Part III - Modular Forms
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1 Introduction

Definition 1.1. We define the following groups:

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$$
$$GL_2(\mathbb{R})^+ = \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\}$$
$$\Gamma(1) = SL_2(\mathbb{Z}) = \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on $\mathfrak{H}$ by Möbius transformations.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{H}$. Then

$$\mathrm{Im}(g\tau) = \frac{1}{2i}\left(\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = \frac{1}{2i}\frac{(ad-bc)(\tau-\bar{\tau})}{|c\tau+d|^2} = \frac{\det(g)\mathrm{Im}(\tau)}{|c\tau+d|^2} > 0,$$

so $g\tau \in \mathfrak{H}$. This action is transitive since

$$x + iy \in \mathfrak{H} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in $\mathfrak{H}$ is conjugate to i. $\qquad\square$

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{H}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{H} \to \mathbb{C}$, then

$$f|_k[g] : \mathfrak{H} \to \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight k action of g on f**.

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

Proof. We compute

$$(f|_k[g])|_k[h](\tau) = \det(h)^{k-1} f|_k[g](h\tau) j(h,\tau)^{-k} =$$
$$\det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g,h\tau)^{-k} j(h,\tau)^{-k} \stackrel{?}{=}$$
$$\det(gh)^{k-1} f(gh\tau) j(gh,\tau)^{-k} = f|_k[gh](\tau).$$

Hence we need to check that $j(gh,\tau) = j(gh,\tau) j(h,\tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g \begin{pmatrix} \tau & 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g,\tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh,\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh \begin{pmatrix} \tau 1 \end{pmatrix} = g \left(j(h,\tau) \begin{pmatrix} h\tau \\ 1 \end{pmatrix} \right) = j(h,\tau) j(g,h\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. □

Formulae. For $g \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{H}$, we have

$$\mathrm{Im}(g\tau) = \det(g) \frac{\mathrm{Im}(\tau)}{|j(g,\tau)|^2} \text{ and } j(g,\tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\gamma \le \Gamma(1)$ of finite index[1]. A **weakly modular function of weight k and level Γ** is a meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ which is invariant under the weight k action of Γ, i.e. such that

$$\forall \tau \in \mathfrak{H}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in $\mathfrak{H}$ and at ∞.

It is a fact that modular forms of fixed weight and level live in finite-dimensional $\mathbb{C}$-vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

(1) They are related to the theory of elliptic functions. Let $E/\mathbb{C}$ be an elliptic curve and ω a holomorphic non–zero 1–form. Then there exists a unique lattice[2] $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \to E$ such that $\phi^*(\omega) = dz$. Then

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[1]In other words, γ is a (finite index) subgroup of $\Gamma(1)$.
[2]i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over $\mathbb{R}$.

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{H}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{H}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over $\mathbb{C}$.

(2) Modular forms f have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n.

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q–expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$.

(3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to $\mathbb{C}$ and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L–function. Modular forms can be used to construct interesting examples of L–functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

(4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves $E/\mathbb{C}$ up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L–functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis".