

Part III - Elliptic Curves

Lectured by Tom Fisher

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0 Introduction

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The best books for the course include *The arithmetic of elliptic curves* by Silverman, Springer 1996, and *Lectures on elliptic curves* by Cassels, CUP 1991.

Lecture 1

1 Fermat's Method of Infinite Descent

A right-angled triangle Δ has $a^2 + b^2 = c^2$ and $\text{area}(\Delta) = \frac{1}{2}ab$.

Definition 1.1. Δ is **rational** if $a, b, c \in \mathbb{Q}$. Δ is **primitive** if $a, b, c \in \mathbb{Z}$ are coprime.

Note that a primitive triangle has pairwise coprime side lengths because $a^2 + b^2 = c^2$.

Lemma 1.1. Every primitive triangle is of the form $(u^2 - v^2, 2uv, u^2 + v^2)$ for some integers $u > v > 0$.

Proof. WLOG let a, b, c be odd, even, odd. Then $(\frac{b}{2})^2 = \frac{c+a}{2} \frac{c-a}{2}$, where we note that the RHS is a product of positive coprime integers. By unique factorization, $\frac{c+a}{2} = u^2, \frac{c-a}{2} = v^2$ for $u, v \in \mathbb{Z}$. This gives the desired result. \square

Definition 1.2. $D \in \mathbb{Q}_{>0}$ is a **congruent** number if there exists a rational triangle Δ with $\text{area}(\Delta) = D$.

Note that it suffices to consider $D \in \mathbb{Z}_{>0}$ squarefree.

Example 1.2. $D = 5, 6$ are congruent.

Lemma 1.3. $D \in \mathbb{Q}_{>0}$ is congruent $\iff Dy^2 = x^3 - x$ for some $x, y \in \mathbb{Q}, y \neq 0$.

Proof. Lemma 1.1 shows that D congruent $\implies Dw^2 = uv(u^2 - v^2)$ for some $u, v, w \in \mathbb{Q}, w \neq 0$. This implication also obviously goes the other way. To finish, divide through by w^4 and take $x = \frac{u}{v}, y = \frac{w}{v^2}$. \square

Fermat showed that 1 is not a congruent number.

Theorem 1.4. There is no solution to $w^2 = uv(u + v)(u - v)$ for $u, v, w \in \mathbb{Z}, w \neq 0$.

Proof. WLOG assume u, v are coprime and that $u, w > 0$. If $v < 0$, then replace (u, v, w) by $(-v, u, w)$. If u, v are both odd, then replace (u, v, w) by $(\frac{u+v}{2}, \frac{u-v}{2}, \frac{w}{2})$. Then $u, v, u+v, u-v$ are pairwise coprime positive integers with their product a square, so by unique factorization in \mathbb{Z} , $u = a^2, v = b^2, u + v = c^2, u - v = d^2$ for $a, b, c, d \in \mathbb{Z}$.

Since $u \not\equiv v \pmod{2}$, both c and d are odd. Then $(\frac{c+d}{2})^2 + (\frac{c-d}{2})^2 = \frac{c^2+d^2}{2} = u = a^2$. This gives a primitive triangle with area $\frac{c^2-d^2}{8} = \frac{v}{4} = (\frac{b^2}{2})$.

Let $w_1 = \frac{b}{2}$, then by Lemma 1.1, $w_1^2 = u_1 v_1 (u_1 + v_1)(u_1 - v_1)$ for some $u_1, v_1 \in \mathbb{Z}$. Hence we have a new solution to our original question, with $4w_1^2 = b^2 = v \mid w^2 \implies w_1 \leq \frac{w}{2}$, so we're done by infinite descent. \square

A variant for polynomials. In the above, K is a field with $\text{char } K \neq 2$. Let \overline{K} be the algebraic closure of K and consider for this whole section K with $\text{char } K \neq 2$.

Lemma 1.5. Let $u, v \in K[t]$ be coprime. If $\alpha u + \beta v$ is a square for 4 distinct $(\alpha : \beta) \in \mathbb{P}^1$, then $u, v \in K$.

Proof. WLOG let $K = \overline{K}$ by extending if necessary. Changing coordinates on \mathbb{P}^1 (i.e. multiplying by a 2×2 invertible matrix), we may assume that the points $(\alpha : \beta)$ are $(1 : 0)$, $(0 : 1)$, $(1 : -1)$, $(1 : -\lambda)$ for $\lambda \in K \setminus \{0, 1\}$. Since our field is algebraically closed, let $\mu = \sqrt{\lambda}$. Then $u = a^2, v = b^2, u - v = (a + b)(a - b), u - \lambda v = (a + \mu b)(a - \mu b)$.

Unique factorization in $K[t]$ implies that $a + b, a - b, a + \mu b, a - \mu b$ are squares (since the necessary terms are coprime up to units, i.e. constants). But $\max(\deg(a), \deg(b)) \leq \frac{1}{2} \max(\deg(u), \deg(v))$, so by Fermat's method of infinite descent, $u, v \in K$. \square

Definition 1.3. (i) An **elliptic curve** E/K is the projective closure of the plane affine curve $y^2 = f(x)$ (this is called a Weierstrass equation) where $f \in K[x]$ is a monic cubic polynomial with distinct roots in \overline{K} .

(ii) For L/K any field extension, $E(L) = \{(x, y) \in L^2 \mid y^2 = f(x)\} \cup \{0\}$ (the point at infinity in the projective closure), it turns out that $E(L)$ is naturally an abelian group.

In this course, we study $E(K)$ for K a finite field, local field, number field.

Lemma 1.3 and Theorem 1.4 show that if $E : y^2 = x^3 - x$, then $E(\mathbb{Q}) = \{0, (0, 0), (\pm 1, 0)\}$.

Corollary 1.6. Let E/K be an elliptic curve. Then $E(K(t)) = E(K)$.

Proof. WLOG $K = \overline{K}$. By a change of coordinates, we may assume $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K \setminus \{0, 1\}$. Suppose $(x, y) \in E(K(t))$. Write $x = \frac{u}{v}$ for $u, v \in K(t)$ coprime. Then $w^2 = uv(u-v)(u-\lambda v)$ for some $w \in K[t]$. Unique factorization in $K[t]$ shows that $u, v, u-v, u-\lambda v$ are all squares, so by Lemma 1.5, $u, v \in K$, so $x, y \in K$. \square

2 Some remarks on algebraic curves

In this section, work over an algebraically closed field $K = \overline{K}$.

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Definition 2.1. A plane curve $C = \{f(x, y) = 0\} \subset \mathbb{A}^2$ (for $f \in K[x, y]$ irreducible) is **rational** if it has a rational parametrization, i.e. $\exists \phi, \psi \in K(t)$ such that

- (i) The map $\mathbb{A}^1 \rightarrow \mathbb{A}^2$ by $t \mapsto (\phi(t), \psi(t))$ is injective on $\mathbb{A}^1 \setminus \{\text{finite set}\}$.
- (ii) $f(\phi(t), \psi(t)) = 0$ in $K(t)$.

Example 2.1. (a) Any nonsingular conic is rational. For example, for $x^2 + y^2 = 1$, take a line with slope t through $(-1, 0)$ (the anchor) and solve to get the rational parametrization $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

(b) Any singular plane cubic is rational, for example $y^2 = x^3$ giving $(x, y) = (t^2, t^3)$ with the anchor at the singularity $(0, 0)$ and $y^2 = x^2(x+1)$ with the parametrization to be computed on Ex. Sheet 1 (anchor still at $(0, 0)$).

(c) Corollary 1.6 shows that elliptic curves are not rational.

Remark. The genus $g(C) \in \mathbb{Z}_{\geq 0}$ is an invariant of a smooth projective curve C . If $K = \mathbb{C}$, then $g(C)$ is the genus of the Riemann surface. A smooth plane curve $C \subset \mathbb{P}^2$ of degree d has genus $g(C) = \frac{(d-1)(d-2)}{2}$.

Proposition 2.2. (Here we still assume $K = \overline{K}$). Let C be a smooth projective curve.

- C is rational (see Definition 2.1) $\iff g(C) = 0$.
- C is an elliptic curve $\iff g(C) = 1$.

Proof. (i) Omitted.

(ii) (\implies): Check C is a smooth plane curve in \mathbb{P}^2 (see Ex. Sheet 1) and use the above remark.

(\impliedby): We will see this later.

□

Order of vanishing. Let C be an algebraic curve with function field $K(C)$ and let $P \in C$ be a smooth point. Write $\text{ord}_P(f)$ for the order of vanishing of $f \in K(C)$ at P (which is negative if f has a pole at P).

Fact. $\text{ord}_P : K(C)^\times \rightarrow \mathbb{Z}$ is a discrete valuation, i.e. $\text{ord}_P(f_1 f_2) = \text{ord}_P(f_1) + \text{ord}_P(f_2)$ and $\text{ord}_P(f_1 + f_2) \geq \min(\text{ord}_P(f_1), \text{ord}_P(f_2))$.

Definition 2.2. We say $t \in K(C)^\times$ is a **uniformizer** at P if $\text{ord}_P(t) = 1$.

Example 2.3. $C = \{g = 0\} \subset \mathbb{A}^2$ for $g \in K[x, y]$. Then $K(C) = \text{Frac} \left(\frac{K[x, y]}{(g)} \right)$. Write $g = g_0 + g_1(x, y) + g_2(x, y) + \dots$ for g_i homogeneous of degree i . Suppose $P = (0, 0)$ is a smooth point, e.g. $g_0 = 0$ and let $g_1(x, y) = \alpha x + \beta y$ with α, β not both zero ($\alpha x + \beta y = 0$ gives a tangent to the curve at P). Let $\gamma, \delta \in K$ and consider also the line $\gamma x + \delta y$ through P . Then it is a fact that $\gamma x + \delta y \in K(C)$ is a uniformizer at P if and only if $\alpha\delta - \beta\gamma \neq 0$.

Example 2.4. Consider $\{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{A}^2$ for $\lambda \neq 0, 1$ and consider its projective closure by taking $x = \frac{X}{Z}, y = \frac{Y}{Z}$ to get $\{Y^2Z = X(X-Z)(X-\lambda Z)\} \subset \mathbb{P}^2$. This has only one point at infinity, $P = (0 : 1 : 0)$. Our aim is to compute $\text{ord}_P(x)$ and $\text{ord}_P(y)$.

For this, put $t = \frac{X}{Y}, w = \frac{Z}{Y}$, so $w \stackrel{(\dagger)}{=} t(t-w)(t-\lambda w)$. Now P is the point $(t, w) = (0, 0)$, which is a smooth point with $\text{ord}_P(t) = \text{ord}_P(t-w) = \text{ord}_P(t-\lambda w) = 1$, so (\dagger) gives $\text{ord}_P(w) = 3$. We now find

$$\begin{aligned} \text{ord}_P(x) &= \text{ord}_P \left(\frac{X}{Z} \right) = \text{ord}_P \left(\frac{t}{w} \right) = 1 - 3 = -2 \\ \text{ord}_P(y) &= \text{ord}_P \left(\frac{Y}{Z} \right) = \text{ord}_P \left(\frac{1}{w} \right) = -3. \end{aligned}$$

Riemann–Roch space. Let C be a smooth projective curve.

Definition 2.3. A **divisor** is a formal sum of points on C , say $D = \sum_{P \in C} n_P P$ where $n_P \in \mathbb{Z}$ and $n_P = 0$ for all but finitely many $P \in C$. We say $\deg D = \sum_{P \in C} n_P$.

D is **effective** (written $D \geq 0$) if $n_P \geq 0 \ \forall P \in C$. If $f \in K(C)^\times$, then $\text{div}(f) = \sum_{P \in C} \text{ord}_P(f) P$. The Riemann–Roch space of $D \in \text{Div}(C)$ is

$$\mathcal{L}(D) = \{f \in K(C)^\times \mid \text{div}(f) + D \geq 0\} \cup \{0\},$$

i.e. the K -vector space of rational functions on C with "poles no worse than specified by D ".

We quote Riemann–Roch for surfaces of genus 1: We have

$$\dim \mathcal{L}(D) = \begin{cases} \deg D & \text{if } \deg D > 0 \\ 0 \text{ or } 1 & \text{if } \deg D = 0 \\ 0 & \text{if } \deg D < 0. \end{cases}$$

Example 2.5. We revisit Example 2.4. We have $\mathcal{L}(2P) = \langle 1, x \rangle$ and $\mathcal{L}(3P) = \langle 1, x, y \rangle$.

We still have $\text{char } K \neq 2$ and $\overline{K} = K$.

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Proposition 2.6. Let $C \subset \mathbb{P}^2$ be a smooth plane cubic and let $P \in C$ be a point of inflection. Then we may change coordinates such that $C : Y^2Z = X(X - Z)(X - \lambda Z)$ and $P = (0 : 1 : 0)$ (for some $\lambda \neq 0, 1$).

Proof. First change coordinates such that $P = (0 : 1 : 0)$. Then change coordinates such that the tangent line becomes $T_P C = \{Z = 0\}$. Say $C = \{F(X, Y, Z) = 0\} \subset \mathbb{P}^2$. A point on the tangent line is of the form $(t : 1 : 0)$ and since $P \in C$ is a point of inflection, we get $F(t, 1, 0) = \text{const} \cdot t^3$, i.e. F has no terms X^2Y, XY^2 or Y^3 .

Hence $F = \langle Y^2Z, XYZ, YZ^2, X^3, X^2Z, XZ^2, Z^3 \rangle$. Notably, Y^2Z has a nonzero coefficient, otherwise $P \in C$ would be singular, a contradiction to C being smooth. The coefficient of X^3 is nonzero as well, otherwise $Z \mid F$. We are free to rescale X, Y, Z, F , so WLOG C is defined by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

Substituting $Y \mapsto Y - \frac{1}{2}a_1X - \frac{1}{2}a_3Z$, we may assume $a_1 = a_3 = 0$. This gives

$$C : Y^2Z = Z^3 f\left(\frac{X}{Z}\right)$$

for a monic cubic polynomial f . Since C is smooth, f has distinct roots, WLOG $0, 1, \lambda$, so $C : Y^2Z = X(X - Z)(X - \lambda Z)$. \square

The form $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ is the Weierstrass form. The form $Y^2Z = X(X - Z)(X - \lambda Z)$ is the Legendre form.

Remark. It can be shown that the points of inflection of a plane curve $C = \{F(X_1, X_2, X_3) = 0\} \subset \mathbb{P}^2$ are given by solving the Hessian:

$$\begin{cases} H = \left(\frac{\partial^2 F}{\partial X_i \partial X_j} \right) = 0 \\ F(X_1, X_2, X_3) = 0. \end{cases}$$

2.1 The degree of a morphism

Let $\phi : C_1 \rightarrow C_2$ be a nonconstant morphism of smooth projective curves. Then $\phi^* : K(C_2) \rightarrow K(C_1)$ by $f \mapsto f \circ \phi$, giving an injective map $\phi^* K(C_2)$ to $K(C_1)$.

Definition 2.4. The **degree** of ϕ is $\deg \phi = [K(C_1) : \phi^* K(C_2)]$.

We say ϕ is **separable** if $K(C_1)/\phi^* K(C_2)$ is a separable field extension.

Suppose $P \in C_1, Q \in C_2$ and $\phi : P \mapsto Q$. Let $t \in K(C_2)$ be a uniformizer at Q .

Definition 2.5. $e_\phi(P) = \text{ord}_P(\phi^* t)$, which is always ≥ 1 and independent of t .

Theorem 2.7. Let $\phi : C_1 \rightarrow C_2$ be a nonconstant morphism of smooth projective curves. Then

$$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi \quad \forall Q \in C_2.$$

Moreover, if ϕ is separable, then $e_\phi(P) = 1$ for all but finitely many $P \in C_1$.

We don't prove this.

In particular, this shows that:

- ϕ is surjective (very important here that we're in \overline{K}).
- $|\phi^{-1}(Q)| \leq \deg \phi$.
- If ϕ is separable, then equality holds in (ii) for all but finitely many points $Q \in C_2$.

Important remark. Let C be an algebraic curve. A rational map is given by

$$\begin{aligned} C &\rightarrow \mathbb{P}^n \\ \phi &\mapsto (f_0, f_1, \dots, f_n) \end{aligned}$$

where $f_0, \dots, f_n \in K(C)$ are not all zero. Then we have a fact: If C is smooth, then ϕ is a morphism. This saves us a lot of time (we can go from a rational map to a morphism immediately).

3 Weierstrass equations

We now drop the assumption that $\overline{K} = K$, but we will still assume that K is perfect.

Definition 3.1. An **elliptic curve** E/K is a smooth projective curve of genus 1 defined over K with a specified K -rational point $O = 0_E$.

Example 3.1. $\{X^3 + pY^3 + p^2Z^3 = 0\} \subset \mathbb{P}^2$ is not an elliptic curve over \mathbb{Q} , since it has no \mathbb{Q} -rational point.

Theorem 3.2. Every elliptic curve E is isomorphic over K to a curve in Weierstrass form via an isomorphism taking 0_E to $(0 : 1 : 0)$.

Remark. Proposition 2.6 treated the special case where E is a smooth plane cubic and 0_E is a point of inflection.

Fact. If $D \in \text{Div}(E)$ is defined over K , then $\mathcal{L}(D)$ has a basis in $K(E)$ (not just in $\overline{K}(E)$). Here D is defined over K if it is fixed by $\text{Gal}(\overline{K}/K)$ (this is unimportant for us and we just write it down to be rigorous).

Proof. $\mathcal{L}(2 \cdot 0_E) \subset \mathcal{L}(3 \cdot 0_E)$. Pick bases $1, x$ and $1, x, y$. Note $\text{ord}_{0_E}(x) = -2$ and $\text{ord}_{0_E}(y) = -3$ (else x, y don't give a basis). The 7 elements $1, x, y, x^2, xy, x^3, y^2$ lie in the 6-dimensional vector space $\mathcal{L}(60_E)$ (as they have at most a sixth order pole), so they must satisfy a linear dependence relation.

Leaving out x^3 or y^2 leaves us with 6 elements, all with different order poles, giving a basis for $\mathcal{L}(60_E)$. Hence the coefficients of x^3 and y^2 are nonzero, so by rescaling x, y (if necessary) we get

$$E' : y^2 + a_1xy + a_2y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_i \in K$. Let E' be the curve defined by this equation (or rather its projective closure). There is a morphism $\phi : E \rightarrow E' \subset \mathbb{P}^2$ by $P \mapsto (x(P) : y(P) : 1) = \left(\frac{x}{y}(P) : 1 : \frac{1}{y}(P)\right)$. (Since E is smooth, we know that this rational map is a morphism). Hence $0_E \mapsto (0 : 1 : 0)$.

We have $E \xrightarrow{x} \mathbb{P}^1$ by $x \mapsto (x : 1)$ (and similarly for y), so

$$\begin{aligned} [K(E) : K(x)] &= \deg(E \xrightarrow{x} \mathbb{P}^1) = \text{ord}_{0_E} \left(\frac{1}{x} \right) = 2 \\ [K(E) : K(y)] &= \deg(E \xrightarrow{y} \mathbb{P}^1) = \text{ord}_{0_E} \left(\frac{1}{y} \right) = 3. \end{aligned}$$

This gives an inclusion of fields $K(x) \leq K(E)$ of degree 2, $K(y) \leq K(E)$ of degree 3, while $K(x), K(y) \leq K(x, y) \leq K(E)$, so tower law gives $[K(E) : K(x, y)] = 1 \implies K(E) = K(x, y) = \phi^* K(E') \implies \deg \phi = 1$. (draw a picture!). This gives us an inverse that is a rational map, which we want to show is a morphism. For this, we just need to show that E' is smooth.

If E' were singular, then E and E' are rational, a contradiction. So E' is smooth and hence ϕ^{-1} is a morphism, so ϕ is an isomorphism. \square

Proposition 3.3. Let E, E' be elliptic curves over K in Weierstrass form. Then $E \cong E'$ over $K \iff$ the equations are related by a change of variables

$$\begin{aligned} x &= u^2x' + r \\ y &= u^3y' + u^2sx' + t \end{aligned}$$

for $r, s, t, u \in K$ with $u \neq 0$.

Proof. $\mathcal{L}(2 \cdot 0_E) = \langle 1, x \rangle = \langle 1, x' \rangle \implies x = \lambda x' + r$ for some $\lambda, r \in K, \lambda \neq 0$. Similarly $\mathcal{L}(3 \cdot 0_E) = \langle 1, x, y \rangle = \langle 1, x', y' \rangle \implies y = \mu y' + \sigma x' + t$ for some $\mu, \sigma, t \in K, \mu \neq 0$.

Looking at the coefficients of x^3 and y^2 tells us that $\lambda^3 = \mu^2$, so $\lambda = u^2, \mu = u^3$ for some $u \in K^\times$. Put $s = \frac{\sigma}{u^2}$ to conclude. \square

A Weierstrass equation defines an elliptic curve \iff it defines a smooth curve $\iff \Delta(a_1, \dots, a_6) \neq 0$, where $\Delta \in \mathbb{Z}[a_1, \dots, a_6]$ is a certain polynomial.

If $\text{char } K \neq 2, 3$, we may reduce to the case $E : y^2 = x^3 + ax + b$. In this case, the discriminant is $\Delta = -16(4a^3 + 27b^2)$.

Corollary 3.4. Assume $\text{char } K \neq 2, 3$. Elliptic curves

$$\begin{aligned} E : y^2 &= x^3 + ax + b \\ E' : y^2 &= x^3 + a'x + b' \end{aligned}$$

are isomorphic over $K \iff \begin{cases} a' = u^4a \\ b' = u^6b \end{cases} \text{ for some } u \in K^\times.$

Proof. E, E' are related by a substitution as in Proposition 3.3 with $r = s = t = 0$. \square

Definition 3.2. The j -invariant is $j(E) = \frac{1728(4a^3)}{4a^3 + 27b^2}$.

Corollary 3.5. $E \cong E' \implies j(E) \cong j(E')$ and the converse holds if $K = \overline{K}$.

Proof. $E \cong E' \iff \begin{cases} a' = u^4a \\ b' = u^6b \end{cases} \text{ for some } u \in K^\times \implies (a^3 : b^2) = ((a')^3 : (b')^2) \iff j(E) = j(E').$ The middle step is reversible if $K = \overline{K}$. \square

4 The Group Law

Let $E \subset \mathbb{P}^2$ be a smooth plane cubic with $0_E \in E(K)$ (not immediately assumed to be in Weierstrass form). E meets any line in 3 points, counted with multiplicity.

For $P, Q \in E$, let S be the 3rd point of intersection of PQ with E and then let R be the 3rd intersection of $0_E S$ with E . We define $P \oplus Q = R$. (Later we drop the circle and just write $+$). If $P = Q$, instead take the tangent line at P , i.e. $T_P E$, etc. This is the "chord and tangent process".

Theorem 4.1. (E, \oplus) is an abelian group.

Remark. Here E means $E(\overline{K})$ since we haven't specified a field yet.

Proof. (i) \oplus is commutative trivially.

(ii) 0_E is the identity, since the line through $0_E P$ meets S for the 3rd time at S and then SP meets E for the 3rd time at 0_E (drawing a picture makes this obvious).

(iii) Inverses: Let S be the 3rd intersection of T_{0_E} with E and Q the 3rd intersection of PS with E . Then $P \oplus Q = 0_E$.

(iv) Associativity is much harder. We have some setup:

Definition 4.1. $D_1, D_2 \in \text{Div}(E)$ are **linearly equivalent** if $\exists f \in K(E)^\times$ such that $\text{div}(f) = D_1 - D_2$. Write $D_1 \sim D_2$ and $[D] = \{D' \mid D' \sim D\}$.

Definition 4.2. The **Picard group** is $\text{Pic}(E) = \text{Div}(E)/\sim$. Also define $\text{Pic}^0(E) = \text{Div}^0(E)/\sim$ where $\text{Div}^0(E) = \{D \in \text{Div}(E) \mid \deg(D) = 0\}$.

We define $\psi : E \rightarrow \text{Pic}^0(E)$ by $P \mapsto [(P) - (0_E)]$.

Proposition 4.2. (i) $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) ψ is a bijection.

Proof. (i) WLOG let the lines PQ and $0_E S$ be given by $l = 0$ and $m = 0$.

Then

$$\text{div}\left(\frac{l}{m}\right) = (P) + (S) + (Q) - (0_E) - (S) - (R),$$

hence $(P) + (Q) \sim (P \oplus Q) + (0_E)$, so $(P \oplus Q) - (0_E) \sim (P) - (0_E) + (Q) - (0_E)$, so $\psi(P \oplus Q) = \psi(P) + \psi(Q)$.

(ii) Injectivity: Suppose $\psi(P) = \psi(Q)$ for $P \neq Q$. Then $\exists f \in \overline{K}(E)^\times$ such that $\text{div}(f) = (P) - (0_E) - (Q) + (0_E) = (P) - (Q) \implies E \xrightarrow{f} \mathbb{P}^1$ has degree 1 (for example since evaluation at 0 on the affine line gives that P has one root and Q has one pole), so $E \cong \mathbb{P}^1$, a contradiction.

Surjectivity: Let $[D] \in \text{Pic}^0(E)$. Then $D + (0_E)$ has degree 1, so by Riemann–Roch, $\dim \mathcal{L}(D + (0_E)) = 1$, so $\exists 0 \neq f \in \overline{K}(E)$ such that $\text{div}(f) + D + (0_E) \geq 0$, but $\text{div}(f) + D + (0_E)$ has degree 1, so $\text{div}(f) + D + (0_E) = (P)$ for some $P \in E \implies (P) - (0_E) \sim D \implies \psi(P) = [D]$.

□

We conclude that ψ identifies (E, \oplus) with $(\text{Pic}^0(E), +)$, so \oplus is associative.

□

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Formulae for E in Weierstrass form. Let $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Choose two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on it. Let the line through P_1 and P_2 be given by $y = \lambda x + \nu$ and let it meet E again at $P' = (x', y')$. We want to find $P_1 \oplus P_2 = P_3 = (x_3, y_3) = \ominus P'$ for $\ominus P$ the reflection of P across the x -axis. We easily compute $\ominus P_1 = (x_1, -(a_1x + a_3) - y_1)$.

Substituting $y = \lambda x + \nu$ into our equation for E and looking at the coefficient of x^2 gives $\lambda^2 + a_1\lambda - a_2 = x_1 + x_2 + x' = x_1 + x_2 + x_3$, so $x_3 = \lambda^2 + a_1\lambda - a_2 - x_1 - x_2$. For y_3 we find

$$y_3 = -(a_1x' + a_3) - y' = -(a_1x_3 + a_3) - (\lambda x_3 + \nu) = -(\lambda + a_1)x_3 - a_3 - \nu.$$

It remains to find formulas for λ and ν .

- Case 1. $x_1 = x_2$, but $P_1 \neq P_2$. Then $P_1 \oplus P_2 = 0_E$.
- Case 2. $x_1 \neq x_2$. Then $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ and $\nu = y_1 - \lambda x_1 = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$.
- Case 3. $P_1 = P_2$. In this case, compute the equation for the tangent line to get λ, ν as rational expressions in x_1, x_2, y_1, y_2 .

Corollary 4.3. $E(K)$ is an abelian group.

Proof. $E(K)$ is a subgroup of (E, \oplus) .

- It has identity 0_E by definition.
- We have closure and inverses through the formulae above.
- Associativity and commutativity is inherited.

□

Theorem 4.4. Elliptic curves are group varieties, i.e.

$$\begin{aligned} [-1] : E &\rightarrow E, P \mapsto \ominus P \\ \oplus : E &\rightarrow E, (P, Q) \mapsto P \oplus Q \end{aligned}$$

are morphisms of algebraic varieties.

Proof. By the above formulae, $[-1] : E \rightarrow E$ is a rational map, i.e. a morphism by our important remark.

For \oplus , note by the above formulae that $\oplus : E \rightarrow E$ is a rational map regular on

$$U = \{(P, Q) \in E \times E \mid 0_E \notin \{P, Q, P \oplus Q, P \ominus Q\}\}.$$

For $P \in E$, let $\tau_P : E \rightarrow E$ be the "translation by P " map, given by $X \mapsto P \oplus X$. τ_P is a rational map, hence a morphism. Now for $A, B \in E$, we factor \oplus as

$$E \times E \xrightarrow{\tau_{\ominus A} \times \tau_{\ominus B}} E \times E \xrightarrow{\oplus} E \xrightarrow{\tau_{A \oplus B}} E.$$

This shows \oplus is regular on $(\tau_A \times \tau_B)(U)$, so \oplus is regular on $E \times E$. □

Statement of results. The following isomorphisms in (i), (ii), (iv) respect the relevant topologies.

(i) $K = \mathbb{C}$. Then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ for Λ a lattice.

(ii) $K = \mathbb{R}$. Then

$$E(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}/\mathbb{Z} & \text{if } \Delta > 0 \\ \mathbb{R}/\mathbb{Z} & \text{if } \Delta < 0. \end{cases}$$

(iii) $K = \mathbb{F}_q$. Then $||E(\mathbb{F}_q)| - (q + 1)| \leq 2\sqrt{q}$. This is Hasse's Theorem.

(iv) For a local field $[K : \mathbb{Q}_p] < \infty$ with ring of integers \mathcal{O}_K , $E(K)$ has a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$.

(v) For a number field $[K : \mathbb{Q}] < \infty$, $E(K)$ is a finitely generated abelian group (this is the Mordell–Weil Theorem). Basic group theory says that if A is a finitely generated abelian group, then $A \cong (\text{finite subgroup}) \times \mathbb{Z}^r$. Here r is called the rank of A . The proof of Mordell–Weil gives an upper bound for rank $E(K)$, but there is no known algorithm to compute the rank in all cases.

Brief remarks on the case $K = \mathbb{C}$. Let $\Lambda = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}\}$ where ω_1, ω_2 are a basis for \mathbb{C} as an \mathbb{R} -vector space. Then meromorphic functions on the Riemann surface \mathbb{C}/Λ correspond bijectively with Λ -invariant meromorphic functions in \mathbb{C} . The function field of \mathbb{C}/Λ is generated by $\wp(z)$ and $\wp'(z)$, where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^3}.$$

These satisfy $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ for some constants $g_2, g_3 \in \mathbb{C}$ depending on Λ . One shows $\mathbb{C}/\Lambda \cong E(\mathbb{C})$, where $E : y^2 = 4x^3 - g_2x - g_3$ which is an isomorphism on both groups (via $z \mapsto (\wp(z), \wp'(z))$) and on Riemann surfaces. We have the following result:

Theorem 4.5 (Uniformization theorem). Every elliptic curve over \mathbb{C} arises in this way.

Definition 4.3. For $n \in \mathbb{Z}$, let $[n] : E \rightarrow E$ be given by $P \mapsto \underbrace{P \oplus P \oplus \dots \oplus P}_{n \text{ copies}}$

if $n > 0$ and $[-n] = [-1] \circ [n]$.

Definition 4.4. The n -torsion subgroup of E is

$$E[n] = \ker(E \xrightarrow{[n]} E).$$

If $K = \mathbb{C}$, then $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, so $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ and $\deg[n] = n^2$. Call these results (1) and (2). We will show that (2) holds over any field K and (1) holds if $\text{char } K \nmid n$.

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Lemma 4.6. Assume $\text{char } K \neq 2$ and $E : y^2 = f(x) = (x - e_1)(x - e_2)(x - e_3)$ (with $e_i \in \bar{K}$). Then $E[2] = \{0, (e_1, 0), (e_2, 0), (e_3, 0)\} \cong (\mathbb{Z}/2\mathbb{Z})^\times$.

Proof. Let $P = (x, y) \in E$. Then $2[P] = 0 \iff P = -P \iff (x, y) = (x, -y) \iff y = 0$. \square

5 Isogenies

Let E_1, E_2 be elliptic curves

Definition 5.1. (i) An **isogeny** $\phi : E_1 \rightarrow E_2$ is a nonconstant morphism with $\phi(0_{E_1}) = 0_{E_2}$.

(ii) We say E_1 and E_2 are **isogenous** if there is an isogeny between them.

In (i), nonconstant is equivalent to surjective on \bar{K} -points. See Theorem 2.7.

Definition 5.2. $\text{Hom}(E_1, E_2) = \{\text{isogenies } E_1 \rightarrow E_2\} \cup \{0\}$ (the constant map at 0_E). This is an abelian group under $(\phi + \psi)(P) = \phi(P) + \psi(P)$.

If $E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3$ are isogenies, then $\psi \circ \phi$ is an isogeny. By tower law, $\deg(\psi \circ \phi) = \deg(\psi)\deg(\phi)$.

Proposition 5.1. If $0 \neq n \in \mathbb{Z}$, then $[n] : E \rightarrow E$ is an isogeny.

Proof. $[n]$ is a morphism by Theorem 4.4. We need to show $[n] \neq [0]$. Assume $\text{char } K \neq 2$.

- Case $n = 2$. Lemma 4.6 implies that $E[2] \neq E$, so $[2] \neq 0$.
- Case n odd. Lemma 4.6 implies that $\exists 0 \neq T \in E[2]$. Then $nT = T \neq 0$, so $[n] \neq [0]$.

Now use $[mn] = [m] \circ [n]$ to conclude.

If $\text{char } K = 2$, then we can replace Lemma 4.6 with an explicit lemma about 3-torsion points. \square

Corollary 5.2. $\text{Hom}(E_1, E_2)$ is a torsion-free \mathbb{Z} -module.

Theorem 5.3. Let $\phi : E_1 \rightarrow E_2$ be an isogeny. Then

$$\phi(P + Q) = \phi(P) + \phi(Q) \quad \forall P, Q \in E.$$

Sketch proof. ϕ induces a map $\phi_* : \text{Div}^0(E_1) \rightarrow \text{Div}^0(E_2)$ by $\sum_{P \in E_1} n_P P \mapsto \sum_{P \in E_2} n_P \phi(P)$. Recall $\phi^* : K(E_2) \hookrightarrow K(E_1)$.

Fact. If $f \in K(E_1)$, then $\text{div}(N_{K(E_1)/K(E_2)}f) = \phi^*(\text{div } f)$. So ϕ_* sends principal divisors to principal divisors. Since $\phi(0_{E_1}) = 0_{E_2}$, the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow f & & \downarrow g \\ \text{Pic}^0(E_1) & \xrightarrow{\phi_*} & \text{Pic}^0(E_2) \end{array}$$

(with $f(P) = [(P) - (0_{E_1})]$, $g(Q) = [(Q) - (0_{E_2})]$). Since ϕ_* is a group homomorphism, ϕ is a group homomorphism. \square

Lemma 5.4. Let $\phi : E_1 \rightarrow E_2$ be an isogeny. Then there exists a morphism ξ making the following diagram commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow x_2 & & \downarrow x_1 \\ \mathbb{P}^1 & \xrightarrow{\xi} & \mathbb{P}^1 \end{array}$$

with x_i the x -coordinate in a Weierstrass equation for E_i . Moreover, if $\xi(t) = \frac{r(t)}{s(t)}$ with $r, s \in K[t]$ coprime, then $\deg(\phi) = \deg(\xi) = \max(\deg(r), \deg(s))$.

Proof. For $i = 1, 2$, $K(E_i)/K(x_i)$ is a degree 2 Galois extension with Galois group generated by $[-1]^*$. By Theorem 5.3, $\phi \circ [-1] = [-1] \circ \phi$, so if $f \in K(x_2)$, then $[-1]^*(\phi^*f) = \phi^*([-1]^*f) = \phi^*f$ and hence $\phi^*f \in K(x_1)$. Hence we find

$$\begin{array}{ccc} & K(E_1) = K(x_1, y_1) & \\ & \swarrow 2 & \downarrow \\ K(x_1) & & K(E_2) = K(x_2, y_2) \\ \downarrow & \swarrow 2 & \\ K(x_2) & & \end{array} \cdot$$

In particular, $\phi^*x_2 = \xi(x_1)$ for some $\xi \in K(t)$. By tower law, $2\deg(\phi) = 2\deg(\xi) \implies \deg(\phi) = \deg(\xi)$. Now $K(x_2) \hookrightarrow K(x_1)$ by $x_2 \mapsto \xi(x_1) = \frac{r(x_1)}{s(x_1)}$ for $r, s \in K[t]$ coprime. Then minimal polynomial of x_1 over $K(x_2)$ is $F(t) = r(t) - s(t)x_2 \in K(x_2)[t]$. This is true as $F(x_1) = 0$, F is irreducible on $K[x_2, t]$ (since r, s are coprime) and by Gauss' Lemma, F is irreducible on $K(x_2)[t]$. Hence $\deg(\phi) = \deg(\xi) = [K(x_1) : K(x_2)] = \deg(F) = \max(\deg(r), \deg(s))$. \square

Lemma 5.5. $\deg[2] = 4$.

Proof. Assume char $K \neq 2, 3$, so $E : y^2 = x^3 + ax + b = f(x)$. If $P = (x, y)$, then $x(2P) = \left(\frac{3x^2+a}{2y}\right)^2 - 2x = \frac{(3x^2+a)^2 - 2xf(x)}{4f(x)}$. The numerator and denominator are coprime, since otherwise $\exists \theta \in \overline{K}$ with $f(\theta) = f'(\theta) = 0$, meaning f has a multiple root, contradiction. We are now done by Lemma 5.4, since $\deg[2] = \max(3, 4) = 4$. \square

Definition 5.3. Let A be an abelian group. Then a map $q : A \rightarrow \mathbb{Z}$ is a quadratic form if

- (i) $q(nx) = n^2q(x) \forall n \in \mathbb{Z}, q \in A$.
- (ii) $(x, y) \mapsto q(x+y) - q(x) - q(y)$ is \mathbb{Z} -bilinear.

Lemma 5.6. $q : A \rightarrow \mathbb{Z}$ is a quadratic form if and only if it satisfies the parallelogram law $q(x+y) + q(x-y) = 2q(x) + 2q(y) \forall x, y \in A$.

Proof. (\implies). Let $\langle x, y \rangle = q(x+y) - q(x) - q(y)$. Then $\langle x, x \rangle = q(2x) - 2q(x) = 2q(x)$ by (i) with $n = 2$. By (ii), $\langle x+y, x+y \rangle + \langle x-y, x-y \rangle = 2\langle x, x \rangle + 2\langle y, y \rangle$, which implies $q(x+y) + q(x-y) = 2q(x) + 2q(y)$.

(\impliedby). This is on Ex. Sheet 2. \square

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Theorem 5.7. $\deg : \text{Hom}(E_1, E_2) \rightarrow \mathbb{Z}$ is a quadratic form (with $\deg(0) = 0$).

Proof. Assume char $K \neq 2, 3$ and write $E_2 = y^2 = x^3 + ax + b$. Let $P, Q \in E_2$ with $P, Q, P+Q, P-Q$ all nonzero and let x_1, x_2, x_3, x_4 be the x -coordinates of these points.

Lemma 5.8. There exist polynomials $W_0, W_1, W_2 \in \mathbb{Z}[a, b][x_1, x_2]$ of degree ≤ 2 in x_1 and of degree ≤ 2 in x_2 such that

$$(1 : x_3 + x_4 : x_3x_4) = (W_0 : W_1 : W_2)$$

Proof. Method 1: Direct calculation (results on the formula sheet) gives the result (e.g. $W_0 = (x_1 - x_2)^2$).

Method 2: Let $y = \lambda x + \nu$ be the line through P and Q . Substituting, we get $x^3 + ax + b - (\lambda x + \nu)^2 = (x - x_1)(x - x_2)(x - x_3) = x^3 - s_1x^2 + s_2x - s_3$ where s_i is the i^{th} symmetric polynomial in x_1, x_2, x_3 . Comparing coefficients gives $\lambda^2 = s_1, -2\lambda\nu = s_2 - a, \nu^2 = s_3 + b$. Eliminating λ and ν gives

$$F(x_1, x_2, x_3) = (s_2 - a)^2 - 4s_1(s_3 + b) = 0,$$

where F has degree at most 2 in each x_i . Hence x_3 is a root of the quadratic $W(t) = F(x_1, x_2, t)$. Repeating this for the line through P and $-Q$ shows that

x_4 is the other root of $W(t)$. Therefore

$$\begin{aligned} W(t) &= W_0(t - x_3)(t - x_4) = W_0t^2 - W_1t + W_2 \\ \implies (1 : x_3 + x_4 : x_3x_4) &= (W_0 : W_1 : W_2). \end{aligned}$$

□

We now show that if $\phi, \psi \in \text{Hom}(E_1, E_2)$, then $\deg(\phi + \psi) + \deg(\phi - \psi) \leq 2\deg(\phi) + 2\deg(\psi)$. We may assume that $\phi, \psi, \phi + \psi, \phi - \psi$ are not the zero maps (otherwise we're done trivially, or use $\deg[-1] = 1$, $\deg[2] = 4$). Now

$$\begin{aligned} \phi : (x, y) &\mapsto (\xi_1(x), \dots) \\ \psi : (x, y) &\mapsto (\xi_2(x), \dots) \\ \phi + \psi : (x, y) &\mapsto (\xi_3(x), \dots) \\ \phi - \psi : (x, y) &\mapsto (\xi_4(x), \dots). \end{aligned}$$

Lemma 5.8 implies $(1 : \xi_3 + \xi_4 : \xi_3\xi_4) = ((\xi_1 - \xi_2)^2 : \dots)$. Say $\xi_i = \frac{r_i}{s_i}$ for $r_i, s_i \in K[t]$ coprime. This gives

$$(s_3s_4 : r_3s_4 + r_4s_3 : r_3r_4) \stackrel{(\star)}{=} ((r_1s_2 - r_2s_1)^2 : \dots)$$

where every term is quadratic in r_3, r_4, s_3 and s_4 . Hence (as the terms on the LHS of (\star) are coprime)

$$\begin{aligned} \deg(\phi + \psi) + \deg(\phi - \psi) &= \max(\deg(r_3), \deg(s_3)) + \max(\deg(r_4), \deg(s_4)) \\ &= \max(\deg(s_3s_4), \deg(r_3s_4 + r_4s_3), \deg(r_3r_4)) \\ &\leq 2\max(\deg(r_1), \deg(s_1)) + 2\max(\deg(r_2), \deg(s_2)) \\ &= 2\deg(\phi) + 2\deg(\psi). \end{aligned}$$

Now replace ϕ and ψ by $\phi + \psi$ and $\phi - \psi$ and use $\deg[2] = 4$ to get

$$4\deg(\phi) + 4\deg(\psi) = \deg(2\phi) + \deg(2\psi) \leq 2\deg(\phi + \psi) + 2\deg(\phi - \psi).$$

This gives the parallelogram law, so \deg is a quadratic form. □

Corollary 5.9. $\deg(n\phi) = n^2\deg(\phi)$. In particular, $\deg[n] = n^2$.

Example 5.10. Let E/K be an elliptic curve. Suppose $\text{char } K \neq 2$ and $0 \neq T \in E(K)[2]$. WLOG let $E : y^2 = x(x^2 + ax + b)$ for $a, b \in K, b(a^2 - 4b) \neq 0$ (by moving a root to zero) and WLOG $T = (0, 0)$.

If $P = (x, y)$ and $P' = P + T = (x', y')$, then

$$\begin{aligned} x' &= \left(\frac{y}{x}\right)^2 - a - x = \frac{x^2 + ax + b}{x} - a - x = \frac{b}{x} \\ y' &= -\left(\frac{y}{x}\right) x' = -\frac{by}{x^2}. \end{aligned}$$

We let $\xi = x + x' + a = \left(\frac{y}{x}\right)^2$, $\eta = y + y' = \frac{y}{x} \left(x - \frac{b}{x}\right)$. Then

$$\eta^2 = \left(\frac{y}{x}\right)^2 \left(\left(x + \frac{b}{x}\right)^2 - 4b \right) = \xi((\xi - a)^2 - 4b) = \xi(\xi^2 - 2a\xi + a^2 - 4b).$$

Let $E' : y^2 = x(x^2 + a'x + b')$ with $a' = -2a$, $b' = a^2 - 4b$. There is an isogeny $\phi : E \rightarrow E'$ given by $(x, y) \mapsto \left(\left(\frac{y}{x}\right)^2 : \frac{y(x^2 - b)}{x^2} : 1\right)$.

Sanity check/finding where 0_E maps to: x is a double pole, y is a triple pole, so $\left(\frac{y}{x}\right)^2$ is a double pole and $\frac{y(x^2 - b)}{x^2}$ is a triple pole (and the last coordinate 1 has degree 0). Multiplying through by a cube of a uniformizer, the degrees go from $(-2, -3, 0)$ to $(1, 0, 3)$, so $0_E \mapsto (0 : 1 : 0)$.

To compute $\deg(\phi)$, $\left(\frac{y}{x}\right)^2 = \frac{x^2 + ax + b}{x}$ with the numerator and denominator coprime as $b \neq 0$, so by Lemma 5.4, $\deg(\phi) = 2$. We say ϕ is a **2-isogeny**.

6 The invariant differential

For C some algebraic curve over $K = \overline{K}$.

Definition 6.1. The space of differentials Ω_C (sometimes called one-forms) is the $K(C)$ -vector space generated by df for all $f \in K(C)$ subject to the relations

- (i) $d(f + g) = df + dg$.
- (ii) $d(fg) = f dg + g df$.
- (iii) $da = 0 \ \forall a \in K$.

Fact. Ω_C is a 1-dimensional $K(C)$ -vector space.