Part III - Commutative Algebra

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0 Introduction

05 Oct 2022, Lecture 1

In this course, a ring R will be a commutative ring with a 1. However, we start with a noncommutative expection:

$$\mathrm{End}(M) = \{f : M \to M \mid f \text{ a group homomorphism}\},$$

the endomorphisms of an abelian group $(M, +)$ with the multiplication being given by composition.

Definition 0.1 (Module)**.** An R-module M is an abelian group M with a fixed ring homomorphism $\rho : R \to \mathrm{End}(M)$, given by $r \cdot m = \rho(r)(m)$.

We have that

(i) $r(m_1 + m_2) = \rho(r)(m_1 + m_2) = \rho(r)(m_1) + \rho(r)(m_2) = rm_1 + rm_2$ since $\rho(r)$ is a group homomorphism $M \to M$.

(ii) $(r_1 + r_2) \cdot m = \rho(r_1 + r_2)m = (\rho(r_1) + \rho(r_2))m = r_1 m + r_2 m$ since ρ is a ring homomorphism.

Example 0.1. (i) For k a field, a k-module is a k-vector space.

(ii) Every abelian group M is a $\mathbb{Z}$-module in a unique way through $\mathbb{Z} \to \mathrm{End}(M)$ given by $1_{\mathbb{Z}} \mapsto \mathrm{id}$.

(iii) Every ring R is an R-module via $R \to \mathrm{End}(R)$, $r_0 \mapsto (r \mapsto r_0 r)$.

(iv) $R^{\oplus \mathbb{N}}$, the direct sum, and $R^{\mathbb{N}}$, the direct product, are R-modules.

1 Chain Conditions

Definition 1.1. An R-module M is called **Noetherian** if

(i) every ascending chain of submodules $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ stabilises, i.e. there exists n such that $M_n = M_{n+1} = \dots$.

(ii) every nonempty set Σ of submodules of M has a maximal element, i.e. there exists $M_0 \in \Sigma$ such that for all $M' \in \Sigma$, $M_0 \subseteq M'$ implies $M' = M_0$.

Definition 1.2. M is called **Artinian** if the same holds, but with descending chains and minimal elements.

Lemma 1.1. An R-module M is Noetherian if and only if every submodule of M is finitely generated.

In particular, every Noetherian module is finitely generated.

Example 1.1. If $R = \mathbb{Z}[T_1, T_2, T_3, \ldots]$ and $M = R$ as an R-module, then is M finitely generated? Yes, by 1_R, as $1_R \in M \implies r \cdot 1_R \in M = R$.

On the other hand, take $M' = \langle T_1, T_2, T_3, \ldots \rangle$, which is not finitely generated, since any finite subset only involves finitely many variables.

Definition 1.3. A ring R is Noetherian (respectively Artinian) if R as an R-module is Noetherian (respectively Artinian).

Example 1.2. $\mathbb{Z}$ is a Noetherian, but not an Artinian module. It is Noetherian since every submodule is finitely generated (by one element, since it is a PID), but $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq \ldots$ is a descending chain of ideals that does not stabilise. The same example works for a Noetherian non-Artinian ring.

For an Artinian non-Noetherian module, we have the non-trivial example $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$, where $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^n} \mid a, n \in \mathbb{Z}\}$.

We will later show that a ring R is Artinian $\iff$ R is Noetherian and R has Krull dimension zero.

Definition 1.4. A sequence

$$\ldots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \ldots$$

of R-modules and R-module homomorphisms is **exact** if $\operatorname{im}(f_i) = \ker(f_{i+1})$ $\forall i$.

Definition 1.5. A **short exact sequence** (SES) is an exact sequence of the form

$$0 \to M' \xrightarrow{i} M \to M'' \to 0.$$

Hence $M'' \cong M/i(M')$. Note that by exactness,

$$0 \to M' \xrightarrow{\text{injective}} M \xrightarrow{\text{surjective}} M'' \to 0.$$

Lemma 1.2. Let $0 \to N \to M \to L \to 0$ be a short exact sequence of R-modules. Then M is Noetherian (Artinian) $\iff$ N and L are Noetherian (Artinian).

In other words, a module is Noetherian if and only if both the submodule and the quotient are Noetherian.

Corollary 1.3. If $M_1, \ldots, M_n$ are Noetherian (Artinian) modules, then so is $M_1 \oplus \ldots \oplus M_n$.

Reminder. A module homomorphism from $M_1 \oplus \ldots \oplus M_n \to L$ (for all the M_i R-modules) is just a collection of homomorphisms $\phi_i : M_i \to L$.

Proposition 1.4. For a Noetherian (Artinian) ring R, every finitely generated R-module is Noetherian (Artinian).

Proof. M is finitely generated $\iff$ there exists $n \geq 1$ and a surjective map $\phi : R^n \to M$. So R^n is Noetherian and being Noetherian passes to quotients. □

Definition 1.6 (Algebra)**.** An R**-algebra** is a ring A together with a fixed ring homomorphism $\rho : R \to A$. We write ra for $\rho(r)a$. Hence $\rho(r) = \rho(r) \cdot 1_A = r \cdot 1_A$, so we don't need to explicitly define ρ.

Example 1.3. A polynomial algebra $k \to k[T_1, \dots, T_n]$. As a module, this is infinite-dimensional. As a k-algebra, it is generated by $T_1, \dots, T_n$.

Definition 1.7. An R-algebra is **finitely generated** if and only if there exists $n \geq 0$ and a surjective R-algebra homomorphism $\phi : R[T_1, \dots, T_n] \to A$.

Definition 1.8. $\phi : A \to B$ is an R**-algebra homomorphism** if and only if ϕ is a ring homomorphism and $\phi(r \cdot 1_A) = r \cdot 1_B$ for all $r \in R$.