Part III - Algebraic Geometry

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Contents

0 Introduction

6 Oct 2022, Lecture 1

The course consists of four parts.

(1) Basics of sheaves on topological spaces.

(2) Definition of schemes and morphisms.

(3) Properties of schemes (e.g. the algebraic geometry notion of compactness and other properties).

(4) A rapid introduction to the cohomology of schemes.

The main reference for the course is Hartshorne's *Algebraic Geometry*.

1 Beyond algebraic varieties

08 Oct 2022, Lecture 2

1.1 Summary of classical algebraic geometry

We let $k = \overline{k}$ be a algebraically closed field and consider $\mathbb{A}^n_k = \mathbb{A}^n = k^n$ as a set.

Definition 1.1. An **affine variety** is a subset $V \subset \mathbb{A}^n$ of the form $\mathbb{V}(S)$ with $S \subset k[x_1, \dots, x_n]$, where $\mathbb{V}$ is the common vanishing locus.

Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$ (the ideal generated by S). By Hilbert Basis Theorem (since $k[x_1, \dots, x_n]$ is noetherian), $\mathbb{V}(I(S)) = \mathbb{V}(S')$ for some finite set $S \subset k[x_1, \dots, x_n]$.

In fact, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in k[x_1, \dots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}$$

is the **radical** of I. For example, in $k[x]$, if $I = (x^2)$, then $\sqrt{I} = (x)$.

Definition 1.2. Given varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, a **morphism** is a (set-theoretic) map $\phi : V \to W \subset \mathbb{A}^m_k$ such that if $\phi = (f_1, \dots, f_m)$, then each f_i is the restriction of a polynomial in $\{x_1, \dots, x_n\}$.

An **isomorphism** is a morphism with a two–sided inverse.

Our basic correspondence is

$$\{\text{Affine varieties over } k\}/\text{up to isomorphism}$$

$$\leftrightarrow$$

$$\{\text{finitely generated } k\text{–algebras } A \text{ without nilpotent elements}\}$$

A finitely generated k–algebra is just a quotient of a polynomial ring in finitely many variables. A nilpotent element is such that some power of it is zero. For example, in $k[x]/(x^2)$, the element x is nilpotent.

How does this correspondence work? Given a variety V (representing an isomorphism class), we write $V = \mathbb{V}(I)$ for $I \subset k[x_1, \dots, x_n]$ a radical ideal[1], and map $V \mapsto k[x_1, \dots, x_n]/I$.

For the reverse, if A is a finitely generated nilpotent free algebra, then $A \cong k[y_1, \dots, y_m]/J$ where we can choose J to be radical (exercise: why?).

We have to check that this is independent of our choice on both sides (exercise: think through this, it should be clear).

Definition 1.3. The algebra associated to V is classically denoted $k[V]$ and called the **coordinate ring of** V.

We have the compatibility of morphisms with our basic correspondence: there is a bijection between

$$\text{Morphisms}(V, W) \leftrightarrow \text{Ring homomorphisms}_k(k[W], k[V])$$

(here RingHom_k means that our homomorphisms preserve k).

We can now make our set into a topological space:

Definition 1.4. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The **Zariski topology** on V is defined such that the closed sets are $\mathbb{V}(S)$, where $S \subset k[V]$.

If $V \cong W$, then the Zariski topological spaces are homeomorphic as varieties (exercise).

Theorem 1.1 (Nullstellensatz)**.** Fix V a variety and let $k[V]$ be its coordinate ring. Given $p \in V$, we can produce a homomorphism $\text{ev}_p : k[V] \to k$ by sending $f \mapsto f(p)$. Note that ev_p is surjective (since we have constant functions), hence $\ker(\text{ev}_p) = m_p$ is a maximal ideal, giving us a map

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}.$$

Nullstellensatz says that this is actually a bijection. For the converse map, given $m \subset k[V]$, we get a quotient $k[V] \to k[V]/m = k$ (Nullstellensatz says this extension is finite, hence must be k). So using/choosing a representation for V in $k[x_1, \dots, x_n]$ gives a surjective homomorphism onto k and specifies a bunch of points.

[1] A radical ideal is an ideal equal to its radical.

1.2 Limitations of classical algebraic geometry

Question. What is an abstract variety, i.e. "some "space" X such that locally as a cover $\{U_i\}$, each U_i is an affine variety, compatible with overlaps".

Example 1.1 (non–algebraically closed fields)**.** Take $I = (x^2+y^2+1) \subset \mathbb{R}[x,y]$. Then $\mathbb{V}(I) = \varnothing \subset \mathbb{R}^2$, but I is prime, so radical, so nullstellensatz fails.

Question. On what topological space is $\mathbb{R}[x,y]/(x^2+y^2+1)$ "naturally" the set of functions? (or $\mathbb{Z}$, or $\mathbb{Z}[x]$).

Example 1.2 (Why restrict to radical ideals?)**.** Take $C = \mathbb{V}(y-x^2) \subset \mathbb{A}^2_k$ and $D = \mathbb{V}(x,y)$, so $C \cap D = \mathbb{V}(y, y-x^2) = \mathbb{V}(x,y) = \{(0,0)\}$. This is a single point, but if $D_\delta = \mathbb{V}(y+\delta)$ for some $\delta \in k$, then $C \cap D_\delta = \{\pm\sqrt{\delta}\}$, which is 2 points for all $\delta \neq 0$. In other words, intersections of varieties don't want to be varieties.

1.3 The spectrum of a ring

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Let A be a commutative ring with identity. We will define a topological space on which A is the ring of functions.

Definition 1.5. The **Zariski spectrum** of A is

$$\text{Spec } A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

A ring homomorphism $\phi : A \to B$ induces a map $\phi^{-1} : \text{Spec } B \to \text{Spec } A$ by $q \mapsto \phi^{-1}(q)$. In general, the preimage of a prime ideal is a prime ideal.

Warning. This would fail if we only considered maximal ideals, since the preimage of a maximal ideal need not be maximal.

Given $f \in A$ and $\mathfrak{p} \in \text{Spec}(A)$, we have an induced $\overline{f} \in A/\mathfrak{p}$ obtained via a quotient. Informally, we can evaluate any $f \in A$ at points $\mathfrak{p} \in \text{Spec}(A)$ with the caveat that the codomain of this evaluation depends on $\mathfrak{p}$.

Example 1.3. Take $A = \mathbb{Z}$. Then $\text{Spec } A = \text{Spec } (\mathbb{Z}) = \{p \mid p \text{ is prime}\} \cup \{(0)\}$. Let's pick an element in $\mathbb{Z}$, say $132 \in \mathbb{Z}$. Given a prime p, we can look at $132 \pmod p \in \mathbb{Z}/p$. The takeaway here is that

$$\begin{gathered} \text{Spec } \mathbb{Z} \to \text{Space} \\ 132 \in \mathbb{Z} \to \text{a function} \\ 132 \pmod p \to \text{value of that function at } p. \end{gathered}$$

Note that based on the value of p, our codomain changes from point to point.

Example 1.4. Take $A = \mathbb{R}[x]$, then $\text{Spec } \mathbb{R}[x] = \mathbb{C}/\text{complex conjugation} \cup \{(0)\}$.

Exercise. Draw Spec $\mathbb{Z}[x]$ and Spec $k[x]$ for k any field (i.e. describe all prime ideals and their containment). This is on example sheet 1.

Example 1.5. If $A = \mathbb{C}[x]$, then Spec $A = \mathbb{C} \cup \{(0)\}$, where given $a \in \mathbb{C}$, we send it to the maximal ideal $\langle z - a \rangle$.

1.4 A topology on Spec A

Fix $f \in A$. Then $\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \equiv 0 \pmod{\mathfrak{p}}\} \subset \text{Spec } A$. (Note that $f \equiv 0 \pmod{\mathfrak{p}}$ is the same as $f \in \mathfrak{p}$).

Similarly for $J \subset A$ an ideal, $\mathbb{V}(J) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\ \forall f \in J\}$.

Proposition 1.2. The sets $\mathbb{V}(J) \subset \text{Spec } A$ ranging over all ideals J form the closed sets of a topology on Spec A. This topology is called the **Zariski topology**.

Proof. Easy fact: $\varnothing$ and Spec A are closed, since we have functions 1 (vanishing nowhere) and 0 (vanishing everywhere). Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ (this is because $I_1 + I_2$ is the smallest ideal containing $I_1 \cup I_2$), arbitrary intersections are closed.

Finally, we claim $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$. The containment $\subset$ is clear: if a prime ideal contains I_1 or I_2, it contains $I_1 \cap I_2$. Conversely, $I_1 I_2 \subset I_1 \cap I_2$, so if $I_1 I_2 \subset I_1 \cap I_2 \subset \mathfrak{p}$, then by primality $I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p}$. $\square$

Example 1.6. Let $k = \mathbb{C}$ and consider Spec $\mathbb{C}[x, y]$. We make a few observations:

- The point $(0) \in \text{Spec } \mathbb{C}[x, y]$ is dense in the Zariski topology, i.e. $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[x, y]$ because every prime ideal contains (0) (because we are in an integral domain).

- Consider the prime ideal $(y^2 - x^3)$ (which is prime since the quotient is an integral domain). Consider a maximal ideal $\mathfrak{m}_{a,b} = (x - a, y - b)$. We can ask: when is $\mathfrak{m}_{a,b} \in \overline{\{(y^2 - x^3)\}}$? The answer: if and only if $b^2 = a^3$, e.g. $(1, 1)$ (see example sheet 1). The lesson here is that points are not closed in the Zariski topology.

1.5 Functions on opens

Definition 1.6. Let $f \in A$. Define the **distinguished open** corresponding to f to be

$$\mathcal{U}_f = (\text{Spec}(A))/\mathbb{V}(f).$$

Example 1.7.
- Let $A = \mathbb{C}[x]$, so Spec $A = \mathbb{C} \cup \{(0)\}$ (with the Zariski topology). Take $f = x$ and consider $\mathcal{U}_x$. Recall the bijection Spec $\mathbb{C} \leftrightarrow \mathbb{C} \cup \{(0)\}$ by $(x - a) \leftarrow\!\shortmid a \in \mathbb{C}$ and $(0) \leftarrow\!\shortmid (0)$. Then $\mathbb{V}(x) = \{\mathfrak{p} \in \text{Spec } A \mid x \in \mathfrak{p}\} = \{(x)\}$, so $\mathcal{U}_f = \text{Spec } A \setminus \{(x)\}$.

- More generally, suppose we fix $a_1, \dots, a_r \in \mathbb{C}$, then Spec $A \backslash \{(x-a_i)\}_{i=1}^r = \mathcal{U}$ and $\mathcal{U} = \mathcal{U}_f$, where $f = \prod_{i=1}^r (x - a_i)$.

Lemma 1.3. The distinguished opens $\mathcal{U}_f$ taken over all $f \in A$ form a basis for the Zariski topology on Spec A.

Proof. Left as an exercise on example sheet 1. □

A bit of commutative algebra:

Definition 1.7. Given $f \in A$, the **localization of** A **at** f is $A_f = A[x]/(xf-1)$, which we can informally think of as $A_f = A[\frac{1}{f}]$.

Lemma 1.4. The distinguished open $\mathcal{U}_f \subset$ Spec A is naturally homeomorphic to Spec A_f via the ring homomorphism $A \xrightarrow{j} A_f$, which produces the inverse j^{-1} : Spec $A_f \to$ Spec A.

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Proof. Primes in the ring A_f are in bijection with primes of A that miss f via j^{-1}. We exhibit this bijection:

- Given $q \subset A_f$ prime, take $j^{-1}(q) \subset A$, which is prime.
- Given $p \subset A$ a prime ideal, take $p_f = j(p)A_f$. We claim p_f is a prime exactly when $f \notin p$.
  - If $f \in p$, then p_f contains f, which is a unit, so $p_f = (1)$ is not prime.
  - If $f \notin p$, then $(A_f/p_f) \cong (A/p)_{\overline{f}}$, where $\overline{f}$ is $f + p$, a coset (exercise: check this formally). Hence $(A/p)_{\overline{f}} \subset FF(A/p)$ (FF stands for fraction field), so it is an integral domain, so p_f is prime.

Finally we need to check that these maps are inverses. This is left as an exercise. □

Facts about distinguished opens:

- $U_f \cap U_g = U_{fg}$ (easy fact).
- $U_{f^n} = U_f$ for all $n \geq 1$ (easy fact).
- The rings A_f and A_{f^n} for $n \geq 1$ are isomorphic. Why? Since $A_f = A[x]/(xf-1)$ and $A_{f^n} = A[y]/(yf^n-1)$, the isomorphism is given by $A_f \to A_{f^n}$ by $x \mapsto f^{n-1}y$ and $A_{f^n} \to A_f$ by $y \mapsto x^n$ (check these are inverses).
- Containment. $U_f \subset U_g \iff f^n$ is a multiple of g for some $n \geq 1$. To orient ourselves: if $f = gf'$, then $U_f \subset U_g$.

Proof. The ($\Longrightarrow$) direction is clear by the orientation above. Conversely, suppose $U_f \subset U_g$, so $\mathbb{V}(f) \supset \mathbb{V}(g)$. The set $\mathbb{V}(f)$ is the set of all primes containing (f). We claim that $\sqrt{(f)} \subset \sqrt{(g)}$. But what is the radical of I? It is the intersection of all primes containing the ideal I. $\square$

Foreshadowing: fix A. We've made an assignment from distinguished opens in Spec A to rings by mapping $U_f \mapsto A_f$. The association is "functorial", i.e. if $U_{f_1} \subset U_{f_2}$, then we can assume that $f_1^n = f_2 f_3$, so $U_{f_1} = U_{f_1^n} = U_{f_2 f_3} \subset U_{f_2}$, so there is a homomorphism $A_{f_2} \to A_{f_1}$. This is the restriction map.

Question: can we extend this association to all open sets? See notes for the answer (yes).

2 Sheaves

2.1 Presheaves

Let X be a topological space.

Definition 2.1. A **presheaf $\mathcal{F}$ on X of abelian groups** is an association from the set of open sets in X to abelian groups given by $U \mapsto \mathcal{F}(U)$ and for $U \subset V$ opens, a homomorphism $\mathrm{res}_u^v : \mathcal{F}(V) \to \mathcal{F}(U)$ (a **restriction map**) such that $\mathrm{res}_u^u = \mathrm{id}$ and $\mathrm{res}_u^v \circ \mathrm{res}_v^w = \mathrm{res}_u^w$ when $U \subset V \subset W$ are opens.

Example 2.1. For any space X, take $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$ with the usual restriction.

Similarly we can get sheaves of rings, sets, modules over a fixed ring, etc.

Definition 2.2. A **morphism** $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves on X is, for each $U \subset X$ open, a homomorphism $\phi(u) : \mathcal{F}(u) \to \mathcal{G}(u)$ compatible with restriction maps, i.e. if $V \subset U$, then the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(u) \arrow[r, "\phi(u)"] \arrow[d, "\mathrm{res}_v^u"] & \mathcal{G}(u) \arrow[d, "\mathrm{res}_v^u"] \\
\mathcal{F}(v) \arrow[r, "\phi(v)"'] & \mathcal{G}(v)
\end{tikzcd}$$

Definition 2.3. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of preshaves is injective (surjective) if $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (surjective) for all $U \subset X$.

2.2 Sheaves

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Definition 2.4. A **sheaf** is a presheaf $\mathcal{F}$ such that

(1) If $U \subset X$ is open and $\{U_i\}$ is an open cover of U, then for $s \in \mathcal{F}(U)$, if $s|_{U_i} = \mathrm{res}^U_{U_i}(s) = 0$ for all i, then $s = 0$.

(2) If U and $\{U_i\}$ are as in (1), then given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Remark. These axioms imply $\mathcal{F}(\varnothing) = 0$ (exercise).

A **morphism** of sheaves is a morphism of the underlying presheaves.

Example 2.2. If X is a topological space, $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$, then f is a sheaf.

Non–example. Let $X = \mathbb{C}$ with the Euclidean topology and take $\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic and bounded}\}$. Then $\mathcal{F}$ is not a sheaf, since bounded functions may glue to unbounded functions. For example, take $U = \mathbb{C}$ and $U_i = D(0, i)$. Then $f(z) = z$ is bounded on each U_i, but not on U. In general, the characterization of elements of a sheaf should be purely local, and being bounded is not a local condition.

Non–example. Fix a group G and a set $\mathcal{F}(U) = G$ (the **constant presheaf**). If U_1, U_2 are disjoint, then $\mathcal{F}(U_1 \cup U_2) = G \times G$.

Example 2.3. Give G the discrete topology (every subset is open and closed) and define

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\} = \{f : U \to G \mid f \text{ is locally constant}\}.$$

This is the **constant sheaf**.

Example 2.4. If V is an irreducible variety, then

$$\mathcal{O}_V(v) = \{f \in k[V] \mid f \text{ is regular at } p \ \forall p \in U\}.$$

Here regular at p means that $f = \frac{g}{h}$ in a neighborhood of p with g, h polynomials and $h(p) \neq 0$. $\mathcal{O}_V$ is the **structure sheaf** of V.

This is a sheaf, since we have a local condition.

2.3 Basic constructions

Terminology. A **section** of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.

Construction of stalks. Fix $p \in X$ and $\mathcal{F}$ a presheaf on X. Then $\mathcal{F}_p$, the **stalk** of $\mathcal{F}$ at p, is defined to be

$$\mathcal{F}_p = \{(U, s) \mid s \in \mathcal{F}(U), p \in U\} / \sim$$

with $(U, s) \sim (V, s')$ if $\exists W \subset U \cap V$ with $p \in W$ such that $s|_W = s'|_W$.

The elements of $\mathcal{F}_p$ are called **germs**.

Example 2.5. Take $\mathbb{A}^1$, the affine line, then $\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \mid g(0) \neq 0 \right\} = k[t]_{(t)} \subset k(t)$.

Proposition 2.1. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X such that for all $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism, then f is an isomorphism.

Here $f_p((U,s)) = (U, f_U(s))$, which is well–defined.

Proof. We will show $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for each U, and we can then define f^{-1} by $(f^{-1})_U = (f_U)^{-1}$.

f_U is injective: suppose $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Since f_p is injective, $(U,s) = 0$ in $\mathcal{F}_p$ for every $p \in U$. Thus for every $p \in U$, there exists an open neighborhood U_p of p such that $s|_{U_p} = 0$. But $\{U_p \mid p \in U\}$ is a cover of U, so $s = 0$ in $\mathcal{F}(U)$ by the first condition of being a sheaf.

f_U is surjective: take $t \in \mathcal{G}(U)$. For each $p \in U$, we have $(U_p, s_p) \in \mathcal{F}_p$ with $f_p(U_p, s_p) = (U,t) \in \mathcal{G}_p$. By shrinking U_p if necessary, we can assume $f_{U_p}(s_p) = t|_{U_p}$. For points $p, p' \in U$,

$$f(_{U_p \cap U_{p'}}) \left(s_p|_{U_p \cap U_{p'}} \setminus s_{p'}|_{U_p \cap U_{p'}} \right) = t|_{U_p \cap U_{p'}} - t|_{U_p \cap U_{p'}} = 0.$$

Thus $s_p|_{U_p \cap U_{p'}} - s_{p'}|_{U_p \cap U_{p'}} = 0$ by the injectivity of $f_{U_p \cap U_{p'}}$. Thus by the second sheaf axiom, $\exists s \in \mathcal{F}(U)$ with $s|_{U_p} = s_p$. Now $f_U(s)|_{U_p} = f_{U_p}(s|_{U_p}) = f_{U_p}(s_p) = t|_{U_p}$. Thus $f_U(s) = t$ by the first sheaf axiom. $\square$

We emphasize that this proof is asymmetric in the sense that we need to first prove injectivity to be able to prove surjectivity.