Part III - Algebraic Geometry

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Contents

0 Introduction

6 Oct 2022, Lecture 1

The course consists of four parts.

(1) Basics of sheaves on topological spaces.

(2) Definition of schemes and morphisms.

(3) Properties of schemes (e.g. the algebraic geometry notion of compactness and other properties).

(4) A rapid introduction to the cohomology of schemes.

The main reference for the course is Hartshorne's *Algebraic Geometry*.

1 Beyond algebraic varieties

08 Oct 2022, Lecture 2

1.1 Summary of classical algebraic geometry

We let $k = \overline{k}$ be a algebraically closed field and consider $\mathbb{A}^n_k = \mathbb{A}^n = k^n$ as a set.

Definition 1.1. An **affine variety** is a subset $V \subset \mathbb{A}^n$ of the form $\mathbb{V}(S)$ with $S \subset k[x_1, \ldots, x_n]$, where $\mathbb{V}$ is the common vanishing locus.

Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$ (the ideal generated by S). By Hilbert Basis Theorem (since $k[x_1, \ldots, x_n]$ is noetherian), $\mathbb{V}(I(S)) = \mathbb{V}(S')$ for some finite set $S \subset k[x_1, \ldots, x_n]$.

In fact, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}$$

is the **radical** of I. For example, in $k[x]$, if $I = (x^2)$, then $\sqrt{I} = (x)$.

Definition 1.2. Given varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, a **morphism** is a (set-theoretic) map $\phi : V \to W \subset \mathbb{A}^m_k$ such that if $\phi = (f_1, \ldots, f_m)$, then each f_i is the restriction of a polynomial in $\{x_1, \ldots, x_n\}$.

An **isomorphism** is a morphism with a two–sided inverse.

Our basic correspondence is

$$\{\text{Affine varieties over } k\}/\text{up to isomorphism}$$
$$\leftrightarrow$$
$$\{\text{finitely generated } k\text{–algebras } A \text{ without nilpotent elements}\}$$

A finitely generated k–algebra is just a quotient of a polynomial ring in finitely many variables. A nilpotent element is such that some power of it is zero. For example, in $k[x]/(x^2)$, the element x is nilpotent.

How does this correspondence work? Given a variety V (representing an isomorphism class), we write $V = \mathbb{V}(I)$ for $I \subset k[x_1, \ldots, x_n]$ a radical ideal[1], and map $V \mapsto k[x_1, \ldots, x_n]/I$.

For the reverse, if A is a finitely generated nilpotent free algebra, then $A \cong k[y_1, \ldots, y_m]/J$ where we can choose J to be radical (exercise: why?).

We have to check that this is independent of our choice on both sides (exercise: think through this, it should be clear).

Definition 1.3. The algebra associated to V is classically denoted $k[V]$ and called the **coordinate ring of** V.

We have the compatibility of morphisms with our basic correspondence: there is a bijection between

$$\text{Morphisms}(V, W) \leftrightarrow \text{Ring homomorphisms}_k(k[W], k[V])$$

(here RingHom_k means that our homomorphisms preserve k).

We can now make our set into a topological space:

Definition 1.4. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The **Zariski topology** on V is defined such that the closed sets are $\mathbb{V}(S)$, where $S \subset k[V]$.

If $V \cong W$, then the Zariski topological spaces are homeomorphic as varieties (exercise).

Theorem 1.1 (Nullstellensatz)**.** Fix V a variety and let $k[V]$ be its coordinate ring. Given $p \in V$, we can produce a homomorphism $\text{ev}_p : k[V] \to k$ by sending $f \mapsto f(p)$. Note that ev_p is surjective (since we have constant functions), hence $\ker(\text{ev}_p) = m_p$ is a maximal ideal, giving us a map

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}.$$

Nullstellensatz says that this is actually a bijection. For the converse map, given $m \subset k[V]$, we get a quotient $k[V] \to k[V]/m = k$ (Nullstellensatz says this extension is finite, hence must be k). So using/choosing a representation for V in $k[x_1, \ldots, x_n]$ gives a surjective homomorphism onto k and specifies a bunch of points.

[1] A radical ideal is an ideal equal to its radical.

1.2 Limitations of classical algebraic geometry

Question. What is an abstract variety, i.e. "some "space" X such that locally as a cover $\{U_i\}$, each U_i is an affine variety, compatible with overlaps".

Example 1.1 (non–algebraically closed fields)**.** Take $I = (x^2+y^2+1) \subset \mathbb{R}[x, y]$. Then $\mathbb{V}(I) = \varnothing \subset \mathbb{R}^2$, but I is prime, so radical, so nullstellensatz fails.

Question. On what topological space is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ "naturally" the set of functions? (or $\mathbb{Z}$, or $\mathbb{Z}[x]$).

Example 1.2 (Why restrict to radical ideals?)**.** Take $C = \mathbb{V}(y - x^2) \subset \mathbb{A}^2_k$ and $D = \mathbb{V}(x, y)$, so $C \cap D = \mathbb{V}(y, y - x^2) = \mathbb{V}(x, y) = \{(0, 0)\}$. This is a single point, but if $D_\delta = \mathbb{V}(y + \delta)$ for some $\delta \in k$, then $C \cap D_\delta = \{\pm\sqrt{\delta}\}$, which is 2 points for all $\delta \neq 0$. In other words, intersections of varieties don't want to be varieties.

1.3 The spectrum of a ring

Let A be a commutative ring with identity. We will define a topological space on which A is the ring of functions. 11 Oct 2022, Lecture 3

Definition 1.5. The **Zariski spectrum** of A is

$$\text{Spec } A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

A ring homomorphism $\phi : A \to B$ induces a map $\phi^{-1} : \text{Spec } B \to \text{Spec } A$ by $q \mapsto \phi^{-1}(q)$. In general, the preimage of a prime ideal is a prime ideal.

Warning. This would fail if we only considered maximal ideals, since the preimage of a maximal ideal need not be maximal.

Given $f \in A$ and $\mathfrak{p} \in \text{Spec}(A)$, we have an induced $\overline{f} \in A/\mathfrak{p}$ obtained via a quotient. Informally, we can evaluate any $f \in A$ at points $\mathfrak{p} \in \text{Spec}(A)$ with the caveat that the codomain of this evaluation depends on $\mathfrak{p}$.

Example 1.3. Take $A = \mathbb{Z}$. Then $\text{Spec } A = \text{Spec } (\mathbb{Z}) = \{p \mid p \text{ is prime}\} \cup \{(0)\}$. Let's pick an element in $\mathbb{Z}$, say $132 \in \mathbb{Z}$. Given a prime p, we can look at $132 \pmod p \in \mathbb{Z}/p$. The takeaway here is that

$$\begin{gathered} \text{Spec } \mathbb{Z} \to \text{Space} \\ 132 \in \mathbb{Z} \to \text{a function} \\ 132 \text{ (mod } p) \to \text{value of that function at } p. \end{gathered}$$

Note that based on the value of p, our codomain changes from point to point.

Example 1.4. Take $A = \mathbb{R}[x]$, then $\text{Spec } \mathbb{R}[x] = \mathbb{C}/\text{complex conjugation} \cup \{(0)\}$.

Exercise. Draw Spec $\mathbb{Z}[x]$ and Spec $k[x]$ for k any field (i.e. describe all prime ideals and their containment). This is on example sheet 1.

Example 1.5. If $A = \mathbb{C}[x]$, then Spec $A = \mathbb{C} \cup \{(0)\}$, where given $a \in \mathbb{C}$, we send it to the maximal ideal $\langle z - a \rangle$.

1.4 A topology on Spec A

Fix $f \in A$. Then $\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \equiv 0 \pmod{\mathfrak{p}}\} \subset \text{Spec } A$. (Note that $f \equiv 0 \pmod{\mathfrak{p}}$ is the same as $f \in \mathfrak{p}$).

Similarly for $J \subset A$ an ideal, $\mathbb{V}(J) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\ \forall f \in J\}$.

Proposition 1.2. The sets $\mathbb{V}(J) \subset \text{Spec } A$ ranging over all ideals J form the closed sets of a topology on Spec A. This topology is called the **Zariski topology**.

Proof. Easy fact: $\varnothing$ and Spec A are closed, since we have functions 1 (vanishing nowhere) and 0 (vanishing everywhere). Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ (this is because $I_1 + I_2$ is the smallest ideal containing $I_1 \cup I_2$), arbitrary intersections are closed.

Finally, we claim $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$. The containment $\subset$ is clear: if a prime ideal contains I_1 or I_2, it contains $I_1 \cap I_2$. Conversely, $I_1 I_2 \subset I_1 \cap I_2$, so if $I_1 I_2 \subset I_1 \cap I_2 \subset \mathfrak{p}$, then by primality $I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p}$. ☐

Example 1.6. Let $k = \mathbb{C}$ and consider Spec $\mathbb{C}[x, y]$. We make a few observations:

- The point $(0) \in \text{Spec } \mathbb{C}[x, y]$ is dense in the Zariski topology, i.e. $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[x, y]$ because every prime ideal contains (0) (because we are in an integral domain).
- Consider the prime ideal $(y^2 - x^3)$ (which is prime since the quotient is an integral domain). Consider a maximal ideal $\mathfrak{m}_{a,b} = (x - a, y - b)$. We can ask: when is $\mathfrak{m}_{a,b} \in \overline{\{(y^2 - x^3)\}}$? The answer: if and only if $b^2 = a^3$, e.g. $(1, 1)$ (see example sheet 1). The lesson here is that points are not closed in the Zariski topology.

1.5 Functions on opens

Definition 1.6. Let $f \in A$. Define the **distinguished open** corresponding to f to be

$$\mathcal{U}_f = (\text{Spec}(A))/\mathbb{V}(f).$$

Example 1.7.

- Let $A = \mathbb{C}[x]$, so Spec $A = \mathbb{C} \cup \{(0)\}$ (with the Zariski topology). Take $f = x$ and consider $\mathcal{U}_x$. Recall the bijection Spec $\mathbb{C} \leftrightarrow \mathbb{C} \cup \{(0)\}$ by $(x - a) \mapsfrom a \in \mathbb{C}$ and $(0) \mapsfrom (0)$. Then $\mathbb{V}(x) = \{\mathfrak{p} \in \text{Spec } A \mid x \in \mathfrak{p}\} = \{(x)\}$, so $\mathcal{U}_f = \text{Spec } A \setminus \{(x)\}$.

- More generally, suppose we fix $a_1, \dots, a_r \in \mathbb{C}$, then Spec $A \backslash \{(x - a_i)\}_{i=1}^r = \mathcal{U}$ and $\mathcal{U} = \mathcal{U}_f$, where $f = \prod_{i=1}^r (x - a_i)$.

Lemma 1.3. The distinguished opens $\mathcal{U}_f$ taken over all $f \in A$ form a basis for the Zariski topology on Spec A.

Proof. Left as an exercise on example sheet 1. □

A bit of commutative algebra:

Definition 1.7. Given $f \in A$, the **localization of** A **at** f is $A_f = A[x]/(xf - 1)$, which we can informally think of as $A_f = A[\frac{1}{f}]$.

Lemma 1.4. The distinguished open $\mathcal{U}_f \subset$ Spec A is naturally homeomorphic to Spec A_f via the ring homomorphism $A \xrightarrow{j} A_f$, which produces the inverse j^{-1} : Spec $A_f \to$ Spec A.

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Proof. Primes in the ring A_f are in bijection with primes of A that miss f via j^{-1}. We exhibit this bijection:

- Given $q \subset A_f$ prime, take $j^{-1}(q) \subset A$, which is prime.
- Given $p \subset A$ a prime ideal, take $p_f = j(p)A_f$. We claim p_f is a prime exactly when $f \notin p$.
  - If $f \in p$, then p_f contains f, which is a unit, so $p_f = (1)$ is not prime.
  - If $f \notin p$, then $(A_f/p_f) \cong (A/p)_{\overline{f}}$, where $\overline{f}$ is $f + p$, a coset (exercise: check this formally). Hence $(A/p)_{\overline{f}} \subset FF(A/p)$ (FF stands for fraction field), so it is an integral domain, so p_f is prime.

Finally we need to check that these maps are inverses. This is left as an exercise. □

Facts about distinguished opens:

- $U_f \cap U_g = U_{fg}$ (easy fact).
- $U_{f^n} = U_f$ for all $n \geq 1$ (easy fact).
- The rings A_f and A_{f^n} for $n \geq 1$ are isomorphic. Why? Since $A_f = A[x]/(xf - 1)$ and $A_{f^n} = A[y]/(yf^n - 1)$, the isomorphism is given by $A_f \to A_{f^n}$ by $x \mapsto f^{n-1}y$ and $A_{f^n} \to A_f$ by $y \mapsto x^n$ (check these are inverses).
- Containment. $U_f \subset U_g \iff f^n$ is a multiple of g for some $n \geq 1$. To orient ourselves: if $f = gf'$, then $U_f \subset U_g$.

Proof. The ($\implies$) direction is clear by the orientation above. Conversely, suppose $U_f \subset U_g$, so $\mathbb{V}(f) \supset \mathbb{V}(g)$. The set $\mathbb{V}(f)$ is the set of all primes containing (f). We claim that $\sqrt{(f)} \subset \sqrt{(g)}$. But what is the radical of I? It is the intersection of all primes containing the ideal I. $\square$

Foreshadowing: fix A. We've made an assignment from distinguished opens in Spec A to rings by mapping $U_f \mapsto A_f$. The association is "functorial", i.e. if $U_{f_1} \subset U_{f_2}$, then we can assume that $f_1^n = f_2 f_3$, so $U_{f_1} = U_{f_1^n} = U_{f_2 f_3} \subset U_{f_2}$, so there is a homomorphism $A_{f_2} \to A_{f_1}$. This is the restriction map.

Question: can we extend this association to all open sets? See notes for the answer (yes).

2 Sheaves

2.1 Presheaves

Let X be a topological space.

Definition 2.1. A **presheaf $\mathcal{F}$ on X of abelian groups** is an association from the set of open sets in X to abelian groups given by $U \mapsto \mathcal{F}(U)$ and for $U \subset V$ opens, a homomorphism $\mathrm{res}_u^v : \mathcal{F}(V) \to \mathcal{F}(U)$ (a **restriction map**) such that $\mathrm{res}_u^u = \mathrm{id}$ and $\mathrm{res}_u^v \circ \mathrm{res}_v^w = \mathrm{res}_u^w$ when $U \subset V \subset W$ are opens.

Example 2.1. For any space X, take $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$ with the usual restriction.

Similarly we can get sheaves of rings, sets, modules over a fixed ring, etc.

Definition 2.2. A **morphism** $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves on X is, for each $U \subset X$ open, a homomorphism $\phi(u) : \mathcal{F}(u) \to \mathcal{G}(u)$ compatible with restriction maps, i.e. if $V \subset U$, then the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(u) \arrow[r, "\phi(u)"] \arrow[d, "\mathrm{res}_v^u"] & \mathcal{G}(u) \arrow[d, "\mathrm{res}_v^u"] \\
\mathcal{F}(v) \arrow[r, "\phi(v)"'] & \mathcal{G}(v)
\end{tikzcd}$$

Definition 2.3. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of preshaves is injective (surjective) if $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (surjective) for all $U \subset X$.

2.2 Sheaves

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Definition 2.4. A **sheaf** is a presheaf $\mathcal{F}$ such that

(1) If $U \subset X$ is open and $\{U_i\}$ is an open cover of U, then for $s \in \mathcal{F}(U)$, if $s|_{U_i} = \mathrm{res}^U_{U_i}(s) = 0$ for all i, then $s = 0$.

(2) If U and $\{U_i\}$ are as in (1), then given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Remark. These axioms imply $\mathcal{F}(\varnothing) = 0$ (exercise).

A **morphism** of sheaves is a morphism of the underlying presheaves.

Example 2.2. If X is a topological space, $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$, then f is a sheaf.

Non–example. Let $X = \mathbb{C}$ with the Euclidean topology and take $\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic and bounded}\}$. Then $\mathcal{F}$ is not a sheaf, since bounded functions may glue to unbounded functions. For example, take $U = \mathbb{C}$ and $U_i = D(0, i)$. Then $f(z) = z$ is bounded on each U_i, but not on U. In general, the characterization of elements of a sheaf should be purely local, and being bounded is not a local condition.

Non–example. Fix a group G and a set $\mathcal{F}(U) = G$ (the **constant presheaf**). If U_1, U_2 are disjoint, then $\mathcal{F}(U_1 \cup U_2) = G \times G$.

Example 2.3. Give G the discrete topology (every subset is open and closed) and define

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\} = \{f : U \to G \mid f \text{ is locally constant}\}.$$

This is the **constant sheaf**.

Example 2.4. If V is an irreducible variety, then

$$\mathcal{O}_V(v) = \{f \in k[V] \mid f \text{ is regular at } p \ \forall p \in U\}.$$

Here regular at p means that $f = \frac{g}{h}$ in a neighborhood of p with g, h polynomials and $h(p) \neq 0$. $\mathcal{O}_V$ is the **structure sheaf** of V.

This is a sheaf, since we have a local condition.

2.3 Basic constructions

Terminology. A **section** of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.

Construction of stalks. Fix $p \in X$ and $\mathcal{F}$ a presheaf on X. Then $\mathcal{F}_p$, the **stalk** of $\mathcal{F}$ at p, is defined to be

$$\mathcal{F}_p = \{(U, s) \mid s \in \mathcal{F}(U), p \in U\}/\sim$$

with $(U, s) \sim (V, s')$ if $\exists W \subset U \cap V$ with $p \in W$ such that $s|_W = s'|_W$.

The elements of $\mathcal{F}_p$ are called **germs**.

Example 2.5. Take $\mathbb{A}^1$, the affine line, then $\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \mid g(0) \neq 0 \right\} = k[t]_{(t)} \subset k(t)$.

Proposition 2.1. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X such that for all $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism, then f is an isomorphism.

Here $f_p((U, s)) = (U, f_U(s))$, which is well–defined.

Proof. We will show $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for each U, and we can then define f^{-1} by $(f^{-1})_U = (f_U)^{-1}$.

f_U is injective: suppose $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Since f_p is injective, $(U, s) = 0$ in $\mathcal{F}_p$ for every $p \in U$. Thus for every $p \in U$, there exists an open neighborhood U_p of p such that $s|_{U_p} = 0$. But $\{U_p \mid p \in U\}$ is a cover of U, so $s = 0$ in $\mathcal{F}(U)$ by the first condition of being a sheaf.

f_U is surjective: take $t \in \mathcal{G}(U)$. For each $p \in U$, we have $(U_p, s_p) \in \mathcal{F}_p$ with $f_p(U_p, s_p) = (U, t) \in \mathcal{G}_p$. By shrinking U_p if necessary, we can assume $f_{U_p}(s_p) = t|_{U_p}$. For points $p, p' \in U$,

$$f_{(U_p \cap U_{p'})} \left(s_p|_{U_p \cap U_{p'}} \setminus s_{p'}|_{U_p \cap U_{p'}} \right) = t|_{U_p \cap U_{p'}} - t|_{U_p \cap U_{p'}} = 0.$$

Thus $s_p|_{U_p \cap U_{p'}} - s_{p'}|_{U_p \cap U_{p'}} = 0$ by the injectivity of $f_{U_p \cap U_{p'}}$. Thus by the second sheaf axiom, $\exists s \in \mathcal{F}(U)$ with $s|_{U_p} = s_p$. Now $f_U(s)|_{U_p} = f_{U_p}(s|_{U_p}) = f_{U_p}(s_p) = t|_{U_p}$. Thus $f_U(s) = t$ by the first sheaf axiom. $\square$

We emphasize that this proof is asymmetric in the sense that we need to first prove injectivity to be able to prove surjectivity. 18 Oct 2022, Lecture 6

Exercises.

(i) There is a map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ mapping $s \mapsto ((U, s))_{p \in U}$. The claim is that this map is injective (by sheaf axiom 1).

(ii) Given two maps $\phi, \psi : \mathcal{F} \to \mathcal{G}$ with $\phi_p = \psi_p \ \forall p \in X$, we have $\phi = \psi$.

Definition 2.5 (Sheafification)**.** If $\mathcal{F}$ is a presheaf on X, then a morphism $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ to the sheaf $\mathcal{F}^{\mathrm{sh}}$ is a **sheafification** if for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ for $\mathcal{G}$ a sheaf there is a unique commutative diagram of the following form:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\phi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

Remark. Since this is a definition by universal property, $\mathcal{F}^{\mathrm{sh}}$ and the map $\mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ are unique (up to unique isomorphism).

A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism of sheaves $\mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$.

Proposition 2.2. Sheafification exists.

Proof. Given a presheaf $\mathcal{F}$ on X, define

$$\mathcal{F}^{\mathrm{sh}}(U) = \{f : U \to \bigsqcup_{p \in U} \mathcal{F}_p \,| f(p) \in \mathcal{F}_p \text{ and for all } p \in U, \text{ there exists an open neighborhood } V_p \subset U \text{ and } s \in \mathcal{F}(V_p) \text{ such that } (V_p, q) = f(q) \in \mathcal{F}_q \ \forall q \in V_p\}.$$

This is clearly a sheaf. Verifying the universal property is left as an exercise. □

Corollary 2.3. The stalks of $\mathcal{F}$ and $\mathcal{F}^{\mathrm{sh}}$ coincide.

Proof. Easy exercise from the definitions. □

Exercise. Find a nonzero presheaf $\mathcal{F}$ with $\mathcal{F}^{\mathrm{sh}} = 0$. (Comment by Dhruv: this is rather stupid).

2.4 Kernels, cokernels, etc.

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then we can define presheaves ker ϕ, coker ϕ, im ϕ by

$$\begin{aligned}
(\ker\, \phi)(u) &= \ker\, \phi_u : \mathcal{F}(U) \to \mathcal{G}(U) \\
(\mathrm{coker}\, \phi)(u) &= \mathrm{coker}\, \phi_u \\
(\mathrm{im}\, \phi)(u) &= \mathrm{im}\, \phi_u.
\end{aligned}$$

These are all presheaves.

Exercise. The presheaf kernel for a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is also a sheaf.

This is not true for coker ϕ in general!

Example 2.6. Take $X = \mathbb{C}$ with the Euclidean topology, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X (with addition as its group operation). Let $\mathcal{O}_X^*$ be the sheaf of nowhere vanishing holomorphic functions (with multiplication as its group operation).

We have a morphism of sheaves $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ by $f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^*(U)$. Thus $\ker(\exp) = 2\pi i \mathbb{Z}$ with $\mathbb{Z}$ the constant sheaf, but coker(exp) is not a sheaf: if we let $U_1 = \mathbb{C} \setminus [0, \infty)$, $U_2 = \mathbb{C} \setminus (-\infty, 0]$ and $U = U_1 \cup U_2 = \mathbb{C} \setminus \{\infty\}$ and we let $f(z) = z \in \mathcal{O}_X^*(U)$, then it is not in the image of $\exp : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ since $\log z$ is not single–valued on U. Thus f defines a nonzero section of (coker exp)(U). But $f|_{U_i}$ is in the image of $\exp_{U_i}$, since we just choose some branch of $\log z$. Thus $f|_{U_i} = 1$ in coker exp, so sheaf axiom 1 fails.

Definition 2.6. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define the **sheaf cokernel** and the **sheaf image** to be the sheafification of the presheaf cokernel and the presheaf image.

Remark. Crucial fact: there is an exact sequence of sheaves

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^* \to 1.$$

In other words, $2\pi i\mathbb{Z} = \ker(\exp)$ and $\mathrm{coker}(\exp) = 1$ (the first of these we showed, the second of this we will show once we've developed the necessary theory).

Remark. $\ker\phi, \mathrm{coker}\,\phi$ satisfy the category theoretic definitions of kernels and cokernels, i.e. they are universal in the appropriate sense. For example, for the kernel, if $\ker\phi : \mathcal{F} \to \mathcal{G}$, then for any other sheaf $\mathcal{L}$ with a map ψ to $\mathcal{F}$ such that $\phi \circ \psi = 0$, this map factors uniquely through the kernel. This is easy to check and left as an exercise.

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, dashed, "\exists!"'] \arrow[d, "\psi"] \arrow[dr, "\phi\circ\psi=0"] & \\
\ker\phi \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{F} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

For the cokernel, reverse all the arrows and check that $\mathrm{coker}\,\phi$ satisfies the universal property (exercise).

Adjacent notions.

(i) **Subsheaves**. $\mathcal{F} \subset \mathcal{G}$ is there exist inclusions $\mathcal{F}(U) \subset \mathcal{G}(U)$ compatible with restrictions. For example, $\ker(\phi : \mathcal{F} \to \mathcal{G}) \subset \mathcal{F}$.

(ii) **Quotient sheaves**. To be added at a later date.

2.5 Moving between spaces

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Definition 2.7. Given $f : X \to Y$ continuous with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, the **presheaf pushforward** $f_\star\mathcal{F}$ is defined by

$$\mathcal{U} \mapsto \mathcal{F}(f^{-1}(\mathcal{U}))$$

for an open set $\mathcal{U} \subset \mathcal{Y}$.

Proposition 2.4. The presheaf pushforward of a sheaf is a sheaf.

Proof. Trivial. □

Definition 2.8. Given $f : X \to Y$ continuous with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, the **inverse image presheaf** $(f^{-1}\mathcal{G})^{\text{pre}}$ is defined by (for V open in X)

$$(f^{-1}\mathcal{G})^{\text{pre}}(V) = \{(s_U, U) \mid U \text{ is an open set containing } V, s_U \in \mathcal{G}(U)\}/\sim$$

where $\sim$ is an equivalence relation that identifies pairs that agree on a smaller open set containing V.

The **inverse image sheaf** is given by $f^{-1}\mathcal{G} = ((f^{-1}\mathcal{G})^{\text{pre}})^{\text{sh}}$.

Example 2.7. Take Y a topological space and set $X = Y \sqcup Y$. Take $\mathcal{G} = \mathbb{Z}$ to be the constant sheaf and $\mathcal{F} = (f^{-1}\mathcal{G})^{\text{pre}}$. Fix $U \subset Y$ open and $V = f^{-1}(U)$. Then $\mathcal{F}(V) = \mathcal{G}(U) = \mathbb{Z}$, constant (assuming U is connected). But $V = U \sqcup U$, so $\mathcal{F}^{\text{sh}}(V) = \mathcal{G}(U) \times \mathcal{G}(U) = \mathbb{Z}^2$. This happens because this isn't a local condition.

Example 2.8. Let $\mathcal{F}$ be a sheaf of X and $\pi : X \to \text{point}$. Then $\pi_\star\mathcal{F}$ is a sheaf on a point, i.e. an abelian group, specifically $\mathcal{F}(\pi^{-1}(\text{point})) = \mathcal{F}(X)$.

Notation. We write $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$, called the global section, and $\mathcal{F}(X) = H^0(X, \mathcal{F})$, the 0^{th} cohomology of coefficients in $\mathcal{F}$.

For $p \in X$, $i : \{p\} \to X$, $\mathcal{G}$ a sheaf on the point, i.e. an abelian group A, we can consider $i_\star\mathcal{G}$. This is the sheaf on X such that $i_\star(\mathcal{G})(U) = \begin{cases} 0 & p \notin U. \\ A & p \in U. \end{cases}$ This is called the skyscraper at p with value A.

3 Schemes

The summary: $\text{Spec}(A)$ has a sheaf $\mathcal{O}_{\text{Spec}(A)}$ such that the value on a distinguished open $\mathcal{U}_f = A_f$, and then globalize this to get a scheme. We now spell this out in detail.

Definition 3.1. Let A be a ring and $S \subset A$ a set that is closed under multiplication. The two examples we should keep in mind are $S = \{1, f, f^2, f^3, \ldots\}$ or $S = A \setminus \mathfrak{p}$ for $\mathfrak{p}$ a prime ideal. The **localization** of A at S is

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\}/\sim$$

where $(a, s) \sim (a', s') \iff \exists s'' \in S$ such that $s''(as' - a's) = 0 \in A$. We read $\frac{a}{s}$ for the equivalence class of (a, s).

Warning. The map $A \to S^{-1}A$ need not be injective, e.g. if S contains a zero divisor.

What's going to happen now? We will define a sheaf $\mathcal{O}_{\text{Spec}(A)}$ on the topological space $\text{Spec}(A)$ with two features:

- The stalk at a prime $\mathfrak{p}$ will be $(A \setminus \mathfrak{p})^{-1}A$.
- If $\mathcal{U}_f$ is a distinguished open, then $\mathcal{O}_{\mathrm{Spec}(A)}(\mathcal{U}_f) = A_f$.

A sheaf on a base. Fix a topological space X and $\mathcal{B}$ a basis for the topology. A **sheaf on the base** $\mathcal{B}$, $\mathfrak{F}$, consists of assignments $B_i \mapsto \mathfrak{F}(B_i)$ on abelian groups/rings/some objects with restriction maps $\mathfrak{F}(B_i) \to \mathfrak{F}(B_j)$ whenever $B_j \subset B_i$. These satisfy the usual commutativity and the identities when $B_i \subset B_j \subset B_k$ or $B_i = B_j$, as well as the following two conditions:

(1) If $B = \cup B_i$ with $B, B_i \in \mathcal{B}$ and $f, g \in \mathfrak{F}(B)$ such that $f|_{B_i} = g|_{B_i}$ $\forall i$, then $f = g$.

(2) If $B = \cup B_i$ as above with $f_i \in \mathfrak{F}(B_i)$ such that they agree where assigned (i.e. $\forall i, j$, if $B' = B_i \cap B_j$, then $f_i|_{B'} = f_j|_{B'}$), then $\exists f \in \mathfrak{F}(B)$ with $f|_{B_i} = f_i$.

Proposition 3.1. Let F be a sheaf on a base $\mathcal{B}$ of a topological space X. Then this uniquely (up to unique isomorphism) determines a sheaf $\mathfrak{F}$ by $\mathfrak{F}(B_i) = F(B_i)$ agreeing with restriction maps.

Proof. Define the stalks of $\mathfrak{F}$ first, i.e. $\mathfrak{F}_{\mathfrak{p}} = \{(S_B, B) \mid B \text{ is a basic open containing } \mathfrak{p}\}/\sim$. Now use the sheafification trick to define

$$\mathfrak{F}(U) = \{(f_p \in \mathfrak{F}_p)_{p \in U} \mid \forall p \in U, \exists \text{ basic open } B \text{ containing } p \text{ and } s \in F(B) \text{ with } s_q = f_q \text{ in } \mathcal{F}_q \, \forall q \in B\}.$$

Thirdly, the natural maps $F(B) \to \mathfrak{F}(B)$ are isomorphic by sheaf axioms. The final fact that this is unique (up to unique isomorphism) is left as an exercise. □

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Setup so far: Spec A is a topological space with base $\{U_f\}$ for $U_f = \mathbb{V}(f)^C$ over $f \in A$. Recall also that $U_f = U_g$ if and only if $\sqrt{(f)} = \sqrt{(g)}$. Also, if $U_f = U_g$, then the localizations $A_f \cong A_g$ are isomorphic. Therefore, the assignment $U_f \mapsto A_f$ is well–defined.

Proposition 3.2. The assignment $U_f \to A_f$ defines a sheaf (of rings) on the base of the topology of Spec A given by distinguished opens.

As a consequence, Spec A inherits a sheaf of rings, denoted $\mathcal{O}_{\mathrm{Spec}\ A}$ and called **the structure sheaf**.

Prelude. Suppose $\{U_{f_i}\}_{i \in I}$ covers Spec A. Then there exists a finite subcover. In other words, Spec A is quasi–compact. Why? Since the U_{f_i} cover, there exists no prime ideal $\mathfrak{p} \subset A$ containing all $(f_i) \iff \sum_{i \in I}(f_i) = (1)$. Hence $1 = \sum_i a_i f_i$, where all but finitely many $a_i = 0$. So if $J \subset I$ are the indices with nonzero coefficient, then $\{U_{f_i}\}_{i \in J}$ cover.

Proof. We need to check that axioms 1 and 2 of a sheaf hold. We will check these for a the basic open $B = \text{Spec } A$ itself (the general case is similar, restrict to a basic open and repeat the proof).

Axiom 1: Suppose $\text{Spec } A = \bigcup_{i=1}^n U_{f_i}$ (finite by the prelude). Given $s \in A$ such that $s|_{U_{f_i}} = 0 \ \forall i$, by the definition of localization $f_i^m s = 0$ for some m large enough. But $(1) = (f_i^m)_{i=1}^n$ for any $m > 0$ because $\{U_{f_i}\}$ cover Spec A, so hence so do $\{U_{f_i^m}\}$. Hence $1 = (\sum r_i f_i^m)$ and multiplying by s on both sides gives us $s = 0$.

Axiom 2: Say $\text{Spec } A = \bigcup_{i \in I} U_{f_i}$ and choose elements in each A_{f_i} that agree in $A_{f_i f_j}$, i.e. if $s_i \in A_{f_i}$, then the images of s_i and s_j in $A_{f_i f_j}$ coincide. We need to build an element in A with these restrictions.

First suppose I is finite. On U_{f_i}, we've chosen an element $\frac{a_i}{f_i^{l_i}} \in A_{f_i}$. Write $g_i = f_i^{l_i}$, noting $U_{f_i} = U_{g_i}$. On overlaps, restrict to $A_{g_i g_j}$. The condition for the second axiom is $(g_i g_j)^{m_j}(a_i g_j - a_j g_i) = 0$. Rewriting this using algebra and $U_f = U_{f^k} \ \forall k \geq 1$, we may assume $m = m_{ij}$ by taking the largest. Write $b_i = a_i g_i^m$ and $h_i = g_i^{m+1}$, so on each U_{h_i} we've chosen an element $\frac{b_i}{h_i}$. [2] But U_{h_i} cover Spec A, so $1 = \sum_i r_i h_i$ for $r_i \in A$. Now we construct $r = \sum r_i b_i$ with r_i as above. This restricts correctly to $\frac{b_i}{h_i}$ on U_{h_i} (i.e. in the localization A_{h_i}).

When I is infinite, pick a finite subcover $(f_1, \ldots, f_n) = A$ such that U_{f_i} form a cover and use the above to build r. But given $(f_1, \ldots, f_n, f_\alpha) = A$, the same construction gives a "new" r'. But $r' = r$ by the first axiom. □

Definition 3.2. The structure sheaf on Spec A is the sheaf associated to the sheaf on the base sending $U_f \mapsto A_f$, denoted $\mathcal{O}_{\text{Spec } A}$.

Observation. The stalk $\mathcal{O}_{\text{Spec } A, \mathfrak{p}} = A_{\mathfrak{p}}$.

Terminology. A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$. An isomorphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the combination of a homomorphism $\pi : X \to Y$ and an isomorphism of sheaves on Y, $\mathcal{O}_Y \xrightarrow{\sim} \pi_\star \mathcal{O}_X$.

Definition 3.3. An **affine scheme** is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic (as a ringed space) to $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

Definition 3.4. A **scheme** is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

Intuitively, every point $p \in X$ has a neighborhood U_p such that the ringed space $(U_p, \mathcal{O}_{U_p})$ is isomorphic to some affine scheme (possibly depending on p). Note that if $U \subset X$ is open, then U is naturally a ringed space with $\mathcal{O}_U(V) = \mathcal{O}_X(V)$.

[2] So far, this is just rewriting everything symbolically with no actual content.

Example 3.1. Spec A for various rings A.

Example 3.2. Take $X = \text{Spec } \mathbb{C}[x,y]$ and $U = \{(x,y)\}^C$. Then the scheme U is not an affine scheme.

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Example 3.3. Open subschemes. Let X be a scheme and $U \subset X$ be open.[3] Write $i : U \to X$ for the inclusion map. Take $\mathcal{O}_U = \mathcal{O}_X|_U = i^{-1}\mathcal{O}_X$ to be the structure sheaf of U.

Proposition 3.3. The ringed space $(U, \mathcal{O}_U)$ is a scheme.

Simple case: take $X = \text{Spec } A$, $U = U_f$ for $f \in A$. Then $(U, \mathcal{O}_U) \cong (\text{Spec } A_f, \mathcal{O}_{\text{Spec } A_f})$.

Proof. Let $p \in U \subset X$ be a point. Since X is a scheme, we can find some $(V_p, \mathcal{O}_X|_{V_p})$ inside X with $p \in V$ such that V_p is isomorphic to an affine scheme. Take $V_p \cap U \subset U$ with the structure sheaf via restriction. However, this may not be affine. But V_p is affine, say $V \cong \text{Spec } B$, and the distinguished opens in Spec B form a basis for the topology. Hence we've reduced to the "simple case" and we're done. ☐

Example 3.4. Define $\mathbb{A}^n_k = \text{Spec } k[x_1, \ldots, x_n]$. Take $U = \mathbb{A}^{n^2} - \{\det(x_{ij}) = 0\}$, i.e. "$U = GL_2(k)$".

Example 3.5. A non–affine scheme. Take $X = \mathbb{A}^2_k = \text{Spec } k[x,y]$ and $U = \mathbb{A}^2_k \setminus \{(x,y)\}$.[4] Then the claim is that U is not affine. We will calculate $\mathcal{O}_U(U)$. Write $U_x = \mathbb{V}(x)^C \subset \mathbb{A}^2$ and $U_y = \mathbb{V}(y)^C \subset \mathbb{A}^2$. Note that $U = U_x \cup U_y$ and $U_x \cap U_y = \mathbb{A}^2 \setminus \mathbb{V}(xy)$.

We have $\mathcal{O}_U(U_x) = k[x, x^{-1}, y]$, $\mathcal{O}_U(U_y) = k[x, y, y^{-1}]$, and $\mathcal{O}_U(U_x \cap U_y) = k[x, x^{-1}, y, y^{-1}]$. Also, the restriction maps $\mathcal{O}_U(U_x) \to \mathcal{O}_U(U_{xy})$ are the obvious ones.

By sheaf axioms, $\mathcal{O}_U(U) = k[x, x^{-1}, y] \cap k[x, y, y^{-1}]$ (inside $k[x, x^{-1}, y, y^{-1}]$). Hence $\mathcal{O}_U(U) = k[x,y]$. This is a contradiction. Why? One way: there exists (in $(U, \mathcal{O}_U)$) a maximal ideal in the global sections ring with empty vanishing locus, namely $(x,y) \subset k[x,y]$. On the other hand, there is no maximal ideal in Spec $k[x,y]$ with empty vanishing locus.

This is a bit of a hack and there is a better conceptual approach that we will discover soon.

A little more on $U = \mathbb{A}^2_k \setminus \{(x,y)\}$ not being affine (this was talked about at the beginning of the following lecture). Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. Fix $p \in X$. Then there's a well–defined stalk $\mathcal{O}_{X,p}$ of $\mathcal{O}_X$ at p. The

[3] From now on, whenever we say "let X be a scheme", we silently take that to mean $(X, \mathcal{O}_X)$.
[4] Illegally, we are allowed to think of this as $\mathbb{R}^2 \setminus \{(0,0)\}$.

stalk is of the form $A_{\mathfrak{p}}$, where A is a ring and $\mathfrak{p} \subset A$ is a prime ideal. In particular, $A_{\mathfrak{p}}$ has a unique maximal ideal, namely $\mathfrak{p}A_{\mathfrak{p}}$. Say f vanishes at $\mathfrak{p}$ if its image in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is 0 (i.e $f \in \mathfrak{p}A_{\mathfrak{p}}$). (Here we're using an isomorphism $p \ni V_p$ open to Spec A). For $f \in \Gamma(X, \mathcal{O}_X)$, $\mathbb{V}(f)$, the vanishing locus of $f \subset X$ is well–defined.

3.1 Interlude: gluing sheaves

Let X be a topological space with cover $\{U_\alpha\}$, sheaves $\{\mathcal{F}_\alpha\}$ on $\{U_\alpha\}$ and isomorphisms (of sheaves) $\phi_{\alpha,\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$ such that $\phi_{\alpha,\alpha} = \mathrm{id}$, $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ and $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (the **cocycle condition**).

Construction. We will build a sheaf $\mathcal{F}$ on X. Given $V \subset X$ open, define

$$\mathcal{F}(V) = \{(S_\alpha)_\alpha, S_\alpha \in \mathcal{F}_\alpha(U_\alpha \cap V) \mid \phi_{\alpha,\beta}(S_\alpha|_{V \cap U_\alpha \cap U_\beta}) = S_\beta|_{V \cap U_\alpha \cap U_\beta}\}.$$

$\mathcal{F}$ is a presheaf, since given $(S_\alpha) \in \mathcal{F}(V)$ and $W \subset V$ open, we can take $(S_a)|_W = \left(\mathrm{res}^{V \cap U_\alpha}_{W \cap U_\alpha}(S_\alpha)\right)_\alpha$. This lies in $\mathcal{F}(W)$ by sheaf axioms.

Proposition 3.4. $\mathcal{F}$ is a sheaf and $\mathcal{F}|_{U_\alpha} = \mathcal{F}_\alpha$ on U_α.

Proof. It is a presheaf, and both sheaf axioms are clear (exercise: check this). But we need to check/build an isomorphism $\mathcal{F}|_{U_\gamma} \to \mathcal{F}_\gamma$. Given $V \subset U_\gamma$ and $S \in \mathcal{F}_\gamma(V)$, define its image in $\mathcal{F}_{U_\gamma}$ to be $(\phi_{\gamma,\alpha}(S|_{V \cap U_\alpha}))_\alpha$. We need to check that this lies in $\mathcal{F}|_{U_\gamma}(V) = \mathcal{F}(V)$, but this follows from the cocycle condition: $\phi_{\alpha,\beta} \circ \phi_{\gamma,\alpha}(S|_{V \cap U_\alpha \cap U_\beta}) = \phi_{\gamma,\beta}(S|_{V \cap U_\alpha \cap U_\beta})$. □

3.2 More schemes

Take schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ with opens $U \subset X, V \subset Y$ and an isomorphism $(U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$. We can glue both the topological spaces and the schemes: $X \sqcup Y/(U \sim V)$ with the sheaf glued as in the previous construction.

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How to glue: Take $(X \sqcup Y)/(U \sim V)$. By definition of the quotient topology, the image of X, Y in S form an open cover and their intersection is the image of U (or V). Now glue the structure sheaves on these opens as in the previous lecture (to get $(S, \mathcal{O}_S)$). Note that there is no cocycle condition, since we only have the intersection of two and not three opens.

Example 3.6. The bug–eyed line, i.e. the line with two origins. Let k be a field and $U \subset X = \mathrm{Spec}\ k[t]$, $V \subset Y = \mathrm{Spec}\ k[u]$, $U = \mathrm{Spec}\ k[t, t^{-1}]$, $V = \mathrm{Spec}\ k[u, u^{-1}]$. We have the isomorphism $U \to V$ by $t \mapsto u$. (Really, this is an isomorphism of rings $k[u, u^{-1}] \to k[t, t^{-1}]$ with $u \mapsto t$ and now take Spec).

On the level of topological spaces, $X = \mathbb{A}^1_k$ and $Y = \mathbb{A}^1_k$ with $U = \mathbb{A}^1 \setminus \{(t)\}$ (i.e. "U minus a point", similarly for V). Hence $X \sqcup Y/\sim$ gives the line with two origins.

What are the types of opens in this scheme?

- W could be contained inside X or Y (inside S). There are nice, easy open sets.
- $W = S \setminus \{p_1, \ldots, p_r\}$ where $p_i \in U$ or $p_i \in V$. The simplest of these is when $W = S$.

What is $\mathcal{O}_S(S)$? Use sheaf axioms to find that $\mathcal{O}_S(S) \cong k[t]$. Conclusion: S is not affine.

Example 3.7. $\mathbb{P}^1_k$. Same setup: $X = \text{Spec } k[t], Y = \text{Spec } k[s], U = \text{Spec } k[t, t^{-1}]$, $V = \text{Spec } k[s, s^{-1}]$. We glue via the isomorphism $s \mapsto t^{-1}$. Then $\mathbb{P}^1_k$ is the result of the gluing (we can consider this as a definition for now).

Proposition 3.5. $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong k$.

Proof. Important exercise: the only elements of $k[t, t^{-1}]$ that are both polynomials in t and t^{-1} are the constants. (Do this!). In particular, $\mathbb{P}^1$ is not affine. □

Example 3.8. Similarly we can build $S =$ "$\mathbb{A}^2_k$ with doubled origin" – this has the interseting property that there exist affine open subschemes $U_1, U_2 \subset S$ such that $U_1 \cap U_2$ is not affine. We flag this example for later.

Gluing schemes. (Example sheet 1). Given schemes X_i for $i \in I$, open subschemes $X_{ij} \subset X_i$ with $X_{ii} = X_i$, isomorphisms $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ with $f_{ii} = \text{id}_{X_i}$, and the cocycle condition $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$, there is a unique scheme X with an open cover given by X_i, glued along $X_{ij} \cong X_{ji}$.

Example 3.9 (Key example)**.** Take A any ring, $X_i = \text{Spec } A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$, $X_{ij} = \mathbb{V}\left(\frac{x_j}{x_i}\right)^C \subset X_i$, and isomorphisms $X_{ij} \to X_{ji}$ by $\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j}\left(\frac{x_i}{x_j}\right)^{-1}$. The resulting glued scheme is called the **projective** n–**space**, denoted $\mathbb{P}^n_A$.

Exercise/calculation. $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

3.3 Proj construction

Definition 3.5. A $\mathbb{Z}$–**grading** on a ring A is a decomposition $A = \bigoplus_{i \in \mathbb{Z}} A_i$ as abelian groups such that $A_i A_j \subset A_{i+j}$.

Example 3.10. Take $A = k[x_0, \ldots, x_n]$ and write $A_d = \{\text{degree } d \text{ homogeneous polynomials}\} \cup \{0\}$.

Also: Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal (i.e. generated by homogeneous elements of possibly different degree). Then $k[x_0, \ldots, x_n]/I$ is also naturally graded. (Exercise: think about how).

Remark. A_0 is always a subring of A.

Assumption. From now on, we will assume that the degree 1 elements generate A as an algebra over A_0. 30 Oct 2022, Lecture 11

Another assumption. We assume $A_i = 0$ for all $i < 0$.

Terminology. $A_+ = \bigoplus_{i \geq 1} A_i \subset A$ is the subgroup of positive degree elements. It forms an ideal, called the **irrelevant ideal**.

A **homogeneous element** $f \in A$ is an element contained in some A_d.

An ideal $I \subset A$ is called **homogeneous** if it is generated by homogeneous elements (possibly of different degrees).

Definition 3.6. The set Proj A is the set of all homogeneous primes in A that do not contain the irrelevant ideal A_+.

If $I \subset A$ is homogeneous, then $\mathbb{V}(I) = \{\mathfrak{p} \in \text{Proj } A \mid \mathfrak{p} \text{ contains } I\}$. Given this, the Zariski topology on Proj A has closed sets $\mathbb{V}(I)$ for $I \subset A$ homogeneous.

Let $f \in A_i$ and $U_f = \text{Proj } A \setminus \mathbb{V}(f)$. Then observe that $\{U_f\}_{f \in A_1}$ covers Proj A. Also, the ring $A[\frac{1}{f}] = A_f$ is naturally $\mathbb{Z}$–graded by $\deg(f^{-1}) = -\deg(f)$.

Example 3.11. Let $A = k[x_0, x_1]$ and $f = x_0$, then $A[\frac{1}{f}] = k[x_0, x_1, x_0^{-1}]$. The degree 0 elements of this include $\lambda (\lambda \in k)$, $\frac{x_1}{x_0}$, $\frac{x_1^2 + x_1 x_0}{x_0^2}$, etc. Similarly, degree 1 elements include $\frac{x_1^2}{x_0}$, etc.

Proposition 3.6. There is a natural bijection between

$$\{\text{homogeneous primes in } A \text{ missing } f\} \leftrightarrow \{\text{primes in } (A_f)_{\text{degree } 0}\}.$$

(Equivalently, the LHS is the homogeneous primes in A_f).

Proof and construction: Primes in A missing f are naturally in bijection with homogeneous primes in A_f. Suppose $\mathfrak{q} \subset (A[\frac{1}{f}])_0$ is a prime. Then let $\Psi(\mathfrak{q})$ be generated by

$$\bigcup_{d \geq 0} \left\{ a \in A_d \mid \frac{a}{f^d} \in \mathfrak{q} \right\} \subset A.$$

Exercise/easy check: this is prime.

Given $\mathfrak{p} \subset A$ homogeneous missing f, take $\phi(\mathfrak{p}) = \left(\mathfrak{p} A[\frac{1}{f}] \cap (A[\frac{1}{f}]_0) \right)$.

We need to check two compositions. $\phi \circ \Psi = \text{id}$ is easy and left as an exercise. However, $\Psi \circ \phi$ is trickier. We will show $\mathfrak{p} = \Psi(\phi(p))$ by exhibiting both containments. Suppose $\mathfrak{p} \in U_f \subset \text{Proj } A$. Then if $a \in \mathfrak{p} \cap A_d$, then $\frac{a}{f^d} \subset \phi(p)$, so $a \in \Psi(\phi(\mathfrak{p}))$. Conversely, if $a \in \Psi(\phi(\mathfrak{p}))$, then $\frac{a}{f^d} \in \phi(\mathfrak{p})$ for some d. Hence there exists $b \in \mathfrak{p}$ such that $\frac{b}{f^e} = \frac{a}{f^d}$ in the localization after inverting f. For some $k \geq 0$, we have $f^k(f^d b - f^e a) = 0$, but $f^{e+k} \not\subset \mathfrak{p}$. Hence by primality, $a \in \mathfrak{p}$, giving the reverse containment. □

Remark. The bijection we constructed is compatible with ideal containment, so it gives a homeomorphism from U_f to $\mathrm{Spec}(A_f)_0$.

Proj A is covered by open sets, each isomorphic to Spec $(A_f)_0$ for some f. If $f, g \in A_1$, then $U_f \cap U_g$ is naturally homeomorphic to $\mathrm{Spec}(A[\frac{1}{f}])_0[\frac{f}{g}] = \mathrm{Spec}(A[f^{-1}, g^{-1}])_0$. Call this property $(\star)$.

Take the open cover $\{U_f\}$ with structure sheaf $\mathcal{O}_{\mathrm{Spec}(A_f)_0}$ on each U_f with isomorphisms on $U_f \cap U_g$ by $(\star)$. The cocycle condition follows immediately from the formal properties of localization (exercise: check this).

Terminology. If $A = k[x_0, \dots, x_n]$ with the standard grading, then Proj A is denoted as $\mathbb{P}^n_k$. This is the same as the projective n–space defined earlier, but we need further work to show this.

4 Morphisms

We have already seen a few "examples". For example, we have a morphism given by inclusion for $U \subset X$. Similarly, if $A \to B$ is a ring homomorphism, then Spec $B \to$ Spec A should be another morphism.

4.1 Morphisms of schemes and locally ringed spaces

Given a scheme $(X, \mathcal{O}_X)$, the stalks $\mathcal{O}_{X,p}$ are **local rings** (i.e. they have a unique maximal ideal). Given a function $f \in \mathcal{O}_X(U)$ with $p \in U$, we can ask whether f vanishes at p, i.e. is the image of f in $\mathcal{O}_{X,p}$ contained in the maximal ideal?

Definition 4.1. A morphism of **ringed spaces** (i.e. a topological space and a sheaf of rings) is $(f, f^\#)$ such that:

- $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$
- $f : X \to Y$ is continuous.
- $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$, a morphism of sheaves of rings on Y.

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Warning. It is possible to find $(f, f^\#)$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ of schemes such that there exists $U \subset Y$ open with $q \in U$ and $h \in \mathcal{O}_Y(U)$ such that h vanishes at q, but $f^\#(h) \in \mathcal{O}_X(f^{-1}(U))$ does not vanish at $p \in X$ such that $f(p) = q$.

Observation. Given $f : X \to Y$ a ringed space morphism and $p \in X$, there is an induced map $f^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$. To spell this out, given $s \in \mathcal{O}_{Y,f(p)}$, we can represent it by (S_U, U) for U open, $f(p) \in U$ and $S_U \in \mathcal{O}_Y(U)$. Therefore $f^\#(S_U) \in \mathcal{O}_X(f^{-1}(U))$, so the pair $(f^\#(S_U), f^{-1}(U))$ defines an element in $\mathcal{O}_{X,p}$.

Definition 4.2. $(X, \mathcal{O}_X)$, a ringed space, is called **locally ringed** if $\forall p \in X$, $\mathcal{O}_{X,p}$ is a local ring (i.e. has a unique maximal ideal). A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism as ringed spaces such that if $\mathfrak{m}_p$ denotes the maximal ideal in $\mathcal{O}_{X,p}$, then $f^\#(\mathfrak{m}_{f(p)}) \subset \mathfrak{m}_p$ (in the stalks).

Definition 4.3. A **morphism of schemes** $X \to Y$ is a morphism as locally ringed spaces.

Theorem 4.1. There is a natural bijection between

$\{$Scheme theoretic morphisms Spec $B \to$ Spec $A\} \leftrightarrow \{$Ring homomorphisms $A \to B\}$.

Prologue. Recall that section of a sheaf $\mathcal{F}$ on U, i.e. $s \in \mathcal{F}(U)$, is a coherent collection of elements $s(p) \in \mathcal{F}_p$ (the stalk) for all $p \in U$.

Proof. We will first show that every ring map $A \to B$ induces a scheme map (we will construct it), and then conversely we will show that every scheme map Spec $B \to$ Spec A arises via our construction.

Given $\phi : A \to B$, we can take ϕ^{-1} : Spec $B \to$ Spec A as the topological part (continuity is formal). Now we build $\phi^\# : \mathcal{O}_{\text{Spec } A} \to \phi_\star^{-1}\mathcal{O}_{\text{Spec } B}$. First, at the stalk level, take $A_{\phi^{-1}(p)} \to B_p$ sending $\frac{a}{s} \mapsto \frac{\phi(a)}{\phi(s)}$ induced by ϕ. This makes sense because if $s \notin \phi^{-1}(p)$, then $\phi(s) \notin p$ (we're treating p as a prime ideal here). Observe that this is automatically a local homomorphism of local rings.

Secondly, on the open set level, given $U \subset$ Spec A, we need to define $\phi^\#$: $\mathcal{O}_{\text{Spec } A}(U) \to \mathcal{O}_{\text{Spec } B}((\phi^{-1})^{-1}(U))$ (this just means take the preimage of U inside Spec B). An element $s \in \mathcal{O}_{\text{Spec } A}(U)$ is a collection of assignments $[p \mapsto s(p)]_{p \in U}$ with $p \in U$, $s(p) \in A_p$. We define $\phi^\#$ by sending $[p \mapsto s(p)]_{p\in U} \mapsto [q \mapsto \phi_q(s(\phi^{-1}(q)))]_{q \in (\phi^{-1})^{-1}(U)}$, where ϕ_q is the map on stalks at q. We can check that this glues (see official notes if we're still in disbelief).

Conversely, suppose we're given $(f, f^\#)$: Spec $B \to$ Spec A. Using $\mathcal{O}_{\text{Spec } A}(\text{Spec } A) \to \mathcal{O}_{\text{Spec } B}(\text{Spec } B)$, we get $g : A \to B$ a ring homomorphism. We need to check that g^{-1} gives the right topological map and that the construction from the first part gives the right map on sheaves.

For the first part, the maps on stalks are compatible with restriction. For instance, $\Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \to \Gamma(\text{Spec } B, \mathcal{O}_{\text{Spec } B})$ is compatible with restricting to the stalks: $\mathcal{O}_{\text{Spec } A, f(p)} \to \mathcal{O}_{\text{Spec } B,p}$ for all $p \in$ Spec B. Hence the following map commutes.

$$\begin{tikzcd}
\Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \arrow[r] \arrow[d] & \Gamma(\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \arrow[d] \\
\mathcal{O}_{\text{Spec } A, f(p)} \arrow[r] & \mathcal{O}_{\text{Spec } B,p}
\end{tikzcd}$$

Equivalently, the following map commutes for all $p \in \text{Spec } B$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ A_{f(p)} \arrow[r, "f^\#"] & B_p \end{tikzcd}$$

Since the morphism is local: $(f^\#)^{-1} pB_p = f(p)A_{f(p)}$. By commutativity, $g^{-1} = f$ topologically. The structure sheaf maps agree at the stalk level by construction, so we're done. ☐

Housekeeping.

Definition 4.4. Let X, Y be schemes. A morphism $f : X \to Y$ is an **open immersion** if f induces an isomorphism of X onto an open subscheme of Y (i.e. $(U, \mathcal{O}_{\mathcal{Y}}|_U)$ for $U \subset Y$ open).

$g : X \to Y$ is a **closed immersion** if the topological map is a homeomorphism onto a closed subset of Y and the map $g^\# : \mathcal{O}_Y \to g_\star \mathcal{O}_X$ is surjective.

Example 4.1. Take $k[t] \to k[t]/t^2$ and take Spec. This is a closed immersion.

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Definition 4.5. Let Y be a scheme. A **closed subscheme** of Y is an equivalence class of closed immersions $\{X \to Y\}$, where $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y$ are equivalent if there is a commuting triangle:

$$\begin{tikzcd} X \arrow[rr, "\text{isom}"] \arrow[dr, "f"] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

A typical example of a closed immersion: if A is a ring and $I \subset A$ is an ideal, then the natural map $\text{Spec } A/I \to \text{Spec } A$ is a closed immersion.

4.2 Fiber products

Fiber products will simultaneously generalize:

- the "product" of schemes.
- if $X_1, X_2 \subset Y$ are closed subschemes,then $X_1 \cap X_2 \subset Y$ is a fibre product.
- Given a morphism $X \xrightarrow{f} Y$ and a subscheme $Z \subset Y$, the preimage $f^{-1}(Z)$ is a subscheme of X, described by the fiber product.

Definition 4.6. Consider a diagram $\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$. The **fiber product** is a scheme $X \times_S Y$ fitting into a diagram $\begin{tikzcd} X \times_S Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$ which commutes such that for any other scheme Z with a commuting diagram of this form ($\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$) there is a unique morphism $Z \to X \times_S Y$ such that

$$\begin{tikzcd} Z \arrow[dr] \arrow[drr, bend left, "q_X"] \arrow[ddr, bend right, "q_Y"'] & & \\ & X \times_S Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

commutes.

Observation. If $X \times_S Y$ exists, then it is unique up to unique isomorphism.

Remarks.

- Similarly we can define these product in Sets, so if X, Y, S are sets with maps $X \stackrel{r_X}{\to} S$ and $Y \stackrel{r_Y}{\to} S$, then

$$X \times_S Y = \{(x, y) \in X \times Y \mid r_X(x) = r_Y(y)\}.$$

  This is left as an important exercise: check that this is the fibre product.

- Fibre products also exist in topological spaces with the same definition as in Sets (given the subspace (of product) topology).

- Say $X \stackrel{r_X}{\to} S$ is a map of sets and say $Y = \{\star\}$ with $r_Y(\star) = s \in S$. Then $X \times_S Y = r_X^{-1}(s)$.

Theorem 4.2. Fibre products of schemes exist.

This is Hartshorne chapter 2, Theorem 3.3, which we should read to understand all the details.

Proof. Affine case: Let X, Y, S be affine schemes with associated rings A, B, R. Then the fibre product $X \times_S Y$ exists and is isomorphic to Spec $(A \otimes_R B)$. We check that the universal property is satisfied, i.e. given any scheme Z with morphisms $\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$, there is a unique morphism $Z \to \text{Spec}\ (A \otimes_R B)$.

Fact (from example sheet 2): a scheme theoretic map $Z \to \text{Spec}\ A \otimes_R B \iff A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$.

Now we start with a general $X \times_S Y$ and "cover by affines".

- If $X \times_S Y$ exists and $U \subset X$ is an open subscheme, then $U \times_S Y$ also exists, since we take the inverse image of U under $X \times_S Y \stackrel{p_X}{\to} X$ with the open subscheme structure.

- If X is covered by opens $\{X_i\}$, then if $X_i \times_S Y$ exists for all i, then $X \times_S Y$ exists, since we can just glue (there's no cocycle condition).

Now for any X and S, Y affine, $X \times_S Y$ exists by above (i.e. we can cover X by affines). But X and Y are interchangable, so $X \times_S Y$ exists for all S affine and X, Y arbitrary. Now we cover S by affines $\{S_i\}$. Let X_i and Y_i be the preimages of S_i in X and Y respectively. Now $X_i \times_{S_i} Y_i$ exists. By the universal property, $X_i \times_{S_i} Y_i = X_i \times_S Y_i$, so glue in to form $X \times_S Y$. $\square$

Example 4.2. (i) $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec}\ \mathbb{Z}} \text{Spec}\ \mathbb{C}$, where $\text{Spec}\ \mathbb{C} \to \text{Spec}\ \mathbb{Z}$ is induced by $\mathbb{Z} \hookrightarrow \mathbb{C}$ and $\mathbb{P}^n_{\mathbb{Z}} \to \text{Spec}\ \mathbb{Z}$ is induced locally by $\mathbb{Z} \hookrightarrow \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$. Compare this with the fact that we already know, $\mathbb{Z}[\overline{X}] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[\overline{X}]$.

(ii) Let $C = \text{Spec}\ \mathbb{C}[x,y]/(y - x^2)$ and $L = \text{Spec}\ \mathbb{C}[x,y]/(y)$. We have natural morphisms $\mathbb{C} \to \mathbb{A}^2_{\mathbb{C}}, L \to \mathbb{A}^2_{\mathbb{C}}$. By algebra, $C \times_{\mathbb{A}^2} L = \text{Spec}(\mathbb{C}[x]/(x^2))$.

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Example sheet 2 is now up, on which many basic definitions are defined for structure sheafs – reduced, integral, irreducible, noetherian, etc. Have a look at the sheet, since we will use these definitions from now on.

Language. In scheme theory, we often fix a "base scheme" S and consider all other schemes X with a fixed morphism $X \to S$, which we often refer to as Sch/S, "schemes over S". A typical case to keep in mind is $S = \text{Spec}\ k$ or $S = \text{Spec}\ \mathbb{Z}$. The product in Sch/S is the fibre product $X \times_S Y$.

4.3 Separated morphisms

Motivation. A topological space X is Hausdorff if and only if the diagonal $\Delta_X = \{(x,x) \mid x \in X\} \subset X \times X$ is closed.

Definition 4.7. Let $X \to S$ be a morphism of schemes. Then the **diagonal** is the morphism $\Delta_{X/S} : X \to X \times_S X$ induced by the univeral property via:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[dr, dotted, "\exists!"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

(We use Δ instead of $\Delta_{X/S}$ when X and S are clear).

Orientation. If $U, V \subset X$ are opens and $S = \text{Spec } k$ for k a field, then $\Delta^{-1}(U \times_S V) = U \cap V$.

Definition 4.8. A morphism $X \to S$ is **separated** if $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion.

Example 4.3. Say $X = \text{Spec } \mathbb{C}[t]$ and $S = \text{Spec } \mathbb{C}$, $X \to S$ is induced by $\mathbb{C} \to \mathbb{C}[t]$. Then $X \times_S X = \text{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t])$, the diagonal Δ applied to $\text{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \to \mathbb{C}[t])$ via the multiplication map (i.e. Δ means take spec of the map of rings given by multiplication).

Δ is closed because $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \to \mathbb{C}[t]$ is surjective.

Clarificational category stuff. There are two ways to describe fiber products, which look different, but are basically the same thing. These two are given by:
$$\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}$$
and
$$\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d] & A \times B \arrow[d] \\
C \arrow[r, "\Delta"] & C \times C
\end{tikzcd}.$$

Proposition 4.3. Let $X \to S$ be a scheme morphism. Then there is a factorization of $\Delta_{X/S}$ given by
$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend right, "\Delta_{X/S}"'] & U \arrow[r] & X \times_s X
\end{tikzcd}$$
such that the first arrow is a closed immersion and the second arrow is an open immersion, i.e. a "locally closed immersion".

Proof. Let $g : X \to S$. Say S is covered by open affine schemes $\{V_i\}$ and suppose X is covered by $\{U_{ij}\}$, where for fixed i, $\{U_{ij}\}$ cover $g^{-1}(V_i)$. We have morphisms $U_{ij} \to V_i$ induced by the fibre product
$$\begin{tikzcd}
U_{ij} \arrow[r] & g^{-1}(V_i) \arrow[r] \arrow[d] & V_i \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}.$$
Now observe $U_{ij} \times_{V_i} U_{ij}$ is affine open in $X \times_S X$ and their union contains the image of $\Delta_{X/S}$. Also, $\Delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij} \subset X$. Take U in the statement

to be the union of $(U_{ij} \times_{V_i} U_{ij})$ over all i, j. The second map is clearly an open immersion. To check that a map $T \to T'$ is a closed immersion, we can do it locally on the codomain. For U_{ij} affine, the diagonal gives a map $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$, which is clearly a closed immersion (if this is not clear, see Example 4.3). $\square$

Proposition 4.4. If $X \to S$ is a morphism of affine schemes, then $\Delta_{X/S}$ is a closed immersion.

Proof. If $X = \text{Spec } A$, $S = \text{Spec } B$, $X \to S = \text{Spec } (B \to A)$, then $A \otimes_B A \to A$ is surjective. $\square$

Example 4.4. Recall the bug–eyed line $X = \mathbb{A}^1_k \sqcup \mathbb{A}^1_k / \sim$ with $\mathbb{A}^1_k \setminus \{0\} = U \subset \mathbb{A}^1_k$ and V similarly and the equivalence $V \xrightarrow{\sim} U$ given by $k[u, u^{-1}] \to k[t, t^{-1}]$ via $u \mapsto t$. We claim this is not separated over Spec k. What is $X \times_S X$? We can compute it via a gluing construction of the fibre product. Hence the output is a plane with doubled axes and four origins. But the diagonal only contains two out of the four origins (exercise), so it is not a closed subset.

Example 4.5. As we will be able to check/work out in a few lectures' time, open and closed immersions are always separated.

An easy consequence of Proposition 4.3: Let $X \to S$ be a morphism of schemes. If $\text{Im}(\Delta_{X/S})$ is closed as a topological subspace, then $X \to S$ is separated.

Proposition 4.5. Let A be any ring. Then the morphism $\mathbb{P}^n_A \to \text{Spec } A$ is separated.

Proposition 4.6. Let k be a field and $X \to \text{Spec } k$ a scheme morphism. Let $U, V \subset X$ be affine opens. Then if $X \to \text{Spec } k$ is separated, then $U \cap V$ is also affine.

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Example 4.6. Another example: $\mathbb{A}^n_k \to \text{Spec } k$ is separated, so $\mathbb{A}^n_k \times_{\text{Spec } k} \mathbb{A}^n_k = \text{Spec } k[x] \otimes_k k[y]$ and the map Δ is induced by multiplication $k[\overline{x}] \otimes k[\overline{y}] \to k[\overline{x}]$ by $f(x) \otimes g(y) \mapsto f(x)g(x)$.

Properties. (All of these are on example sheet 3).

- Open and closed immersions are always separated. For example, for closed immersions, the key observation is $A/J \otimes_A A/I \cong A/(I + J)$, so Δ is the identity and so closed (see example sheet 3).
- Compositions of separated morphisms are separated.

- Base extensions: suppose $X \to S$ is separated and $S' \to S$ is arbitrary. Then the natural map $X \times_S S' \to S'$ coming from
$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$
through the fibre product is also separated.

Proposition 4.7. Let R be any ring. Then $\mathbb{P}^n_R \to \text{Spec } R$ is always separated.

Proof. We want to show that in our following fibre product Δ is closed.

$$\begin{array}{ccccc} \mathbb{P}^n_R & \longrightarrow & \mathbb{P}^n_R \times_{R=\text{Spec } R} \mathbb{P}^n_R & \longrightarrow & \mathbb{P}^n_R \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^n_R & \longrightarrow & \text{Spec } R \end{array}$$

It suffices to check this on an open cover of $\mathbb{P}^n \times_R \mathbb{P}^n$ (note that here and before we abuse notation by writing R instead of Spec R as a subscript). Let $A = R[x_0, \ldots, x_n]$ with the usual grading. Let $U_i = \text{Spec}\left(A[\frac{1}{x_i}]\right)_0$. From our proj discussion, $\{U_i\}_{i=0}^n$ cover $\mathbb{P}^n_R$.

Now $U_i \times_R U_j = \text{Spec } R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \ldots, \frac{y_n}{y_j}\right]$. Now observe (exercise) that the restriction of Δ to $\Delta^{-1}(U_i \times_R U_j)$ is $U_i \cap U_j \to U_i \times_R U_j$, given on rings by $R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]\left[\frac{x_i}{x_j}\right] \leftarrow R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \ldots, \frac{y_n}{y_j}\right]$ with the map "change y to x". This map is clearly surjective and $U_i \times_R U_j$ cover $\mathbb{P}^n \times_R \mathbb{P}^n$, so Δ is closed. ☐

Let $k = \overline{k}$ be an algebraically closed field and let $X \to \text{Spec } k$ be a scheme over Spec k. Recall (from ES2) that X is of finite type over Spec k if there is a cover of X by affines $\{U_\alpha\}_\alpha$ such that $\mathcal{O}_X(U_\alpha)$ is a finitely generated k–algebra (i.e. a quotient of a polynomial ring by an ideal in finitely many variables). Also from ES2, X is reduced if and only if for all opens $U \subset X$, $\mathcal{O}_X(U)$ has no nilpotent elements.

Definition 4.9. $X \to \text{Spec } k$ is a **variety** if it is reduced, of finite type, and separated.

Example 4.7. For many examples, see the Part II Algebraic Geometry course or Chapter 1 of Hartshorne.

4.4 Properness

Let $f : X \to S$ be a morphism of schemes. Then f is of **finite type** if there exists an affine cover of S by opens $\{V_\alpha\}$ with each $V_\alpha = \text{Spec}(A_\alpha)$ and corresponding covers $\{U_{\alpha,\beta}\}_\beta$ of $f^{-1}(V_\alpha)$ by open affines such that $U_{\alpha,\beta} = \text{Spec}(B_{\alpha,\beta})$ such that $B_{\alpha,\beta}$ is a finitely generated A_α–module and $\{U_{\alpha,\beta}\}_\beta$ can be chosen to be finite.

Definition 4.10. $f : X \to S$ is **closed** if it is a closed topological map. It is **universally closed** if for any $S' \to S$, the induced $X \times_S S' \to S'$ is also closed.

Definition 4.11. $f : X \to S$ is **proper** if it is separated, of finite type, and universally closed.

Example 4.8. Closed immersions are proper (check or wait to find out why this is the case later).

Non–example. The obvious map $\mathbb{A}^1_k \to \text{Spec } k$ is not proper.

Proof. It is separated and of finite type, but we show it's not universally closed. $\mathbb{A}^1_k \to \text{Spec } k$ is closed (as it consists of one point). Consider the fibre product

$$\begin{tikzcd} \mathbb{A}^2_k \arrow[r] \arrow[d, "f'"] & \mathbb{A}^1_k \arrow[d, "f"] \\ \mathbb{A}^1_k \arrow[r] & \text{Spec } k \end{tikzcd}$$

Take $\mathbb{V}(xy-1) = Z \subset \mathbb{A}^2_k = \text{Spec } k[x,y]$. Then $f'(Z)$ is not Zariski closed. □

Observation. If $X \to S$ is proper, then any base extension $X \times_S S' \to S'$ is also proper. 10 Nov 2022,

We have two notions "separated" and "proper" for morphisms. If $X \to \text{Spec } k$ is a morphism, then the terminology is to say "X is separated" or "X is proper". Lecture 16

Example 4.9.
- $\mathbb{A}^1_k$ is separated, but not proper.
- The line with two origins is neither separated nor proper (as its not universally closed.)

Proposition 4.8. Let R be a commutative ring. Then $\mathbb{P}^n_R \to \text{Spec } R$ is proper.

Proof. Universal closedness of $X \to S$ is stable under base extension, i.e. for $S' \to S$ universally closed, $X \times_S S' \to S'$ is also universally closed. Since we already checked that $\mathbb{P}^n_R \to \text{Spec } R$ is separated, and finite type is immediate by construction, it suffices to prove the case $R = \mathbb{Z}$ since $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } R$.

We must show that for any $Y \to \text{Spec } \mathbb{Z}$, the base extension $\mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} Y \to Y$ is closed. But Y is covered by affine schemes of the form Spec R, and closedness is local on the target, it suffices to show that $\mathbb{P}^n_R \to \text{Spec } R$ is closed.

Let $Z \subset \mathbb{P}^n_R$ be Zariski closed, i.e. it is the vanishing locus of homogeneous polynomials $g_1, g_2, \ldots$. Our goal: if $\pi : \mathbb{P}^n_R \to \text{Spec } R$, then we need to show $\pi(Z)$ is closed. Let $K(p) = FF(R/p)$. We have a morphism Spec $K(p) \to$

Spec R. We want to know for which p is $Z_p = Z \times_{\text{Spec } R} K(p)$ nonempty. But Z_p is nonempty $\iff$ $\overline{g}_1, \overline{g}_2, \ldots$ cut out the origin in $\mathbb{A}^{n+1}_{K(p)}$ in $\mathbb{P}^n_{K(p)}$. Thus Z_p is nonempty $\iff$ $\sqrt{(\overline{g}_1, \overline{g}_2, \ldots)} \not\supset (x_0, \ldots, x_n)$ (where $\mathbb{P}^n_R = \text{Proj } R[x_0, \ldots, x_n]$). Equivalently, for all positive integers d, $(x_0, \ldots, x_n)^d \not\subset (\overline{g}_1, \overline{g}_2, \ldots)$.

Write $A = R[\overline{x}]$ with the usual grading. The non–containment is equivalent to the map $\bigoplus A_{d-\deg(g_i)} \to A_d$ given by $f_i \mapsto f_i g_i$ in the i^{th} factor to be non–surjective mod p (or equivalently in $K(p)$) for all degrees d. This condition is given by the vanishing of maximal minors of the matrix associated to $\bigoplus A_{d-\deg(g_i)} \to A_d$, which is infinitely many polynomials, each in the coefficients of the g_i. $\square$

From this point onwards, we will assume that all schemes are Noetherian (i.e. they have a finite cover by Noetherian rings).

4.5 Valuative criteria (for separatedness and properness)

A discrete valuation ring is a local PID.

Example 4.10.

- $\mathbb{C}[[t]]$ is a DVR.
- $\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \mid g(0) \neq 0 \right\}$ is a DVR.
- $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ for p–adic integers are DVRs.

Terminology/observations. Let A be a valuation ring (it is a discrete valuation ring, since it is noetherian). Then Spec A consists of two points $(0) \subset A$ and $\mathfrak{m} \subset A$, the unique maximal ideal.

The topology on Spec $A = \{(0), \mathfrak{m}\}$. (0) is dense, and the closure of $\{(0)\} =$ Spec A, and $\mathfrak{m}$ is closed.

Any generator π for the maximal ideal $\mathfrak{m}$ is called either a **uniformizer** or **uniformizing parameter**.

Any element $a \in A$ can be written as $u\pi^k$ where u is a unit and k is unique. The integer k is called the **valuation** of a. This gives a map val $: A \setminus \{0\} \to \mathbb{N}$ by $a \mapsto k$ (independent of the choice of π). 13 Nov 2022,

This allows us to construct a **valued field** on $K = FF(A)$, where the valuation extends to $K \setminus \{0\} \to \mathbb{Z}$ by $\frac{a}{b} \to \text{val}(a) - \text{val}(b)$. Lecture 17

Example 4.11. $A = K[[t]]$, so $FF(A) = K((t))$ and the valuation of $a = \alpha_k t^k + \ldots + \alpha_0$ is k.

Theorem 4.9. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any (discrete) valuation ring A with factor field K, given the

diagram of solid arrows
$$\begin{tikzcd} \mathrm{Spec}\ K \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\ A \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$
, there exists at most one choice of lift to fill in the dotted arrow.

Similarly, f is universally closed if and only if there exists at least one choice of lift for the dotted arrow.

Proof. Omitted and non–examinable. $\square$

Corollary 4.10. (i) $\mathbb{P}^n_R \to \mathrm{Spec}\ R$ is proper.

(ii) $\mathbb{A}^n_R \to \mathrm{Spec}\ R$ is not proper, but it is separated.

(iii) Closed immersions are proper, so in particular if $Z \to \mathbb{P}^n_R$ is closed, then $Z \to \mathrm{Spec}\ R$ is proper.

(iv) Compositions of proper (or separated) morphisms remain so.

(v) (Base extension) If $X \xrightarrow{f} Y$ is proper and $Y' \to Y$ is arbitrary, then $X \times_Y Y' \to Y'$ is also proper.

Some sample verifications: (i) $\mathbb{A}^1_k \to \mathrm{Spec}\ k$ is not proper (not universally closed). Write $\mathbb{A}^1_k = \mathrm{Spec}\ k[x]$ and consider $A = k[[t]], K = k((t))$. We have the diagram
$$\begin{tikzcd} \mathrm{Spec}\ k((t)) \arrow[r, "\phi"] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\ \mathrm{Spec}\ k[[t]] \arrow[r] & \mathrm{Spec}\ k \end{tikzcd}$$
where we let ϕ be induced by $k[x] \to k((t))$ via $x \mapsto \frac{1}{t}$.

Exercise. Use the valuative criterion to show that if Spec A is proper, then Spec A is finite as a topological space.

Observe that if $\mathbb{A}^1_k$ is replaced with $\mathbb{P}^1_k$, then this is always an affine chart in $\mathbb{P}^1$ such that the map above looks like $x \mapsto t$.

5 Modules over $\mathcal{O}_X$

Example 5.1. Let $\mathbb{CP}^n$ be the variety $C^{n+1} \setminus \{0\}/\sim$ modulo scaling. We have the structure sheaf: $\mathcal{O}_{\mathbb{CP}^n}$, if $U \subset \mathbb{CP}^n$ is Zariski open, then $\mathcal{O}_{\mathbb{CP}^n}(U) = \left\{\frac{P(X)}{Q(X)} \mid P, Q \text{ homogeneous of same degree and the ratio is regular at all } p \in U\right\}$.

Also, for $d \in \mathbb{Z}$, consider a sheaf $\mathcal{O}_{\mathbb{CP}^n}(d)$ with

$$\mathcal{O}_{\mathbb{CP}^n}(d)(U) = \left\{\frac{P(X)}{Q(X)} \mid P, Q \text{ homogeneous with } \deg(P) = \deg(Q) = d \text{ and regularity at all } p \in U\right\}.$$

Notice that $\mathcal{O}_{\mathbb{CP}^n}(d)(U)$ is a $\mathcal{O}_{\mathbb{CP}^n}(U)$–module.

Example 5.2. Let A be a ring and M an A–module. Define a sheaf $\mathcal{F}_M$ on Spec A: if $U \subset \text{Spec } A$ is a distinguished open $U = U_f$, then set $\mathcal{F}_M(U) = M_f$ (on general opens use sheaf on a base construction).

5.1 Definition of $\mathcal{O}_X$–modules

Fix $(X, \mathcal{O}_X)$ a ringed space.

Definition 5.1. A sheaf of $\mathcal{O}_X$–modules on X is a sheaf $\mathcal{F}$ of groups together with a multiplication $\mathcal{F}(U) \times \mathcal{O}_X(U) \to \mathcal{F}(U)$ giving a module (compatible with restriction).

Similarly we can build a sheaf of $\mathcal{O}_X$–algebras.

Definition 5.2. A morphism between sheaves of modules $\phi : \mathcal{F} \to \mathcal{G}$ on X is a homomorphism of sheaves of abelian groups compatible with multiplication.

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Example 5.3. Apparently this example is from last time. Let $X = \text{Spec } A$ and M be a A–module, then M^{sh} is sheaf on X with $M^{\text{sh}}(U_f) = M; e[\frac{1}{f}]$. We refer to this as the sheaf associated to a module M.

Basic operations: given a morphism $f : \mathcal{F} \to \mathcal{G}$ of (sheaves of) $\mathcal{O}_X$–modules, we can define (for modules) the kernel, cokernel*, image*, direct sum, tensor product*, Hom, etc., and we can extend these to sheaves of modules. Note that everything with an asterisk requires sheafification.

Example 5.4. The sheaf tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ associated to open $U \subset X$ by $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ and then sheafify.

$f : X \to Y$ is a morphism of ringed spaces/schemes. Given a $\mathcal{O}_X$–module $\mathcal{F}$, the push forward $f_\star(\mathcal{F})$ is a sheaf of abelian groups. But we have $f^\# : \mathcal{O}_Y \to f_\star \mathcal{O}_X$, so this gives $f_\star(\mathcal{F})$ an $\mathcal{O}_Y$–module structure. The module structure here: given $U \subset Y$ open and $a \in \mathcal{O}_Y(U)$ and $m \in f_\star(\mathcal{F}(U)) = \mathcal{F}(f^{-1}(U))$, we define $a \cdot m = \underbrace{f^\#(a)}_{\in \mathcal{O}_X(f^{-1}(U))} \cdot m.$

Conversely, if $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$–modules, then we define the pullback sheaf $f^\star(\mathfrak{G}) = f^{-1}\mathcal{G} \otimes_{\mathcal{F}^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where the $(f^{-1}\mathcal{O}_Y)$–module structure on $\mathcal{O}_X$ is defined via the adjoint to $f^\#$ (by example sheet 1 Q14).

5.2 $\mathcal{O}_X$–modules on schemes and quasi–coherence

Definition 5.3. A **quasi–coherent** scheme $\mathcal{F}$ (on a scheme X) of $\mathcal{O}_X$–modules is a sheaf of $\mathcal{O}_X$–modules $\mathcal{F}$ such that there exists a cover of X by affines $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is the sheaf associated to a module over the ring $\mathcal{O}_X(U_i)$.

If the modules over these $\mathcal{O}_X(U_i)$ can be taken as finitely generated, then $\mathcal{F}$ is **coherent**.

Example 5.5.
- On any scheme X, $\mathcal{O}_X$ is quasi–coherent (and coherent).
- $\mathcal{O}_X^{\oplus n}$ is also quasi–coherent and coherent.
- $\bigoplus_I \mathcal{O}_X$ for $|I| = \infty$ is quasi–coherent, but not coherent.
- If $i : X \hookrightarrow Y$ is a closed immersion, then $i_\star \mathcal{O}_X$ is a quasicoherent $\mathcal{O}_X$–module. Why? Say $U \subset X$ is affine and $U = \text{Spec } A$, then $X \cap U \hookrightarrow U$ gives an ideal $I \subset A$, i.e. the kernel of $\mathcal{O}_Y(U) \twoheadrightarrow \mathcal{O}_X(X, U)$. On U, $i_\star \mathcal{O}_X|_U$ is the sheaf associated to the A–module A/I.

Proposition 5.1. An $\mathcal{O}_X$–module $\mathcal{F}$ is quasi–coherent if and only if on any open $U \subset X$ with $U = \text{Spec } A$, $\mathcal{F}|_U$ is the sheaf associated to a module over A. Similarly we have the claim for coherence with finitely generated modules.

For this, we have a key lemma:

Lemma 5.2. Suppose $X = \text{Spec } A, f \in A$ and $\mathcal{F}$ is a quasi–coherent $\mathcal{O}_X$–module. Let $s \in \Gamma(X, \mathcal{F})$. Then

(i) If s restricts to 0 on U_f, then $f^n s = 0$ for some $n \geq 1$.

(ii) If $t \in \mathcal{F}(U_f)$, then $f^m \cdot t$ is the restriction of a global section of $\mathcal{F}$ (on X) for some $m \geq 1$.

Proof. There exists some cover of X by schemes of the form Spec $B = V$ such that $\mathcal{F}|_V = \mathcal{M}^{\text{sh}}$ for $\mathcal{M}$ a B–module. Now we can cover V by distinguished affines of the form U_g for $g \in A$. But $\mathcal{F}|_{U_g} = (\mathcal{M} \otimes_B A_g)^{\text{sh}}$, since $\mathcal{F}|_V$ is already quasi–coherent. Recall Spec A is "quasi–compact" (i.e. every open cover has a finite subcover), so finitely many g_i, U_{g_i} and $\mathcal{M}_i$ will suffice to cover X by opens such that $\mathcal{F}$ restricts to $\mathcal{M}_i^{\text{sh}}$ on U_{g_i}. Now the lemma follows from formal properties of localization (See Hartshorne II.5 for details). □

Proof of Proposition 5.1. $\mathcal{F}$ is quasi–coherent on X, so given $U \subset X$, $\mathcal{F}|_U$ is also quasi–coherent by (ideals similar to) Lemma 5.2. Hence we reduce to the case $X = \text{Spec } A$. Now take $\mathcal{M} = \Gamma(X, \mathcal{F})$ and let $\mathcal{M}^{\text{sh}}$ be the associated sheaf.

We claim that $\mathcal{M}^{\text{sh}}$ and $\mathcal{F}$ are isomorphic. For this, let $\alpha : \mathcal{M}^{\text{sh}} \to \mathcal{F}$ given by restriction of global sections (e.g. via stalks). Then α is an isomorphism at the stalk level – this is the content of Lemma 5.2. Hence α is an isomorphism of sheaves. □