \leq k - 1$. Then a concatenation of walks is again a walk.

Proposition 1.1. If W is an xy walk, then W contains a xy path.

Proof. Let $W' \subseteq W$ be a minimal xy walk. We claim this is a path. If not, then some vertex x_i must be visited at least twice, say $W' = x_1 x_2 \ldots x_i \ldots x_i x_l \ldots x_k$. Then take $W'' = x_1 x_2 \ldots x_i x_l \ldots x_k$. This contradicts the minimality of W', so we're done. □

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Remark. We may define a **distance** on $V(G)$: for $x, y \in V(G)$, let $d(x, y)$ be the length of the shortest xy path. If G is connected, then this distance defines a metric on $V(G)$.

1.1 Trees

Definition 1.5. A graph G is **acyclic** if it does not contain a cycle as a subgraph.

Definition 1.6. A graph G is a **tree** if it is acyclic and connected.

Proposition 1.2. The following are equivalent:

1. G is a tree;
2. G is minimally connected ($\forall xy \in E(G)$, $G - xy$ is not connected);
3. G is maximally acyclic ($\forall xy \notin E(G)$, $G + xy$ contains a cycle).

Proof. (a) $\implies$ (b): A tree is connected. Assume for contradiction that $\exists xy \in E(G)$ such that $G - xy$ is connected. Let P be a xy path in $G - xy$. But then P defines a cycle in G, contradiction.

(b) $\implies$ (a): Minimally connected implies connected. For acyclicness, assume for contradiction that G contains a cycle C. Let $xy \in E(C)$. We claim that $G - xy$ is connected. Choose $u \neq v \in V(G - xy)$. Let P be a uv path in G. If P does not contain xy, we're done. If P does contain xy, then take paths $u \to x$; $x \to y$ along our cycle without using xy; $y \to v$. The concatenation of these gives a uv walk, which contains a uv path. Hence $G - xy$ is connected, contradiction.

(a) $\implies$ (c): A tree is acyclic. Let $xy \notin E(G), x \neq y$. Let P be a xy path. Then P defines a cycle in $G + xy$.

(c) $\implies$ (a): We have acyclicity. If G is not connected, $\exists x \neq y \in V(G)$ with no xy path. Then $G + xy$ is acyclic. $\square$

Definition 1.7. If T is a tree and $v \in V(T)$ with $\deg(v) = 1$, we call v a **leaf**.

Definition 1.8. Let G be a graph and $X \subseteq V(G)$. Then

$$G[X] = (X, E(G) \cap \{(u, v) \mid u, v \in X\}).$$

We call this **the graph induced on** X.

Definition 1.9. If $x \in V(G)$, define $G - x = G[V(G) \setminus \{x\}]$.

Proposition 1.3. Let T be a tree, $|T| \geq 2$. Then T has a leaf.

Proof. Let $P = x_1 \dots x_k$ be the a longest possible path in T. Note $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$. If $x_i \sim x_k$ for some $1 \leq i \leq k-2$, there is a cycle in T, contradiction. Thus $N(x_k) = \{x_{k-1}\} \implies X_k$ is a leaf. $\square$

Remark. We can show that any T has two leaves, but we can't do any better (consider a path).

Remark. We could have also proved this by taking a non-backtracking walk in G (i.e. go out of a different edge then you came in on). Either we return (hence have a cycle) or get stuck (hence have a leaf).

Proposition 1.4. Let T be a tree on $n \geq 1$ vertices. Then $e(G) = n - 1$.

Proof. By induction. $n = 1$ is trivial. Assume the claim holds for n. Take a tree T with $n + 1$ vertices. Let $x \in V(T)$ be a leaf. Then $T - x$ is connected and acyclic, therefore a tree, thus $e(T - x) = n - 1$. But $e(G) = e(G - x) + 1$ and $|V(G)| = |V(G - x)| + 1$, hence we're done. $\square$

Definition 1.10. Let G be a connected graph. Then a subgraph T of G is a **spanning tree** if T is a tree on $V(G)$.

Proposition 1.5. Every connected graph contains a spanning tree.

Proof. Start with the graph G, then throw away edges of $E(G)$ one by one subject to keeping the graph connected. At some point, the removal of any further edge will disconnect the graph, at which point we have a minimal connected subgraph of G, which by Prop. 1.2 is a tree. □

1.2 Bipartite graphs

Definition 1.11. Let $G = (V, E)$ be a graph. G is **bipartite** if there exists a partition $V = A \cup B$ such that $E(G) \subseteq \{uv \mid u \in A, v \in B\}$.

Definition 1.12. The **complete bipartite graph** $K_{n,m}$ is the graph with vertex set $A \cup B$, $A = \{x_1, \ldots, x_n\}, B = \{y_1, \ldots, y_m\}$ and edge set $E(K_{n,m}) = \{x_i y_j \mid x_i \in A, y_j \in B\}$.

Remark. There obviously exist non-bipartite graphs: odd cycles are not bipartite.

Definition 1.13. A **circuit** is a sequence $x_1, x_2, \ldots x_l x_{l+1}$, where $x_i x_{i+1} \in E(G)$ and $x_{l+1} = x_1$. The length of this circuit is l. We say a circuit is **odd** if its length is odd.

Proposition 1.6. Let C be an odd circuit in a graph G. Then C contains an odd cycle.

Proof. Let $x_1 x_2 \ldots x_i x_{i+1} \ldots x_i x_k \ldots x_l x_1$ be an odd circuit. Consider the circuits $C_1 = x_1 \ldots x_i x_k \ldots x_l x_1$ and $C_2 = x_i x_{i+1} \ldots x_{k-2} x_i$. Then one of C_1, C_2 has odd length and is strictly shorter, so we're done by induction. □

Theorem 1.7. Let G be a graph. Then

$$G \text{ is bipartite} \iff G \text{ does not contain an odd cycle.}$$

Proof. ($\implies$): If G contains an odd cycle, then as odd cycles are not bipartite, G cannot be bipartite.

($\impliedby$): We may assume that G is connected. Let us fix $x_0 \in V(G)$. Let

$$V_0 = \{x \in V(G) \mid d(x, x_0) \equiv 0 \pmod 2\}$$
$$V_1 = \{x \in V(G) \mid d(x, x_0) \equiv 1 \pmod 2\}.$$

We claim this is a bipartition of G. Assume for contradiction that $\exists u, v \in V_0$ s.t. $uv \in E(G)$. But there is an even ux_0 path and and an even vx_0 path, thus putting these three paths together gives an odd circuit in G. By Prop 1.6, G contains an odd cycle, contradiction. (Analogous proof for V_1). □

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1.3 Planar graphs

Definition 1.14. A **planar graph** is a graph that can be drawn in the plane with no edge crossings.

Example 1.5. K_4 is planar. A path P_n is planar.

Definition 1.15. A **plane graph** is one such drawing of a graph in the plane without edge crossings.

Note that this is important, since we can draw K_4 in a way that it does have edges crossing.

Example 1.6. $K_{2,3}$ is planar. $K_{3,3}$ is not planar. K_5 is not planar (we don't prove this right now).

Question. What graphs are planar? Is there a (simple) method to decide if a graph is planar?

Definition 1.16. Let G be a plane graph. Consider $\mathbb{R}^2 \setminus G$. This is broken into finitely many regions. These are called the **faces** of the plane graph.

Definition 1.17. The **boundary** of a face F is the collection of vertices and edges on the topological boundary.

Remark. The boundary of a face need not be a cycle. It need also not be connected. In fact, the boundary need not contain a cycle at all (consider a tree).

Remark. We also note that two different drawings of a graph in the plane can be fundamentally different.

Theorem 1.8 (Euler)**.** Let G be a connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

Proof. We induct on m. $m = 1$ is clear. If G is acyclic, then G is a tree, so $m = n - 1$, $f = 1$ and we're done.

So assume G contains a cycle and let e be an edge on this cycle. Delete e. Then n stays fixed, m decreases by 1, and f decreases by 1, so by induction, $n - (m - 1) + (f - 1) = 2$ and we're done. □

Remark. We really do need the graph to be connected, consider t triangles in the plane as a counterexample.

Corollary 1.9. Let G be a planar graph, $|G| \geq 3$. Then $e(G) \leq 3|G| - 6$.

Proof. Draw G in the plane. We may assume that G is connected. Let F be a face, let $\deg(F)$ = the number of edges in G that touch F. Note $\deg(F) \geq 3$. Now note that since every edge touches at most two faces, we get

$$3f \leq \sum_{F \text{ a face}} \deg(F) \leq 2e(G) \implies f \leq \frac{2}{3}e(G).$$

Put this into Euler's formula to get

$$n - \frac{1}{3}e(G) \geq n - e(G) + f = 2 \implies 3(n-2) \geq e(G).$$

$\square$

Remarks. (i): This is a statement about planar graphs only.

(ii): This is quite restrictive. K_n has $\binom{n}{2} \approx n^2/2$ edges, while our above corollary says the number of edges of a planar graph is linear in n.

Corollary 1.10. K_5 is not planar.

Proof. We have $e(K_5) = 10, n = 5$, so $10e(G) \not\leq 3|G| - 6 = 9$, so we're done by the above corollary. $\square$

But $K_{3,3}$ does not fail this test. So we need to improve our argument:

Corollary 1.11. Let G be a planar graph, $|G| \geq 4$ and G has no cycles of length 3. Then $e(G) \leq 2|G| - 4$.

Proof. Repeat the proof of Corollary 1.9, but use $\deg(F) \geq 4$ for every face. $\square$

Now we can see that $K_{3,3}$ is not planar. $K_{3,3}$ has no cycle of length 3 by definition, $n = 6$, $e(G) = 9$, so $9 = e(G) \not\leq 2 \cdot (6-2) = 8$. 14 Oct 2022, Lecture 4

Definition 1.18. A **subdivision** of a graph G is a subgraph where we replace some of the edges of G with disjoint paths.

Observation. A subdivision of a **non-planar** graph is non-planar.

Observation. If G contains a $K_{3,3}$ or K_5 subdivision as a subgraph, then G is non-planar.

Theorem 1.12 (Kuratowski's theorem)**.** G is planar $\iff$ G does not contain a subdivided $K_{3,3}$ or K_5.

We do not prove this, but the proof is actually not too hard.

2 Connectivity & matching

2.1 Matching in bipartite graphs

Let $G = (X \sqcup Y, E)$ be bipartite with bipartition X, Y.

Definition 2.1. A **matching from** X **to** Y is a set of edges $\{xy_x \mid x \in X, y_x \in Y\}$ and $x \to y_x$ is an injection.

Question. When does a bipartite graph have a X to Y matching?

We can first think about examples where we do not have a matching. For example, we clearly have no matching if $|X| > |Y|$.

Definition 2.2. Let G be a graph, $A \subseteq V(G)$. Define $N_G(A) = \bigcup_{x \in A} N(x)$.

Then we clearly also don't have a matching if we have $A \subset X$ such that $|N(A)| < |A|$. But this is actually the only obstruction:

Theorem 2.1 (Hall's Marriage Theorem)**.** Let G be a bipartite graph $G = (X \sqcup Y, E)$. Then

$$G \text{ has a matching from } X \text{ to } Y \iff \forall A \subseteq X,\ |N(A)| \geq A.$$

The right-hand side is called Hall's criterion.

Proof. ($\Longrightarrow$) is the easy direction.

Now let M be a matching and let $A \subseteq X$. Then if $\{y_1, \ldots, y_{|A|}\}$ are matched to A, we show $|N(A)| \geq |\{y_1, \ldots, y_{|A|}\}| \geq |A|$.

($\Longleftarrow$): Apply induction on $|X|$. If $|X| = 1$, we're done. For the induction step, consider the following question: is there $\emptyset \neq A \subsetneq X$ such that $|N(A)| = |A|$?

If the answer is no, then $\forall A \subsetneq X$ we have $|N(A)| \geq |A| + 1$. Let $xy \in E(G)$ and let $G' = G[X \setminus \{x\} \cup Y \setminus \{y\}]$. We now check Hall's criterion for G'. If $B \subseteq X \setminus \{x\}$, then $|N_{G'}(B)| \geq |N_G(B)| - 1 \geq |B|$, so done by induction.

If the answer is yes, then let $G_1 = G[A \cup N(A)]$ and $G_2 = G[X \setminus A \cup Y \setminus N(A)]$.

Claim 1: G_1 satisfies Hall's criterion. Let $B \subseteq A$, then

$$|N_{G_1}(B)| = |N_G(B)| \geq B.$$

Claim 2: G_2 satisfies Hall's criterion. Let $B \subset X \setminus A$. Consider $N_G(A \cup B)$. One the one hand, $|N_G(A \cup B)| \geq |A| + |B|$. On the other hand, $|N_G(A)| + |N_{G_2}(B)| = |N_G(A \cup B)|$. As $|N(A)| = |A|$, we get $|N_{G_2}(B)| \geq |B|$.

From claims 1 and 2 we can apply induction in G_1, G_2 to get a matching in these graphs, and then put them together to get a matching in G. ◻

Definition 2.3. A matching of deficiency of d from X to Y is a matching from X' to Y where $X' \subseteq X$, $|X| - d = |X'|$.

Theorem 2.2 (Defect Hall's Theorem)**.**

$$G \text{ contains a matching of deficiency } d \iff \forall A \subseteq X,\ |N(A)| \geq |A| - d.$$

Proof. ($\Longrightarrow$) : easy.

($\Longleftarrow$) : Add d phantom vertices to Y, which we join to all vertices in X, so we now satisfy Hall's condition. Apply Hall to get a matching, and then remove the d vertices we added, which removes at most d elements of X. □

Definition 2.4. Let G be a graph. The **minimum degree** in G is $\delta(G) = \min_{x \in V(G)} d(x)$, and the **maximal degree** in G is $\Delta(G) = \max_{x \in V(G)} d(x)$.

Definition 2.5. A graph is **regular** if $\delta(G) = \Delta(G)$. It is **k-regular** if $k = \delta(G) = \Delta(G)$.

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Corollary 2.3. For $k \geq 1$, if $G = (X \sqcup Y, E)$ is a k-regular bipartite graph, then there exists a matching from X to Y.

Proof. We check Hall's criterion. Let $A \subseteq X$. On the one hand,

$$e(G[A \cup N(A)]) = \sum_{v \in A} \deg(v) = k|A|.$$

On the other hand,

$$e(G[A \cup N(A)]) \leq \sum_{v \in N(A)} \deg(v) \leq k|N(A)|.$$

Hence $|N(A)| \geq |A|$ and we're done. □

Let Γ be a finite group and let H be a subgroup of Γ. Let $L_1, \ldots, L_n$ be the set of left cosets and $R_1, \ldots, R_n$ be the right cosets (of the forms gH and Hg respectively).

Question. Is there $g_1, \ldots, g_n \in \Gamma$ such that $g_1H, \ldots, g_nH$ are the left cosets and $Hg_1, \ldots, Hg_n$ are the right cosets?

Corollary 2.4. There exist $g_1, \ldots, g_n \in \Gamma \geq H$ such that $g_1H, \ldots, g_nH$ are the left cosets and $Hg_1, \ldots, Hg_n$ are the right cosets.

Proof. It is enough to find a pairing $L_i \leftrightarrow R_{\sigma(i)}$ such that $L_i \cap R_{\sigma(i)} \neq \emptyset\ \forall i$. Then choose $g_i \in L_i \cap R_{\sigma(i)}$ and we have $g_iH = L_i$, $Hg_i = R_{\sigma(i)}$.

Define $X = \{R_1, \ldots, R_n\}$ and $Y = \{L_1, \ldots, L_n\}$, and define $R_i \sim L_j$ when $R_i \cap L_j \neq \emptyset \; \forall i, j$. Let $A = \{r_{i_1}, \ldots, R_{i_k}\}$. Note

$$\left| \bigcup_{j=1}^{k} R_{i_j} \right| = k|H|.$$

But $L_1, \ldots, L_n$ partition Γ and $|L_i| = |H|$, so at least k left cosets must intersect $\bigcup R_{i_j}$. Thus Hall's criterion is satisfied and we're done. $\square$

2.2 Connectivity

For a tree, $G - x$ (where x is any non-leaf) is disconnected. On the other hand, remove any 2 vertices from the Petersen graph and it stays connected (but if you remove any 3, you disconnect it).

Notation. Let $S \subseteq V(G)$, and let $G - S = G[V(G) \setminus S]$.

Definition 2.6. Let G be a graph, $|G| \geq 1$. Define

$$\kappa(G) = \min\{|S| \mid \exists S \subseteq V(G), G - S \text{ is disconnected or a single vertex}\}.$$

We say a graph G is **k-connected** if $\kappa(G) \geq k$.

In other words, G is k-connected if and only if $G - S$ is connected for all $S \subseteq V(G), |S| \leq k - 1$.

Example 2.1.

- $\kappa(\text{Tree}) = 1$.
- $\kappa(\text{Petersen graph}) = 3$, so we can say the Petersen graph is 3-connected.
- $\kappa(\text{Cycle}) = 2$.
- $\kappa(K_n) = n - 1$.

We have another natural definition of connectivity.

Definition 2.7. Let G be a graph and let $a, b \in V(G)$. Say that ab paths $P_1, \ldots, P_k$ are **disjoint** if $V(P_i) \cap V(P_j) = \{a, b\} \; \forall i \neq j$.

Amazingly, we have Menger's theorem: These two notions of connectivity (# of disjoint paths and $\kappa(G)$) are equivalent.

Remarks:

- We have $\delta(G) \geq \kappa(G)$. To see this, delete $N(x)$ for $x \in V(G)$ of minimal degree, then $G - N(x)$ is disconnected (or a single vertex).
- We have $\kappa(G - x) \geq \kappa(G) - 1$. This is clear: if $S \subset V(G - x)$ disconnects $G - x$ with $|S| \leq \kappa(G) - 2$, then $S \cup \{x\}$ disconnects G, contradiction.

- We can have $\kappa(G-x) > \kappa(G)$. For example, a cycle is 2-connected, but a cycle with one protruding edge is 1-connected.

Definition 2.8. A **component** in G is a maximal connected subgraph.

Definition 2.9. Let G be a graph, let $a, b \in V(G), a \neq b, a \not\sim b$. Say $S \subseteq V(G) \setminus \{a,b\}$is a $a-b$ **separator** if $G-S$ disconnects a from b (i.e. a, b are in different components of $G-S$).

Theorem 2.5 (Menger's theorem, form 1)**.** Let G be a connected graph and fix $a, b \in V(G), a \neq b, a \not\sim b$. Then the minimum size of an $a-b$ separator is equal to the maximal number of disjoint paths from a to b.

In other words, if all $a-b$ separators have size $\geq k$, then there exist $P_1, \ldots, P_k$, disjoint paths between a and b.

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Note. Define $\kappa_{a,b}(G)$ be the size of the minimal $a-b$ separator.

Note. Recall $\kappa(G-x) \geq \kappa(G)-1$, and also $\kappa(G-xy) \geq \kappa(G)-1$. We also have $\kappa_{a,b}(G-x) \geq \kappa_{a,b}(G)-1$ and $\kappa_{a,b}(G-xy) \geq \kappa_{a,b}(G)-1$ (exercise, not hard).

Proof. Assume for contradiction that the statement of the theorem is false. Let G be a minimal counterexample to the theorem that

(a) minimizes k;

(b) subject to (a), choose G to minimize $e(G)$.

Now let S be a minimal a, b separator in G. We have $|S| = k$. Note that the theorem is true for $k = 1$, so assume $k \geq 2$.

If $S \neq N(A)$ and $S \neq N(B)$, consider $G-S$ and let A be the component containing a and B be the component containing B.

Define $G_a = G[A \cup S]$ along with a vertex c joined to each vertex in S, and $G_b = G[B \cup S]$ along with a vertex c joined to each vertex in S. Note that $\kappa_{a,c}(G_a) \geq k$, since any $a-c$ separator in G_a is a a, b separator in G. Likewise, $\kappa_{b,c}(G_b) \geq k$.

Note that $e(G_a) < e(G), e(G_b) < e(G)$ since $N(a) \not\subset S, N(b) \not\subset S$. So there exists a neighbor x of b in B with $\deg(x) \geq 2$, else we can remove x and apply minimality.

So by minimality of G, we can find k disjoint a, c paths, say $P_1, \ldots, P_k$ in G_a, and likewise we can find k b, c paths $Q_1, \ldots, Q_k$ in G_b. We can put these paths together to get paths $P_1 Q_{\sigma(1)}, \ldots, P_k Q_{\sigma(k)}$, which are k disjoint a, b paths, contradiction, done.

Let us now assume WLOG that $S = N(a)$.

Claim: $N(a) \cap N(b) = \emptyset$.

Indeed, if $\exists x \in N(a) \cap N(b)$, consider $G - x$. We have $\kappa_{a,b}(G - x) \geq k - 1$. Thus, by minimality, we can find $k-1$ disjoint ab paths in $G-x$, so all of these, plus axb, gives us k disjoint ab paths in G, contradiction.

Let $ax_1 \ldots x_l b$ be a shortest ab path. Note that $l \geq 2$ and $x_2 \neq b$. Consider $G - x_1x_2$. We must have $\kappa_{a,b}(G - x_1x_2) \leq k - 1$ by minimality, so $\kappa_{a,b}(G - x_1x_2) = k - 1$. So there is a a, b separator $\tilde{S}$, $|\tilde{S}| = k - 1$ in $G - x_1x_2$. We see that $\tilde{S} \cup \{x_1\}$ and $\tilde{S} \cup \{x_2\}$ are a, b separators in G of size at most k. Now either $\tilde{S} \cup \{x_1\} \neq N(a), N(b)$ or $\tilde{S} \cup \{x_2\} \neq N(a), N(b)$, so we're done. □

Corollary 2.6 (Menger's theorem, form 2)**.** Let G be a connected graph, $|G| \geq 2$. Then G is k-connected $\iff$ $\forall a, b \in V(G), a \neq b$, there exist k disjoint ab paths in G.

Proof. $\Longleftarrow$ is the easy direction. Say $G - S$ is disconnected and let a, b be in different components of $G - S$. Note $a \not\sim b$. Then $\exists$ k disjoint $a - b$ paths and S must intersect each of these, so $|S| \geq k$.

$\Longrightarrow$. Let $a, b \in V(G), a \neq b$. If $a \not\sim b$, then just apply Menger form 1 and we're done. If $a \sim b$, then consider $G - ab$. We have $\kappa_{a,b}(G - ab) \geq k - 1$, so apply Menger form 1 to get $k - 1$ disjoint paths and add back ab as a k^{th} path. □

2.2.1 Edge connectivity

Let G be a graph. Let $\lambda(G) = \min\{|W| \mid W \subseteq E(G), G - W \text{ is disconnected}\}$. We say that G is k**-edge connected** if $\lambda(G) \geq k$.

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Example 2.2.

- A cycle has $\kappa(C_n) = 2, \lambda(C_n) = 2$.
- A "bowtie graph" has $\kappa(G) = 1, \lambda(G) = 2$. We can generalize this and take two copies of K_n which intersect in one vertex, then $\kappa(G) = 1$ and $\lambda(G) = n - 1$.

Definition 2.10. We say paths $P_1, \ldots, P_k$ are **edge disjoint** if

$$E(P_i) \cap E(P_j) = \emptyset \; \forall i \neq j.$$

Theorem 2.7 (Menger, edge version)**.** Let G be a connected graph and $a, b \in V(G), a \neq b$. Then, every $W \subseteq E(G)$ that separates a from b having size $\geq k \implies \exists k$ edge disjoint $a - b$ paths $P_1, \ldots, P_k$.

Definition 2.11. Let G be a graph. The **line graph** of G, denoted $L(G)$, is defined to be the graph

$$V(L(G)) = E(G);$$
$$\text{If } e, f \in E(G), \text{ then } e \sim f \text{ if they share a vertex.}$$

Proof of Thorem 2.7. Given G, define a new graph G' by taking the line graph of G and adding a vertex a', which we join to all edges incident to $a \in G$, and similarly adding a vertex b', which we join to all edges incident to $b \in G$.

Note that there is a ab path in G if and only if there is an $a'b'$ path in G'. Thus $W \subseteq V(G') \setminus \{a, b\}$ is a $a'b'$ separator if and only if $W \subseteq E(G)$ is an ab separator. Hence $\kappa_{a,b}(G') \geq k$.

Now apply Menger (form 1) to find k disjoint ab paths $P_1, \ldots, P_k$ in G'. These describe edge disjoint walks in G from a to b. Thus there are disjoint paths $\tilde{P}_1 \subseteq P_1, \ldots, \tilde{P}_k \subseteq P_k$ and we're done. □

Theorem 2.8 (Menger, edge version 2)**.** Let G be a connected graph. Then $\lambda(G) \geq k \iff \forall a, b \in V(G)$ with $a \neq b, \exists k$ edge disjoint ab paths.

Proof. $\Longleftarrow$ is the easy direction. To separate any two vertices, say a, b, we must remove an edge from each of the ab paths, so $\lambda(G) \geq k$. $\Longrightarrow$ follows from Menger, edge version 1. □

3 Graph coloring

Definition 3.1. We say that $c : V(G) \to \{1, \ldots, k\}$ is a k**-coloring** (or a proper k-coloring) if $c(x) \neq c(y)$ $\forall x \sim y$.

Definition 3.2. The **chromatic number of** G is

$$\chi(G) = \min\{k \mid \exists \text{ a } k\text{-coloring of } G\}.$$

Example 3.1.
- A path has chromatic number 2. $\chi(P_n) = 2$.
- For a cycle, $\chi(G) = \begin{cases} 2 & \text{if } n \text{ is even.} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$
- A tree has chromatic number 2 by induction.
- A complete graph has $\chi(K_n) = n$.
- A bipartite graph has $\chi(K_{m,n}) = 2$. In fact, a graph G is bipartite if and only if $\chi(G) = 2$.

Proposition 3.1. Let G be a graph. Then $\chi(G) \leq \Delta(G) + 1$.

Proof. Let $x_1, \ldots, x_n$ be an ordering of $V(G)$. We color the (x_i) one at a time in this order. When we come to vertex x_i, at most Δ colors have been used in $N(x_i)$, so there is a free color for x_i. □

Remark. This proposition is sharp (e.g. on K_n).

Remark. This is sometimes called a greedy coloring. But a greedy coloring may produce a coloring that is not optimal!

3.1 Coloring planar graphs

Observation. Let G be a planar graph. Then $\delta(G) \leq 5$.

Proof. The average degree of G is

$$\frac{1}{n} \sum_{v \in V(G)} \deg(v) = \frac{2e(G)}{n} \leq \frac{2(3n-6)}{n} = 6 - \frac{12}{n} < 6.$$

But all degrees of vertices are integers, so the minimal degree is ≤ 5. □

Proposition 3.2. If G is planar, then $\chi(G) \leq 6$.

Proof. We induct. Base step: if $|G| \leq 6$, then we're clearly done.

Induction step: Given a graph G, let x be a vertex with $\deg(x) \leq 5$. Apply induction to $G - x$, which gives a coloring of $G - x$ with 6 colors. But x has degree ≤ 5, thus there is a free color to color x with. □

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Theorem 3.3. If G is planar, then $\chi(G) \leq 5$.

Proof. We apply induction on $|G|$. If $|G| \leq 5$, we're done.

Now let G be a planar graph and $x \in V(G)$ with $\deg(x) \leq 5$. Apply induction to $G - x$. Let neighbor x_i of x have color i, ordered clockwise around x.

Question: Can we get from x_1 to x_3 only walking along vertices colored 1 and 3?

If no, let C be the component of G of vertices colored 1 or 3 that contains x_1, so $x_3 \notin C$. "Swap" the colours 1 and 3 on C. This is a proper coloring of $G - x$, so we can color x with color 1 and we're done.

If yes, ask the same question for x_2 and x_4 - if the answer is no, swap colors on the 2–4 component containing x_2, and we can color x with color 2.

If the answer is again yes, then we get a contradiction to planarity since the 1–3 path and the 2–4 path have to intersect somewhere and we're done. □

Theorem 3.4 (Four color theorem, non-examinable)**.** If G is planar, then $\chi(G) \leq 4$.

To see this is equivalent to the map version, take the dual of our graph (place a vertex inside each face (and one for the infinite face) and connect two vertices by an edge if the two faces have any common boundary).

Kempe "proved" the four colour theorem in 1879, but his proof had a mistake. It was then proved in 1976 by reducing the problem to about 2000 configurations and checking them by computer.

Also, this is the best we can do, since K_4 is planar - there is no "three color theorem".

3.2 Coloring graphs

Proposition 3.5. Let G be connected and $\delta(G) < \Delta(G)$. Then $\chi(G) \leq \Delta(G)$.

Proof. Let x_n have $\deg(x_n) < \Delta(G)$. Then choose x_{n-1} to be adjacent to x_n, x_{n-2} to be adjacent to one of $\{x_n, x_{n-1}\}$, etc. Since G is connected, we eventually order everything and the ordering has the property that all vertices have less than $\Delta(G)$ edges going to vertices with smaller index, so color greedily and we're done. □

Theorem 3.6 (Brooks)**.** Let G be a connected graph which is not an odd cycle or a complete graph. Then $\chi(G) \leq \Delta(G)$.

Proof. Apply induction on $|G|$. The claim is clearly is true for $|G| \leq 3$.

Note that we may assume that $\Delta \geq 3$ (where $\Delta = \Delta(G)$).

Claim 1. If G is 3-connected, then we're done.

Proof. Define an ordering of G as in the proposition above. Let x_n have $\deg(x_n) = \Delta$ and choose $x_1, x_2 \in N(x_n)$ with $x_1 \neq x_2, x_1 \not\sim x_2$. This is possible, since G is not $K_{\Delta+1}$.

Now consider $G \setminus \{x_1, x_2\}$. We order this in the same way as above: connect x_{n-1} to x_n, x_{n-2} to x_{n-1} or x_n, etc. Since $G \setminus \{x_1, x_2\}$ is connected, we eventually order every vertex. Hence we're done by coloring greedily.

Claim 2. If $\kappa(G) = 1$, we're done.

Proof. Let x be a cut vertex and let $C_1, \ldots, C_k$ be the components of $G - x$. By induction, we can color each $G[C_i \cup x]$. Then permute colors to have x always be the same color, and put everything back together to get a coloring of G.

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Claim 3. If $\kappa(G) = 2$, we're done.

Proof. Let $S \subset V(G)$, $S = \{x, y\}, x \neq y$ be our separator. Let $C_1, \ldots, C_k$ be the components of $G - S$ and let $G_i = G[C_i \cup S] + xy$ for each i.

- Case 1: $\delta(G_i) < \Delta(G)\ \forall i$. In this case we can always color $C_1, \ldots, C_k$ by Proposition 3.5. Note that in each of these colorings, x, y have different colors, so we can permute colors in a way that x, y always have the same color, and put everything back together to get a coloring of G.

- Case 2: $\delta(G_1) = \Delta(G)$ (wlog assume $i = 1$). In this case, $k = 2$ and $|N(x) \cap C_1| = \Delta - 1$. Also, $|N(x) \cap C_2| = 1$ and $|N(y) \cap C_2| = 1$. Thus let $x', y' \in C_2$ be such that $x' \sim x, y' \sim y$. Now observe that $\tilde{S} = \{x, y'\}$ is not of this bad form, so we're done by Case 1.

□

3.3 Chromatic polynomials

Definition 3.3. For a graph G, define the **chromatic polynomial** of G to be the function $P_G : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $P_G(t)$ = the number of t–colorings of G.

In particular, the minimal value of t for which $P_G(t) > 0$ is $t = \chi(G)$.

Example 3.2.

- $P_{\overline{K_n}}(t) = t^n$.
- $P_{K_n}(t) = t(t-1)\dots(t-(n-1))$.
- $P_{P_{n-1}}(t) = t(t-1)^{n-1}$.
- $P_T(t) = t(t-1)^{n-1}$ where T is a tree with n vertices (proof: take a leaf, induct).

Definition 3.4. Let G be a graph and $e \in E(G)$. Define G/e to be the **contraction** of G along e, which is the graph

$$\begin{aligned} V(G/e) &= V(G) \setminus \{x, y\} \cup \{xy\}, \\ E(G) &= E(G[V \setminus \{x, y\}]) \cup \{ez \mid x \sim z\} \cup \{ez \mid y \sim z\}. \end{aligned}$$

(In other words, you just "squish" the edge e and its endpoints become a single vertex.)

Proposition 3.7. Let G be a graph. We have $P_G(t) = P_{G-e}(t) - P_{G/e}(t)$.

Proof. Let $e = xy$. A t–coloring of $G - e$ where x, y get different colors corresponds exactly to a t–coloring of G. On the other hand, a t–coloring of $G - e$ where x, y get the same color corresponds exactly to a t–coloring of G/e. □

Proposition 3.8. For a graph G, P_G is a polynomial.

Proof. Apply induction on $e(G)$. For the base case $e(G) = 0$, $P_{\overline{K_n}}(t) = t^n$. Induction step: For $e \in E(G)$, $P_G(t) = P_{G-e}(t) - P_{G/e}(t)$ is a polynomial (as it is the difference of two polynomials), so we're done by induction. □

Proposition 3.9. Let G be a graph with n vertices and m edges. Then

$$P_G(t) = t^n - mt^{n-1} + \dots.$$

Proof. Induction on the number of edges. If $e(G) = 0$, then we're done.

Induction step: Let $e \in E(G)$ and apply the deletion-contraction relation, $P_G(t) = P_{G-e}(t) - P_{G/e}(t)$. By induction, the leading terms on the RHS are

$$(t^n - (m-1)t^{n-1} + \dots) - (t^{n-1} - e(G/e)t^{n-2} + \dots) = t^n - mt^{n-1} + \dots.$$

□

Remarks. We're going to stop here, but chromatic polynomials have a lot of other properties. For example:

- Other coefficients of P_G contain other information about G, e.g.

$$P_G = t^n - mt^{n-1} + \left(\binom{m}{2} - \text{the number of triangles in } G\right) t^{n-2} + \dots.$$

- If G is planar, then $P_G(2 + \frac{1+\sqrt{5}}{2}) \neq 0$.
- The coefficients of $c_0, \dots, c_n$ of $P_G(t)$ are log-concave, i.e. $c_i^2 \geq c_{i-1}c_{i+1}$. June Huh won a Fields medal for this in 2021.

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3.4 Edge coloring

Definition 3.5. For a graph G, a k–edge coloring is a function $c : E(G) \to [k]$ such that $c(e) \neq c(f)$ whenever $e, f \in E(G)$ share an endpoint.

Define the **edge-chromatic number** (sometimes called the **chromatic index**) to be

$$\chi'(G) = \min\{k \mid \exists k\text{-edge coloring of } G\}.$$

Note that an edge coloring of G corresponds exactly to a vertex coloring of the line graph of G, and so $\chi'(G) = \chi(L(G))$.

Remarks.

- $\chi'(G) = \begin{cases} 2 & \text{if } n \text{ even.} \\ 3 & \text{if } n \text{ odd.} \end{cases}$
- $\Delta(G) \leq \chi'(G)$.
- We can have $\Delta(G) < \chi'(G)$. For example, $\chi'(\text{Petersen}) = 4$.
- $\chi'(G) \leq 2\Delta(G) - 1$ by coloring greedily.
- χ and χ' can be very different. For example, $\chi'(K_{t,1}) = t, \chi(K_{t,1}) = 2$.

We now prove Vizing's theorem. First, some setup:

Definition 3.6. Given an edge-coloring $c : E(G) \to [k]$, define the **color classes** of c to be

$$C_i = \{e : c(e) = i\}.$$

Question. What does $(V(G), C_i \cup C_j)$ look like? It is the union of disjoint paths, isolated vertices and even cycles.

We say the components of this graph are $\{i, j\}$–components.

Definition 3.7. Let $y \in V(G), c \in [\Delta + 1]$. We say that c is **missing** at y if none of the edges incident at y have color c.

Theorem 3.10 (Vizing's theorem)**.** Let G be a graph. Then

$$\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}.$$

Proof. Induction on $e(G)$. If $e(G) = 0$, we're done.

Induction step: given G, $e(G) > 0$, let $xv \in e(G)$. Apply induction to the graph $G - xv$ to get a $\Delta + 1$ edge coloring. We want to extend this coloring to G.

Note that there is a color missing at every vertex and let c_0 be the color missing at x. Define a sequence of $v_1, \dots, v_k \in N(x)$ and corresponding colors $c_1, \dots, c_k$ such that c_i is missing at v_i as follows:

First set $v_1 = v$ and let c_1 be any color missing at v_1. Now say we have defined v_i and c_i. Define v_{i+1} such that xv_{i+1} is colored c_i. Continue this way until one of two things happens:

- Case 1: Say we have defined v_k, c_k and c_k is missing at x. Then recolor xv_k with color c_k, so now c_{k-1} is missing at x. Hence recolor xv_{k-1} with color c_{k-1}, so now c_{k-2} is missing at x, etc, until we color xv_i with color c_i for all $i \leq k$. This includes $i = 1$, so we're done.

- Case 2: Say $c_k = c_i$ for some $i < k$. Note that we may assume $i = 1$ by recoloring as in Case 1 (i.e. uncolor xv_i and color $xv_j \to c_j$ for all $j < i$). The question we ask is whether v_1 is in the same $\{c_0, c_1\}$–component as x.

  – If not, swap colors on the $\{c_0, c_1\}$–component containing v_1. Then c_0 is missing at both x and v_1, and so we can color xv_1 with c_0 and we're done.

  – If yes, we need another question: Is x in the same $\{c_0, c_1\}$–component as v_k?

    ∗ If the answer is no, then swap colors in the $\{c_0, c_1\}$–component containing v_k. So c_0 is missing at v_k and x, so we are done by Case 1.

    ∗ If yes, then x, v_1, v_k are all in the same $\{c_0, c_1\}$–component. But one of c_0, c_1 is missing at each of x, v_1, v_k. Thus x, v_1, v_k all have to be endpoints of a path in the same $\{c_0, c_1\}$–component, which is a contradiction.

□

3.5 Coloring graphs on surfaces

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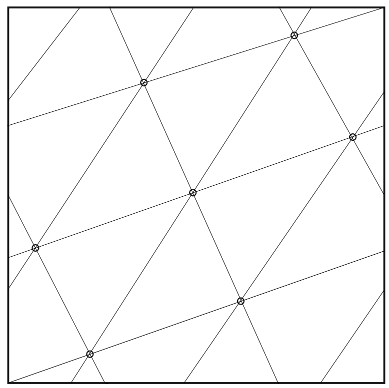
If G is planar, we know that$\chi(G) \leq 5$. (And really $\chi(G) \leq 4$). If G is drawn on a surface S, can we say anything about $\chi(G)$?

K_7 can be drawn on a torus without edge crossings, so we definitely don't have a 4 color theorem for the torus:



Question. Is there a "7 color theorem" for the torus?

Definition 3.8. We define the **surface of genus** g (more formally, a compact orientable surface of genus g) to be the sphere with g handles attached (or equivalently, the sphere with g holes).

Do we have an Euler-type theorem in this setting? Different cycles on the torus give different values for $V - E + F$, we get all of 2, 1, 0. *picture*

Proposition 3.11. Let G be a graph drawn on a surface of genus g with n vertices, m edges, and f faces. Then

$$n - m + f \geq 2 - 2g.$$

Remarks.

- $E = 2 - 2g$ is the **Euler characteristic** of the surface of genus g.
- This is the direction of Euler's theorem ($\geq$, not $\leq$) that we need to bound the number of edges of G.
- We do not assume that the graph is connected here.
- The proof is easy: we induct on $e(G)$. Removing an edge reduces m by 1, and reduces f by either 0 or 1.

Corollary 3.12. Let G be a graph with $|G| \geq 3$, drawn on a surface of Euler characteristic E. Then

$$e(G) \leq 3(|G| - E).$$

Proof. If $e(G) \leq 2$, we can check the claim holds. When $e(G) \geq 3$, then every face in G has length at least 3, so

$$3f \leq \sum_{F \text{ a face}} \deg(F) \leq 2e(G).$$

Thus $f \leq \frac{2}{3}e(G)$, so $e(G) \leq n + f - E \implies e(G) \leq 3(n - E)$. $\square$

Theorem 3.13 (Heawood's theorem)**.** Let G be a graph drawn on a surface of Euler characteristic $E \leq 0$. Then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

Proof. Let G be a graph drawn on a surface of Euler characteristic E and let $k = \chi(G)$. We may assume that G is minimal, i.e. all proper subgraphs of G have $\chi \leq k - 1$.

We esimate $\delta(G)$ in two ways. Note $\delta(G) \geq k - 1$: if not, let x be a vertex with $\deg(x) \leq k - 2$. We have $\chi(G - x) \leq k - 1$ by minimality, so we can put x back and color it $\implies \chi(G) \leq k - 1$, contradiction.

Note

$$\delta(G) \leq \text{the average degree of the graph} = \frac{1}{n} \sum_{x \in V(G)} \deg(x) =$$

$$\frac{2e(G)}{n} \leq \frac{6(n - E)}{n} = 6 - \frac{6E}{n} \leq 6 - \frac{6E}{k}$$

because $n \geq k$ and $E \leq 0$. So

$$k - 1 \leq 6 - \frac{6E}{k} \implies k^2 - 7k + 6E \leq 0 \implies k \leq \frac{7 + \sqrt{49 - 24E}}{2}.$$

As k is an integer, we're done. $\square$

Remarks.

- Define $H(E) = \left\lfloor \frac{7+\sqrt{49-24E}}{2} \right\rfloor$.
- Did we just prove the four color theorem, since the above gives $H(2) = 4$? No, since we assumed $E \leq 0$.
- Amazingly, Heawood's theorem is sharp. In particular, $K_{H(E)}$ can be drawn in the surface of Euler characteristic E (but this is very hard!).

4 Extremal graph theory

Question. What is the minimum number of edges a graph must have before it is forced to have a triangle? In other words, what is the value of k such that if $e(G) > k$, then $K_3 \subset G$?

Question. How large does $\delta(G)$ have to be before G is forced to have a cycle of length $|G|$?

02 Nov 2022, Lecture 12

Definition 4.1. A graph is said to be **Hamiltonian** if it contains a cycle that contains all vertices. Such a cycle is called a **Hamilton** cycle.

Theorem 4.1 (Dirac)**.** Let G be a graph on n vertices, $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Remark. This theorem is sharp. If n is even, take two disjoint $K_{n/2}$'s. If n is odd, take two copies of $K_{(n+1)/2}$ that overlap at exactly one vertex.

Proof. First note that G is connected. If $x \not\sim y$, then $N(x), N(y) \subset G - x - y$, but $|N(x)|, |N(y)| \geq \frac{n}{2}$, so $N(x) \cap N(y) \neq \emptyset$ by the pigeonhole principle, so x and y have a common neighbor.

Now take a path of maximal length $x_1 \dots x_l$. The claim is that there is no cycle in G of length l. Indeed, if such a cycle did exist, then either $l = n$ and we're done, or $l < n$ and we can use this cycle to find a path of length $l + 1$ in G, contradicting maximality.

Observe that $N(x_1) \subset \{x_2, \dots, x_{l-1}\}$ and $N(x_l) \subset \{x_2, \dots, x_{l-1}\}$. Now define $N^-(x_1) = \{x_i \mid x_{i+1} \in N(x_1)\}$. Note

$$|N^-(X_1) \cup N^-(x_l)| \leq \{x_1, x_2, \dots, x_{l-1}\} \leq l - 1 \leq n - 1,$$

but $|N^-(x_1)|, |N(x_2)| \geq \frac{n}{2}$. So there must exist x_i in their intersection.

So we have $x_1 \dots x_i x_l x_{l-1} \dots x_{i+1} x_1$, a cycle of length l, a contradiction and we're done. $\square$

Remark. Note that there's no interesting theorem of the form "if $e(G) > K$, then G is Hamiltonian", since $K_{n-1} + x$ where x has degree 1 is not Hamiltonian.

Theorem 4.2. Let G be a graph on n vertices. Then $e(G) > \frac{n(k-1)}{2} \implies G$ contains a path of length k.

Remark. If $k \mid n$, then this is sharp: take the disjoint union of $\frac{n}{k}$ copies of K_k.

Lemma 4.3. Let G be a graph on n vertices, $n \geq 3$, and let $k < n$. If G is connected and $\delta(G) \geq \frac{k}{2}$, then G contains a path of length k.

Remarks.

- We need the assumption that $k < n$, otherwise take K_k.
- We need the assumption that G is connected, otherwise take a bunch of disjoint K_k.
- The $\frac{k}{2}$ is sharp. Take a bunch of K_k's that overlap at a single vertex.

Proof. Let $x_1 \dots x_l$ be a path of maximum length. We have that there is no cycle of length l: If $l = n$, then we're done since $k < n$, and if $l < n$, then use a cycle of length l to build a path of length $l + 1$ (since G is connected), a contradiction.

So we get $N(x_1) \subset \{x_2, \dots, x_{l-1}\}, N(x_l) \subset \{x_2, \dots, x_{l-1}\}$, and $N^-(x_1) = \{x_i \mid x_{i+1} \in N(x_1)\} \subset \{x_1, \dots, x_{l-2}\}$. Thus

$$|N^-(x_1) \cup N(x_l)| \leq |\{x_1, \dots, x_{l-1}\}| \leq l - 1 \overset{(\star)}{\leq} k - 1$$

(where $(\star)$ follows since else we're immediately done), but $|N^-(x_1)|, |N(x_l)| \geq \frac{k}{2}$, so $N^-(x_1) \cap N(x_l) \neq \emptyset$. So as before, we can build a cycle of length l, contradiction, done. $\square$

Proof of Theorem 4.2. If $k = 1$, we're done. Now assume $k \geq 2$ and apply induction on n, $n = 2$ being clear.

For the induction step, given a graph G on $n \geq 3$ vertices, first note that $\frac{n(k-1)}{2} < e(G) \leq \frac{n(n-1)}{2} \implies k \leq n$.

We may assume that G is connected: say $C_1, \dots, C_r$ are the components of G and $|C_i| = n_i$. Then $\sum_{i=1}^r e(G[C_i]) = e(G) > \frac{n(k-1)}{2}$, so since $\sum_{i=1}^r n_i = n$,

$$\sum_{i=1}^r \left(e(G[C_i]) - \frac{n_i(k-1)}{2} \right) > 0.$$

So there exists some component C_i that has $e(G[C_i]) > \frac{n_i(k-1)}{2}$, so apply induction to this component and we're done.

If $\delta(G) \geq \frac{k}{2}$, then apply Lemma 4.3 to see that G contains a path of length k. Otherwise, there exists a vertex $x \in V(G)$ with $\deg(x) \leq \frac{k-1}{2}$. Note that we have

$$e(G - x) > \frac{n(k-1)}{2} - \frac{k-1}{2} = \frac{(n-1)(k-1)}{2},$$

hence G contains a path of length k and we're done. $\square$