

# Part II - Graph Theory

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## 0 Introduction

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Lecture 1

**Notation.** We write  $[n]$  for  $\{1, 2, \dots, n\}$ . For a set  $X$  and  $k \in \mathbb{N}$ , define  $X^{(k)} = \{S \subset X \mid |S| = k\}$ , i.e. the set of all subsets of size  $k$ .

## 1 Fundamentals

**Definition 1.1.** A **graph** is an object  $G = (V, E)$  where  $V$  is a set and  $E \subseteq V^{(2)}$ .

$V$  is the set of vertices, and  $E$  is the set of edges.

$V(G)$  will denote  $V$ ,  $E(G)$  will denote  $E$ , and we define  $|G| = |V(G)|$  (sometimes called the order) and  $e(G) = |E(G)|$  (sometimes called the size).

**Example 1.1.** The **complete graph** on  $n$  vertices is denoted  $K_n$ . This is the graph where  $V(K_n) = [n]$  and  $E(K_n) = [n]^{(2)}$ .

**Remark.** We assume the following:

- We don't order edges;
- We don't allow loops (an edge joining a vertex to itself);
- We don't allow multiple edges;
- Most of the time,  $V(G)$  will be finite (we will explicitly say when it's not).

**Example 1.2.** The **empty graph** on  $n$  vertices, denoted  $\overline{K_n}$ , has  $V(\overline{K_n}) = [n]$  and  $E(\overline{K_n}) = \emptyset$ .

**Example 1.3.** The path of length  $n$ , denoted  $P_n$ , is a path: it has  $V(P_n) = [n+1]$  and  $E(P_n) = \{\{i, i+1\} \mid 1 \leq i \leq n\}$ .

**Example 1.4.** The cycle of length  $n$ , denoted  $C_n$ , has  $V(C_n) = [n]$  and  $E(C_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$ .

Let  $G$  be a graph and  $x \in V(G)$ . The **neighborhood** of  $x$  is  $N(x) = \{y \mid xy \in E(G)\}$ , i.e. all the vertices connected to  $x$ . If  $y \in N(x)$ , we write  $x \sim y$  and say  $y$  is a **neighbor** of  $x$  or that  $y$  is **adjacent** to  $x$ .

The **degree** of  $x$  is  $\deg(x) = |N(x)|$ .

Just as a formality, we define graph isomorphism: let  $G, H$  be graphs. A graph isomorphism is a bijection  $\phi : V(G) \rightarrow V(H)$  such that it maps edges to edges, i.e.  $uv \in E(G) \iff \phi(u)\phi(v) \in E(H)$ .

**Definition 1.2** (Subgraph). We say  $H$  is a **subgraph** of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .

Two subgraph types that are important enough to have their own notation:

- If  $G$  is a graph, and  $xy \in E(G)$ , define  $G - xy$  to be the graph  $(V(G), E(G) \setminus \{xy\})$ .
- For  $x, y \in V(G)$ , define  $G + xy$  to be the graph  $(V(G), E(G) \cup \{xy\})$ .

**Definition 1.3** (Path). Let  $G$  be a graph,  $x, y \in V(G)$ . A  $x - y$  **path** in  $G$  is a sequence  $x_1, \dots, x_k$  where  $x_1 = x$ ,  $x_k = y$  and  $x_i x_{i+1} \in E(G) \forall 1 \leq i \leq k - 1$  and all the  $x_i$  are distinct.

**Definition 1.4.** A graph is **connected** if  $\forall x \neq y \in V(G)$ , there exists an  $x - y$  path in  $G$ .

**Remark.** A little annoyingly, if  $P$  is a  $x - y$  path and  $P'$  is a  $y - z$  path, then the concatenation  $PP'$  may not be a path (since the vertices of the new path might not be unique).

So let an  $x - y$  **walk** in a graph  $G$  be a sequence  $x_1, \dots, x_k$  where  $x_1 = x$ ,  $x_k = y$  and  $x_i x_{i+1} \in E(G) \forall 1 \leq i \leq k - 1$ . Then a concatenation of walks is again a walk.

**Proposition 1.1.** If  $W$  is an  $xy$  walk, then  $W$  contains a  $xy$  path.

*Proof.* Let  $W' \subseteq W$  be a minimal  $xy$  walk. We claim this is a path. If not, then some vertex  $x_i$  must be visited at least twice, say  $W' = x_1 x_2 \dots x_i \dots x_i x_l \dots x_k$ . Then take  $W'' = x_1 x_2 \dots x_i x_l \dots x_k$ . This contradicts the minimality of  $W'$ , so we're done.  $\square$

**Remark.** We may define a **distance** on  $V(G)$ : for  $x, y \in V(G)$ , let  $d(x, y)$  be the length of the shortest  $xy$  path. If  $G$  is connected, then this distance defines a metric on  $V(G)$ .

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Lecture 2

## 1.1 Trees

**Definition 1.5.** A graph  $G$  is **acyclic** if it does not contain a cycle as a subgraph.

**Definition 1.6.** A graph  $G$  is a **tree** if it is acyclic and connected.

**Proposition 1.2.** The following are equivalent:

1.  $G$  is a tree;
2.  $G$  is minimally connected (  $\forall xy \in E(G)$ ,  $G - xy$  is not connected);
3.  $G$  is maximally acyclic (  $\forall xy \notin E(G)$ ,  $G + xy$  contains a cycle).

*Proof.* (a)  $\implies$  (b): A tree is connected. Assume for contradiction that  $\exists xy \in E(G)$  such that  $G - xy$  is connected. Let  $P$  be a  $xy$  path in  $G - xy$ . But then  $P$  defines a cycle in  $G$ , contradiction.

(b)  $\implies$  (a): Minimally connected implies connected. For acyclicity, assume for contradiction that  $G$  contains a cycle  $C$ . Let  $xy \in E(C)$ . We claim that  $G - xy$  is connected. Choose  $u \neq v \in V(G - xy)$ . Let  $P$  be a  $uv$  path in  $G$ . If  $P$  does not contain  $xy$ , we're done. If  $P$  does contain  $xy$ , then take paths  $u \rightarrow x$ ;  $x \rightarrow y$  along our cycle without using  $xy$ ;  $y \rightarrow v$ . The concatenation of these gives a  $uv$  walk, which contains a  $uv$  path. Hence  $G - xy$  is connected, contradiction.

(a)  $\implies$  (c): A tree is acyclic. Let  $xy \notin E(G)$ ,  $x \neq y$ . Let  $P$  be a  $xy$  path. Then  $P$  defines a cycle in  $G + xy$ .

(c)  $\implies$  (a): We have acyclicity. If  $G$  is not connected,  $\exists x \neq y \in V(G)$  with no  $xy$  path. Then  $G + xy$  is acyclic.  $\square$

**Definition 1.7.** If  $T$  is a tree and  $v \in V(T)$  with  $\deg(v) = 1$ , we call  $v$  a **leaf**.

**Definition 1.8.** Let  $G$  be a graph and  $X \subseteq V(G)$ . Then

$$G[X] = (X, E(G) \cap \{(u, v) \mid u, v \in X\}).$$

We call this **the graph induced on  $X$** .

**Definition 1.9.** If  $x \in V(G)$ , define  $G - x = G[V(G) \setminus \{x\}]$ .

**Proposition 1.3.** Let  $T$  be a tree,  $|T| \geq 2$ . Then  $T$  has a leaf.

*Proof.* Let  $P = x_1 \dots x_k$  be the a longest possible path in  $T$ . Note  $N(x_k) \subseteq \{x_1, \dots, x_{k-1}\}$ . If  $x_i \sim x_k$  for some  $1 \leq i \leq k-2$ , there is a cycle in  $T$ , contradiction. Thus  $N(x_k) = \{x_{k-1}\} \implies x_k$  is a leaf.  $\square$

**Remark.** We can show that any  $T$  has two leaves, but we can't do any better (consider a path).

**Remark.** We could have also proved this by taking a non-backtracking walk in  $G$  (i.e. go out of a different edge then you came in on). Either we return (hence have a cycle) or get stuck (hence have a leaf).

**Proposition 1.4.** Let  $T$  be a tree on  $n \geq 1$  vertices. Then  $e(G) = n - 1$ .

*Proof.* By induction.  $n = 1$  is trivial. Assume the claim holds for  $n$ . Take a tree  $T$  with  $n + 1$  vertices. Let  $x \in V(T)$  be a leaf. Then  $T - x$  is connected and acyclic, therefore a tree, thus  $e(T - x) = n - 1$ . But  $e(G) = e(G - x) + 1$  and  $|V(G)| = |V(G - x)| + 1$ , hence we're done.  $\square$

**Definition 1.10.** Let  $G$  be a connected graph. Then a subgraph  $T$  of  $G$  is a **spanning tree** if  $T$  is a tree on  $V(G)$ .

**Proposition 1.5.** Every connected graph contains a spanning tree.

*Proof.* Start with the graph  $G$ , then throw away edges of  $E(G)$  one by one subject to keeping the graph connected. At some point, the removal of any further edge will disconnect the graph, at which point we have a minimal connected subgraph of  $G$ , which by Prop. 1.2 is a tree.  $\square$

## 1.2 Bipartite graphs

**Definition 1.11.** Let  $G = (V, E)$  be a graph.  $G$  is **bipartite** if there exists a partition  $V = A \cup B$  such that  $E(G) \subseteq \{uv \mid u \in A, v \in B\}$ .

**Definition 1.12.** The **complete bipartite graph**  $K_{n,m}$  is the graph with vertex set  $A \cup B$ ,  $A = \{x_1, \dots, x_n\}$ ,  $B = \{y_1, \dots, y_m\}$  and edge set  $E(K_{n,m}) = \{x_i y_j \mid x_i \in A, y_j \in B\}$ .

**Remark.** There obviously exist non-bipartite graphs: odd cycles are not bipartite.

**Definition 1.13.** A **circuit** is a sequence  $x_1, x_2, \dots, x_l x_{l+1}$ , where  $x_i x_{i+1} \in E(G)$  and  $x_{l+1} = x_1$ . The length of this circuit is  $l$ . We say a circuit is **odd** if its length is odd.

**Proposition 1.6.** Let  $C$  be an odd circuit in a graph  $G$ . Then  $C$  contains an odd cycle.

*Proof.* Let  $x_1 x_2 \dots x_i x_{i+1} \dots x_i x_k \dots x_l x_1$  be an odd circuit. Consider the circuits  $C_1 = x_1 \dots x_i x_k \dots x_l x_1$  and  $C_2 = x_i x_{i+1} \dots x_k x_{k-1} x_i$ . Then one of  $C_1, C_2$  has odd length and is strictly shorter, so we're done by induction.  $\square$

**Theorem 1.7.** Let  $G$  be a graph. Then

$$G \text{ is bipartite} \iff G \text{ does not contain an odd cycle.}$$

*Proof.* ( $\implies$ ): If  $G$  contains an odd cycle, then as odd cycles are not bipartite,  $G$  cannot be bipartite.

( $\impliedby$ ): We may assume that  $G$  is connected. Let us fix  $x_0 \in V(G)$ . Let

$$V_0 = \{x \in V(G) \mid d(x, x_0) \equiv 0 \pmod{2}\}$$

$$V_1 = \{x \in V(G) \mid d(x, x_0) \equiv 1 \pmod{2}\}.$$

We claim this is a bipartition of  $G$ . Assume for contradiction that  $\exists u, v \in V_0$  s.t.  $uv \in E(G)$ . But there is an even  $ux_0$  path and an even  $vx_0$  path, thus putting these three paths together gives an odd circuit in  $G$ . By Prop 1.6,  $G$  contains an odd cycle, contradiction. (Analogous proof for  $V_1$ ).  $\square$

### 1.3 Planar graphs

**Definition 1.14.** A **planar graph** is a graph that can be drawn in the plane with no edge crossings.

**Example 1.5.**  $K_4$  is planar. A path  $P_n$  is planar.

**Definition 1.15.** A **plane graph** is one such drawing of a graph in the plane without edge crossings.

Note that this is important, since we can draw  $K_4$  in a way that it does have edges crossing.

**Example 1.6.**  $K_{2,3}$  is planar.  $K_{3,3}$  is not planar.  $K_5$  is not planar (we don't prove this right now).

**Question.** What graphs are planar? Is there a (simple) method to decide if a graph is planar?

**Definition 1.16.** Let  $G$  be a plane graph. Consider  $\mathbb{R}^2 \setminus G$ . This is broken into finitely many regions. These are called the **faces** of the plane graph.

**Definition 1.17.** The **boundary** of a face  $F$  is the collection of vertices and edges on the topological boundary.

**Remark.** The boundary of a face need not be a cycle. It need also not be connected. In fact, the boundary need not contain a cycle at all (consider a tree).

**Remark.** We also note that two different drawings of a graph in the plane can be fundamentally different.

**Theorem 1.8** (Euler). Let  $G$  be a connected plane graph with  $n$  vertices,  $m$  edges and  $f$  faces. Then  $n - m + f = 2$ .

*Proof.* We induct on  $m$ .  $m = 1$  is clear. If  $G$  is acyclic, then  $G$  is a tree, so  $m = n - 1$ ,  $f = 1$  and we're done.

So assume  $G$  contains a cycle and let  $e$  be an edge on this cycle. Delete  $e$ . Then  $n$  stays fixed,  $m$  decreases by 1, and  $f$  decreases by 1, so by induction,  $n - (m - 1) + (f - 1) = 2$  and we're done.  $\square$

**Remark.** We really do need the graph to be connected, consider  $t$  triangles in the plane as a counterexample.

**Corollary 1.9.** Let  $G$  be a planar graph,  $|G| \geq 3$ . Then  $e(G) \leq 3|G| - 6$ .

*Proof.* Draw  $G$  in the plane. We may assume that  $G$  is connected. Let  $F$  be a face, let  $\deg(F)$  = the number of edges in  $G$  that touch  $F$ . Note  $\deg(F) \geq 3$ . Now note that since every edge touches at most two faces, we get

$$3f \leq \sum_{F \text{ a face}} \deg(F) \leq 2e(G) \implies f \leq \frac{2}{3}e(G).$$

Put this into Euler's formula to get

$$n - \frac{1}{3}e(G) \geq n - e(G) + f = 2 \implies 3(n - 2) \geq e(G).$$

□

**Remarks.** (i): This is a statement about planar graphs only.

(ii): This is quite restrictive.  $K_n$  has  $\binom{n}{2} \approx n^2/2$  edges, while our above corollary says the number of edges of a planar graph is linear in  $n$ .

**Corollary 1.10.**  $K_5$  is not planar.

*Proof.* We have  $e(K_5) = 10, n = 5$ , so  $10e(G) \not\leq 3|G| - 6 = 9$ , so we're done by the above corollary. □

But  $K_{3,3}$  does not fail this test. So we need to improve our argument:

**Corollary 1.11.** Let  $G$  be a planar graph,  $|G| \geq 4$  and  $G$  has no cycles of length 3. Then  $e(G) \leq 2|G| - 4$ .

*Proof.* Repeat the proof of Corollary 1.9, but use  $\deg(F) \geq 4$  for every face. □

Now we can see that  $K_{3,3}$  is not planar.  $K_{3,3}$  has no cycle of length 3 by definition,  $n = 6, e(G) = 9$ , so  $9 = e(G) \not\leq 2 \cdot (6 - 2) = 8$ .

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Lecture 4

**Definition 1.18.** A **subdivision** of a graph  $G$  is a subgraph where we replace some of the edges of  $G$  with disjoint paths.

**Observation.** A subdivision of a **non-planar** graph is non-planar.

**Observation.** If  $G$  contains a  $K_{3,3}$  or  $K_5$  subdivision as a subgraph, then  $G$  is non-planar.

**Theorem 1.12** (Kuratowski's theorem).  $G$  is planar  $\iff G$  does not contain a subdivided  $K_{3,3}$  or  $K_5$ .

We do not prove this, but the proof is actually not too hard.

## 2 Connectivity & matching

### 2.1 Matching in bipartite graphs

Let  $G = (X \sqcup Y, E)$  be bipartite with bipartition  $X, Y$ .

**Definition 2.1.** A **matching from  $X$  to  $Y$**  is a set of edges  $\{xy_x \mid x \in X, y_x \in Y\}$  and  $x \rightarrow y_x$  is an injection.

**Question.** When does a bipartite graph have a  $X$  to  $Y$  matching?

We can first think about examples where we do not have a matching. For example, we clearly have no matching if  $|X| > |Y|$ .

**Definition 2.2.** Let  $G$  be a graph,  $A \subseteq V(G)$ . Define  $N_G(A) = \bigcup_{x \in A} N(x)$ .

Then we clearly also don't have a matching if we have  $A \subset X$  such that  $|N(A)| < |A|$ . But this is actually the only obstruction:

**Theorem 2.1** (Hall's Marriage Theorem). Let  $G$  be a bipartite graph  $G = (X \sqcup Y, E)$ . Then

$$G \text{ has a matching from } X \text{ to } Y \iff \forall A \subseteq X, |N(A)| \geq |A|.$$

The right-hand side is called Hall's criterion.

*Proof.* ( $\implies$ ) is the easy direction.

Now let  $M$  be a matching and let  $A \subseteq X$ . Then if  $\{y_1, \dots, y_{|A|}\}$  are matched to  $A$ , we show  $|N(A)| \geq |\{y_1, \dots, y_{|A|}\}| \geq |A|$ .

( $\impliedby$ ): Apply induction on  $|X|$ . If  $|X| = 1$ , we're done. For the induction step, consider the following question: is there  $\emptyset \neq A \subsetneq X$  such that  $|N(A)| = |A|$ ?

If the answer is no, then  $\forall A \subsetneq X$  we have  $|N(A)| \geq |A| + 1$ . Let  $xy \in E(G)$  and let  $G' = G[X \setminus \{x\} \cup Y \setminus \{y\}]$ . We now check Hall's criterion for  $G'$ . If  $B \subseteq X \setminus \{x\}$ , then  $|N_{G'}(B)| \geq |N_G(B)| - 1 \geq |B|$ , so done by induction.

If the answer is yes, then let  $G_1 = G[A \cup N(A)]$  and  $G_2 = G[X \setminus A \cup Y \setminus N(A)]$ .

Claim 1:  $G_1$  satisfies Hall's criterion. Let  $B \subseteq A$ , then

$$|N_{G_1}(B)| = |N_G(B)| \geq |B|.$$

Claim 2:  $G_2$  satisfies Hall's criterion. Let  $B \subset X \setminus A$ . Consider  $N_G(A \cup B)$ . On the one hand,  $|N_G(A \cup B)| \geq |A| + |B|$ . On the other hand,  $|N_G(A)| + |N_{G_2}(B)| = |N_G(A \cup B)|$ . As  $|N(A)| = |A|$ , we get  $|N_{G_2}(B)| \geq |B|$ .

From claims 1 and 2 we can apply induction in  $G_1, G_2$  to get a matching in these graphs, and then put them together to get a matching in  $G$ .  $\square$



**Definition 2.3.** A matching of deficiency of  $d$  from  $X$  to  $Y$  is a matching from  $X'$  to  $Y$  where  $X' \subseteq X$ ,  $|X| - d = |X'|$ .

**Theorem 2.2** (Defect Hall's Theorem).

$G$  contains a matching of deficiency  $d \iff \forall A \subseteq X, |N(A)| \geq |A| - d$ .

*Proof.* ( $\implies$ ) : easy.

( $\impliedby$ ) : Add  $d$  phantom vertices to  $Y$ , which we join to all vertices in  $X$ , so we now satisfy Hall's condition. Apply Hall to get a matching, and then remove the  $d$  vertices we added, which removes at most  $d$  elements of  $X$ .  $\square$

**Definition 2.4.** Let  $G$  be a graph. The **minimum degree** in  $G$  is  $\delta(G) = \min_{x \in V(G)} d(x)$ , and the **maximal degree** in  $G$  is  $\Delta(G) = \max_{x \in V(G)} d(x)$ .

**Definition 2.5.** A graph is **regular** if  $\delta(G) = \Delta(G)$ .