Part III - Modular Forms Lectured by Jack Thorne

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Contents

1	Introduction	2
2	Modular Forms on $\Gamma(1)$	5

1 Introduction

06 Oct 2022, Lecture 1

Definition 1.1. We define the following groups:

$$\mathfrak{h} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0 \}$$

$$GL_2(\mathbb{R})^+ = \{ g \in GL_2(\mathbb{R}) \mid \det(g) > 0 \}$$

$$\Gamma(1) = SL_2(\mathbb{Z}) = \{ g \in M_2(\mathbb{Z}) \mid \det(g) = 1 \}.$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on \mathfrak{h} by Möbius transformations.

Proof. Let
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{h}$$
. Then

$$\operatorname{Im}(g\tau) = \frac{1}{2i} \left(\frac{a\tau + b}{c\tau + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(\tau - \overline{\tau})}{|c\tau + d|^2} = \frac{\det(g)\operatorname{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{h}$. This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in \mathfrak{h} is conjugate to i.

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, then define

$$j(q,\tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{h} \to \mathbb{C}$, then

$$f|_k[g]:\mathfrak{h}\to\mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g,\tau)^{-k}.$$

This is the weight k action of g on f.

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_{k}[gh] = (f|_{k}[g])|_{k}[h].$$

Proof. We compute

$$(f|_{k}[g])|_{k}[h](\tau) = \det(h)^{k-1}f|_{k}[g](h\tau)j(h,\tau)^{-k} = \det(h)^{k-1}\det(g)^{k-1}f(gh\tau)j(g,h\tau)^{-k}j(h,\tau)^{-k} \stackrel{?}{=} \det(gh)^{k-1}f(gh\tau)j(gh,\tau)^{-k} = f|_{k}[gh](\tau).$$

Hence we need to check that $j(gh,\tau)=j(gh,\tau)j(h,\tau)$. Note that if $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh,\tau)\begin{pmatrix}gh\tau\\1\end{pmatrix}=gh\begin{pmatrix}\tau\\1\end{pmatrix}=g\left(j(h,\tau)\begin{pmatrix}h\tau\\1\end{pmatrix}\right)=j(h,\tau)j(g,h\tau)\begin{pmatrix}gh\tau\\1\end{pmatrix},$$

which finishes the computation and proof.

Formulae. For $g \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{h}$, we have

$$\operatorname{Im}(g\tau) = \det(g) \frac{\operatorname{Im}(\tau)}{|j(g,\tau)|^2} \text{ and } j(g,\tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\gamma \leq \Gamma(1)$ of finite index¹. A weakly modular function of weight k and level Γ is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ which is invariant under the weight k action of Γ , i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in \mathfrak{h} and at ∞ .

It is a fact that modular forms of fixed weight and level live in finitedimensional \mathbb{C} -vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

(1) They are related to the theory of elliptic functions. Let E/\mathbb{C} be an elliptic curve and ω a holomorphic non–zero 1–form. Then there exists a unique lattice² $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \to E$ such that $\phi^*(\omega) = dz$. Then

¹In other words, γ is a (finite index) subgroup of $\Gamma(1)$.

²i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over \mathbb{R} .

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for k > 2. If $\tau \in \mathfrak{h}$, then $\Lambda \tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

 $\mathfrak{h}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over \mathbb{C} .

- (2) Modular forms f have Fourier expansions $\sum_{n\in\mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n .
 - For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q-expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n.4\nmid d} d$.
- (3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:
 - The Euler product $\zeta(s) = \prod_{p} (1 p^{-s})^{-1}$.
 - It has a meromorphic continuation to $\mathbb C$ and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L-function. Modular forms can be used to construct interesting examples of L-functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

(4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves E/\mathbb{C} up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L-functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

09 Oct 2022,

Lecture 2

2 Modular Forms on $\Gamma(1)$

Reminder. A **meromorphic** function in an open subset $U \subset \mathbb{C}$ is a closed subset $A \subset U$ and a holomorphic function $f: U \setminus A \to \mathbb{C}$ such that $\forall a \in A$, $\exists \delta > 0$ such that $D^*(a, \delta) \subset U \setminus A$ and $\exists n \geq 0$ such that $(z - a)^n f(z)$ extends to a holomorphic function in $D(a, \delta)$.

f then has a Laurent expansion $\sum_{m\in\mathbb{Z}} a_m(z-a)^m$ valid on $D^*(a,\delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a meromorphic function \tilde{f} in $D^*(0,1)$ (the "q-disk") such that

$$f(\tau) = \tilde{f}(e^{2\pi i \tau}).$$

Proof. f is meromorphic in \mathfrak{h} by assumption. Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$, as f is invariant under the weight k action of γ . But also $f(\gamma\tau) = f(\tau+1)$, so f is periodic.

Now map a strip of \mathfrak{h} of width 1 to $D^*(0,1)$ by $\tau \mapsto e^{2\pi i \tau}$. Let $a \in D^*(0,1)$ and $\delta > 0$ be such that $D(a,\delta) \subset D^*(0,1)$. Define \tilde{f} on $D(a,\delta)$ by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i}\log q\right),$$

for any branch of log defined in $D(a, \delta)$. This is meromorphic and independent of the choice of the branch of log, as f is periodic with period 1. This defines \tilde{f} in $D^*(0, 1)$. Finally, \tilde{f} is unique since $\tau \mapsto e^{2\pi i \tau}$ is surjective.

If \tilde{f} extends to a meromorphic function³ in D(0,1), then $\exists \delta > 0$ such that \tilde{f} has a Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0,\delta)$.

In the region $\{\tau \in \mathfrak{h} \mid \operatorname{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$, we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q=e^{2\pi i \tau}$. This is called the q-expansion of the weakly modular function f.

Definition 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is **meromorphic at** ∞ if \tilde{f} extends to a meromorphic function in D(0,1).

We say f is **holomorphic at** ∞ if \tilde{f} is meromorphic at ∞ and has a

³This might not be the case if the set of poles has a limit inside the disk.

removable singularity at q = 0. In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\mathrm{Im}(\tau) \to \infty} f(\tau).$$

We say f vanishes at ∞ if f is holomorphic at ∞ and $f(\infty) = 0$.

Definition 2.2. A modular function (of weight k and level $\Gamma(1)$) is a weakly modular function (of weight k and level $\Gamma(1)$) which is meromorphic at ∞ .

A **modular form** is a weakly modular function which is holomorphic in \mathfrak{h} and holomorphic at ∞ .

A cuspidal modular form is a modular form that vanishes at ∞ .

Remark. We let $M_k(\Gamma(1))$ denote the set of modular forms of weight k and level $\Gamma(1)$. We write $S_k(\Gamma(1))$ for the set of cuspidal modular forms of weight k, level $\Gamma(1)$. Note $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$. These are \mathbb{C} -vector spaces. If k is odd, then these both only contain the zero function, since taking $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ gives $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$.

We now consider even weights only. If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_{\tau} \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k},$$

where $\Lambda_{\tau} = \mathbb{Z}\tau \oplus \mathbb{Z}$ for any $\tau \in \mathfrak{h}$.

If $\gamma \in \Gamma(1)$, then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau)j(\gamma,\tau)^{-k} = \sum_{\lambda \in \Lambda_{\alpha} \setminus 0} \lambda^{-k}j(\gamma,\tau)^{-k},$$

but $\Lambda_{\gamma\tau} = \mathbb{Z} \frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1} (\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)) = (c\tau+d)^{-1} \Lambda_{\tau}$. Hence

$$G_k|_k[g](\tau) = \sum_{\lambda \in (c\tau+d)^{-1}\Lambda_\tau \setminus 0} \lambda^{-k} (c\tau+d)^{-k}$$
$$= \sum_{\lambda \in \Lambda_\tau \setminus 0} ((c\tau+d)^{-1}\lambda)^{-k} (c\tau+d)^{-k} = G_k(\tau).$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely. Hence:

Proposition 2.2. Let k > 2 be an even integer. Then $G_k(\tau)$ converges absolutely and defines a modular form of weight k and level $\Gamma(1)$ which has

 $G_k(\infty) = 2\zeta(k)$. G_k is the weight k Eisenstein series.

We will later see that $M_2(\Gamma(1)) = 0$.

Proof. We want to show absolute and locally uniform convergence in \mathfrak{h} . This will show that G_k is holomorphic by complex analysis. Let $A \geq 2$ and define $\Omega_A = \{ \tau \in \mathfrak{h} \mid \operatorname{Im}(\tau) \geq \frac{1}{A}, \operatorname{Re}(\tau) \in [-A, A] \}$. We show uniform convergence in

$$\Omega_A$$
. If $\tau \in \Omega_A$, $x \in \mathbb{R}$, then $|\tau + x| \ge \begin{cases} \frac{1}{A} & |x| \le 2A \\ \frac{|x|}{2} & |x| \ge 2A. \end{cases}$ Hence

$$|\tau + x| \stackrel{(\dagger)}{\ge} \sup\left(\frac{1}{A}, \frac{|x|}{2A^2}\right) \ge \sup\left(\frac{1}{2A^2}, \frac{|x|}{2A^2}\right) = \frac{1}{2A^2} \sup(1, |x|).$$

(†) follows by drawing a diagram with the lines $y=\frac{1}{A}$ and $y=\frac{x}{2A^2}$ and marking the point $(2A,\frac{1}{A})$ on it, then noticing that out supremum always lies above the supremum of these two lines. If $(m,n)\in\mathbb{Z}^2, m\neq 0$, then

$$|m\tau+n|=|m|\left|\tau+\frac{n}{m}\right|\geq |m|\frac{1}{2A^2}\sup\left(1,\left|\frac{n}{m}\right|\right)=\frac{1}{2A^2}\sup\left(|m|,|n|\right).$$

This is also valid when m=0 by inspection. If $\tau \in \Omega_A$, then

$$\sum_{(m,n)\in\mathbb{Z}^2\backslash 0} |m\tau + n|^{-k}$$

$$\leq \left(\frac{1}{2A^2}\right)^{-k} \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \sup(|m|,|n|)^{-k}$$

$$= (2A^2)^k \sum_{d\in\mathbb{N}} d^{-k} \cdot \left| \{(m,n)\in\mathbb{Z}^2 \mid \sup(|m|,|n|) = d \} \right|$$

$$= (2A^2)^k \sum_{d\in\mathbb{N}} d^{-k}8d = 8(2A^2)^k \sum_{d\in\mathbb{N}} d^{1-k}$$

$$< \infty$$

whenever k-1>1, i.e. k>2. This shows absolute convergence, and uniform convergence in Ω_A by the Weierstrass M-test⁴. Hence G_k is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$. It remains to show that G_k is holomorphic at ∞ with $G_k(\infty)=2\zeta(k)$. For this, it suffices to check that

$$\lim_{\mathrm{Im}(\tau)\to\infty} G_k(\tau) = 2\zeta(k).$$

⁴If we have a sequence of functions $f_n: \Omega \to \mathbb{C}$ and values $M_n > 0$ with $|f_n(x)| < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges absolutely and uniformly on Ω . Here, replace n with d and sum d over $\sum_{(m,n)\in\mathbb{Z}^2\setminus 0,\sup(|m|,|n|)=d}|m\tau+n|^{-k}$.

This follows from uniform convergence in Ω_A : we get

$$\lim_{\mathrm{Im}(\tau)\to\infty}G_k(\tau)=\sum_{(m,n)\in\mathbb{Z}^2\backslash 0}\lim_{\mathrm{Im}(\tau)\to\infty}(m\tau+n)^{-k}=\sum_{n\in\mathbb{Z}\backslash 0}n^{-k}=2\sum_{n\geq 1}n^{-k}=2\zeta(k).$$

11 Oct 2022,

Lecture 3

Recap. We defined what it means for a function $f:\mathfrak{h}\to\mathbb{C}$ to be a modular form of weight k and level $\Gamma(1)$. $M_k(\Gamma(1))$ is the \mathbb{C} -vector space of such forms. If $f\in M_k(\Gamma(1))$, then there exists a holomorphic $\tilde{f}:D(0,1)\to\mathbb{C}$ (here we call D(0,1) the q-disk) such that $\forall \tau\in\mathfrak{h}, f(\tau)=\tilde{f}(e^{2\pi i \tau})$. The Taylor expansion of \tilde{f} gives the q-expansion

$$f(\tau) = \sum_{n>0} a_n q^n, \ q = e^{2\pi i \tau}.$$

We have $f(\infty) = \tilde{f}(0) = a_0$. If k > 2 is even, then $G_k(\tau) = \sum_{\lambda \in \Lambda_{\tau} \setminus 0} \lambda^{-k}$ converges absolutely and defines an element of $M_k(\Gamma(1))$ with $G_k(\infty) = 2\zeta(k)$.

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n>1} a_n q^n.$$

We will soon show that we have $a_n \in \mathbb{Q} \ \forall n \geq 1$.

We can construct more modular forms: if $f \in M_k(\Gamma(1))$ and $g \in M_l(\Gamma(1))$, then $fg \in M_{k+l}(\Gamma(1))$. To check this is a modular form, we need to check that:

- fg is holomorphic, which is true as f, g are holomorphic.
- fg is invariant under the weight k+l action of $\Gamma(1)$, which is true as f,g are invariant under the weight k and l actions of $\Gamma(1)$ this is just a computation.
- fg is holomorphic at ∞ . This is true as the q-expansions multiply, so since f, g have no negative terms, the same is true for fg.

Hence we get e.g. $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$ and $E_4^3 - E_6^2 \in S_{12}(\Gamma(1))$ (i.e. it is cuspidal since zero at infinity). This difference is Ramanujan's Δ -function. We will show it is nonzero later.

We now want to show that $M_k(\Gamma(1))$ is finite-dimensional. We first study $\Gamma(1)/\mathfrak{h}$. For this, introduce a fundamental set $\mathfrak{f}' \subset \mathfrak{h}$ for the $\Gamma(1)$ -action. We define⁵ a fundamental set to be a set that intersects each $\Gamma(1)$ -orbit in exactly

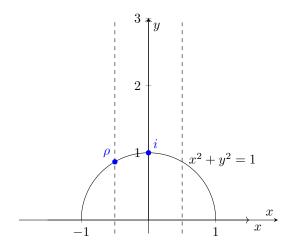
⁵Definitions in literature may vary, so we omit a formal definition.

one element. Define

$$\mathfrak{f} = \left\{ \tau \in \mathfrak{h} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right], |\tau| \ge 1 \right\}.$$

$$\mathfrak{f}' = \left\{ \tau \in \mathfrak{f} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right), |\tau| = 1 \implies \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, 0 \right] \right\}.$$

Introduce $T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\Gamma(1)$. We observe that every element of $\mathfrak f$ is conjugate under S or T^{-1} to an element of $\mathfrak f'$, which is true since $T(\tau)=\tau+1$ and $S(\tau)=-\frac{1}{\tau}$.



Proposition 2.3. Let $G = \Gamma(1)/\{\pm I\}$. Then

- (i) $\forall \tau \in \mathfrak{h}, \tau$ is $\Gamma(1)$ -conjugate to an element of \mathfrak{f}' .
- (ii) If $\tau, \tau' \in \mathfrak{f}'$ are $\Gamma(1)$ -conjugate, then $\tau = \tau'$.
- (iii) If $\tau \in \mathfrak{f}'$, then $\operatorname{Stab}_G(\tau)$ is trivial, except in the two cases $\operatorname{Stab}_G(i) = \langle S \rangle$ and $\operatorname{Stab}_G(\rho) = \langle ST \rangle$, where $\rho = e^{2\pi i/3}$.
- (iv) $\Gamma(1)$ is generated by S and T.

Proof. Let H be the subgroup of G generated by S and T.

Claim. Every $\tau \in \mathfrak{h}$ is H-conjugate to an element of \mathfrak{f}' .

Proof. By our above observation and since $S,T\in H$, it suffices to prove that every $\tau\in\mathfrak{h}$ is H-conjugate to \mathfrak{f} . Take $\tau\in\mathfrak{h}$. Recall that if $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z})$, then $\mathrm{Im}(\gamma\tau)=\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$.

In particular, $\forall R \geq 0$, the intersection $H\tau \cap \{\operatorname{Im}(\tau') > R\}$ is finite, since $\operatorname{Im}(\gamma\tau) > R \iff |c\tau + d|^2 < \frac{\operatorname{Im}(\tau)}{R}$, but $\Lambda_{\tau} = \mathbb{Z}\tau \oplus \mathbb{Z}$ is a lattice, so the set $\{(c,d) \in \mathbb{Z}^2 \mid |c\tau + d| < R'\}$ is finite.

So there exists $h \in H$ such that $\operatorname{Im}(h\tau) \geq \operatorname{Im}(h'\tau) \ \forall h' \in H$. After replacing τ by $h\tau$, we can assume $\operatorname{Im}(\tau) \geq \operatorname{Im}(h\tau) \ \forall h \in H$. Since acting by T does not change $\operatorname{Im}(\tau)$, we can also assume $\operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We have $\operatorname{Im}(\tau) \geq \operatorname{Im}(S\tau) = \frac{\operatorname{Im}(\tau)}{|\tau|^2} \implies |\tau| \geq 1$, proving the claim and (i).

Now take $\tau, \tau' \in \mathfrak{f}'$ and suppose $\gamma \tau = \tau'$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. We want to show that either $\gamma = \pm I$ or $\tau = i, \rho$.

WLOG assume $\operatorname{Im}(\tau') = \operatorname{Im}(\gamma\tau) \geq \operatorname{Im}(\tau)$, i.e. $\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau+d|^2} \geq \operatorname{Im}(\tau)$, so $|c\tau+d| \leq 1$. However, if $\tau \in \mathfrak{f}'$, then $\operatorname{Im}(\tau) \geq \frac{\sqrt{3}}{2}$ with equality if and only if $\tau = \rho$. Hence $|c\tau+d| \geq |c|\operatorname{Im}(\tau) \geq |c|\frac{\sqrt{3}}{2} \implies |c| \leq \frac{2}{\sqrt{3}} \implies |c| = 0, 1 \implies c = 0$ or $c = \pm 1$.

- If c = 0, then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, so $ad = 1 \implies a = d = \pm 1$, so $\gamma = \pm T^m$ for $m \in \mathbb{Z}$. However, T acts on \mathfrak{f}' by shifting the real part, so it can only stay in \mathfrak{f}' if m = 0 (as $\operatorname{Re}(\mathfrak{f}') \in \left[-\frac{1}{2}, \frac{1}{2}\right]$), so $\gamma = \pm I$ and $\tau' = \tau$.
- If c=1, then $\gamma=\begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ and $|\tau+d|\leq 1$. By drawing another picture, we see that the only circles centered at integers of radius 1 which intersect \mathfrak{f}' are centered at -d=0, -d=-1. Hence either d=0, whence $|\tau|=1$, or d=1, whence $\tau=\rho$.
 - If $c=1, d=0, |\tau|=1$, then $\gamma=\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ since the determinant must be 1. Then $\gamma\tau=\frac{a\tau-1}{\tau}=a-\frac{1}{\tau}=a-\overline{\tau}$, so $\operatorname{Re}(\gamma\tau)=a-\operatorname{Re}(\tau)\in\operatorname{Re}(\mathfrak{f}'\cap\{|\tau|=1\})=\left[-\frac{1}{2},0\right]$. However, we also have $\operatorname{Re}(\gamma\tau)\in a-\left[-\frac{1}{2},0\right]=a+\left[0,\frac{1}{2}\right]$.

The intersection $\left[-\frac{1}{2},0\right] \cap \left(a+\left[0,\frac{1}{2}\right]\right)$ can be nonempty only if either a=0, whence $\operatorname{Re}(\gamma\tau)=\operatorname{Re}(\tau)=0$, so $\tau=\gamma\tau=i$, or a=-1, whence $\operatorname{Re}(\tau)=\operatorname{Re}(\gamma\tau)=-\frac{1}{2}$, so $\tau=\gamma\tau=\rho$.

If a = 0, then $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$, which stabilizes i, and $\langle -S \rangle = \langle S \rangle$.

If a=-1, then $\gamma=\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}=(ST)^2$, which stabilizes ρ , and $(ST)^3=I$, so $\langle (ST)^2\rangle=\langle ST\rangle$.

- If $c=1, d=1, \tau=\rho$, then $\gamma=\begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$, so $\rho=\gamma\rho=\frac{a\rho+b}{\rho+1}$. We have $\rho^2+\rho+1=0$, so $\rho^2+\rho=-1$, so $a\rho+b=\rho^2+\rho=-1$. But $a,b\in\mathbb{Z}$ and $1,\rho$ are linearly independent over \mathbb{R} , so a=0,b=-1, so $\gamma=\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}=-ST$, which stabilizes ρ .
- If c = -1, we can reduce this to the case c = 1 by replacing γ with $-\gamma$.

We have now shown the first three parts of the proposition. It remains to show the last part, i.e. $\Gamma(1) = \langle S, T \rangle$. Since $S^2 = -I$, it is enough to show that H = G. Choose $\tau \in \text{Int}(f)$, so $\text{Stab}_G(\tau) = \{I\}$. Let $g \in G$. By our claim proving (i), $\exists h \in H$ such that $hg\tau \in \mathfrak{f}'$. We must therefore have $hg\tau = \tau$, hence $hg \in \text{Stab}_G(\tau) = \{I\}$, so $g = h^{-1} \in H$.