Part III - Elliptic Curves

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0 Introduction

19 Jan 2024, Lecture 1

The best books for the course include *The arithmetic of elliptic curves* by Silverman, Springer 1996, and *Lectures on elliptic curves* by Cassels, CUP 1991.

1 Fermat's Method of Infinite Descent

A right–angled triangle Δ has $a^2 + b^2 = c^2$ and $\text{area}(\Delta) = \frac{1}{2}ab$.

Definition 1.1. Δ is **rational** if $a, b, c \in \mathbb{Q}$. Δ is **primitive** if $a, b, c \in \mathbb{Z}$ are coprime.

Note that a primitive triangle has pairwise coprime side lengths because $a^2 + b^2 = c^2$.

Lemma 1.1. Every primitive triangle is of the form $(u^2 - v^2, 2uv, u^2 + v^2)$ for some integers $u > v > 0$.

Proof. WLOG let a, b, c be odd, even, odd. Then $\left(\frac{b}{2}\right)^2 = \frac{c+a}{2}\frac{c-a}{2}$, where we note that the RHS is a product of positive coprime integers. By unique factorization, $\frac{c+a}{2} = u^2$, $\frac{c-a}{2} = v^2$ for $u, v \in \mathbb{Z}$. This gives the desired result. $\square$

Definition 1.2. $D \in \mathbb{Q}_{>0}$ is a **congruent** number if there exists a rational triangle Δ with $\text{area}(\Delta) = D$.

Note that it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

Example 1.2. $D = 5, 6$ are congruent.

Lemma 1.3. $D \in \mathbb{Q}_{>0}$ is congruent $\iff$ $Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}, y \neq 0$.

Proof. Lemma 1.1 shows that D congruent $\implies$ $Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}, w \neq 0$. This implication also obviously goes the other way. To finish, divide through by w^4 and take $x = \frac{u}{v}, y = \frac{w}{v^2}$. $\square$

Fermat showed that 1 is not a congruent number.

Theorem 1.4. There is no solution to $w^2 = uv(u + v)(u - v)$ for $u, v, w \in \mathbb{Z}, w \neq 0$.

Proof. WLOG assume u, v are coprime and that $u, w > 0$. If $v < 0$, then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $\left(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2}\right)$. Then $u, v, u+v, u-v$ are pairwise coprime positive integers with their product a square, so by unique factorization in $\mathbb{Z}$, $u = a^2, v = b^2, u + v = c^2, u - v = d^2$ for $a, b, c, d \in \mathbb{Z}$.

Since $u \not\equiv v \pmod 2$, both c and d are odd. Then $\left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 = \frac{c^2+d^2}{2} = u = a^2$. This gives a primitive triangle with area $\frac{c^2-d^2}{8} = \frac{v}{4} = \left(\frac{b^2}{2}\right)$.

Let $w_1 = \frac{b}{2}$, then by Lemma 1.1, $w_1^2 = u_1 v_1 (u_1 + v_1)(u_1 - v_1)$ for some $u_1, v_1 \in \mathbb{Z}$. Hence we have a new solution to our original question, with $4w_1^2 = b^2 = v \mid w^2 \implies w_1 \le \frac{w}{2}$, so we're done by infinite descent. $\square$

A variant for polynomials. In the above, K is a field with char $K \neq 2$. Let $\overline{K}$ be the algebraic closure of K and consider for this whole section K with char $K \neq 2$.

Lemma 1.5. Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for 4 distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.

Proof. WLOG let $K = \overline{K}$ by extending if necessary. Changing coordinates on $\mathbb{P}^1$ (i.e. multiplying by a 2×2 invertible matrix), we may assume that the points $(\alpha : \beta)$ are $(1 : 0)$, $(0 : 1)$, $(1 : -1)$, $(1 : -\lambda)$ for $\lambda \in K \setminus \{0, 1\}$. Since our field is algebraically closed, let $\mu = \sqrt{\lambda}$. Then $u = a^2, v = b^2, u - v = (a+b)(a-b), u - \lambda v = (a + \mu b)(a - \mu b)$.

Unique factorization in $K[t]$ implies that $a + b, a - b, a + \mu b, a - \mu b$ are squares (since the necessary terms are coprime up to units, i.e. constants). But $\max(\deg(a), \deg(b)) \le \frac{1}{2}\max(\deg(u), \deg(v))$, so by Fermat's method of infinite descent, $u, v \in K$. $\square$

Definition 1.3.

(i) An **elliptic curve** E/K is the projective closure of the plane affine curve $y^2 = f(x)$ (this is called a Weierstrass equation) where $f \in K[x]$ is a monic cubic polynomial with distinct roots in $\overline{K}$.

(ii) For L/K any field extension, $E(L) = \{(x, y) \in L^2 \mid y^2 = f(x)\} \cup \{0\}$ (the point at infinity in the projective closure), it turns out that $E(L)$ is naturally an abelian group.

In this course, we study $E(K)$ for K a finite field, local field, number field.

Lemma 1.3 and Theorem 1.4 show that if $E : y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$.

Corollary 1.6. Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.

Proof. WLOG $K = \overline{K}$. By a change of coordinates, we may assume $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ for $u, v \in K(t)$ coprime. Then $w^2 = uv(u-v)(u-\lambda v)$ for some $w \in K[t]$. Unique factorization in $K[t]$ shows that $u, v, u - v, u - \lambda v$ are all squares, so by Lemma 1.5, $u, v \in K$, so $x, y \in K$. $\square$

2 Some remarks on algebraic curves

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In this section, work over an algebraically closed field $K = \overline{K}$.

Definition 2.1. A plane curve $C = \{f(x,y) = 0\} \subset \mathbb{A}^2$ (for $f \in K[x,y]$ irreducible) is **rational** if it has a rational parametrization, i.e. $\exists \phi, \psi \in K(t)$ such that

(i) The map $\mathbb{A}^1 \to \mathbb{A}^2$ by $t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{\text{finite set}\}$.

(ii) $f(\phi(t), \psi(t)) = 0$ in $K(t)$.

Example 2.1. (a) Any nonsingular conic is rational. For example, for $x^2 + y^2 = 1$, take a line with slope t through $(-1, 0)$ (the anchor) and solve to get the rational parametrization $(x,y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

(b) Any singular plane cubic is rational, for example $y^2 = x^3$ giving $(x,y) = (t^2, t^3)$ with the anchor at the singularity $(0,0)$ and $y^2 = x^2(x+1)$ with the parametrization to be computed on Ex. Sheet 1 (anchor still at $(0,0)$).

(c) Corollary 1.6 shows that elliptic curves are not rational.

Remark. The genus $g(C) \in \mathbb{Z}_{\geq 0}$ is an invariant of a smooth projective curve C. If $K = \mathbb{C}$, then $g(C)$ is the genus of the Riemann surface. A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

Proposition 2.2. (Here we still assume $K = \overline{K}$). Let C be a smooth projective curve.

- C is rational (see Definition 2.1) $\iff g(C) = 0$.
- C is an elliptic curve $\iff g(C) = 1$.

Proof. (i) Omitted.

(ii) ($\implies$): Check C is a smooth plane curve in $\mathbb{P}^2$ (see Ex. Sheet 1) and use the above remark.

($\impliedby$): We will see this later.

$\square$

Order of vanishing. Let C be an algebraic curve with function field $K(C)$ and let $P \in C$ be a smooth point. Write $\operatorname{ord}_P(f)$ for the order of vanishing of $f \in K(C)$ at P (which is negative if f has a pole at P).

Fact. $\operatorname{ord}_P : K(C)^\times \to \mathbb{Z}$ is a discrete valuation, i.e. $\operatorname{ord}_P(f_1 f_2) = \operatorname{ord}_P(f_1) + \operatorname{ord}_P(f_2)$ and $\operatorname{ord}_P(f_1 + f_2) \geq \min(\operatorname{ord}_P(f_1), \operatorname{ord}_P(f_2))$.

Definition 2.2. We say $t \in K(C)^\times$ is a **uniformizer** at P if $\operatorname{ord}_P(t) = 1$.

Example 2.3. $C = \{g = 0\} \subset \mathbb{A}^2$ for $g \in K[x, y]$. Then $K(C) = \operatorname{Frac}\left(\frac{K[x,y]}{(g)}\right)$. Write $g = g_0 + g_1(x, y) + g_2(x, y) + \dots$ for g_i homogeneous of degree i. Suppose $P = (0, 0)$ is a smooth point, e.g. $g_0 = 0$ and let $g_1(x, y) = \alpha x + \beta y$ with α, β not both zero ($\alpha x + \beta y = 0$ gives a tangent to the curve at P). Let $\gamma, \delta \in K$ and consider also the line $\gamma x + \delta y$ through P. Then it is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\alpha\delta - \beta\gamma \neq 0$.

Example 2.4. Consider $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$ for $\lambda \neq 0, 1$ and consider its projective closure by taking $x = \frac{X}{Z}, y = \frac{Y}{Z}$ to get $\{Y^2 Z = X(X-Z)(X-\lambda Z)\} \subset \mathbb{P}^2$. This has only one point at infinity, $P = (0 : 1 : 0)$. Our aim is to compute $\operatorname{ord}_P(x)$ and $\operatorname{ord}_P(y)$.

For this, put $t = \frac{X}{Y}, w = \frac{Z}{Y}$, so $w \overset{(\dagger)}{=} t(t - w)(t - \lambda w)$. Now P is the point $(t, w) = (0, 0)$, which is a smooth point with $\operatorname{ord}_P(t) = \operatorname{ord}_P(t - w) = \operatorname{ord}_P(t - \lambda w) = 1$, so ($\dagger$) gives $\operatorname{ord}_P(w) = 3$. We now find

$$\begin{aligned} \operatorname{ord}_P(x) &= \operatorname{ord}_P\left(\frac{X}{Z}\right) = \operatorname{ord}_P\left(\frac{t}{w}\right) = 1 - 3 = -2 \\ \operatorname{ord}_P(y) &= \operatorname{ord}_P\left(\frac{Y}{Z}\right) = \operatorname{ord}_P\left(\frac{1}{w}\right) = -3. \end{aligned}$$

Riemann–Roch space. Let C be a smooth projective curve.

Definition 2.3. A **divisor** is a formal sum of points on C, say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. We say $\deg D = \sum_{P \in C} n_P$.

D is **effective** (written $D \geq 0$) if $n_P \geq 0 \ \forall P \in C$. If $f \in K(C)^\times$, then $\operatorname{div}(f) = \sum_{P \in C} \operatorname{ord}_P(f) P$. The Riemann–Roch space of $D \in \operatorname{Div}(C)$ is

$$\mathcal{L}(D) = \{f \in K(C)^\times \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\},$$

i.e. the K–vector space of rational functions on C with ”poles no worse than specified by D” (i.e. every coefficient of $\operatorname{div}(f) + D$ is nonnegative).

We quote Riemann–Roch for surfaces of genus 1: We have

$$\dim \mathcal{L}(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ 0 \text{ or } 1 & \text{if } \deg D = 0 \\ 0 & \text{if } \deg D < 0. \end{cases}$$

Example 2.5. We revisit Example 2.4. We have $\mathcal{L}(2P) = \langle 1, x \rangle$ and $\mathcal{L}(3P) = \langle 1, x, y \rangle$.

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We still have char $K \neq 2$ and $\overline{K} = K$.

Proposition 2.6. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic and let $P \in C$ be a point of inflection. Then we may change coordinates such that $C : Y^2Z = X(X-z)(X-\lambda Z)$ and $P = (0:1:0)$ (for some $\lambda \neq 0, 1$).

Proof. First change coordinates such that $P = (0:1:0)$. Then change coordinates such that the tangent line becomes $T_pC = \{Z = 0\}$. Say $C = \{F(X,Y,Z) = 0\} \subset \mathbb{P}^2$. A point on the tangent line is of the form $(t:1:0)$ and since $P \in C$ is a point of inflection, we get $F(t,1,0) = \text{const} \cdot t^3$, i.e. F has no terms X^2Y, XY^2 or Y^3.

Hence $F = \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3 \rangle$. Notably, Y^2Z has a nonzero coefficient, otherwise $P \in C$ would be singular, a contradition to C being smooth. The coefficient of X^3 is nonzero as well, otherwise $Z \mid F$. We are free to rescale X, Y, Z, F, so WLOG C is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

Substituting $Y \mapsto Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$. This gives

$$C : Y^2Z = Z^3 f\left(\frac{X}{Z}\right)$$

for a monic cubic polynomial f. Since C is smooth, f has distinct roots, WLOG $0, 1, \lambda$, so $C : Y^2Z = X(X-Z)(X-\lambda Z)$. $\square$

The form $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ is the Weierstrass form. The form $Y^2Z = X(X-Z)(X-\lambda Z)$ is the Legendre form.

Remark. It can be shown that the points of inflection of a plane curve $C = \{F(X_1, X_2, X_3) = 0\} \subset \mathbb{P}^2$ are given by solving the Hessian:

$$\begin{cases} H = \left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right) = 0 \\ F(X_1, X_2, X_3) = 0. \end{cases}$$

2.1 The degree of a morphism

Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then $\phi^* : K(C_2) \to K(C_1)$ by $f \mapsto f \circ \phi$, giving an injective map $\phi^* K(C_2)$ to $K(C_1)$.

Definition 2.4. The **degree** of ϕ is $\deg \phi = [K(C_1) : \phi^* K(C_2)]$.

We say ϕ is **separable** if $K(C_1)/\phi^* K(C_2)$ is a separable field extension.

Suppose $P \in C_1, Q \in C_2$ and $\phi : P \mapsto Q$. Let $t \in K(C_2)$ be a uniformizer at Q.

Definition 2.5. $e_\phi(P) = \text{ord}_P(\phi^* t)$, which is always ≥ 1 and independent of t.

Theorem 2.7. Let $\phi : C_1 \to C_2$ be a nonconstant morphism of smooth projective curves. Then

$$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi \ \forall Q \in C_2.$$

Moreover, if ϕ is separable, then $e_\phi(P) = 1$ for all but infitely many $P \in C_1$.

We don't prove this.

In particular, this shows that:

- ϕ is surjective (very important here that we're in $\overline{K}$).
- $|\phi^{-1}(Q)| \leq \deg \phi$.
- If ϕ is separable, then equality holds in (ii) for all but finitely many points $Q \in C_2$.

Important remark. Let C be an algebraic curve. A rational map is given by

$$\begin{aligned} C &\to \mathbb{P}^n \\ \phi &\mapsto (f_0, f_1, \dots, f_n) \end{aligned}$$

where $f_0, \dots, f_n \in K(C)$ are not all zero. Then we have a fact: If C is smooth, then ϕ is a morphism. This saves us a lot of time (we can go from a rational map to a morphism immediately).

3 Weierstrass equations

We now drop the assumption that $\overline{K} = K$, but we will still assume that K is perfect.

Definition 3.1. An **elliptic curve** E/K is a smooth projective curve of genus 1 defined over K with a specified K–rational point $O = 0_E$.

Example 3.1. $\{X^3 + pY^3 + p^2Z^3 = 0\} \subset \mathbb{P}^2$ is not an elliptic curve over $\mathbb{Q}$, since it has no $\mathbb{Q}$–rational point.

Theorem 3.2. Every elliptic curve E is isomorphic over K to a curve in Weierstrass form via an isomorphism taking 0_E to $(0 : 1 : 0)$.

Remark. Proposition 2.6 treated the special case where E is a smooth plane cubic and 0_E is a point of inflection.

Fact. If $D \in \mathrm{Div}(E)$ is defined over K, then $\mathcal{L}(D)$ has a basis in $K(E)$ (not just in $\overline{K}(E)$). Here D is defined over K if it is fixed by $\mathrm{Gal}(\overline{K}/K)$ (this is unimportant for us and we just write it down to be rigorous).

Proof. $\mathcal{L}(2\cdot 0_E) \subset \mathcal{L}(3\cdot 0_E)$. Pick bases $1, x$ and $1, x, y$. Note $\mathrm{ord}_{0_E}(x) = -2$ and $\mathrm{ord}_{0_E}(y) = -3$ (else x, y don't give a basis). The 7 elements $1, x, y, x^2, xy, x^3, y^2$ lie in the 6–dimensional vector space $\mathcal{L}(60_E)$ (as they have at most a sixth order pole), so they must satisfy a linear dependence relation.

Leaving out x^3 or y^2 leaves us with 6 elements, all with different order poles, giving a basis for $\mathcal{L}(60_E)$. Hence the coefficients of x^3 and y^2 are nonzero, so by rescaling x, y (if necessary) we get

$$E' : y^2 + a_1xy + a_2y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_i \in K$. Let E' be the curve defined by this equation (or rather its projective closure). There is a morphism $\phi : E \to E' \subset \mathbb{P}^2$ by $P \mapsto (x(P) : y(P) : 1) = \left(\frac{x}{y}(P) : 1 : \frac{1}{y}(P)\right)$. (Since E is smooth, we know that this rational map is a morphism). Hence $0_E \mapsto (0 : 1 : 0)$.

We have $E \xrightarrow{x} \mathbb{P}^1$ by $x \mapsto (x : 1)$ (and similarly for y), so

$$\begin{aligned}
[K(E) : K(x)] &= \deg(E \xrightarrow{x} \mathbb{P}^1) = \mathrm{ord}_{0_E}\left(\frac{1}{x}\right) = 2 \\
[K(E) : K(y)] &= \deg(E \xrightarrow{y} \mathbb{P}^1) = \mathrm{ord}_{0_E}\left(\frac{1}{y}\right) = 3.
\end{aligned}$$

This gives an inclusion of fields $K(x) \leq K(E)$ of degree 2, $K(y) \leq K(E)$ of degree 3, while $K(x), K(y) \leq K(x, y) \leq K(E)$, so tower law gives $[K(E) : K(x, y)] = 1 \implies K(E) = K(x, y) = \phi^* K(E') \implies \deg\phi = 1$. (draw a picture!). This gives us an inverse that is a rational map, which we want to show is a morphism. For this, we just need to show that E' is smooth.

If E' were singular, then E and E' are rational, a contradiction. So E' is smooth and hence ϕ^{-1} is a morphism, so ϕ is an isomorphism. □

Proposition 3.3. Let E, E' be elliptic curves over K in Weierstrass form. Then $E \cong E'$ over $K \iff$ the equations are related by a change of variables

$$\begin{aligned}
x &= u^2x' + r \\
y &= u^3y' + u^2sx' + t
\end{aligned}$$

for $r, s, t, u \in K$ with $u \neq 0$.

Proof. $\mathcal{L}(2\cdot 0_E) = \langle 1, x\rangle = \langle 1, x'\rangle \implies x = \lambda x' + r$ for some $\lambda, r \in K, \lambda \neq 0$. Similarly $\mathcal{L}(3\cdot 0_E) = \langle 1, x, y\rangle = \langle 1, x', y'\rangle \implies y = \mu y' + \sigma x' + t$ for some $\mu, \sigma, t \in K, \mu \neq 0$.

Looking at the coefficients of x^3 and y^2 tells us that $\lambda^3 = \mu^2$, so $\lambda = u^2, \mu = u^3$ for some $u \in K^\times$. Put $s = \frac{\sigma}{u^2}$ to conclude. □

A Weierstrass equation defines an elliptic curve $\iff$ it defines a smooth curve $\iff$ $\Delta(a_1, \ldots, a_6) \neq 0$, where $\Delta \in \mathbb{Z}[a_1, \ldots, a_6]$ is a certain polynomial.

If char $K \neq 2, 3$, we may reduce to the case $E : y^2 = x^3 + ax + b$. In this case, the discriminant is $\Delta = -16(4a^3 + 27b^2)$.

Corollary 3.4. Assume char $K \neq 2, 3$. Elliptic curves

$$\begin{aligned} E &: y^2 = x^3 + ax + b \\ E' &: y^2 = x^3 + a'x + b' \end{aligned}$$

are isomorphic over $K \iff \begin{cases} a' = u^4 a \\ b' = u^6 b \end{cases}$ for some $u \in K^\times$.

Proof. E, E' are related by a substitution as in Proposition 3.3 with $r = s = t = 0$. □

Definition 3.2. The j–**invariant** is $j(E) = \frac{1728(4a^3)}{4a^3+27b^2}$.

Corollary 3.5. $E \cong E' \implies j(E) \cong j(E')$ and the converse holds if $K = \overline{K}$.

Proof. $E \cong E' \iff \begin{cases} a' &= u^4 a \\ b' &= u^6 b \end{cases}$ for some $u \in K^\times \implies (a^3 : b^2) = ((a')^3 : (b')^2) \iff j(E) = j(E')$. The middle step is reversible if $K = \overline{K}$. □

4 The Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic with $0_E \in E(K)$ (not immediately assumed to be in Weierstrass form). E meets any line in 3 points, counted with multiplicity.

For $P, Q \in E$, let S be the 3^{rd} point of intersction of PQ with E and then let R be the 3^{rd} intersection of $0_E S$ with E. We define $P \oplus Q = R$. (Later we drop the circle and just write $+$). If $P = Q$, instead take the tangent line at P, i.e. $T_P E$, etc. This is the "chord and tangent process".

Theorem 4.1. $(E, \oplus)$ is an abelian group.

Remark. Here E means $E(\overline{K})$ since we haven't specified a field yet.

Proof. (i) $\oplus$ is commutative trivially.

(ii) 0_E is the identity, since the line through $0_E P$ meets S for the 3^{rd} time at S and then SP meets E for the 3^{rd} time at 0_E (drawing a picture makes this obvious).

(iii) Inverses: Let S be the 3^{rd} intersection of T_{0_E} with E and Q the 3^{rd} intersection of PS with E. Then $P \oplus Q = 0_E$.

(iv) Associativity is much harder. We have some setup:

Definition 4.1. $D_1, D_2 \in \operatorname{Div}(E)$ are **linearly equivalent** if $\exists f \in K(E)^\times$ such that $\operatorname{div}(f) = D_1 - D_2$. Write $D_1 \sim D_2$ and $[D] = \{D' \mid D' \sim D\}$.

Definition 4.2. The **Picard group** is $\operatorname{Pic}(E) = \operatorname{Div}(E)/\sim$. Also define $\operatorname{Pic}^0(E) = \operatorname{Div}^0(E)/\sim$ where $\operatorname{Div}^0(E) = \{D \in \operatorname{Div}(E) \mid \deg(D) = 0\}$.

We define $\psi : E \to \operatorname{Pic}^0(E)$ by $P \mapsto [(P) - (0_E)]$.

Proposition 4.2. (i) $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) ψ is a bijection.

Proof. (i) WLOG let the lines PQ and $0_E S$ be given by $l = 0$ and $m = 0$. Then

$$\operatorname{div}\left(\frac{l}{m}\right) = (P) + (S) + (Q) - (0_E) - (S) - (R),$$

hence $(P) + (Q) \sim (P \oplus Q) + (0_E)$, so $(P \oplus Q) - (0_E) \sim (P) - (0_E) + (Q) - (0_E)$, so $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) Injectivity: Suppose $\psi(P) = \psi(Q)$ for $P \neq Q$. Then $\exists f \in \overline{K}(E)^\times$ such that $\operatorname{div}(f) = (P) - (0_E) - (Q) + (0_E) = (P) - (Q) \implies E \xrightarrow{f} \mathbb{P}^1$ has degree 1 (for example since evaluation at 0 on the affine line gives that P has one root and Q has one pole), so $E \cong \mathbb{P}^1$, a contradiction.

Surjectivity: Let $[D] \in \operatorname{Pic}^0(E)$. Then $D + (0_E)$ has degree 1, so by Riemann–Roch, $\dim \mathcal{L}(D + (0_E)) = 1$, so $\exists 0 \neq f \in \overline{K}(E)$ such that $\operatorname{div}(f) + D + (0_E) \geq 0$, but $\operatorname{div}(f) + D + (0_E)$ has degree 1, so $\operatorname{div}(f) + D + (0_E) = (P)$ for some $P \in E \implies (P) - (0_E) \sim D \implies \psi(P) = [D]$.

□

We conclude that ψ identifies $(E, \oplus)$ with $(\operatorname{Pic}^0(E), +)$, so $\oplus$ is associative.

□

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Formulae for E in Weierstrass form. Let $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Choose two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on it. Let the line through P_1 and P_2 be given by $y = \lambda x + \nu$ and let it meet E again at $P' = (x', y')$. We want to find $P_1 \oplus P_2 = P_3 = (x_3, y_3) = \ominus P'$ for $\ominus P$ the reflection of P across the x–axis. We easily compute $\ominus P_1 = (x_1, -(a_1x + a_3) - y_1)$.

Substituting $y = \lambda x + \nu$ into our equation for E and looking at the coefficient of x^2 gives $\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x' = x_1 + x_2 + x_3$, so $x_3 = \lambda^2 + a_1\lambda - a_2 - x_1 - x_2$. For y_3 we find

$$y_3 = -(a_1 x' + a_3) - y' = -(a_1 x_3 + a_3) - (\lambda x_3 + \nu) = -(\lambda + a_1)x_3 - a_3 - \nu.$$

It remains to find formulas for λ and ν.

- Case 1. $x_1 = x_2$, but $P_1 \neq P_2$. Then $P_1 \oplus P_2 = 0_E$.
- Case 2. $x_1 \neq x_2$. Then $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ and $\nu = y_1 - \lambda x_1 = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.
- Case 3. $P_1 = P_2$. In this case, compute the equation for the tangent line to get λ, ν as rational expressions in x_1, x_2, y_1, y_2.

Corollary 4.3. $E(K)$ is an abelian group.

Proof. $E(K)$ is a subgroup of $(E, \oplus)$.

- It has identity 0_E by definition.
- We have closure and inverses through the formulae above.
- Associativity and commutativity is inherited.

□

Theorem 4.4. Elliptic curves are group varieties, i.e.

$$\begin{aligned} [-1] : E \to E, P \mapsto \ominus P \\ \oplus : E \to E, (P, Q) \mapsto P \oplus Q \end{aligned}$$

are morphisms of algebraic varieties.

Proof. By the above formulae, $[-1] : E \to E$ is a rational map, i.e. a morphism by our important remark.

For $\oplus$, note by the above formulae that $\oplus : E \to E$ is a rational map regular on

$$U = \{(P, Q) \in E \times E \mid 0_E \notin \{P, Q, P \oplus Q, P \ominus Q\}\}.$$

For $P \in E$, let $\tau_P : E \to E$ be the "translation by P" map, given by $X \mapsto P \oplus X$. τ_P is a rational map, hence a morphism. Now for $A, B \in E$, we factor $\oplus$ as

$$E \times E \overset{\tau_{\ominus A} \times \tau_{\ominus B}}{\to} E \times E \overset{\oplus}{\to} E \overset{\tau_{A \oplus B}}{\to} E.$$

This shows $\oplus$ is regular on $(\tau_A \times \tau_B)(U)$, so $\oplus$ is regular on $E \times E$. □

Statement of results. The following isomorphisms in (i), (ii), (iv) respect the relevant topologies.

(i) $K = \mathbb{C}$. Then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ for Λ a lattice.

(ii) $K = \mathbb{R}$. Then
$$E(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \text{if } \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \text{if } \Delta < 0. \end{cases}$$

(iii) $K = \mathbb{F}_q$. Then $||E(\mathbb{F}_q)| - (q+1)| \leq 2\sqrt{q}$. This is Hasse's Theorem.

(iv) For a local field $[K : \mathbb{Q}_p] < \infty$ with ring of integers $\mathcal{O}_K$, $E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$.

(v) For a number field $[K : Q] < \infty$, $E(K)$ is a finitely generated abelian group (this is the Mordell–Weil Theorem). Basic group theory says that if A is a finitely generated abelian group, then $A \cong (\text{finite subgroup}) \times \mathbb{Z}^r$. Here r is called the rank of A. The proof of Mordell–Weil gives an upper bound for rank $E(K)$, but there is no known algorithm to compute the rank in all cases.

Brief remarks on the case $K = \mathbb{C}$. Let $\Lambda = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}\}$ where ω_1, ω_2 are a basis for $\mathbb{C}$ as an $\mathbb{R}$–vector space. Then meromorphic functions on the Riemann surface $\mathbb{C}/\Lambda$ correspond bijectively with Λ–invariant meromorphic functions in $\mathbb{C}$. The function field of $\mathbb{C}/\Lambda$ is generated by $\wp(z)$ and $\wp'(z)$, where

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \\ \wp'(z) &= -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}. \end{aligned}$$

These satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ for some constants $g_2, g_3 \in \mathbb{C}$ depending on Λ. One shows $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where $E : y^2 = 4x^3 - g_2 x - g_3$ which is an isomorphism on both groups (via $z \mapsto (\wp(z), \wp'(z))$) and on Riemann surfaces. We have the following result:

Theorem 4.5 (Uniformization theorem)**.** Every elliptic curve over $\mathbb{C}$ arises in this way.

Definition 4.3. For $n \in \mathbb{Z}$, let $[n] : E \to E$ be given by $P \mapsto \underbrace{P \oplus P \oplus \ldots \oplus P}_{n \text{ copies}}$ if $n > 0$ and $[-n] = [-1] \circ [n]$.

Definition 4.4. The n–**torsion subgroup** of E is

$$E[n] = \ker(E \xrightarrow{[n]} E).$$

If $K = \mathbb{C}$, then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, so $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ and $\deg[n] = n^2$. Call these results (1) and (2). We will show that (2) holds over any field K and (1) holds if char $K \nmid n$. 31 Jan 2024, Lecture 6

Lemma 4.6. Assume char $K \neq 2$ and $E : y^2 = f(x) = (x-e_1)(x-e_2)(x-e_3)$ (with $e_i \in \overline{K}$). Then $E[2] = \{0, (e_1, 0), (e_2, 0), (e_3, 0)\} \cong (\mathbb{Z}/2\mathbb{Z})^\times$.

Proof. Let $P = (x, y) \in E$. Then $2[P] = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. □

5 Isogenies

Let E_1, E_2 be elliptic curves

Definition 5.1. (i) An **isogeny** $\phi : E_1 \to E_2$ is a nonconstant morphism with $\phi(0_{E_1}) = 0_{E_2}$.

(ii) We say E_1 and E_2 are **isogenous** if there is an isogeny between them.

In (i), nonconstant is equivalent to surjective on $\overline{K}$–points. See Theorem 2.7.

Definition 5.2. $\mathrm{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \to E_2\} \cup \{0\}$ (the constant map at 0_E). This is an abelian group under $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 \stackrel{\phi}{\to} E_2 \stackrel{\psi}{\to} E_3$ are isogenies, then $\psi \circ \phi$ is an isogeny. By tower law, $\deg(\psi \circ \phi) = \deg(\psi)\deg(\phi)$.

Proposition 5.1. If $0 \neq n \in \mathbb{Z}$, then $[n] : E \to E$ is an isogeny.

Proof. $[n]$ is a morphism by Theorem 4.4. We need to show $[n] \neq [0]$. Assume char $K \neq 2$.

- Case $n = 2$. Lemma 4.6 implies that $E[2] \neq E$, so $[2] \neq 0$.
- Case n odd. Lemma 4.6 implies that $\exists 0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n] \neq [0]$.

Now use $[mn] = [m] \circ [n]$ to conclude.

If char $K = 2$, then we can replace Lemma 4.6 with an explicit lemma about 3–torsion points. □

Corollary 5.2. $\mathrm{Hom}(E_1, E_2)$ is a torsion–free $\mathbb{Z}$–module.

Theorem 5.3. Let $\phi : E_1 \to E_2$ be an isogeny. Then

$$\phi(P + Q) = \phi(P) + \phi(Q)\ \forall P, Q \in E.$$

Sketch proof. ϕ induces a map $\phi_* : \mathrm{Div}^0(E_1) \to \mathrm{Div}^0(E_2)$ by $\sum_{P \in E_1} n_P P \mapsto \sum_{p \in E_1} n_P \phi(P)$. Recall $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

Fact. If $f \in K(E_1)$, then $\mathrm{div}(N_{K(E_1)/K(E_2)} f) = \phi^*(\mathrm{div}\, f)$. So ϕ_* sends principal divisors to principal divisors. Since $\phi(0_{E_1}) = 0_{E_2}$, the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "f"] & E_2 \arrow[d, "g"] \\ \mathrm{Pic}^0(E_1) \arrow[r, "\phi_*"] & \mathrm{Pic}^0(E_2) \end{tikzcd}$$

(with $f(P) = [(P) - (0_{E_1})], g(Q) = [(Q) - (0_{E_2})]$). Since ϕ_* is a group homomorphism, ϕ is a group homomorphism. $\square$

Lemma 5.4. Let $\phi : E_1 \to E_2$ be an isogeny. Then there exists a morphism ξ making the following diagram commute:

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "x_2"] & E_2 \arrow[d, "x_1"] \\ \mathbb{P}^1 \arrow[r, "\xi"] & \mathbb{P}^1 \end{tikzcd}$$

with x_i the x–coordinate in a Weierstrass equation for E_i. Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ with $r, s \in K[t]$ coprime, then $\deg(\phi) = \deg(\xi) = \max(\deg(r), \deg(s))$.

Proof. For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 Galois extension with Galois group generated by $[-1]^*$. By Theorem 5.3, $\phi \circ [-1] = [-1] \circ \phi$, so if $f \in K(x_2)$, then $[-1]^*(\phi^* f) = \phi^*([-1]^* f) = \phi^* f$ and hence $\phi^* f \in K(x_1)$. Hence we find

$$\begin{tikzcd} & K(E_1) = K(x_1, y_1) \arrow[dl, dash, "2"'] \arrow[d, dash] \\ K(x_1) \arrow[d, dash] & K(E_2) = K(x_2, y_2) \arrow[dl, dash, "2"'] \\ K(x_2) & \end{tikzcd} .$$

In particular, $\phi^* x_2 = \xi(x_1)$ for some $\xi \in K(t)$. By tower law, $2\deg(\phi) = 2\deg(\xi) \implies \deg(\phi) = \deg(\xi)$. Now $K(x_2) \hookrightarrow K(x_1)$ by $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_1)}$ for $r, s \in K[t]$ coprime. Then minimal polynomial of x_1 over $K(x_2)$ is $F(t) = r(t) - s(t)x_2 \in K(x_2)[t]$. This is true as $F(x_1) = 0$, F is irreducible on $K[x_2, t]$ (since r, s are coprime) and by Gauss' Lemma, F is irreducible on $K(x_2)[t]$. Hence $\deg(\phi) = \deg(\xi) = [K(x_1) : K(x_2)] = \deg(F) = \max(\deg(r), \deg(s))$. $\square$

Lemma 5.5. $\deg[2] = 4$.

Proof. Assume char $K \neq 2, 3$, so $E : y^2 = x^3+ax+b = f(x)$. If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2-2xf(x)}{4f(x)}$. The numerator and denominator are coprime, since otherwise $\exists \theta \in \overline{K}$ with $f(\theta) = f'(\theta) = 0$, meaning f has a multiple root, contradiction. We are now done by Lemma 5.4, since $\deg[2] = \max(3, 4) = 4$. □

Definition 5.3. Let A be an abelian group. Then a map $q : A \to \mathbb{Z}$ is a quadratic form if

(i) $q(nx) = n^2q(x)\ \forall n \in \mathbb{Z}, q \in A$.

(ii) $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$–bilinear.

Lemma 5.6. $q : A \to \mathbb{Z}$ is a quadratic form if and only if it satisfies the parallelogram law $q(x + y) + q(x - y) = 2q(x) + 2q(y)\ \forall x, y \in A$.

Proof. ($\implies$). Let $\langle x, y\rangle = q(x+y)-q(x)-q(y)$. Then $\langle x, x\rangle = q(2x)-2q(x) = 2q(x)$ by (i) with $n = 2$. By (ii), $\langle x+y, x+y\rangle + \langle x-y, x-y\rangle = 2\langle x, x\rangle + 2\langle y, y\rangle$, which implies $q(x + y) + q(x - y) = 2q(x) + 2q(y)$.

($\impliedby$). This is on Ex. Sheet 2. □

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Theorem 5.7. $\deg : \mathrm{Hom}(E_1, E_2) \to \mathbb{Z}$ is a quadratic form (with $\deg(0) = 0$).

Proof. Assume char $K \neq 2, 3$ and write $E_2 = y^2 = x^3 + ax + b$. Let $P, Q \in E_2$ with $P, Q, P + Q, P - Q$ all nonzero and let x_1, x_2, x_3, x_4 be the x–coordinates of these points.

Lemma 5.8. There exist polynomials $W_0, W_1, W_2 \in \mathbb{Z}[a, b][x_1, x_2]$ of degree ≤ 2 in x_1 and of degree ≤ 2 in x_2 such that

$$(1 : x_3 + x_4 : x_3x_4) = (W_0 : W_1 : W_2)$$

Proof. Method 1: Direct calculation (results on the formula sheet) gives the result (e.g. $W_0 = (x_1 - x_2)^2$).

Method 2: Let $y = \lambda x + \nu$ be the line through P and Q. Substituting, we get $x^3 + ax + b - (\lambda x + \nu)^2 = (x - x_1)(x - x_2)(x - x_3) = x^3 - s_1x^2 + s_2x - s_3$ where s_i is the i^{th} symmetric polynomial in x_1, x_2, x_3. Comparing coefficients gives $\lambda^2 = s_1, -2\lambda\nu = s_2 - a, \nu^2 = s_3 + b$. Eliminating λ and ν gives

$$F(x_1, x_2, x_3) = (s_2 - a)^2 - 4s_1(s_3 + b) = 0,$$

where F has degree at most 2 in each x_i. Hence x_3 is a root of the quadratic $W(t) = F(x_1, x_2, t)$. Repeating this for the line through P and $-Q$ shows that

x_4 is the other root of $W(t)$. Therefore

$$\begin{aligned}
&W(t) = W_0(t - x_3)(t - x_4) = W_0 t^2 - W_1 t + W_2 \\
&\implies (1 : x_3 + x_4 : x_3 x_4) = (W_0 : W_1 : W_2).
\end{aligned}$$

□

We now show that if $\phi, \psi \in \mathrm{Hom}(E_1, E_2)$, then $\deg(\phi + \psi) + \deg(\phi - \psi) \leq 2\deg(\phi) + 2\deg(\psi)$. We may assume that $\phi, \psi, \phi + \psi, \phi - \psi$ are not the zero maps (otherwise we're done trivially, or use $\deg[-1] = 1,\ \deg[2] = 4$). Now

$$\begin{aligned}
\phi &: (x, y) \mapsto (\xi_1(x), \ldots) \\
\psi &: (x, y) \mapsto (\xi_2(x), \ldots) \\
\phi + \psi &: (x, y) \mapsto (\xi_3(x), \ldots) \\
\phi - \psi &: (x, y) \mapsto (\xi_4(x), \ldots).
\end{aligned}$$

Lemma 5.8 implies $(1 : \xi_3 + \xi_4 : \xi_3 \xi_4) = ((\xi_1 - \xi_2)^2 : \ldots)$. Say $\xi_i = \frac{r_i}{s_i}$ for $r_i, s_i \in K[t]$ coprime. This gives

$$(s_3 s_4 : r_3 s_4 + r_4 s_3 : r_3 r_4) \overset{(\star)}{=} ((r_1 s_2 - r_2 s_1)^2 : \ldots)$$

where every term is quadratic in r_3, r_4, s_3 and s_4. Hence (as the terms on the LHS of $(\star)$ are coprime)

$$\begin{aligned}
\deg(\phi + \psi) + \deg(\phi - \psi) &= \max(\deg(r_3), \deg(s_3)) + \max(\deg(r_4), \deg(s_4)) \\
&= \max(\deg(s_3 s_4), \deg(r_3 s_4 + r_4 s_3), \deg(r_3 r_4)) \\
&\leq 2\max(\deg(r_1), \deg(s_1)) + 2\max(\deg(r_2), \deg(s_2)) \\
&= 2\deg(\phi) + 2\deg(\psi).
\end{aligned}$$

Now replace ϕ and ψ by $\phi + \psi$ and $\phi - \psi$ and use $\deg[2] = 4$ to get

$$4\deg(\phi) + 4\deg(\psi) = \deg(2\phi) + \deg(2\psi) \leq 2\deg(\phi + \psi) + 2\deg(\phi - \psi).$$

This gives the parallelogram law, so deg is a quadratic form. □

Corollary 5.9. $\deg(n\phi) = n^2 \deg(\phi)$. In particular, $\deg[n] = n^2$.

Example 5.10. Let E/K be an elliptic curve. Suppose char $K \neq 2$ and $0 \neq T \in E(K)[2]$. WLOG let $E : y^2 = x(x^2 + ax + b)$ for $a, b \in K, b(a^2 - 4b) \neq 0$ (by moving a root to zero) and WLOG $T = (0, 0)$.

If $P = (x, y)$ and $P' = P + T = (x', y')$, then

$$x' = \left(\frac{y}{x}\right)^2 - a - x = \frac{x^2 + ax + b}{x} - a - x = \frac{b}{x}$$
$$y' = -\left(\frac{y}{x}\right) x' = -\frac{by}{x^2}.$$

We let $\xi = x + x' + a = \left(\frac{y}{x}\right)^2, \eta = y + y' = \frac{y}{x}\left(x - \frac{b}{x}\right)$. Then

$$\eta^2 = \left(\frac{y}{x}\right)^2 \left(\left(x + \frac{b}{x}\right)^2 - 4b\right) = \xi((\xi - a)^2 - 4b) = \xi(\xi^2 - 2a\xi + a^2 - 4b).$$

Let $E' : y^2 = x(x^2 + a'x + b')$ with $a' = -2a, b' = a^2 - 4b$. There is an isogeny $\phi : E \to E'$ given by $(x, y) \mapsto \left(\left(\frac{y}{x}\right)^2 : \frac{y(x^2-b)}{x^2} : 1\right)$.

Sanity check/finding where 0_E maps to: x is a double pole, y is a triple pole, so $\left(\frac{y}{x}\right)^2$ is a double pole and $\frac{y(x^2-b)}{x^2}$ is a triple pole (and the last coordinate 1 has degree 0). Multiplying through by a cube of a uniformizer, the degrees go from $(-2, -3, 0)$ to $(1, 0, 3)$, so $0_E \mapsto (0 : 1 : 0)$.

To compute $\deg(\phi)$, $\left(\frac{y}{x}\right)^2 = \frac{x^2+ax+b}{x}$ with the numerator and denominator coprime as $b \neq 0$, so by Lemma 5.4, $\deg(\phi) = 2$. We say ϕ is a **2–isogeny**.

6 The invariant differential

For C some algebraic curve over $K = \overline{K}$.

Definition 6.1. The space of differentials Ω_C (sometimes called one–forms) is the $K(C)$–vector space generated by $\mathrm{d}f$ for all $f \in K(C)$ subject to the relations

(i) $\mathrm{d}(f + g) = \mathrm{d}f + \mathrm{d}g$.

(ii) $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$.

(iii) $\mathrm{d}a = 0\ \forall a \in K$.

Fact. Ω_C is a 1–dimensional $K(C)$–vector space. 05 Feb 2024, Lecture 8

Let $0 \neq \omega \in \Omega_C$, let $P \in C$ be a smooth point and let $t \in K(C)$ be a uniformizer at P. Then $\omega = f\mathrm{d}t$ for some $f \in K(C)^\times$. We define $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$, which is independent of the choice of t.

Fact. Suppose $f \in K(C)^\times$ with $\mathrm{ord}_P(f) = n \neq 0$. If $\mathrm{char}\, K \nmid n$, then $\mathrm{ord}_P(\mathrm{d}f) = n - 1$.

We assume that C is a smooth projective curve.

Definition 6.2. We define $\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega) P \in \mathrm{Div}(C)$. Here we use the fact that $\mathrm{ord}_P(\omega) = 0$ for all but finitely many $P \in C$.

Definition 6.3. A differential $\omega \in \Omega_C$ is regular if $\operatorname{div}(\omega) \geq 0$. We define the genus $g(C)$ of C to be

$$g(C) = \dim_K \{\omega \in \Omega_C \mid \operatorname{div}(\omega) \geq 0\},$$

where the set on the RHS is the set of regular differentials.

As a consequence of Riemann–Roch, we have that if $0 \neq \omega \in \Omega_C$, then $\deg(\operatorname{div}(\omega)) = 2g(C) - 2$.

Lemma 6.1. Assume char $K \neq 2$ and let $E : y^2 = (x - e_1)(x - e_2)(x - e_3)$ for e_1, e_2, e_3 distinct. Then $\omega = \frac{\mathrm{d}x}{y}$ is a differential on E with no zeroes or poles, which implies $g(E) = 1$. In particular, the K–vector space of regular differentials on E is 1–dimensional, spanned by ω.

Proof. Let $T_i = (e_i, 0)$. Then $E[2] = \{0, T_1, T_2, T_3\}$ and $\operatorname{div}(y) \overset{(\dagger)}{=} (T_1) + (T_2) + (T_3) - 3(0)$. For $0 \neq P \in E$, $\operatorname{div}(x - x_P) = (P) + (-P) - 2(0)$.

- If $P \in E \setminus E[2]$, then $\operatorname{ord}(x - x_P) = 1 \implies \operatorname{ord}_P(\mathrm{d}x) = 0$.
- If $P = T_i$, then $\operatorname{ord}_P(x - x_P) = 2 \implies \operatorname{ord}_P(\mathrm{d}x) = 1$.
- If $P = 0$, then $\operatorname{ord}_P(x) = -2 \implies \operatorname{ord}_P(\mathrm{d}x) = -3$.

Hence $\operatorname{div}(\mathrm{d}x) = (T_1) + (T_2) + (T_3) - 3(0)$, which with $(\dagger)$ gives $\operatorname{div}\left(\frac{\mathrm{d}x}{y}\right) = 0$. □

Definition 6.4. For $\phi : C_1 \to C_2$ a nonconstant morphism, we define

$$\begin{aligned} \phi^* : \Omega_{C_2} &\to \Omega_{C_1} \\ f\mathrm{d}g &\mapsto \phi^* f d(\phi^* g). \end{aligned}$$

Lemma 6.2. Let $P \in E$, $\tau_P : E \to E$ by $X \mapsto X + P$ and $\omega = \frac{\mathrm{d}x}{y}$ as above. Then $\tau_P^* \omega = \omega$. We say ω is the **invariant differential**.

Proof. $\tau_P^* \omega$ is a regular differential on E, so $\tau_P^* \omega = \lambda_P \omega$ for some $\lambda_P \in K^\times$. The map $E \to \mathbb{P}^1$ by $P \mapsto \lambda_P$ is a morphism of smooth projectives curves, but it is not surjective (as it misses 0 and ∞). Hence it is constant by Theorem 2.7, i.e. $\exists \lambda \in K^\times$ such that $\tau_P^* \omega = \lambda \omega \ \forall P \in E$. Taking $P = 0$ shows $\lambda = 1$. □

Remark. If $K = \mathbb{C}$ and $\mathbb{C}/\Lambda \cong E(\mathbb{C})$ by $z \mapsto (\wp(z), \wp'(z)) := (x, y)$, then $\frac{\mathrm{d}x}{y} = \frac{\wp'(z)\mathrm{d}z}{\wp'(z)} = \mathrm{d}z$, which is invariant under $z \mapsto z + \text{const}$.

Lemma 6.3. Let $\phi, \psi \in \operatorname{Hom}(E_1, E_2)$. Let ω be the invariant differential on E_2. Then $(\phi + \psi)^* \omega = \phi^* \omega + \psi^* \omega$.

Proof. Write E for E_2. We have the maps

$$\begin{aligned} E \times E &\to E \\ \mu : (P, Q) &\mapsto P + Q \\ \mathrm{pr}_1 : (P, Q) &\mapsto P \\ \mathrm{pr}_2 : (P, Q) &\mapsto Q. \end{aligned}$$

Fact. $\Omega_{E \times E}$ is a 2–dimensional $K(E \times E)$–vector space with basis $\mathrm{pr}_1^* \omega$ and $\mathrm{pr}_2^* \omega$. Consequently, $\mu^* \omega \overset{(\dagger)}{=} f \mathrm{pr}_1^* \omega + g \mathrm{pr}_2^* \omega$ for some $f, g \in K(E \times E)$.

For fixed $Q \in E$, let $i_Q : E \to E \times E$ by $P \mapsto (P, Q)$. Applying i_Q^* to $(\dagger)$ gives

$$\begin{aligned} \underbrace{(\mu \circ i_Q)}_{\tau_Q}{}^* \omega &= (i_Q^* f)(\underbrace{\mathrm{pr}_1 \circ i_Q}_{\text{identity map}})^* \omega + (i_Q^* g)(\underbrace{\mathrm{pr}_2 \circ i_Q}_{\text{constant map}})^* \omega \\ \implies \tau_Q^* \omega &= (i_Q^* f)\omega + 0. \end{aligned}$$

As $\tau_Q^* \omega = \omega$ by the previous lemma, we conclude $i_Q^* f = 1 \ \forall q \in E$, so $f(P, Q) = 1 \ \forall P, Q \in E$. Similarly $g(P, Q) = 1 \ \forall P, Q \in E$, so $(\dagger)$ gives $\mu^* \omega = \mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega$. Now pull back using

$$\begin{aligned} E_1 &\to E \times E \\ P &\mapsto (\phi(P), \psi(P)) \end{aligned}$$

to get $(\phi + \psi)^* \omega = \phi^* \omega + \psi^* \omega$. $\square$

Lemma 6.4. Let $\phi : C_1 \to C_2$ be a nonconstant morphism. Then ϕ is separable if and only if $\phi^* : \Omega_{C_2} \to \Omega_{C_1}$ is nonzero.

Proof. Omitted. $\square$

Example 6.5. Let $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ be the multiplicative group. For $n \geq 2$ an integer, consider $\phi : \mathbb{G}_m \to \mathbb{G}_m$ by $x \mapsto x^n$. Then $\phi^*(\mathrm{d}x) = \mathrm{d}(x^n) = n x^{n-1} \mathrm{d}x$. So if char $K \nmid n$, then ϕ is separable, so $\left|\phi^{-1}(Q)\right| = \deg \phi$ for all but at most finitely many $Q \in \mathbb{G}_m$.

But ϕ is a group homomorphism, so $\left|\phi^{-1}(Q)\right| = |\ker(Q)| \ \forall Q \in \mathbb{G}_m$. Hence $|\ker Q| = \deg \phi = n$. This shows that $K = \overline{K}$ contains exactly n distinct n^{th} roots of unity.

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Theorem 6.6. If char $K \nmid n$, then $E[n] = (\mathbb{Z}/n\mathbb{Z})^2$.

Proof. Lemma 6.3 and induction imply $[n]^* \omega = n \omega$ where char $K \nmid n$, so $[n]$ is separable by Lemma 6.4. Hence $\left|[n]^{-1}(Q)\right| = \deg[n]$ for all but finitely many points $Q \in E$. But $[n]$ is a group homomorphism, so $\left|[n]^{-1}Q\right| = |E[n]|\ \forall Q \in E$. We conclude that $|E[n]| = \deg[n] = n^2$ by Corollary 5.9.

By classification of finite abelian groups, $E[n] \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times d_t\mathbb{Z}/\mathbb{Z}$ with $d_1 \mid d_2 \mid \ldots \mid d_t$, but $d_t \mid n$, and if p is a prime with $p \mid d_1$, then $E[p] \cong (\mathbb{Z}/p\mathbb{Z})^t$, so $|E[p]| = p^2$, so $t = 2$. Hence $d_1 \mid d_2 \mid n$ with $d_1 d_2 = n^2$, so $d_1 = d_2 = n$ and so $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$. $\square$

Remark. If char $K = p$, then $[p]$ is inseparable. It can be shown that either $E[p^r] \cong \mathbb{Z}/p^r\mathbb{Z}\ \forall r \geq 1$ or $E[p] = 0$ (the "ordinary" case and the "supersingular" case).

Remark about the remark. Do not use this remark to trivialize a question on Ex. Sheet 2.

7 Elliptic curves over finite fields

Lemma 7.1. Let A be an abelian group. Let $q : A \to \mathbb{Z}$ be a positive definite quadratic form. Then

$$\underbrace{|q(x+y) - q(x) - q(y)|}_{\langle x,y\rangle} \leq 2\sqrt{q(x)q(y)}.$$

Proof. We may assume $x \neq 0$, otherwise the result is clear. Hence $q(x) \neq 0$. Let $m, n \in \mathbb{Z}$, then

$$\begin{aligned} 0 \leq q(mx+ny) &= \frac{1}{2}\langle mx+ny, mx+ny\rangle \\ &= m^2 q(x) + mn\langle x,y\rangle + n^2 q(y) \\ &= q(x)\left(m + \frac{\langle x,y\rangle}{2q(x)}n\right)^2 + \left(q(y) - \frac{\langle x,y\rangle^2}{4q(x)}\right)n^2. \end{aligned}$$

Get rid of the first term by taking $m = -\langle x,y\rangle$ and $n = 2q(x)$ to deduce $\langle x,y\rangle^2 \leq 4q(x)q(y)$, so the result follows. $\square$

Theorem 7.2 (Hasse)**.** Let $E/\mathbb{F}_q$ be an elliptic curve. Then

$$|\#E(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}.$$

Proof. Recall $\mathrm{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)$ is cyclic of order r, generated by the Frobenius map $x \mapsto x^q$. Let E have Weierstrass equation with coefficients $a_1, \ldots, a_6 \in \mathbb{F}_q$ (and note that $a_i^q = a_i\ \forall i$).

Define the Frobenius endomorphism $\phi : E \to E$ by $(x,y) \mapsto (x^q, y^q)$, which is an isogeny of degree q. Then $E(\mathbb{F}_q) = \{P \in E \mid \phi(P) = P\} = \ker(1-\phi)$. We have

$$\phi^*\omega = \phi^*\left(\frac{\mathrm{d}x}{y}\right) = \frac{d(x^q)}{y^q} = \frac{qx^{q-1}\mathrm{d}x}{y^q} = 0$$

as $q = p^n$, so $p \mid q$. By Lemma 6.3,

$$(1-\phi)^*\omega = \omega - \phi^*\omega = \omega \neq 0,$$

so $1-\phi$ is separable. By Theorem 2.7 and the fact that $1-\phi$ is a group homomorphism, we argue in the proof of Theorem 6.6 that

$$\underbrace{|\ker(1-\phi)|}_{|E(\mathbb{F}_q)|} = \deg(1-\phi).$$

The map $\deg : \mathrm{Hom}(E,E) \to \mathbb{Z}$ is a positive definite quadratic form by Theorem 5.7. Hence by Lemma 7.1,

$$\begin{aligned} &|\deg(1-\phi) - 1 - \deg\phi| \leq 2\sqrt{\deg\phi} \\ \implies &|\#E(\mathbb{F}_q) - q - 1| \leq 2\sqrt{q}. \end{aligned}$$

□

Definition 7.1. For $\phi, \psi \in \mathrm{End}(E) = \mathrm{Hom}(E,E)$, we put $\langle \phi, \psi \rangle = \deg(\phi + \psi) - \deg(\phi) - \deg(\psi)$ and $\mathrm{tr}(\phi) = \langle \phi, 1 \rangle$.

Corollary 7.3. Let $E/\mathbb{F}_q$ be an elliptic curve and let $\phi \in \mathrm{End}(E)$ be the q^{th} power Frobenius map. Then $\#E(\mathbb{F}_q) = q + 1 - \mathrm{tr}(\phi)$ and $|\mathrm{tr}(\phi)| \leq 2\sqrt{q}$.

Zeta functions. For K a number field,

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{(N(\mathfrak{a}))^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K, \mathfrak{p} \text{ prime}} \left(1 - \frac{1}{(N(\mathfrak{p}))^s}\right)^{-1}.$$

For K a function field, i.e. $K = \mathbb{F}_q(C)$ where C is a smooth projective curve,

$$\zeta_K(s) = \prod_{x \in |C|} \left(1 - \frac{1}{(Nx))^2}\right)^{-1},$$

where $|C| = \{\text{closed points of } C\} = \{\text{orbits for the action of } \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \text{ on } C(\overline{\mathbb{F}_q})\}$ and $Nx = q^{\deg x}$, where $\deg x$ is the size of the corresponding orbit (these definitions are borrowed from scheme theory). We have $\zeta_K(s) = F(q^{-s})$ for some $F \in \mathbb{Q}[[T]]$. We have

$$\begin{aligned}
F(T) &= \prod_{x \in |C|} \left(1 - T^{\deg x}\right)^{-1} \\
\implies \log F(T) &= \sum_{x \in |C|} \sum_{m=1}^{\infty} \frac{1}{m} T^{m \deg x} \\
\implies T \frac{\mathrm{d}}{\mathrm{d}T} \log F(T) &= \sum_{x \in |C|} \sum_{m=1}^{\infty} \deg x T^{m \deg x} \\
&\overset{n = m \deg x}{=} \sum_{n=1}^{\infty} \left(\sum_{x \in |C|, \deg x \mid n} \deg x \right) T^n \\
&= \sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n}) T^n \\
\implies F(T) &= \exp\left(\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} T^n \right).
\end{aligned}$$

Definition 7.2. The zeta function of a smooth projective curve $C/\mathbb{F}_q$ is

$$Z_C(T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#C(\mathbb{F}_{q^n})}{n} T^n \right).$$