Part III - Elliptic Curves

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0 Introduction

19 Jan 2024, Lecture 1

The best books for the course include *The arithmetic of elliptic curves* by Silverman, Springer 1996, and *Lectures on elliptic curves* by Cassels, CUP 1991.

1 Fermat's Method of Infinite Descent

A right–angled triangle Δ has $a^2 + b^2 = c^2$ and $\text{area}(\Delta) = \frac{1}{2}ab$.

Definition 1.1. Δ is **rational** if $a, b, c \in \mathbb{Q}$. Δ is **primitive** if $a, b, c \in \mathbb{Z}$ are coprime.

Note that a primitive triangle has pairwise coprime side lengths because $a^2 + b^2 = c^2$.

Lemma 1.1. Every primitive triangle is of the form $(u^2 - v^2, 2uv, u^2 + v^2)$ for some integers $u > v > 0$.

Proof. WLOG let a, b, c be odd, even, odd. Then $\left(\frac{b}{2}\right)^2 = \frac{c+a}{2}\frac{c-a}{2}$, where we note that the RHS is a product of positive coprime integers. By unique factorization, $\frac{c+a}{2} = u^2$, $\frac{c-a}{2} = v^2$ for $u, v \in \mathbb{Z}$. This gives the desired result. $\square$

Definition 1.2. $D \in \mathbb{Q}_{>0}$ is a **congruent** number if there exists a rational triangle Δ with $\text{area}(\Delta) = D$.

Note that it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

Example 1.1. $D = 5, 6$ are congruent.

Lemma 1.2. $D \in \mathbb{Q}_{>0}$ is congruent $\iff$ $Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}, y \neq 0$.

Proof. Lemma 1.1 shows that D congruent $\implies$ $Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}, w \neq 0$. This implication also obviously goes the other way. To finish, divide through by w^4 and take $x = \frac{u}{v}, y = \frac{w}{v^2}$. $\square$

Fermat showed that 1 is not a congruent number.

Theorem 1.3. There is no solution to $w^2 = uv(u + v)(u - v)$ for $u, v, w \in \mathbb{Z}, w \neq 0$.

Proof. WLOG assume u, v are coprime and that $u, w > 0$. If $v < 0$, then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $\left(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2}\right)$. Then $u, v, u+v, u-v$ are pairwise coprime positive integers with their product a square, so by unique factorization in $\mathbb{Z}$, $u = a^2, v = b^2, u + v = c^2, u - v = d^2$ for $a, b, c, d \in \mathbb{Z}$.

Since $u \not\equiv v \pmod 2$, both c and d are odd. Then $\left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 = \frac{c^2+d^2}{2} = u = a^2$. This gives a primitive triangle with area $\frac{c^2-d^2}{8} = \frac{v}{4} = \left(\frac{b^2}{2}\right)$.

Let $w_1 = \frac{b}{2}$, then by Lemma 1.1, $w_1^2 = u_1 v_1 (u_1 + v_1)(u_1 - v_1)$ for some $u_1, v_1 \in \mathbb{Z}$. Hence we have a new solution to our original question, with $4w_1^2 = b^2 = v \mid w^2 \implies w_1 \le \frac{w}{2}$, so we're done by infinite descent. □

A variant for polynomials. In the above, K is a field with char $K \neq 2$. Let $\overline{K}$ be the algebraic closure of K and consider for this whole section K with char $K \neq 2$.

Lemma 1.4. Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for 4 distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.

Proof. WLOG let $K = \overline{K}$ by extending if necessary. Changing coordinates on $\mathbb{P}^1$ (i.e. multiplying by a 2×2 invertible matrix), we may assume that the points $(\alpha : \beta)$ are $(1 : 0)$, $(0 : 1)$, $(1 : -1)$, $(1 : -\lambda)$ for $\lambda \in K \setminus \{0, 1\}$. Since our field is algebraically closed, let $\mu = \sqrt{\lambda}$. Then $u = a^2, v = b^2, u - v = (a+b)(a-b), u - \lambda v = (a + \mu b)(a - \mu b)$.

Unique factorization in $K[t]$ implies that $a + b, a - b, a + \mu b, a - \mu b$ are squares (since the necessary terms are coprime up to units, i.e. constants). But $\max(\deg(a), \deg(b)) \le \frac{1}{2} \max(\deg(u), \deg(v))$, so by Fermat's method of infinite descent, $u, v \in K$. □

Definition 1.3.
(i) An **elliptic curve** E/K is the projective closure of the plane affine curve $y^2 = f(x)$ (this is called a Weierstrass equation) where $f \in K[x]$ is a monic cubic polynomial with distinct roots in $\overline{K}$.
(ii) For L/K any field extension, $E(L) = \{(x, y) \in L^2 \mid y^2 = f(x)\} \cup \{0\}$ (the point at infinity in the projective closure), it turns out that $E(L)$ is naturally an abelian group.

In this course, we study $E(K)$ for K a finite field, local field, number field.

Lemma 1.2 and Theorem 1.3 show that if $E : y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$.

Corollary 1.5. Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.

Proof. WLOG $K = \overline{K}$. By a change of coordinates, we may assume $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ for $u, v \in K(t)$ coprime. Then $w^2 = uv(u - v)(u - \lambda v)$ for some $w \in K[t]$. Unique factorization in $K[t]$ shows that $u, v, u - v, u - \lambda v$ are all squares, so by Lemma 1.4, $u, v \in K$, so $x, y \in K$. □

2 Some remarks on algebraic curves

22 Jan 2024, Lecture 2

In this section, work over an algebraically closed field $K = \overline{K}$.

Definition 2.1. A plane curve $C = \{f(x,y) = 0\} \subset \mathbb{A}^2$ (for $f \in K[x,y]$ irreducible) is **rational** if it has a rational parametrization, i.e. $\exists \phi, \psi \in K(t)$ such that

(i) The map $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{\text{finite set}\}$.

(ii) $f(\phi(t), \psi(t)) = 0$ in $K(t)$.

Example 2.1. (a) Any nonsingular conic is rational. For example, for $x^2 + y^2 = 1$, take a line with slope t through $(-1, 0)$ (the anchor) and solve to get the rational parametrization $(x,y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

(b) Any singular plane cubic is rational, for example $y^2 = x^3$ giving $(x,y) = (t^2, t^3)$ with the anchor at the singularity $(0,0)$ and $y^2 = x^2(x+1)$ with the parametrization to be computed on Ex. Sheet 1 (anchor still at $(0,0)$).

(c) Corollary 1.5 shows that elliptic curves are not rational.

Remark. The genus $g(C) \in \mathbb{Z}_{\geq 0}$ is an invariant of a smooth projective curve C. If $K = \mathbb{C}$, then $g(C)$ is the genus of the Riemann surface. A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

Proposition 2.1. (Here we still assume $K = \overline{K}$). Let C be a smooth projective curve.

- C is rational (see Definition 2.1) $\iff g(C) = 0$.
- C is an elliptic curve $\iff g(C) = 1$.

Proof. (i) Omitted.

(ii) ($\implies$): Check C is a smooth plane curve in $\mathbb{P}^2$ (see Ex. Sheet 1) and use the above remark.

($\impliedby$): We will see this later.

□

Order of vanishing. Let C be an algebraic curve with function field $K(C)$ and let $P \in C$ be a smooth point. Write $\mathrm{ord}_P(f)$ for the order of vanishing of $f \in K(C)$ at P (which is negative if f has a pole at P).

Fact. $\mathrm{ord}_P : K(C)^\times \to \mathbb{Z}$ is a discrete valuation, i.e. $\mathrm{ord}_P(f_1 f_2) = \mathrm{ord}_P(f_1) + \mathrm{ord}_P(f_2)$ and $\mathrm{ord}_P(f_1 + f_2) \geq \min(\mathrm{ord}_P(f_1), \mathrm{ord}_P(f_2))$.

Definition 2.2. We say $t \in K(C)^\times$ is a **uniformizer** at P if $\mathrm{ord}_P(t) = 1$.

Example 2.2. $C = \{g = 0\} \subset \mathbb{A}^2$ for $g \in K[x, y]$. Then $K(C) = \text{Frac}\left(\frac{K[x,y]}{(g)}\right)$. Write $g = g_0 + g_1(x, y) + g_2(x, y) + \dots$ for g_i homogeneous of degree i. Suppose $P = (0, 0)$ is a smooth point, e.g. $g_0 = 0$ and let $g_1(x, y) = \alpha x + \beta y$ with α, β not both zero ($\alpha x + \beta y = 0$ gives a tangent to the curve at P). Let $\gamma, \delta \in K$ and consider also the line $\gamma x + \delta y$ through P. Then it is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\alpha\delta - \beta\gamma \neq 0$.

Example 2.3. Consider $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$ for $\lambda \neq 0, 1$ and consider its projective closure by taking $x = \frac{X}{Z}, y = \frac{Y}{Z}$ to get $\{Y^2Z = X(X-Z)(X-\lambda Z)\} \subset \mathbb{P}^2$. This has only one point at infinity, $P = (0 : 1 : 0)$. Our aim is to compute $\text{ord}_P(x)$ and $\text{ord}_P(y)$.

For this, put $t = \frac{X}{Y}, w = \frac{Z}{Y}$, so $w \overset{(\dagger)}{=} t(t - w)(t - \lambda w)$. Now P is the point $(t, w) = (0, 0)$, which is a smooth point with $\text{ord}_P(t) = \text{ord}_P(t - w) = \text{ord}_P(t - \lambda w) = 1$, so ($\dagger$) gives $\text{ord}_P(w) = 3$. We now find

$$\text{ord}_P(x) = \text{ord}_P\left(\frac{X}{Z}\right) = \text{ord}_P\left(\frac{t}{w}\right) = 1 - 3 = -2$$
$$\text{ord}_P(y) = \text{ord}_P\left(\frac{Y}{Z}\right) = \text{ord}_P\left(\frac{1}{w}\right) = -3.$$

Riemann–Roch space. Let C be a smooth projective curve.

Definition 2.3. A **divisor** is a formal sum of points on C, say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. We say $\deg D = \sum_{P \in C} n_P$.

D is **effective** (written $D \geq 0$) if $n_P \geq 0\ \forall P \in C$. If $f \in K(C)^\times$, then $\text{div}(f) = \sum_{P \in C} \text{ord}_P(f)P$. The Riemann–Roch space of $D \in \text{Div}(C)$ is

$$\mathcal{L}(D) = \{f \in K(C)^\times \mid \text{div}(f) + D \geq 0\} \cup \{0\},$$

i.e. the K–vector space of rational functions on C with "poles no worse than specified by D".

We quote Riemann–Roch for surfaces of genus 1: We have

$$\dim \mathcal{L}(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ 0 \text{ or } 1 & \text{if } \deg D = 0 \\ 0 & \text{if } \deg D < 0. \end{cases}$$

Example 2.4. We revisit Example 2.3. We have $\mathcal{L}(2P) = \langle 1, x\rangle$ and $\mathcal{L}(3P) = \langle 1, x, y\rangle$.