Part III - Algebraic Geometry

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Contents

0 Introduction

6 Oct 2022, Lecture 1

The course consists of four parts.

(1) Basics of sheaves on topological spaces.

(2) Definition of schemes and morphisms.

(3) Properties of schemes (e.g. the algebraic geometry notion of compactness and other properties).

(4) A rapid introduction to the cohomology of schemes.

The main reference for the course is Hartshorne's *Algebraic Geometry*.

1 Beyond algebraic varieties

08 Oct 2022, Lecture 2

1.1 Summary of classical algebraic geometry

We let $k = \overline{k}$ be a algebraically closed field and consider $\mathbb{A}^n_k = \mathbb{A}^n = k^n$ as a set.

Definition 1.1. An **affine variety** is a subset $V \subset \mathbb{A}^n$ of the form $\mathbb{V}(S)$ with $S \subset k[x_1, \ldots, x_n]$, where $\mathbb{V}$ is the common vanishing locus.

Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$ (the ideal generated by S). By Hilbert Basis Theorem (since $k[x_1, \ldots, x_n]$ is noetherian), $\mathbb{V}(I(S)) = \mathbb{V}(S')$ for some finite set $S \subset k[x_1, \ldots, x_n]$.

In fact, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}$$

is the **radical** of I. For example, in $k[x]$, if $I = (x^2)$, then $\sqrt{I} = (x)$.

Definition 1.2. Given varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, a **morphism** is a (set-theoretic) map $\phi : V \to W \subset \mathbb{A}^m_k$ such that if $\phi = (f_1, \ldots, f_m)$, then each f_i is the restriction of a polynomial in $\{x_1, \ldots, x_n\}$.

An **isomorphism** is a morphism with a two–sided inverse.

Our basic correspondence is

$$\{\text{Affine varieties over } k\}/\text{up to isomorphism}$$
$$\leftrightarrow$$
$$\{\text{finitely generated } k\text{–algebras } A \text{ without nilpotent elements}\}$$

A finitely generated k–algebra is just a quotient of a polynomial ring in finitely many variables. A nilpotent element is such that some power of it is zero. For example, in $k[x]/(x^2)$, the element x is nilpotent.

How does this correspondence work? Given a variety V (representing an isomorphism class), we write $V = \mathbb{V}(I)$ for $I \subset k[x_1, \ldots, x_n]$ a radical ideal[1], and map $V \mapsto k[x_1, \ldots, x_n]/I$.

For the reverse, if A is a finitely generated nilpotent free algebra, then $A \cong k[y_1, \ldots, y_m]/J$ where we can choose J to be radical (exercise: why?).

We have to check that this is independent of our choice on both sides (exercise: think through this, it should be clear).

Definition 1.3. The algebra associated to V is classically denoted $k[V]$ and called the **coordinate ring of V**.

We have the compatibility of morphisms with our basic correspondence: there is a bijection between

$$\text{Morphisms}(V, W) \leftrightarrow \text{Ring homomorphisms}_k(k[W], k[V])$$

(here RingHom_k means that our homomorphisms preserve k).

We can now make our set into a topological space:

Definition 1.4. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The **Zariski topology** on V is defined such that the closed sets are $\mathbb{V}(S)$, where $S \subset k[V]$.

If $V \cong W$, then the Zariski topological spaces are homeomorphic as varieties (exercise).

Theorem 1.1 (Nullstellensatz)**.** Fix V a variety and let $k[V]$ be its coordinate ring. Given $p \in V$, we can produce a homomorphism $\text{ev}_p : k[V] \to k$ by sending $f \mapsto f(p)$. Note that ev_p is surjective (since we have constant functions), hence $\ker(\text{ev}_p) = m_p$ is a maximal ideal, giving us a map

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}.$$

Nullstellensatz says that this is actually a bijection. For the converse map, given $m \subset k[V]$, we get a quotient $k[V] \to k[V]/m = k$ (Nullstellensatz says this extension is finite, hence must be k). So using/choosing a representation for V in $k[x_1, \ldots, x_n]$ gives a surjective homomorphism onto k and specifies a bunch of points.

[1] A radical ideal is an ideal equal to its radical.

1.2 Limitations of classical algebraic geometry

Question. What is an abstract variety, i.e. "some "space" X such that locally as a cover $\{U_i\}$, each U_i is an affine variety, compatible with overlaps".

Example 1.1 (non–algebraically closed fields)**.** Take $I = (x^2+y^2+1) \subset \mathbb{R}[x,y]$. Then $\mathbb{V}(I) = \varnothing \subset \mathbb{R}^2$, but I is prime, so radical, so nullstellensatz fails.

Question. On what topological space is $\mathbb{R}[x,y]/(x^2+y^2+1)$ "naturally" the set of functions? (or $\mathbb{Z}$, or $\mathbb{Z}[x]$).

Example 1.2 (Why restrict to radical ideals?)**.** Take $C = \mathbb{V}(y-x^2) \subset \mathbb{A}^2_k$ and $D = \mathbb{V}(x,y)$, so $C \cap D = \mathbb{V}(y, y-x^2) = \mathbb{V}(x,y) = \{(0,0)\}$. This is a single point, but if $D_\delta = \mathbb{V}(y+\delta)$ for some $\delta \in k$, then $C \cap D_\delta = \{\pm\sqrt{\delta}\}$, which is 2 points for all $\delta \neq 0$. In other words, intersections of varieties don't want to be varieties.

1.3 The spectrum of a ring

11 Oct 2022, Lecture 3

Let A be a commutative ring with identity. We will define a topological space on which A is the ring of functions.

Definition 1.5. The **Zariski spectrum** of A is

$$\text{Spec } A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

A ring homomorphism $\phi : A \to B$ induces a map $\phi^{-1} : \text{Spec } B \to \text{Spec } A$ by $q \mapsto \phi^{-1}(q)$. In general, the preimage of a prime ideal is a prime ideal.

Warning. This would fail if we only considered maximal ideals, since the preimage of a maximal ideal need not be maximal.

Given $f \in A$ and $\mathfrak{p} \in \text{Spec}(A)$, we have an induced $\overline{f} \in A/\mathfrak{p}$ obtained via a quotient. Informally, we can evaluate any $f \in A$ at points $\mathfrak{p} \in \text{Spec}(A)$ with the caveat that the codomain of this evaluation depends on $\mathfrak{p}$.

Example 1.3. Take $A = \mathbb{Z}$. Then $\text{Spec } A = \text{Spec }(\mathbb{Z}) = \{p \mid p \text{ is prime}\} \cup \{(0)\}$. Let's pick an element in $\mathbb{Z}$, say $132 \in \mathbb{Z}$. Given a prime p, we can look at $132 \pmod p \in \mathbb{Z}/p$. The takeaway here is that

$$\begin{gathered} \text{Spec } \mathbb{Z} \to \text{Space} \\ 132 \in \mathbb{Z} \to \text{a function} \\ 132 \ (\text{mod } p) \to \text{value of that function at } p. \end{gathered}$$

Note that based on the value of p, our codomain changes from point to point.

Example 1.4. Take $A = \mathbb{R}[x]$, then $\text{Spec } \mathbb{R}[x] = \mathbb{C}/\text{complex conjugation} \cup \{(0)\}$.

Exercise. Draw Spec $\mathbb{Z}[x]$ and Spec $k[x]$ for k any field (i.e. describe all prime ideals and their containment). This is on example sheet 1.

Example 1.5. If $A = \mathbb{C}[x]$, then Spec $A = \mathbb{C} \cup \{(0)\}$, where given $a \in \mathbb{C}$, we send it to the maximal ideal $\langle z - a \rangle$.

1.4 A topology on Spec A

Fix $f \in A$. Then $\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \equiv 0 \pmod{\mathfrak{p}}\} \subset \text{Spec } A$. (Note that $f \equiv 0 \pmod{\mathfrak{p}}$ is the same as $f \in \mathfrak{p}$).

Similarly for $J \subset A$ an ideal, $\mathbb{V}(J) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p} \ \forall f \in J\}$.

Proposition 1.2. The sets $\mathbb{V}(J) \subset \text{Spec } A$ ranging over all ideals J form the closed sets of a topology on Spec A. This topology is called the **Zariski topology**.

Proof. Easy fact: $\varnothing$ and Spec A are closed, since we have functions 1 (vanishing nowhere) and 0 (vanishing everywhere). Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ (this is because $I_1 + I_2$ is the smallest ideal containing $I_1 \cup I_2$), arbitrary intersections are closed.

Finally, we claim $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$. The containment $\subset$ is clear: if a prime ideal contains I_1 or I_2, it contains $I_1 \cap I_2$. Conversely, $I_1 I_2 \subset I_1 \cap I_2$, so if $I_1 I_2 \subset I_1 \cap I_2 \subset \mathfrak{p}$, then by primality $I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p}$. $\square$

Example 1.6. Let $k = \mathbb{C}$ and consider Spec $\mathbb{C}[x, y]$. We make a few observations:

- The point $(0) \in \text{Spec } \mathbb{C}[x, y]$ is dense in the Zariski topology, i.e. $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[x, y]$ because every prime ideal contains (0) (because we are in an integral domain).
- Consider the prime ideal $(y^2 - x^3)$ (which is prime since the quotient is an integral domain). Consider a maximal ideal $\mathfrak{m}_{a,b} = (x - a, y - b)$. We can ask: when is $\mathfrak{m}_{a,b} \in \overline{\{(y^2 - x^3)\}}$? The answer: if and only if $b^2 = a^3$, e.g. $(1, 1)$ (see example sheet 1). The lesson here is that points are not closed in the Zariski topology.

1.5 Functions on opens

Definition 1.6. Let $f \in A$. Define the **distinguished open** corresponding to f to be

$$\mathcal{U}_f = (\text{Spec}(A))/\mathbb{V}(f).$$

Example 1.7.

- Let $A = \mathbb{C}[x]$, so Spec $A = \mathbb{C} \cup \{(0)\}$ (with the Zariski topology). Take $f = x$ and consider $\mathcal{U}_x$. Recall the bijection Spec $\mathbb{C} \leftrightarrow \mathbb{C} \cup \{(0)\}$ by $(x - a) \leftarrow\!\shortmid a \in \mathbb{C}$ and $(0) \leftarrow\!\shortmid (0)$. Then $\mathbb{V}(x) = \{\mathfrak{p} \in \text{Spec } A \mid x \in \mathfrak{p}\} = \{(x)\}$, so $\mathcal{U}_f = \text{Spec } A \setminus \{(x)\}$.

- More generally, suppose we fix $a_1, \ldots, a_r \in \mathbb{C}$, then Spec $A \backslash \{(x-a_i)\}_{i=1}^r = \mathcal{U}$ and $\mathcal{U} = \mathcal{U}_f$, where $f = \prod_{i=1}^r (x - a_i)$.

Lemma 1.3. The distinguished opens $\mathcal{U}_f$ taken over all $f \in A$ form a basis for the Zariski topology on Spec A.

Proof. Left as an exercise on example sheet 1. $\square$

A bit of commutative algebra:

Definition 1.7. Given $f \in A$, the **localization of** A **at** f is $A_f = A[x]/(xf-1)$, which we can informally think of as $A_f = A[\frac{1}{f}]$.

Lemma 1.4. The distinguished open $\mathcal{U}_f \subset$ Spec A is naturally homeomorphic to Spec A_f.