

Part III - Modular Forms

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1 Introduction

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Lecture 1

Definition 1.1. We define the following groups:

$$\begin{aligned}\mathfrak{h} &= \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\} \\ GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\ \Gamma(1) &= SL_2(\mathbb{Z}) = \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.\end{aligned}$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on \mathfrak{h} by Möbius transformations.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$. Then

$$\operatorname{Im}(g\tau) = \frac{1}{2i} \left(\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\det(g)\operatorname{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{h}$. This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in \mathfrak{h} is conjugate to i . □

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{h} \rightarrow \mathbb{C}$, then

$$f|_k[g] : \mathfrak{h} \rightarrow \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight k action of g on f** .

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

Proof. We compute

$$\begin{aligned} (f|_k[g])|_k[h](\tau) &= \det(h)^{k-1} f|_k[g](h\tau) j(h, \tau)^{-k} = \\ &= \det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g, h\tau)^{-k} j(h, \tau)^{-k} \stackrel{?}{=} \\ &= \det(gh)^{k-1} f(gh\tau) j(gh, \tau)^{-k} = f|_k[gh](\tau). \end{aligned}$$

Hence we need to check that $j(gh, \tau) = j(gh, \tau)j(h, \tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh, \tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g \left(j(h, \tau) \begin{pmatrix} h\tau \\ 1 \end{pmatrix} \right) = j(h, \tau) j(g, h\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. \square

Formulae. For $g \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$, we have

$$\mathrm{Im}(g\tau) = \det(g) \frac{\mathrm{Im}(\tau)}{|j(g, \tau)|^2} \text{ and } j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\gamma \leq \Gamma(1)$ of finite index¹. A **weakly modular function of weight k and level Γ** is a meromorphic function $f : \mathfrak{h} \rightarrow \mathbb{C}$ which is invariant under the weight k action of Γ , i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in \mathfrak{h} and at ∞ .

It is a fact that modular forms of fixed weight and level live in finite-dimensional \mathbb{C} -vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

- (1) They are related to the theory of elliptic functions. Let E/\mathbb{C} be an elliptic curve and ω a holomorphic non-zero 1-form. Then there exists a unique lattice² $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \rightarrow E$ such that $\phi^*(\omega) = dz$. Then

¹In other words, γ is a (finite index) subgroup of $\Gamma(1)$.

²i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over \mathbb{R} .

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{h}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{h}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over \mathbb{C} .

- (2) Modular forms f have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n .

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q -expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$.

- (3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to \mathbb{C} and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L -function. Modular forms can be used to construct interesting examples of L -functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

- (4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves E/\mathbb{C} up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L -functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

2 Modular Forms on $\Gamma(1)$

09 Oct 2022,
Lecture 2

Reminder. A **meromorphic** function in an open subset $U \subset \mathbb{C}$ is a closed subset $A \subset U$ and a holomorphic function $f : U \setminus A \rightarrow \mathbb{C}$ such that $\forall a \in A$, $\exists \delta > 0$ such that $D^*(a, \delta) \subset U \setminus A$ and $\exists n \geq 0$ such that $(z - a)^n f(z)$ extends to a holomorphic function in $D(a, \delta)$.

f then has a Laurent expansion $\sum_{m \in \mathbb{Z}} a_m (z - a)^m$ valid on $D^*(a, \delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a meromorphic function \tilde{f} in $D^*(0, 1)$ such that $f(\tau) = \tilde{f}(e^{2\pi i \tau})$.

Proof. f is meromorphic in \mathfrak{h} by assumption. Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$, as f is invariant under the weight k action of γ . But also $f(\gamma\tau) = f(\tau + 1)$, so f is periodic.

Now map a strip of \mathfrak{h} of width 1 to $D^*(0, 1)$ by $\tau \mapsto e^{2\pi i \tau}$. Existence of \tilde{f} : Let $a \in D^*(0, 1)$ and $\delta > 0$ be such that $D(a, \delta) \subset D^*(0, 1)$. Define \tilde{f} on $D(a, \delta)$ by $\tilde{f}(q) = f(\frac{1}{2\pi i} \log q)$, for any branch of \log defined in $D(a, \delta)$. This is meromorphic and independent of the choice of the branch of \log , as f is periodic with period 1. This defines \tilde{f} in $D^*(0, 1)$.

\tilde{f} is unique since $\tau \mapsto e^{2\pi i \tau}$ is surjective. \square

If \tilde{f} extends to a meromorphic function³ in $D(0, 1)$, then $\exists \delta > 0$ such that \tilde{f} has a Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0, \delta)$.

In the region $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$, we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q = e^{2\pi i \tau}$. This is called the **q -expansion** of the weakly modular function f .

Definition 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is **meromorphic at ∞** if \tilde{f} extends to a meromorphic function in $D(0, 1)$.

We say f is **holomorphic at ∞** if \tilde{f} is meromorphic at ∞ and has a removable singularity at $q = 0$. In this case, we define $f(\infty) = \tilde{f}(0) = \lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau)$

We say f **vanishes at ∞** if f is holomorphic at ∞ and $f(\infty) = 0$.

³This might not be the case if the set of poles has a limit inside the disk.

Definition 2.2. A **modular function** (of weight k and level $\Gamma(1)$) is a weakly modular function (of weight k and level $\Gamma(1)$) which is meromorphic at ∞ .

A **modular form** is a weakly modular function which is holomorphic in \mathfrak{h} and holomorphic at ∞ .

A **cuspidal modular form** is a modular form that vanishes at ∞ .

Remark. We let $M_k(\Gamma(1))$ denote the set of modular forms of weight k and level $\Gamma(1)$. We write $S_k(\Gamma(1))$ for the set of cuspidal modular forms of weight k , level $\Gamma(1)$. Note $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$. These are \mathbb{C} -vector spaces. If k is odd, then these both only contain the zero function, since taking $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ gives $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$.

We now consider even weights only. If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k},$$

where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ for any $\tau \in \mathfrak{h}$.

If $\gamma \in \Gamma(1)$, then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau)j(\gamma, \tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k}j(\gamma, \tau)^{-k},$$

but $\Lambda_{\gamma\tau} = \mathbb{Z} \frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1} (\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)) = (c\tau+d)^{-1} \Lambda_\tau$.
Hence

$$\begin{aligned} G_k|_k[g](\tau) &= \sum_{\lambda \in (c\tau+d)^{-1} \Lambda_\tau \setminus 0} \lambda^{-k} (c\tau+d)^{-k} \\ &= \sum_{\lambda \in (c\tau+d)^{-1} \Lambda_\tau \setminus 0} ((c\tau+d)^{-1} \lambda)^{-k} (c\tau+d)^{-k} = G_k(\tau). \end{aligned}$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely. Hence:

Proposition 2.2. Let $k > 2$ be an even integer. Then $G_k(\tau)$ converges absolutely and defines a modular form of weight k and level $\Gamma(1)$ with $G_k(\infty) = 2\zeta(k)$. G_k is the **weight k Eisenstein series**.

We will later see that $M_2(\Gamma(1)) = 0$.

Proof. We want to show absolute and locally uniform convergence in \mathfrak{h} . This will show that G_k is holomorphic by complex analysis. Let $A \geq 2$ and define

$\Omega_A = \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq \frac{1}{A}, \text{Re}(\tau) \in [-A, A]\}$. We show uniform convergence in Ω_A . If $\tau \in \Omega_A, x \in \mathbb{R}$, then $|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A. \end{cases}$ Hence

$$|\tau + x| \geq \sup \left(\frac{1}{A}, \frac{|x|}{2A^2} \right) \geq \sup \left(\frac{1}{2A^2}, \frac{|x|}{2A^2} \right) = \frac{1}{2A^2} \sup(1, |x|).$$

If $(m, n) \in \mathbb{Z}^2, m \neq 0$, then

$$|m\tau + n| = |m| \left| \tau + \frac{n}{m} \right| \geq \frac{1}{2A^2} \sup \left(1, \left| \frac{n}{m} \right| \right) \cdot |m| = \frac{1}{2A^2} \sup(|m|, |n|).$$

This is also valid when $m = 0$ by inspection. If $\tau \in \Omega_A$, then

$$\begin{aligned} & \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} |m\tau + n|^{-k} \\ & \leq \left(\frac{1}{2n^2} \right)^{-k} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \sup(|m|, |n|)^{-k} \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} \cdot |\{(m, n) \in \mathbb{Z}^2 \mid \sup(|m|, |n|) = d\}| \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} 8d = 8(2A^2)^k \sum_{d \in \mathbb{N}} d^{1-k} \\ & < \infty \end{aligned}$$

whenever $k - 1 > 1$, i.e. $k > 2$. This shows absolute convergence, and uniform convergence in Ω_A by the Weierstrass M-test⁴. Hence G_k is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$. It remains to show that G_k is holomorphic at ∞ with $G_k(\infty) = 2\zeta(k)$. For this, it suffices to check that

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = 2\zeta(k).$$

This follows from uniform convergence in Ω_A : we get

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \lim_{\text{Im}(\tau) \rightarrow \infty} (m\tau + n)^{-k} = \sum_{n \in \mathbb{Z} \setminus 0} n^{-k} = 2 \sum_{n \geq 1} n^{-k} = 2\zeta(k).$$

□

⁴If we have a sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$, $M_n > 0$, $|f_n(x)| < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges uniformly