

Part III - Modular Forms

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1 Introduction

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Lecture 1

Definition 1.1. We define the following groups:

$$\begin{aligned}\mathfrak{h} &= \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\} \\ GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\ \Gamma(1) &= SL_2(\mathbb{Z}) = \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.\end{aligned}$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on \mathfrak{h} by Möbius transformations.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$. Then

$$\operatorname{Im}(g\tau) = \frac{1}{2i} \left(\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\det(g)\operatorname{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{h}$. This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in \mathfrak{h} is conjugate to i . □

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{h} \rightarrow \mathbb{C}$, then

$$f|_k[g] : \mathfrak{h} \rightarrow \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight k action of g on f** .

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

Proof. We compute

$$\begin{aligned} (f|_k[g])|_k[h](\tau) &= \det(h)^{k-1} f|_k[g](h\tau) j(h, \tau)^{-k} = \\ \det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g, h\tau)^{-k} j(h, \tau)^{-k} &\stackrel{?}{=} \\ \det(gh)^{k-1} f(gh\tau) j(gh, \tau)^{-k} &= f|_k[gh](\tau). \end{aligned}$$

Hence we need to check that $j(gh, \tau) = j(gh, \tau)j(h, \tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh, \tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g \left(j(h, \tau) \begin{pmatrix} h\tau \\ 1 \end{pmatrix} \right) = j(h, \tau) j(g, h\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. \square

Formulae. For $g \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$, we have

$$\operatorname{Im}(g\tau) = \det(g) \frac{\operatorname{Im}(\tau)}{|j(g, \tau)|^2} \text{ and } j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\gamma \leq \Gamma(1)$ of finite index¹. A **weakly modular function of weight k and level Γ** is a meromorphic function $f : \mathfrak{h} \rightarrow \mathbb{C}$ which is invariant under the weight k action of Γ , i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in \mathfrak{h} and at ∞ .

It is a fact that modular forms of fixed weight and level live in finite-dimensional \mathbb{C} -vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

- (1) They are related to the theory of elliptic functions. Let E/\mathbb{C} be an elliptic curve and ω a holomorphic non-zero 1-form. Then there exists a unique lattice² $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \rightarrow E$ such that $\phi^*(\omega) = dz$. Then

¹In other words, γ is a (finite index) subgroup of $\Gamma(1)$.

²i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over \mathbb{R} .

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{h}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{h}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over \mathbb{C} .

- (2) Modular forms f have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n .

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q -expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$.

- (3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to \mathbb{C} and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L -function. Modular forms can be used to construct interesting examples of L -functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

- (4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves E/\mathbb{C} up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L -functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

2 Modular Forms on $\Gamma(1)$

09 Oct 2022,
Lecture 2

Reminder. A **meromorphic** function in an open subset $U \subset \mathbb{C}$ is a closed subset $A \subset U$ and a holomorphic function $f : U \setminus A \rightarrow \mathbb{C}$ such that $\forall a \in A$, $\exists \delta > 0$ such that $D^*(a, \delta) \subset U \setminus A$ and $\exists n \geq 0$ such that $(z - a)^n f(z)$ extends to a holomorphic function in $D(a, \delta)$.

f then has a Laurent expansion $\sum_{m \in \mathbb{Z}} a_m (z - a)^m$ valid on $D^*(a, \delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a meromorphic function \tilde{f} in $D^*(0, 1)$ (the "q-disk") such that

$$f(\tau) = \tilde{f}(e^{2\pi i \tau}).$$

Proof. f is meromorphic in \mathfrak{h} by assumption. Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$, as f is invariant under the weight k action of γ . But also $f(\gamma\tau) = f(\tau + 1)$, so f is periodic.

Now map a strip of \mathfrak{h} of width 1 to $D^*(0, 1)$ by $\tau \mapsto e^{2\pi i \tau}$. Let $a \in D^*(0, 1)$ and $\delta > 0$ be such that $D(a, \delta) \subset D^*(0, 1)$. Define \tilde{f} on $D(a, \delta)$ by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i} \log q\right),$$

for any branch of \log defined in $D(a, \delta)$. This is meromorphic and independent of the choice of the branch of \log , as f is periodic with period 1. This defines \tilde{f} in $D^*(0, 1)$. Finally, \tilde{f} is unique since $\tau \mapsto e^{2\pi i \tau}$ is surjective. \square

If \tilde{f} extends to a meromorphic function³ in $D(0, 1)$, then $\exists \delta > 0$ such that \tilde{f} has a Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0, \delta)$.

In the region $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$, we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q = e^{2\pi i \tau}$. This is called the **q-expansion** of the weakly modular function f .

Definition 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is **meromorphic at ∞** if \tilde{f} extends to a meromorphic function in $D(0, 1)$.

We say f is **holomorphic at ∞** if \tilde{f} is meromorphic at ∞ and has a

³This might not be the case if the set of poles has a limit inside the disk.

removable singularity at $q = 0$. In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau).$$

We say f **vanishes at ∞** if f is holomorphic at ∞ and $f(\infty) = 0$.

Definition 2.2. A **modular function** (of weight k and level $\Gamma(1)$) is a weakly modular function (of weight k and level $\Gamma(1)$) which is meromorphic at ∞ .

A **modular form** is a weakly modular function which is holomorphic in \mathfrak{h} and holomorphic at ∞ .

A **cuspidal modular form** is a modular form that vanishes at ∞ .

Remark. We let $M_k(\Gamma(1))$ denote the set of modular forms of weight k and level $\Gamma(1)$. We write $S_k(\Gamma(1))$ for the set of cuspidal modular forms of weight k , level $\Gamma(1)$. Note $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$. These are \mathbb{C} -vector spaces. If k is odd, then these both only contain the zero function, since taking $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ gives $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$.

We now consider even weights only. If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k},$$

where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ for any $\tau \in \mathfrak{h}$.

If $\gamma \in \Gamma(1)$, then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau)j(\gamma, \tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k}j(\gamma, \tau)^{-k},$$

but $\Lambda_{\gamma\tau} = \mathbb{Z} \frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1} (\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)) = (c\tau+d)^{-1} \Lambda_\tau$.
Hence

$$\begin{aligned} G_k|_k[g](\tau) &= \sum_{\lambda \in (c\tau+d)^{-1} \Lambda_\tau \setminus 0} \lambda^{-k} (c\tau+d)^{-k} \\ &= \sum_{\lambda \in \Lambda_\tau \setminus 0} ((c\tau+d)^{-1} \lambda)^{-k} (c\tau+d)^{-k} = G_k(\tau). \end{aligned}$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely. Hence:

Proposition 2.2. Let $k > 2$ be an even integer. Then $G_k(\tau)$ converges absolutely and defines a modular form of weight k and level $\Gamma(1)$ which has

$G_k(\infty) = 2\zeta(k)$. G_k is the **weight k Eisenstein series**.

We will later see that $M_2(\Gamma(1)) = 0$.

Proof. We want to show absolute and locally uniform convergence in \mathfrak{h} . This will show that G_k is holomorphic by complex analysis. Let $A \geq 2$ and define $\Omega_A = \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq \frac{1}{A}, \text{Re}(\tau) \in [-A, A]\}$. We show uniform convergence in Ω_A . If $\tau \in \Omega_A, x \in \mathbb{R}$, then $|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A. \end{cases}$ Hence

$$|\tau + x| \stackrel{(\dagger)}{\geq} \sup \left(\frac{1}{A}, \frac{|x|}{2A^2} \right) \geq \sup \left(\frac{1}{2A^2}, \frac{|x|}{2A^2} \right) = \frac{1}{2A^2} \sup(1, |x|).$$

(\dagger) follows by drawing a diagram with the lines $y = \frac{1}{A}$ and $y = \frac{x}{2A^2}$ and marking the point $(2A, \frac{1}{A})$ on it, then noticing that our supremum always lies above the supremum of these two lines. If $(m, n) \in \mathbb{Z}^2, m \neq 0$, then

$$|m\tau + n| = |m| \left| \tau + \frac{n}{m} \right| \geq |m| \frac{1}{2A^2} \sup \left(1, \left| \frac{n}{m} \right| \right) = \frac{1}{2A^2} \sup(|m|, |n|).$$

This is also valid when $m = 0$ by inspection. If $\tau \in \Omega_A$, then

$$\begin{aligned} & \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} |m\tau + n|^{-k} \\ & \leq \left(\frac{1}{2A^2} \right)^{-k} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \sup(|m|, |n|)^{-k} \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} \cdot |\{(m, n) \in \mathbb{Z}^2 \mid \sup(|m|, |n|) = d\}| \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} 8d = 8(2A^2)^k \sum_{d \in \mathbb{N}} d^{1-k} \\ & < \infty \end{aligned}$$

whenever $k - 1 > 1$, i.e. $k > 2$. This shows absolute convergence, and uniform convergence in Ω_A by the Weierstrass M-test⁴. Hence G_k is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$. It remains to show that G_k is holomorphic at ∞ with $G_k(\infty) = 2\zeta(k)$. For this, it suffices to check that

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = 2\zeta(k).$$

⁴If we have a sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ and values $M_n > 0$ with $|f_n(x)| < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges absolutely and uniformly on Ω . Here, replace n with d and sum d over $\sum_{(m,n) \in \mathbb{Z}^2 \setminus 0, \sup(|m|, |n|) = d} |m\tau + n|^{-k}$.

This follows from uniform convergence in Ω_A : we get

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \lim_{\text{Im}(\tau) \rightarrow \infty} (m\tau + n)^{-k} = \sum_{n \in \mathbb{Z} \setminus 0} n^{-k} = 2 \sum_{n \geq 1} n^{-k} = 2\zeta(k).$$

□

11 Oct 2022,
Lecture 3

Recap. We defined what it means for a function $f : \mathfrak{h} \rightarrow \mathbb{C}$ to be a modular form of weight k and level $\Gamma(1)$. $M_k(\Gamma(1))$ is the \mathbb{C} -vector space of such forms. If $f \in M_k(\Gamma(1))$, then there exists a holomorphic $\tilde{f} : D(0, 1) \rightarrow \mathbb{C}$ (here we call $D(0, 1)$ the q -disk) such that $\forall \tau \in \mathfrak{h}$, $f(\tau) = \tilde{f}(e^{2\pi i \tau})$. The Taylor expansion of \tilde{f} gives the q -expansion

$$f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}.$$

We have $f(\infty) = \tilde{f}(0) = a_0$. If $k > 2$ is even, then $G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k}$ converges absolutely and defines an element of $M_k(\Gamma(1))$ with $G_k(\infty) = 2\zeta(k)$.

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n.$$

We will soon show that we have $a_n \in \mathbb{Q} \forall n \geq 1$.

We can construct more modular forms: if $f \in M_k(\Gamma(1))$ and $g \in M_l(\Gamma(1))$, then $fg \in M_{k+l}(\Gamma(1))$. To check this is a modular form, we need to check that:

- fg is holomorphic, which is true as f, g are holomorphic.
- fg is invariant under the weight $k + l$ action of $\Gamma(1)$, which is true as f, g are invariant under the weight k and l actions of $\Gamma(1)$ – this is just a computation.
- fg is holomorphic at ∞ . This is true as the q -expansions multiply, so since f, g have no negative terms, the same is true for fg .

Hence we get e.g. $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$ and $\frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$ (i.e. it is cuspidal since zero at infinity). This difference is Ramanujan's Δ -function. We will show it is nonzero later.

We now want to show that $M_k(\Gamma(1))$ is finite-dimensional. We first study $\Gamma(1)/\mathfrak{h}$. For this, introduce a fundamental set $\mathfrak{f}' \subset \mathfrak{h}$ for the $\Gamma(1)$ -action. We define⁵ a fundamental set to be a set that intersects each $\Gamma(1)$ -orbit in exactly

⁵Definitions in literature may vary, so we omit a formal definition.

one element. Define

$$\mathfrak{f} = \left\{ \tau \in \mathfrak{h} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right], |\tau| \geq 1 \right\}.$$

$$\mathfrak{f}' = \left\{ \tau \in \mathfrak{f} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right), |\tau| = 1 \implies \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, 0 \right] \right\}.$$

Introduce $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\Gamma(1)$. We observe that every element of \mathfrak{f} is conjugate under S or T^{-1} to an element of \mathfrak{f}' , which is true since $T(\tau) = \tau + 1$ and $S(\tau) = -\frac{1}{\tau}$.



Proposition 2.3. Let $G = \Gamma(1)/\{\pm I\}$. Then

- (i) $\forall \tau \in \mathfrak{h}, \tau$ is $\Gamma(1)$ -conjugate to an element of \mathfrak{f}' .
- (ii) If $\tau, \tau' \in \mathfrak{f}'$ are $\Gamma(1)$ -conjugate, then $\tau = \tau'$.
- (iii) If $\tau \in \mathfrak{f}'$, then $\operatorname{Stab}_G(\tau)$ is trivial, except in the two cases $\operatorname{Stab}_G(i) = \langle S \rangle$ and $\operatorname{Stab}_G(\rho) = \langle ST \rangle$, where $\rho = e^{2\pi i/3}$.
- (iv) $\Gamma(1)$ is generated by S and T .

Proof. Let H be the subgroup of G generated by S and T .

Claim. Every $\tau \in \mathfrak{h}$ is H -conjugate to an element of \mathfrak{f}' .

Proof. By our above observation and since $S, T \in H$, it suffices to prove that every $\tau \in \mathfrak{h}$ is H -conjugate to \mathfrak{f} . Take $\tau \in \mathfrak{h}$. Recall that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}$.

In particular, $\forall R \geq 0$, the intersection $H\tau \cap \{\text{Im}(\tau') > R\}$ is finite, since $\text{Im}(\gamma\tau) > R \iff |c\tau + d|^2 < \frac{\text{Im}(\tau)}{R}$, but $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ is a lattice, so the set $\{(c, d) \in \mathbb{Z}^2 \mid |c\tau + d| < R'\}$ is finite.

So there exists $h \in H$ such that $\text{Im}(h\tau) \geq \text{Im}(h'\tau) \forall h' \in H$. After replacing τ by $h\tau$, we can assume $\text{Im}(\tau) \geq \text{Im}(h\tau) \forall h \in H$. Since acting by T does not change $\text{Im}(\tau)$, we can also assume $\text{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$. We have $\text{Im}(\tau) \geq \text{Im}(S\tau) = \frac{\text{Im}(\tau)}{|\tau|^2} \implies |\tau| \geq 1$, proving the claim and (i). \square

Now take $\tau, \tau' \in \mathfrak{f}'$ and suppose $\gamma\tau = \tau'$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. We want to show that either $\gamma = \pm I$ or $\tau = i, \rho$.

WLOG assume $\text{Im}(\tau') = \text{Im}(\gamma\tau) \geq \text{Im}(\tau)$, i.e. $\text{Im}(\gamma\tau) = \frac{\text{Im}(\tau)}{|c\tau + d|^2} \geq \text{Im}(\tau)$, so $|c\tau + d| \leq 1$. However, if $\tau \in \mathfrak{f}'$, then $\text{Im}(\tau) \geq \frac{\sqrt{3}}{2}$ with equality if and only if $\tau = \rho$. Hence $|c\tau + d| \geq |c|\text{Im}(\tau) \geq |c|\frac{\sqrt{3}}{2} \implies |c| \leq \frac{2}{\sqrt{3}} \implies |c| = 0, 1 \implies c = 0$ or $c = \pm 1$.

- If $c = 0$, then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, so $ad = 1 \implies a = d = \pm 1$, so $\gamma = \pm T^m$ for $m \in \mathbb{Z}$. However, T acts on \mathfrak{f}' by shifting the real part, so it can only stay in \mathfrak{f}' if $m = 0$ (as $\text{Re}(\mathfrak{f}') \in [-\frac{1}{2}, \frac{1}{2}]$), so $\gamma = \pm I$ and $\tau' = \tau$.
- If $c = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ and $|\tau + d| \leq 1$. By drawing another picture, we see that the only circles centered at integers of radius 1 which intersect \mathfrak{f}' are centered at $-d = 0, -d = -1$. Hence either $d = 0$, whence $|\tau| = 1$, or $d = 1$, whence $\tau = \rho$.

– If $c = 1, d = 0, |\tau| = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ since the determinant must be 1. Then $\gamma\tau = \frac{a\tau - 1}{\tau} = a - \frac{1}{\tau} = a - \bar{\tau}$, so $\text{Re}(\gamma\tau) = a - \text{Re}(\tau) \in \text{Re}(\mathfrak{f}' \cap \{|\tau| = 1\}) = [-\frac{1}{2}, 0]$. However, we also have $\text{Re}(\gamma\tau) \in a - [-\frac{1}{2}, 0] = a + [0, \frac{1}{2}]$.

The intersection $[-\frac{1}{2}, 0] \cap (a + [0, \frac{1}{2}])$ can be nonempty only if either $a = 0$, whence $\text{Re}(\gamma\tau) = \text{Re}(\tau) = 0$, so $\tau = \gamma\tau = i$, or $a = -1$, whence $\text{Re}(\tau) = \text{Re}(\gamma\tau) = -\frac{1}{2}$, so $\tau = \gamma\tau = \rho$.

If $a = 0$, then $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$, which stabilizes i , and $\langle -S \rangle = \langle S \rangle$.

If $a = -1$, then $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$, which stabilizes ρ , and $(ST)^3 = I$, so $\langle (ST)^2 \rangle = \langle ST \rangle$.

- If $c = 1, d = 1, \tau = \rho$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$, so $\rho = \gamma\rho = \frac{a\rho+b}{\rho+1}$. We have $\rho^2 + \rho + 1 = 0$, so $\rho^2 + \rho = -1$, so $a\rho + b = \rho^2 + \rho = -1$. But $a, b \in \mathbb{Z}$ and $1, \rho$ are linearly independent over \mathbb{R} , so $a = 0, b = -1$, so $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -ST$, which stabilizes ρ .

- If $c = -1$, we can reduce this to the case $c = 1$ by replacing γ with $-\gamma$.

We have now shown the first three parts of the proposition. It remains to show the last part, i.e. $\Gamma(1) = \langle S, T \rangle$. Since $S^2 = -I$, it is enough to show that $H = G$. Choose $\tau \in \text{Int}(f)$, so $\text{Stab}_G(\tau) = \{I\}$. Let $g \in G$. By our claim proving (i), $\exists h \in H$ such that $hg\tau \in \mathfrak{f}'$. We must therefore have $hg\tau = \tau$, hence $hg \in \text{Stab}_G(\tau) = \{I\}$, so $g = h^{-1} \in H$. \square

Notation. We write $e_\tau = |\text{Stab}_G(\tau)|$.

Let f be a nonzero modular function of weight k , level $\Gamma(1)$. If $\tau \in \mathfrak{h}$, then $v_\tau(f)$ is the order of f at τ (the unique $n \in \mathbb{Z}$ such that $f(z) = (z - \tau)^n g(z)$ for some meromorphic g that is holomorphic and non-vanishing at τ). We define $v_\infty(f)$ to be the order of f at infinity, i.e. $v_\infty(f) = v_0(\tilde{f})$ for \tilde{f} the meromorphic function in $D(0, 1)$ with $f(\tau) = \tilde{f}(e^{2\pi i \tau})$.

Proposition 2.4. Let f be a nonzero modular function of weight k , level $\Gamma(1)$. Then

$$\sum_{\tau \in \Gamma(1) \backslash \mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.$$

Proof. We first check that the sum is well-defined:

- If $\tau \in \mathfrak{h}$, then $e_\tau, v_\tau(f)$ only depend on the $\Gamma(1)$ -orbit of τ . This is because if $\gamma \in \Gamma(1)$ and $\tau \in \mathfrak{h}$, then $\text{Stab}_{\Gamma(1)}(\tau)$ and $\text{Stab}_{\Gamma(1)}(\gamma\tau)$ are conjugate subgroups of $\Gamma(1) \implies e_\tau = e_{\gamma\tau}$. On the other hand, $f(\gamma\tau) = f(\tau)j(\gamma, \tau)^k$ and $j(\gamma, \tau)$ is holomorphic and non-vanishing on \mathfrak{h} , so $v_{\gamma\tau}(f) = v_\tau(f)$.
- The sum only has a finite number of nonzero terms, since if f is a modular function and \tilde{f} is a meromorphic function on $D(0, 1)$, then $\exists \delta > 0$ such that \tilde{f} is holomorphic and non-vanishing in $D^*(0, \delta)$. Thus $\exists R > 0$ such that f is holomorphic and non-vanishing in $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > R\}$. Hence to show the sum is finite, it suffices to show that f only has a finite number of zeroes and poles in \mathfrak{f} (as f intersects every $\Gamma(1)$ -orbit), for which it suffices to show that f has a finite number of zeroes and poles in $\mathfrak{f} \cap \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \leq R\}$, which is true as the set is compact (closed and bounded) and the zeroes and poles of f are discrete.

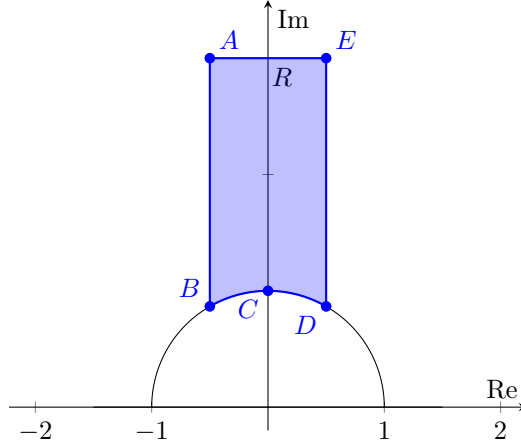
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Lecture 4

To prove the identity, we use contour integration. Setup: if $U \subset \mathbb{C}$ is an open subset, $f : U \rightarrow \mathbb{C}$ is holomorphic and $\gamma : [0, 1] \rightarrow U$ is a path, then $\int_{\gamma} f(z) dz = \int_{t=0}^1 f(\gamma(t)) \gamma'(t) dt$. We have the pullback formula: if $u : U \rightarrow V$ is a holomorphic map between open subsets of \mathbb{C} , $g : V \rightarrow \mathbb{C}$ is holomorphic and γ is a path in U , then $\int_{u \circ \gamma} g(z) dz = \int_{\gamma} u^*(g(z)) dz = \int_{\gamma} g(u(z)) u'(z) dz$. A particularly nice case: if $g(z) = h'(z)/h(z)$, then $g(z) dz = d \log h$, so $\int_{u \circ \gamma} d \log h = \int_{\gamma} u^*(d \log h) = \int_{\gamma} d(\log h \circ u) = \int_{\gamma} \frac{(h \circ u)'(z)}{(h \circ u)(z)} dz$.

We also have (Cauchy's) argument principle: if $U \subset \mathbb{C}$ is a simply connected open subset, $\gamma \subset U$ is a simple positively oriented closed path and g a meromorphic function in U with no zeroes or poles on γ , then

$$\frac{1}{2\pi i} \oint_{\gamma} d \log g = \frac{1}{2\pi i} \oint_{\gamma} \frac{g'(z)}{g(z)} dz = \sum_{a \in \text{Int}(\gamma)} v_a(g).$$

We now apply this to our modular function f . Choose $R > 0$ such that f has no zeroes or poles in $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq R\}$. We consider $\frac{1}{2\pi i} \oint_{\gamma} d \log f$, where γ is the contour $ABCDE$.



By choice of R , there are no zeroes or poles of f on AE . We first consider the case where f has no zeroes or poles at all on γ . Then the argument principle gives

$$\frac{1}{2\pi i} \oint_{\gamma} d \log f = \frac{1}{2\pi i} \int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d \log f = \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_{\tau}} v_{\tau}(f)$$

(as $v_{\tau}(f) \neq 0$, $e_{\tau} = 1$ under our assumptions).

Apply the pullback formula with $u(\tau) = \tau + 1$. Then $u(AB) = ED$, $f \circ u = f$,

so

$$\int_{u(AB)} d\log f = \int_{AB} d\log f \circ u = \int_{AB} d\log f = \int_{ED} d\log f = - \int_{DE} d\log f.$$

Hence $\int_{AB} + \int_{DE} d\log f = 0$.

Now take $q = e^{2\pi i \tau}$, so $f = \tilde{f} \circ q$ and $q(AE)$ is a positively oriented circle around 0 in $D(0, 1)$. So

$$\frac{1}{2\pi i} \int_{q(AE)} d\log \tilde{f} = v_\infty(f) = \frac{1}{2\pi i} \int_{AE} d\log \tilde{f} \circ q = \frac{1}{2\pi i} \int_{AE} d\log f.$$

Now take $v(\tau) = S(\tau) = -\frac{1}{\tau}$. Then $v(BC) = DC$ and we know $f|_k[S](\tau) = f(-\frac{1}{\tau})\tau^{-k} = f(\tau)$, so $f \circ v = f(\tau)\tau^k$. Hence

$$\begin{aligned} \int_{DC} d\log f &= \int_{v(BC)} d\log f = \int_{BC} d\log(f \circ v) = \int_{BC} d\log(f(\tau)\tau^k) \\ &= \int_{BC} d\log f + k d\log \tau = \int_{BC} d\log f + k(\log C - \log B) \end{aligned}$$

where here \log is any branch of the logarithm defined on BC . But $B = \rho, C = i$, so $\log B = i\frac{2\pi}{3}$ and $\log C = i\frac{\pi}{2}$. Hence

$$\int_{CD} d\log f = - \int_{DC} d\log f + k \left(\frac{2\pi i}{3} - \frac{2\pi i}{4} \right),$$

giving

$$\int_{BC} + \int_{CD} d\log f = 2\pi i k \frac{1}{12}.$$

We have

$$\begin{aligned} \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) &= \frac{1}{2\pi i} \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d\log f \right) \\ &= \frac{1}{2\pi i} \left(0 + \frac{k}{12} + 0 - v_\infty(f) \right) \\ &\implies \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}. \end{aligned}$$

This finishes the proof in the case where there are no zeroes or poles. If there are zeroes or poles on γ , we need to modify the contour. For example, if there's a zero or a pole at a point P on AB , then consider the contour γ' , which is just γ but with a small semicircle around our (discrete) pole, which satisfies the property that f has no zeroes or poles on γ' . The trickiest case is when there is

a zero or pole at $B = \rho$ or $C = i$. This is Q3 on example sheet 1. \square

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Lecture 5

Example 2.1. Take $k = 4$, $f = E_4 \in M_4(\Gamma(1))$. Hence $\forall \tau \in \mathfrak{h}$, $v_\tau(E_4) \geq 0$ (as it is holomorphic in \mathfrak{h}). We know $E_4(\tau) = 1 + \sum_{n \geq 1} a_n q^n$, so $E_4(\infty) \neq 0$ and $v_\infty(E_4) = 0$. Hence our formula gives

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_\tau} v(E_4) = \frac{1}{3} v_\rho(E_4) + \frac{1}{2} v_i(E_4) + \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}, \tau \not\sim \rho, i} v_\tau(E_4) = \frac{1}{3}.$$

So we have $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{3}$, where $a, b, c \in \mathbb{Z}_{\geq 0}$, which gives the only solution $a = 1, b = c = 0$, so $E_4(\rho) = 0$ and $E_4(\tau) \neq 0$ if $\tau \notin \Gamma(1)\rho$.

If $k = 6$, $f = E_6$, then we get

$$\frac{1}{3} v_\rho(E_6) + \frac{1}{2} v_i(E_6) + \sum_{\tau \not\sim \rho, i} v_\tau(E_6) = \frac{6}{12} = \frac{1}{2},$$

so this forces $v_\rho(E_6) = 0$, $v_i(E_6) = 1$, $v_\tau(E_6) \neq 0$ if $\tau \not\sim \rho, i$, so $E_6(1) = 0$, $E_6(\tau) \neq 0$ if $\tau \not\sim \rho, i$.

Recall $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$. This is nonzero since $\Delta(\rho) = \frac{E_4(\rho)^3 - E_6(\rho)^2}{1728} = -\frac{E_6(\rho)^2}{1728} \neq 0$. We also have $v_\infty(\Delta) \geq 1$ by construction, so plug in Δ to our formula to get

$$\sum_{\tau} \frac{1}{e_\tau} v_\tau(\Delta) + v_\infty(\Delta) = 1,$$

so $v_\infty(\Delta) = 1$, so Δ has a simple zero at ∞ and Δ is nonvanishing in \mathfrak{h} .

Theorem 2.5. Let $k \in 2\mathbb{Z}$. Then:

- (1) If $k < 0$ or $k = 2$, then $M_k(\Gamma(1)) = 0$, $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.
- (2) If $4 \leq k \leq 10$, then $M_k(\Gamma(1)) = \mathbb{C} \cdot E_k$.
- (3) For any k , multiplication by Δ gives an isomorphism $M_k(\Gamma(1)) \xrightarrow{\times \Delta} S_{k+12}(\Gamma(1))$.

Proof. (1) Let $f \in M_k(\Gamma(1))$ be nonzero. Then $\sum_{\tau} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$. Note the LHS is ≥ 0 , but for $k < 0$, the RHS is < 0 . If $k = 2$, then we get the equation $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{6}$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, which has no solutions.

Suppose $f \in M_0(\Gamma(1)) \setminus \mathbb{C} \cdot 1$. Then $f - f(\infty) \cdot 1 \in S_0(\Gamma(1))$ is a nonzero function (here 1 is the constant function 1). Then $\sum_{\tau} \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot 1) + \underbrace{v_\infty(f - f(\infty) \cdot 1)}_{\geq 1} = 0$, a contradiction, so $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.

- (2) Let $4 \leq k \leq 10$ and $f \in M_k(\Gamma(1))$. Consider $f - f(\infty) \cdot E_k \in S_k(\Gamma(1))$. If this is nonzero, then

$$\sum_{\tau} \frac{1}{e_{\tau}} v_{\tau}(f - f(\infty) \cdot E_k) + \underbrace{v_{\infty}(f - f(\infty) \cdot E_k)}_{\geq 1} = \frac{k}{12} < 1,$$

a contradiction. So $f = f(\infty) \cdot E_k$.

- (3) Our map $\times \Delta : M_k(\Gamma(1)) \rightarrow S_{k+12}(\Gamma(1))$ is a well-defined \mathbb{C} -linear map. It is injective, since if $\Delta f = 0$, then $f = 0$ (as Δ is nonvanishing in \mathfrak{h}). For surjectivity, if $f \in S_{k+12}(\Gamma(1))$, then $\frac{f}{\Delta}$ is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$.

We need to show $\frac{f}{\Delta}$ is holomorphic at ∞ , as then $\frac{f}{\Delta} \in M_k(\Gamma(1))$, so $f = \frac{f}{\Delta} f \in \text{Im}(\times \Delta)$. Hence we need $v_{\infty}\left(\frac{f}{\Delta}\right) \geq 0$. But $v_{\infty}\left(\frac{f}{\Delta}\right) = \underbrace{v_{\infty}(f)}_{\geq 1} - \underbrace{v_{\infty}(\Delta)}_{=1} \geq 0$, so we're done.

□

Corollary 2.6. If $k \in 2\mathbb{Z}$, $k \geq 0$, then $M_k(\Gamma(1))$ is finite-dimensional and

$$\dim_{\mathbb{C}} M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12}. \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}. \end{cases}$$

Proof. We proved this for $0 \leq k \leq 10$. In general, use induction on k : we need to show that for $k \geq 0$, $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = \dim_{\mathbb{C}} M_k(\Gamma(1)) + 1$.

We know $E_{k+12} \in M_{k+12}(\Gamma(1))$, so $M_{k+12}(\Gamma(1)) = \mathbb{C}E_{k+12} \oplus S_{k+12}(\Gamma(1))$. But this equals $\mathbb{C}E_{k+12} \oplus \Delta M_k(\Gamma(1))$, so $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$. □

Example 2.2. We have $E_4^2 \in M_8(\Gamma(1)) = \mathbb{C}E_8$. So there is a relation between E_4^2 and E_8 , but $E_8(\infty) = 1 = E_4(\infty)^2 \implies E_4^2 = E_8$.

Similarly, $E_4 E_6 \in M_{10}(\Gamma(1)) = \mathbb{C}E_{10}$, so the same argument gives $E_4 E_6 = E_{10}$.

Corollary 2.7. If $k \in 2\mathbb{N}$, then $M_k(\Gamma(1))$ is spanned as a \mathbb{C} -vector space by $\{E_4^a, E_6^b \mid a, b \in \mathbb{Z}_{\geq 0}, 4a + 6b = k\}$. In other words, if $\mathcal{M} = \oplus_{k \in \mathbb{Z}} M_k(\Gamma(1))$, then \mathcal{M} is a graded \mathbb{C} -algebra generated by E_4 and E_6 .

Proof. We proved this for $0 \leq k \leq 10$. If $k \geq 12$, then

$$M_k(\Gamma(1)) = \mathbb{C}E_k \oplus \Delta M_{k-12}(\Gamma(1)) = \mathbb{C}f \oplus \Delta M_{k-12}(\Gamma(1))$$

for any $f \in M_k(\Gamma(1))$ such that $f(\infty) \neq 0$ by the same argument. We can always find some $A, B \in \mathbb{Z}_{\geq 0}$ such that $4A + 6B = k$, so $E_4^A E_6^B \in M_k(\Gamma(1))$ and $(E_4^A E_6^B)(\infty) \neq 0$. Now by induction, $M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$, so $\Delta M_{k-12}(\Gamma(1)) = \langle \Delta E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$. But $\Delta \in \langle E_4^3, E_6^2 \rangle$, so

$$\Delta M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$$

and $E_4^A E_6^B \in \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$, so $M_k(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$. \square

Theorem 2.8. Let $j(\tau) = \frac{E_4(t)^3}{\Delta}$. Then j is a modular function of weight 0, level $\Gamma(1)$ which is holomorphic on \mathfrak{h} and has a simple pole at ∞ . It defines a bijection $\Gamma(1) \setminus \mathfrak{h} \rightarrow \mathbb{C}$ given by $\tau \rightarrow j(\tau)$. Moreover, every modular function of weight 0, level $\Gamma(1)$ is a rational function of j .⁶

The interpretation of this is that it is possible to define a Riemann surface structure on $\Gamma(1) \setminus \mathfrak{h} \sqcup \{\infty\}$ such that we get a compact Riemann surface whose meromorphic functions are exactly the modular functions of weight 0. So the theorem says that this Riemann surface, called $X(1)$, is isomorphic to the Riemann sphere, and our formula says that if \mathcal{L} is an invertible sheaf on a compact Riemann surface and S is a meromorphic section, then $\sum_a v_a(S) = \deg(\mathcal{L})$. This is useful if we are also taking algebraic geometry.

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Lecture 6

Proof. We showed that Δ is nonvanishing in \mathfrak{h} and has a simple zero at ∞ . Hence j is holomorphic in \mathfrak{h} and $v_\infty(j) = 3v_\infty(E_4) - v_\infty(\Delta) = -1$. Note that if $\gamma \in \Gamma(1)$, then $j|_0[\gamma](\tau) = j(\gamma\tau) = j(\tau)$. To show $\Gamma(1) \setminus \mathfrak{h}$ is a bijection, we need to show that $\forall z \in \mathbb{C}$, there exists a unique orbit $\Gamma(1) \cdot \tau$ such that $j(\tau) = z$, i.e. $v_\tau(j - z) > 0$.

We know

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_\tau} \underbrace{v_\tau(j - z)}_{\geq 0, \text{ as } j - z \text{ is holomorphic in } \mathfrak{h}} = 1,$$

(since $v_\infty(j - z) = -1$ and $\frac{k}{12} = 0$) again giving $a + \frac{b}{2} + \frac{c}{3} = 1$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, $a = v_\rho(j - z)$, $b = v_i(j - z)$, $c = \sum_{\tau \neq \rho, i} v_\tau(j - z)$. This gives the solutions

- $(a, b, c) = (0, 0, 1)$, so $j - z$ vanishes at a unique $\Gamma(1) \cdot \tau$.
- $(a, b, c) = (0, 2, 0)$, so $j - z$ vanishes at i .
- $(a, b, c) = (3, 0, 0)$, so $j - z$ vanishes at ρ .

Hence our map is bijective. Consider a nonzero modular function f of weight 0. We can consider a product $f \cdot \prod_{i=0}^n (j(\tau) - j(a_i))^{b_i}$ for $a_i \in \mathfrak{h}$, $b_i \in \mathbb{Z}_{\geq 0}$, where the a_i are among the poles of f in \mathfrak{h} . Hence to show f is a rational function

⁶Remember that $\Gamma(1) \setminus \mathfrak{h}$ is the set of orbits of $\Gamma(1)$ under \mathfrak{h} .

of j , it is enough to consider the case where f is holomorphic in \mathfrak{h} . Then there exists $m \geq 0$ such that $\Delta^m f$ is holomorphic at ∞ , so $\Delta^m f$ is holomorphic in \mathfrak{h} and at ∞ , so $\Delta^m f \in M_{12m}(\Gamma(1))$. We showed that $M_{12m}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = 12m \rangle$, so f is a linear combination of functions of the form $\frac{E_4^a E_6^b}{\Delta^m}$, where $4a + 6b = 12m$.

Hence it is enough to show that $\frac{E_4^a E_6^b}{\Delta^m}$ is a rational function of j where $4a + 6b = 12m$, $a, b \in \mathbb{Z}_{\geq 0}$. But then $2a + 3b = 6m$, which gives $p, q \in \mathbb{Z}_{\geq 0}$ such that $a = 3p, b = 2q$, so $p + q = m$. Then

$$\frac{E_4^a E_6^b}{\Delta^m} = \left(\frac{E_4^3}{\Delta} \right)^p \left(\frac{E_6^2}{\Delta} \right)^q = j^p \left(\frac{E_6^2}{\Delta} \right)^q.$$

As $E_4^3 - E_6^2 = 1728\Delta$, we get $j = \frac{E_6^2}{\Delta} + 1728$. So this is a rational function of j . \square

Proposition 2.9. Let $k \geq 4$ be an even integer. Then

$$G_k(\tau) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \cdot \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where $q = e^{2\pi i \tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Proof. We start from the identity

$$\pi \cot(\pi \tau) = \frac{1}{\tau} + \sum_{n \geq 1} \left(\frac{1}{\tau + n} + \frac{1}{\tau - n} \right).$$

This is true for $\tau \in \mathfrak{h}$ and it is even locally uniformly convergent in \mathfrak{h} . We can write

$$\pi \cot(\pi \tau) = i\pi \frac{e^{\pi i \tau} + e^{-\pi i \tau}}{e^{\pi i \tau} - e^{-\pi i \tau}} = \pi i \frac{q + 1}{q - 1} = -\pi i (1+q)(1-q)^{-1} = -\pi i \left(1 + 2 \sum_{n \geq 1} q^n \right).$$

Differentiate term-by-term $k-1$ times. The LHS is

$$-2\pi i \left(\frac{d}{d\tau} \right)^{k-1} \left(\sum_{n \geq 1} q^n \right) = -(2\pi i)^k \sum_{n \geq 1} n^{k-1} q^n,$$

while the RHS is

$$(-1)^{k-1} (k-1)! \left(\tau^{-k} + \sum_{n \geq 1} (\tau + n)^{-k} + (\tau - n)^{-k} \right) = (-1)^{k-1} (k-1)! \sum_{n \in \mathbb{Z}} (\tau + n)^{-k}.$$

Rearranging and using the fact that k is even (to make the sign go away) gives

$$\sum_{n \in \mathbb{Z}} (\tau + n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^n, \tau \in \mathfrak{h}.$$

Then

$$G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k} = 2\zeta(k) + \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus 0, \\ m \neq 0}} (m\tau + n)^{-k} = 2\zeta(k) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-k}.$$

Plug in our identity to get

$$G_k(\tau) = 2\zeta(k) + \sum_{m \geq 1} \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^{mn} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{N \geq 1} \underbrace{\left(\sum_{n|N} n^{k-1} \right)}_{=\sigma_{k-1}(N)} q^N.$$

□

Corollary 2.10. $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n$ has all $a_n \in \mathbb{Q}$. Moreover, if $k = 4$ or $k = 6$, then $a_n \in \mathbb{Z}$.

Proof. We have

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

Hence we need to show that $\frac{\zeta(k)}{\pi^k}$ is rational. This is on example sheet 1 (when k is even). One can show that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$, so

$$\begin{aligned} E_4(\tau) &= 1 + \frac{2^4 \pi^4 \cdot 90}{\pi^4 \cdot 6} \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \\ E_6(\tau) &= 1 - \frac{2^6 \pi^6 \cdot 3^3 \cdot 5 \cdot 7}{\pi^6 \cdot 5!} \sum_{n \geq 1} \sigma_5(n) q^n = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n. \end{aligned}$$

□

Corollary 2.11. If $\Delta = \sum_{n \geq 1} \tau(n) q^n$ is the q -expansion of Δ , then $\tau(n) \in \mathbb{Z} \forall n \geq 1$ and $\tau(1) = 1$.

Proof. Write $E_4 = 1 + 240U$ and $E_6 = 1 - 504V$ for $U, V = q + \dots \in \mathbb{Z}[[q]]$.

Then

$$\begin{aligned}\Delta &= \frac{E_4^3 - E_6^2}{1728} = \frac{(1 + 240U)^3 - (1 - 504V)^2}{1728} \\ &= \frac{3 \cdot 240U + 3 \cdot 240^2 U^2 + 240^3 U^3 + 2 \cdot 504V - 504^2 V^2}{1728} \\ &= \frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} + R,\end{aligned}$$

where we claim $R \in q^2\mathbb{Z}[[q]]$, but for this we just need to check that $1728 \mid 3 \cdot 240^2, 1728 \mid 240^3, 1728 \mid 504^2$.

We need to check that

$$\frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} = \frac{2^4 \cdot 3^2 \cdot 5 \cdot U + 2^4 \cdot 3^2 \cdot 7 \cdot V}{2^6 \cdot 3^3} \in \mathbb{Z}[[q]].$$

But this equals

$$\frac{5U + 7V}{12} = \frac{5(U - V)}{12} + V.$$

Hence we need to check that

$$\frac{5}{12}(\sigma_3(n) - \sigma_5(n)) \in \mathbb{Z} \quad \forall n \geq 1,$$

i.e. we need to check that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod{12} \quad \forall n \geq 1.$$

But this is true as $d^3 \equiv d^5 \pmod{12} \quad \forall d \in \mathbb{N}$.

Finally, we compute $\tau(1) = \frac{3 \cdot 240 + 2 \cdot 504}{1728} = 1$. □