# Part III - Elliptic Curves
### Lectured by Tom Fisher

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# Contents

# 0   Introduction

The best books for the course include *The arithmetic of elliptic curves* by Silverman, Springer 1996, and *Lectures on elliptic curves* by Cassels, CUP 1991.

# 1   Fermat's Method of Infinite Descent

A right–angled triangle $\Delta$ has $a^2 + b^2 = c^2$ and $\text{area}(\Delta) = \frac{1}{2}ab$.

**Definition 1.1.** $\Delta$ is **rational** if $a, b, c \in \mathbb{Q}$. $\Delta$ is **primitive** if $a, b, c \in \mathbb{Z}$ are coprime.

Note that a primitive triangle has pairwise coprime side lengths because $a^2 + b^2 = c^2$.

**Lemma 1.1.** Every primitive triangle is of the form $(u^2 - v^2, 2uv, u^2 + v^2)$ for some integers $u > v > 0$.

*Proof.* WLOG let $a, b, c$ be odd, even, odd. Then $\left(\frac{b}{2}\right)^2 = \frac{c+a}{2} \frac{c-a}{2}$, where we note that the RHS is a product of positive coprime integers. By unique factorization, $\frac{c+a}{2} = u^2, \frac{c-a}{2} = v^2$ for $u, v \in \mathbb{Z}$. This gives the desired result. $\quad\square$

**Definition 1.2.** $D \in \mathbb{Q}_{>0}$ is a **congruent** number if there exists a rational triangle $\Delta$ with $\text{area}(\Delta) = D$.

Note that it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

**Example 1.1.** $D = 5, 6$ are congruent.

**Lemma 1.2.** $D \in \mathbb{Q}_{>0}$ is congruent $\iff Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}, y \neq 0$.

*Proof.* Lemma 1.1 shows that $D$ congruent $\implies Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}, w \neq 0$. This implication also obviously goes the other way. To finish, divide through by $w^4$ and take $x = \frac{u}{v}, y = \frac{w}{v^2}$. $\quad\square$

Fermat showed that 1 is not a congruent number.

**Theorem 1.3.** There is no solution to $w^2 = uv(u + v)(u - v)$ for $u, v, w \in \mathbb{Z}, w \neq 0$.

*Proof.* WLOG assume $u, v$ are coprime and that $u, w > 0$. If $v < 0$, then replace $(u, v, w)$ by $(-v, u, w)$. If $u, v$ are both odd, then replace $(u, v, w)$ by $\left(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2}\right)$. Then $u, v, u+v, u-v$ are pairwise coprime positive integers with their product a square, so by unique factorization in $\mathbb{Z}$, $u = a^2, v = b^2, u + v = c^2, u - v = d^2$ for $a, b, c, d \in \mathbb{Z}$.

Since $u \not\equiv v \pmod 2$, both $c$ and $d$ are odd. Then $\left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 = \frac{c^2+d^2}{2} = u = a^2$. This gives a primitive triangle with area $\frac{c^2-d^2}{8} = \frac{v}{4} = \left(\frac{b^2}{2}\right)$.

Let $w_1 = \frac{b}{2}$, then by Lemma 1.1, $w_1^2 = u_1 v_1 (u_1 + v_1)(u_1 - v_1)$ for some $u_1, v_1 \in \mathbb{Z}$. Hence we have a new solution to our original question, with $4w_1^2 = b^2 = v \mid w^2 \implies w_1 \leq \frac{w}{2}$, so we're done by infinite descent. $\square$

**A variant for polynomials.** In the above, $K$ is a field with char $K \neq 2$. Let $\overline{K}$ be the algebraic closure of $K$ and consider for this whole section $K$ with char $K \neq 2$.

**Lemma 1.4.** Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for 4 distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.

*Proof.* WLOG let $K = \overline{K}$ by extending if necessary. Changing coordinates on $\mathbb{P}^1$ (i.e. multiplying by a $2 \times 2$ invertible matrix), we may assume that the points $(\alpha : \beta)$ are $(1 : 0), (0 : 1), (1 : -1), (1 : -\lambda)$ for $\lambda \in K \setminus \{0, 1\}$. Since our field is algebraically closed, let $\mu = \sqrt{\lambda}$. Then $u = a^2, v = b^2, u - v = (a + b)(a - b), u - \lambda v = (a + \mu b)(a - \mu b)$.

Unique factorization in $K[t]$ implies that $a + b, a - b, a + \mu b, a - \mu b$ are squares (since the necessary terms are coprime up to units, i.e. constants). But $\max(\deg(a), \deg(b)) \leq \frac{1}{2} \max(\deg(u), \deg(v))$, so by Fermat's method of infinite descent, $u, v \in K$. $\square$

**Definition 1.3.**   (i) An **elliptic curve** $E/K$ is the projective closure of the plane affine curve $y^2 = f(x)$ (this is called a Weierstrass equation) where $f \in K[x]$ is a monic cubic polynomial with distinct roots in $\overline{K}$.

  (ii) For $L/K$ any field extension, $E(L) = \{(x, y) \in L^2 \mid y^2 = f(x)\} \cup \{0\}$ (the point at infinity in the projective closure), it turns out that $E(L)$ is naturally an abelian group.

In this course, we study $E(K)$ for $K$ a finite field, local field, number field.

Lemma 1.2 and Theorem 1.3 show that if $E : y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$.

**Corollary 1.5.** Let $E/K$ be an elliptic curve. Then $E(K(t)) = E(K)$.

*Proof.* WLOG $K = \overline{K}$. By a change of coordinates, we may assume $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ for $u, v \in K(t)$ coprime. Then $w^2 = uv(u - v)(u - \lambda v)$ for some $w \in K[t]$. Unique factorization in $K[t]$ shows that $u, v, u - v, u - \lambda v$ are all squares, so by Lemma 1.4, $u, v \in K$, so $x, y \in K$. $\square$

# 2   Some remarks on algebraic curves

In this section, work over an algebraically closed field $K = \overline{K}$.

**Definition 2.1.** A plane curve $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$ (for $f \in K[x, y]$ irreducible) is **rational** if it has a rational parametrization, i.e. $\exists \phi, \psi \in K(t)$ such that

(i) The map $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{\text{finite set}\}$.

(ii) $f(\phi(t), \psi(t)) = 0$ in $K(t)$.

**Example 2.1.** (a) Any nonsingular conic is rational. For example, for $x^2 + y^2 = 1$, take a line with slope $t$ through $(-1, 0)$ (the anchor) and solve to get the rational parametrization $(x, y) = \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$.

(b) Any singular plane cubic is rational, for example $y^2 = x^3$ giving $(x, y) = (t^2, t^3)$ with the anchor at the singularity $(0,0)$ and $y^2 = x^2(x+1)$ with the parametrization to be computed on Ex. Sheet 1 (anchor still at $(0,0)$).

(c) Corollary 1.5 shows that elliptic curves are not rational.

**Remark.** The genus $g(C) \in \mathbb{Z}_{\geq 0}$ is an invariant of a smooth projective curve $C$. If $K = \mathbb{C}$, then $g(C)$ is the genus of the Riemann surface. A smooth plane curve $C \subset \mathbb{P}^2$ of degree $d$ has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

**Proposition 2.1.** (Here we still assume $K = \overline{K}$). Let $C$ be a smooth projective curve.

- $C$ is rational (see Definition 2.1) $\iff g(C) = 0$.

- $C$ is an elliptic curve $\iff g(C) = 1$.

*Proof.*   (i) Omitted.

(ii) ( $\implies$ ): Check $C$ is a smooth plane curve in $\mathbb{P}^2$ (see Ex. Sheet 1) and use the above remark.

( $\impliedby$ ): We will see this later.

$\square$

**Order of vanishing.** Let $C$ be an algebraic curve with function field $K(C)$ and let $P \in C$ be a smooth point. Write $\mathrm{ord}_P(f)$ for the order of vanishing of $f \in K(C)$ at $P$ (which is negative if $f$ has a pole at $P$).

**Fact.** $\mathrm{ord}_P : K(C)^\times \to \mathbb{Z}$ is a discrete valuation, i.e. $\mathrm{ord}_P(f_1 f_2) = \mathrm{ord}_P(f_1) + \mathrm{ord}_P(f_2)$ and $\mathrm{ord}_P(f_1 + f_2) \geq \min(\mathrm{ord}_P(f_1), \mathrm{ord}_P(f_2))$.

**Definition 2.2.** We say $t \in K(C)^\times$ is a **uniformizer** at $P$ if $\mathrm{ord}_P(t) = 1$.

**Example 2.2.** $C = \{g = 0\} \subset \mathbb{A}^2$ for $g \in K[x, y]$. Then $K(C) = \operatorname{Frac}\left(\frac{K[x,y]}{(g)}\right)$. Write $g = g_0 + g_1(x, y) + g_2(x, y) + \ldots$ for $g_i$ homogeneous of degree $i$. Suppose $P = (0, 0)$ is a smooth point, e.g. $g_0 = 0$ and let $g_1(x, y) = \alpha x + \beta y$ with $\alpha, \beta$ not both zero ($\alpha x + \beta y = 0$ gives a tangent to the curve at $P$). Let $\gamma, \delta \in K$ and consider also the line $\gamma x + \delta y$ through $P$. Then it is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at $P$ if and only if $\alpha\delta - \beta\gamma \neq 0$.

**Example 2.3.** Consider $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$ for $\lambda \neq 0, 1$ and consider its projective closure by taking $x = \frac{X}{Z}, y = \frac{Y}{Z}$ to get $\{Y^2 Z = X(X-Z)(X-\lambda Z)\} \subset \mathbb{P}^2$. This has only one point at infinity, $P = (0 : 1 : 0)$. Our aim is to compute $\operatorname{ord}_P(x)$ and $\operatorname{ord}_P(y)$.

For this, put $t = \frac{X}{Y}, w = \frac{Z}{Y}$, so $w \stackrel{(\dagger)}{=} t(t - w)(t - \lambda w)$. Now $P$ is the point $(t, w) = (0, 0)$, which is a smooth point with $\operatorname{ord}_P(t) = \operatorname{ord}_P(t - w) = \operatorname{ord}_P(t - \lambda w) = 1$, so $(\dagger)$ gives $\operatorname{ord}_P(w) = 3$. We now find

$$\operatorname{ord}_P(x) = \operatorname{ord}_P\left(\frac{X}{Z}\right) = \operatorname{ord}_P\left(\frac{t}{w}\right) = 1 - 3 = -2$$

$$\operatorname{ord}_P(y) = \operatorname{ord}_P\left(\frac{Y}{Z}\right) = \operatorname{ord}_P\left(\frac{1}{w}\right) = -3.$$

**Riemann–Roch space.** Let $C$ be a smooth projective curve.

**Definition 2.3.** A **divisor** is a formal sum of points on $C$, say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. We say $\deg D = \sum_{P \in C} n_P$.

$D$ is **effective** (written $D \geq 0$) if $n_P \geq 0 \ \forall P \in C$. If $f \in K(C)^\times$, then $\operatorname{div}(f) = \sum_{P \in C} \operatorname{ord}_P(f)P$. The Riemann–Roch space of $D \in \operatorname{Div}(C)$ is

$$\mathcal{L}(D) = \{f \in K(C)^\times \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\},$$

i.e. the $K$–vector space of rational functions on $C$ with "poles no worse than specified by $D$" (i.e. every coefficient of $\operatorname{div}(f) + D$ is nonnegative).

We quote Riemann–Roch for surfaces of genus 1: We have

$$\dim \mathcal{L}(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ 0 \text{ or } 1 & \text{if } \deg D = 0 \\ 0 & \text{if } \deg D < 0. \end{cases}$$

**Example 2.4.** We revisit Example 2.3. We have $\mathcal{L}(2P) = \langle 1, x \rangle$ and $\mathcal{L}(3P) = \langle 1, x, y \rangle$.

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We still have char $K \neq 2$ and $\overline{K} = K$.

**Proposition 2.2.** Let $C \subset \mathbb{P}^2$ be a smooth plane cubic and let $P \in C$ be a point of inflection. Then we may change coordinates such that $C : Y^2Z = X(X - Z)(X - \lambda Z)$ and $P = (0 : 1 : 0)$ (for some $\lambda \neq 0, 1$).

*Proof.* First change coordinates such that $P = (0 : 1 : 0)$. Then change coordinates such that the tangent line becomes $T_p C = \{Z = 0\}$. Say $C = \{F(X, Y, Z) = 0\} \subset \mathbb{P}^2$. A point on the tangent line is of the form $(t : 1 : 0)$ and since $P \in C$ is a point of inflection, we get $F(t, 1, 0) = \text{const} \cdot t^3$, i.e. $F$ has no terms $X^2Y, XY^2$ or $Y^3$.

Hence $F = \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3 \rangle$. Notably, $Y^2Z$ has a nonzero coefficient, otherwise $P \in C$ would be singular, a contradition to $C$ being smooth. The coefficient of $X^3$ is nonzero as well, otherwise $Z \mid F$. We are free to rescale $X, Y, Z, F$, so WLOG $C$ is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

Substituting $Y \mapsto Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$. This gives

$$C : Y^2Z = Z^3 f\left(\frac{X}{Z}\right)$$

for a monic cubic polynomial $f$. Since $C$ is smooth, $f$ has distinct roots, WLOG $0, 1, \lambda$, so $C : Y^2Z = X(X - Z)(X - \lambda Z)$. $\qquad\square$

The form $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ is the Weierstrass form. The form $Y^2Z = X(X - Z)(X - \lambda Z)$ is the Legendre form.

**Remark.** It can be shown that the points of inflection of a plane curve $C = \{F(X_1, X_2, X_3) = 0\} \subset \mathbb{P}^2$ are given by solving the Hessian:

$$\begin{cases} \det H = \det\left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right) = 0 \\ F(X_1, X_2, X_3) = 0. \end{cases}$$

## 2.1 The degree of a morphism

Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then $\phi^* : K(C_2) \to K(C_1)$ by $f \mapsto f \circ \phi$, giving an injective map $\phi^* K(C_2)$ to $K(C_1)$.

**Definition 2.4.** The **degree** of $\phi$ is $\deg \phi = [K(C_1) : \phi^* K(C_2)]$.

We say $\phi$ is **separable** if $K(C_1)/\phi^* K(C_2)$ is a separable field extension.

Suppose $P \in C_1, Q \in C_2$ and $\phi : P \mapsto Q$. Let $t \in K(C_2)$ be a uniformizer at $Q$.

**Definition 2.5.** $e_\phi(P) = \text{ord}_P(\phi^* t)$, which is always $\geq 1$ and independent of $t$.

**Theorem 2.3.** Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then

$$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi \ \forall Q \in C_2.$$

Moreover, if $\phi$ is separable, then $e_\phi(P) = 1$ for all but finitely many $P \in C_1$.

We don't prove this.

In particular, this shows that:

(i) $\phi$ is surjective (very important here that we're in $\overline{K}$).

(ii) $|\phi^{-1}(Q)| \le \deg \phi$.

(iii) If $\phi$ is separable, then equality holds in (ii) for all but finitely many points $Q \in C_2$.

**Important remark.** Let $C$ be an algebraic curve. A rational map is given by

$$C \to \mathbb{P}^n$$
$$\phi \mapsto (f_0, f_1, \ldots, f_n)$$

where $f_0, \ldots, f_n \in K(C)$ are not all zero. Then we have a fact: If $C$ is smooth, then $\phi$ is a morphism. This saves us a lot of time (we can go from a rational map to a morphism immediately).

# 3   Weierstrass equations

We now drop the assumption that $\overline{K} = K$, but we will still assume that $K$ is perfect.

**Definition 3.1.** An **elliptic curve** $E/K$ is a smooth projective curve of genus 1 defined over $K$ with a specified $K$–rational point $O = 0_E$.

**Example 3.1.** $\{X^3 + pY^3 + p^2 Z^3 = 0\} \subset \mathbb{P}^2$ is not an elliptic curve over $\mathbb{Q}$, since it has no $\mathbb{Q}$–rational point.

**Theorem 3.1.** Every elliptic curve $E$ is isomorphic over $K$ to a curve in Weierstrass form via an isomorphism taking $0_E$ to $(0 : 1 : 0)$.

**Remark.** Proposition 2.2 treated the special case where $E$ is a smooth plane cubic and $0_E$ is a point of inflection.

**Fact.** If $D \in \mathrm{Div}(E)$ is defined over $K$, then $\mathcal{L}(D)$ has a basis in $K(E)$ (not just in $\overline{K}(E)$). Here $D$ is defined over $K$ if it is fixed by $\mathrm{Gal}(\overline{K}/K)$ (this is unimportant for us and we just write it down to be rigorous).

*Proof.* $\mathcal{L}(2{\cdot}0_E) \subset \mathcal{L}(3{\cdot}0_E)$. Pick bases $1, x$ and $1, x, y$. Note $\mathrm{ord}_{0_E}(x) = -2$ and $\mathrm{ord}_{0_E}(y) = -3$ (else $x, y$ don't give a basis). The 7 elements $1, x, y, x^2, xy, x^3, y^2$ lie in the 6–dimensional vector space $\mathcal{L}(60_E)$ (as they have at most a sixth order pole), so they must satisfy a linear dependence relation.

Leaving out $x^3$ or $y^2$ leaves us with 6 elements, all with different order poles, giving a basis for $\mathcal{L}(60_E)$. Hence the coefficients of $x^3$ and $y^2$ are nonzero, so by rescaling $x, y$ (if necessary) we get

$$E' : y^2 + a_1 xy + a_2 y = x^3 + a_2 x^2 + a_4 x + a_6$$

for some $a_i \in K$. Let $E'$ be the curve defined by this equation (or rather its projective closure). There is a morphism $\phi : E \to E' \subset \mathbb{P}^2$ by $P \mapsto (x(P) : y(P) : 1) = \left( \frac{x}{y}(P) : 1 : \frac{1}{y}(P) \right)$. (Since $E$ is smooth, we know that this rational map is a morphism). Hence $0_E \mapsto (0 : 1 : 0)$.

We have $E \overset{x}{\to} \mathbb{P}^1$ by $x \mapsto (x : 1)$ (and similarly for $y$), so

$$[K(E) : K(x)] = \deg(E \overset{x}{\to} \mathbb{P}^1) = \mathrm{ord}_{0_E}\left( \frac{1}{x} \right) = 2$$

$$[K(E) : K(y)] = \deg(E \overset{y}{\to} \mathbb{P}^1) = \mathrm{ord}_{0_E}\left( \frac{1}{y} \right) = 3.$$

This gives an inclusion of fields $K(x) \leq K(E)$ of degree 2, $K(y) \leq K(E)$ of degree 3, while $K(x), K(y) \leq K(x, y) \leq K(E)$, so tower law gives $[K(E) : K(x, y)] = 1 \implies K(E) = K(x, y) = \phi^* K(E') \implies \deg \phi = 1$. (draw a picture!). This gives us an inverse that is a rational map, which we want to show is a morphism. For this, we just need to show that $E'$ is smooth.

If $E'$ were singular, then $E$ and $E'$ are rational, a contradiction. So $E'$ is smooth and hence $\phi^{-1}$ is a morphism, so $\phi$ is an isomorphism. $\qquad\square$

**Proposition 3.2.** Let $E, E'$ be elliptic curves over $K$ in Weierstrass form. Then $E \cong E'$ over $K \iff$ the equations are related by a change of variables

$$x = u^2 x' + r$$
$$y = u^3 y' + u^2 s x' + t$$

for $r, s, t, u \in K$ with $u \neq 0$.

*Proof.* $\mathcal{L}(2 \cdot 0_E) = \langle 1, x \rangle = \langle 1, x' \rangle \implies x = \lambda x' + r$ for some $\lambda, r \in K, \lambda \neq 0$. Similarly $\mathcal{L}(3 \cdot 0_E) = \langle 1, x, y \rangle = \langle 1, x', y' \rangle \implies y = \mu y' + \sigma x' + t$ for some $\mu, \sigma, t \in K, \mu \neq 0$.

Looking at the coefficients of $x^3$ and $y^2$ tells us that $\lambda^3 = \mu^2$, so $\lambda = u^2, \mu = u^3$ for some $u \in K^\times$. Put $s = \frac{\sigma}{u^2}$ to conclude. $\qquad\square$

A Weierstrass equation defines an elliptic curve $\iff$ it defines a smooth curve $\iff \Delta(a_1,\ldots,a_6) \neq 0$, where $\Delta \in \mathbb{Z}[a_1,\ldots,a_6]$ is a certain polynomial.

If char $K \neq 2,3$, we may reduce to the case $E : y^2 = x^3 + ax + b$. In this case, the discriminant is $\Delta = -16(4a^3 + 27b^2)$.

**Corollary 3.3.** Assume char $K \neq 2,3$. Elliptic curves

$$E : y^2 = x^3 + ax + b$$
$$E' : y^2 = x^3 + a'x + b'$$

are isomorphic over $K \iff \begin{cases} a' = u^4 a \\ b' = u^6 b \end{cases}$ for some $u \in K^\times$.

*Proof.* $E, E'$ are related by a substitution as in Proposition 3.2 with $r = s = t = 0$. $\qquad\qquad\square$

**Definition 3.2.** The $j$–**invariant** is $j(E) = \frac{1728(4a^3)}{4a^3 + 27b^2}$.

**Corollary 3.4.** $E \cong E' \implies j(E) \cong j(E')$ and the converse holds if $K = \overline{K}$.

*Proof.* $E \cong E' \iff \begin{cases} a' & = u^4 a \\ b' & = u^6 b \end{cases}$ for some $u \in K^\times \implies (a^3 : b^2) = ((a')^3 : (b')^2) \iff j(E) = j(E')$. The middle step is reversible if $K = \overline{K}$. $\qquad\square$

# 4   The Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic with $0_E \in E(K)$ (not immediately assumed to be in Weierstrass form). $E$ meets any line in 3 points, counted with multiplicity.

For $P, Q \in E$, let $S$ be the 3rd point of intersction of $PQ$ with $E$ and then let $R$ be the 3rd intersection of $0_E S$ with $E$. We define $P \oplus Q = R$. (Later we drop the circle and just write $+$). If $P = Q$, instead take the tangent line at $P$, i.e. $T_P E$, etc. This is the "chord and tangent process".

**Theorem 4.1.** $(E, \oplus)$ is an abelian group.

**Remark.** Here $E$ means $E(\overline{K})$ since we haven't specified a field yet.

*Proof.*   (i) $\oplus$ is commutative trivially.

 (ii) $0_E$ is the identity, since the line through $0_E P$ meets $S$ for the 3rd time at $S$ and then $SP$ meets $E$ for the 3rd time at $0_E$ (drawing a picture makes this obvious).

(iii) Inverses: Let $S$ be the $3^{\text{rd}}$ intersection of $T_{0_E}$ with $E$ and $Q$ the $3^{\text{rd}}$ intersection of $PS$ with $E$. Then $P \oplus Q = 0_E$.

(iv) Associativity is much harder. We have some setup:

**Definition 4.1.** $D_1, D_2 \in \operatorname{Div}(E)$ are **linearly equivalent** if $\exists f \in K(E)^{\times}$ such that $\operatorname{div}(f) = D_1 - D_2$. Write $D_1 \sim D_2$ and $[D] = \{D' \mid D' \sim D\}$.

**Definition 4.2.** The **Picard group** is $\operatorname{Pic}(E) = \operatorname{Div}(E)/\sim$. Also define $\operatorname{Pic}^0(E) = \operatorname{Div}^0(E)/\sim$ where $\operatorname{Div}^0(E) = \{D \in \operatorname{Div}(E) \mid \deg(D) = 0\}$.

We define $\psi : E \to \operatorname{Pic}^0(E)$ by $P \mapsto [(P) - (0_E)]$.

**Proposition 4.2.**    (i) $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) $\psi$ is a bijection.

*Proof.*    (i) WLOG let the lines $PQ$ and $0_E S$ be given by $l = 0$ and $m = 0$. Then
$$\operatorname{div}\left(\frac{l}{m}\right) = (P) + (S) + (Q) - (0_E) - (S) - (R),$$
hence $(P) + (Q) \sim (P \oplus Q) + (0_E)$, so $(P \oplus Q) - (0_E) \sim (P) - (0_E) + (Q) - (0_E)$, so $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) Injectivity: Suppose $\psi(P) = \psi(Q)$ for $P \neq Q$. Then $\exists f \in \overline{K}(E)^{\times}$ such that $\operatorname{div}(f) = (P) - (0_E) - (Q) + (0_E) = (P) - (Q) \implies E \xrightarrow{f} \mathbb{P}^1$ has degree 1 (for example since evaluation at 0 on the affine line gives that $P$ has one root and $Q$ has one pole), so $E \cong \mathbb{P}^1$, a contradiction.

Surjectivity: Let $[D] \in \operatorname{Pic}^0(E)$. Then $D + (0_E)$ has degree 1, so by Riemann–Roch, $\dim \mathcal{L}(D + (0_E)) = 1$, so $\exists 0 \neq f \in \overline{K}(E)$ such that $\operatorname{div}(f) + D + (0_E) \geq 0$, but $\operatorname{div}(f) + D + (0_E)$ has degree 1, so $\operatorname{div}(f) + D + (0_E) = (P)$ for some $P \in E \implies (P) - (0_E) \sim D \implies \psi(P) = [D]$. $\square$

We conclude that $\psi$ identifies $(E, \oplus)$ with $(\operatorname{Pic}^0(E), +)$, so $\oplus$ is associative. $\square$

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**Formulae for $E$ in Weierstrass form.** Let $E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. Choose two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on it. Let the line through $P_1$ and $P_2$ be given by $y = \lambda x + \nu$ and let it meet $E$ again at $P' = (x', y')$. We want to find $P_1 \oplus P_2 = P_3 = (x_3, y_3) = \ominus P'$ for $\ominus P$ the reflection of $P$ across the $x$–axis. We easily compute $\ominus P_1 = (x_1, -(a_1 x + a_3) - y_1)$.

Substituting $y = \lambda x + \nu$ into our equation for $E$ and looking at the coefficient of $x^2$ gives $\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x' = x_1 + x_2 + x_3$, so $x_3 = \lambda^2 + a_1\lambda - a_2 - x_1 - x_2$. For $y_3$ we find

$$y_3 = -(a_1 x' + a_3) - y' = -(a_1 x_3 + a_3) - (\lambda x_3 + \nu) = -(\lambda + a_1)x_3 - a_3 - \nu.$$

It remains to find formulas for $\lambda$ and $\nu$.

- Case 1. $x_1 = x_2$, but $P_1 \neq P_2$. Then $P_1 \oplus P_2 = 0_E$.

- Case 2. $x_1 \neq x_2$. Then $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ and $\nu = y_1 - \lambda x_1 = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.

- Case 3. $P_1 = P_2$. In this case, compute the equation for the tangent line to get $\lambda, \nu$ as rational expressions in $x_1, x_2, y_1, y_2$.

**Corollary 4.3.** $E(K)$ is an abelian group.

*Proof.* $E(K)$ is a subgroup of $(E, \oplus)$.

- It has identity $0_E$ by definition.

- We have closure and inverses through the formulae above.

- Associativity and commutativity is inherited.

$\square$

**Theorem 4.4.** Elliptic curves are group varieties, i.e.

$$[-1] : E \to E, P \mapsto \ominus P$$
$$\oplus : E \to E, (P, Q) \mapsto P \oplus Q$$

are morphisms of algebraic varieties.

*Proof.* By the above formulae, $[-1] : E \to E$ is a rational map, i.e. a morphism by our important remark.

For $\oplus$, note by the above formulae that $\oplus : E \to E$ is a rational map regular on

$$U = \{(P, Q) \in E \times E \mid 0_E \notin \{P, Q, P \oplus Q, P \ominus Q\}\}.$$

For $P \in E$, let $\tau_P : E \to E$ be the "translation by $P$" map, given by $X \mapsto P \oplus X$. $\tau_P$ is a rational map, hence a morphism. Now for $A, B \in E$, we factor $\oplus$ as

$$E \times E \overset{\tau_{\ominus A} \times \tau_{\ominus B}}{\to} E \times E \overset{\oplus}{\to} E \overset{\tau_{A \oplus B}}{\to} E.$$

This shows $\oplus$ is regular on $(\tau_A \times \tau_B)(U)$, so $\oplus$ is regular on $E \times E$. $\square$

**Statement of results.** The following isomorphisms in (i), (ii), (iv) respect the relevant topologies.

(i) $K = \mathbb{C}$. Then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ for $\Lambda$ a lattice.

(ii) $K = \mathbb{R}$. Then

$$E(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \text{if } \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \text{if } \Delta < 0. \end{cases}$$

(iii) $K = \mathbb{F}_q$. Then $||E(\mathbb{F}_q)| - (q+1)| \leq 2\sqrt{q}$. This is Hasse's Theorem.

(iv) For a local field $[K : \mathbb{Q}_p] < \infty$ with ring of integers $\mathcal{O}_K$, $E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$.

(v) For a number field $[K : \mathbb{Q}] < \infty$, $E(K)$ is a finitely generated abelian group (this is the Mordell–Weil Theorem). Basic group theory says that if $A$ is a finitely generated abelian group, then $A \cong (\text{finite subgroup}) \times \mathbb{Z}^r$. Here $r$ is called the rank of $A$. The proof of Mordell–Weil gives an upper bound for rank $E(K)$, but there is no known algorithm to compute the rank in all cases.

**Brief remarks on the case $K = \mathbb{C}$.** Let $\Lambda = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}\}$ where $\omega_1, \omega_2$ are a basis for $\mathbb{C}$ as an $\mathbb{R}$–vector space. Then meromorphic functions on the Riemann surface $\mathbb{C}/\Lambda$ correspond bijectively with $\Lambda$–invariant meromorphic functions in $\mathbb{C}$. The function field of $\mathbb{C}/\Lambda$ is generated by $\wp(z)$ and $\wp'(z)$, where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^3}.$$

These satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ for some constants $g_2, g_3 \in \mathbb{C}$ depending on $\Lambda$. One shows $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where $E : y^2 = 4x^3 - g_2x - g_3$ which is an isomorphism on both groups (via $z \mapsto (\wp(z), \wp'(z))$) and on Riemann surfaces. We have the following result:

**Theorem 4.5** (Uniformization theorem)**.** Every elliptic curve over $\mathbb{C}$ arises in this way.

**Definition 4.3.** For $n \in \mathbb{Z}$, let $[n] : E \to E$ be given by $P \mapsto \underbrace{P \oplus P \oplus \ldots \oplus P}_{n \text{ copies}}$

if $n > 0$ and $[-n] = [-1] \circ [n]$.

**Definition 4.4.** The $n$–**torsion subgroup** of $E$ is

$$E[n] = \ker(E \xrightarrow{[n]} E).$$

If $K = \mathbb{C}$, then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, so $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ and $\deg[n] = n^2$. Call these results (1) and (2). We will show that (2) holds over any field $K$ and (1) holds if char $K \nmid n$.

**Lemma 4.6.** Assume char $K \neq 2$ and $E : y^2 = f(x) = (x - e_1)(x - e_2)(x - e_3)$ (with $e_i \in \overline{K}$). Then $E[2] = \{0, (e_1, 0), (e_2, 0), (e_3, 0)\} \cong (\mathbb{Z}/2\mathbb{Z})^\times$.

*Proof.* Let $P = (x, y) \in E$. Then $2[P] = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. $\qquad\square$

# 5   Isogenies

Let $E_1, E_2$ be elliptic curves.

**Definition 5.1.**    (i) An **isogeny** $\phi : E_1 \rightarrow E_2$ is a nonconstant morphism with $\phi(0_{E_1}) = 0_{E_2}$.

(ii) We say $E_1$ and $E_2$ are **isogenous** if there is an isogeny between them.

In (i), nonconstant is equivalent to surjective on $\overline{K}$–points. See Theorem 2.3.

**Definition 5.2.** $\mathrm{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \rightarrow E_2\} \cup \{0\}$ (the constant map at $0_E$). This is an abelian group under $(\phi + \psi)(P) := \phi(P) \oplus \psi(P)$.

If $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$ are isogenies, then $\psi \circ \phi$ is an isogeny. By tower law, $\deg(\psi \circ \phi) = \deg(\psi)\deg(\phi)$.

**Proposition 5.1.** If $0 \neq n \in \mathbb{Z}$, then $[n] : E \rightarrow E$ is an isogeny.

*Proof.* $[n]$ is a morphism by Theorem 4.4. We need to show $[n] \neq [0]$. Assume char $K \neq 2$.

- Case $n = 2$. Lemma 4.6 implies that $E[2] \neq E$, so $[2] \neq 0$.

- Case $n$ odd. Lemma 4.6 implies that $\exists 0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n] \neq [0]$.

Now use $[mn] = [m] \circ [n]$ to conclude.

If char $K = 2$, then we can replace Lemma 4.6 with an explicit lemma about 3–torsion points. $\qquad\square$

**Corollary 5.2.** $\mathrm{Hom}(E_1, E_2)$ is a torsion–free $\mathbb{Z}$–module.

**Theorem 5.3.** Let $\phi : E_1 \rightarrow E_2$ be an isogeny. Then

$$\phi(P + Q) = \phi(P) + \phi(Q) \ \forall P, Q \in E.$$

*Sketch proof.* $\phi$ induces a map $\phi_* : \text{Div}^0(E_1) \to \text{Div}^0(E_2)$ by $\sum_{P \in E_1} n_P P \mapsto \sum_{p \in E_1} n_P \phi(P)$. Recall $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

**Fact.** If $f \in K(E_1)$, then $\text{div}(N_{K(E_1)/K(E_2)}f) = \phi^*(\text{div } f)$. So $\phi_*$ sends principal divisors to principal divisors. Since $\phi(0_{E_1}) = 0_{E_2}$, the following diagram commutes:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \phi\ } & E_2 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
\text{Pic}^0(E_1) & \xrightarrow{\ \phi_*\ } & \text{Pic}^0(E_2)
\end{array}
$$

(with $f(P) = [(P)-(0_{E_1})], g(Q) = [(Q)-(0_{E_2})]$). Since $\phi_*$ is a group homomorphism, $\phi$ is a group homomorphism.  $\square$

**Lemma 5.4.** Let $\phi : E_1 \to E_2$ be an isogeny. Then there exists a morphism $\xi$ making the following diagram commute:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \phi\ } & E_2 \\
\downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle x_2} \\
\mathbb{P}^1 & \xrightarrow{\ \xi\ } & \mathbb{P}^1
\end{array}
$$

with $x_i$ the $x$–coordinate in a Weierstrass equation for $E_i$. Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ with $r, s \in K[t]$ coprime, then $\deg(\phi) = \deg(\xi) = \max(\deg(r), \deg(s))$.

*Proof.* For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 Galois extension with Galois group generated by $[-1]^*$. By Theorem 5.3, $\phi \circ [-1] = [-1] \circ \phi$, so if $f \in K(x_2)$, then $[-1]^*(\phi^*f) = \phi^*([-1]^*f) = \phi^*f$ and hence $\phi^*f \in K(x_1)$. Hence we find

$$
\begin{array}{c}
K(E_1) = K(x_1, y_1) \\[1em]
\end{array}
$$

$$
\begin{array}{ccc}
& {\scriptstyle 2} & \\
K(x_1) & & K(E_2) = K(x_2, y_2) \\
\\
{\scriptstyle |} & {\scriptstyle 2} & \\
K(x_2) & &
\end{array}
\quad \cdot
$$

In particular, $\phi^* x_2 = \xi(x_1)$ for some $\xi \in K(t)$. By tower law, $2\deg(\phi) = 2\deg(\xi) \implies \deg(\phi) = \deg(\xi)$. Now $K(x_2) \hookrightarrow K(x_1)$ by $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_1)}$ for $r, s \in K[t]$ coprime. Then minimal polynomial of $x_1$ over $K(x_2)$ is $F(t) = r(t) - s(t)x_2 \in K(x_2)[t]$. This is true as $F(x_1) = 0$, $F$ is irreducible on $K[x_2, t]$ (since $r, s$ are coprime) and by Gauss' Lemma, $F$ is irreducible on $K(x_2)[t]$. Hence $\deg(\phi) = \deg(\xi) = [K(x_1) : K(x_2)] = \deg(F) = \max(\deg(r), \deg(s))$.  $\square$

**Lemma 5.5.** $\deg[2] = 4$.

*Proof.* Assume char $K \neq 2, 3$, so $E : y^2 = x^3 + ax + b = f(x)$. If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2 - 2xf(x)}{4f(x)}$. The numerator and denominator are coprime, since otherwise $\exists \theta \in \overline{K}$ with $f(\theta) = f'(\theta) = 0$, meaning $f$ has a multiple root, contradiction. We are now done by Lemma 5.4, since $\deg[2] = \max(3, 4) = 4$. $\qquad \square$

**Definition 5.3.** Let $A$ be an abelian group. Then a map $q : A \to \mathbb{Z}$ is a quadratic form if

(i) $q(nx) = n^2 q(x) \ \forall n \in \mathbb{Z}, q \in A$.

(ii) $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$–bilinear.

**Lemma 5.6.** $q : A \to \mathbb{Z}$ is a quadratic form if and only if it satisfies the parallelogram law $q(x + y) + q(x - y) = 2q(x) + 2q(y) \ \forall x, y \in A$.

*Proof.* ( $\implies$ ). Let $\langle x, y \rangle = q(x+y) - q(x) - q(y)$. Then $\langle x, x \rangle = q(2x) - 2q(x) = 2q(x)$ by (i) with $n = 2$. By (ii), $\langle x + y, x + y \rangle + \langle x - y, x - y \rangle = 2\langle x, x \rangle + 2\langle y, y \rangle$, which implies $q(x + y) + q(x - y) = 2q(x) + 2q(y)$.

( $\impliedby$ ). This is on Ex. Sheet 2. $\qquad \square$

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Lecture 7

**Theorem 5.7.** $\deg : \mathrm{Hom}(E_1, E_2) \to \mathbb{Z}$ is a quadratic form (with $\deg(0) = 0$).

*Proof.* Assume char $K \neq 2, 3$ and write $E_2 = y^2 = x^3 + ax + b$. Let $P, Q \in E_2$ with $P, Q, P + Q, P - Q$ all nonzero and let $x_1, x_2, x_3, x_4$ be the $x$–coordinates of these points.

**Lemma 5.8.** There exist polynomials $W_0, W_1, W_2 \in \mathbb{Z}[a, b][x_1, x_2]$ of degree $\leq 2$ in $x_1$ and of degree $\leq 2$ in $x_2$ such that

$$(1 : x_3 + x_4 : x_3 x_4) = (W_0 : W_1 : W_2)$$

*Proof.* Method 1: Direct calculation (results on the formula sheet) gives the result (e.g. $W_0 = (x_1 - x_2)^2$).

Method 2: Let $y = \lambda x + \nu$ be the line through $P$ and $Q$. Substituting, we get $x^3 + ax + b - (\lambda x + \nu)^2 = (x - x_1)(x - x_2)(x - x_3) = x^3 - s_1 x^2 + s_2 x - s_3$ where $s_i$ is the $i^{\text{th}}$ symmetric polynomial in $x_1, x_2, x_3$. Comparing coefficients gives $\lambda^2 = s_1, -2\lambda\nu = s_2 - a, \nu^2 = s_3 + b$. Eliminating $\lambda$ and $\nu$ gives

$$F(x_1, x_2, x_3) = (s_2 - a)^2 - 4s_1(s_3 + b) = 0,$$

where $F$ has degree at most 2 in each $x_i$. Hence $x_3$ is a root of the quadratic $W(t) = F(x_1, x_2, t)$. Repeating this for the line through $P$ and $-Q$ shows that

$x_4$ is the other root of $W(t)$. Therefore

$$W(t) = W_0(t - x_3)(t - x_4) = W_0 t^2 - W_1 t + W_2$$
$$\implies (1 : x_3 + x_4 : x_3 x_4) = (W_0 : W_1 : W_2).$$

$\square$

We now show that if $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$, then $\deg(\phi + \psi) + \deg(\phi - \psi) \leq 2\deg(\phi) + 2\deg(\psi)$. We may assume that $\phi, \psi, \phi + \psi, \phi - \psi$ are not the zero maps (otherwise we're done trivially, or use $\deg[-1] = 1$, $\deg[2] = 4$). Now

$$\phi : (x, y) \mapsto (\xi_1(x), \ldots)$$
$$\psi : (x, y) \mapsto (\xi_2(x), \ldots)$$
$$\phi + \psi : (x, y) \mapsto (\xi_3(x), \ldots)$$
$$\phi - \psi : (x, y) \mapsto (\xi_4(x), \ldots).$$

Lemma 5.8 implies $(1 : \xi_3 + \xi_4 : \xi_3 \xi_4) = ((\xi_1 - \xi_2)^2 : \ldots)$. Say $\xi_i = \frac{r_i}{s_i}$ for $r_i, s_i \in K[t]$ coprime. This gives

$$(s_3 s_4 : r_3 s_4 + r_4 s_3 : r_3 r_4) \overset{(\star)}{=} ((r_1 s_2 - r_2 s_1)^2 : \ldots)$$

where every term is quadratic in $r_3, r_4, s_3$ and $s_4$. Hence (as the terms on the LHS of $(\star)$ are coprime)

$$\deg(\phi + \psi) + \deg(\phi - \psi) = \max(\deg(r_3), \deg(s_3)) + \max(\deg(r_4), \deg(s_4))$$
$$= \max(\deg(s_3 s_4), \deg(r_3 s_4 + r_4 s_3), \deg(r_3 r_4))$$
$$\leq 2\max(\deg(r_1), \deg(s_1)) + 2\max(\deg(r_2), \deg(s_2))$$
$$= 2\deg(\phi) + 2\deg(\psi).$$

Now replace $\phi$ and $\psi$ by $\phi + \psi$ and $\phi - \psi$ and use $\deg[2] = 4$ to get

$$4\deg(\phi) + 4\deg(\psi) = \deg(2\phi) + \deg(2\psi) \leq 2\deg(\phi + \psi) + 2\deg(\phi - \psi).$$

This gives the parallelogram law, so deg is a quadratic form. $\square$

**Corollary 5.9.** $\deg(n\phi) = n^2 \deg(\phi)$. In particular, $\deg[n] = n^2$.

**Example 5.1.** Let $E/K$ be an elliptic curve. Suppose char $K \neq 2$ and $0 \neq T \in E(K)[2]$. WLOG let $E : y^2 = x(x^2 + ax + b)$ for $a, b \in K, b(a^2 - 4b) \neq 0$ (by moving a root to zero) and WLOG $T = (0, 0)$.

If $P = (x, y)$ and $P' = P + T = (x', y')$, then

$$x' = \left(\frac{y}{x}\right)^2 - a - x = \frac{x^2 + ax + b}{x} - a - x = \frac{b}{x}$$

$$y' = -\left(\frac{y}{x}\right)x' = -\frac{by}{x^2}.$$

We let $\xi = x + x' + a = \left(\frac{y}{x}\right)^2, \eta = y + y' = \frac{y}{x}\left(x - \frac{b}{x}\right)$. Then

$$\eta^2 = \left(\frac{y}{x}\right)^2 \left(\left(x + \frac{b}{x}\right)^2 - 4b\right) = \xi((\xi - a)^2 - 4b) = \xi(\xi^2 - 2a\xi + a^2 - 4b).$$

Let $E' : y^2 = x(x^2 + a'x + b')$ with $a' = -2a, b' = a^2 - 4b$. There is an isogeny $\phi : E \to E'$ given by $(x, y) \mapsto \left(\left(\frac{y}{x}\right)^2 : \frac{y(x^2-b)}{x^2} : 1\right)$.

Sanity check/finding where $0_E$ maps to: $x$ is a double pole, $y$ is a triple pole, so $\left(\frac{y}{x}\right)^2$ is a double pole and $\frac{y(x^2-b)}{x^2}$ is a triple pole (and the last coordinate 1 has degree 0). Multiplying through by a cube of a uniformizer, the degrees go from $(-2, -3, 0)$ to $(1, 0, 3)$, so $0_E \mapsto (0 : 1 : 0)$.

To compute $\deg(\phi)$, $\left(\frac{y}{x}\right)^2 = \frac{x^2+ax+b}{x}$ with the numerator and denominator coprime as $b \neq 0$, so by Lemma 5.4, $\deg(\phi) = 2$. We say $\phi$ is a **2–isogeny**.

# 6   The invariant differential

For $C$ some algebraic curve over $K = \overline{K}$.

**Definition 6.1.** The space of differentials $\Omega_C$ (sometimes called one–forms) is the $K(C)$–vector space generated by $df$ for all $f \in K(C)$ subject to the relations

  (i)  $d(f + g) = df + dg$.

 (ii)  $d(fg) = f dg + g df$.

(iii)  $da = 0 \ \forall a \in K$.

    **Fact.** $\Omega_C$ is a 1–dimensional $K(C)$–vector space.

    Let $0 \neq \omega \in \Omega_C$, let $P \in C$ be a smooth point and let $t \in K(C)$ be a uniformizer at $P$. Then $\omega = f dt$ for some $f \in K(C)^\times$. We define $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$, which is independent of the choice of $t$.

    **Fact.** Suppose $f \in K(C)^\times$ with $\mathrm{ord}_P(f) = n \neq 0$. If char $K \nmid n$, then $\mathrm{ord}_P(df) = n - 1$.

    We assume that $C$ is a smooth projective curve.

**Definition 6.2.** We define $\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega)P \in \mathrm{Div}(C)$. Here we use the fact that $\mathrm{ord}_P(\omega) = 0$ for all but finitely many $P \in C$.

**Definition 6.3.** A differential $\omega \in \Omega_C$ is regular if $\mathrm{div}(\omega) \geq 0$. We define the genus $g(C)$ of $C$ to be

$$g(C) = \dim_K\{\omega \in \Omega_C \mid \mathrm{div}(\omega) \geq 0\},$$

where the set on the RHS is the set of regular differentials.

As a consequence of Riemann–Roch, we have that if $0 \neq \omega \in \Omega_C$, then $\deg(\mathrm{div}(\omega)) = 2g(C) - 2$.

**Lemma 6.1.** Assume char $K \neq 2$ and let $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$ for $e_1, e_2, e_3$ distinct. Then $\omega = \frac{\mathrm{d}x}{y}$ is a differential on $E$ with no zeroes or poles, which implies $g(E) = 1$. In particular, the $K$–vector space of regular differentials on $E$ is 1–dimensional, spanned by $\omega$.

*Proof.* Let $T_i = (e_i, 0)$. Then $E[2] = \{0, T_1, T_2, T_3\}$ and $\mathrm{div}(y) \overset{(\dagger)}{=} (T_1) + (T_2) + (T_3) - 3(0)$. For $0 \neq P \in E$, $\mathrm{div}(x - x_P) = (P) + (-P) - 2(0)$.

- If $P \in E \setminus E[2]$, then $\mathrm{ord}(x - x_P) = 1 \implies \mathrm{ord}_P(\mathrm{d}x) = 0$.

- If $P = T_i$, then $\mathrm{ord}_P(x - x_P) = 2 \implies \mathrm{ord}_P(\mathrm{d}x) = 1$.

- If $P = 0$, then $\mathrm{ord}_P(x) = -2 \implies \mathrm{ord}_P(\mathrm{d}x) = -3$.

Hence $\mathrm{div}(\mathrm{d}x) = (T_1) + (T_2) + (T_3) - 3(0)$, which with $(\dagger)$ gives $\mathrm{div}\left(\frac{\mathrm{d}x}{y}\right) = 0$. $\quad\square$

**Definition 6.4.** For $\phi : C_1 \to C_2$ a nonconstant morphism, we define

$$\phi^* : \Omega_{C_2} \to \Omega_{C_1}$$
$$f\mathrm{d}g \mapsto \phi^* f d(\phi^* g).$$

**Lemma 6.2.** Let $P \in E$, $\tau_P : E \to E$ by $X \mapsto X + P$ and $\omega = \frac{\mathrm{d}x}{y}$ as above. Then $\tau_P^* \omega = \omega$. We say $\omega$ is the **invariant differential**.

*Proof.* $\tau_P^* \omega$ is a regular differential on $E$, so $\tau_P^* \omega = \lambda_P \omega$ for some $\lambda_P \in K^\times$. The map $E \to \mathbb{P}^1$ by $P \mapsto \lambda_P$ is a morphism of smooth projectives curves, but it is not surjective (as it misses $0$ and $\infty$). Hence it is constant by Theorem 2.3, i.e. $\exists \lambda \in K^\times$ such that $\tau_P^* \omega = \lambda \omega \ \forall P \in E$. Taking $P = 0$ shows $\lambda = 1$. $\quad\square$

**Remark.** If $K = \mathbb{C}$ and $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ by $z \mapsto (\wp(z), \wp'(z)) := (x, y)$, then $\frac{\mathrm{d}x}{y} = \frac{\wp'(z)\mathrm{d}z}{\wp'(z)} = \mathrm{d}z$, which is invariant under $z \mapsto z + \mathrm{const}$.

**Lemma 6.3.** Let $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$. Let $\omega$ be the invariant differential on $E_2$. Then $(\phi + \psi)^* \omega = \phi^* \omega + \psi^* \omega$.

*Proof.* Write $E$ for $E_2$. We have the maps

$$E \times E \to E$$
$$\mu : (P, Q) \mapsto P + Q$$
$$\mathrm{pr}_1 : (P, Q) \mapsto P$$
$$\mathrm{pr}_2 : (P, Q) \mapsto Q.$$

**Fact.** $\Omega_{E \times E}$ is a 2–dimensional $K(E \times E)$–vector space with basis $\mathrm{pr}_1^*\omega$ and $\mathrm{pr}_2^*\omega$. Consequently, $\mu^*\omega \overset{(\dagger)}{=} f\mathrm{pr}_1^*\omega + g\mathrm{pr}_2^*\omega$ for some $f, g \in K(E \times E)$.

For fixed $Q \in E$, let $i_Q : E \to E \times E$ by $P \mapsto (P, Q)$. Applying $i_Q^*$ to $(\dagger)$ gives

$$(\underbrace{\mu \circ i_Q}_{\tau_Q})^*\omega = (i_Q^*f)(\underbrace{\mathrm{pr}_1 \circ i_Q}_{\text{identity map}})^*\omega + (i_Q^*g)(\underbrace{\mathrm{pr}_2 \circ i_Q}_{\text{constant map}})^*\omega$$
$$\implies \tau_Q^*\omega = (i_Q^*f)\omega + 0.$$

As $\tau_Q^*\omega = \omega$ by the previous lemma, we conclude $i_Q^*f = 1 \ \forall q \in E$, so $f(P, Q) = 1 \ \forall P, Q \in E$. Similarly $g(P, Q) = 1 \ \forall P, Q \in E$, so $(\dagger)$ gives $\mu^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$. Now pull back using

$$E_1 \to E \times E$$
$$P \mapsto (\phi(P), \psi(P))$$

to get $(\phi + \psi)^*\omega = \phi^*\omega + \psi^*\omega$. $\qquad\square$

**Lemma 6.4.** Let $\phi : C_1 \to C_2$ be a nonconstant morphism. Then $\phi$ is separable if and only if $\phi^* : \Omega_{C_2} \to \Omega_{C_1}$ is nonzero.

*Proof.* Omitted. $\qquad\square$

**Example 6.1.** Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ be the multiplicative group. For $n \geq 2$ an integer, consider $\phi : \mathbb{G}_m \to \mathbb{G}_m$ by $x \mapsto x^n$. Then $\phi^*(\mathrm{d}x) = \mathrm{d}(x^n) = nx^{n-1}\mathrm{d}x$. So if char $K \nmid n$, then $\phi$ is separable, so $\left|\phi^{-1}(Q)\right| = \deg\phi$ for all but at most finitely many $Q \in \mathbb{G}_m$.

But $\phi$ is a group homomorphism, so $\left|\phi^{-1}(Q)\right| = |\ker(Q)| \ \forall Q \in \mathbb{G}_m$. Hence $|\ker Q| = \deg\phi = n$. This shows that $K = \overline{K}$ contains exactly $n$ distinct $n^{\text{th}}$ roots of unity.

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**Theorem 6.5.** If char $K \nmid n$, then $E[n] = (\mathbb{Z}/n\mathbb{Z})^2$.

*Proof.* Lemma 6.3 and induction imply $[n]^*\omega = n\omega$ where char $K \nmid n$, so $[n]$ is separable by Lemma 6.4. Hence $\left|[n]^{-1}(Q)\right| = \deg[n]$ for all but finitely many

points $Q \in E$. But $[n]$ is a group homomorphism, so $\left|[n]^{-1}Q\right| = |E[n]| \ \forall Q \in E$. We conclude that $|E[n]| = \deg[n] = n^2$ by Corollary 5.9.

By classification of finite abelian groups, $E[n] \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times d_t\mathbb{Z}/\mathbb{Z}$ with $d_1 \mid d_2 \mid \ldots \mid d_t$, but $d_t \mid n$, and if $p$ is a prime with $p \mid d_1$, then $E[p] \cong (\mathbb{Z}/p\mathbb{Z})^t$, so $|E[p]| = p^2$, so $t = 2$. Hence $d_1 \mid d_2 \mid n$ with $d_1 d_2 = n^2$, so $d_1 = d_2 = n$ and so $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$. $\qquad\qquad\square$

**Remark.** If char $K = p$, then $[p]$ is inseparable. It can be shown that either $E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z} \ \forall r \geq 1$ or $E[p] = 0$ (the "ordinary" case and the "supersingular" case).

**Remark about the remark.** Do not use this remark to trivialize a question on Ex. Sheet 2.

# 7   Elliptic curves over finite fields

**Lemma 7.1.** Let $A$ be an abelian group. Let $q : A \to \mathbb{Z}$ be a positive definite quadratic form. Then

$$\underbrace{|q(x+y) - q(x) - q(y)|}_{\langle x,y\rangle} \leq 2\sqrt{q(x)q(y)}.$$

*Proof.* We may assume $x \neq 0$, otherwise the result is clear. Hence $q(x) \neq 0$. Let $m, n \in \mathbb{Z}$, then

$$0 \leq q(mx + ny) = \frac{1}{2}\langle mx + ny, mx + ny\rangle$$
$$= m^2 q(x) + mn\langle x, y\rangle + n^2 q(y)$$
$$= q(x)\left(m + \frac{\langle x,y\rangle}{2q(x)}n\right)^2 + \left(q(y) - \frac{\langle x,y\rangle^2}{4q(x)}\right)n^2.$$

Get rid of the first term by taking $m = -\langle x, y\rangle$ and $n = 2q(x)$ to deduce $\langle x, y\rangle^2 \leq 4q(x)q(y)$, so the result follows. $\qquad\qquad\square$

**Theorem 7.2** (Hasse). Let $E/\mathbb{F}_q$ be an elliptic curve. Then

$$|\#E(\mathbb{F}_q) - (q + 1)| \leq 2\sqrt{q}.$$

*Proof.* Recall $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ is cyclic of order $r$, generated by the Frobenius map $x \mapsto x^q$. Let $E$ have Weierstrass equation with coefficients $a_1, \ldots, a_6 \in \mathbb{F}_q$ (and note that $a_i^q = a_i \ \forall i$).

Define the Frobenius endomorphism $\phi : E \to E$ by $(x, y) \mapsto (x^q, y^q)$, which is an isogeny of degree $q$. Then $E(\mathbb{F}_q) = \{P \in E \mid \phi(P) = P\} = \ker(1 - \phi)$. We

have

$$\phi^*\omega = \phi^*\left(\frac{\mathrm{d}x}{y}\right) = \frac{d(x^q)}{y^q} = \frac{qx^{q-1}\mathrm{d}x}{y^q} = 0$$

as $q = p^n$, so $p \mid q$. By Lemma 6.3,

$$(1 - \phi)^*\omega = \omega - \phi^*\omega = \omega \neq 0,$$

so $1 - \phi$ is separable. By Theorem 2.3 and the fact that $1 - \phi$ is a group homomorphism, we argue in the proof of Theorem 6.5 that

$$\underbrace{|\ker(1 - \phi)|}_{|E(\mathbb{F}_q)|} = \deg(1 - \phi).$$

The map $\deg : \mathrm{Hom}(E, E) \to \mathbb{Z}$ is a positive definite quadratic form by Theorem 5.7. Hence by Lemma 7.1,

$$|\deg(1 - \phi) - 1 - \deg\phi| \leq 2\sqrt{\deg\phi}$$
$$\implies |\#E(\mathbb{F}_q) - q - 1| \leq 2\sqrt{q}. \qquad \square$$

**Definition 7.1.** For $\phi, \psi \in \mathrm{End}(E) = \mathrm{Hom}(E, E)$, we put $\langle \phi, \psi \rangle = \deg(\phi + \psi) - \deg(\phi) - \deg(\psi)$ and $\mathrm{tr}(\phi) = \langle \phi, 1 \rangle$.

**Corollary 7.3.** Let $E/\mathbb{F}_q$ be an elliptic curve and let $\phi \in \mathrm{End}(E)$ be the $q^{\mathrm{th}}$ power Frobenius map. Then $\#E(\mathbb{F}_q) = q + 1 - \mathrm{tr}(\phi)$ and $|\mathrm{tr}(\phi)| \leq 2\sqrt{q}$.

**Zeta functions.** For $K$ a number field,

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{(N(\mathfrak{a}))^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K, \mathfrak{p} \text{ prime}} \left(1 - \frac{1}{(N(\mathfrak{p}))^s}\right)^{-1}.$$

For $K$ a function field, i.e. $K = \mathbb{F}_q(C)$ where $C$ is a smooth projective curve,

$$\zeta_K(s) = \prod_{x \in |C|} \left(1 - \frac{1}{(Nx)^2}\right)^{-1},$$

where $|C| = \{\text{closed points of } C\} = \{\text{orbits for the action of } \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \text{ on } C(\overline{\mathbb{F}_q})\}$ and $Nx = q^{\deg x}$, where $\deg x$ is the size of the corresponding orbit (these definitions are borrowed from scheme theory). We have $\zeta_K(s) = F(q^{-s})$ for

some $F \in \mathbb{Q}[[T]]$. We have

$$F(T) = \prod_{x \in |C|} \left(1 - T^{\deg x}\right)^{-1}$$

$$\implies \log F(T) = \sum_{x \in |C|} \sum_{m=1}^{\infty} \frac{1}{m} T^{m \deg x}$$

$$\implies T \frac{\mathrm{d}}{\mathrm{d}T} \log F(T) = \sum_{x \in |C|} \sum_{m=1}^{\infty} \deg x T^{m \deg x}$$

$$\overset{n = m \deg x}{=} \sum_{n=1}^{\infty} \left( \sum_{x \in |C|, \deg x | n} \deg x \right) T^n$$

$$= \sum_{n=1}^{\infty} \# C(\mathbb{F}_{q^n}) T^n$$

$$\implies F(T) = \exp \left( \sum_{n=1}^{\infty} \frac{\# C(\mathbb{F}_{q^n})}{n} T^n \right).$$

**Definition 7.2.** The zeta function of a smooth projective curve $C/\mathbb{F}_q$ is

$$Z_C(T) = \exp \left( \sum_{n=1}^{\infty} \frac{\# C(\mathbb{F}_{q^n})}{n} T^n \right).$$

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**Theorem 7.4.** Let $E/\mathbb{F}_q$ be an elliptic curve with $\# E(\mathbb{F}_q) = q + 1 - a$. Then

$$Z_E(T) = \frac{1 - aT + qT^2}{(1 - T)(1 - qT)}.$$

*Proof.* Let $\phi : E \to E$ be the $q$–power Frobenius map. By Corollary 7.3, $\# E(\mathbb{F}_q) = q + 1 - \mathrm{tr}(q)$, so $\mathrm{tr}(\phi) = a$ and $\deg(\phi) = q$. By a result from Ex. Sheet 2, $\phi^2 - a\phi + q = 0$. Hence $\phi^{n+2} - a\phi^{n+1} + q\phi^n = 0$. As the trace is linear, $\mathrm{tr}(\phi^{n+2}) - a\mathrm{tr}(\phi^{n+1}) + q\mathrm{tr}(\phi^n) = 0$. The second order difference equation with initial conditions $\mathrm{tr}(1) = \langle 1, 1 \rangle = 2^2 - 1^2 - 1^2 = 2$ and $\mathrm{tr}(\phi) = a$ has solution

$$\mathrm{tr}(\phi^n) = \alpha^n + \beta^n$$

for $\alpha, \beta \in \mathbb{C}$ are roots of $X^2 - aX + q = 0$.[1] Apply Corollary 7.3 again to get

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[1] We don't need to worry about the case where the roots are equal, since we don't want a general solution, just a solution satisfying our initial conditions.

that $\#E(\mathbb{F}_{q^n}) = q^n + 1 - \text{tr}(\phi^n) = 1 + q^n - \alpha^n - \beta^n$. Hence

$$
\begin{aligned}
Z_E(T) &= \exp \sum_{n=1}^{\infty} \left( \frac{T^n}{n} + \frac{(qT)^n}{n} - \frac{(\alpha T)^n}{n} - \frac{(\beta T)^n}{n} \right) \\
&= \exp \left( -\log(1 - T) - \log(1 - qT) + \log(1 - \alpha T) + \log(1 - \beta T) \right) \\
&= \frac{(1 - \alpha T)(1 - \beta T)}{(1 - T)(1 - qT)} \\
&= \frac{1 - aT + qT^2}{(1 - T)(1 - qT)}.
\end{aligned}
$$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

**Remark.** Hasse's theorem tells us that $|a| \leq 2\sqrt{q}$, so the discriminant $a^2 - q$ is nonpositive, so the roots are complex conjugates, i.e. $\alpha = \bar{\beta}$, and $|\alpha| = |\beta| \overset{(\dagger)}{=} \sqrt{q}$.

Let $K = \mathbb{F}_q(E)$, then $\zeta_K(s) = 0 \implies Z_E(q^{-s}) = 0 \implies q^{-s} \in \{\frac{1}{\alpha}, \frac{1}{\beta}\} \implies q^s \in \{\alpha, \beta\} \implies q^{\text{Re}(s)} = |\alpha| = |\beta| \implies \text{Re}(s) = \frac{1}{2}$. This proves the Riemann hypothesis for elliptic curves over finite fields.

# 8   Formal groups

**Definition 8.1.** Let $R$ be a ring and $I \subset R$ an ideal. The $I$–**adic topology** on $R$ has basis $\{r + I^n \mid r \in R, n \geq 1\}$.

**Definition 8.2.** A sequence $(x_n)$ in $R$ is **Cauchy** if $\forall k$, $\exists N$ such that $x_m - x_n \in I^k$ $\forall m, n \geq N$.

**Definition 8.3.** $R$ is **complete** if

(i) $\bigcap_{n \geq 0} I^n = \{0\}$ (this is a Hausdorff–type condition).

(ii) Every Cauchy sequence converges.

**Useful remark.** If $x \in I$, then $\frac{1}{1-x} = 1 + x + x^2 + \dots$. This exists as the sequence of partial sums form a Cauchy sequence, and then we check that the result it converges to is an inverse for $\frac{1}{1-x}$. Hence $1 - x \in R^{\times}$.

**Example 8.1.** Basically the only two examples we care about in this course are:

- $R = \mathbb{Z}_p$, the $p$–adic integers, and $I = p\mathbb{Z}_p$.

- $R = \mathbb{Z}[[t]]$ and $I = (t)$.

**Lemma 8.1** (Hensel's lemma)**.** Let $R$ be complete with respect to an ideal $I$. Let $F \in R[X]$, $s \geq 1$ with $s \in \mathbb{Z}$. Suppose $a \in R$ satisfies

$$F(a) \equiv 0 \pmod{I^s}$$
$$F'(a) \in R^\times$$

Then there exists a unique $b \in R$ such that $F(b) = 0$ and $b \equiv a \pmod{I^s}$.

*Proof.* Let $u \in R^\times$ be such that $F'(a) = u \pmod I$ (e.g. we could take $u = F'(a)$). Replacing $F(X)$ by $\frac{F(X+a)}{u}$ we may assume $a = 0$ and $F'(0) \equiv 1 \pmod I$. We put $x_0 = 0$ and $x_{n+1} \overset{(\dagger)}{=} x_n - F(x_n)$. Each induction shows that $x_n \equiv 0 \pmod{I^s}\ \forall n$ ($\ddagger$). Now use the useful identity

$$F(X) - F(Y) = (X - Y)(F'(0) + XG(X,Y) + YH(X,Y))$$

for some $G, H \in R[X, Y]$. Call this identity ($\star$).

We claim that $x_{n+1} \equiv x_n \pmod{I^{n+s}}\ \forall n \geq 0$. To prove this, use induction. The case $n = 0$ is clear. Suppose $x_n \equiv x_{n-1} \pmod{I^{n+s-1}}$. By ($\star$),

$$F(x_n) - F(x_{n-1}) = (x_n - x_{n-1})(1 + c)$$

for some $c \in I$. Modulo $I^{n+s}$ we now use ($\dagger$) to get

$$F(x_n) - F(x_{n-1}) \equiv x_n - x_{n-1} \pmod{I^{n+s}}$$
$$\implies x_n - F(x_n) = x_{n-1} - F(x_{n-1}) \pmod{I^{n+s}}$$
$$\implies x_{n+1} \equiv x_n \pmod{I^{n+s}}.$$

Hence $(x_n)_{n \geq 0}$ is Cauchy, and $R$ is complete, so $x_n \to b$ as $n \to \infty$ for some $b \in R$. Taking the limit in ($\dagger$) gives $b = b - F(b)$ (as the polynomial is continuous in our topology), so $F(b) = 0$. Taking the limit in ($\ddagger$) gives $b \equiv 0 \equiv a \pmod{I^s}$.

For uniqueness, if $b_1, b_2$ work, then plug them into ($\star$) and use the useful remark that $1 - x$ is a unit to get that $b_1 = b_2$. $\qquad\square$

Write $E : Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3$ and look at its affine piece $Y \neq 0$ with $t = -\frac{X}{Y}, w = -\frac{Z}{Y}$ (the minus signs are here to match Silverman's book). We get

$$w = t^3 + a_1 tw + a_2 t^2 w + a_3 w^2 + a_4 tw^2 + a_6 w^3 = f(t, w).$$

We apply Hensel's lemma (Lemma 8.1) with $R = \mathbb{Z}[a_1, \ldots, a_6][[t]]$, $I = (t)$ and $F(X) = X - f(t, X) \in R[X]$. We take $s = 3$, $a = 0$ and check that $F(a) = F(0) = -f(t, 0) = -t^3 \equiv 0 \pmod{I^3}$ and $F'(0) = 1 - a_1 t - a_2 t^2 \in R^x$

by our useful remark, so the assumptions hold. Hence there exists a unique $\omega(t) \in R = \mathbb{Z}[a_1, \ldots, a_6][[t]]$ such that $\omega(t) = f(t, w(t))$ and $w(t) \equiv 0 \pmod{t^3}$.

**Remarks.**

(i) Taking $u = 1$ in the proof of Hensel's lemma gives $w(t) = \lim_{n \to \infty} w_n(t)$ where $w_0(t) = 0$, $w_{n+1}(t) = f(t, w_n(t))$.

(ii) In fact, $w(t) = t^3(1 + A_1 t + A_2 t^2 + \ldots)$ where $A_1 = a_1$, $A_2 = a_1^2 + a_2$, $A_3 = a_1^3 + 2a_1 a_2 + 2a_3$, etc. (i.e. we can compute the series explicitly).

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**Lemma 8.2.** Let $R$ be an integral domain, complete with respect to an ideal $I$. Let $a_0, \ldots, a_6 \in R$ and let $K = \mathrm{Frac}(R)$. Then

$$\widehat{E}(I) := \{(t, w) \in E(K) \mid t, w \in I\}$$

is a subgroup of $E(K)$.

**Remark.** By uniqueness in Hensel's lemma, $\widehat{E}(I) = \{(t, w(t)) \in E(K) \mid t \in I\}$.

*Proof.* Taking $(t, w) = (0, 0)$ shows $0_E \in \widehat{E}(I)$. So it suffices to show that if $P_1, P_2 \in \widehat{E}(I)$, then $P_3 := -P_1 - P_2 \in \widehat{E}(I)$. Since we're working over an affine piece with the identity at 0, we know three points sum to zero if and only if they lie on the same line. Say $P_i = (t_i, w_i)$ with the line $P_1 P_2$ given by $w = \lambda t + \nu$. We have $P_1, P_2 \in \widehat{E}(I) \implies t_1, t_2 \in I$ and $w_1 = w(t_1), w_2 = w(t_2)$. Write $w(t) = \sum_{n=2}^{\infty} A_{n-2} t^{n+1}$ with $A_0 = 1$. We have

$$\lambda = \begin{cases} \frac{w(t_2) - w(t_1)}{t_2 - t_1} & \text{if } t_1 \neq t_2 \\ w'(t_1) & \text{if } t_1 = t_2 \end{cases} = \sum_{n=2}^{\infty} A_{n-2}(t_1^n + t_1^{n-1} t_2 + \ldots + t_2^n) \in I,$$

$$\nu = w_1 - \lambda t_1 \in I.$$

Substituting $w = \lambda t + \nu$ into $w = f(t, w)$ gives

$$\lambda t + \nu = t^3 + a_1 t(\lambda t + \nu) + a_2 t^2(\lambda t + \nu) + a_3(\lambda t + \nu)^2 + a_4 t(\lambda t + \nu)^2 + a_6(\lambda t + \nu)^3.$$

Let

$$A = (\text{coeff. of } t^3) = 1 + a_2 \lambda + a_4 \lambda^2 + a_6 \lambda^3,$$
$$B = (\text{coeff. of } t^2) = a_1 \lambda + a_2 \nu + a_3 \lambda^2 + 2a_4 \lambda \nu + 3a_6 \lambda^2 \nu.$$

We have $A \in R^\times$, $B \in I$. Hence $t_3 = \frac{-B}{A} - t_1 - t_2 \in I$ and $w_3 = \lambda t_3 + \nu \in I$.  $\square$

Taking $R = \mathbb{Z}[a_1, \ldots, a_6][[t]]$ and $I = (t)$ and using Lemma 8.2 implies $\exists \iota \in \mathbb{Z}[a_1, \ldots, a_6][[t]]$ with $\iota(0) = 0$ such that $[-1](t, w(t)) = (\iota(t), w(\iota(t)))$.

Taking $R = \mathbb{Z}[a_1, \ldots, a_6][[t_1, t_2]]$ and $I = (t_1, t_2)$ and using Lemma 8.2 implies $\exists F \in \mathbb{Z}[a_1, \ldots, a_6][[t_1, t_2]]$ with $F(0,0) = 0$ and

$$(t_1, w(t_1)) + (t_2, w(t_2)) = (F(t_1, t_2), w(F(t_1, t_2))).$$

In fact, $F(X, Y) = X + Y - a_1 XY - a_2(X^2 Y + XY^2) + \ldots$.

By properties of the group law, we deduce

(i) $F(X, Y) = F(Y, X)$,

(ii) $F(X, 0) = X$ and $F(0, Y) = Y$,

(iii) $F(X, F(Y, Z)) = F(F(X, Y), Z)$,

(iv) $F(X, \iota(X)) = 0$.

**Definition 8.4.** Let $R$ be a ring. A **formal group** over $R$ is a power series $F(X, Y) \in R[[X, Y]]$ satisfying the first three axioms above.

An exercise on Ex. Sheet 2 asks us to show that the first three conditions imply the fourth, i.e. there is a unique $\iota(X) = -X + \ldots \in R[[X]]$ such that $F(X, \iota(X)) = 0$.

**Example 8.2.**  (i) The additive formal group $F(X, Y) = X + Y$, called $\widehat{\mathbb{G}_a}$.

(ii) The multiplicative formal group $F(X, Y) = X + Y + XY = (1 + X)(1 + Y) - 1$, called $\widehat{\mathbb{G}_m}$.

(iii) The formal group of an elliptic curve, $F(X, Y) = $ [see above], called $\widehat{E}$.

**Definition 8.5.** Let $\mathcal{F}$ and $\mathcal{G}$ be formal groups over $R$ given by power serise $F$ and $G$.

(i) A **morphism** $\mathcal{F} \to \mathcal{G}$ is a power series $f \in R[[T]]$ such that $f(0) = 0$ satisfying $f(F(X, Y)) = G(f(X), f(Y))$.

(ii) We say $\mathcal{F}$ is **isomorphic** to $\mathcal{G}$, i.e. $\mathcal{F} \cong \mathcal{G}$ if there exist morphisms $\mathcal{F} \xrightarrow{f} \mathcal{G}$ and $\mathcal{G} \xrightarrow{g} \mathcal{F}$ such that $f(g(T)) = g(f(T)) = T$.

**Theorem 8.3.** If char $R = 0$, then any formal group $\{$ over $R$ is isomorphic to $\widehat{\mathbb{G}_a}$ over $R \otimes \mathbb{Q}$. (In other words, our conditions are char $R = 0$ and "the integers are invertible"). More precisely:

(i) There is a unique power series $\log(T) = T + \frac{a_2}{2} T^2 + \frac{a_3}{3} T^3 + \ldots$ with $a_i \in R$ such that
$$\log(F(X, Y)) = \log(X) + \log(Y). \ (\star)$$

(ii) There is a unique power series $\exp(T) = T + \frac{b_2}{2!}T^2 + \frac{b_3}{3!}T^3 + \dots$ with $b_i \in R$ such that

$$\exp(\log(T)) = \log(\exp(T)) = T.$$

*Proof.*   (i) Notation: Write $F_1(X,Y) = \frac{\partial F}{\partial X}(X,Y)$. Uniqueness: Let $p(T) = \frac{\mathrm{d}}{\mathrm{d}T}\log T = 1 + a_2 T + 1_3 T^2 + \dots$. Differentiating $(\star)$ with respect to $X$ gives $p(F(X,Y))F_1(X,Y) = p(X) + 0$. Putting $X = 0$ gives $P(Y)F_1(0,Y) = 1$, so $p(Y) = \frac{1}{F_1(0,Y)}$, proving uniqueness.

Existence: Let $p(T) = F_1(0,T)^{-1} = 1 + a_2 T + a_3 T^2 + \dots$ for some $a_i \in R$. Define $\log T = T + \frac{a_2}{2}T^2 + \frac{a_3}{3}T^3 + \dots$, so $p(T) = \frac{\mathrm{d}}{\mathrm{d}T}\log T$. Then

$$F(F(X,Y),Z) = F(X,F(Y,Z))$$

$$\overset{\frac{\mathrm{d}}{\mathrm{d}X}}{\Longrightarrow} F_1(F(X,Y),Z)F_1(X,Y) = F_1(X,F(Y,Z))$$

$$\overset{X=0}{\Longrightarrow} F_1(Y,Z)p(Y)^{-1} = p(F(Y,Z))^{-1}$$

$$\Longrightarrow F_1(Y,Z)p(F(Y,Z)) = p(Y)$$

$$\overset{\text{intg. wrt } Y}{\Longrightarrow} \log(F(Y,Z)) = \log(Y) + h(Z)$$

for some power series $H$. But the symmetry in $Y$ and $Z$ implies that $h(Z) = \log Z$, so we're done.

(ii)

$\square$