Part III - Algebraic Geometry

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Contents

0 Introduction

The course consists of four parts.

(1) Basics of sheaves on topological spaces.

(2) Definition of schemes and morphisms.

(3) Properties of schemes (e.g. the algebraic geometry notion of compactness and other properties).

(4) A rapid introduction to the cohomology of schemes.

The main reference for the course is Hartshorne's *Algebraic Geometry*.

1 Beyond algebraic varieties

1.1 Summary of classical algebraic geometry

We let $k = \overline{k}$ be a algebraically closed field and consider $\mathbb{A}^n_k = \mathbb{A}^n = k^n$ as a set.

Definition 1.1. An **affine variety** is a subset $V \subset \mathbb{A}^n$ of the form $\mathbb{V}(S)$ with $S \subset k[x_1, \ldots, x_n]$, where $\mathbb{V}$ is the common vanishing locus.

Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$ (the ideal generated by S). By Hilbert Basis Theorem (since $k[x_1, \ldots, x_n]$ is noetherian), $\mathbb{V}(I(S)) = \mathbb{V}(S')$ for some finite set $S \subset k[x_1, \ldots, x_n]$.

In fact, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}$$

is the **radical** of I. For example, in $k[x]$, if $I = (x^2)$, then $\sqrt{I} = (x)$.

Definition 1.2. Given varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, a **morphism** is a (set-theoretic) map $\phi : V \to W \subset \mathbb{A}^m_k$ such that if $\phi = (f_1, \ldots, f_m)$, then each f_i is the restriction of a polynomial in $\{x_1, \ldots, x_n\}$.

An **isomorphism** is a morphism with a two–sided inverse.

Our basic correspondence is

$$\{\text{Affine varieties over } k\}/\text{up to isomorphism}$$

$$\leftrightarrow$$

$$\{\text{finitely generated } k\text{–algebras } A \text{ without nilpotent elements}\}$$

A finitely generated k–algebra is just a quotient of a polynomial ring in finitely many variables. A nilpotent element is such that some power of it is zero. For example, in $k[x]/(x^2)$, the element x is nilpotent.

How does this correspondence work? Given a variety V (representing an isomorphism class), we write $V = \mathbb{V}(I)$ for $I \subset k[x_1, \ldots, x_n]$ a radical ideal[1], and map $V \mapsto k[x_1, \ldots, x_n]/I$.

For the reverse, if A is a finitely generated nilpotent free algebra, then $A \cong k[y_1, \ldots, y_m]/J$ where we can choose J to be radical (exercise: why?).

We have to check that this is independent of our choice on both sides (exercise: think through this, it should be clear).

Definition 1.3. The algebra associated to V is classically denoted $k[V]$ and called the **coordinate ring of** V.

We have the compatibility of morphisms with our basic correspondence: there is a bijection between

$$\text{Morphisms}(V, W) \leftrightarrow \text{Ring homomorphisms}_k(k[W], k[V])$$

(here RingHom_k means that our homomorphisms preserve k).

We can now make our set into a topological space:

Definition 1.4. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The **Zariski topology** on V is defined such that the closed sets are $\mathbb{V}(S)$, where $S \subset k[V]$.

If $V \cong W$, then the Zariski topological spaces are homeomorphic as varieties (exercise).

Theorem 1.1 (Nullstellensatz)**.** Fix V a variety and let $k[V]$ be its coordinate ring. Given $p \in V$, we can produce a homomorphism $\text{ev}_p : k[V] \to k$ by sending $f \mapsto f(p)$. Note that ev_p is surjective (since we have constant functions), hence $\ker(\text{ev}_p) = m_p$ is a maximal ideal, giving us a map

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}.$$

Nullstellensatz says that this is actually a bijection. For the converse map, given $m \subset k[V]$, we get a quotient $k[V] \to k[V]/m = k$ (Nullstellensatz says this extension is finite, hence must be k). So using/choosing a representation for V in $k[x_1, \ldots, x_n]$ gives a surjective homomorphism onto k and specifies a bunch of points.

[1] A radical ideal is an ideal equal to its radical.

1.2 Limitations of classical algebraic geometry

Question. What is an abstract variety, i.e. "some "space" X such that locally as a cover $\{U_i\}$, each U_i is an affine variety, compatible with overlaps".

Example 1.1 (non–algebraically closed fields)**.** Take $I = (x^2+y^2+1) \subset \mathbb{R}[x, y]$. Then $\mathbb{V}(I) = \varnothing \subset \mathbb{R}^2$, but I is prime, so radical, so nullstellensatz fails.

Question. On what topological space is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ "naturally" the set of functions? (or $\mathbb{Z}$, or $\mathbb{Z}[x]$).

Example 1.2 (Why restrict to radical ideals?)**.** Take $C = \mathbb{V}(y - x^2) \subset \mathbb{A}^2_k$ and $D = \mathbb{V}(x, y)$, so $C \cap D = \mathbb{V}(y, y - x^2) = \mathbb{V}(x, y) = \{(0, 0)\}$. This is a single point, but if $D_\delta = \mathbb{V}(y + \delta)$ for some $\delta \in k$, then $C \cap D_\delta = \{\pm\sqrt{\delta}\}$, which is 2 points for all $\delta \neq 0$. In other words, intersections of varieties don't want to be varieties.