Part III - Modular Forms

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Contents

1 Introduction

06 Oct 2022, Lecture 1

Definition 1.1. We define the following groups:

$$\begin{aligned}
\mathfrak{h} &= \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \\
GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\
\Gamma(1) = SL_2(\mathbb{Z}) &= \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.
\end{aligned}$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on $\mathfrak{h}$ by Möbius transformations.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{h}$. Then

$$\mathrm{Im}(g\tau) = \frac{1}{2i}\left(\frac{a\tau + b}{c\tau + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right) = \frac{1}{2i}\frac{(ad - bc)(\tau - \overline{\tau})}{|c\tau + d|^2} = \frac{\det(g)\mathrm{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{h}$. This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in $\mathfrak{h}$ is conjugate to i. □

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{h} \to \mathbb{C}$, then

$$f|_k[g] : \mathfrak{h} \to \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight k action of g on f**.

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

Proof. We compute

$$(f|_k[g])|_k[h](\tau) = \det(h)^{k-1} f|_k[g](h\tau) j(h,\tau)^{-k} =$$
$$\det(h)^{k-1}\det(g)^{k-1} f(gh\tau) j(g,h\tau)^{-k} j(h,\tau)^{-k} \stackrel{?}{=}$$
$$\det(gh)^{k-1} f(gh\tau) j(gh,\tau)^{-k} = f|_k[gh](\tau).$$

Hence we need to check that $j(gh,\tau) = j(gh,\tau)j(h,\tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh\begin{pmatrix} \tau \\ 1 \end{pmatrix} = g\left(j(h,\tau)\begin{pmatrix} h\tau \\ 1 \end{pmatrix}\right) = j(h,\tau)j(g,h\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. □

Formulae. For $g \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{h}$, we have

$$\mathrm{Im}(g\tau) = \det(g)\frac{\mathrm{Im}(\tau)}{|j(g,\tau)|^2} \text{ and } j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g\begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ of finite index[1]. A **weakly modular function of weight** k **and level** Γ is a meromorphic function $f : \mathfrak{h} \to \mathbb{C}$ which is invariant under the weight k action of Γ, i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in $\mathfrak{h}$ and at ∞.

It is a fact that modular forms of fixed weight and level live in finite-dimensional $\mathbb{C}$-vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

(1) They are related to the theory of elliptic functions. Let $E/\mathbb{C}$ be an elliptic curve and ω a holomorphic non–zero 1–form. Then there exists a unique lattice[2] $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \to E$ such that $\phi^*(\omega) = dz$. Then

[1] In other words, Γ is a (finite index) subgroup of $\Gamma(1)$.

[2] i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over $\mathbb{R}$.

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{h}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{h}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over $\mathbb{C}$.

(2) Modular forms f have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n.

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q–expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4\nmid d} d$.

(3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to $\mathbb{C}$ and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L–function. Modular forms can be used to construct interesting examples of L–functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

(4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves $E/\mathbb{C}$ up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L–functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

2 Modular Forms on $\Gamma(1)$

09 Oct 2022, Lecture 2

Reminder. A **meromorphic** function in an open subset $U \subset \mathbb{C}$ is a closed subset $A \subset U$ and a holomorphic function $f : U \setminus A \to \mathbb{C}$ such that $\forall a \in A$, $\exists \delta > 0$ such that $D^*(a, \delta) \subset U \setminus A$ and $\exists n \geq 0$ such that $(z-a)^n f(z)$ extends to a holomorphic function in $D(a, \delta)$.

f then has a Laurent expansion $\sum_{m \in \mathbb{Z}} a_m (z-a)^m$ valid on $D^*(a, \delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a meromorphic function $\tilde{f}$ in $D^*(0,1)$ (the "q-disk") such that

$$f(\tau) = \tilde{f}(e^{2\pi i \tau}).$$

Proof. f is meromorphic in $\mathfrak{h}$ by assumption. Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$, as f is invariant under the weight k action of γ. But also $f(\gamma\tau) = f(\tau + 1)$, so f is periodic.

Now map a strip of $\mathfrak{h}$ of width 1 to $D^*(0,1)$ by $\tau \mapsto e^{2\pi i \tau}$. Let $a \in D^*(0,1)$ and $\delta > 0$ be such that $D(a, \delta) \subset D^*(0,1)$. Define $\tilde{f}$ on $D(a, \delta)$ by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i} \log q\right),$$

for any branch of log defined in $D(a, \delta)$. This is meromorphic and independent of the choice of the branch of log, as f is periodic with period 1. This defines $\tilde{f}$ in $D^*(0,1)$. Finally, $\tilde{f}$ is unique since $\tau \mapsto e^{2\pi i \tau}$ is surjective. $\square$

If $\tilde{f}$ extends to a meromorphic function[3] in $D(0,1)$, then $\exists \delta > 0$ such that $\tilde{f}$ has a Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0, \delta)$.

In the region $\{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$, we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q = e^{2\pi i \tau}$. This is called the q–**expansion** of the weakly modular function f.

Definition 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is **meromorphic at** ∞ if $\tilde{f}$ extends to a meromorphic function in $D(0,1)$.

We say f is **holomorphic at** ∞ if $\tilde{f}$ is meromorphic at ∞ and has a

[3] This might not be the case if the set of poles has a limit inside the disk.

removable singularity at $q = 0$. In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\operatorname{Im}(\tau)\to\infty} f(\tau).$$

We say f **vanishes at** ∞ if f is holomorphic at ∞ and $f(\infty) = 0$.

Definition 2.2. A **modular function** (of weight k and level $\Gamma(1)$) is a weakly modular function (of weight k and level $\Gamma(1)$) which is meromorphic at ∞.

A **modular form** is a weakly modular function which is holomorphic in $\mathfrak{h}$ and holomorphic at ∞.

A **cuspidal modular form** is a modular form that vanishes at ∞.

Remark. We let $M_k(\Gamma(1))$ denote the set of modular forms of weight k and level $\Gamma(1)$. We write $S_k(\Gamma(1))$ for the set of cuspidal modular forms of weight k, level $\Gamma(1)$. Note $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$. These are $\mathbb{C}$–vector spaces. If k is odd, then these both only contain the zero function, since taking $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ gives $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$.

We now consider even weights only. If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda\in\Lambda_\tau\setminus 0} \lambda^{-k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} (m\tau+n)^{-k},$$

where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ for any $\tau \in \mathfrak{h}$.

If $\gamma \in \Gamma(1)$, then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau) j(\gamma,\tau)^{-k} = \sum_{\lambda\in\Lambda_{\gamma\tau}\setminus 0} \lambda^{-k} j(\gamma,\tau)^{-k},$$

but $\Lambda_{\gamma\tau} = \mathbb{Z}\frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1}\left(\mathbb{Z}(a\tau+b)\oplus\mathbb{Z}(c\tau+d)\right) = (c\tau+d)^{-1}\Lambda_\tau$. Hence

$$\begin{aligned} G_k|_k[g](\tau) &= \sum_{\lambda\in(c\tau+d)^{-1}\Lambda_\tau\setminus 0} \lambda^{-k}(c\tau+d)^{-k} \\ &= \sum_{\lambda\in\Lambda_\tau\setminus 0} ((c\tau+d)^{-1}\lambda)^{-k}(c\tau+d)^{-k} = G_k(\tau). \end{aligned}$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely. Hence:

Proposition 2.2. Let $k > 2$ be an even integer. Then $G_k(\tau)$ converges absolutely and defines a modular form of weight k and level $\Gamma(1)$ which has

$G_k(\infty) = 2\zeta(k)$. G_k is the **weight k Eisenstein series**.

We will later see that $M_2(\Gamma(1)) = 0$.

Proof. We want to show absolute and locally uniform convergence in $\mathfrak{h}$. This will show that G_k is holomorphic by complex analysis. Let $A \geq 2$ and define $\Omega_A = \{\tau \in \mathfrak{h} \mid \operatorname{Im}(\tau) \geq \frac{1}{A}, \operatorname{Re}(\tau) \in [-A, A]\}$. We show uniform convergence in Ω_A. If $\tau \in \Omega_A, x \in \mathbb{R}$, then $|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A. \end{cases}$ Hence

$$|\tau + x| \overset{(\dagger)}{\geq} \sup\left(\frac{1}{A}, \frac{|x|}{2A^2}\right) \geq \sup\left(\frac{1}{2A^2}, \frac{|x|}{2A^2}\right) = \frac{1}{2A^2}\sup(1, |x|).$$

($\dagger$) follows by drawing a diagram with the lines $y = \frac{1}{A}$ and $y = \frac{x}{2A^2}$ and marking the point $(2A, \frac{1}{A})$ on it, then noticing that out supremum always lies above the supremum of these two lines. If $(m,n) \in \mathbb{Z}^2, m \neq 0$, then

$$|m\tau + n| = |m|\left|\tau + \frac{n}{m}\right| \geq |m|\frac{1}{2A^2}\sup\left(1, \left|\frac{n}{m}\right|\right) = \frac{1}{2A^2}\sup\left(|m|, |n|\right).$$

This is also valid when $m = 0$ by inspection. If $\tau \in \Omega_A$, then

$$\begin{aligned}
&\sum_{(m,n)\in\mathbb{Z}^2\setminus 0} |m\tau + n|^{-k} \\
&\leq \left(\frac{1}{2A^2}\right)^{-k} \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \sup\left(|m|, |n|\right)^{-k} \\
&= (2A^2)^k \sum_{d\in\mathbb{N}} d^{-k} \cdot \left|\{(m,n) \in \mathbb{Z}^2 \mid \sup\left(|m|, |n|\right) = d\}\right| \\
&= (2A^2)^k \sum_{d\in\mathbb{N}} d^{-k} 8d = 8(2A^2)^k \sum_{d\in\mathbb{N}} d^{1-k} \\
&< \infty
\end{aligned}$$

whenever $k - 1 > 1$, i.e. $k > 2$. This shows absolute convergence, and uniform convergence in Ω_A by the Weierstrass M-test[4]. Hence G_k is holomorphic in $\mathfrak{h}$ and invariant under the weight k action of $\Gamma(1)$. It remains to show that G_k is holomorphic at ∞ with $G_k(\infty) = 2\zeta(k)$. For this, it suffices to check that

$$\lim_{\operatorname{Im}(\tau)\to\infty} G_k(\tau) = 2\zeta(k).$$

[4]If we have a sequence of functions $f_n : \Omega \to \mathbb{C}$ and values $M_n > 0$ with $|f_n(x)| < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges absolutely and uniformly on Ω. Here, replace n with d and sum d over $\sum_{(m,n)\in\mathbb{Z}^2\setminus 0, \sup(|m|,|n|)=d} |m\tau + n|^{-k}$.

This follows from uniform convergence in Ω_A: we get

$$\lim_{\mathrm{Im}(\tau)\to\infty} G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \lim_{\mathrm{Im}(\tau)\to\infty} (m\tau+n)^{-k} = \sum_{n\in\mathbb{Z}\setminus 0} n^{-k} = 2\sum_{n\geq 1} n^{-k} = 2\zeta(k).$$

□

11 Oct 2022, Lecture 3

Recap. We defined what it means for a function $f : \mathfrak{h} \to \mathbb{C}$ to be a modular form of weight k and level $\Gamma(1)$. $M_k(\Gamma(1))$ is the $\mathbb{C}$–vector space of such forms. If $f \in M_k(\Gamma(1))$, then there exists a holomorphic $\tilde{f} : D(0,1) \to \mathbb{C}$ (here we call $D(0,1)$ the q–disk) such that $\forall \tau \in \mathfrak{h}$, $f(\tau) = \tilde{f}(e^{2\pi i\tau})$. The Taylor expansion of $\tilde{f}$ gives the q–expansion

$$f(\tau) = \sum_{n\geq 0} a_n q^n,\ q = e^{2\pi i\tau}.$$

We have $f(\infty) = \tilde{f}(0) = a_0$. If $k > 2$ is even, then $G_k(\tau) = \sum_{\lambda\in\Lambda_\tau\setminus 0} \lambda^{-k}$ converges absolutely and defines an element of $M_k(\Gamma(1))$ with $G_k(\infty) = 2\zeta(k)$.

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n\geq 1} a_n q^n.$$

We will soon show that we have $a_n \in \mathbb{Q}\ \forall n \geq 1$.

We can construct more modular forms: if $f \in M_k(\Gamma(1))$ and $g \in M_l(\Gamma(1))$, then $fg \in M_{k+l}(\Gamma(1))$. To check this is a modular form, we need to check that:

- fg is holomorphic, which is true as f, g are holomorphic.
- fg is invariant under the weight $k + l$ action of $\Gamma(1)$, which is true as f, g are invariant under the weight k and l actions of $\Gamma(1)$ – this is just a computation.
- fg is holomorphic at ∞. This is true as the q–expansions multiply, so since f, g have no negative terms, the same is true for fg.

Hence we get e.g. $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$ and $\frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$ (i.e. it is cuspidal since zero at infinity). This difference is Ramanujan's Δ–function. We will show it is nonzero later.

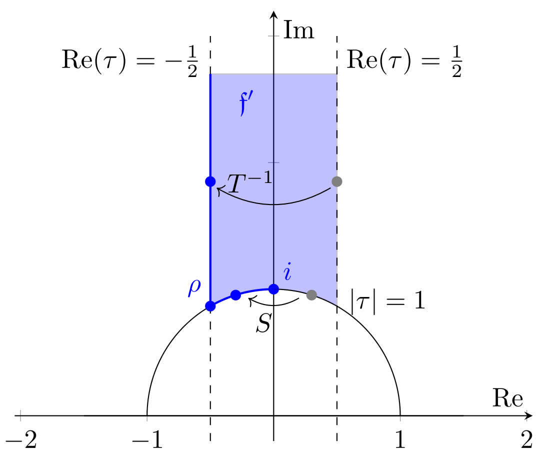
We now want to show that $M_k(\Gamma(1))$ is finite–dimensional. We first study $\Gamma(1)/\mathfrak{h}$. For this, introduce a fundamental set $\mathfrak{f}' \subset \mathfrak{h}$ for the $\Gamma(1)$–action. We define[5] a fundamental set to be a set that intersects each $\Gamma(1)$–orbit in exactly

[5] Definitions in literature may vary, so we omit a formal definition.

one element. Define

$$\mathfrak{f} = \left\{ \tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2}\right], |\tau| \geq 1 \right\}.$$
$$\mathfrak{f}' = \left\{ \tau \in \mathfrak{f} \mid \mathrm{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2}\right), |\tau| = 1 \implies \mathrm{Re}(\tau) \in \left[-\frac{1}{2}, 0\right] \right\}.$$

Introduce $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\Gamma(1)$. We observe that every element of $\mathfrak{f}$ is conjugate under S or T^{-1} to an element of $\mathfrak{f}'$, which is true since $T(\tau) = \tau + 1$ and $S(\tau) = -\frac{1}{\tau}$.



Proposition 2.3. Let $G = \Gamma(1)/\{\pm I\}$. Then

(i) $\forall \tau \in \mathfrak{h}, \tau$ is $\Gamma(1)$–conjugate to an element of $\mathfrak{f}'$.

(ii) If $\tau, \tau' \in \mathfrak{f}'$ are $\Gamma(1)$–conjugate, then $\tau = \tau'$.

(iii) If $\tau \in \mathfrak{f}'$, then $\mathrm{Stab}_G(\tau)$ is trivial, except in the two cases $\mathrm{Stab}_G(i) = \langle S \rangle$ and $\mathrm{Stab}_G(\rho) = \langle ST \rangle$, where $\rho = e^{2\pi i/3}$.

(iv) $\Gamma(1)$ is generated by S and T.

Proof. Let H be the subgroup of G generated by S and T.

Claim. Every $\tau \in \mathfrak{h}$ is H–conjugate to an element of $\mathfrak{f}'$.

Proof. By our above observation and since $S, T \in H$, it suffices to prove that every $\tau \in \mathfrak{h}$ is H–conjugate to $\mathfrak{f}$. Take $\tau \in \mathfrak{h}$. Recall that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$.

In particular, $\forall R \geq 0$, the intersection $H\tau \cap \{\mathrm{Im}(\tau') > R\}$ is finite, since $\mathrm{Im}(\gamma\tau) > R \iff |c\tau + d|^2 < \frac{\mathrm{Im}(\tau)}{R}$, but $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ is a lattice, so the set $\{(c,d) \in \mathbb{Z}^2 \mid |c\tau + d| < R'\}$ is finite.

So there exists $h \in H$ such that $\mathrm{Im}(h\tau) \geq \mathrm{Im}(h'\tau)\ \forall h' \in H$. After replacing τ by $h\tau$, we can assume $\mathrm{Im}(\tau) \geq \mathrm{Im}(h\tau)\ \forall h \in H$. Since acting by T does not change $\mathrm{Im}(\tau)$, we can also assume $\mathrm{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. We have $\mathrm{Im}(\tau) \geq \mathrm{Im}(S\tau) = \frac{\mathrm{Im}(\tau)}{|\tau|^2} \implies |\tau| \geq 1$, proving the claim and (i). $\square$

Now take $\tau, \tau' \in \mathfrak{f}'$ and suppose $\gamma\tau = \tau'$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. We want to show that either $\gamma = \pm I$ or $\tau = i, \rho$.

WLOG assume $\mathrm{Im}(\tau') = \mathrm{Im}(\gamma\tau) \geq \mathrm{Im}(\tau)$, i.e. $\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2} \geq \mathrm{Im}(\tau)$, so $|c\tau + d| \leq 1$. However, if $\tau \in \mathfrak{f}'$, then $\mathrm{Im}(\tau) \geq \frac{\sqrt{3}}{2}$ with equality if and only if $\tau = \rho$. Hence $|c\tau + d| \geq |c|\mathrm{Im}(\tau) \geq |c|\frac{\sqrt{3}}{2} \implies |c| \leq \frac{2}{\sqrt{3}} \implies |c| = 0, 1 \implies c = 0$ or $c = \pm 1$.

- If $c = 0$, then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, so $ad = 1 \implies a = d = \pm 1$, so $\gamma = \pm T^m$ for $m \in \mathbb{Z}$. However, T acts on $\mathfrak{f}'$ by shifting the real part, so it can only stay in $\mathfrak{f}'$ if $m = 0$ (as $\mathrm{Re}(\mathfrak{f}') \in \left[-\frac{1}{2}, \frac{1}{2}\right)$), so $\gamma = \pm I$ and $\tau' = \tau$.

- If $c = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ and $|\tau + d| \leq 1$. By drawing another picture, we see that the only circles centered at integers of radius 1 which intersect $\mathfrak{f}'$ are centered at $-d = 0, -d = -1$. Hence either $d = 0$, whence $|\tau| = 1$, or $d = 1$, whence $\tau = \rho$.

  - If $c = 1, d = 0, |\tau| = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ since the determinant must be 1. Then $\gamma\tau = \frac{a\tau - 1}{\tau} = a - \frac{1}{\tau} = a - \overline{\tau}$, so $\mathrm{Re}(\gamma\tau) = a - \mathrm{Re}(\tau) \in \mathrm{Re}(\mathfrak{f}' \cap \{|\tau| = 1\}) = \left[-\frac{1}{2}, 0\right]$. However, we also have $\mathrm{Re}(\gamma\tau) \in a - \left[-\frac{1}{2}, 0\right] = a + \left[0, \frac{1}{2}\right]$.

    The intersection $\left[-\frac{1}{2}, 0\right] \cap \left(a + \left[0, \frac{1}{2}\right]\right)$ can be nonempty only if either $a = 0$, whence $\mathrm{Re}(\gamma\tau) = \mathrm{Re}(\tau) = 0$, so $\tau = \gamma\tau = i$, or $a = -1$, whence $\mathrm{Re}(\tau) = \mathrm{Re}(\gamma\tau) = -\frac{1}{2}$, so $\tau = \gamma\tau = \rho$.

    If $a = 0$, then $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$, which stabilizes i, and $\langle -S \rangle = \langle S \rangle$.

    If $a = -1$, then $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$, which stabilizes ρ, and $(ST)^3 = I$, so $\langle (ST)^2 \rangle = \langle ST \rangle$.

  - If $c=1, d=1, \tau=\rho$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$, so $\rho = \gamma\rho = \frac{a\rho+b}{\rho+1}$. We have $\rho^2+\rho+1=0$, so $\rho^2+\rho=-1$, so $a\rho+b=\rho^2+\rho=-1$. But $a, b \in \mathbb{Z}$ and $1, \rho$ are linearly independent over $\mathbb{R}$, so $a=0, b=-1$, so $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -ST$, which stabilizes ρ.

- If $c=-1$, we can reduce this to the case $c=1$ by replacing γ with $-\gamma$.

We have now shown the first three parts of the proposition. It remains to show the last part, i.e. $\Gamma(1) = \langle S, T\rangle$. Since $S^2 = -I$, it is enough to show that $H = G$. Choose $\tau \in \mathrm{Int}(f)$, so $\mathrm{Stab}_G(\tau) = \{I\}$. Let $g \in G$. By our claim proving (i), $\exists h \in H$ such that $hg\tau \in \mathfrak{f}'$. We must therefore have $hg\tau = \tau$, hence $hg \in \mathrm{Stab}_G(\tau) = \{I\}$, so $g = h^{-1} \in H$. $\square$

13 Oct 2022, Lecture 4

Notation. We write $e_\tau = |\mathrm{Stab}_G(\tau)|$.

Let f be a nonzero modular function of weight k, level $\Gamma(1)$. If $\tau \in \mathfrak{h}$, then $v_\tau(f)$ is the order of f at τ (the unique $n \in \mathbb{Z}$ such that $f(z) = (z-\tau)^n g(z)$ for some meromorphic g that is holomorphic and non–vanishing at τ). We define $v_\infty(f)$ to be the order of f at infinity, i.e. $v_\infty(f) = v_0(\tilde{f})$ for $\tilde{f}$ the meromorphic function in $D(0,1)$ with $f(\tau) = \tilde{f}(e^{2\pi i\tau})$.

Proposition 2.4. Let f be a nonzero modular function of weight k, level $\Gamma(1)$. Then

$$\sum_{\tau \in \Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.$$

Proof. We first check that the sum is well–defined:

- If $\tau \in \mathfrak{h}$, then $e_\tau, v_\tau(f)$ only depend on the $\Gamma(1)$–orbit of τ. This is because if $\gamma \in \Gamma(1)$ and $\tau \in \mathfrak{h}$, then $\mathrm{Stab}_{\Gamma(1)}(\tau)$ and $\mathrm{Stab}_{\Gamma(1)}(\gamma\tau)$ are conjugate subgroups of $\Gamma(1)$, so $e_\tau = e_{\gamma\tau}$. On the other hand, $f(\gamma\tau) = f(\tau)j(\gamma,\tau)^k$ and $j(\gamma,\tau)$ is holomorphic and non–vanishing on $\mathfrak{h}$, so $v_{\gamma\tau}(f) = v_\tau(f)$.

- The sum only has a finite number of nonzero terms, since if f is a modular function and $\tilde{f}$ is a meromorphic function on $D(0,1)$, then $\exists\delta > 0$ such that $\tilde{f}$ is holomorphic and non–vanishing in $D^*(0,\delta)$. Thus $\exists R > 0$ such that f is holomorphic and non–vanishing in $\{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) > R\}$. Hence to show the sum is finite, it suffices to show that f only has a finite number of zeroes and poles in $\mathfrak{f}$ (as $\mathfrak{f}$ intersects every $\Gamma(1)$–orbit), for which it suffices to show that f has a finite number of zeroes and poles in $\mathfrak{f} \cap \{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) \le R\}$, which is true as the set is compact (closed and bounded) and the zeroes and poles of f are discrete.

To prove the identity, we use contour integration. Setup: if $U \subset \mathbb{C}$ is an open subset, $f : U \to \mathbb{C}$ is holomorphic and $\gamma : [0,1] \to U$ is a path, then

$$\int_\gamma f(z)\mathrm{d}z = \int_{t=0}^1 f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

We have the pullback formula: if $u : U \to V$ is a holomorphic map between open subsets of $\mathbb{C}$, $g : V \to \mathbb{C}$ is holomorphic and γ is a path in U, then

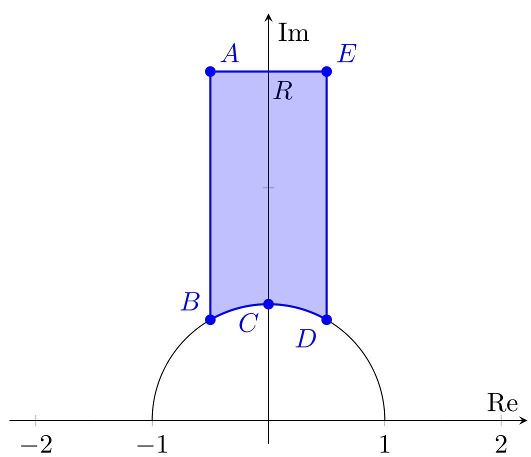
$$\int_{u\circ\gamma} g(z)\mathrm{d}z = \int_\gamma u^*(g(z)\mathrm{d}z) = \int_\gamma g(u(z))u'(z)\mathrm{d}z.$$

A particularly nice case: if $g(z) = h'(z)/h(z)$, then $g(z)\mathrm{d}z = d\log h$, so $\int_{u\circ\gamma} d\log h = \int_\gamma u^*(d\log h) = \int_\gamma d(\log h\circ u) = \int_\gamma \frac{(h\circ u)'(z)}{(h\circ u)(z)}\mathrm{d}z$.

We also have (Cauchy's) argument principle: if $U \subset \mathbb{C}$ is a simply connected open subset, $\gamma \subset U$ is a simple positvely oriented closed path and g is a meromorphic function in U with no zeroes or poles on γ, then

$$\frac{1}{2\pi i}\oint_\gamma d\log g = \frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z = \sum_{a\in\mathrm{Int}(\gamma)} v_a(g).$$

We now apply this to our modular function f. Choose $R > 0$ such that f has no zeroes or poles in $\{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) \geq R\}$. We consider $\frac{1}{2\pi i}\oint_\gamma d\log f$, where γ is the contour $ABCDE$.



By choice of R, there are no zeroes or poles of f on AE. We first consider the case where f has no zeroes or poles at all on γ. Then the argument principle gives

$$\frac{1}{2\pi i}\oint_{\gamma} d\log f = \frac{1}{2\pi i}\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d\log f = \sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f)$$

(as $v_\tau(f) \neq 0$, $e_\tau = 1$ under our assumptions).

Apply the pullback formula with $u(\tau) = \tau+1$. Then $u(AB) = ED$, $f\circ u = f$, so

$$\int_{u(AB)} d\log f = \int_{AB} d\log f\circ u = \int_{AB} d\log f = \int_{ED} d\log f = -\int_{DE} d\log f.$$

Hence $\int_{AB} + \int_{DE} d\log f = 0$.

Now take $q = e^{2\pi i\tau}$, so $f = \tilde{f}\circ q$ and $q(AE)$ is a positively oriented circle around 0 in $D(0,1)$. So

$$\frac{1}{2\pi i}\int_{q(AE)} d\log\tilde{f} = v_\infty(f) = \frac{1}{2\pi i}\int_{AE} d\log\tilde{f}\circ q = \frac{1}{2\pi i}\int_{AE} d\log f.$$

Now take $v(\tau) = S(\tau) = -\frac{1}{\tau}$. Then $v(BC) = DC$ and we know $f|_k[S](\tau) = f\left(-\frac{1}{\tau}\right)\tau^{-k} = f(\tau)$, so $f\circ v = f(\tau)\tau^k$. Hence

$$\int_{DC} d\log f = \int_{v(BC)} d\log f = \int_{BC} d\log(f\circ v) = \int_{BC} d\log(f(\tau)\tau^k)$$
$$= \int_{BC} d\log f + k d\log\tau = \int_{BC} d\log f + k(\log C - \log B)$$

where here log is any branch of the logarithm defined on BC. But $B = \rho, C = i$, so $\log B = i\frac{2\pi}{3}$ and $\log C = i\frac{\pi}{2}$. Hence

$$\int_{CD} d\log f = -\int_{DC} d\log f + k\left(\frac{2\pi i}{3} - \frac{2\pi i}{4}\right),$$

giving

$$\int_{BC} + \int_{CD} d\log f = 2\pi i k\frac{1}{12}.$$

We have

$$\begin{aligned}\sum_{\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e^{\tau}} v_\tau(f) =& \frac{1}{2\pi i}\left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d\log f\right) \\ =& \frac{1}{2\pi i}\left(0 + \frac{k}{12} + 0 - v_\infty(f)\right) \\ \implies & \sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.\end{aligned}$$

This finishes the proof in the case where there are no zeroes or poles. If there are zeroes or poles on γ, we need to modify the contour. For example, if there's a zero or a pole at a point P on AB, then consider the contour γ', which is just γ but with a small semicircle around our (discrete) pole, which satisfies the property that f has no zeroes or poles on γ'. The trickiest case is when there is a zero or pole at $B = \rho$ or $C = i$. This is Q3 on example sheet 1. □

16 Oct 2022, Lecture 5

Example 2.1. Take $k = 4$, $f = E_4 \in M_4(\Gamma(1))$. Hence $\forall \tau \in \mathfrak{h}, v_\tau(E_4) \geq 0$ (as it is holomorphic in $\mathfrak{h}$). We know $E_4(\tau) = 1 + \sum_{n\geq 1} a_n q^n$, so $E_4(\infty) \neq 0$ and $v_\infty(E_4) = 0$. Hence our formula gives

$$\sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}} \frac{1}{e_\tau} v(E_4) = \frac{1}{3} v_\rho(E_4) + \frac{1}{2} v_i(E_4) + \sum_{\tau\in\Gamma(1)\backslash\mathfrak{h}, \tau\not\sim\rho,i} v_\tau(E_4) = \frac{1}{3}.$$

So we have $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{3}$, where $a, b, c \in \mathbb{Z}_{\geq 0}$, which gives the only solution $a = 1, b = c = 0$, so $E_4(\rho) = 0$ and $E_4(\tau) \neq 0$ if $\tau \notin \Gamma(1)\rho$.

If $k = 6$, $f = E_6$, then we get

$$\frac{1}{3} v_\rho(E_6) + \frac{1}{2} v_i(E_6) + \sum_{\tau\not\sim\rho,i} v_\tau(E_6) = \frac{6}{12} = \frac{1}{12},$$

so this forces $v_\rho(E_6) = 0$, $v_i(E_6) = 1$, $v_\tau(E_6) \neq 0$ if $\tau \not\sim \rho$ and $\tau \not\sim i$, so $E_6(i) = 0$, $E_6(\tau) \neq 0$ if $\tau \not\sim \rho, i$.

Recall $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$. This is nonzero since $\Delta(\rho) = \frac{E_4(\rho)^3 - E_6(\rho)^2}{1728} = -\frac{E_6(\rho)^2}{1728} \neq 0$. We also have $v_\infty(\Delta) \geq 1$ by construction, so plug in Δ to our formula to get

$$\sum_\tau \frac{1}{e_\tau} v_\tau(\Delta) + v_\infty(\Delta) = 1,$$

so $v_\infty(\Delta) = 1$, so Δ has a simple zero at ∞ and Δ is nonvanishing in $\mathfrak{h}$.

Theorem 2.5. Let $k \in 2\mathbb{Z}$. Then:

(1) If $k < 0$ or $k = 2$, then $M_k(\Gamma(1)) = 0$; and $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.

(2) If $4 \leq k \leq 10$, then $M_k(\Gamma(1)) = \mathbb{C} \cdot E_k$.

(3) For any k, multiplication by Δ gives an isomorphism $M_k(\Gamma(1)) \xrightarrow{\times\Delta} S_{k+12}(\Gamma(1))$.

Proof. (1) Let $f \in M_k(\Gamma(1))$ be nonzero. Then $\sum \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$. Note the LHS is ≥ 0, but for $k < 0$, the RHS is < 0. If $k = 2$, then we get the equation $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{6}$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, which has no solutions.

Suppose $f \in M_0(\Gamma(1)) \setminus \mathbb{C} \cdot 1$. Then $f - f(\infty) \cdot 1 \in S_0(\Gamma(1))$ is a nonzero function (here 1 is the constant function 1). Then $\sum_\tau \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot 1) + \underbrace{v_\infty(f - f(\infty) \cdot 1)}_{\geq 1} = 0$, a contradiction, so $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.

(2) Let $4 \leq k \leq 10$ and $f \in M_k(\Gamma(1))$. Consider $f - f(\infty) \cdot E_k \in S_k(\Gamma(1))$. If this is nonzero, then

$$\sum_\tau \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot E_k) + \underbrace{v_\infty(f - f(\infty) \cdot E_k)}_{\geq 1} = \frac{k}{12} < 1,$$

a contradiction. So $f = f(\infty) \cdot E_k$.

(3) Our map $\times\Delta : M_k(\Gamma(1)) \to S_{k+12}(\Gamma(1))$ is a well–defined $\mathbb{C}$–linear map. It is injective, since if $\Delta f = 0$, then $f = 0$ (as Δ is nonvanishing in $\mathfrak{h}$). For surjectivity, if $f \in S_{k+12}(\Gamma(1))$, then $\frac{f}{\Delta}$ is holomorphic in $\mathfrak{h}$ and invariant under the weight k action of $\Gamma(1)$.

We need to show $\frac{f}{\Delta}$ is holomorphic at ∞, as then $\frac{f}{\Delta} \in M_k(\Gamma(1))$, so $f = \frac{f}{\Delta} f \in \mathrm{Im}(\times\Delta)$. Hence we need $v_\infty\left(\frac{f}{\Delta}\right) \geq 0$. But $v_\infty\left(\frac{f}{\Delta}\right) = \underbrace{v_\infty(f)}_{\geq 1} - \underbrace{v_\infty(\Delta)}_{=1} \geq 0$, so we're done.

∎

Corollary 2.6. If $k \in 2\mathbb{Z}$, $k \geq 0$, then $M_k(\Gamma(1))$ is finite–dimensional and

$$\dim_{\mathbb{C}} M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12}. \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}. \end{cases}$$

Proof. We proved this for $0 \leq k \leq 10$. In general, use induction on k: we need to show that for $k \geq 0$, $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = \dim_{\mathbb{C}} M_k(\Gamma(1)) + 1$.

We know $E_{k+12} \in M_{k+12}(\Gamma(1))$, so $M_{k+12}(\Gamma(1)) = \mathbb{C}E_{k+12} \oplus S_{k+12}(\Gamma(1))$. But this equals $\mathbb{C}E_{k+12} \oplus \Delta M_k(\Gamma(1))$, so $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$. ∎

Example 2.2. We have $E_4^2 \in M_8(\Gamma(1)) = \mathbb{C}E_8$. So there is a relation between E_4^2 and E_8 (in this case, one is a scalar multiple of the other), but we have $E_8(\infty) = 1 = E_4(\infty)^2 \implies E_4^2 = E_8$.

Similarly, $E_4E_6 \in M_{10}(\Gamma(1)) = \mathbb{C}E_{10}$, so we find $E_4E_6 = E_{10}$.

Corollary 2.7. If $k \in 2\mathbb{N}$, then $M_k(\Gamma(1))$ is spanned as a $\mathbb{C}$–vector space by $\{E_4^aE_6^b \mid a, b \in \mathbb{Z}_{\geq 0}, 4a+6b = k\}$. In other words, if $\mathcal{M} = \oplus_{k\in\mathbb{Z}} M_k(\Gamma(1))$, then $\mathcal{M}$ is a graded $\mathbb{C}$–algebra generated by E_4 and E_6.

Proof. We proved this for $0 \leq k \leq 10$. If $k \geq 12$, then

$$M_k(\Gamma(1)) = \mathbb{C}E_k \oplus \Delta M_{k-12}(\Gamma(1)) = \mathbb{C}f \oplus \Delta M_{k-12}(\Gamma(1))$$

for any $f \in M_k(\Gamma(1))$ such that $f(\infty) \neq 0$ by the same argument. We can always find some $A, B \in \mathbb{Z}_{\geq 0}$ such that $4A+6B = k$, so $E_4^AE_6^B \in M_k(\Gamma(1))$ and $(E_4^AE_6^B)(\infty) \neq 0$. Now by induction, $M_{k-12}(\Gamma(1)) = \langle E_4^aE_6^b \mid 4a+6b = k-12\rangle$, so $\Delta M_{k-12}(\Gamma(1)) = \langle \Delta E_4^aE_6^b \mid 4a+6b = k-12\rangle$. But $\Delta \in \langle E_4^3, E_6^2\rangle$, so

$$\Delta M_{k-12}(\Gamma(1)) = \langle E_4^aE_6^b \mid 4a+6b = k\rangle$$

and $E_4^AE_6^B \in \langle E_4^aE_6^b \mid 4a+6b = k\rangle$, so $M_k(\Gamma(1)) = \langle E_4^aE_6^b \mid 4a+6b = k\rangle$. □

Theorem 2.8. Let $j(\tau) = \frac{E_4(\tau)^3}{\Delta}$. Then j is a modular function of weight 0, level $\Gamma(1)$ which is holomorphic on $\mathfrak{h}$ and has a simple pole at ∞. It defines a bijection $\Gamma(1) \setminus \mathfrak{h} \to \mathbb{C}$ given by $\tau \to j(\tau)$. Moreover, every modular function of weight 0, level $\Gamma(1)$ is a rational function of j.[6]

The interpretation of this is that it is possible to define a Riemann surface structure on $\Gamma(1) \setminus \mathfrak{h} \sqcup \{\infty\}$ such that we get a compact Riemann surface whose meromorphic functions are exactly the modular functions of weight 0. So the theorem says that this Riemann surface, called $X(1)$, is isomorphic to the Riemann sphere, and our formula says that if $\mathcal{L}$ is an invertible sheaf on a compact Riemann surface and S is a meromorphic section, then $\sum_a v_a(S) = \deg(\mathcal{L})$. This is useful if we are also taking algebraic geometry.

18 Oct 2022, Lecture 6

Proof. We showed that Δ is nonvanishing in $\mathfrak{h}$ and has a simple zero at ∞. Hence j is holomorphic in $\mathfrak{h}$ and $v_\infty(j) = 3v_\infty(E_4) - v_\infty(\Delta) = -1$. Note that if $\gamma \in \Gamma(1)$, then $j|_0[\gamma](\tau) = j(\gamma\tau) = j(\tau)$ since the map is constant on $\Gamma(1)$–orbits. To show the map is a bijection, we need to show that $\forall z \in \mathbb{C}$, there exists a unique orbit $\Gamma(1)\cdot\tau$ such that $j(\tau) = z$, i.e. $v_\tau(j-z) > 0$.

We know

$$\sum_{\tau\in\Gamma(1)\setminus\mathfrak{h}} \frac{1}{e_\tau} \underbrace{v_\tau(j-z)}_{\geq 0,\text{ as } j-z \text{ is holomorphic in } \mathfrak{h}.} = 1,$$

[6] Remember that $\Gamma(1) \setminus \mathfrak{h}$ is the set of orbits of $\Gamma(1)$ under $\mathfrak{h}$.

(since $v_\infty(j-z) = -1$ and $\frac{k}{12} = 0$) again giving $\frac{a}{3} + \frac{b}{2} + c = 1$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, $a = v_\rho(j-z), b = v_i(j-z), c = \sum_{\tau \not\sim \rho, i} v_\tau(j-z)$. This gives the solutions

- $(a, b, c) = (0, 0, 1)$, so $j - z$ vanishes at a unique $\Gamma(1) \cdot \tau$.
- $(a, b, c) = (0, 2, 0)$, so $j - z$ vanishes at i.
- $(a, b, c) = (3, 0, 0)$, so $j - z$ vanishes at ρ.

Hence our map is bijective. Consider a nonzero modular function f of weight 0. To get rid of all the poles, we can consider a product $f \cdot \prod_{i=0}^{n} (j(\tau) - j(a_i))^{b_i}$ for $a_i \in \mathfrak{h}$, $b_i \in \mathbb{Z}_{\geq 0}$, where the a_i are among the poles of f in $\mathfrak{h}$. Hence to show f is a rational function of j, it is enough to consider the case where f is holomorphic in $\mathfrak{h}$. Then there exists $m \geq 0$ such that $\Delta^m f$ is holomorphic at ∞, so $\Delta^m f$ is holomorphic in $\mathfrak{h}$ and at ∞, so $\Delta^m f \in M_{12m}(\Gamma(1))$. We showed that $M_{12m}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = 12m \rangle$, so f is a linear combination of functions of the form $\frac{E_4^a E_6^b}{\Delta^m}$, where $4a + 6b = 12m$.

Hence it is enough to show that $\frac{E_4^a E_6^b}{\Delta^m}$ is a rational function of j where $4a + 6b = 12m$, $a, b \in \mathbb{Z}_{\geq 0}$. But then $2a + 3b = 6m$, which gives $p, q \in \mathbb{Z}_{\geq 0}$ such that $a = 3p, b = 2q$, so $p + q = m$. Then

$$\frac{E_4^a E_6^b}{\Delta^m} = \left(\frac{E_4^3}{\Delta}\right)^p \left(\frac{E_6^2}{\Delta}\right)^q = j^p \left(\frac{E_6^2}{\Delta}\right)^q.$$

As $E_4^3 - E_6^2 = 1728\Delta$, we get $j = \frac{E_6^2}{\Delta} + 1728$. So this is a rational function of j. ☐

Proposition 2.9. Let $k \geq 4$ be an even integer. Then

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where $q = e^{2\pi i \tau}$ and $\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$.

Proof. We start from the identity

$$\pi \cot(\pi\tau) = \frac{1}{\tau} + \sum_{n \geq 1} \left(\frac{1}{\tau + n} + \frac{1}{\tau - n}\right).$$

This is true for $\tau \in \mathfrak{h}$ and it is even locally uniformly convergent in $\mathfrak{h}$. We can write

$$\pi \cot(\pi\tau) = i\pi \frac{e^{\pi i\tau} + e^{-\pi i\tau}}{e^{\pi i\tau} - e^{-\pi i\tau}} = \pi i \frac{q+1}{q-1} = -\pi i(1+q)(1-q)^{-1} = -\pi i \left(1 + 2\sum_{n \geq 1} q^n\right).$$

Differentiate term–by–term $k-1$ times. The RHS of the bottom expression is

$$-2\pi i\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{k-1}\left(\sum_{n\geq 1}q^n\right) = -\left(2\pi i\right)^k\sum_{n\geq 1}n^{k-1}q^n,$$

while the RHS of the top expression is

$$(-1)^{k-1}(k-1)!\left(\tau^{-k}+\sum_{n\geq 1}(\tau+n)^{-k}+(\tau-n)^{-k}\right) = (-1)^{k-1}(k-1)!\sum_{n\in\mathbb{Z}}(\tau+n)^{-k}.$$

Rearranging and using the fact that k is even (to make the sign go away) gives

$$\sum_{n\in\mathbb{Z}}(\tau+n)^{-k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}n^{k-1}q^n, \tau\in\mathfrak{h}.$$

Then

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0}(m\tau+n)^{-k} = 2\zeta(k)+\sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus 0,\\ m\neq 0}}(m\tau+n)^{-k} = 2\zeta(k)+2\sum_{m\geq 1}\sum_{n\in\mathbb{Z}}(m\tau+n)^{-k}.$$

Plug in our identity to get

$$G_k(\tau) = 2\zeta(k)+\sum_{m\geq 1}\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}n^{k-1}q^{mn} = 2\zeta(k)+\frac{2(2\pi i)^k}{(k-1)!}\sum_{N\geq 1}\underbrace{\left(\sum_{n\mid N}n^{k-1}\right)}_{=\sigma_{k-1}(N)}q^N.$$

□

Corollary 2.10. $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1+\sum_{n\geq 1}a_nq^n$ has all $a_n\in\mathbb{Q}$. Moreover, if $k=4$ or $k=6$, then $a_n\in\mathbb{Z}$.

Proof. We have

$$E_k(\tau) = 1+\frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Hence we need to show that $\frac{\zeta(k)}{\pi^k}$ is rational. This is on example sheet 1 (when

k is even). One can show that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$, so

$$E_4(\tau) = 1 + \frac{2^4\pi^4 \cdot 90}{\pi^4 \cdot 6} \sum_{n\geq 1} \sigma_3(n)q^n = 1 + 240 \sum_{n\geq 1} \sigma_3(n)q^n$$
$$E_6(\tau) = 1 - \frac{2^6\pi^6 \cdot 3^3 \cdot 5 \cdot 7}{\pi^6 \cdot 5!} \sum_{n\geq 1} \sigma_5(n)q^n = 1 - 504 \sum_{n\geq 1} \sigma_5(n)q^n.$$

□

Corollary 2.11. If $\Delta(\tau) = \sum_{n\geq 1} \tau(n)q^n$ is the q–expansion of Δ, then $\tau(1) = 1$ and $\tau(n) \in \mathbb{Z}\ \forall n \geq 1$.

Proof. Write $E_4 = 1 + 240U$ and $E_6 = 1 - 504V$ for $U, V = q + \ldots \in \mathbb{Z}[[q]]$. Then

$$\begin{aligned}\Delta &= \frac{E_4^3 - E_6^2}{1728} = \frac{(1+240U)^3 - (1-504V)^2}{1728} \\ &= \frac{3\cdot 240U + 3\cdot 240^2U^2 + 240^3U^3 + 2\cdot 504V - 504^2V^2}{1728} \\ &= \frac{(3\cdot 240U + 2\cdot 504V)}{1728} + R,\end{aligned}$$

where we claim $R \in q^2\mathbb{Z}[[q]]$, but for this we just need to check that $1728 \mid 3\cdot 240^2, 1728 \mid 240^3, 1728 \mid 504^2$, which is true.

We need to check that

$$\frac{(3\cdot 240U + 2\cdot 504V)}{1728} = \frac{2^4\cdot 3^2\cdot 5\cdot U + 2^4\cdot 3^2\cdot 7\cdot V}{2^6\cdot 3^3} \in \mathbb{Z}[[q]].$$

But this equals

$$\frac{5U + 7V}{12} = \frac{5(U-V)}{12} + V.$$

Hence we need to check that

$$\frac{5}{12}(\sigma_3(n) - \sigma_5(n)) \in \mathbb{Z}\ \forall n \geq 1,$$

i.e. we need to check that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod{12}\ \forall n \geq 1.$$

But this is true as $d^3 \equiv d^5 \pmod{12}\ \forall d \in \mathbb{N}$.

Finally, we compute $\tau(1) = \frac{3\cdot 240 + 2\cdot 504}{1728} = 1$. □

20 Oct 2022, Lecture 7

Theorem 2.12. Let $k \geq 4$ be even and $N = \dim_{\mathbb{C}} S_k(\Gamma(1))$. Then there exists a unique basis $f_0, \ldots, f_N$ for $M_k(\Gamma(1))$ as a $\mathbb{C}$–vector space such that

(a) $\forall 0 \leq i \leq N$, $f_i = \sum_{n \geq 0} a_n(f_i) q^n$ for $a_n(f_i) \in \mathbb{Z}$ $\forall n \geq 0$.

(b) If $0 \leq i, n \leq N$, then $a_n(f_i) = \delta_{in}$.

So in other words, $f_i = q^i + O(q^{N+1})$. This is important because $M_k(\Gamma(1))$ has a $\mathbb{Z}$–structure, i.e. we can realize it as a tensor product $M_k(\Gamma(1)) = M_k(\Gamma(1), \mathbb{Z}) \oplus \mathbb{C}$, where $M_k(\Gamma(1), \mathbb{Z}) = \{f \in M_k(\Gamma(1)) \mid \forall n \geq 0, a_n(f) \in \mathbb{Z}\}$.

Proof. We first construct $f_0, \ldots, f_N \in M_k(\Gamma(1))$ with properties (a) and (b). Write $k = 12a + d$, for $a, d \in \mathbb{Z}_{\geq 0}$ such that $d = 14$ if $k \equiv 2 \pmod{12}$, or $0 \leq d \leq 10$ if $d \not\equiv 2 \pmod{12}$.

Then

$$\left\lfloor \frac{k}{12} \right\rfloor = \begin{cases} a & k \not\equiv 2 \pmod{12} \\ a+1 & k \equiv 2 \pmod{12} \end{cases} \implies \lfloor a \rfloor = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor - 1 & k \equiv 2 \pmod{12}. \end{cases}$$

We have $\dim_{\mathbb{C}} M_k(\Gamma(1)) = N + 1 = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}, \end{cases}$ so $a = N$, $k = 12N + d$.

Now consider $A, B \in \mathbb{Z}_{\geq 0}$ such that $d = 4A + 6B$. Consider the modular forms

$$g_i = \Delta^i E_4^A E_6^B E_6^{2(N-i)}$$

for $0 \leq i \leq N$. Each g_i has weight $12i + 4A + 6B + 12(N - i) = 12N + d = k$, so $g_i \in M_k(\Gamma(1))$. As E_4, E_6, Δ have q–expansions in $\mathbb{Z}[[q]]$, so does g_i. The leading term of g_i is q^i, so the q–expansions look like

$$\begin{aligned} g_0 &= 1 + a_1(g_0)q + \ldots + a_N(g_0)q^N + O(q^{N+1}) \\ &\vdots \\ g_{N-1} &= 0 + \ldots + q_{N-1} + a_N(g_{N-1})q^N + O(q^{N+1}) \\ g_N &= 0 + \ldots + 0 + q^N + O(q^{N+1}) \end{aligned}$$

We can now carry out row reduction on the g_i to obtain $f_0, \ldots, f_N$ satisfying (a) and (b). For uniqueness, consider the linear functionals

$$\begin{aligned} a_0, \ldots, a_N &: M_k(\Gamma(1)) \to \mathbb{C} \\ f &\mapsto a_i(f), \ f = \sum_{n \geq 0} a_n(f) q^n. \end{aligned}$$

Then $a_i(f_j) = \delta_{ij}$, which forces $a_0, \ldots, a_n$ to be linearly independent. Hence they form a basis of the dual vector space $M_k(\Gamma(1))^*$. So $f_0, \ldots, f_N$ is the dual basis of $M_k(\Gamma(1))$, and they form the unique basis with this property. $\square$

3 Hecke operators

Hecke operators are just symmetries (linear endomorphisms) of spaces of modular forms. They can arise from either representation theory: $\Gamma(1) \leq GL_2(\mathbb{Q})^+$, which acts on $\{f : \mathfrak{h} \to \mathbb{C}\}$ by $f \mapsto f|_k[g]$. But $M_k(\Gamma(1)) \leq \{f : \mathfrak{h} \to \mathbb{C}\}^{\Gamma(1)}$, and a general group theory fact says that under suitable conditions, there's an action by a big class of operators; or from geometry: we can think of modular forms as functions on the set of lattices $\mathcal{L}$ in $\mathbb{C}$. In this course, we will follow the second point of view.

Recall. If V is a finite–dimensional $\mathbb{R}$–vector space, then a lattice Λ in V is a subgroup $\Lambda \subset V$ which is discrete and cocompact (i.e. V/Λ is compact).

Lemma 3.1. A subgroup $\Lambda \leq V$ is a lattice if and only if there exists a basis $e_1, \dots, e_n$ for V as a $\mathbb{R}$–vector space such that $\Lambda = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$.

Proof. This is a question on example sheet 2. □

We study $\mathcal{L} = \{\Lambda \leq \mathbb{C} \text{ a lattice}\}$ with its action by $\mathbb{C}^\times$, i.e. $z\Lambda = \{z\lambda \mid \lambda \in \Lambda\}$ for $z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}$.

Proposition 3.2. The map $\tau \mapsto \Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ induces a bijection between

$$\Gamma(1) \setminus \mathfrak{h} \leftrightarrow \mathbb{C}^\times \setminus \mathcal{L}$$

(orbits of $\Gamma(1)$ in $\mathfrak{h}$ and the set of lattices in $\mathbb{C}$ modulo scalar multiplication).

Proof. This map is well–defined, since if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, $\tau \in \mathfrak{h}$, then

$$\Lambda_{\gamma\tau} = \mathbb{Z}\left(\frac{a\tau + b}{c\tau + d}\right) \oplus \mathbb{Z} = (c\tau + d)^{-1}\left(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)\right) = (c\tau + d)^{-1}\Lambda_\tau.$$

For surjectivity, if Λ is a lattice, then $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\operatorname{Im}\left(\frac{e_1}{e_2}\right) \neq 0$. Swapping e_1, e_2 if necessary, we may assume that $\operatorname{Im}\left(\frac{e_1}{e_2}\right) > 0$. Then $\Lambda = e_2(\mathbb{Z}e_1/e_2 \oplus \mathbb{Z}) = e_2\Lambda_\tau$ for $\tau = \frac{e_1}{e_2}$.

For injectivity, if τ, τ' have the same image, then $\exists z \in \mathbb{C}^\times$ such that $z\Lambda_\tau = \Lambda_{\tau'}$, i.e. $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $\tau' = az\tau + bz, 1 = cz\tau + dz$. Then $\tau' = \frac{az\tau + bz}{cz\tau + dz} = \frac{a\tau + b}{c\tau + d}$. But $\operatorname{Im}(\tau') = \operatorname{Im}(\gamma\tau) = \det(\gamma)\frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}$ and $\operatorname{Im}(\tau) > 0, \operatorname{Im}(\tau') > 0$, hence $\det(\gamma) > 0$, so $\det(\gamma) = 1$ and so $\gamma \in \Gamma(1)$. □

Definition 3.1. If $k \in \mathbb{Z}$, say a function $F : \mathcal{L} \to \mathbb{C}$ is **of weight** k if $\forall z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}$, $F(z\Lambda) = z^{-k}F(\Lambda)$.

Proposition 3.3. Let

$$V_k = \{F : \mathcal{L} \to \mathbb{C} \text{ of weight } k\}.$$
$$W_k = \{f : \mathfrak{h} \to \mathbb{C} \mid \forall \gamma \in \Gamma(1), f|_k[\gamma] = f\}.$$

Then the map $F \mapsto (f : \tau \mapsto F(\Lambda\tau))$ induces a $\mathbb{C}$-vector space isomorphism $V_k \to W_k$.

Proof. We first check that if $F \in V_k$, $f(\tau) = F(\Lambda\tau)$, then $f \in W_k$. If $\gamma \in \Gamma(1)$,

$$f|_k[g](\tau) = f(\gamma\tau)j(\gamma,\tau)^{-k} = F(\lambda\gamma\tau)j(\gamma,\tau)^{-k} = F(j(\gamma,\tau)\Lambda_{\gamma\tau}) = F(\Lambda\tau) = f(\tau),$$

so $j(\gamma,\tau)\Lambda_{\gamma\tau} = \Lambda_\tau$.

To show that the map is an isomorphism, we write down its inverse: define $\alpha : W_k \to V_k$ by $\alpha(f)(\Lambda) = e_2^{-k} f(e_1/e_2)$ if $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\operatorname{Im}(e_1/e_2) > 0$. This is well–defined, since if e_1', e_2' is another basis with $\operatorname{Im}(e_1'/e_2') > 0$, then $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ such that $e_1' = ae_1 + be_2, e_2' = ce_1 + de_2$. Then

$$\begin{aligned} e_2'^{-k} f(e_1'/e_2') &= (ce_1 + de_2)^{-k} f\left(\frac{ae_1 + be_2}{ce_1 + de_2}\right) \\ &= e_2^{-k}(ce_1/e_2 + d)^{-k} f\left(\frac{ae_1/e_2 + b}{ce_1/e_2 + d}\right) = e_2^{-k} f\left(\frac{e_1}{e_2}\right). \end{aligned}$$

Exercise: check that the two maps are inverse to each other. □

23 Oct 2022, Lecture 8

Definition 3.2. Let $n \in \mathbb{N}$. The n^{th} Hecke operator $T_n : V_k \to V_k$ is defined by the formula

$$(T_n F)(\Lambda) = n^{k-1} \sum_{\Lambda' \underset{n}{\leq} \Lambda} F(\Lambda').$$

Here $\sum_{\Lambda' \underset{n}{\leq} \Lambda}$ means summing over all subgroups Λ' of Λ of index n.

We also write $T_n : W_k \to W_k$ for the endomorphism arising from the isomorphism $V_k \xrightarrow{\sim} W_k$.

Why is T_n a well–defined endomorphism of V_k? First of all, the sum is finite since there's a bijection

$$\begin{aligned} \{\Lambda' \leq \Lambda\} &\leftrightarrow \{H \leq \Lambda/n\Lambda \text{ of index } n\} \\ \Lambda' &\mapsto \Lambda'/n\Lambda \\ H + n\Lambda &\leftarrow H \end{aligned}$$

This is well–defined, since Lagrange's theorem implies that

$$\Lambda' \underset{n}{\leq} \Lambda \implies n(\Lambda/\Lambda') = 0 \implies n\Lambda \leq \Lambda'.$$

But $\Lambda/n\Lambda \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is finite, so it has finitely many subgroups of index n.

If $\Lambda' \underset{n}{\leq} \Lambda$, then $n\Lambda \leq \Lambda' \leq \Lambda$, so Λ' is also discrete and cocompact in $\mathbb{C}$.

We next check that T_nF is of weight k, i.e. that $(T_nF)(z\Lambda) = z^{-k}(T_nF)(\Lambda)$. We have an isomorphism $\{\Lambda' \underset{n}{\leq} z\Lambda\} \leftrightarrow \{\Lambda' \underset{n}{\leq} \Lambda\}$ given by $\Lambda' \mapsto z^{-1}\Lambda'$, so

$$(T_nF)(z\Lambda) = n^{k-1} \sum_{\Lambda' \underset{n}{\leq} z\Lambda} F(\Lambda') = n^{k-1} \sum_{\Lambda' \underset{n}{\leq} \Lambda} F(z\Lambda') = n^{k-1} \sum_{\Lambda' \underset{n}{\leq} \Lambda} z^{-k} F(\Lambda') = z^{-k}(T_nF)(\Lambda).$$

Proposition 3.4. (1) If $m, n \in \mathbb{N}$ with $(m,n) = 1$, then $T_mT_n = T_{mn}$.

(2) If p is a prime number and $n \in \mathbb{N}$, then $T_{p^n}T_p = T_{p^{n+1}} + p^{k-1}T_{p^{n-1}}$ (acting on V_k).

Proof. Let $m, n \in \mathbb{N}$, not necessarily coprime. Then

$$\begin{aligned}(T_m(T_nF))(\Lambda) &= m^{k-1} \sum_{\Lambda' \underset{m}{\leq} \Lambda} (T_nF)(\Lambda') = (mn)^{k-1} \sum_{\Lambda' \underset{m}{\leq} \Lambda} \sum_{\Lambda'' \underset{n}{\leq} \Lambda'} F(\Lambda'') \\ &= (mn)^{k-1} \sum_{\Lambda'' \underset{mn}{\leq} \Lambda} a(\Lambda, \Lambda'')F(\Lambda''),\end{aligned}$$

where $a(\Lambda, \Lambda'') = |\{\Lambda \underset{m}{\geq} \Lambda' \underset{n}{\geq} \Lambda''\}| = |H \leq \Lambda/\Lambda'' \mid |H| = n|$ is the number of ways to express Λ' as an intermediate subgroup. If $(m,n) = 1$, then $a(\Lambda, \Lambda'') = 1$ for all $\Lambda'' \leq \Lambda$ as any finite abelian group of order mn has a unique subgroup of order n.

(1) In this case, we find

$$T_mT_nF(\Lambda) = (mn)^{k-1} \sum_{\Lambda'' \underset{mn}{\leq} \Lambda} F(\Lambda'') = (T_{mn}F)(\Lambda) \implies T_mT_n = T_{mn}.$$

(2) The same computation gives (for p prime, $n \in \mathbb{N}$)

$$(T_{p^n}(T_pF))(\Lambda) = p^{(n+1)(k-1)} \sum_{\Lambda'' \underset{p^{n+1}}{\leq} \Lambda} a(\Lambda, \Lambda'')F(\Lambda''),$$

where $a(\Lambda, \Lambda'') = |\{H \subset \Lambda/\Lambda'' \mid |H| = p\}|$. But if $\Lambda'' \underset{p^{n+1}}{\leq} \Lambda$, then Λ/Λ'' need not have a unique subgroup of order p, as $\Lambda \cong \mathbb{Z}^2$, so Λ/Λ'' is a finite abelian group of order p^{n+1} that can be generated by 2 elements. But any such group is isomorphic to $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$, where $a \geq b \geq 0$ are integers such that $a + b = n + 1$. We now split into two cases:

- $b = 0$, so $a = n + 1$ and $\Lambda/\Lambda'' \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$. This group is cyclic and has a unique subgroup of order p, so $a(\Lambda, \Lambda'') = 1$.
- $b > 0$, so $\Lambda/\Lambda'' \cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$. Let $\Lambda/\Lambda''[p] = \{x \in \Lambda/\Lambda'' \mid px = 0\}$. This is a subgroup of Λ/Λ'', and

$$\{H \leq \Lambda/\Lambda'' \mid |H| = p\} = \{H \leq \Lambda/\Lambda''[p] \mid |H| = p\}.$$

  Hence $\Lambda/\Lambda''[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ from our above isomorphism. So in this case, $a(\Lambda, \Lambda'') = |\{H \leq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \mid |H| = p\}|$. In other words,

$$a(\Lambda, \Lambda') = |\mathbb{P}^1(\mathbb{F}_p)| = |\mathbb{A}^1(\mathbb{F}_p) \cup \{\infty\}| = p + 1.$$

How do we distinguish between these two cases? We will show on example sheet 2 that if $\Lambda'' \underset{p^{n+1}}{\leq} \Lambda$, then there exists a $\mathbb{Z}$–basis e_1, e_2 for Λ such that $\Lambda'' = \mathbb{Z}p^a e_1 \oplus \mathbb{Z}p^b e_2$ for the same a, b satisfying $a \geq b \geq 0, a + b = n + 1$ as before (this is a consequence of Smith normal form).

Hence we see that we are in case 2 if and only if $\Lambda'' \leq p\Lambda$. Thus we find

$$(T_{p^n}(T_pF)(\Lambda)) = p^{(n+1)(k-1)} \sum_{\Lambda'' \underset{p^{n+1}}{\leq} \Lambda} F(\Lambda'') + p^{(n+1)(k-1)} \sum_{\Lambda'' \underset{p^{n-1}}{\leq} p\Lambda} pF(\Lambda'').$$

Here each Λ'' in case 1 goes once into the first sum and each Λ'' in case 2 goes once into the first sum and p times into the second sum. We have

$$\begin{aligned} p^{(n+1)(k-1)} \sum_{\Lambda'' \underset{p^{n-1}}{\leq} p\Lambda} pF(\Lambda'') &= p^{(n-1)(k-1)} p^{2(k-1)} \sum_{\Lambda'' \underset{p^{n-1}}{\leq} \Lambda} pF(p\Lambda'') \\ =& p^{(n-1)(k-1)} p^{2(k-1)} p^{1-k} \sum_{\Lambda'' \underset{p^{n-1}}{\leq} \Lambda} F(\Lambda'') = p^{k-1} T_{p^{n-1}} F(\Lambda). \end{aligned}$$

Hence $T_{p^n} T_p F(\Lambda) = T_{p^{n+1}} F(\Lambda) + p^{k-1} T_{p^{n-1}} F(\Lambda)$.

□

Corollary 3.5. $\forall m, n \in \mathbb{N}$, $T_m T_n = T_n T_m$ as endomorphisms of V_k, i.e. all Hecke operators commute.

Proof. If we write $m = \prod_{i=1}^r p_i^{a_i}$ for $a_i \geq 1$, p_i distinct, then $T_m = T_{p_1^{a_1}} \dots T_{p_r^{a_r}}$. We've shown that if p, q are distinct primes, then T_{p^a}, T_{q^b} commute $\forall a, b \geq 1$.

We need to show that if p is a prime and $a, b \in \mathbb{N}$, then T_{p^a} and T_{p^b} commute. But we have a stronger claim that $\forall a \in \mathbb{N}$, T_{p^a} is a polynomial in T_p. We prove this by induction on a, the case $a = 1$ being trivial.

In general, $T_{p^{a+1}} = T_{p^a} T_p - p^{k-1} T_{p^{a-1}}$, which proves the claim. $\square$

25 Oct 2022, Lecture 9

Lemma 3.6. Let $n \in \mathbb{N}$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \leq \mathbb{C}$ a lattice. Then $\{\Lambda' \underset{n}{\leq} \Lambda\} = \{\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2 \mid a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, b < d\}$, where this is isomorphic to the set $\{a, b, d \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$.

Proof. Consider the short exact sequence

$$0 \to \mathbb{Z}e_2/\mathbb{Z}e_2 \cap \Lambda' \to \Lambda/\Lambda' \to \underbrace{\Lambda/\mathbb{Z}e_2 + \Lambda'}_{\cong \mathbb{Z}e_1/\mathbb{Z}e_1 \cap (\mathbb{Z}e_2 + \Lambda)} \to 0.$$

Then $|\Lambda/\Lambda'| = n$. We let $d = |\mathbb{Z}e_2/\mathbb{Z}e_2 \cap \Lambda'| = \inf\{d \geq 1 \mid de_2 \in \Lambda'\}$ and $a = |\Lambda/\mathbb{Z}e_2 + \Lambda'| = \inf\{a \geq 1 \mid \exists b \in \mathbb{Z} \text{ s.t. } ae_1 + be_2 \in \Lambda'\}$. Then $n = ad$ and there exists a unique $0 \leq b < d$ such that $ae_1 + be_2 \in \Lambda'$.

We now claim that $\Lambda' = \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$. The inclusion $\geq$ is clear. On the other hand, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z})$, $\alpha\delta - \beta\gamma = N \in \mathbb{Z}$ is nonzero, then $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = |N|$. So $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = n = [\Lambda : \Lambda']$, so $[\Lambda' : \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2] = 1$, so they're equal.

We've defined a map $\{\Lambda' \underset{n}{\leq} \Lambda\} \to \{(a, b, d) \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$. This map has an inverse, given by $(a, b, d) \mapsto \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$, so it's a bijection. $\square$

Lemma 3.7. Let $f \in W_k$. Then we have the two formulas

$$(T_n f)(\tau) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) = \sum_{\substack{ad=n \\ 0 \leq b < d}} f|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right].$$

Proof. $f \leftrightarrow F \in V_k$ with $f(\tau) = F(\Lambda_\tau)$. By definition,

$$(T_n f)(\tau) = (T_n F)(\Lambda_\tau) = n^{k-1} \sum_{\Lambda' \underset{n}{\leq} \Lambda_\tau} F(\Lambda') = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} F(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}d).$$

This equals

$$n^{k-1} \sum_{a,b,d} F(d(\mathbb{Z}(\frac{a\tau + b}{d} \oplus \mathbb{Z}))) = n^{k-1} \sum_{a,b,d} d^{-k} F(\Lambda_{\frac{a\tau+b}{d}}) = n^{k-1} \sum_{a,b,d} d^{-k} f(\frac{a\tau + b}{d}).$$

For the second formula, recall that if $g \in GL_2(\mathbb{R})^+$, then $f|_k[g] = \det(g)^{k-1} f(g\tau) j(g,\tau)^{k-1}$, so

$$f|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right] (\tau) = n^{k-1} f\left(\frac{a\tau + b}{d} \right) d^{-k}.$$

Hence $(T_n f)(\tau) = \sum_{a,b,d} f|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]$. □

Corollary 3.8. If $f \in W_k$ and f is holomorphic, then $T_n f$ is also holomorphic.

Proof. Look at the formula above: $T_n f$ is a finite sum of holomorphic functions. □

Proposition 3.9. Let $f \in W_k$ be holomorphic in $\mathfrak{h}$ with q–expansion $f(\tau) = \sum_{m \in \mathbb{Z}} b_m q^m$. Then $T_n f$ has q–expansion $T_n f = \sum_{m \in \mathbb{Z}} c_m q^m$, where

$$c_m = \sum_{\substack{a \in \mathbb{N} \\ a|(m,n)}} a^{k-1} b_{(mn/a^2)}.$$

Proof.

$$\begin{aligned} T_n f =& n^{k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} d^{-k} f\left(\frac{a\tau + b}{d} \right) = n^{k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a\tau/d} e^{2\pi i m b\tau/d} \\ =& n^{k-1} \sum_{ad=n} d^{-k} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a \tau/d} \left(\sum_{0 \le b < d} e^{2\pi i m b/d} \right). \end{aligned}$$

Note that $\sum_{0 \le b < d} e^{2\pi i m b/d} = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$. Hence

$$T_n f = n^{k-1} \sum_{ad=n} d^{1-k} \sum_{m \in \mathbb{Z}} b_{dm} e^{2\pi i a m \tau}.$$

This gives

$$T_n f = \sum_{ad=n} \left(\frac{n}{d} \right)^{k-1} \sum_{m \in \mathbb{Z}} b_{dm} q^{am} = \sum_{a|n} a^{k-1} \sum_{m \in \mathbb{Z}} b_{nm/a} q^{am}.$$

This equals $\sum_{N \in \mathbb{Z}} c_N q^N$, where $c_N = \sum_{\substack{a|m \\ a|n}} a^{k-1} b_{nN/a^2}$. □

Theorem 3.10. T_n preserves the subspaces $S_k(\Gamma(1)) \le M_k(\Gamma(1)) \le W_k \forall n \ge 1$. Moreover, if $f \in M_k(\Gamma(1))$, then $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$ and $a_1(T_n f) = a_n(f)$.

Proof. To show that T_n preserves $M_k(\Gamma(1))$, we need to show that if $f \in M_k(\Gamma(1))$, then $T_n f$ is holomorphic in $\mathfrak{h}$ (then we're done by the previous corollary) and at ∞, i.e. $a_N(T_n f) = 0$ if $N < 0$.

But $a_N(T_n f) = \sum_{a|(N,n)} a^{k-1} a_{Nn/a^2}(f)$. Since $Nn/a^2 < 0$ and f is holomorphic at ∞, all summands are 0, so $T_n f$ is holomorphic at in ∞.

We have $a_0(T_n f) = \sum_{a|(n,0)} a^{k-1} a_{n\cdot 0/a^2}(f) = \sum_{a|n} a^{k-1} a_0(f) = \sigma_{k-1}(n) a_0(f)$.

Also $a_1(T_n f) = \sum_{a|(n,1)} a^{k-1} a_{n\cdot 1/a^2}(f) = a_n(f)$.

Finally, if $f \in S_k(\Gamma(1))$, then $a_0(f) = 0$, and then $T_n f \in M_k(\Gamma(1))$ and $a_0(T_n f) = \sigma_{k-1}(n) a_0(f) = 0 \implies T_n f \in S_k(\Gamma(1))$. □

Our next goal is to study the spectral decomposition of Hecke operators on $M_k(\Gamma(1))$, i.e. the decomposition of $M_k(\Gamma(1))$ as a sum of (simultaneous) generalized eigenspaces for the T_n.

The simplest case is when $M_k(\Gamma(1))$ or $S_k(\Gamma(1))$ is 1–dimensional (as then every nonzero element is an eigenvector). For example, $S_{12}(\Gamma(1))$ is 1–dimensional, spanned by $\Delta(\tau) = \sum_{n\geq 1} \tau(n) q^n$. So Δ is a T_n–eigenvector for all $n \geq 1$. If $T_n\Delta = \alpha_n \Delta$ for some $\alpha_n \in \mathbb{C}$, then $a_1(T_n\Delta) = a_1(\alpha_n\Delta) = \alpha_n a_1(\Delta) = \alpha_n$ (as we proved $a_1(\Delta) = 1$). But we also have $a_1(T_n\Delta) = a_n(\Delta) = \tau(n)$. Hence $\alpha_n =$ Hecke eigenvalue $= \tau(n) =$ coefficient of q^n.

Ramanujan conjectured in 1916 that τ is multiplicative and $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$ for p prime, $n \in \mathbb{N}$. These identities are true for Hecke operators (i.e. $T_{mn} = T_m T_n$ and $T_{p^{n+1}} = T_p T_{p^n} - p^{k-1} T_{p^{n-1}}$), hence also for the eigenvalues α_n, hence for the numbers $\tau(n)$.

Our goal now is to study the spectral decomposition of $M_k(\Gamma(1))$ and the arithmetic properties of Hecke eigenvalues. 27 Oct 2022, Lecture 10

Definition 3.3. If $f \in M_k(\Gamma(1))$, we say f is an **eigenform** if f is a T_n–eigenvector $\forall n \geq 1$.

We say f is a **normalized eigenform** if $a_1(f) = 1$.

Lemma 3.11. Suppose $k > 0$. Then any eigenform $f \in M_k(\Gamma(1))$ is a scalar multiple of a unique normalized eigenform. Moreover, if f is normalized, then $T_n(f) = a_n(f) f$ $\forall n \geq 1$. (In other words, the n^{th} Hecke eigenvalue = the n^{th} q–expansion coefficient).

For example, Δ is a normalized eigenform and $\tau(n)\Delta = T_n\Delta$.

Proof. We know $a_1(T_n f) = a_n(f)$. We need to show that if f is an eigenform, then $a_1(f) \neq 0$ (as then $f/a_1(f)$ is normalized). But if $a_1 = 0$ and α_n is the eigenvalue of T_n on f, then $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = 0$ $\forall n \geq 1$.

Then $f = \sum_{n\geq 0} a_n(f)q^n = a_0(f)$, which is a contradiction as constants are not modular forms of weights $k > 0$.

If f is normalized, then $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = \alpha_n$. $\square$

Proposition 3.12. Let $k \geq 4$ be even. Then $G_k(\tau)$ is an eigenform.

Proof. We need to show that G_k is a T_n–eigenvector $\forall n \geq 1$. We know T_n is a polynomial in T_p for p ranging over $p \mid n$ for p prime. Hence it is enough to show that G_k is a T_p–eigenvector $\forall p$ prime.

$G_k(\tau) = G_k(\Lambda_\tau)$ for $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$. Then

$$(T_pG_k)(\Lambda) = p^{k-1} \sum_{\Lambda' \underset{p}{\leq} \Lambda} G_k(\Lambda') = p^{k-1} \sum_{\Lambda' \underset{p}{\leq} \Lambda} \sum_{\lambda \in \Lambda' \setminus 0} \lambda^{-k} = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} a(\Lambda, \lambda)\lambda^{-k}$$

where $a(\Lambda, \lambda) = |\{\Lambda' \underset{p}{\leq} \Lambda \mid \lambda \in \Lambda'\}|$. We know that if $\Lambda' \underset{p}{\leq} \Lambda$, then $p\Lambda \leq \Lambda' \leq \Lambda$ and we have a bijection $\{\Lambda' \underset{p}{\leq} \Lambda\} \leftrightarrow \{H \leq \Lambda/p\Lambda \mid |H| = p\}$.

If $\lambda \in p\Lambda$, then $\{\Lambda' \underset{p}{\leq} \Lambda \mid \lambda \in \Lambda'\} = \{\Lambda' \underset{p}{\leq} \Lambda\}$, so $a(\Lambda, \lambda) = p + 1$.

If $\lambda \notin p\Lambda$, then $\lambda \neq 0$ modulo $p\Lambda$ and there exists a unique subgroup $H \leq \Lambda/p\Lambda$ of order p such that $\lambda \in H$. Hence in this case, $\{\Lambda' \underset{p}{\leq} \Lambda\} = \{\mathbb{Z}\lambda + p\Lambda\}$ and $a(\Lambda, \lambda) = 1$. Hence

$$(T_pG_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in p\Lambda \setminus 0} p\lambda^{-k}.$$

We get

$$(T_pG_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} p(p\lambda)^{-k} = p^{k-1}G_k(\Lambda) + G_k(\Lambda) = \sigma_{k-1}(p)G_k(\Lambda).$$

$\square$

We can compute the T_n–eigenvalues on G_k for all n now using $a_0(T_nf) = \sigma_{k-1}(n)a_0(f)$. So if f is an eigenform and $a_0(f) \neq 0$, then this forces the eigenvalue to be equal to $\sigma_{k-1}(n)$. So $T_nG_k = \sigma_{k-1}(n)G_k$ $\forall n \geq 1$. The q–expansion of G_k is $2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n)q^n$ and we also defined $E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n)q^n$. Hence $a_0(E_k) = 1$, but E_k is not a normalized eigenform. Hence the associated normalized eigenform is

$$F_k(\tau) = \frac{\zeta(k)(k-1)!}{(2\pi i)^k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n = \frac{-B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}(n)q^n$$

(here we gave multiple equivalent expressions).

We have a decomposition $M_k(\Gamma(1)) = \mathbb{C}F_k \oplus S_k(\Gamma(1))$ (for $k \geq 4$). Both summands are T_n–invariant, so it's enough to study the action of T_n on S_k.

Remark. T_n do not usually respect multiplication. In particular, the product of eigenforms is not usually an eigenform. For example, $E_4^2 = E_8$, but $E_4^3 \in M_{12}(\Gamma(1)) = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$ requires both E_{12} and $\mathbb{C}$ to be expressed and hence is not an eigenform.

Proposition 3.13. If $f \in S_k(\Gamma(1))$ is a cuspidal eigenform, then all of the T_n–eigenvalues on f are algebraic integers. If f is normalized, then $\mathbb{Q}(\{a_n(f)\}_{n=1}^{\infty})$ has finite degree over $\mathbb{Q}$ (i.e. it is a number field).

Proof. We will show that for all $n \geq 1$, all eigenvalues of T_n on $S_k(\Gamma(1))$ are algebraic integers. We will do this by showing that the characteristic polynomial of T_n acting on S_k has integer coefficients (and it is of course monic).

We consider the basis $f_1, \dots, f_N$ for $S_k(SL_2(\mathbb{Z}))$ characterized by:

- $\forall 1 \leq i \leq N$ and $\forall n \geq 1$, $a_n(f_i) \in \mathbb{Z}$.
- $\forall 1 \leq i, n \leq N$, $a_n(f_i) = \delta_{in}$.

Recall that this meant that $f_1, \dots, f_N$ was the dual basis to the basis of functionals $a_1, \dots, a_N$ of $S_k(\Gamma(1))^*$. Hence $\forall f \in S_k(\Gamma(1))$, $f = \sum_{i=1}^N a_i(f) f_i$ (this identity holds for any elements of a finite dimensional vector space with its basis and dual basis)

The claim is that if A denotes the matrix of T_n in the basis of $f_1, \dots, f_N$, then A has integer entries. As the characteristic polynomial of T_n is $\det(X \cdot I - A)$, this will show that the characteristic polynomial has coefficients in $\mathbb{Z}$.

By definition, $T_n(j) = \sum_{i=1}^N A_{ij} f_i$. Then for $1 \leq m \leq N$,

$$a_m(T_n f_j) = \sum_{i=1}^N A_{ij} a_m(f_j) = \sum_{i=1}^N A_{ij} \delta_{im} = A_{mj}.$$

But $a_m(T_n f_j) = \sum_{a|(m,n)} a^{k-1} a_{mn/a^2}(f_j)$ by the formula from the last lecture. Note that each $a_{mn/a^2}(f_j)$ is in $\mathbb{Z}$ by the definition of f_j, so $\forall m, j$, $A_{mj} \in \mathbb{Z}$.

If f is a normalized eigenform, $f = \sum_{i=1}^N a_i(f) f_i$, then $\forall n \geq 1$, $a_n(f) = \sum_{i=1}^N a_i(f) \underbrace{a_n(f_i)}_{\in \mathbb{Z}}$. Hence $\mathbb{Q}(\{a_n(f)\}_{n \geq 1}) = \mathbb{Q}(\{a_n(f)\}_{n=1}^N)$ has finite degree over $\mathbb{Q}$. ∎

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We can use this argument to compute Hecke eigenvalues.

Example 3.1. Take $k = 24$. We will compute the eigenvalues of T_{24} acting on $S_{24}(\Gamma(1))$. $S_{24}(\Gamma(1))$ has a unique basis f_1, f_2 with $f_1 = q + O(q^3)$ and $f_2 = q^2 +$

$O(q^3)$. For any $f \in S_{24}(\Gamma(1))$, we have $f = a_1(f)f_1 + a_2(f)f_2$. So in particular, $T_2 f_1 = a_1(T_2 f_1)f_1 + a_2(T_2 f_1)f_2$. We know $a_m(T_n f) = \sum_{a|(m,n)} a^{k-1} a_{mn/a^2}(f)$, so

$$T_2 f_1 = a_1(T_2 f_1)f_1 + a_2(T_2 f_1)f_2 = a_2(f_1)f_1 + (a_4(f_1) + 2^{23}a_1(f_2))f_2 = (a_4(f_1) + 2^{23})f_2.$$

Similarly we get

$$T_2 f_2 = a_2(f_2)f_1 + (a_4(f_2) + 2^{23}a_1(f_2))f_2 = f_1 + a_4(f_2)f_2.$$

In fact,

$$\begin{aligned} f_1 &= \Delta E_6^2 + 1032\Delta^2 = q + 195660q^3 + 12080128q^4 + \dots \\ f_2 &= \Delta^2 = q^2 - 48q^3 + 1080q^4 + \dots \end{aligned}$$

So the matrix of f_2 is

$$\begin{pmatrix} 0 & 1 \\ 20468736 & 1080 \end{pmatrix},$$

so the eigenvalues of T_2 on $S_{24}(\Gamma(1))$ are $12(45 \pm \sqrt{144169})$. Hence $S_{24}(\Gamma(1))$ has a basis of normalized eigenforms g_1, g_2 with q–expansion coefficients in $K_{g_i} = \mathbb{Q}(\sqrt{144169})$ (sidenote: this is a prime number).

Definition 3.4. Let $f : \mathfrak{h} \to \mathbb{C}$ be a continuous function that is invariant under the weight 0 action of $\Gamma(1)$, i.e. $f(\gamma\tau) = f(\tau)\ \forall \gamma \in \gamma(1)$. We define

$$\int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\mathfrak{f}'} f(\tau)\frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

(where $\tau = x + iy$).

The motivation for this is that the area form $\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$ on $\mathfrak{h}$ is invariant under $GL_2(\mathbb{R})^+$ (i.e. $g^*(\omega) = \omega\ \forall g \in GL_2(\mathbb{R})^+$). We'd like to say that $\Gamma(1) \setminus \mathfrak{h} \cong \mathbb{C}$ is a manifold where $\omega = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$ descends to $\Gamma(1) \setminus \mathfrak{h}$, so we could use integration on manifolds. This has the following problems:

- We don't assume any knowledge of differential geometry. (In general, if we have a manifold (M, ω), then we have a volume form $\int_M \omega$).
- ω does not descend to $\Gamma(1) \setminus \mathfrak{h}$, because $\Gamma(1)/\{\pm I\}$ has fixed points in $\mathfrak{h}$. The solution for this is to choose a finite order subgroup $\Gamma \leq \Gamma(1)$ with no nontrivial elements of finite order. Then ω will descend to ω_Γ on $\Gamma \setminus \mathfrak{h}$ and $\frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma\setminus\mathfrak{h}} f\omega_\Gamma$ will be independent of the choice of Γ.

Lemma 3.14. Let $f, g \in S_k(\Gamma(1))$. Then the function $f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k$ is invariant under the weight 0 action of $\Gamma(1)$ and the integral

$$\int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\overline{\gamma(\tau)}\mathrm{Im}(\tau)^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

converges absolutely.

Proof. If $\gamma \in \Gamma(1)$, $f(\gamma\tau) = f(\tau)j(\gamma,\tau)^k$ and $\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|j(\gamma,\tau)|^2}$. So

$$f(\gamma\tau)\overline{g(\gamma\tau)}\mathrm{Im}(\gamma\tau)^k = f(\tau)\overline{g(\tau)}j(\gamma,\tau)^k\overline{j(\gamma,\tau)}^k\mathrm{Im}(\tau)^k\frac{1}{|j(\gamma,\tau)|^{2k}} = f(\tau)g(\tau)\mathrm{Im}(\tau)^k.$$

If $f(\tau) = \tilde{f}(q)$ for $\tilde{f} : D(0,1) \to \mathbb{C}$ holomorphic and vanishing at 0, then $\tilde{f}(q) = qh(q)$ for $h : D(0,1) \to \mathbb{C}$ holomorphic. Hence $\forall\delta \in (0,1)$, $\exists C_\delta > 0$ such that $|h(q)| \le C_\delta$ if $0 \le |q| \le \delta$. Hence $|\tilde{f}(q)| \le |q|C_\delta$ if $0 \le |q| \le \delta$.

So $\forall R \ge 0$, $\exists C_{f,R} > 0$ such that $\forall\tau \in \mathfrak{h}$ such that $\mathrm{Im}(\tau) \ge R$, $|f(\tau)| \le |q|C_{f,R} = e^{-2\pi\mathrm{Im}(\tau)}C_{f,R}$. So

$$\int_{\Gamma(1)\backslash\mathfrak{h}} \left|f(\tau)\overline{\gamma(\tau)}\mathrm{Im}(\tau)^k\right| \frac{\mathrm{d}x\mathrm{d}y}{y^2} \le \int_{\mathfrak{f}'} C_{f,\frac{\sqrt{3}}{2}}C_{g,\frac{\sqrt{3}}{2}}e^{-2\pi y}e^{-2\pi y}y^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Furthermore, $\mathfrak{f}' \subset \left\{x + iy \mid x \in \left[-\frac{1}{2},\frac{1}{2}\right], y \in \left[\frac{\sqrt{3}}{2},\infty\right)\right\}$. Hence our integral is

$$\le \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=\frac{\sqrt{3}}{2}}^{\infty} e^{-4\pi y}y^{k-2}\mathrm{d}x\mathrm{d}y = \int_{y=\frac{\sqrt{3}}{2}}^{\infty} e^{-4\pi y}y^{k-2}\mathrm{d}y < \infty.$$

$\square$

Remark. The second part of the lemma does not hold if f, g are not assumed to be cuspidal.

Definition 3.5. The **Petersson inner product** on $S_k(\Gamma(1))$ is given by the formula

$$\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

This is an inner product as $\langle f, f\rangle = \int_{\Gamma(1)\backslash\mathfrak{h}} |f(\tau)|^2\mathrm{Im}(\tau)^k\frac{\mathrm{d}x\mathrm{d}y}{y^2}$. So if $\langle f, f\rangle = 0$, then $|f|^2y^k = 0$, hence $f = 0$.

Theorem 3.15. For all $n \ge 1$, T_n is Hermitian with respect to the Peterson inner product, i.e. $\forall f, g \in S_k(\Gamma(1))$, $\langle T_nf, g\rangle = \langle f, T_ng\rangle$.

We will give a sketch proof of this next time.

Theorem 3.16. For all $k \ge 12$ even, there exists a basis $f_1, \dots, f_N$ of normalized eigenforms for $S_k(\Gamma(1))$, unique up to reordering, with the following property:

$\forall 1 \leq i \leq N$, $K_{f_i} = \mathbb{Q}(\{a_n(f_i)\}_{n\geq 1})$ is a number field, contained in $\mathbb{R}$, and $\forall n \geq 1$, $a_n(f_i) \in \mathcal{O}_{K_{f_i}}$ (the algebraic integers in K_{f_i}).

Proof. We know from linear algebra that if $(V, (\cdot,\cdot))$ is an inner product space over $\mathbb{C}$, and $T : V \to V$ is a Hermitian endomorphism, then all eigenvalues of T are real and T is diagonalizable. We also know that if $A_1, A_2, A_3, \ldots$ is an infinite family of commuting Hermitian endomorphisms, then they can be diagonalized simultaneously. So in our case, we find a basis $f_1, \ldots, f_N$ of $S_k(\Gamma(1))$ of eigenforms, which we may assume are normalized. We only need to show that this basis is unique up to reordering , i.e. that all simultaneous eigenspaces are 1–dimensional. But if $f, g \in S_k(\Gamma(1))$ are normalized eigenforms with the same T_n–eigenvalues $\forall n \geq 1$, then $a_n(f) = a_n(g)$ $\forall n \geq 1 \implies f = g$. □

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These sequences $(a_1(f), a_2(f), \ldots)_{n\geq 1}$ of eigenvalues of Hecke operators on normalized eigenforms f are among the most interesting objects in number theory. One reason for this is that the sequences $(a_p(f))_{p \text{ prime}}$ are exactly what we need in order to formulate the main conjectures of the Langlands program.

Ramanujan made conjectures concerning $\tau(n) = a_n(\Delta)$. One of them was $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$ and another was $\tau(p)\tau(p^n) = t(p^{n+1}) + p^{11}\tau(p^{n-1})$. These properties follow from basic properties of Hecke operators (and these properties also hold for general $a_n(f)$) for f a normalized eigenform.

While these two conjectures were proved the year after Ramanujan stated them, there is also a third conjecture that was only proved in the 1970s and Deligne won a Fields medal for it. To motivate this, let us prove:

Lemma 3.17. If p is prime, then

$$\sum_{n\geq 0} \tau(p^n) X^n = \frac{1}{(1 - \tau(p)X + p^{11}X^2)}.$$

Proof. We compute

$$\begin{aligned}
&(1 - \tau(p)X + p^{11}X^2) \sum_{n\geq 0} \tau(p^n) X^n \\
=&1 + \sum_{n\geq 2} \left(\tau(p^n)X^n - \tau(p)X\tau(p^{n-1})X^{n-1} + p^{11}X^2\tau(p^{n-2})X^{n-2}\right) \\
=&1 + \sum_{n\geq 2} \left(\tau(p^n) - \tau(p)\tau(p^{n-1}) + p^{11}\tau(p^{n-2})\right) X^n = 1.
\end{aligned}$$

□

Let us factor $1-\tau(p)X+p^{11}X^2 = (1-\alpha_p X)(1-\beta_p X)$ for $\alpha_p, \beta_p \in \mathbb{C}$. There are two possibilities:

- If $\tau(p)^2 - 4p^{11} > 0$, then α_p, β_p are distinct real numbers which hence have distinct absolute values.
- If $\tau(p)^2 - 4p^{11} \leq 0$, then α_p, β_p are conjugate complex numbers of the same absolute value $\sqrt{p^{11}}$.

Ramanujan conjectured that we always have the second case, i.e. $|\tau(p)| \leq 2p^{11/2}$ for any prime number p. The general form of this conjecture is:

Conjecture. Let $f \in S_k(\Gamma(1))$ be a normalized eigenform. Then $\forall p$ prime,

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}.$$

This is what Deligne proved in the 1970s.

Ramanujan proved the formula (for all p an odd prime)

$$r_{24}(p) = \left| \left\{ (x_1, \ldots, x_{24}) \in \mathbb{Z}^{24} \mid \sum_{i=1}^{24} x_i^2 = p \right\} \right| = \frac{16}{691}(1+p^{11}) + \frac{33152}{691}\tau(p).$$

A consequence of the Ramanujan conjecture is that

$$r_{24}(p) = \frac{16}{691}p^{11} + O(p^{11/2}).$$

We will now present a **non–examinable** sketch proof of Theorem 3.15. In particular, everything from now until the end of the lecture is non–examinable.

Sketch of proof, non–examinable. Recall $\langle f, g\rangle = \int_{\Gamma(1)\backslash\mathfrak{h}} f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$. Hence we want to show that

$$\int_{\Gamma(1)\backslash\mathfrak{h}} (T_n f)\ \overline{g}\mathrm{Im}(\tau)^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\Gamma(1)\backslash\mathfrak{h}} f(\overline{T_n g})\mathrm{Im}(\tau)^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

Initial reduction: it is enough to prove the theorem for $n = p$ a prime, since any T_n is a polynomial in T_p for $p \mid n$ with coefficients in $\mathbb{Z}$. We proved last time that the function $f(\tau)\overline{g(\tau)}\mathrm{Im}(\tau)^k$ is invariant under the weight 0 action of $\Gamma(1)$, so it therefore corresponds to a function $\mathcal{L} \to \mathbb{C}$ invariant under $\mathbb{C}^\times$. We claim that this function is $\Lambda \mapsto F(\Lambda)\overline{G(\Lambda)}\mathrm{covol}(\Lambda)^k$, where $F(\Lambda_\tau) = f(\tau), G(\Lambda_\tau) = g(\tau)$, and $\mathrm{covol}(\Lambda) = \int_{\mathbb{C}/\Lambda} \mathrm{d}x\mathrm{d}y = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right|$ where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $e_j = x_j + iy_j$.

Indeed, we can check that $\Lambda_\tau \mapsto F(\Lambda_\tau)\overline{G(\Lambda_\tau)}\mathrm{covol}(\Lambda_\tau)^k$ and $\Lambda_\tau = \mathbb{Z}_\tau \oplus \mathbb{Z}$, so $\mathrm{covol}\Lambda_\tau = y = \mathrm{Im}(\tau)$. Now, if $A : \mathbb{C}^\times/\mathcal{L} \to \mathbb{C}$ is a continuous function, we define $\int_{\mathbb{C}^\times/\mathcal{L}} A(\Lambda)\mathrm{d}\Lambda = \int_{\Gamma(1)/\mathfrak{h}} a(\tau)\frac{\mathrm{d}x\mathrm{d}xy}{y^2}$, where $a(\tau) = a(\Lambda\tau)$. Hence

$$\begin{aligned}\langle f, g\rangle &= \int_{\mathbb{C}^x/\mathcal{L}} F(\Lambda)\overline{G(\Lambda)}\mathrm{covol}(\Lambda)^k \mathrm{d}\Lambda \\ \langle T_p f, g\rangle &= p^{k-1}\int_{\mathbb{C}^x/\mathcal{L}} \sum_{\Lambda' \underset{p}{\leq} \Lambda} F(\Lambda')\overline{G(\Lambda)}\mathrm{covol}(\Lambda)^k \mathrm{d}\Lambda \\ &\overset{?}{=} p^{k-1}\int_{\mathbb{C}^x/\mathcal{L}} \sum_{\Lambda' \underset{p}{\leq} \Lambda} F(\Lambda)\overline{G(\Lambda')}\mathrm{covol}(\Lambda)^k \mathrm{d}\Lambda = \langle f, T_p g\rangle.\end{aligned}$$

Define $\mathcal{L}_p = \{(\Lambda', \Lambda) \mid \Lambda \in \mathcal{L}, \Lambda' \underset{p}{\leq} \Lambda\} \to \Lambda$ by $(\Lambda', \Lambda) \mapsto \Lambda$.

Fact. There is a bijection $\mathfrak{h}/\Gamma_0(p) \to \mathbb{C}^\times/\mathcal{L}_p$ where $\Gamma_0(p) = \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod p\}$ given by $\tau \mapsto (\mathbb{Z}p_\tau \oplus \mathbb{Z} \underset{p}{\leq} \mathbb{Z}\tau \oplus \mathbb{Z})$. (proving this fact is left as an exercise for the especially motivated).

If $A : C^\times/\mathcal{L}_p \to \mathbb{C}$ is a continuous function, then we define

$$\int_{\mathbb{C}^\times/\mathcal{L}_p} A(\Lambda', \Lambda) d(\Lambda', \Lambda) = \int_{\Gamma_0(p)/\mathfrak{h}} a(\tau)\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

We can rewrite

$$\begin{aligned}\langle T_p f, g\rangle &= p^{k-1}\int_{\mathbb{C}^x/\mathcal{L}} \sum_{\Lambda' \underset{p}{\leq} \Lambda} F(\Lambda')\overline{G(\Lambda)}\mathrm{covol}(\Lambda)^k \mathrm{d}\Lambda \\ &= p^{k-1}\int_{\mathbb{C}^\times/\mathcal{L}_p} F(\Lambda')\overline{G(\Lambda)}\mathrm{covol}(\Lambda)^k d(\Lambda', \Lambda) \\ &\overset{?}{=} p^{k-1}\int_{\mathbb{C}^\times/\mathcal{L}_p} F(\Lambda)\overline{G(\Lambda')}\mathrm{covol}(\Lambda)^k d(\Lambda', \Lambda) = \langle f, T_p g\rangle.\end{aligned}$$

Observe that if $\Lambda' \underset{p}{\leq} \Lambda$, then $p\Lambda \underset{p}{\leq} \Lambda'$, so there's a map $\iota : \mathcal{L}_p \to \mathcal{L}_p$ by $(\Lambda', \Lambda) \mapsto (p\Lambda, \Lambda')$, so $\iota^2(\Lambda', \Lambda) = (p\Lambda', p\Lambda)$, so ι descends to a map $\bar{\iota}\colon \mathbb{C}^\times/\mathcal{L}_p \to \mathbb{C}^\times/\mathcal{L}_p$. The key point is that this map $\bar{\iota}$ is measure–preserving and transforms $\langle T_p f, g\rangle$ into $\langle f, T_p g\rangle$ (exercise).

Why is $\bar{\iota}$ measure–preserving? Under the bijection $\mathbb{C}^\times/\mathcal{L}_p \xrightarrow{\sim} \Gamma_0(p)/\mathfrak{h}$, it corresponds to the action of $\begin{pmatrix} 0 & -1 \\ p & 0\end{pmatrix} \in GL_2(\mathbb{Q})^+$. We defined integration on $\mathfrak{h}$ using $\omega = \frac{\mathrm{d}x\mathrm{d}y}{y^2}$, which is invariant even in $GL_2(\mathbb{R})^+$. □

Proposition 3.18. Let $f : \mathfrak{h} \to \mathbb{C}$ be continuous and Γ_∞–invariant, where

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$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c = 0 \right\}$, i.e. $f(\tau) = f(\tau+1)$. Suppose that $\forall \tau \in \mathfrak{h}$,

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} |f(\gamma\tau)| < \infty.$$

Also suppose that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} |f(x+iy)| \frac{\mathrm{d}x\mathrm{d}y}{y^2} < \infty.$$

Then $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ is a measurable function, $\Gamma(1)$–invariant and

$$\int_{\Gamma(1)\backslash\mathfrak{h}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} |f(x+iy)| \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

An application of this proposition is called "unfolding". This is because $\{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]\}$ is a fundamental set for $\Gamma_\infty \setminus \{\pm I\}$, so

$$\int_{\Gamma(1)\backslash\mathfrak{h}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\Gamma_\infty\backslash\mathfrak{h}} f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \text{RHS of the prop above.}$$

Proof. We want to show that $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ is measurable on f and the equality

$$\int_{\mathfrak{f}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} f(x+iy) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Fubini's theorem says: Suppose $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\mathfrak{f}} |f(\gamma\tau)| \frac{\mathrm{d}x\mathrm{d}y}{y^2} < \infty$. Then $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau)$ is measurable and absolutely integrable in $\mathfrak{f}$ and

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\mathfrak{f}} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\mathfrak{f}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

We'll be done if we can show

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\mathfrak{f}} f(\gamma\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} f(x+iy) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

But the LHS is equal to

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \int_{\gamma\mathfrak{f}} f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

by using the change of variable $\mathfrak{f} \mapsto \gamma\mathfrak{f}$ and invariance of $\frac{\mathrm{d}x\mathrm{d}y}{y^2}$ under the pullback by γ. Recall from the first example sheet that if $\mathfrak{f}^0 = \mathrm{Int}(\mathfrak{f})$, then $\forall \gamma \in \Gamma(1)$, $\gamma\mathfrak{f}^0 \cap \{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in \frac{1}{2} + \mathbb{Z}\} = \varnothing$. Hence for $\gamma \in \Gamma(1)$, $\gamma\mathfrak{f}^0$ is contained in $\{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in (-\frac{1}{2}, \frac{1}{2}) + a\}$ for some $a \in \mathbb{Z}$. Also, there's a unique $\delta \in \Gamma_\infty \setminus \{\pm I\}$ such that $\delta\gamma\mathfrak{f}^0 \subset \{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in (-\frac{1}{2}, \frac{1}{2})\} = U$. Hence the set $\{\gamma \in \Gamma(1) \setminus \{\pm I\} \mid \gamma\mathfrak{f}^0 \subset U\}$ is a set of coset representatives for $\Gamma_\infty \setminus \Gamma(1)$. Thus

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\gamma\mathfrak{f}} f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \sum_{\substack{\gamma \in \Gamma(1) \setminus \{\pm I\} \\ \gamma\mathfrak{f}^0 \subset U}} \int_{\gamma\mathfrak{f}} f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} \stackrel{?}{=} \int_U f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

But we know that $\mathfrak{h} = \left(\bigsqcup_{\gamma \in \Gamma(1) \setminus \{\pm I\}} \gamma\mathfrak{f}^0\right) \sqcup W$ for W of measure zero, e.g. the union of all the $\Gamma(1)$–translates of the vertical line $\mathrm{Re}(\tau) = \frac{1}{2}$. Hence

$$U = \bigsqcup_{\gamma \in \Gamma(1) \setminus \{\pm I\}} (\gamma\mathfrak{f}^0 \cap U) \sqcup (W \cap U) = \bigsqcup_{\substack{\gamma \in \Gamma(1) \setminus \{\pm I\} \\ \gamma\mathfrak{f}^0 \subset U}} (\gamma\mathfrak{f}^0) \sqcup (W \cap U)$$

for $W \cap U$ of measure zero. Hence

$$\int_U f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\gamma\mathfrak{f}^0} f(\tau) \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

which concludes the proof. □

4 L–functions

Normalized eigenforms can be used to construct L–functions. What is an L–function? Motivation: the Riemann zeta function, $\zeta(s) = \sum_{n \geq 1} n^{-s}$. This converges absolutely in $\{s \mid \mathrm{Re}(s) > 1\}$ and defines a holomorphic function in that region. Key properties:

- The Euler product: $\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$ (converges absolutely when $\mathrm{Re}(s) > 1$).
- Meromorphic continuation: $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ and no other pole.
- Functional equation: Define $\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. Then $\xi(s) = \xi(1 - s)$.
- Special values of $\zeta(s)$ at $s \in \mathbb{Z}$ shall have arithmetic meaning.

Other examples of functions of similar properties:

- Dirichlet L–functions $L(\chi, s) = \sum_{\substack{n \in \mathbb{N} \\ (n,N)=1}} \chi(n \pmod N) n^{-s}$ associated to $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

- If $E/\mathbb{Q}$ is an elliptic curve, then the Hasse–Weil L–function $L(E,s) = \sum_{n\geq 1} a_n n^{-s}$.

In general, an L–function is a Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$, $a_n \in \mathbb{C}$ which either provably has or is expected to have properties analogous to $\zeta(s)$.

Definition 4.1. Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma(1))$. Then its associated Dirichlet series is

$$L(f,s) = \sum_{n\geq 1} a_n n^{-s}.$$

We will consider separately the case of Eisenstein series and the case of cuspidal modular forms.

Let $F_k(\tau)$ be the normalized eigenform associated to G_k (for $k \geq 4$ even). Then

$$\begin{aligned} L(F_k,s) &= \sum_{n\geq 1} \sigma_{k-1}(n) n^{-s} = \sum_{n\geq 1}\sum_{d|n} d^{k-1} n^{-s} = \sum_{n\geq 1}\sum_{d|n} d^{k-1} d^{-s} \left(\frac{n}{d}\right)^{-s} \\ &= \sum_{a,d\geq 1} d^{k-1-s} a^{-s} = \zeta(s)\zeta(s+1-k). \end{aligned}$$

Lemma 4.1. Let $f \in S_k(\Gamma(1))$. Then $L(f,s)$ converges absolutely in the region $\{\mathrm{Re}(s) > 1 + \frac{k}{2}\}$ and defines a holomorphic function there.

Proof. We use a fact from the second example sheet: $\exists C_f > 0$ such that for all $n \geq 1, |a_n| \leq C_f n^{k/2}$. We then claim that $\forall \delta > 0, \sum_{n\geq 1} a_n n^{-s}$ converges absolutely and uniformly in $\{\mathrm{Re}(s) > 1 + \frac{k}{2} + \delta\}$. To prove this, we use the Weierstrass M–test.

Write $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$. Then $n^{-s} = \exp(-s\log n) \implies |n^{-s}| = \exp(-\sigma \log n) = n^{-\sigma}$. If $\sigma > 1 + \frac{k}{2} + \delta$, then

$$\sum_{n\geq 1} |a_n n^{-s}| \leq \sum_{n\geq 1} C_f n^{k/2} n^{-(1+k/2+\delta)} = \sum_{n\geq 1} C_f n^{-(1+\delta)} < \infty.$$

$\square$

Remark. If we assume the Ramanujan–Petersson conjecture, we can get absolute convergence when $\mathrm{Re}(s) > \frac{1+k}{2}$.

06 Nov 2022, Lecture 14

Theorem 4.2. Let $f \in S_k(\Gamma(1))$ be a cuspidal modular form. Then:

(1) $L(f,s)$ admits an analytic continuation to $\mathbb{C}$.

(2) If $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$, then $\Lambda(f,s) = i^k \Lambda(f,k-s)$.

To warm up, we consider the gamma function

$$\Gamma(s) = \int_{y=0}^{\infty} e^{-y} y^s \frac{\mathrm{d}y}{y}$$

(when the integral converges absolutely).

Proposition 4.3. (i) $\Gamma(s)$ converges absolutely when $\mathrm{Re}(s) > 0$ and is a holomorphic function in $\{\mathrm{Re}(s) > 0\}$.

(ii) $\Gamma(s)$ admits a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and no other poles.

Proof. $\Gamma(s)$ converges absolutely when $\mathrm{Re}(s) > 0$, i.e. $\int_{y=0}^{\infty} |e^{-y} y^s| \frac{\mathrm{d}y}{y} < \infty$. Checking this is left as an easy exercise.

Next we show $\Gamma(s)$ is continuous in $\{\mathrm{Re}(s) > 0\}$. If $N > 1$, then define $\Gamma_N(s) = \int_{y=\frac{1}{N}}^{N} e^{-y} y^s \frac{\mathrm{d}y}{y}$. We claim that $\Gamma_N(s)$ is continuous in $\{\mathrm{Re}(s) > 0\}$. But if $\mathrm{Re}(s) \geq 0$ and $\epsilon > 0$, then $\exists \delta > 0$ such that if $s' \in \mathbb{C}$ with $|s - s'| < \delta$ and $y \in \left[\frac{1}{N}, N\right]$, then $|y^s - y^{s'}| < \epsilon$ (since $(y,s) \mapsto y^s : \left[\frac{1}{N}, N\right] \times \mathbb{C} \to \mathbb{C}$ is continuous and $\left[, \frac{1}{N}, N\right]$ is compact). Then

$$|\Gamma_N(s) - \Gamma_N(s')| \leq \int_{y=\frac{1}{N}}^{N} e^{-y} |y^s - y^{s'}| \frac{\mathrm{d}y}{y} \leq \epsilon \int_{y=\frac{1}{N}}^{N} e^{-y} \frac{\mathrm{d}y}{y} = C_N \epsilon,$$

so $\Gamma_N(s)$ is continuous.

To show $\Gamma(s)$ is holomorphic, we recall Morera's theorem: If $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is continuous and $\oint_\gamma f(z)\mathrm{d}z = 0$ for all closed continuous paths γ in U, then f is holomorphic. We have

$$\oint_\gamma \Gamma_N(s)\mathrm{d}s = \oint_\gamma \int_{y=\frac{1}{N}}^{N} e^{-y} y^s \frac{\mathrm{d}y}{y} \mathrm{d}s = \int_{y=\frac{1}{N}}^{N} e^{-y} \underbrace{\oint_\gamma y^s \mathrm{d}s}_{0, \text{ as } y^s \text{ is holomorphic}} \frac{\mathrm{d}y}{y} = 0.$$

Hence $\Gamma_N(s)$ is holomorphic by Morera's theorem.

To show Γ is holomorphic, we show $\Gamma_N \to \Gamma$ locally uniformly. In fact, we show uniform convergence in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \in [\sigma_0, \sigma_1]\}$ $\forall 0 < \sigma_0 < \sigma_1$, i.e. "uniform convergence in vertical strips". If s lies in this set, then

$$\begin{aligned} |\Gamma(s) - \Gamma_N(s)| &\leq \int_{y=0}^{\frac{1}{N}} |y^s e^{-y}| \frac{\mathrm{d}y}{y} + \int_{y=N}^{\infty} |y^s e^{-y}| \frac{\mathrm{d}y}{y} \\ &\leq \int_{y=0}^{\frac{1}{N}} y^{\sigma_0 - 1} e^{-y} \mathrm{d}y + \int_{y=N}^{\infty} y_1^{\sigma_1 - 1} e^{-y} \mathrm{d}y \overset{n\to\infty}{\to} 0 \end{aligned}$$

at a rate independent of s, giving us uniform convergence and showing (i).

To prove (ii), we use the equation $s\Gamma(s) = \Gamma(s+1)$ (which we can prove from the definition by integrating by parts). This can be used to extend $\Gamma(s)$ into a meromorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > -k\}$ for any $k \in \mathbb{N}$ by induction on k, and the description of the poles also follows. $\square$

Proof of Theorem 4.2. Consider

$$F(s) = \int_{y=0}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y},$$

called the Mellin transform of $f(iy)$. We claim that $F(s)$ converges for any $s \in \mathbb{C}$ and defines a holomorphic function. For absolute convergence, write

$$\int_{y=0}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y} = \int_{y=0}^{1} f(iy) y^s \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y}.$$

We know $|f(\tau)| \leq C_f |e^{2\pi i \tau}|$, so $|f(iy)| \leq C_f e^{-2\pi y}$, so $\int_{y=1}^{\infty} |f(iy) y^s| \frac{\mathrm{d}y}{y} < \infty$ for any $s \in \mathbb{C}$. For the first integral we have

$$\int_{y=0}^{1} f(iy) y^s \frac{\mathrm{d}y}{y} = \int_{y=1}^{\infty} f\left(\frac{1}{y}\right) y^{-s} \frac{\mathrm{d}y}{y}.$$

But also $f(\tau) = f(-1/\tau) t^{-k}$, so $f(iy) = f\left(\frac{i}{y}\right)(iy)^{-k}$. Hence

$$\int_{y=1}^{\infty} \left| f\left(\frac{i}{y}\right) y^{-s} \right| \frac{\mathrm{d}y}{y} = \int_{y=1}^{\infty} |f(iy)| y^{k-s} \frac{\mathrm{d}y}{y} < \infty$$

for any $s \in \mathbb{C}$, since $f(iy)$ decays exponentially as $y \to \infty$.

The fact that F is holomorphic is left as an exercise: it is similar to the proof of holomorphicity of $\Gamma(s)$ above, but easier, since we don't have to worry about blowing up anywhere.

What is $F(s)$? We have

$$F(s) = \int_{y=0}^{\infty} \sum_{n=1}^{\infty} a_n e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} \overset{(\star)}{=} \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{2\pi n y} y^s \frac{\mathrm{d}y}{y}.$$

$(\star)$ is justified by Fubini's theorem provided that

$$\sum_{n=1}^{\infty} |a_n| \int_{y=0}^{\infty} |e^{-2\pi n y} y^s| \frac{\mathrm{d}y}{y} < \infty.$$

If we assume this holds, then we get

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} &= \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-y} y^s (2\pi n)^{-s} \frac{\mathrm{d}y}{y} \\ = \sum_{n=1}^{\infty} (2\pi)^{-s} a_n n^{-s} \int_{y=0}^{\infty} e^{-y} y^s \frac{\mathrm{d}y}{y} = \Lambda(f,s).\end{aligned}$$

To justify $(\star)$, we have

$$\sum_{n=1}^{\infty} |a_n| \int_{y=0}^{\infty} |e^{-2\pi n y} y^s| \frac{\mathrm{d}y}{y} = (2\pi)^{-\sigma}\Gamma(\sigma) \sum_{n=1}^{\infty} |a_n| n^{-\sigma},$$

where $\sigma = \mathrm{Re}(s)$ (so $|y|^s = |y|^\sigma$, $|n^{-s}| = n^{-\sigma}$), i.e. whenever $L(f,s)$ is absolutely convergent.

We conclude that $F(s)$ is holomorphic in $\mathbb{C}$ and equals $\Lambda(f,s)$ when $\mathrm{Re}(s) > 1 + \frac{k}{2}$, i.e. $\Lambda(f,s)$ has an analytic continuation to $\mathbb{C}$. We can write $L(f,s) = \frac{\Lambda(f,s)}{(2\pi)^{-s}\Gamma(s)}$, which is also analytic in $\mathbb{C}$, since $\frac{1}{\Gamma(s)}$ is entire.

For the last part, we have

$$\Lambda(f,s) = \int_{y=0}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y} = \int_{y=1}^{\infty} f\left(\frac{i}{y}\right) y^{-s} \frac{\mathrm{d}y}{y} + \int_{y=1}^{\infty} f(iy) y^s \frac{\mathrm{d}y}{y}.$$

Use $f(i/y) = f(iy)(iy)^k$ to find that

$$\Lambda(f,s) = \int_{y=1}^{\infty} f(iy)\left(i^k y^{k-s} + y^s\right) \frac{\mathrm{d}y}{y}$$

If $f \neq 0$, then k is even, so $i^k \in \{\pm 1\}$. Hence

$$\Lambda(f, k-s) = i^k \Lambda(f,s).$$

□

Theorem 4.4. Let $f \in S_k(\Gamma(1))$ be a normalized eigenform. Then

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p \text{ prime}} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

We can interpret this as either an equality of formal Dirichlet series or as an equality of complex numbers when $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent.

Proof. By an exercise on the third example sheet, it is enough to consider the formal identity. But we know $a_{mn} = a_m a_n$ if $(m,n) = 1$ (a property of Hecke operators inherited by their eigenvalues). Hence

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p \text{ prime}} (1 + a_p p^{-s} + a_{p^2} p^{-2s} + a_{p^3} p^{-3s} + \ldots).$$

So we need to show

$$1 + a_p p^{-s} + a_{p^2} p^{-2s} + a_{p^3} p^{-3s} + \ldots = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

But this is equivalent to $a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}}$, which we showed for Hecke operators. □

08 Nov 2022, Lecture 15

We will now use the following theorem as a black box result. For a proof, see Lang's *Algebraic Number Theory*.

Theorem 4.5 (Wiener–Ikehara Tauberian theorem)**.** Consider a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s} = f(s)$, absolutely convergent when $\mathrm{Re}(s) > 1$ (so f is holomorphic in this region). Suppose f admits a meromorphic continuation to an open neighborhood of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$ which is holomorphic on the line $\mathrm{Re}(s) = 1$, except possibly for a simple pole at $s = 1$ of residue α. Then

$$\sum_{1 \leq n \leq X} a_n = \alpha X + o(X) \text{ as } X \to \infty.$$

Here $o(X)$ denotes any function $g(x)$ such that $g(X)/X \to 0$ as $X \to \infty$, and $O(X)$ denotes any function $h(X)$ such that $h(X)/X$ is bounded as $X \to \infty$.

As an illustration, we prove:

Proposition 4.6. Suppose that the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ admits a meromorphic continuation to an open neighborhood of $\{\mathrm{Re}(s) \geq 1\}$ which is holomorphic and non–vanishing on the line $\mathrm{Re}(s) = 1$ except for a simple pole at $s = 1$. Then the Prime Number Theorem holds, i.e.

$$\pi(X) = \sum_{\substack{p \text{ prime} \\ p \leq X}} 1 = \frac{X}{\log X} + o\left(\frac{X}{\log X}\right)$$

as $X \to \infty$.

Proof. Note that we have the Taylor series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ for $-\log(1-z)$ valid for $|z| < 1$. A branch of $\log \zeta(s) = \log \prod_p (1 - p^{-s})^{-1}$ is given by

$$\sum_p \log(1 - p^{-s})^{-1} = \sum_p \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}.$$

This Dirichlet series is absolutely convergent when $\mathrm{Re}(s) > 1$, hence locally uniformly convergent, so we can compute the derivative term–by–term to find

$$-\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{p^{-ks}}{k}\right) = \sum_p \sum_{k=1}^{\infty} (\log p) p^{-ks}$$
$$\implies -\frac{\zeta'(s)}{\zeta(s)} = \sum_p (\log p) p^{-s} + \sum_p \sum_{k\geq 2} (\log p) p^{-ks}.$$

Note the second term is absolutely convergent when $\mathrm{Re}(s) > \frac{1}{2}$. But $-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s)$, so if $\zeta(s)$ has a zero or a pole of order k at s_0, then $-\frac{\zeta'(s)}{\zeta(s)}$ will have a simple pole at s_0 of residue $-k$.

We are assuming that $\zeta(s)$ has a meromorphic continuation which is holomorphic and non–vanishing on $\{\mathrm{Re}(s) = 1\}$ except for a simple pole at $s = 1$. Hence $-\frac{\zeta'(s)}{\zeta(s)}$ has a meromorphic continuation defined where ζ is defined, holomorphic on $\{\mathrm{Re}(s) = 1\}$ except for a simple pole at $s = 1$ of residue 1.

We conclude that $\sum_p (\log p)p^{-s}$ has a meromorphic continuation to a neighborhood of $\{\mathrm{Re}(s) \geq 1\}$, holomorphic on the line $\{\mathrm{Re}(s) = 1\}$ except for a simple pole at $s = 1$ of residue 1. Hence applying the Wiener–Ikehara Tauberian theorem to $\sum_p (\log p)p^{-s}$ gives

$$\theta(X) = \sum_{p\leq X} \log p = X + o(X).$$

To get back to $\pi(x)$, we use Lemma 4.7 (partial summation, to be proved after this proof). We take $a_n = \begin{cases} 0 & n \text{ not prime} \\ \log p & n = p \text{ prime} \end{cases}$ and $f(t) = \frac{1}{\log t}$. By partial summation,

$$\begin{aligned}\pi(X) &= 1 + \sum_{e<n\leq x} 1_{n \text{ is prime}} \\ &= 1 + \sum_{e<n\leq x} a_n f(n) = A(X)f(X) - A(e)f(e) + \int_{t=e}^{X} \frac{A(t)}{t(\log t)^2}\mathrm{d}t.\end{aligned}$$

Note that $A(x) = \sum_{p\leq x} \log p = \theta(X) = X + o(X)$, so the above is

$$\begin{aligned}&= \frac{\theta(X)}{\log X} - A(e)f(e) + \int_{t=e}^{X} \frac{\theta(t)}{t(\log t)^2}\mathrm{d}t \\ &= \frac{X}{\log X} + o\left(\frac{X}{\log X}\right) + \int_{t=e}^{X} \frac{\theta(t)}{t(\log t)^2}\mathrm{d}t.\end{aligned}$$

To finish, we need to show that the last term can be absorbed into the error term. But $\theta(X) = X + o(X)$, so $\theta(X) = O(X)$, so $\exists C > 0$ such that $\theta(t) \leq Ct$ $\forall t > 0$, so our integral is

$$
\begin{aligned}
&\leq C \int_{t=e}^{X} \frac{1}{(\log t)^2} \mathrm{d}t = C \int_{t=e}^{\sqrt{X}} \frac{1}{(\log t)^2} \mathrm{d}t + C \int_{t=\sqrt{X}}^{X} \frac{1}{(\log t)^2} \mathrm{d}t \\
&\leq C\sqrt{X} + C \frac{X}{(\log \sqrt{X})^2} \\
&= C\sqrt{X} + \frac{4CX}{(\log X)^2} = o\left(\frac{X}{\log X}\right)
\end{aligned}
$$

as desired.

$\square$

Lemma 4.7 (Partial summation)**.** Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Let $0 < X < Y$ be real numbers and let $f : [X, Y] \to \mathbb{R}$ be a continuously differentiable function. Let $A(t) = \sum_{0\leq n\leq t} a_n$. Then

$$
\sum_{X<n\leq Y} a_n f(n) = A(Y)f(Y) - A(X)f(X) - \int_{t=X}^{Y} A(t)f'(t)\mathrm{d}t.
$$

Proof. Elementary exercise. $\square$

We will establish all required properties of $\zeta(s)$ for the proof later in the course using modular forms.

Theorem 4.8. Fix $n \geq 1$. Suppose we're given for all primes p a matrix $\Phi_p \in M_n(\mathbb{C})$ which is either zero or whose eigenvalues have absolute value 1. Define

$$
L(\{\Phi_p\}, s) = \prod_p \det(1_n - p^{-s}\Phi_p)^{-1}.
$$

Then $L(\{\Phi_p\}, s)$ is absolutely convergent when $\mathrm{Re}(s) > 1$. Furthermore, suppose $L(\{\Phi_p\}, s)$ admits a meromorphic continuation to an open neighborhood of $\{\mathrm{Re}(s) \geq 1\}$ and is holomorphic and nonvanishing on the line $\{\mathrm{Re}(s) = 1\}$ except possibly for a pole at $s = 1$ of order δ. Then

$$
\sum_{\substack{p\leq X \\ p \text{ prime}}} \mathrm{tr}\ \Phi_p = \frac{\delta X}{\log X} + o\left(\frac{X}{\log X}\right).
$$

Proof. Left as an exercise on the third example sheet. This is just a generalization of the case $n = 1, \Phi_p = 1, L = \zeta$ that we just did. $\square$

Example 4.1. Dirichlet's theorem on primes in arithmetic progressions. Fix $N \in \mathbb{N}, N \geq 1$. For any homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, consider

$$L(\chi, s) = \sum_{\substack{(n,N)=1 \\ n \in \mathbb{N}}} \chi(n \bmod N) n^{-s} = \prod_{p \nmid N} (1 - \chi(p \bmod N) p^{-s})^{-1}.$$

These are called the Dirichlet L–functions. It is a fact that the hypotheses of Theorem 4.8 apply to $L(\chi, s)$, so we conclude that for any χ,

$$\sum_{p \leq X} \chi(p \bmod N) = \mathrm{ord}_{s=1} L(\chi, s) \cdot \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$

One can show that $\mathrm{ord}_{s=1} L(\chi, s) = \begin{cases} -1 & \chi \text{ trivial.} \\ 0 & \chi \text{ nontrivial.} \end{cases}$

If $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, then $1_{a \bmod N}(g) = \frac{1}{\phi(N)} = \sum_\chi \overline{\chi(a)} \chi(g)$ for any $g \in (\mathbb{Z}/N\mathbb{Z})^\times$. Hence

$$\sum_{\substack{p \leq x \\ p \nmid N}} 1_{a \bmod N}(p) = \frac{1}{\phi(N)} \sum_\chi \overline{\chi(n)} \sum_{p \leq X} \chi(p \bmod N) = \frac{1}{\phi(N)} \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$

Let $f \in S_k(\Gamma(1))$ be a normalized eigenform. Then 10 Nov 2022, Lecture 16

$$L(f, s) = \prod_p (1 - a_p p^{-s} p^{k-1-2s})^{-1}.$$

Factor $(1 - a_p X + p^{k-1} X^2) = (1 - \alpha_p X)(1 - \beta_p X)$ for $\alpha_p, \beta_p \in \mathbb{C}$ and let $\Phi_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$, sometimes called the Satake parameter of f at p. We can factor

$$\begin{aligned} L(\{\Phi_p\}, s) &= \prod_p \det \begin{pmatrix} 1 - \alpha_p p^{-s} & 0 \\ 0 & 1 - \beta_p p^{-s} \end{pmatrix}^{-1} \\ &= \prod_p ((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}))^{-1} = L(f, s). \end{aligned}$$

The Ramanujan conjecture says that $|\alpha_p| = |\beta_p| = p^{\frac{k-1}{2}}$. If this holds, then $L(\{p^{\frac{1-k}{2}} \Phi_p\}, s)$ fits into our framework of our theorem. Here the above expression equals

$$\prod_p ((1-\alpha_p p^{-s} p^{-(k-1)/2})(1-\beta_p p^{-s} p^{-(k-1)/2}))^{-1} = L\left(f, s+\frac{k-1}{2}\right).$$

Corollary 4.9. Suppose $f \in S_k(\Gamma(1))$ is a normalized eigenform and that the Ramanujan–Petersson conjecture holds for f, and that $L(f,s)$ is nonvanishing on $\{\mathrm{Re}(s) = \frac{k+1}{2}\}$. Then

$$\lim_{X\to\infty} \left(\sum_{p\le X} \frac{a_p(f)}{p^{(k-1)/2}}\right) / \pi(X) = 0.$$

This says that the average value of $\frac{a_p}{p^{(k-1)/2}} \in [-2,2]$ is 0.

Recall. For p an odd prime, $r_{24}(p) = \frac{16}{691}(1+p^{11}) + \frac{33152}{691}\tau(p)$. If the hypotheses of Corollary 4.9 hold (which they do), then the average of

$$\frac{r_{24}(p) - \frac{16}{691}(1+p^{11})}{p^{11/2}}$$

is 0.

In fact, we can go much farther. We can introduce a family of L–functions associated to the normalized eigenform f:

Definition 4.2. If $n \ge 1$, then

$$L(f, \mathrm{Sym}^n, s) = L(\{\mathrm{Sym}^n \Phi_p\}, s) = \prod_p \prod_{i=0}^{n} (1-\alpha_p^i \beta_p^{n-i} p^{-s})^{-1},$$

where $\mathrm{Sym}^n : GL_2 \to GL_{n+1}$ is the n^{th} symmetric power of the standard representation. A priori, we know these converge absolutely in some right half–plane. If $n=1$, $L(f, \mathrm{Sym}^1, s) = L(f,s)$.

Proposition 4.10. (i) (Langlands, 1967). If $\forall n \ge 1$, $L(f, \mathrm{Sym}^n, s)$ admits an analytic continuation to $\mathbb{C}$, then the Ramanujan–Petersson conjecture holds for f.

(ii) (Serre, 1967). If the Ramanujan-Petersson conjecture holds for f and if $\forall n \ge 1$ $L(f, \mathrm{Sym}^n, s)$ admits an analytic continuation which is non–vanishing on the line $\{\mathrm{Re}(s) = 1 + \frac{n(k-1)}{2}\}$, then the Sato–Tate conjecture holds for f, i.e. the numbers $a_p(f)/2p^{(k-1)/2} \in [-1,1]$ are equidistributed with respect to the Sato–Tate density $\frac{2}{\pi}\sqrt{1-t^2}\mathrm{d}t$. This means that for any $g \in C([-1,1])$,

$$\lim_{X\to\infty} \frac{1}{\pi(X)} \sum_{p\leq X} g(a_p/2p^{(k-1)/2}) = \int_{t=-1}^{1} g(t)\frac{2}{\pi}\sqrt{1-t^2}\mathrm{d}t.$$

This says that

$$\frac{691}{66304}\left(r_{24}(p) - \frac{16}{691}(1+p^{11})\right)\frac{1}{p^{11/2}}$$

are distributed according to the density $\frac{2}{\pi}\sqrt{1-t^2}\mathrm{d}t$.

We now know that $L(f, \mathrm{Sym}^n, s)$ does have the required properties. There is a nice article *Finding meaning in error terms* by Mazur uploaded on Moodle, which we can take a look at.

5 Modular forms on congruence subgroups of $\Gamma(1)$

Definition 5.1. A **congruence subgroup** $\Gamma \leq \Gamma(1)$ is any subgroup containing $\ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}))$ for some $N \geq 1$.

The main examples are:

- $\Gamma(N) = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}))$.
- $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod N\right\}$.
- $\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod N, a \equiv d \equiv 1 \pmod N\right\}$.

Remark. If $\Gamma \leq \Gamma(1)$ is a congruence subgroup, then $[\Gamma(1) : \Gamma]$ is finite, as $[\Gamma : \Gamma(N)] \leq |SL_2(\mathbb{Z}/N\mathbb{Z})|$ is finite.

Many of the most interesting modular forms only exist at level $\Gamma < \Gamma(1)$ for Γ a proper subgroup of $\Gamma(1)$. One example we will see is the θ–function of a lattice Λ, and also the normalized eigenforms associated to elliptic curves over $\mathbb{Q}$ (defined on $\Gamma_0(N_E)$, where N_E is the conductor of E).

Definition 5.2. Let $k \in \mathbb{Z}$, $\Gamma \leq \Gamma(1)$ a congruence subgroup. A **weakly modular function** of weight k, level Γ is a meromorphic function f in $\mathfrak{h}$ such that $\forall \gamma \in \Gamma, f|_k[\gamma] = f$.

Fact. $\mathfrak{f}_0(2) = \{\tau \in \mathfrak{h} \mid \mathrm{Re}(\tau) \in [0,1], |\tau - \frac{1}{2}| \geq \frac{1}{2}\}$ is (the closure of) a fundamental set for the action of $\Gamma_0(2)$ acting on $\mathfrak{h}$ (draw a picture!).

Note that there is more than one way to "go to infinity", and also note that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(2)$ sending 0 to 1, so the "infinities" at 0 and 1 are similar, but different to the one at $\mathrm{Im}(\tau) \to \infty$.

Definition 5.3. Let $\Gamma \leq \Gamma(1)$ be a congruence subgroup. A **cusp** of Γ is a Γ–orbit in $\mathbb{P}^1(\mathbb{Q})$.

Here $\mathbb{P}^1(\mathbb{Q})$ comes from

$$\begin{array}{ccccc} GL_2(\mathbb{C}) & \curvearrowright & \mathbb{P}^1(\mathbb{C}) & = & \mathbb{C} \cup \{\infty\} \\ \rotatebox{90}{$\leq$} & & & & \\ GL_2(\mathbb{Q}) & \curvearrowright & \mathbb{P}^1(\mathbb{Q}) & = & \mathbb{Q} \cup \{\infty\}. \\ \rotatebox{90}{$\leq$} & & & & \\ \Gamma & & & & \end{array}$$

Lemma 5.1. $\Gamma(1)$ has a unique cusp.

Proof. We need to show that $\Gamma(1)$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$. We will show that any $\frac{a}{c} \in \mathbb{Q}$ with $(a, c) = 1$ is $\Gamma(1)$–conjugate to ∞. By Bezout, $\exists r, s \in \mathbb{Z}$ such that $ar + cs = 1$. Let $\gamma = \begin{pmatrix} a & -s \\ c & r \end{pmatrix} \in \Gamma(1)$. Then

$$\gamma\infty = \frac{a\infty - s}{c\infty + r} = \frac{a}{c}.$$

$\square$

Corollary 5.2. If Γ is a congruence subgroup, then it has finitely many cusps. 13 Nov 2022, Lecture 17

Proof. We know by the orbit–stabilizer theorem that there's a $\Gamma(1)$–equivalent bijection $\Gamma(1)/\Gamma_\infty \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q})$ by $\gamma\Gamma_\infty \mapsto \gamma\infty$, where $\Gamma_\infty = \mathrm{Stab}_{\Gamma(1)}(\infty) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma(1) \right\}$. If $\Gamma \leq \Gamma(1)$ is a congruence, then there's an induced bijection $\Gamma \setminus \Gamma(1)/\Gamma_\infty \xrightarrow{\sim} \Gamma \setminus \mathbb{P}^1(\mathbb{Q})$ (where $\Gamma \setminus \Gamma(1)/\Gamma_\infty$ is a double coset, the set of Γ–Γ_∞-double cosets, i.e. subsets of $\Gamma(1)$ of the form $\Gamma_\gamma\Gamma_\infty = \{g\gamma h \mid g \in \Gamma, h \in \Gamma_\infty\}$). $\Gamma \setminus \Gamma(1)/\Gamma_\infty$ is finite as it's the set of right Γ_∞–orbits on $\Gamma \setminus \Gamma(1)$, which is finite, as $[\Gamma(1) : \Gamma] < \infty$. $\square$

Idea. $Y(\Gamma) = \Gamma \setminus \mathfrak{h}$ is a non–compact Riemann surface, which we can compactify by adding finitely many points, one for each cusp in $\Gamma \setminus \mathbb{P}^1(\mathbb{Q})$. We know how to define cusps around ∞, and deal with the general case by transforming to this case (details to follow).

Let f be a weakly modular function of weight k and level $\Gamma \leq \Gamma(1)$. The index $[\Gamma_\infty : \Gamma \cap \Gamma_\infty]$ is finite, since if $\Gamma(N) \leq \Gamma$, then $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \cap \Gamma$.

Definition 5.4. The **width** of ∞ (as a cusp of Γ) is $\min\left(h \in \mathbb{N} \mid \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma \cap \Gamma_\infty\right)$.

If h is the width, then $f|_h\left[\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\right] = f(\tau + h) = f(\tau)$ (as f has level Γ). The same argument as in the case of level $\Gamma(1)$ shows us that there exists a unique meromorphic function $\tilde{f}$ in $D^*(0,1)$ such that $f(\tau) = \tilde{f}(e^{2\pi i\tau/h})$. We say that f is $\begin{cases} \text{meromorphic at } \infty \\ \text{holomorphic at } \infty \\ \text{vanishes at } \infty \end{cases}$ if $\begin{cases} \tilde{f} \text{ extends to a meromorphic function in } D(0,1). \\ f \text{ is meromorphic at } \infty, \tilde{f} \text{ has a removable singularity at } 0. \\ f \text{ is holomorphic at } \infty \text{ and } \tilde{f}(0) = 0. \end{cases}$

If f is meromorphic at ∞, then it has a q–expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q_h^n$$

for $q_h = e^{2\pi i\tau/h}$, derived from the Laurent expansion of $\tilde{f}$. Hence this is absolutely convergent in $\{\tau \mid \mathrm{Im}(\tau) > R\}$ for some $R > 0$, with only finitely many nonzero a_n with $n < 0$.

Now take a general cusp $\Gamma \cdot z$, $z \in \mathbb{P}^1(\mathbb{Q})$. Choose $\alpha \in \Gamma(1)$ such that $\alpha\infty = z$. We say that f is $\begin{cases} \text{meromorphic} \\ \text{holomorphic} \\ \text{vanishing} \end{cases}$ at $\Gamma \cdot z$ if $f|_k[\alpha]$ is $\begin{cases} \text{meromorphic} \\ \text{holomorphic} \\ \text{vanishing} \end{cases}$ at ∞, when we consider $f|_k[\alpha]$ as a weakly modular function of weight k and level $\alpha^{-1}\Gamma\alpha$. Note that $\alpha^{-1}\Gamma\alpha$ is a congruence subgroup, as $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, as it arises through a kernel. For the weight, we verify

$$f|_k[\alpha]|_h[\alpha^{-1}\gamma\alpha] = f|_k[\alpha\alpha^{-1}\gamma\alpha] = f|_k[\gamma\alpha] = f|_k[\alpha].$$

Lemma 5.3. The property of being holomorphic/meromorphic/vanishing at $\Gamma \cdot z$ is independent of the choice of α with $\alpha\infty = z$ and of the choice of z.

Proof. First we show that the choice of α doesn't matter. If $\alpha, \beta \in \Gamma(1)$ with $\alpha\infty = \beta\infty = z$, then $\beta = \alpha\delta$ for some $\delta \in \mathrm{Stab}_{\Gamma(1)}\infty = \Gamma_\infty$. Then $\delta = \pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$ and $f|_k[\beta] = f|_k[\alpha]|_k[\delta] = f|_k[\alpha](\tau + m)(-1)^k$. We want to show that $f|_k[\alpha]$ is holomorphic at $\infty \iff f|_k[\beta]$ is holomorphic as ∞ (the left on the group $\alpha^{-1}\Gamma\alpha$ and the right on $\beta^{-1}\Gamma\beta$).

We claim that the width of the cusp at ∞ for $\alpha^{-1}\Gamma\alpha$ is the width of the cusp at ∞ for $\beta^{-1}\Gamma\beta$. The LHS is $\min\left(h \in \mathbb{N} \mid \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha \cap \Gamma_\infty\right)$. We show this is the same as the corresponding object with β instead of α. Compute

$$\begin{aligned}
&\beta^{-1}\Gamma\beta \cap \Gamma_\infty \\
=&\beta^{-1}(\Gamma \cap \beta\Gamma_\infty\beta^{-1})\beta \\
=&\delta^{-1}\alpha^{-1}(\Gamma \cap \alpha\delta\Gamma_\infty\delta^{-1}\alpha^{-1})\alpha\delta \\
\overset{\delta\in\Gamma_\infty}{=}&\delta^{-1}\alpha^{-1}(\Gamma \cap \alpha\Gamma_\infty\alpha^{-1})\alpha\delta \\
=&\delta^{-1}(\alpha^{-1}\Gamma\alpha \cap \Gamma_\infty)\delta \\
\overset{(\star)}{=}&\alpha^{-1}\Gamma\alpha \cap \Gamma_\infty,
\end{aligned}$$

where $(\star)$ follows as Γ_∞ is abelian. Now $\widetilde{f|_k[\alpha]}(e^{2\pi i\tau/h}) = f|_k[\alpha](\tau)$ and

$$\widetilde{f|_k[\beta]}(e^{2\pi i\tau/h}) = f|_k[\beta](\tau) = f|_k[\alpha](\tau+m)(-1)^k = (-1)^k\widetilde{f|_k[\alpha]}(e^{2\pi i\tau/h}e^{2\pi im/h}).$$

In particular, $\widetilde{f|_k[\alpha]}$ is holomorphic at 0 $\iff$ $\widetilde{f|_k[\beta]}$ is holomorphic at 0, with the same holding for the other conditions. This shows the choice of α does not matter.

Next we show the choice of z does not matter. If $\Gamma \cdot z = \Gamma \cdot z'$ with $z, z' \in \mathbb{P}^1(\mathbb{Q})$, then $z' = \gamma z$ for $\gamma \in \Gamma$. If $\alpha \in \Gamma(1)$, $\alpha\infty = z$, then $\gamma\alpha\infty = \gamma z = z'$. We need to show that $f|_k[\alpha]$ is holomorphic at $\infty \iff f|_k[\gamma\alpha]$ is holomorphic at ∞. This is true as $f|_k[\gamma\alpha] = f|_k[\alpha]$ and $\alpha^{-1}\Gamma\alpha = \alpha^{-1}\gamma^{-1}\Gamma\gamma\alpha$, as $\gamma \in \Gamma$. □

We can define the width of a cusp $\Gamma \cdot z$ to be the width of ∞ as a cusp of $\alpha^{-1}\Gamma\alpha$. The proof of Lemma 5.3 shows that this is well–defined.

Definition 5.5. Let f be a weakly modular function of weight k and level Γ. We say that:

- f is a **modular function** if f is meromorphic at every cusp of Γ.
- f is a **modular form** (of weight k and level Γ) if f is holomorphic in $\mathfrak{h}$ and at every cusp of Γ.
- f is a **cuspidal modular form** if it's a modular form vanishing at every cusp.

Notation. $M_k(\Gamma)$ is the $\mathbb{C}$–vector space of modular forms of weight k and level Γ. We write $S_k(\Gamma) \leq M_k(\Gamma)$ for the $\mathbb{C}$–vector subspace of cuspidal modular forms.

Exercise. If f is a weakly modular function holomorphic in $\mathfrak{h}$, then f is a modular form $\iff$ $\forall\alpha \in \Gamma(1)$, $\exists R > 0$ such that $f|_k[\alpha]$ is bounded in $\{\tau \in \mathfrak{h} \mid \mathrm{Im}(\tau) > R\}$.

Lemma 5.4. Let $k, l \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ a congruence subgroup. Then

(1) If $f \in M_k(\Gamma), g \in M_l(\Gamma)$, then $fg \in M_{k+l}(\Gamma)$.

(2) If $\Gamma' \leq \Gamma$ is another congruence subgroup and $f \in M_k(\Gamma)$, then $f \in M_k(\Gamma')$.

(3) If $\Gamma' \leq \Gamma(1)$ is a congruence subgroup, $\alpha \in GL_2(\mathbb{Q})^+$, $\Gamma' \leq \alpha^{-1}\Gamma\alpha$, and $f \in M_k(\Gamma)$, then $f|_k[\alpha] \in M_k(\Gamma')$.

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Proof. (1) Follows from the definitions as in the case $\Gamma = \Gamma(1)$.

(2) This is a special case of (3) with $\alpha = 1$.

(3) $f|_k[\alpha]$ is holomorphic in $\mathfrak{h}$ and weakly modular of level Γ': if $\gamma' \in \Gamma'$, then $f|_k[\alpha]|_k[\gamma'] = f|_k[\alpha\gamma'\alpha^{-1}\alpha] = f|_k[\alpha\gamma'\alpha^{-1}]|_k[\alpha] = f|_k[\alpha]$ with the last step following from $\alpha\gamma'\alpha^{-1} \in \Gamma$. We need to show that $\forall\beta \in \Gamma(1)$, $f|_k[\alpha\beta](\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$. This is not immediate, since $\alpha\beta \in GL_2(\mathbb{Q})+$, but $\alpha\beta$ is not necessarily in $\Gamma(1)$. We know $GL_2(\mathbb{Q})^+$ acts on $\mathbb{P}^1(\mathbb{Q})$ and $\Gamma(1) \leq GL_2(\mathbb{Q})^+$ acts transitively, so $\exists\gamma \in \Gamma(1)$ such that $\alpha\beta\infty = \gamma\infty$. Thus $\alpha\beta = \gamma\delta$ for some $\delta \in \mathrm{Stab}_{GL_2(\mathbb{Q})^+}(\infty)$, so $\delta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for $a, b, d \in \mathbb{Q}, ad > 0$. Then

$$f|_k[\alpha\beta](\tau) = f|_k[\gamma\delta](\tau) = f|_k[\gamma]|_k[\delta](\tau) = f|_h[\gamma]\left(\frac{a\tau + b}{d}\right) d^{-k}(ad)^{k-1}.$$

We know $f|_k[\gamma](\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$. Suppose $f|_k[\gamma](\tau)$ is bounded in $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > R\}$. Then $f|_k[\gamma]\left(\frac{a\tau+b}{d}\right)$ is bounded in $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > \frac{dR}{a}\}$, concluding the proof.

□

Corollary 5.5. Suppose $M, d \in \mathbb{N}$ and let $N = dM$. If $f \in M_k(\Gamma_0(M))$, then $f(d\tau) \in M_k(\Gamma_0(N))$.

Proof. $\Gamma_0(M) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod{M}\right\}$. The lemma says that if $f \in M_k(\Gamma_0(M))$, then $f|_k\left[\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right] \in M_k(\Gamma')$ for any $\Gamma' \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(M) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$.

First note $f|_k\left[\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\right](\tau) = f(d\tau)d^{k-1} \in M_k(\Gamma') \iff f(d\tau) \in M_k(\Gamma')$.

Claim: $\Gamma_0(N) \leq \Gamma' \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(M) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \iff \Gamma' \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \leq$ $\Gamma_0(M)$ by conjugation.

This is true as $\Gamma' \leq \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} A & dB \\ d^{-1}C & D \end{pmatrix}$, and if $c \equiv 0 \pmod{N}$, then $d^{-1}C \equiv 0 \pmod{M}$. □

Example 5.1. If $k \geq 4$ is even, then $M_k(\Gamma_0(N))$ contains $G_k(d\tau)$ $\forall d \mid N$.

We now show how to construct modular forms using θ–functions.

Example 5.2. The Jacobi θ function for $\tau \in \mathfrak{h}$ is

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n \in \mathbb{Z}} q_2^{n^2}$$

for $q_2 = e^{\pi i \tau}$. The power series

$$1 + 2 \sum_{n \geq 1} q_2^{n^2}$$

is absolutely convergent when $|q_2| < 1$, so θ is holomorphic in $\mathfrak{h}$.

We will show that certain powers of θ are modular forms. These are interesting generating functions: for $k \in \mathbb{N}$,

$$\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) q_2^n,$$

where $r_k(n) = |\{\overline{x} \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$.

Proposition 5.6 (Poisson summation formula)**.** Consider $f : \mathbb{R} \to \mathbb{C}$ continuous such that $\exists C, \delta > 0$ such that $\forall t \in \mathbb{R}$,

$$|f(t)| \leq \frac{C}{(1 + |t|)^{\delta+1}}.$$

Let $\hat{f}(s) = \int_{t=-\infty}^{\infty} f(t) e^{-2\pi i s t} \mathrm{d}t$ and suppose $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$. Then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Proof. Define $F : \mathbb{R} \to \mathbb{C}$ by $F(t) = \sum_{n \in \mathbb{Z}} f(n+t)$. This is uniformly convergent in any compact interval $[a, b]$ (Exercise: prove using Weierstrass M–test and the bound on $|f(t)|$). Thus F is continuous on the real line and $\mathbb{Z}$–periodic.

Define $\hat{F}(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n t}$. This is uniformly convergent in $\mathbb{R}$ using Weierstrass M–test and the fact that $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$. Hence $\hat{F}$ is continuous and $\mathbb{Z}$–periodic.

We claim that $F = \hat{F}$, which will imply the proposition by plugging in $t = 0$. For this, we will prove that $\forall m \in \mathbb{Z}$, $\int_{t=0}^{1} F(t) e^{-2\pi i m t} \mathrm{d}t = \int_{t=0}^{1} \hat{F}(t) e^{-2\pi i m t} \mathrm{d}t$, i.e. the Fourier transform coefficients are equal. The LHS is

$$\int_{t=0}^{1}\sum_{\in\mathbb{Z}} f(n+t)e^{-2\pi imt}\mathrm{d}t \overset{(\star)}{=} \sum_{n\in\mathbb{Z}}\int_{t=0}^{1} f(n+t)e^{-2\pi imt}\mathrm{d}t = \int_{t=-\infty}^{\infty} f(t)e^{-2\pi imt} = \hat{f}(m),$$

where $(\star)$ is justified by uniform convergence. To conclude, the RHS is

$$\int_{t=0}^{1}\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi i(n-m)dt}\mathrm{d}t = \sum_{n\in\mathbb{Z}}\int_{t=0}^{1}\hat{f}(n)e^{2\pi i(n-m)t}\mathrm{d}t = \hat{f}(m).$$

□

We apply this to $f_y(t) = e^{-\pi t^2 y}$ for $y > 0$ fixed. Then

$$\theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = \sum_{n\in\mathbb{Z}} f_y(n).$$

To apply the Poisson summation formula, we compute

$$\begin{aligned}\hat{f}_y(s) = &\int_{t=-\infty}^{\infty} e^{-\pi t^2 y}e^{-2\pi ist}\mathrm{d}t = \int_{t=-\infty}^{\infty} e^{-\pi\left(t\sqrt{y}+is/\sqrt{y}\right)^2 e^{-\pi s^2/y}}\mathrm{d}t\\ =&e^{-\pi s^2/y}\frac{1}{\sqrt{y}}\int_{x=-\infty}^{\infty} e^{-\pi(x+is/\sqrt{y})^2\mathrm{d}x} = \frac{1}{\sqrt{y}}e^{-\pi s^2 y}\int_{-\infty+is/\sqrt{y}}^{\infty+is\sqrt{y}} e^{-\pi x^2}\mathrm{d}x\end{aligned}$$

(i.e. we take the contour integral over the horizontal line intersecting the imaginary axis at $s/\sqrt{y}$). By moving the contour, this equals

$$\frac{1}{\sqrt{y}}e^{-\pi s^2/y}\underbrace{\int_{x=-\infty}^{\infty} e^{-\pi x^2}\mathrm{d}x}_{1} = \frac{1}{\sqrt{y}}e^{-\pi s^2/y} = \frac{1}{\sqrt{y}}f_{y^{-1}}(s).$$

The fact that moving the contour is justified is left as an exercise. For this, we need to show that for $R\to\infty$, over the vertical line segments connecting R to $R + is/\sqrt{y}$ and $-R$ to $-R + is\sqrt{y}$, the contour integrals go to zero.

The Poisson summation formula now gives

$$\theta(iy) = \sum_{n\in\mathbb{Z}} f_y(n) = \sum_{n\in\mathbb{Z}}\hat{f}(n) = \frac{1}{\sqrt{y}}\sum_{n\in\mathbb{Z}} f_{y^{-1}}(n) = \frac{1}{\sqrt{y}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/y} = \frac{1}{\sqrt{y}}\theta\left(\frac{1}{y}\right).$$

The functions $\theta(\tau)$ and $\sqrt{\frac{\tau}{i}}^{-1}\theta\left(-\frac{1}{\tau}\right)$ are holomorphic in $\mathfrak{h}$ and equal on the line $\tau = iy$ (which has a limit point, so the identity principle applies). Thus by the identity principle,

$$\theta(\tau) = \sqrt{\frac{\tau}{i}}^{-1}\theta\left(-\frac{1}{\tau}\right).$$

Here $\sqrt{\frac{\tau}{i}}$ is the unique branch of the square root defined in $\mathfrak{h}$ which takes the value $\sqrt{y} > 0$ when $\tau = iy$.

Proposition 5.7. If $k \in 8\mathbb{N}$, then $\theta^k \in M_{k/2}(\Gamma)$, where $\Gamma = \Gamma(2) \sqcup S\Gamma(2)$ (i.e. all matrices that modulo 2 are congruent to the identity or $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$).

Proof. We know θ^k is holomorphic in $\mathfrak{h}$ and θ is a function of q^k, hence we find $\theta(\tau+2) = \theta(\tau)$. Hence $\theta^k(\tau+2) = \theta^k(\tau) = \theta^k|_{k/2}[T^2] = \theta^k$, as $T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

We next claim $\theta^k|_{k/2}[S] = \theta^k$. The LHS is

$$\theta^k\left(-\frac{1}{\tau}\right)(\tau)^{-k/2} = \left(\theta(\tau)\sqrt{\frac{\tau}{i}}\right)^k \tau^{-k/2} = \theta^k(\tau)\left(\frac{\tau}{i}\right)^{k/2}\tau^{-k/2} = \theta^k(\tau)$$

Fact: $\Gamma = \langle S, T^2 \rangle$. This is similar to $\Gamma(1) = \langle S, T \rangle$, but requires a lot (a lecture's worth) of details. Using this we get

$$\theta^k|_{k/2}[\gamma] = \theta^k \ \forall \gamma \in \Gamma,$$

hence θ^k is weakly modular of weight $\frac{k}{2}$ and level Γ. $\square$