Part II - Graph Theory

Lectured by Dr J. Sahasrabudhe

Artur Avameri

Michaelmas 2022

Contents

0 Introduction **2**

1 Fundamentals **2**
- 1.1 Trees . . . 3
- 1.2 Bipartite graphs . . . 5
- 1.3 Planar graphs . . . 6

2 Connectivity & matching **8**
- 2.1 Matching in bipartite graphs . . . 8
- 2.2 Connectivity . . . 10
  - 2.2.1 Edge connectivity . . . 12

3 Graph coloring **13**
- 3.1 Coloring planar graphs . . . 14

0 Introduction

07 Oct 2022, Lecture 1

Notation. We write $[n]$ for $\{1, 2, \ldots, n\}$. For a set X and $k \in \mathbb{N}$, define $X^{(k)} = \{S \subset X \mid |S| = k\}$, i.e. the set of all subsets of size k.

1 Fundamentals

Definition 1.1. A **graph** is an object $G = (V, E)$ where V is a set and $E \subseteq V^{(2)}$.

V is the set of vertices, and E is the set of edges.

$V(G)$ will denote V, $E(G)$ will denote E, and we define $|G| = |V(G)|$ (sometimes called the order) and $e(G) = |E(G)|$ (sometimes called the size).

Example 1.1. The **complete graph** on n vertices is denoted K_n. This is the graph where $V(K_n) = [n]$ and $E(K_n) = [n]^{(2)}$.

Remark. We assume the following:

- We don't order edges;
- We don't allow loops (an edge joining a vertex to itself);
- We don't allow multiple edges;
- Most of the time, $V(G)$ will be finite (we will explicitly say when it's not).

Example 1.2. The **empty graph** on n vertices, denoted $\overline{K_n}$, has $V(\overline{K_n}) = [n]$ and $E(\overline{K_n}) = \emptyset$.

Example 1.3. The path of length n, denoted P_n, is a path: it has $V(P_n) = [n+1]$ and $E(P_n) = \{\{i, i+1\} \mid 1 \leq i \leq n\}$.

Example 1.4. The cycle of length n, denoted C_n, has $V(C_n) = [n]$ and $E(C_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.

Let G be a graph and $x \in V(G)$. The **neighborhood** of x is $N(x) = \{y \mid xy \in E(G)\}$, i.e. all the vertices connected to x. If $y \in N(x)$, we write $x \sim y$ and say y is a **neighbor** of x or that y is **adjacent** to x.

The **degree** of x is $\deg(x) = |N(x)|$.

Just as a formality, we define graph isomorphism: let G, H be graphs. A graph isomorphism is a bijection $\phi : V(G) \to V(H)$ such that it maps edges to edges, i.e. $uv \in E(G) \iff \phi(u)\phi(v) \in E(H)$.

Definition 1.2 (Subgraph)**.** We say H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Two subgraph types that are important enough to have their own notation:

- If G is a graph, and $xy \in E(G)$, define $G - xy$ to be the graph $(V(G), E(G) \setminus \{xy\})$.
- For $x, y \in V(G)$, define $G + xy$ to be the graph $(V(G), E(G) \cup \{xy\})$.

Definition 1.3 (Path)**.** Let G be a graph, $x, y \in V(G)$. A $x - y$ **path in** G is a sequence $x_1, \dots, x_k$ where $x_1 = x,\ x_k = y$ and $x_i x_{i+1} \in E(G)\ \forall 1 \le i \le k-1$ and all the x_i are distinct.

Definition 1.4. A graph is **connected** if $\forall x \neq y \in V(G)$, there exists an $x - y$ path in G.

Remark. A little annoyingly, if P is a $x - y$ path and P' is a $y - z$ path, then the concatenation PP' may not be a path (since the vertices of the new path might not be unique).

So let an $x - y$ **walk** in a graph G be a sequence $x_1, \dots, x_k$ where $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E(G)\ \forall 1 \le i \le k-1$. Then a concatenation of walks is again a walk.

Proposition 1.1. If W is an xy walk, then W contains a xy path.

Proof. Let $W' \subseteq W$ be a minimal xy walk. We claim this is a path. If not, then some vertex x_i must be visited at least twice, say $W' = x_1 x_2 \dots x_i \dots x_i x_l \dots x_k$. Then take $W'' = x_1 x_2 \dots x_i x_l \dots x_k$. This contradicts the minimality of W', so we're done. $\square$

10 Oct 2022, Lecture 2

Remark. We may define a **distance** on $V(G)$: for $x, y \in V(G)$, let $d(x, y)$ be the length of the shortest xy path. If G is connected, then this distance defines a metric on $V(G)$.

1.1 Trees

Definition 1.5. A graph G is **acyclic** if it does not contain a cycle as a subgraph.

Definition 1.6. A graph G is a **tree** if it is acyclic and connected.

Proposition 1.2. The following are equivalent:

1. G is a tree;
2. G is minimally connected ($\forall xy \in E(G)$, $G - xy$ is not connected);
3. G is maximally acyclic ($\forall xy \notin E(G)$, $G + xy$ contains a cycle).

Proof. (a) $\implies$ (b): A tree is connected. Assume for contradiction that $\exists xy \in E(G)$ such that $G - xy$ is connected. Let P be a xy path in $G - xy$. But then P defines a cycle in G, contradiction.

(b) $\implies$ (a): Minimally connected implies connected. For acyclicness, assume for contradiction that G contains a cycle C. Let $xy \in E(C)$. We claim that $G - xy$ is connected. Choose $u \neq v \in V(G - xy)$. Let P be a uv path in G. If P does not contain xy, we're done. If P does contain xy, then take paths $u \to x$; $x \to y$ along our cycle without using xy; $y \to v$. The concatenation of these gives a uv walk, which contains a uv path. Hence $G - xy$ is connected, contradiction.

(a) $\implies$ (c): A tree is acyclic. Let $xy \notin E(G), x \neq y$. Let P be a xy path. Then P defines a cycle in $G + xy$.

(c) $\implies$ (a): We have acyclicity. If G is not connected, $\exists x \neq y \in V(G)$ with no xy path. Then $G + xy$ is acyclic. $\square$

Definition 1.7. If T is a tree and $v \in V(T)$ with $\deg(v) = 1$, we call v a **leaf**.

Definition 1.8. Let G be a graph and $X \subseteq V(G)$. Then

$$G[X] = (X, E(G) \cap \{(u, v) \mid u, v \in X\}).$$

We call this **the graph induced on** X.

Definition 1.9. If $x \in V(G)$, define $G - x = G[V(G) \setminus \{x\}]$.

Proposition 1.3. Let T be a tree, $|T| \geq 2$. Then T has a leaf.

Proof. Let $P = x_1 \dots x_k$ be the a longest possible path in T. Note $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$. If $x_i \sim x_k$ for some $1 \leq i \leq k-2$, there is a cycle in T, contradiction. Thus $N(x_k) = \{x_{k-1}\} \implies X_k$ is a leaf. $\square$

Remark. We can show that any T has two leaves, but we can't do any better (consider a path).

Remark. We could have also proved this by taking a non-backtracking walk in G (i.e. go out of a different edge then you came in on). Either we return (hence have a cycle) or get stuck (hence have a leaf).

Proposition 1.4. Let T be a tree on $n \geq 1$ vertices. Then $e(G) = n - 1$.

Proof. By induction. $n = 1$ is trivial. Assume the claim holds for n. Take a tree T with $n + 1$ vertices. Let $x \in V(T)$ be a leaf. Then $T - x$ is connected and acyclic, therefore a tree, thus $e(T - x) = n - 1$. But $e(G) = e(G - x) + 1$ and $|V(G)| = |V(G - x)| + 1$, hence we're done. $\square$

Definition 1.10. Let G be a connected graph. Then a subgraph T of G is a **spanning tree** if T is a tree on $V(G)$.

Proposition 1.5. Every connected graph contains a spanning tree.

Proof. Start with the graph G, then throw away edges of $E(G)$ one by one subject to keeping the graph connected. At some point, the removal of any further edge will disconnect the graph, at which point we have a minimal connected subgraph of G, which by Prop. 1.2 is a tree. □

1.2 Bipartite graphs

Definition 1.11. Let $G = (V, E)$ be a graph. G is **bipartite** if there exists a partition $V = A \cup B$ such that $E(G) \subseteq \{uv \mid u \in A, v \in B\}$.

Definition 1.12. The **complete bipartite graph** $K_{n,m}$ is the graph with vertex set $A \cup B$, $A = \{x_1, \ldots, x_n\}, B = \{y_1, \ldots, y_m\}$ and edge set $E(K_{n,m}) = \{x_i y_j \mid x_i \in A, y_j \in B\}$.

Remark. There obviously exist non-bipartite graphs: odd cycles are not bipartite.

Definition 1.13. A **circuit** is a sequence $x_1, x_2, \ldots x_l x_{l+1}$, where $x_i x_{i+1} \in E(G)$ and $x_{l+1} = x_1$. The length of this circuit is l. We say a circuit is **odd** if its length is odd.

Proposition 1.6. Let C be an odd circuit in a graph G. Then C contains an odd cycle.

Proof. Let $x_1 x_2 \ldots x_i x_{i+1} \ldots x_i x_k \ldots x_l x_1$ be an odd circuit. Consider the circuits $C_1 = x_1 \ldots x_i x_k \ldots x_l x_1$ and $C_2 = x_i x_{i+1} \ldots x_{k-2} x_i$. Then one of C_1, C_2 has odd length and is strictly shorter, so we're done by induction. □

Theorem 1.7. Let G be a graph. Then

$$G \text{ is bipartite} \iff G \text{ does not contain an odd cycle.}$$

Proof. ($\implies$): If G contains an odd cycle, then as odd cycles are not bipartite, G cannot be bipartite.

($\impliedby$): We may assume that G is connected. Let us fix $x_0 \in V(G)$. Let

$$V_0 = \{x \in V(G) \mid d(x, x_0) \equiv 0 \pmod 2\}$$
$$V_1 = \{x \in V(G) \mid d(x, x_0) \equiv 1 \pmod 2\}.$$

We claim this is a bipartition of G. Assume for contradiction that $\exists u, v \in V_0$ s.t. $uv \in E(G)$. But there is an even ux_0 path and and an even vx_0 path, thus putting these three paths together gives an odd circuit in G. By Prop 1.6, G contains an odd cycle, contradiction. (Analogous proof for V_1). □

12 Oct 2022, Lecture 3

1.3 Planar graphs

Definition 1.14. A **planar graph** is a graph that can be drawn in the plane with no edge crossings.

Example 1.5. K_4 is planar. A path P_n is planar.

Definition 1.15. A **plane graph** is one such drawing of a graph in the plane without edge crossings.

Note that this is important, since we can draw K_4 in a way that it does have edges crossing.

Example 1.6. $K_{2,3}$ is planar. $K_{3,3}$ is not planar. K_5 is not planar (we don't prove this right now).

Question. What graphs are planar? Is there a (simple) method to decide if a graph is planar?

Definition 1.16. Let G be a plane graph. Consider $\mathbb{R}^2 \setminus G$. This is broken into finitely many regions. These are called the **faces** of the plane graph.

Definition 1.17. The **boundary** of a face F is the collection of vertices and edges on the topological boundary.

Remark. The boundary of a face need not be a cycle. It need also not be connected. In fact, the boundary need not contain a cycle at all (consider a tree).

Remark. We also note that two different drawings of a graph in the plane can be fundamentally different.

Theorem 1.8 (Euler)**.** Let G be a connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

Proof. We induct on m. $m = 1$ is clear. If G is acyclic, then G is a tree, so $m = n - 1$, $f = 1$ and we're done.

So assume G contains a cycle and let e be an edge on this cycle. Delete e. Then n stays fixed, m decreases by 1, and f decreases by 1, so by induction, $n - (m - 1) + (f - 1) = 2$ and we're done. □

Remark. We really do need the graph to be connected, consider t triangles in the plane as a counterexample.

Corollary 1.9. Let G be a planar graph, $|G| \geq 3$. Then $e(G) \leq 3|G| - 6$.

Proof. Draw G in the plane. We may assume that G is connected. Let F be a face, let $\deg(F) =$ the number of edges in G that touch F. Note $\deg(F) \geq 3$. Now note that since every edge touches at most two faces, we get

$$3f \leq \sum_{F \text{ a face}} \deg(F) \leq 2e(G) \implies f \leq \frac{2}{3}e(G).$$

Put this into Euler's formula to get

$$n - \frac{1}{3}e(G) \geq n - e(G) + f = 2 \implies 3(n-2) \geq e(G).$$

□

Remarks. (i): This is a statement about planar graphs only.

(ii): This is quite restrictive. K_n has $\binom{n}{2} \approx n^2/2$ edges, while our above corollary says the number of edges of a planar graph is linear in n.

Corollary 1.10. K_5 is not planar.

Proof. We have $e(K_5) = 10, n = 5$, so $10e(G) \not\leq 3|G| - 6 = 9$, so we're done by the above corollary. □

But $K_{3,3}$ does not fail this test. So we need to improve our argument:

Corollary 1.11. Let G be a planar graph, $|G| \geq 4$ and G has no cycles of length 3. Then $e(G) \leq 2|G| - 4$.

Proof. Repeat the proof of Corollary 1.9, but use $\deg(F) \geq 4$ for every face. □

Now we can see that $K_{3,3}$ is not planar. $K_{3,3}$ has no cycle of length 3 by definition, $n = 6$, $e(G) = 9$, so $9 = e(G) \not\leq 2 \cdot (6-2) = 8$.

14 Oct 2022, Lecture 4

Definition 1.18. A **subdivision** of a graph G is a subgraph where we replace some of the edges of G with disjoint paths.

Observation. A subdivision of a **non-planar** graph is non-planar.

Observation. If G contains a $K_{3,3}$ or K_5 subdivision as a subgraph, then G is non-planar.

Theorem 1.12 (Kuratowski's theorem)**.** G is planar $\iff$ G does not contain a subdivided $K_{3,3}$ or K_5.

We do not prove this, but the proof is actually not too hard.

2 Connectivity & matching

2.1 Matching in bipartite graphs

Let $G = (X \sqcup Y, E)$ be bipartite with bipartition X, Y.

Definition 2.1. A **matching from** X **to** Y is a set of edges $\{xy_x \mid x \in X, y_x \in Y\}$ and $x \to y_x$ is an injection.

Question. When does a bipartite graph have a X to Y matching?

We can first think about examples where we do not have a matching. For example, we clearly have no matching if $|X| > |Y|$.

Definition 2.2. Let G be a graph, $A \subseteq V(G)$. Define $N_G(A) = \bigcup_{x \in A} N(x)$.

Then we clearly also don't have a matching if we have $A \subset X$ such that $|N(A)| < |A|$. But this is actually the only obstruction:

Theorem 2.1 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph $G = (X \sqcup Y, E)$. Then*

$$G \text{ has a matching from } X \text{ to } Y \iff \forall A \subseteq X,\ |N(A)| \geq A.$$

The right-hand side is called Hall's criterion.

Proof. ($\Longrightarrow$) is the easy direction.

Now let M be a matching and let $A \subseteq X$. Then if $\{y_1, \ldots, y_{|A|}\}$ are matched to A, we show $|N(A)| \geq |\{y_1, \ldots, y_{|A|}\}| \geq |A|$.

($\Longleftarrow$): Apply induction on $|X|$. If $|X| = 1$, we're done. For the induction step, consider the following question: is there $\emptyset \neq A \subsetneq X$ such that $|N(A)| = |A|$?

If the answer is no, then $\forall A \subsetneq X$ we have $|N(A)| \geq |A| + 1$. Let $xy \in E(G)$ and let $G' = G[X \setminus \{x\} \cup Y \setminus \{y\}]$. We now check Hall's criterion for G'. If $B \subseteq X \setminus \{x\}$, then $|N_{G'}(B)| \geq |N_G(B)| - 1 \geq |B|$, so done by induction.

If the answer is yes, then let $G_1 = G[A \cup N(A)]$ and $G_2 = G[X \setminus A \cup Y \setminus N(A)]$.

Claim 1: G_1 satisfies Hall's criterion. Let $B \subseteq A$, then

$$|N_{G_1}(B)| = |N_G(B)| \geq B.$$

Claim 2: G_2 satisfies Hall's criterion. Let $B \subset X \setminus A$. Consider $N_G(A \cup B)$. One the one hand, $|N_G(A \cup B)| \geq |A| + |B|$. On the other hand, $|N_G(A)| + |N_{G_2}(B)| = |N_G(A \cup B)|$. As $|N(A)| = |A|$, we get $|N_{G_2}(B)| \geq |B|$.

From claims 1 and 2 we can apply induction in G_1, G_2 to get a matching in these graphs, and then put them together to get a matching in G. $\square$

Definition 2.3. A matching of deficiency of d from X to Y is a matching from X' to Y where $X' \subseteq X$, $|X| - d = |X'|$.

Theorem 2.2 (Defect Hall's Theorem)**.**

$$G \text{ contains a matching of deficiency } d \iff \forall A \subseteq X,\ |N(A)| \geq |A| - d.$$

Proof. ($\implies$) : easy.

($\impliedby$) : Add d phantom vertices to Y, which we join to all vertices in X, so we now satisfy Hall's condition. Apply Hall to get a matching, and then remove the d vertices we added, which removes at most d elements of X. □

Definition 2.4. Let G be a graph. The **minimum degree** in G is $\delta(G) = \min_{x \in V(G)} d(x)$, and the **maximal degree** in G is $\Delta(G) = \max_{x \in V(G)} d(x)$.

Definition 2.5. A graph is **regular** if $\delta(G) = \Delta(G)$. It is **k-regular** if $k = \delta(G) = \Delta(G)$.

17 Oct 2022, Lecture 5

Corollary 2.3. For $k \geq 1$, if $G = (X \sqcup Y, E)$ is a k-regular bipartite graph, then there exists a matching from X to Y.

Proof. We check Hall's criterion. Let $A \subseteq X$. On the one hand,

$$e(G[A \cup N(A)]) = \sum_{v \in A} \deg(v) = k|A|.$$

On the other hand,

$$e(G[A \cup N(A)]) \leq \sum_{v \in N(A)} \deg(v) \leq k|N(A)|.$$

Hence $|N(A)| \geq |A|$ and we're done. □

Let Γ be a finite group and let H be a subgroup of Γ. Let $L_1, \ldots, L_n$ be the set of left cosets and $R_1, \ldots, R_n$ be the right cosets (of the forms gH and Hg respectively).

Question. Is there $g_1, \ldots, g_n \in \Gamma$ such that $g_1H, \ldots, g_nH$ are the left cosets and $Hg_1, \ldots, Hg_n$ are the right cosets?

Corollary 2.4. There exist $g_1, \ldots, g_n \in \Gamma \geq H$ such that $g_1H, \ldots, g_nH$ are the left cosets and $Hg_1, \ldots, Hg_n$ are the right cosets.

Proof. It is enough to find a pairing $L_i \leftrightarrow R_{\sigma(i)}$ such that $L_i \cap R_{\sigma(i)} \neq \emptyset\ \forall i$. Then choose $g_i \in L_i \cap R_{\sigma(i)}$ and we have $g_iH = L_i$, $Hg_i = R_{\sigma(i)}$.

Define $X = \{R_1, \ldots, R_n\}$ and $Y = \{L_1, \ldots, L_n\}$, and define $R_i \sim L_j$ when $R_i \cap L_j \neq \emptyset \; \forall i, j$. Let $A = \{r_{i_1}, \ldots, R_{i_k}\}$. Note

$$\left| \bigcup_{j=1}^{k} R_{i_j} \right| = k|H|.$$

But $L_1, \ldots, L_n$ partition Γ and $|L_i| = |H|$, so at least k left cosets must intersect $\bigcup R_{i_j}$. Thus Hall's criterion is satisfied and we're done. $\square$

2.2 Connectivity

For a tree, $G - x$ (where x is any non-leaf) is disconnected. On the other hand, remove any 2 vertices from the Petersen graph and it stays connected (but if you remove any 3, you disconnect it).

Notation. Let $S \subseteq V(G)$, and let $G - S = G[V(G) \setminus S]$.

Definition 2.6. Let G be a graph, $|G| \geq 1$. Define

$$\kappa(G) = \min\{|S| \mid \exists S \subseteq V(G), G - S \text{ is disconnected or a single vertex}\}.$$

We say a graph G is k**-connected** if $\kappa(G) \geq k$.

In other words, G is k-connected if and only if $G - S$ is connected for all $S \subseteq V(G), |S| \leq k - 1$.

Example 2.1.

- $\kappa(\text{Tree}) = 1$.
- $\kappa(\text{Petersen graph}) = 3$, so we can say the Petersen graph is 3-connected.
- $\kappa(\text{Cycle}) = 2$.
- $\kappa(K_n) = n - 1$.

We have another natural definition of connectivity.

Definition 2.7. Let G be a graph and let $a, b \in V(G)$. Say that ab paths $P_1, \ldots, P_k$ are **disjoint** if $V(P_i) \cap V(P_j) = \{a, b\} \; \forall i \neq j$.

Amazingly, we have Menger's theorem: These two notions of connectivity (# of disjoint paths and $\kappa(G)$) are equivalent.

Remarks:

- We have $\delta(G) \geq \kappa(G)$. To see this, delete $N(x)$ for $x \in V(G)$ of minimal degree, then $G - N(x)$ is disconnected (or a single vertex).
- We have $\kappa(G - x) \geq \kappa(G) - 1$. This is clear: if $S \subset V(G - x)$ disconnects $G - x$ with $|S| \leq \kappa(G) - 2$, then $S \cup \{x\}$ disconnects G, contradiction.

- We can have $\kappa(G-x) > \kappa(G)$. For example, a cycle is 2-connected, but a cycle with one protruding edge is 1-connected.

Definition 2.8. A **component** in G is a maximal connected subgraph.

Definition 2.9. Let G be a graph, let $a, b \in V(G), a \neq b, a \not\sim b$. Say $S \subseteq V(G) \setminus \{a, b\}$is a $a-b$ **separator** if $G-S$ disconnects a from b (i.e. a, b are in different components of $G-S$).

Theorem 2.5 (Menger's theorem, form 1)**.** Let G be a connected graph and fix $a, b \in V(G), a \neq b, a \not\sim b$. Then the minimum size of an $a-b$ separator is equal to the maximal number of disjoint paths from a to b.

In other words, if all $a-b$ separators have size $\geq k$, then there exist $P_1, \ldots, P_k$, disjoint paths between a and b.

19 Oct 2022, Lecture 6

Note. Define $\kappa_{a,b}(G)$ be the size of the minimal $a-b$ separator.

Note. Recall $\kappa(G-x) \geq \kappa(G)-1$, and also $\kappa(G-xy) \geq \kappa(G)-1$. We also have $\kappa_{a,b}(G-x) \geq \kappa_{a,b}(G)-1$ and $\kappa_{a,b}(G-xy) \geq \kappa_{a,b}(G)-1$ (exercise, not hard).

Proof. Assume for contradiction that the statement of the theorem is false. Let G be a minimal counterexample to the theorem that

(a) minimizes k;

(b) subject to (a), choose G to minimize $e(G)$.

Now let S be a minimal a, b separator in G. We have $|S| = k$. Note that the theorem is true for $k = 1$, so assume $k \geq 2$.

If $S \neq N(A)$ and $S \neq N(B)$, consider $G - S$ and let A be the component containing a and B be the component containing B.

Define $G_a = G[A \cup S]$ along with a vertex c joined to each vertex in S, and $G_b = G[B \cup S]$ along with a vertex c joined to each vertex in S. Note that $\kappa_{a,c}(G_a) \geq k$, since any $a-c$ separator in G_a is a a, b separator in G. Likewise, $\kappa_{b,c}(G_b) \geq k$.

Note that $e(G_a) < e(G), e(G_b) < e(G)$ since $N(a) \not\subset S, N(b) \not\subset S$. So there exists a neighbor x of b in B with $\deg(x) \geq 2$, else we can remove x and apply minimality.

So by minimality of G, we can find k disjoint a, c paths, say $P_1, \ldots, P_k$ in G_a, and likewise we can find k b, c paths $Q_1, \ldots, Q_k$ in G_b. We can put these paths together to get paths $P_1Q_{\sigma(1)}, \ldots, P_kQ_{\sigma(k)}$, which are k disjoint a, b paths, contradiction, done.

Let us now assume WLOG that $S = N(a)$.

Claim: $N(a) \cap N(b) = \emptyset$.

Indeed, if $\exists x \in N(a) \cap N(b)$, consider $G - x$. We have $\kappa_{a,b}(G - x) \geq k - 1$. Thus, by minimality, we can find $k - 1$ disjoint ab paths in $G - x$, so all of these, plus axb, gives us k disjoint ab paths in G, contradiction.

Let $ax_1 \dots x_l b$ be a shortest ab path. Note that $l \geq 2$ and $x_2 \neq b$. Consider $G - x_1x_2$. We must have $\kappa_{a,b}(G - x_1x_2) \leq k - 1$ by minimality, so $\kappa_{a,b}(G - x_1x_2) = k - 1$. So there is a a, b separator $\tilde{S}$, $|\tilde{S}| = k - 1$ in $G - x_1x_2$. We see that $\tilde{S} \cup \{x_1\}$ and $\tilde{S} \cup \{x_2\}$ are a, b separators in G of size at most k. Now either $\tilde{S} \cup \{x_1\} \neq N(a), N(b)$ or $\tilde{S} \cup \{x_2\} \neq N(a), N(b)$, so we're done. $\square$

Corollary 2.6 (Menger's theorem, form 2)**.** Let G be a connected graph, $|G| \geq 2$. Then G is k-connected $\iff$ $\forall a, b \in V(G), a \neq b$, there exist k disjoint ab paths in G.

Proof. $\Longleftarrow$ is the easy direction. Say $G - S$ is disconnected and let a, b be in different components of $G - S$. Note $a \not\sim b$. Then $\exists$ k disjoint $a - b$ paths and S must intersect each of these, so $|S| \geq k$.

$\Longrightarrow$. Let $a, b \in V(G), a \neq b$. If $a \not\sim b$, then just apply Menger form 1 and we're done. If $a \sim b$, then consider $G - ab$. We have $\kappa_{a,b}(G - ab) \geq k - 1$, so apply Menger form 1 to get $k - 1$ disjoint paths and add back ab as a k^{th} path. $\square$

2.2.1 Edge connectivity

Let G be a graph. Let $\lambda(G) = \min\{|W| \mid W \subseteq E(G), G - W \text{ is disconnected}\}$. We say that G is k**-edge connected** if $\lambda(G) \geq k$. 21 Oct 2022, Lecture 7

Example 2.2.

- A cycle has $\kappa(C_n) = 2, \lambda(C_n) = 2$.
- A "bowtie graph" has $\kappa(G) = 1, \lambda(G) = 2$. We can generalize this and take two copies of K_n which intersect in one vertex, then $\kappa(G) = 1$ and $\lambda(G) = n - 1$.

Definition 2.10. We say paths $P_1, \dots, P_k$ are **edge disjoint** if

$$E(P_i) \cap E(P_j) = \emptyset \ \forall i \neq j.$$

Theorem 2.7 (Menger, edge version)**.** Let G be a connected graph and $a, b \in V(G), a \neq b$. Then, every $W \subseteq E(G)$ that separates a from b having size $\geq k \implies \exists k$ edge disjoint $a - b$ paths $P_1, \dots, P_k$.

Definition 2.11. Let G be a graph. The **line graph** of G, denoted $L(G)$, is defined to be the graph

$$V(L(G)) = E(G);$$
$$\text{If } e, f \in E(G), \text{ then } e \sim f \text{ if they share a vertex.}$$

Proof of Thorem 2.7. Given G, define a new graph G' by taking the line graph of G and adding a vertex a', which we join to all edges incident to $a \in G$, and similarly adding a vertex b', which we join to all edges incident to $b \in G$.

Note that there is a ab path in G if and only if there is an $a'b'$ path in G'. Thus $W \subseteq V(G') \setminus \{a, b\}$ is a $a'b'$ separator if and only if $W \subseteq E(G)$ is an ab separator. Hence $\kappa_{a,b}(G') \geq k$.

Now apply Menger (form 1) to find k disjoint ab paths $P_1, \ldots, P_k$ in G'. These describe edge disjoint walks in G from a to b. Thus there are disjoint paths $\tilde{P}_1 \subseteq P_1, \ldots, \tilde{P}_k \subseteq P_k$ and we're done. □

Theorem 2.8 (Menger, edge version 2)**.** Let G be a connected graph. Then $\lambda(G) \geq k \iff \forall a, b \in V(G)$ with $a \neq b, \exists k$ edge disjoint ab paths.

Proof. $\Longleftarrow$ is the easy direction. To separate any two vertices, say a, b, we must remove an edge from each of the ab paths, so $\lambda(G) \geq k$. $\Longrightarrow$ follows from Menger, edge version 1. □

3 Graph coloring

Definition 3.1. We say that $c : V(G) \to \{1, \ldots, k\}$ is a k**-coloring** (or a proper k-coloring) if $c(x) \neq c(y)\ \forall x \sim y$.

Definition 3.2. The **chromatic number of** G is

$$\mathcal{X}(G) = \min\{k \mid \exists \text{ a } k\text{coloring of } G\}.$$

Example 3.1.

- A path has chromatic number 2. $\mathcal{X}(P_n) = 2$.
- For a cycle, $\mathcal{X}(G) = \begin{cases} 2 & \text{if } n \text{ is even.} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$
- A tree has chromatic number 2 by induction.
- A complete graph has $\mathcal{X}(K_n) = n$.
- A bipartite graph has $\mathcal{X}(K_{m,n}) = 2$. In fact, a graph G is bipartite if and only if $\mathcal{X}(G) = 2$.

Proposition 3.1. Let G be a graph. Then $\mathcal{X}(G) \geq \Delta(G) + 1$.

Proof. Let $x_1, \ldots, x_n$ be an ordering of $V(G)$. We color the (x_i) one at a time in this order. When we come to vertex x_i, at most Δ colors have been used in $N(x_i)$, so there is a free color for x_i. □

Remark. This proposition is sharp (e.g. on K_n).

Remark. This is sometimes called a greedy coloring. But a greedy coloring may produce a coloring that is not optimal!

3.1 Coloring planar graphs

Observation. Let G be a planar graph. Then $\delta(G) \leq 5$.

Proof. The average degree of G is

$$\frac{1}{n} \sum_{v \in V(G)} \deg(v) = \frac{2e(G)}{n} \leq \frac{2(3n-6)}{n} = 6 - \frac{12}{n} < 6.$$

But all degrees of vertices are integers, so the minimal degree is ≤ 5. $\square$

Proposition 3.2. If G is planar, then $\mathcal{X}(G) \leq 6$.

Proof. We induct. Base step: if $|G| \leq 6$, then we're clearly done.

Induction step: Given a graph G, let x be a vertex with $\deg(x) \leq 5$. Apply induction to $G - x$, which gives a coloring of $G - x$ with 6 colors. But x has degree ≤ 5, thus there is a free color to color x with. $\square$