Introduction to Additive Combinatorics
Part III

Lectured by Julia Wolf

Artur Avameri

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1 Fourier–analytic techniques

19 Jan 2024, Lecture 1

Let $G = \mathbb{F}_p^n$ for p a small fixed prime (usually $p = 2, 3, 5$) and n is large (often we consider $n \to \infty$).

Notation. Given a finite set B and any function $f : B \to \mathbb{C}$, we write $\mathbb{E}_{x\in B} f(x)$ to mean $\frac{1}{B}\sum_{x\in B} f(x)$. Also write $\omega = e^{2\pi i/p}$ for the p^{th} root of unity. Note that $\sum_{a\in\mathbb{F}_p}\omega^a = 0$.

Definition 1.1. Given $f : \mathbb{F}_p^n : \mathbb{C}$, we define its **Fourier transform** $\hat{f} : \mathbb{F}_p^n \to \mathbb{C}$ by

$$\hat{f}(t) = \mathbb{E}_{x\in\mathbb{F}_p^n} f(x)\omega^{x\cdot t} \ \forall t \in \mathbb{F}_p^n$$

where $x \cdot t$ is the standard scalar product.

It is easy to verify the **inversion formula**:

$$f(x) = \sum_{t\in\mathbb{F}_p^n} \hat{f}(t)\omega^{-x\cdot t} \ \forall x \in \mathbb{F}_p^n.$$

Indeed,

$$\sum_{t\in\mathbb{F}_p^n} \hat{f}(t)\omega^{-x\cdot t} = \sum_{t\in\mathbb{F}_p^n} \left(\mathbb{E}_y f(y)\omega^{y\cdot t}\right)\omega^{-x\cdot t}$$
$$= \mathbb{E}_y f(y) \underbrace{\sum_{t\in\mathbb{F}_p^n} \omega^{(y-x)\cdot t}}_{p^n 1_{\{y=x\}}} = f(x).$$

Remark. We could use an unnormalized sum in our definition and a normalized sum in the inversion formula, or a minus sign in our definition and a plus sign in the inversion formula – this doesn't matter as long as we're consistent.

Given a subset A of a finite group G, write:

- 1_A for the **characteristic function** of A, i.e. $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$.
  This is also called the **indicator function**.
- f_A for the **balanced function** of A, i.e. $f_A(x) = 1_A(x) - \alpha$, where $\alpha = \frac{|A|}{|G|}$.
- μ_A for the **characteristic measure** of A, i.e. $\mu_A(x) = \alpha^{-1}1_A(x)$.

Note $\mathbb{E}_{x\in G} f_A(x) = 0$ and $\mathbb{E}_{x\in G}\mu_A(x) = 1$. Given $A \subset \mathbb{F}_p^n$, we have

$$\hat{1}_A(t) = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_A(x)\omega^{x\cdot t}.$$

At $t = 0$, we get $\hat{1}_A(0) = \mathbb{E}_{x \in \mathbb{F}_p^n} 1_A(x) = \alpha$.

Writing $-A = \{-a \mid a \in A\}$, we have

$$\begin{aligned}\hat{1}_{-A}(t) &= \mathbb{E}_{x \in \mathbb{F}_p^n} 1_{-A}(x)\omega^{x \cdot t} = \mathbb{E}_{x \in \mathbb{F}_p^n} 1_A(-x)\omega^{x \cdot t} \\ &\overset{y=-x}{=} \mathbb{E}_{y \in \mathbb{F}_p^n} 1_A(y)\omega^{-y \cdot t} = \overline{\mathbb{E}_{y \in \mathbb{F}_p^n} 1_A(y)\omega^{y \cdot t}} = \overline{\hat{1}_A(t)}.\end{aligned}$$

Example 1.2. Let $V \leq \mathbb{F}_p^n$. Then

$$\hat{1}_V(t) = \mathbb{E}_{x \in \mathbb{F}_p^n} 1_V(x)\omega^{x \cdot t} = \frac{|V|}{p^n} 1_{\{x \cdot t = 0 \ \forall x \in V\}} = \frac{|V|}{p^n} 1_{V^\perp}(t),$$

so $\hat{\mu}_V(t) = 1_{V^\perp}(t)$. (Here we use the fact that if $t \notin \{x \cdot t = 0 \ \forall x \in V\}$, then $x \cdot t$ runs over the values uniformly and the sum is zero - details left as exercise).

Example 1.3. Let $R \subset \mathbb{F}_p^n$ be such that each $x \in \mathbb{F}_p^n$ lies in R independently with probability $\frac{1}{2}$. Then with high probability (i.e. $\mathbb{P} \to 1$ as $n \to \infty$),

$$\sup_{t \neq 0} |\hat{1}_R(t)| = O\left(\sqrt{\frac{\log(p^n)}{p^n}}\right).$$

Proving this is on Ex. Sheet 1. This is proved using a Chernoff–type bound: given complex–valued independent random variables $X_1, \ldots, X_n$ with mean 0, $\forall \theta \geq 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \theta \sqrt{\sum_{i=1}^n ||X_i||_{L^\infty(\mathbb{P})}^2}\right) \leq 4\exp\left(-\theta^2/4\right).$$

Example 1.4. Let $Q = \{x \in \mathbb{F}_p^n \mid x \cdot x = 0\}$. Then $|Q| = \left(\frac{1}{p} + O(p^{-n})\right) p^n$ and $\sup_{t \neq 0} |\hat{1}_Q(t)| = O(p^{-n/2})$. This is again on Ex. Sheet 1.

Notation. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, write

$$\langle f, g \rangle = \mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\overline{g(x)}$$

and

$$\langle \hat{f}, \hat{g} \rangle = \sum_{t \in \mathbb{F}_p^n} \hat{f}(t)\overline{\hat{g}(t)}.$$

Consequently, $||f||_2^2 = \mathbb{E}_x |f(x)|^2$ and $||\hat{f}||_2^2 = \sum_t |\hat{f}(t)|^2$.

Lemma 1.5. The following hold for all $f, g : \mathbb{F}_p^n \to \mathbb{C}$:

(i) $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ (Plancherel's identity).

(ii) $||f||_2 = ||\hat{f}||_2$ (Parseval's identity).

Proof. (ii) follows from (i). For (i), compute

$$\begin{aligned}\langle \hat{f}, \hat{g}\rangle &= \sum_{t\in\mathbb{F}_p^n} \hat{f}(t)\overline{\hat{g}(t)} = \sum_{t\in\mathbb{F}_p^n} \frac{1}{p^{2n}} \sum_{x\in\mathbb{F}_p^n} f(x)\omega^{x\cdot t} \sum_{y\in\mathbb{F}_p^n} \overline{g(y)\omega^{y\cdot t}} \\ &= \frac{1}{p^{2n}} \sum_{x,y\in\mathbb{F}_p^n} f(x)\overline{g(y)} \sum_{t\in\mathbb{F}_p^n} \omega^{(x-y)t} = \frac{1}{p^{2n}} \sum_{x\in\mathbb{F}_p^n} p^n f(x)\overline{g(x)} = \langle f, g\rangle.\end{aligned}$$

□

Definition 1.6. Let $\rho > 0$ and $f : \mathbb{F}_p^n \to \mathbb{C}$. Define the ρ–**large spectrum** of f to be

$$\mathrm{Spec}_\rho(f) = \{t \in \mathbb{F}_p^n \mid |\hat{f}(t)| \geq \rho||f||_1\}.$$

Example 1.7. By Example 1.2, if $f = 1_V$ with $V \leq \mathbb{F}_p^n$, then $\forall \rho > 0$, $\mathrm{Spec}_\rho(f) = V^\perp$. [1]

Lemma 1.8. For all $\rho > 0$, $|\mathrm{Spec}_\rho(f)| \leq \rho^{-2}\frac{||f||_2^2}{||f||_1^2}$.

Proof. By Parseval,

$$||f||_2^2 = ||\hat{f}||_2^2 \geq \sum_{t\in\mathrm{Spec}_\rho(f)} |\hat{f}(t)^2| \geq |\mathrm{Spec}_\rho(f)|(\rho||f||_1)^2.$$

□

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Definition 1.9. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, define their **convolution** $f * g : \mathbb{F}_p^n \to \mathbb{C}$ by

$$f * g(x) = \mathbb{E}_{y\in\mathbb{F}_p^n} f(y)g(x-y) \ \forall x \in \mathbb{F}_p^n.$$

Example 1.10. Given $A, B \subset \mathbb{F}_p^n$,

$$\begin{aligned}1_A * 1_B(x) &= \mathbb{E}_{y\in\mathbb{F}_p^n} 1_A(y)1_B(x-y) = \frac{1}{p^n}|A\cap(x-B)| \\ &= \frac{1}{p^n}\#\text{ways } x \text{ can be written as } x = a+b \text{ with } a\in A, b\in B.\end{aligned}$$

In particular, the support of $1_A * 1_B$ is the **sum set**

$$A + B = \{a + b \mid a \in A, b \in B\}$$

of A and B.

Lemma 1.11. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$,

$$\widehat{f * g}(t) = \hat{f}(t)\hat{g}(t) \ \forall t \in \mathbb{F}_p^n.$$

[1] Here we have $0 < \rho \leq 1$, since it is clear by triangle inequality that $||f||_1 \geq |\hat{f}(t)|$.

Proof. Set $u = x - y$ to get

$$\begin{aligned}\widehat{f * g}(t) &= \mathbb{E}_{x\in\mathbb{F}_p^n}\left(\mathbb{E}_{y\in\mathbb{F}_p^n} f(y)g(x-y)\right)\omega^{x\cdot t}\\ &= \mathbb{E}_y f(y)\mathbb{E}_u g(u)\omega^{(u+y)\cdot t}\\ &= \hat{f}(t)\hat{g}(t).\end{aligned}$$

$\square$

Example 1.12. $||\hat{f}||_4^4 = \mathbb{E}_{x+y=z+w} f(x)f(y)\overline{f(z)f(w)}$. This is on Ex. Sheet 1.

Lemma 1.13 (Bogolyubov's Lemma)**.** Given $A \subset \mathbb{F}_p^n$ of density $\alpha > 0$, there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension at most $2\alpha^{-2}$ s.t. $A + A - A - A \supset V$.

Proof. Observe that

$$A + A - A - A = \mathrm{supp}(\underbrace{1_A * 1_A * 1_{-A} * 1_{-A}}_{:=g}).$$

Hence we wish to find $V \leq \mathbb{F}_p^n$ such that $g(x) > 0\ \forall x \in V$. Let $K = \mathrm{Spec}_\rho(1_A)$ with ρ to be determined later and let $V = \langle K\rangle^\perp$. By Lemma 1.8[2], $|K| \leq \rho^{-2}\alpha^{-1}$ and hence $\mathrm{codim}(V) \leq |K| \leq \rho^{-2}\alpha^{-1}$. By the inversion formula,

$$\begin{aligned}g(x) &= \sum_{t\in\mathbb{F}_p^n} (1_A * \widehat{1_A * 1_{-A}} * 1_{-A})(t)\omega^{-x\cdot t}\\ &= \sum_{t\in\mathbb{F}_p^n} |\hat{1}_A(t)|^4\omega^{-x\cdot t}\\ &= \alpha^4 + \underbrace{\sum_{t\in K\setminus\{0\}} |\hat{1}_A(t)|^4\omega^{-x\cdot t}}_{(1)} + \underbrace{\sum_{t\notin K} |\hat{1}_A(t)|^4\omega^{-x\cdot t}}_{(2)}.\end{aligned}$$

For (1), we see it is ≥ 0 since $x\cdot t = 0\ \forall t \in K, x \in V$. (Note we could give better lower bounds but we don't need them).

For (2), we have

$$\begin{aligned}|(2)| &\leq \sum_{t\notin K} |\hat{1}_A(t)|^4 \leq \sup_{t\notin K} |\hat{1}_A(t)|^2 \sum_{t\notin K} |\hat{1}_A(t)|^2 \leq \sup_{t\notin K} |\hat{1}_A(t)|^2 \sum_t |\hat{1}_A(t)|^2\\ &\leq (\rho\alpha)^2||1_A||_2^2 = \rho^2\alpha^3.\end{aligned}$$

Now pick ρ such that $\rho^2\alpha^3 \leq \frac{\alpha^4}{2}$, e.g. $\rho = \sqrt{\frac{\alpha}{2}}$, so $g(x) \geq \frac{\alpha^4}{2} > 0\ \forall x \in V$. $\square$

[2]Here $f = 1_A$ and $\alpha = \frac{||f||_1^2}{||f||_2^2} = \frac{\left(\frac{1}{p^n}\sum|f|\right)^2}{\left(\frac{1}{p^n}\sum|f|^2\right)} = \frac{|A|}{p^n} = \alpha$.

Example 1.14. The set $A = \{x \in \mathbb{F}_2^n \mid |x| \ge \frac{n}{2} + \frac{\sqrt{n}}{2}\}$ has density at least $\frac{1}{4}$, and there is no coset C of any subspace of codimension at most $\sqrt{n}$ such that $C \subset A + A$. This is on Ex. Sheet 1.

Lemma 1.15. Let $A \subset \mathbb{F}_p^n$ of density α be such that $\exists t \neq 0$ in $\mathrm{Spec}_\rho(1_A)$. Then $\exists V \le \mathbb{F}_p^n$ of codimension 1 and $\exists x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \ge \alpha \left(1 + \frac{\rho}{2}\right) |V|.$$

Proof. Let $t \neq 0$ be such that $|\hat{1}_A(t)| \ge \rho\alpha$ and let $V = \langle t \rangle^\perp$. Write $v_j + V$ for $j \in [p] := \{1, 2, \dots, p\}$ for the cosets of V such that $v_j + V = \{x \in \mathbb{F}_p^n \mid x \cdot t = j\}$. Then

$$\begin{aligned} \rho\alpha \le \hat{1}_A(t) &= \hat{f}_A(t) \\ &= \mathbb{E}_{x \in \mathbb{F}_p^n} (1_A(x) - \alpha) \omega^{x \cdot t} \\ &= \mathbb{E}_{j \in [p]} \underbrace{\mathbb{E}_{x \in v_j + V} (1_A(x) - \alpha)}_{:= a_j = \frac{|A \cap (v_j + V)|}{|V|} - \alpha} \omega^j. \end{aligned}$$

By the triangle inequality, $\mathbb{E}_{j \in [p]} |a_j| \ge \rho\alpha$. Since $\mathbb{E}_{j \in [p]} a_j = \frac{|A|}{p^{n-1}} - p\alpha = 0$, $\mathbb{E}_{j \in [p]} (a_j + |a_j|) \ge \rho\alpha$, so $\exists j \in [p]$ such that $a_j + |a_j| \ge \rho\alpha \implies a_j \ge \frac{\rho\alpha}{2}$. □

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Lemma 1.16. Let $p \ge 3$ and $A \subset \mathbb{F}_p^n$ of density $\alpha > 0$ be such that

$$\sup_{t \neq 0} |\hat{1}_A(t)| = o(1).$$

Then A contains $(\alpha^3 + o(1))(p^n)^2$ 3–term arithmetic progressions (3–APs).

In other words, a set with small Fourier coefficients has the same number of 3–APs as a truly random set of the same density.

Notation. Given $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$, $T_3(f, g, h) = \mathbb{E}_{x,d} f(x) g(x + d) h(x + 2d)$.

Given $A \subset \mathbb{F}_p^n$, write $2 \cdot A = \{2a \mid a \in A\}$. This is different from $2A = A + A = \{a + a' \mid a, a' \in A\}$.

Proof. The number of 3–APs in A is $(p^n)^2$ times $T_3(1_A, 1_A, 1_A)$, where

$$\begin{aligned} T_3(1_A, 1_A, 1_A) &= \mathbb{E}_{x,d} 1_A(x) 1_A(x + d) 1_A(x + 2d) \\ &= \mathbb{E}_{x,y} 1_A(x) 1_A(y) 1_A(2y - x) && y = x + d \\ &= \mathbb{E}_y 1_A(y) (1_A * 1_A)(2y) \\ &= \langle 1_{2 \cdot A}, 1_A * 1_A \rangle && z = 2y \\ &= \langle \widehat{1_{2 \cdot A}}, \widehat{1_A * 1_A} \rangle. && \text{by Plancherel.} \end{aligned}$$

Continue the last manipulation to get

$$\begin{aligned} &= \langle \widehat{1_{2\cdot A}}, \hat{1}_A^2 \rangle \\ &= \alpha^3 + \sum_{t\neq 0} \widehat{1_A}(t)^2 \overline{\widehat{1_{2\cdot A}}(t)}. \end{aligned}$$

The last sum in absolute value is at most

$$\begin{aligned} &\leq \sup_{t\neq 0} |\widehat{1_A}(t)| \sum_{t\neq 0} |\widehat{1_A}(t)\overline{\widehat{1_{2\cdot A}}(t)}| \\ &\leq \sup_{t\neq 0} |\widehat{1_A}(t)| \left(\sum_t |\widehat{1_A}(t)|^2 \right)^{1/2} \left(\sum_t |\widehat{1_{2\cdot A}}(t)|^2 \right)^{1/2} \\ &\leq \sup_{t\neq 0} |\widehat{1_A}(t)| \cdot \alpha^{1/2} \cdot \alpha^{1/2} \\ &\leq \sup_{t\neq 0} |\widehat{1_A}(t)| \end{aligned}$$

by C–S and Parseval. □

Using the above two results, we prove:

Theorem 1.17 (Meshulam's Theorem)**.** Let $p \geq 3$ and let $A \subset \mathbb{F}_p^n$ be a set containing no non–trivial 3–APs. Then $|A| = O\left(\frac{p^n}{n \log p}\right)$.

Proof. By assumption, $T_3(1_A, 1_A, 1_A) = \frac{\alpha}{p^n}$, but as in Lemma 1.16,

$$T_3(1_A, 1_A, 1_A) = \alpha^3 + \sum_{t\neq 0} \hat{1}_A(t)^2 \overline{\hat{1}_{2\cdot A}(t)},$$

so $\left|\frac{\alpha}{p^n} - \alpha^3\right| \leq \sup_{t\neq 0} |\hat{1}_A(t)| \cdot \alpha$, which gives $\sup_{t\neq 0} |\hat{1}_A(t)| \geq \left|\frac{1}{p^n} - \alpha^2\right| \geq \frac{\alpha^2}{2}$ provided $p^n \geq 2\alpha^{-2}$. By Lemma 1.15 with $\rho = \frac{\alpha}{2}$, $\exists V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that $|A \cap (x+V)| \geq \left(\alpha + \frac{\alpha^2}{4}\right)|V|$.

We iterate this observation. Let $A_0 = A, V_0 = \mathbb{F}_p^n$, $\alpha_0 = \alpha = \frac{|A_0|}{|V_0|}$. At step i of this iteration, we are given a set $A_{i-1} \subset V_{i-1}$ of density α_{i-1} with no nontrivial 3–APs. Provided that $p^{\dim(V_{i-1})} \geq 2\alpha_{i-1}^{-2}$, $\exists V_i \leq V_{i-1}$ of codimension 1 and $x_i \in V_{i-1}$ such that $|A_{i-1} \cap (x_i + V_i)| \geq \left(\alpha_{i-1} + \frac{\alpha_{i-1}^2}{4}\right)|V_i|$. Set $A_i = A_{i-1} - x$. Note $\alpha_i \geq \alpha_{i-1} + \frac{\alpha_{i-1}^2}{4}$ and A_i is free of nontrivial 3–APs. Through this iteration, the density of A increases from α to 2α in at most $\frac{\alpha}{\alpha^2/4} = 4\alpha^{-1}$ steps, from 2α to 4α in at most $\frac{2\alpha}{(2\alpha)^2/4} = 2\alpha^{-1}$ steps, etc, which reaches 1 in at most

$$(4\alpha^{-1} + 2\alpha^{-1} + \alpha^{-1} + \dots) = 8\alpha^{-1}$$

steps. The argument must therefore end with $\dim(V_i) \geq n - 8\alpha^{-1}$, at which point we must have had $p^{\dim(V_i)} \leq 2\alpha_i^{-2} \leq 2\alpha^{-2}$ (or else we could have continued). But we may assume that $\alpha \geq \sqrt{2}p^{-n/4}$ (else we're done), whence $p^{n-8\alpha^{-1}} \leq p^{n/2}$, i.e. $\frac{n}{2} \leq 8\alpha^{-1}$, so $\alpha \leq \frac{16}{n}$, finishing the proof (in fact, we can now take $C = 16\log p$ as an explicit constant in the big O notation). $\square$

So for $A \subset \mathbb{F}_3^n$ containing no nontrivial 3–APs, we have $|A| = O\left(\frac{3^n}{n}\right)$. The largest known subset of $\mathbb{F}_3^n$ containing no notrivial 3–APs has size $\geq (2.218)^n$. (Proving 2^n is trivial: take all combinations of zeroes and ones with no twos). 26 Jan 2024, Lecture 4

From now on, let G be a finite abelian group. G comes equipped with a set of **characters**, i.e. group homomorphisms $\gamma : G \to \mathbb{C}^\times$, which themselves form a group, denoted by $\hat{G}$, often referred to as the **dual** of G. It turns out that if G is finite and abelian, then $\hat{G} \cong G$. For instance:

- If $G = \mathbb{F}_p^n$, then $\hat{G} = \{\gamma_t : x \mapsto \omega^{x \cdot t} \mid t \in G\}$.
- If $G = \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$, then $\hat{G} = \{\gamma_t : x \mapsto \omega^{xt} \mid t \in G\}$.

Definition 1.18. Given $f : G \to \mathbb{C}$, define its **Fourier transform** $\hat{f} : \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\gamma) = \mathbb{E}_{x\in G} f(x)\gamma(x) \ \forall \gamma \in \hat{G}.$$

It is easy to verify that we have an inversion formula, given by

$$f(x) = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma)\overline{\gamma(x)}.$$

We can also check that Definition 1.6 and 1.9, Examples 1.3 and 1.10 and Lemmas 1.5, 1.8 and 1.11 go through in this general context.

Example 1.19. Let p be a prime, let $L \leq p - 1$ be even and consider $J = \left[-\frac{L}{2}, \frac{L}{2}\right] \subset \mathbb{Z}_p$. Then $\forall t \neq 0$,

$$|\hat{1}_J(t)| \leq \min\left\{\frac{L+1}{p}, \frac{1}{2|t|}\right\}.$$

This is on Ex. Sheet 1.

Theorem 1.20 (Roth's Theorem)**.** Let $A \subset [N] := \{1, 2, \ldots, N\}$ be a set containing no non–trivial 3–APs. Then $|A| = O\left(\frac{N}{\log\log N}\right)$.

Lemma 1.21. Let $A \subset [N]$ be of density $\alpha > 0$ satisfying $N > 50\alpha^{-2}$ containing no nontrivial 3–APs. Let p be a prime in $\left[\frac{N}{3}, \frac{2N}{3}\right]$ and write $A' = A \cap [p] \subset \mathbb{Z}_p$. Then either

(i) $\sup_{t\neq 0} |\hat{1}_{A'}(t)| \geq \frac{\alpha^2}{10}$ (where the Fourier coefficient is computed in $\mathbb{Z}_p$); or

(ii) $\exists$ interval $J \subset [N]$ of length $\geq \frac{N}{3}$ such that $|A \cap J| \geq \alpha\left(1 + \frac{\alpha}{400}\right)|J|$.

Proof. We may assume that $|A'| = |A \cap [p]| \geq \alpha\left(1 - \frac{\alpha}{200}\right)p$, since otherwise $|A \cap [p+1, N]| \geq \alpha N - \alpha\left(1 - \frac{\alpha}{200}\right)p = \alpha(N-p) + \frac{\alpha^2 p}{200} \geq \alpha\left(1 + \frac{\alpha}{400}\right)(N-p)$, so case (ii) holds with $J = [p+1, N]$.

Let $A'' = A' \cap \left[\frac{p}{3}, \frac{2p}{3}\right]$. Note that all 3–APs of the form $(x, x+d, x+2d) \in A' \times A'' \times A''$ are in fact proper APs in $[N]$ (and not only in $\mathbb{Z}_p$, since there's no "wrapping around", since $x+d, x+2d \in \left[\frac{p}{3}, \frac{2p}{3}\right]$).

If $|A' \cap [p/3]|$ or $|A' \cap [2p/3, p]|$ are at least $\frac{2|A'|}{5}$, then we are again in case (ii) (details left as exercise). Hence we may assume that $|A''| \geq \frac{|A'|}{5}$. Now as in Lemma 1.16 and Theorem 1.17 with $\alpha' = |A'|/p, \alpha'' = |A''|/p$,

$$\frac{\alpha''}{p} = \frac{|A''|}{p^2} = T_3(1_{A'}, 1_{A''}, 1_{A''}) = \alpha' \cdot \alpha''^2 + \sum_{t \neq 0} \hat{1}_{A'}(t)\hat{1}_{A''}(t)\overline{\hat{1}_{2 \cdot A''}(t)},$$

so as before,

$$\left|\frac{\alpha''}{p} - \alpha'\alpha''^2\right| \leq \frac{\alpha' \cdot \alpha''^2}{2} \leq \sup_{t \neq 0} |\hat{1}_{A'}(t)| \cdot \alpha''$$
$$\implies \sup |\hat{1}_{A'}(t)| \geq \frac{\alpha' \cdot \alpha''}{2} \geq \frac{(\alpha')^2}{10}$$

provided that $\frac{\alpha''}{p} \leq \frac{\alpha'(\alpha'')^2}{2}$ which holds since (using $p \geq \frac{N}{3}$ and $N > 50\alpha^{-2}$)

$$\alpha'\alpha'' p \geq \alpha'\alpha''\frac{N}{3} > \frac{\alpha'}{\alpha}\frac{\alpha''}{\alpha} \cdot 50 \geq \left(\frac{\alpha'}{\alpha}\right)^2 \cdot 10 = \left(1 - \frac{\alpha}{200}\right)^2 \cdot 10 \geq \frac{1}{2},$$

where the last step holds for $\alpha = 1$ and hence for any $\alpha \leq 1$. $\square$

We first now convert the large Fourier coefficient into a density increment. 29 Nov 2024, Lecture 5

Lemma 1.22. Let $m \in \mathbb{N}$ and let $\phi : [m] \to \mathbb{Z}_p$ by $x \mapsto xt$ for some nonzero t. Given $\epsilon > 0$, there exists a partition of $[m]$ into progressions P_i of length $\in [\epsilon\sqrt{m}/2, \epsilon\sqrt{m}]$ such that $\operatorname{diam}(\phi(P_i)) = \max_{x,y \in P_i} |\phi(x) - \phi(y)| \leq \epsilon p \ \forall i$.

Proof. Set $u = \lfloor \sqrt{m} \rfloor$ and consider $0, t, 2t, \ldots, ut$. By pigeonhole, we can find $0 \leq v < w \leq u$ such that $|wt - vt| \leq \frac{p}{u}$. Divide $[m]$ into residue classes mod s, where $s = w - v$ (so $|st| \leq \frac{p}{u}$). Each of these has size at least $\frac{m}{s} \geq \frac{m}{u}$. But each residue class can be divided into progressions of the form $a, a+s, a+2s, a+ds$ with $\frac{\epsilon u}{2} < d \leq \epsilon u$. The diameter of the image of each progression under ϕ is $|dst| \leq \epsilon p$. $\square$

Lemma 1.23. Let $A \subset [N]$ be of density $\alpha > 0$. Let p be a prime in $\left[\frac{N}{3}, \frac{2N}{3}\right]$ and write $A' = A \cap [p]$ as a subset of $\mathbb{Z}_p$. Suppose $\exists t \neq 0$ such that $\left|\widehat{1'_A}(t)\right| \geq \frac{\alpha^2}{10}$.

Then there exists a progression P of length at least $\frac{\alpha^2\sqrt{N}}{500}$ such that $|A \cap P| \geq \alpha\left(1+\frac{\alpha}{80}\right)|P|$.

Proof. Let $\epsilon = \frac{\alpha^2}{40\pi}$ and use Lemma 1.22 to partition $[p]$ into progressions P_i of length at least $\frac{\epsilon\sqrt{p}}{2} \geq \frac{\alpha^2}{40\pi}\sqrt{\frac{N}{3}} \cdot \frac{1}{2} \geq \alpha^2\sqrt{N} \cdot \frac{1}{500}$ and $\operatorname{diam}(\phi(P_i)) \leq \epsilon p$. Fix one x_i from each P_i. Now work with the balanced function: since $t \neq 0$, the Fourier coefficient at t is the same for the indicator function and the balanced function.

$$\begin{aligned}
\frac{\alpha^2}{10} \leq \left|\widehat{f_{A'}}(t)\right| &= \frac{1}{p}\left|\sum_{x\in\mathbb{Z}_p} f_{A'}(x)\omega^{xt}\right| = \frac{1}{p}\left|\sum_i \sum_{x\in P_i} f_{A'}(x)\omega^{xt}\right| \\
&= \frac{1}{p}\left|\sum_i \sum_{x\in P_i} f_{A'}(x)\omega^{x_i t} + \sum_i \sum_{x\in P_i} f_{A'}(x)\left(\omega^{xt} - \omega^{x_i t}\right)\right| \\
&\leq \frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| + \frac{1}{p}\sum_i \sum_{x\in P_i} |f_{A'}(x)|\, 2\pi\epsilon \\
&\leq \frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| + \frac{\alpha^2}{20}
\end{aligned}$$

since $|t(x_i - x)| \leq \epsilon p\ \forall x \in P_i$. Hence

$$\frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| \geq \frac{\alpha^2}{20}.$$

Since $f_{A'}$ has mean zero,

$$\sum_i \left(\left|\sum_{x\in P_i} f_{A'}(x)\right| + \sum_{x\in P_i} f_{A'}(x)\right) \geq \frac{\alpha^2 p}{20},$$

so $\exists i$ such that $\left|\sum_{x\in P_i} f_{A'}(x)\right| + \sum_{x\in P_i} f_{A'}(x) \geq \frac{\alpha^2|P_i|}{40}$ and so

$$\sum_{x\in P_i} f_{A'}(x) \geq \frac{\alpha^2|P_i|}{80}.$$

□

This is about as technical as we get in this course.

Proof of Roth's Theorem, theorem 1.20. This is on Ex. Sheet 1. □

Example 1.24 (Behrend's example)**.** There exists a set $A \subset [N]$ containing no nontrivial 3–APs of size $|A| \geq C\exp\left(-c\sqrt{\log N}\right)N$, where c and C are absolute constants. This is again on Ex. Sheet 1.

Definition 1.25. Let $\Gamma \subset \widehat{G}$ and $\rho > 0$. By the **Bohr set**, written $B(\Gamma, \rho)$, we mean

$$B(\Gamma, \rho) = \{x \in G \mid |\gamma(x) - 1| \leq \rho \ \forall \gamma \in \Gamma\}.$$

We call $|\Gamma|$ the **rank** and ρ the **radius** of the Bohr set.

Example 1.26. When $G = \mathbb{F}_p^n$ and $p = 3$, we have $B(\Gamma, \rho) = \langle \Gamma \rangle^{\perp} \ \forall \rho < 1$ (draw a picture!). For larger p, the same holds for smaller ρ.

Lemma 1.27. Let $\Gamma \subset \widehat{G}$ be of size d and let $\rho > 0$. Then $|B(\Gamma, \rho)| \geq \left(\frac{\rho}{2\pi}\right)^d |G|$.

Proof. This is on Ex. Sheet 2. □

Lemma 1.28 (Bogolyubov's lemma, again)**.** Given $A \subset \mathbb{Z}_p$ of density $\alpha > 0$, $\exists \Gamma \subset \widehat{\mathbb{Z}_p}$ of size at most $2\alpha^{-2}$ such that $B\left(\Gamma, \frac{1}{2}\right) \subset A + A - A - A$.

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Proof. Recall $1_A * 1_A * 1_{-A} * 1_{-A}(x) = \sum_{t \in \widehat{Z_p}} \left|\widehat{1_A}(t)\right|^4 \omega^{-xt}$. Let $\Gamma = \mathrm{Spec}_{\sqrt{\frac{\alpha}{2}}}(1_A)$ and note that for all $x \in B\left(\Gamma, \frac{1}{2}\right)$ and $t \in \Gamma$, $\cos(2\pi xt/p) > 0$. Hence

$$\mathrm{Re}\left(\sum_{t \in \widehat{Z_p}} \left|\widehat{1_A}(t)\right|^4 \omega^{-xt}\right) = \underbrace{\sum_{t \in \Gamma} \left|\widehat{1_A}(t)\right|^4 \cos\left(2\pi xt/p\right)}_{\geq \alpha^4} + \underbrace{\sum_{t \notin \Gamma} \left|\widehat{1_A}(t)\right|^4 \cos(2\pi xt/p)}_{\text{in absolute value } \leq \sup_{t \notin \Gamma} \left|\widehat{1_A}(t)\right|^2 \sum \left|\widehat{1_A}(t)\right|^2 \leq \left(\sqrt{\frac{\alpha}{2}} \cdot \alpha\right)^2 \cdot \alpha = \frac{\alpha^4}{2}}.$$

□

2 Combinatorial methods

For now, let G be an abelian group. Given $A, B \subset G$. We defined $A + B = \{a + b \mid a \in A, b \in B\}$ and can define $A - B = \{a - b \mid a \in A, b \in B\}$. If A and B are finite, then

$$\max(|A|, |B|) \leq |A \pm B| \leq |A| |B|$$

(and better bounds are available in certain settings).

Example 2.1. Let $V \leq \mathbb{F}_p^n$ be a subspace. Then $V + V = V$, so $|V + V| = |V|$. In fact, if $A \subset \mathbb{F}_p^n$ is such that $|A + A| = |A|$, then A must be a coset of a subspace.

Example 2.2. Let $A \subset \mathbb{F}_p^n$ be such that $|A + A| < \frac{3}{2}|A|$. Then $\exists V \leq \mathbb{F}_p^n$ such that $A \subset V$ and $|V| < \frac{3}{2}|A|$. This is on Ex. Sheet 2.

Example 2.3. Let $A \subset \mathbb{F}_p^n$ be a set of linearly independent vectors. Then $A+A$ has size $\binom{|A|}{2}$. However, $|A| \leq n$, which is a small set.

Let $A \subset \mathbb{F}_p^n$ be a set chosen randomly with probability $p^{-\theta n}$ with $\theta \in \left(\frac{1}{2}, 1\right]$. Then with high probability, $|A+A| = (1-o(1))\frac{|A|^2}{2}$.

Definition 2.4. Given finite sets $A, B \subset G$, we define the **Rusza distance** $d(A,B)$ between A and B by

$$d(A,B) = \log \frac{|A-B|}{\sqrt{|A|\,|B|}}.$$

Observe that $d(A,B)$ is nonnegative and symmetric.

Lemma 2.5 (Rusza's triangle inequality)**.** Given finite sets A, B, C, we have

$$d(A,C) \leq d(A,B) + d(B,C).$$

Proof. Observe that $|B|\,|A-C| \leq |A-B|\,|B-C|$. Indeed, writing each $d \in A-C$ as $d = a_d - c_d$ for some $a_d \in A, c_d \in C$, the map

$$\begin{aligned} \phi : B \times (A-C) &\to (A-B) \times (B-C) \\ (b,d) &\mapsto (a_d - b) \times (b - c_d) \end{aligned}$$

is injective (easy exercise). The triangle inequality now follows from the definition of the Rusza distance. □

Definition 2.6. Given a finite set $A \subset G$, we write $\sigma(A) = \frac{|A+A|}{|A|}$ for the **doubling constant** and $\delta(A) = \frac{|A-A|}{|A|}$ for the **difference constant**.

Then by Lemma 2.5,

$$\log \delta(A) = d(A,A) \leq d(A,-A) + d(A,-A) = 2\log \sigma(A),$$

so $\delta(A) \leq \sigma(A)^2$, i.e. $|A-A| \leq \frac{|A+A|^2}{|A|}$.

Notation. Given $A \subset G$ and $l, m \in \mathbb{Z}_{\geq 0}$, write $lA - mA$ for the set

$$\underbrace{A + A + \ldots + A}_{l \text{ times}} - \underbrace{A - A - \ldots - A}_{m \text{ times}}.$$

Theorem 2.7 (Plünnecke's inequality)**.** Let $A, B \subset G$ be finite sets such that $|A+B| \leq K\,|A|$ for some $K > 0$. Then for any $l, m \in \mathbb{Z}_{\geq 0}$,

$$|lB - mB| \leq K^{l+m}\,|A|.$$

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Proof. WLOG assume that $|A+B| = K|A|$. Choose a nonempty subset $A' \subset A$ such that the ratio $\frac{|A'+B|}{|A'|}$ is minimized, and call this ratio K'. Then $|A'+B| = K'|A'|$, $K' \leq K$ and $|A''+B| \geq K'|A''|$ $\forall A'' \subset A$.

Claim. For any finite $C \subset G$, $|A'+B+C| \leq K'|A'+C|$.

We first finish the proof assuming this claim, and then prove it. We first show that $|A'+mB| \leq (K')^m|A|$ $\forall m \in \mathbb{Z}_{\geq 0}$. The cases $m=0$ and $m=1$ are clear. Now suppose that $m>1$ and the result holds for $m-1$. By the claim with $C=(m-1)B$,

$$|A'+mB| = |A'+B+(m-1)B| \leq K'|A'+(m-1)B| \leq K' \cdot (K')^{m-1}|A'|.$$

But as in the proof of Rusza's triangle inequality,

$$\begin{aligned} &|A'|\,|lB-mB| \leq |A'+lB|\,|A'+mB| \leq (K')^l|A'|\,(K')^m|A'| \\ \implies &|lB-mB| \leq (K')^{l+m}|A'| \leq K^{l+m}|A|. \end{aligned}$$

Finally, we prove the claim by induction on $|C|$. For $|C|=1$, we are just translating sets, so the claim holds. Now suppose the claim holds for some $|C|$ and consider $C' = C \cup \{x\}$ for some $x \notin C$. Observe

$$A'+B+C' = (A'+B+C) \cup (A'+B+x)$$

and in fact

$$A'+B+C' = (A'+B+C) \cup (A'+B+x) \setminus (D+B+x)$$

where $D = \{a \in A' \mid A'+B+x \subset A'+B+C\}$. By the definition of K, $|D+B| \geq K'|D|$, so

$$\begin{aligned} |A'+B+C'| &\leq |A'+B+C| + |(A'+B+x) \setminus (D+B+x)| \\ &\leq |A'+B+C| + |A'+B| - |D+B| \\ &\leq K'|A'+C| + K'|A'| - K'|D| \\ &= K'(|A'+C| + |A'| - |D|). \end{aligned}$$

Now apply the same argument again for $A'+C' = (A'+C) \sqcup ((A'+x) \setminus (E+x))$, where $E = \{a \in A' \mid a+x \in A'+C\} \subset D$. Notice that the union is disjoint in this case. We conclude that

$$\begin{aligned} &|A'+C'| = |A'+C| + |A'| - |E| \geq |A'+C| + |A'| - |D| \\ \implies &|A'+B+C'| \leq K'(|A'+C| + |A'| - |D|) \leq K'|A'+C'|, \end{aligned}$$

proving the claim and hence the proof. $\square$

We are now in a position to generalize Example 2.2.

Theorem 2.8 (Freiman–Rusza theorem)**.** Let $A \subset \mathbb{F}_p^n$ be such that $|A+A| \leq K|A|$ (i.e. $\sigma(A) = K$) for some $K > 0$. Then A is contained in a coset of a subspace $H \leq \mathbb{F}_p^n$ of size $|H| \leq K^2 p^{K^4} |A|$.

Proof. Choose maximal $X \subset 2A - A$ such that the translates $x + A$ for $x \in X$ are disjoint. X cannot be too large: $\forall x \in X, x + A \subset 3A - A$ and by Plünnecke, $|3A - A| \leq K^4 |A|$. But the translates $x + A$ for $x \in X$ are isjoint and each of size $|A|$, so

$$|X|\,|A| = \left|\bigcup_{x \in X} (x + A)\right| \leq |3A - A| \leq K^4 |A|,$$

hence $|X| \leq K^4$. We next show that $2A - A \overset{(\star)}{\subset} X + A - A$. Indeed, if $y \in 2A - A$ and $y \notin X$, then $y + A \cap (x + A) \neq \varnothing$ for some $x \in X$ by maximality of X, so $y \in X + A - A$. If $y \in X$, then trivially $y \in X + A - A$. It follows by induction from $(\star)$ that for all $l \geq 2$,

$$lA - A \overset{(\star\star)}{\subset} (l-1)X + A - A,$$

since using the induction hypothesis,

$$\begin{aligned} lA - A = A + (l-1)A - A &\overset{\text{hyp}}{\subset} A + (l-2)X + A - A \\ &= (l-2)X + 2A - A \overset{(\star)}{\subset} (l-2)X + X + (A - A) = (l-1)X + A - A. \end{aligned}$$

Now let H be the subgroup of $\mathbb{F}_p^n$ generated by A, which we can write in the form $H = \cup_{l \geq 1}(lA - A) \overset{(\star\star)}{\subset} Y + A - A$, where Y is the subgroup generated by X. Then $|Y| \leq p^{|X|} \leq p^{K^4}$, so

$$|H| \leq |Y + A - A|\,|Y|\,|A - A| \leq p^{K^4} K^2 |A|.$$

$\square$

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Example 2.9. This example shows that we need a constant that is exponential in K in the previous result. Let $A = H \cup R \subset \mathbb{F}_p^n$ where $H \leq \mathbb{F}_p^n$ is a subspace of dimension $K \lll d \lll n - K$, and R consists of $K - 1$ linearly independent vectors in $H^\perp$. Then $|A| = |H \cup R| \approx |H|$ and

$$|A + A| = |(H \cup R) + (H \cup R)| = |(H + H) \cup (H + R) \cup (R + R)| \approx K|H| \approx K|A|$$

since $H + H = H$ and $H + R$ gives us $K - 1$ cosets of H, while $R + R$ has tiny size.

However, a subspace $V \leq \mathbb{F}_p^n$ containing A must have size $\geq p^{d+(K-1)} = |H| \cdot p^{K-1} \approx |A| \cdot p^{K-1}$, where the constant is exponential in K.

Conjecture 2.10 (Polynomial Freiman–Rusza)**.** Let $A \subset \mathbb{F}_p^n$ be such that $|A+A| \leq K\,|A|$. Then there is a subspace $H \leq \mathbb{F}_p^n$ of size at most $C_1(K)\,|A|$ such that for some $x \in \mathbb{F}_p^n$,

$$|A \cap (x+H)| \geq \frac{|A|}{C_2(K)}$$

where $C_1(K)$ and $C_2(K)$ are polynomials in K. For $p=2$, this is now a theorem since November 2023 (by Gowers, Green, Manning, Tao).

Definition 2.11. Given an abelian group G and finite sets $A, B \subset G$, define the **additive energy** between A and B to be

$$E(A,B) = \frac{\#\{(a,a',b,b') \in A \times A \times B \times B \mid a+b = a'+b'\}}{|A|^{3/2}\,|B|^{3/2}}.$$

We refer to quadruples $(a,a',b,b') \in A^2 \times B^2$ such that $a+b=a'+b'$ as **additive quadruples**.

Observe that if G is finite and abelian, then

$$\left|A^3\right| E(A,A) = |G|^3\,\mathbb{E}_{x+y=z+w} 1_A(x)1_A(y)1_A(z)1_A(w) \overset{(\star)}{=} |G|^3\,||\widehat{1_A}||_4^4$$

where $(\star)$ follows from Ex. Sheet 1, Q3.

Example 2.12. When $H \leq \mathbb{F}_p^n$, then $E(V,V)=1$, i.e. the additive energy achieves its maximum. Exercise on Ex. Sheet 2: think of an example where the additive energy is small.

Lemma 2.13. Let G be abelian and let $A, B \subset G$ be finite. Then

$$E(A,B) \geq \frac{\sqrt{|A|\,|B|}}{|A+B|}.$$

Proof. Note that for some x in G,

$$|A|^{3/2}\,|B|^{3/2}\,E(A,B) = \#\{(a,a',b,b') \in A \times A \times B \times B \mid a+b=a'+b'\} = x = \sum_{x \in G} r_{A+B}(x)^2,$$

where $r_{A+B}(x) = \#$ways of writing $x=a+b$ with $a \in A, b \in B$. Observe that

$$\sum_{x \in G} r_{A+B}(x) = |A|\,|B|\,,$$

so

$$|A|^{3/2} |B|^{3/2} E(A,B) = \sum_{x \in G} r_{A+B}(x)^2 \geq \frac{\left(\sum_{x \in G} r_{A+B}(x)\right)^2}{\sum_{x \in G} 1_{A+B}(x)^2} = \frac{(|A|\,|B|)^2}{|A+B|}$$

using Cauchy–Schwarz and the fact that we're only summing over $x \in G$ that are in $A+B$. $\square$

In particular, if $A \subset G$ such that $|A+A| \leq K|A|$, then $E(A) \geq \frac{1}{K}$. The converse is not true.

Remark. The same proof goes through for $A - B$ instead of $A + B$.

Example 2.14. Let G be our favorite abelian group (really our favorite class of abelian groups, e.g. $\mathbb{Z}_p$ for p running over primes). Then there exist constants $\eta, \theta > 0$ such that for all sufficiently large n, there exists $A \subset G$ with $|A| = n$ satisfying $E(A,A) \geq \eta$ and $|A+A| \geq \theta |A|^2$. This is on Ex. Sheet 2.

Theorem 2.15 (Balog–Szemeredi–Gowers)**.** Let G be an abelian group and let $A \subset G$ be finite such that $E(A,A) \geq \eta$ for some $\eta > 0$. Then $\exists A' \subset A$ of size at least $c(\eta)|A|$ such that

$$|A' + A'| \leq C(\eta)\,|A|\,.$$

Furthermore, here $c(\eta)$ and $C(\eta)$ are polynomials in η.[3]

We first prove a technical lemma using a method called "dependent random choice".

Lemma 2.16. Let $A_1, A_2, \ldots, A_m \subset [n]$ and suppose $\sum_{i,j \in [m]} |A_i \cap A_j| \geq \delta^2 n m^2$. Then there exists $X \subset [m]$ of size at least $\frac{\delta^5 m}{\sqrt{2}}$ such that $|A_i \cap A_j| \geq \frac{\delta^2 n}{2}$ for at least 90% of the pairs $(i,j) \in X^2$.

Proof. First choose x_1, x_2, x_3, x_4, x_5 at random from $[n]$, and then define the set $X = \{i \in [m] \mid x_j \in A_i \ \forall j \in [5]\}$. Observe that if $|A_i \cap A_j| = \gamma n$, then $\mathbb{P}\left((i,j) \in X^2\right) = \gamma^5$, and hence (by convexity or Hölder)

$$\mathbb{E}\,|X|^2 = \sum_{i,j} \mathbb{P}\left((i,j) \in X^2\right) \geq \delta^{10} m^2.$$

Call a pair (i,j) "bad" if $|A_i \cap A_j| < \frac{\delta^2 n}{2}$. As before,

$$\mathbb{E}(\#\text{bad pairs in } X^2) \leq \frac{\delta^{10}}{2^5} m^2.$$

[3]TODO: see beginning of lec 9 - shoudl it be $C(\eta)\,|A'|$ in the above?

Hence $\mathbb{E}\left(\left|X^2\right| - 16 \cdot \#\text{bad pairs in } X^2\right) = \frac{\delta^{10}}{2^5}m^2$,[4] so there must be a choice of $x_1, x_2, \ldots, x_5$ such that $|X| \geq \frac{\delta^5 m}{\sqrt{2}}$ and the proportion of bad pairs in X is at most $\frac{1}{16} < 10\%$. □

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Proof of Theorem 2.15. We call a difference d "popular" if d can be written as $d = x - y$ with $x, y \in A$ in at least $\eta |A|/2$ ways, i.e. $r_{A-A}(d) \geq \eta |A|/2$. There must be at least $\eta |A| 2$ popular differences, for if not, we get a contradiction through

$$\begin{aligned}\sum_d r_{A-A}(d)^2 &= \sum_{d \text{ popular}} r_{A-A}(d)^2 + \sum_{d \text{ not popular}} r_{A-A}(d)^2 \\ &< \eta\frac{|A|}{2}|A|^2 + \eta\frac{|A|}{2}\sum_d r_{A-A}(d) \\ &\leq \eta\frac{|A|}{2}|A|^2 + \eta\frac{|A|}{2}|A|^2.\end{aligned}$$

Define a graph with vertex set A, joining x and y by an edge if $y - x$ is a popular difference. Then

$$\mathbb{E}_{x\in A}|N(x)| = \frac{1}{|A|}\sum_{x\in A}|N(x)| \geq \frac{\eta|A|}{2}.$$

We also have $\mathbb{E}_{x,y\in A}|N(x)\cap N(y)| \geq \frac{\eta^2|A|}{4}$. Indeed, by Cauchy–Schwarz,

$$\mathbb{E}_{x,y\in A}|N(x)\cap N(y)| = \mathbb{E}_{x,y\in A}\sum_{z\in A}1_{N(x)}(z)1_{N(y)}(z) = \sum_{z\in A}\left(\mathbb{E}_{x\in A}1_{N(x)}(z)\right)^2$$
$$\geq \frac{1}{|A|}\left(\sum_{z\in A}\mathbb{E}_{x\in A}1_{N(x)}(z)\right)^2 = \frac{1}{|A|}\left(\mathbb{E}_{x\in A}|N(x)|\right)^2 \geq \frac{1}{|A|}\left(\frac{\eta|A|}{2}\right)^2 = \frac{\eta^2|A|}{4}.$$

We apply Lemma 2.16 with $m = n = |A|$ and $\delta^2 = \frac{\eta^2}{4}$ to find a subset $A' \subset A$ of size $\geq \eta^{10}\frac{|A|}{2^{11}}$ with the property that $|N(x)\cap N(y)| \geq \frac{\eta^2|A|}{8}$ for at least 90% of $(x,y) \in A'^2$. But then for at least 10% of $x \in A'$, $|N(x)\cap N(y)| \geq \frac{\eta^2|A|}{8}$ for at least 80% of $y \in A'$. Hence $\exists A'' \subset A'$ of size $\geq \frac{\eta^{10}|A|}{2^{15}}$ such that $\forall x \in A''$, at least 80% of $z \in A'$ satisfy $|N(x)\cap N(z)| \geq \frac{\eta^2|A|}{8}$. In particular, if $x, y \in A''$, then there are at least $\frac{\eta^{10}|A|}{2^{12}}$ values of $z \in A'$ such that $|N(x)\cap N(z)| \geq \frac{\eta^2|A|}{8}$ and $|N(y)\cap N(z)| \geq \frac{eta^2|A|}{8}$.

[We shall prove an upper bound of $|A''-A''|$ by showing that each element of $A''-A''$ can be written as a linear combination of distinct octuples from A.]

[4] TODO: This 2^5 should just be 2, right?

For each such z, there are thus $\geq \left(\frac{\eta^2|A|}{8}\right)^2$ pairs (u,v) such that $u \in N(x) \cap N(y)$ and $v \in N(y) \cap N(z)$. For each such pair (u,v), the elements $u-x, z-u, v-z, y-v$ are all popular differences. Hence, for each pair (u,v), there are at least $\left(\frac{\eta|A|}{2}\right)^4$ octuples $(a_1, a_2, \ldots, a_8) \in A^8$ such that

$$u - x = a_2 - a_1,\ z - u = a_4 - a_3,\ v - z = a_6 - a_5,\ y - v = a_8 - a_7.$$

In other words, there are at least

$$\underbrace{\left(\frac{\eta^{10}|A|}{2^{12}}\right)}_{z}\underbrace{\left(\frac{\eta^2|A|}{8}\right)^2}_{u,v}\underbrace{\left(\frac{\eta|A|}{2}\right)^4}_{(a_1,\ldots,a_8)} = \frac{\eta^{18}}{2^{22}}|A|^7$$

octuples $(a_1, \ldots, a_8) \in A^8$ such that

$$\begin{aligned} y - x &= (u-x) + (z-u) + (v-z) + (y-v) \\ &= a_2 - a_1 + a_4 - a_3 + a_6 - a_5 + a_8 - a_7. \end{aligned}$$

But distinct $y - x$ give rise to distinct octuples, so

$$\begin{aligned} &\frac{\eta^{18}}{2^{12}}|A|^7 \cdot |A'' - A''| \leq |A|^8 \\ \implies &|A'' - A''| \leq 2^{12}\eta^{-18}|A| \leq 2^{27}\eta^{-28}|A''| \end{aligned}$$

(and $|A'' + A''|$ follows from Plünnecke). $\square$

3 Probabilistic tools

Remark. Assume in this chapter that all our probability spaces are finite, so we don't need to worry about convergence issues.

Proposition 3.1 (Khintchine's inequality)**.** Let $X_1, X_2, \ldots, X_n$ be independent random variables taking values $\pm x_i$ with probability $\frac{1}{2}$ $\forall i = 1, \ldots, n$. Then $\forall p \in [2, \infty)$,

$$||\sum_{i=1}^n X_i||_{L^p(\mathbb{P})} = O\left(p^{1/2}\left(\sum_{i=1}^n ||X_i||^2_{L^2(\mathbb{P})}\right)^{1/2}\right)$$

Proof. By nesting of norms, it suffices to prove the case $p = 2k$ with $k \in \mathbb{N}$. For simplicity, write $X = \sum_{i=1}^n X_i$ and WLOG assume that $\sum_{i=1}^n ||X_i||^2_\infty =$ 09 Feb 2024, Lecture 10

$\sum_{i=1}^{n} ||X_i||_2^2 = 1$. By Chernoff (Example 1.3), which states that $\forall \theta > 0$,

$$\mathbb{P}\left(|X \geq \theta|\right) \leq 4\exp(-\theta^2/4),$$

we have (using integration by parts, this is the alternative something formula, rewatch lecture to find out the name)

$$||X||_{2k}^{2k} = \int_0^\infty 2kt^{2k-1}\mathbb{P}\left(|X| \geq t\right) \mathrm{d}t \leq 8k \underbrace{\int_0^\infty t^{2k-1}\exp(-t^2/4)\mathrm{d}t}_{:=I(k)}.$$

We shall prove by induction that $I(k) \leq C^{2k}(2k)^k/4k$ for some constant $C > 0$. For $k = 1$,

$$\int_0^\infty t\exp\left(-t^2/4\right) \mathrm{d}t = \left[-2\exp\left(-t^2/4\right)\right]_0^\infty = 2 \leq C^2\frac{2}{4}$$

for $C \geq 2$. For $k > 1$, we have

$$\begin{aligned}
I(k) &= \int_0^\infty t^{2k-2} \cdot t\exp\left(-t^2/4\right) \mathrm{d}t \\
&= \left[t^{2k-2}(-2)\exp\left(-t^2/4\right)\right]_0^\infty - \int_0^\infty (2k-2)t^{2k-3}(-2)\exp\left(-t^2/4\right) \mathrm{d}t \\
&= 4(k-1)\int_0^\infty t^{2(k-1)-1}\exp(-t^2/4)\mathrm{d}t \\
&= 4(k-1)I(k-1) \\
&\leq 4(k-1)C^{2(k-1)}\frac{(2(k-1))^{k-1}}{4(k-1)} \\
&\leq C^{2k}\frac{(2k)^k}{4k}
\end{aligned}$$

for some C, where $C \geq \sqrt{2}$ is claimed to work. $\square$

Corollary 3.2 (Rudin's inequality). Let $\Lambda \subset \widehat{\mathbb{F}_2^n}$ be a linearly independent set and let $p \in [2, \infty)$. Then $\forall \widehat{f} \in \ell^2(\Lambda)$, i.e. $\widehat{f} : \Lambda \to \mathbb{C}$,

$$||\sum_{\gamma\in\Lambda} \widehat{f}(\gamma)\gamma||_{L^p(\mathbb{F}_2^n)} = O\left(\sqrt{p}||\widehat{f}||_{\ell^2(\Lambda)}\right)$$

Remark. Note that here the LHS uses L^p for the normalized counting measure (i.e. $\mathbb{E}$), while the RHS uses ℓ^2 for the counting measure (i.e. $\sum$). In other words, these are the same, except one is normalized.

Corollary 3.3 (Dual form of Rudin's inequality). Let $\Lambda \subset \widehat{\mathbb{F}_2^n}$ be linearly independent and let $p \in (1,2]$. Then $\forall f \in L^p(\mathbb{F}_2^n)$,

$$||\widehat{f}||_{\ell^2(\Lambda)} = O\left(\sqrt{\frac{p}{p-1}}||f||_{L^p(\mathbb{F}_2^n)}\right).$$

Proof. Let $f \in L^p(\mathbb{F}_2^n)$ and write $g = \sum_{\gamma\in\Lambda} \widehat{f}(\gamma)\gamma$. Then, as g has the same Fourier coefficients as f,

$$||\widehat{f}||_{\ell^2(\Lambda)}^2 = \sum_{\gamma\in\Lambda} \left|\widehat{f}(\gamma)\right|^2 = \sum_{\gamma\in\Lambda} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} = \langle \widehat{f}, \widehat{g}\rangle_{\ell^2(\mathbb{F}_2^n)} = \langle f, g\rangle_{L^2(\mathbb{F}_2^n)},$$

but by Hölder, $\langle f, g\rangle_{L^2(\mathbb{F}_2^n)} \leq ||f||_{L^p(\mathbb{F}_2^n)}||g||_{L^{p'}(\mathbb{F}_2^n)}$, where $\frac{1}{p}+\frac{1}{p'} = 1$. By Rudin's inequality for $p' = \frac{p}{p-1}$, we get

$$||g||_{L^{p'}(\mathbb{F}_2^n)} = O\left(\sqrt{p'}||\widehat{g}||_{\ell^2(\Lambda)}\right) = O\left(\sqrt{\frac{p}{p-1}}||\widehat{f}||_{\ell^2(\Lambda)}\right),$$

so

$$\begin{aligned} ||\widehat{f}||_{\ell^2(\Lambda)}^2 &= ||f||_{L^p(\mathbb{F}_2^n)} O\left(\sqrt{\frac{p}{p-1}}||\widehat{f}||_{\ell^2(\Lambda)}\right) \\ \implies ||\widehat{f}||_{\ell^2(\Lambda)} &= O\left(\sqrt{\frac{p}{p-1}}||f||_{L^p(\mathbb{F}_2^n)}\right). \end{aligned}$$

□

Recall that given $A \subset \mathbb{F}_2^n$ of density $\alpha > 0$, $\left|\mathrm{Spec}_\rho(1_A)\right| \leq \rho^{-2}\alpha^{-1}$. This is the best possible, as the example of a subspace $H \leq \mathbb{F}_2^n$ shows $\mathrm{Spec}_1(1_H) = H^\perp$, so $|\mathrm{Spec}_1(1_H)| = \left|H^\perp\right| = \frac{|\mathbb{F}_2^n|}{|H|} = \left(\frac{|H|}{|\mathbb{F}_2^n|}\right)^{-1} = \alpha^{-1}$.

Theorem 3.4 (Special case of Chen's theorem)**.** LEt $A \subset \mathbb{F}_2^n$ with density $\alpha > 0$. Then $\forall \rho > 0$, there exists a subspace $H \leq \mathbb{F}_2^n$ of dimension at most $O\left(\rho^{-2}\log\alpha^{-1}\right)$ such that $\mathrm{Spec}_\rho(1_A) \subset H$.

Proof. Let $\Lambda \subset \mathrm{Spec}_\rho(1_A)$ be a maximal linearly independent subset of $\mathrm{Spec}_\rho(1_A)$ and let $H = \langle \mathrm{Spec}_\rho(1_A)\rangle$. Then $\dim(H) = |\Lambda|$. By dual Rudin (Corollary 3.3), $\forall p \in (1,2]$,

$$(\rho\alpha)^2|\Lambda| \leq \sum_{\gamma\in\Lambda}\left|\widehat{1_A}(\gamma)\right|^2 = ||\widehat{1_A}||_{\ell^2(\Lambda)}^2 = O\left(\frac{p}{p-1}||1_A||_{L^p(\mathbb{F}_2^n)}^2\right).$$

We can explicitly compute

$$||1_A||_{L^p(\mathbb{F}_2^n)}^2 = \left(\mathbb{E}_y|1_A(y)|^p\right)^{2/p} = \alpha^{2/p}.$$

Thus $|\Lambda| \leq \rho^{-2}\alpha^{-2}O\left(\frac{p}{p-1}\alpha^{2/p}\right)$. We want to choose p very close to 1, so choose

$p = 1 + \left(\log \alpha^{-1}\right)^{-1}$ to conclude that

$$|\Lambda| \leq O\left(\rho^{-2} \log \alpha^{-1}\right)$$

(calculation details omitted). □

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Theorem 3.5 (Chang's Theorem)**.** Let G be a finite abelian group and let $A \subset G$ have density $\alpha > 0$. If $\Lambda \subset \operatorname{Spec}_\rho(1_A)$ is dissociated, then $|\Lambda| = O\left(\rho^{-2} \log \alpha^{-1}\right)$.

Remark. Last lecture, we wrote $f \in L^p(G)$ to mean that f is a function on G with bounded L^p–norm and then said $||f||_{L^p(G)} = \left(\mathbb{E}_{x \in G} f(x)^p\right)^{1/p}$. Since we assumed that our groups are finite, the condition "with bounded L^p–norm" is unnecessary here, but we keep it as it is in line with the usual notation. We also said that $\widehat{f} \in \ell^2(\Lambda)$ if $\widehat{f}$ is a function supported on $\Lambda \subset \widehat{G}$ with bounded ℓ^2–norm: $||\widehat{f}||_{\ell^2(\Lambda)} = \left(\sum_{\gamma \in \Lambda} \left|\widehat{f}(\gamma)\right|^2\right)^{1/2}$. Finally, $X \in L^p(\mathbb{P})$ means that the random variable X has bounded p^{th} moment, i.e. $\mathbb{E}\,|X|^p < \infty$ (with expectation taken with respect to $\mathbb{P}$).

Remark. The proofs of these probabilistic inequalities are nonexaminable. However, we are expected to be able to state them and apply them.

We may boostrap Khintchine's inequality to obtain the following:

Theorem 3.6 (Marcinkiewicz–Zygmund Inequality)**.** Let $p \in [2, \infty)$ and let $X_1, X_2, \ldots, X_n \in L^P(\mathbb{P})$ be independent random variables with $\mathbb{E} \sum_{i=1}^n X_i = 0$. Then

$$|| \sum_{i=1}^n X_i ||_{L^p(\mathbb{P})} = O\left(p^{1/2} || \sum_{i=1}^n |X_i|^2 ||_{L^{p/2}(\mathbb{P})}^{1/2}\right).$$

Proof. For $\mathbb{C}$–valued random variables, the result follows from the real caes by taking real and imaginary parts and applying the triangle inequality.

Next assume that the distribution of the X_i's is symmetric, i.e. $\mathbb{P}(X_i = a) = \mathbb{P}(X_i = -a)$ $\forall a \in \mathbb{R}$. Partition the probability space Ω into sets $\Omega_1, \Omega_2, \ldots, \Omega_M$, writing $\mathbb{P}_j$ for the induced probability measure on Ω_j, such that all X_i's are symmetric and take at most two values on each Ω_j. Applying Khintchine, for each $j \in [M]$,

$$|| \sum_{i=1}^n X_i ||_{L^p(\mathbb{P}_j)}^p = O(p^{p/2} \underbrace{\left(\sum_{i=1}^n ||X_i||_{L^2(\mathbb{P}_j)}^2\right)^{p/2}}_{= || \sum_{i=1}^n |X_i|^2 ||_{L^{p/2}(\mathbb{P}_j)}^{p/2}})$$

so summing over all $j \in [M]$ and taking the p^{th} roots gives the symmetric case.

Now suppose the X_i's are arbitrary and let $Y_1, \ldots, Y_n$ be such that $X_i \sim Y_i \; \forall i$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are independent. Applying the symmetric result to $X_i - Y_i$,

$$\begin{aligned}||\sum_{i=1}^{n}(X_i - Y_i)||_{L^p(\mathbb{P}\times\mathbb{P})} &= O\left(p^{1/2}||\sum_{i=1}^{n}|X_i - Y_i|^2\,||^{1/2}_{L^{p/2}(\mathbb{P}\times\mathbb{P})}\right) \\ &= O\left(p^{1/2}||\sum_{i=1}^{n}|X_i|^2\,||^{1/2}_{L^{p/2}(\mathbb{P})}\right)\end{aligned}$$

by expanding $|X_i - Y_i|^2$ and bounding above by $4\,|X_i|^2$. But also

$$\begin{aligned}||\sum_{i=1}^{n}X_i||_{L^p(\mathbb{P})} &= ||\sum_{i=1}^{n}X_i - \mathbb{E}\sum_{i=1}^{n}Y_i||_{L^p(\mathbb{P})} \\ &\leq ||\sum_{i=1}^{n}(X_i - Y_i)||_{L^p(\mathbb{P}\times\mathbb{P})}\end{aligned}$$

by convexity/Jensen. □

Theorem 3.7 (Croot–Sisask Almost Periodicity)**.** Let G be a finite abelian group, let $\epsilon > 0$ and let $p \in [2, \infty)$. Let $A, B \subset G$ be such that $|A + B| \leq K\,|A|$ and let $f : G \to \mathbb{C}$. Then $\exists b \in B$ and a set $X \subset B - b$ such that

$$|X| \geq (2K)^{-O(\epsilon^{-2}p)}\,|B|$$

and

$$||\tau_x(f * \mu_A) - f * \mu_A||_{L^p(G)} \leq \epsilon||f||_{L^p(G)} \;\forall x \in X,$$

where $\tau_x g(y) = g(y + x)$ and μ_A is the characteristic measure of A, defined by $\mu_A(x) = \alpha^{-1}1_A(x)$.

Remark. We only need G to be discrete for the result to hold, but we consider the case ”finite and abelian” as we don't want to introduce too much notation in the proof.

Remark. For intuition, work through the example $f = 1_{A-A}$.

Proof. The main idea is to approximate $f * \mu_A(y) = \mathbb{E}_x\mu_A(x)f(y - x) = \mathbb{E}_{x\in A}f(y - x)$ by $\frac{1}{k}\sum_{i=1}^{k} f(y - z_i)$ with z_i samped independently at random from A for some suitable choice k.

For each $y \in G$, define $Z_i(y) = \tau_{-z_i}(f)(y) - f * \mu_A(y)$ for $i \in [k]$. For fixed $y \in G$, these are independent and have mean 0, so by Marcinkiewicz–Zygmond, for each $y \in G$,

$$
\begin{aligned}
|| \sum_{i=1}^{k} Z_i(y) ||_{L^p(\mathbb{P})}^p &= O\left(p^{p/2} || \sum_{i=1}^{k} |Z_i(y)|^2 \, ||_{L^{p/2}(\mathbb{P})}^{p/2} \right) \\
&= O\left(p^{p/2} \mathbb{E} \left| \sum_{i=1}^{n} |Z_i(y)|^2 \right|^{p/2} \right)
\end{aligned}
$$

TODO: finish this expression. So for each $y \in G$, $||Z_i(y)||_{L^p(\mathbb{P})}^p = O\left(p^{p/2} k^{\frac{p}{2}-1} \sum_{i=1}^{k} |Z_i(y)|^p \right)$.

$\square$