

# Part III - Modular Forms

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# 1 Introduction

06 Oct 2022,  
Lecture 1

**Definition 1.1.** We define the following groups:

$$\begin{aligned}\mathfrak{h} &= \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\} \\ GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\ \Gamma(1) &= SL_2(\mathbb{Z}) = \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.\end{aligned}$$

Note that  $\Gamma(1)$  is a subgroup of  $GL_2(\mathbb{R})^+$ .

**Lemma 1.1.**  $GL_2(\mathbb{R})^+$  acts transitively on  $\mathfrak{h}$  by Möbius transformations.

*Proof.* Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ ,  $\tau \in \mathfrak{h}$ . Then

$$\operatorname{Im}(g\tau) = \frac{1}{2i} \left( \frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\det(g)\operatorname{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so  $g\tau \in \mathfrak{h}$ . This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in  $\mathfrak{h}$  is conjugate to  $i$ . □

**Definition 1.2.** If  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$  and  $\tau \in \mathfrak{h}$ , then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If  $k \in \mathbb{Z}$  and  $f : \mathfrak{h} \rightarrow \mathbb{C}$ , then

$$f|_k[g] : \mathfrak{h} \rightarrow \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight  $k$  action of  $g$  on  $f$** .

**Lemma 1.2.** This is a right action of  $GL_2(\mathbb{R})^+$ : if  $g, h \in GL_2(\mathbb{R})^+$ , then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

*Proof.* We compute

$$\begin{aligned} (f|_k[g])|_k[h](\tau) &= \det(h)^{k-1} f|_k[g](h\tau) j(h, \tau)^{-k} = \\ \det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g, h\tau)^{-k} j(h, \tau)^{-k} &\stackrel{?}{=} \\ \det(gh)^{k-1} f(gh\tau) j(gh, \tau)^{-k} &= f|_k[gh](\tau). \end{aligned}$$

Hence we need to check that  $j(gh, \tau) = j(gh, \tau)j(h, \tau)$ . Note that if  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , then

$$g \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh, \tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g \left( j(h, \tau) \begin{pmatrix} h\tau \\ 1 \end{pmatrix} \right) = j(h, \tau) j(g, h\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof.  $\square$

**Formulae.** For  $g \in GL_2(\mathbb{R})^+$ ,  $\tau \in \mathfrak{h}$ , we have

$$\operatorname{Im}(g\tau) = \det(g) \frac{\operatorname{Im}(\tau)}{|j(g, \tau)|^2} \text{ and } j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

**Definition 1.3.** Let  $k \in \mathbb{Z}$  and  $\Gamma \leq \Gamma(1)$  of finite index<sup>1</sup>. A **weakly modular function of weight  $k$  and level  $\Gamma$**  is a meromorphic function  $f : \mathfrak{h} \rightarrow \mathbb{C}$  which is invariant under the weight  $k$  action of  $\Gamma$ , i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in  $\mathfrak{h}$  and at  $\infty$ .

It is a fact that modular forms of fixed weight and level live in finite-dimensional  $\mathbb{C}$ -vector spaces called  $M_k(\Gamma)$ . These form the main objects of study in this course.

**Motivation.** Why study modular forms?

- (1) They are related to the theory of elliptic functions. Let  $E/\mathbb{C}$  be an elliptic curve and  $\omega$  a holomorphic non-zero 1-form. Then there exists a unique lattice<sup>2</sup>  $\Lambda \in \mathbb{C}$  and isomorphism  $\phi : \mathbb{C}/\Lambda \rightarrow E$  such that  $\phi^*(\omega) = dz$ . Then

<sup>1</sup>In other words,  $\Gamma$  is a (finite index) subgroup of  $\Gamma(1)$ .

<sup>2</sup>i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over  $\mathbb{R}$ .

$E$  is isomorphic to the elliptic curve  $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$  where if  $k \in \mathbb{Z}$ , then  $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$ . This converges absolutely for  $k > 2$ .

If  $\tau \in \mathfrak{h}$ , then  $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$  is a lattice and  $G_k(\tau) = G_k(\Lambda_\tau)$ . This is a modular form of weight  $k$  and level  $\Gamma(1)$ , called an Eisenstein series.

$\mathfrak{h}/SL_2(\mathbb{Z})$  can be identified with the set of (isomorphism classes of) elliptic curves over  $\mathbb{C}$ .

- (2) Modular forms  $f$  have Fourier expansions  $\sum_{n \in \mathbb{Z}} a_n g^n$ ,  $a_n \in \mathbb{C}$  and they often serve as a generating functions for arithmetically interesting sequences  $a_n$ .

For example, take  $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$ . If  $k \in 2\mathbb{N}$ , then  $\theta^k$  is a modular form with  $q$ -expansion  $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$ , where  $r_k(n)$  is the number of ways of writing  $n$  as a sum of  $k$  squares, i.e.  $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$ . By expressing  $\theta^k$  in terms of other modular forms, we can prove formulae such as  $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$ .

- (3) The Riemann zeta function  $\zeta(s)$  is an important object of study. Its pleasant features include:

- The Euler product  $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ .
- It has a meromorphic continuation to  $\mathbb{C}$  and has a functional equation relating  $\zeta(s)$  and  $\zeta(1-s)$ .

A Dirichlet series  $\sum_{n \geq 1} a_n n^{-s}$  which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an  $L$ -function. Modular forms can be used to construct interesting examples of  $L$ -functions. In practice, we take  $M_k(\Gamma)$  and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

- (4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves  $E/\mathbb{C}$  up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and  $L$ -functions.

**Homework.** There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

## 2 Modular Forms on $\Gamma(1)$

09 Oct 2022,  
Lecture 2

**Reminder.** A **meromorphic** function in an open subset  $U \subset \mathbb{C}$  is a closed subset  $A \subset U$  and a holomorphic function  $f : U \setminus A \rightarrow \mathbb{C}$  such that  $\forall a \in A$ ,  $\exists \delta > 0$  such that  $D^*(a, \delta) \subset U \setminus A$  and  $\exists n \geq 0$  such that  $(z - a)^n f(z)$  extends to a holomorphic function in  $D(a, \delta)$ .

$f$  then has a Laurent expansion  $\sum_{m \in \mathbb{Z}} a_m (z - a)^m$  valid on  $D^*(a, \delta)$ .

**Lemma 2.1.** Let  $f$  be a weakly modular function of weight  $k$  and level  $\Gamma(1)$ . Then there exists a meromorphic function  $\tilde{f}$  in  $D^*(0, 1)$  (the "q-disk") such that

$$f(\tau) = \tilde{f}(e^{2\pi i \tau}).$$

*Proof.*  $f$  is meromorphic in  $\mathfrak{h}$  by assumption. Take  $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$ . Then  $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$ , as  $f$  is invariant under the weight  $k$  action of  $\gamma$ . But also  $f(\gamma\tau) = f(\tau + 1)$ , so  $f$  is periodic.

Now map a strip of  $\mathfrak{h}$  of width 1 to  $D^*(0, 1)$  by  $\tau \mapsto e^{2\pi i \tau}$ . Let  $a \in D^*(0, 1)$  and  $\delta > 0$  be such that  $D(a, \delta) \subset D^*(0, 1)$ . Define  $\tilde{f}$  on  $D(a, \delta)$  by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i} \log q\right),$$

for any branch of  $\log$  defined in  $D(a, \delta)$ . This is meromorphic and independent of the choice of the branch of  $\log$ , as  $f$  is periodic with period 1. This defines  $\tilde{f}$  in  $D^*(0, 1)$ . Finally,  $\tilde{f}$  is unique since  $\tau \mapsto e^{2\pi i \tau}$  is surjective.  $\square$

If  $\tilde{f}$  extends to a meromorphic function<sup>3</sup> in  $D(0, 1)$ , then  $\exists \delta > 0$  such that  $\tilde{f}$  has a Laurent expansion  $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$  valid in  $D^*(0, \delta)$ .

In the region  $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$ , we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where  $q = e^{2\pi i \tau}$ . This is called the **q-expansion** of the weakly modular function  $f$ .

**Definition 2.1.** Let  $f$  be a weakly modular function of weight  $k$  and level  $\Gamma(1)$ . We say that  $f$  is **meromorphic at  $\infty$**  if  $\tilde{f}$  extends to a meromorphic function in  $D(0, 1)$ .

We say  $f$  is **holomorphic at  $\infty$**  if  $\tilde{f}$  is meromorphic at  $\infty$  and has a

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<sup>3</sup>This might not be the case if the set of poles has a limit inside the disk.

removable singularity at  $q = 0$ . In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau).$$

We say  $f$  **vanishes at  $\infty$**  if  $f$  is holomorphic at  $\infty$  and  $f(\infty) = 0$ .

**Definition 2.2.** A **modular function** (of weight  $k$  and level  $\Gamma(1)$ ) is a weakly modular function (of weight  $k$  and level  $\Gamma(1)$ ) which is meromorphic at  $\infty$ .

A **modular form** is a weakly modular function which is holomorphic in  $\mathfrak{h}$  and holomorphic at  $\infty$ .

A **cuspidal modular form** is a modular form that vanishes at  $\infty$ .

**Remark.** We let  $M_k(\Gamma(1))$  denote the set of modular forms of weight  $k$  and level  $\Gamma(1)$ . We write  $S_k(\Gamma(1))$  for the set of cuspidal modular forms of weight  $k$ , level  $\Gamma(1)$ . Note  $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$ . These are  $\mathbb{C}$ -vector spaces. If  $k$  is odd, then these both only contain the zero function, since taking  $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$  gives  $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$ .

We now consider even weights only. If  $k \in \mathbb{Z}$  is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k},$$

where  $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$  for any  $\tau \in \mathfrak{h}$ .

If  $\gamma \in \Gamma(1)$ , then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau)j(\gamma, \tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k}j(\gamma, \tau)^{-k},$$

but  $\Lambda_{\gamma\tau} = \mathbb{Z} \frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1} (\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)) = (c\tau+d)^{-1} \Lambda_\tau$ .  
Hence

$$\begin{aligned} G_k|_k[g](\tau) &= \sum_{\lambda \in (c\tau+d)^{-1} \Lambda_\tau \setminus 0} \lambda^{-k} (c\tau+d)^{-k} \\ &= \sum_{\lambda \in \Lambda_\tau \setminus 0} ((c\tau+d)^{-1} \lambda)^{-k} (c\tau+d)^{-k} = G_k(\tau). \end{aligned}$$

This is justified only when the series defining  $G_k(\tau)$  converges absolutely. Hence:

**Proposition 2.2.** Let  $k > 2$  be an even integer. Then  $G_k(\tau)$  converges absolutely and defines a modular form of weight  $k$  and level  $\Gamma(1)$  which has

$G_k(\infty) = 2\zeta(k)$ .  $G_k$  is the **weight  $k$  Eisenstein series**.

We will later see that  $M_2(\Gamma(1)) = 0$ .

*Proof.* We want to show absolute and locally uniform convergence in  $\mathfrak{h}$ . This will show that  $G_k$  is holomorphic by complex analysis. Let  $A \geq 2$  and define  $\Omega_A = \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq \frac{1}{A}, \text{Re}(\tau) \in [-A, A]\}$ . We show uniform convergence in  $\Omega_A$ . If  $\tau \in \Omega_A, x \in \mathbb{R}$ , then  $|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A. \end{cases}$  Hence

$$|\tau + x| \stackrel{(\dagger)}{\geq} \sup \left( \frac{1}{A}, \frac{|x|}{2A^2} \right) \geq \sup \left( \frac{1}{2A^2}, \frac{|x|}{2A^2} \right) = \frac{1}{2A^2} \sup(1, |x|).$$

( $\dagger$ ) follows by drawing a diagram with the lines  $y = \frac{1}{A}$  and  $y = \frac{x}{2A^2}$  and marking the point  $(2A, \frac{1}{A})$  on it, then noticing that our supremum always lies above the supremum of these two lines. If  $(m, n) \in \mathbb{Z}^2, m \neq 0$ , then

$$|m\tau + n| = |m| \left| \tau + \frac{n}{m} \right| \geq |m| \frac{1}{2A^2} \sup \left( 1, \left| \frac{n}{m} \right| \right) = \frac{1}{2A^2} \sup(|m|, |n|).$$

This is also valid when  $m = 0$  by inspection. If  $\tau \in \Omega_A$ , then

$$\begin{aligned} & \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} |m\tau + n|^{-k} \\ & \leq \left( \frac{1}{2A^2} \right)^{-k} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \sup(|m|, |n|)^{-k} \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} \cdot |\{(m, n) \in \mathbb{Z}^2 \mid \sup(|m|, |n|) = d\}| \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} 8d = 8(2A^2)^k \sum_{d \in \mathbb{N}} d^{1-k} \\ & < \infty \end{aligned}$$

whenever  $k - 1 > 1$ , i.e.  $k > 2$ . This shows absolute convergence, and uniform convergence in  $\Omega_A$  by the Weierstrass M-test<sup>4</sup>. Hence  $G_k$  is holomorphic in  $\mathfrak{h}$  and invariant under the weight  $k$  action of  $\Gamma(1)$ . It remains to show that  $G_k$  is holomorphic at  $\infty$  with  $G_k(\infty) = 2\zeta(k)$ . For this, it suffices to check that

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = 2\zeta(k).$$

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<sup>4</sup>If we have a sequence of functions  $f_n : \Omega \rightarrow \mathbb{C}$  and values  $M_n > 0$  with  $|f_n(x)| < M_n$  and  $\sum M_n < \infty$ , then  $\sum f_n$  converges absolutely and uniformly on  $\Omega$ . Here, replace  $n$  with  $d$  and sum  $d$  over  $\sum_{(m,n) \in \mathbb{Z}^2 \setminus 0, \sup(|m|, |n|) = d} |m\tau + n|^{-k}$ .

This follows from uniform convergence in  $\Omega_A$ : we get

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \lim_{\text{Im}(\tau) \rightarrow \infty} (m\tau + n)^{-k} = \sum_{n \in \mathbb{Z} \setminus 0} n^{-k} = 2 \sum_{n \geq 1} n^{-k} = 2\zeta(k).$$

□

11 Oct 2022,  
Lecture 3

**Recap.** We defined what it means for a function  $f : \mathfrak{h} \rightarrow \mathbb{C}$  to be a modular form of weight  $k$  and level  $\Gamma(1)$ .  $M_k(\Gamma(1))$  is the  $\mathbb{C}$ -vector space of such forms. If  $f \in M_k(\Gamma(1))$ , then there exists a holomorphic  $\tilde{f} : D(0, 1) \rightarrow \mathbb{C}$  (here we call  $D(0, 1)$  the  $q$ -disk) such that  $\forall \tau \in \mathfrak{h}$ ,  $f(\tau) = \tilde{f}(e^{2\pi i \tau})$ . The Taylor expansion of  $\tilde{f}$  gives the  $q$ -expansion

$$f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}.$$

We have  $f(\infty) = \tilde{f}(0) = a_0$ . If  $k > 2$  is even, then  $G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k}$  converges absolutely and defines an element of  $M_k(\Gamma(1))$  with  $G_k(\infty) = 2\zeta(k)$ .

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n.$$

We will soon show that we have  $a_n \in \mathbb{Q} \forall n \geq 1$ .

We can construct more modular forms: if  $f \in M_k(\Gamma(1))$  and  $g \in M_l(\Gamma(1))$ , then  $fg \in M_{k+l}(\Gamma(1))$ . To check this is a modular form, we need to check that:

- $fg$  is holomorphic, which is true as  $f, g$  are holomorphic.
- $fg$  is invariant under the weight  $k + l$  action of  $\Gamma(1)$ , which is true as  $f, g$  are invariant under the weight  $k$  and  $l$  actions of  $\Gamma(1)$  – this is just a computation.
- $fg$  is holomorphic at  $\infty$ . This is true as the  $q$ -expansions multiply, so since  $f, g$  have no negative terms, the same is true for  $fg$ .

Hence we get e.g.  $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$  and  $\frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$  (i.e. it is cuspidal since zero at infinity). This difference is Ramanujan's  $\Delta$ -function. We will show it is nonzero later.

We now want to show that  $M_k(\Gamma(1))$  is finite-dimensional. We first study  $\Gamma(1)/\mathfrak{h}$ . For this, introduce a fundamental set  $\mathfrak{f}' \subset \mathfrak{h}$  for the  $\Gamma(1)$ -action. We define<sup>5</sup> a fundamental set to be a set that intersects each  $\Gamma(1)$ -orbit in exactly

<sup>5</sup>Definitions in literature may vary, so we omit a formal definition.



one element. Define

$$\mathfrak{f} = \left\{ \tau \in \mathfrak{h} \mid \operatorname{Re}(\tau) \in \left[ -\frac{1}{2}, \frac{1}{2} \right], |\tau| \geq 1 \right\}.$$

$$\mathfrak{f}' = \left\{ \tau \in \mathfrak{f} \mid \operatorname{Re}(\tau) \in \left[ -\frac{1}{2}, \frac{1}{2} \right), |\tau| = 1 \implies \operatorname{Re}(\tau) \in \left[ -\frac{1}{2}, 0 \right] \right\}.$$

Introduce  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  in  $\Gamma(1)$ . We observe that every element of  $\mathfrak{f}$  is conjugate under  $S$  or  $T^{-1}$  to an element of  $\mathfrak{f}'$ , which is true since  $T(\tau) = \tau + 1$  and  $S(\tau) = -\frac{1}{\tau}$ .



**Proposition 2.3.** Let  $G = \Gamma(1)/\{\pm I\}$ . Then

- (i)  $\forall \tau \in \mathfrak{h}, \tau$  is  $\Gamma(1)$ -conjugate to an element of  $\mathfrak{f}'$ .
- (ii) If  $\tau, \tau' \in \mathfrak{f}'$  are  $\Gamma(1)$ -conjugate, then  $\tau = \tau'$ .
- (iii) If  $\tau \in \mathfrak{f}'$ , then  $\operatorname{Stab}_G(\tau)$  is trivial, except in the two cases  $\operatorname{Stab}_G(i) = \langle S \rangle$  and  $\operatorname{Stab}_G(\rho) = \langle ST \rangle$ , where  $\rho = e^{2\pi i/3}$ .
- (iv)  $\Gamma(1)$  is generated by  $S$  and  $T$ .

*Proof.* Let  $H$  be the subgroup of  $G$  generated by  $S$  and  $T$ .

**Claim.** Every  $\tau \in \mathfrak{h}$  is  $H$ -conjugate to an element of  $\mathfrak{f}'$ .

*Proof.* By our above observation and since  $S, T \in H$ , it suffices to prove that every  $\tau \in \mathfrak{h}$  is  $H$ -conjugate to  $\mathfrak{f}$ . Take  $\tau \in \mathfrak{h}$ . Recall that if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ , then  $\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}$ .

In particular,  $\forall R \geq 0$ , the intersection  $H\tau \cap \{\text{Im}(\tau') > R\}$  is finite, since  $\text{Im}(\gamma\tau) > R \iff |c\tau + d|^2 < \frac{\text{Im}(\tau)}{R}$ , but  $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$  is a lattice, so the set  $\{(c, d) \in \mathbb{Z}^2 \mid |c\tau + d| < R'\}$  is finite.

So there exists  $h \in H$  such that  $\text{Im}(h\tau) \geq \text{Im}(h'\tau) \forall h' \in H$ . After replacing  $\tau$  by  $h\tau$ , we can assume  $\text{Im}(\tau) \geq \text{Im}(h\tau) \forall h \in H$ . Since acting by  $T$  does not change  $\text{Im}(\tau)$ , we can also assume  $\text{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$ . We have  $\text{Im}(\tau) \geq \text{Im}(S\tau) = \frac{\text{Im}(\tau)}{|\tau|^2} \implies |\tau| \geq 1$ , proving the claim and (i).  $\square$

Now take  $\tau, \tau' \in \mathfrak{f}'$  and suppose  $\gamma\tau = \tau'$  for some  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ . We want to show that either  $\gamma = \pm I$  or  $\tau = i, \rho$ .

WLOG assume  $\text{Im}(\tau') = \text{Im}(\gamma\tau) \geq \text{Im}(\tau)$ , i.e.  $\text{Im}(\gamma\tau) = \frac{\text{Im}(\tau)}{|c\tau + d|^2} \geq \text{Im}(\tau)$ , so  $|c\tau + d| \leq 1$ . However, if  $\tau \in \mathfrak{f}'$ , then  $\text{Im}(\tau) \geq \frac{\sqrt{3}}{2}$  with equality if and only if  $\tau = \rho$ . Hence  $|c\tau + d| \geq |c|\text{Im}(\tau) \geq |c|\frac{\sqrt{3}}{2} \implies |c| \leq \frac{2}{\sqrt{3}} \implies |c| = 0, 1 \implies c = 0$  or  $c = \pm 1$ .

- If  $c = 0$ , then  $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ , so  $ad = 1 \implies a = d = \pm 1$ , so  $\gamma = \pm T^m$  for  $m \in \mathbb{Z}$ . However,  $T$  acts on  $\mathfrak{f}'$  by shifting the real part, so it can only stay in  $\mathfrak{f}'$  if  $m = 0$  (as  $\text{Re}(\mathfrak{f}') \in [-\frac{1}{2}, \frac{1}{2}]$ ), so  $\gamma = \pm I$  and  $\tau' = \tau$ .
- If  $c = 1$ , then  $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$  and  $|\tau + d| \leq 1$ . By drawing another picture, we see that the only circles centered at integers of radius 1 which intersect  $\mathfrak{f}'$  are centered at  $-d = 0, -d = -1$ . Hence either  $d = 0$ , whence  $|\tau| = 1$ , or  $d = 1$ , whence  $\tau = \rho$ .

– If  $c = 1, d = 0, |\tau| = 1$ , then  $\gamma = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$  since the determinant must be 1. Then  $\gamma\tau = \frac{a\tau - 1}{\tau} = a - \frac{1}{\tau} = a - \bar{\tau}$ , so  $\text{Re}(\gamma\tau) = a - \text{Re}(\tau) \in \text{Re}(\mathfrak{f}' \cap \{|\tau| = 1\}) = [-\frac{1}{2}, 0]$ . However, we also have  $\text{Re}(\gamma\tau) \in a - [-\frac{1}{2}, 0] = a + [0, \frac{1}{2}]$ .

The intersection  $[-\frac{1}{2}, 0] \cap (a + [0, \frac{1}{2}])$  can be nonempty only if either  $a = 0$ , whence  $\text{Re}(\gamma\tau) = \text{Re}(\tau) = 0$ , so  $\tau = \gamma\tau = i$ , or  $a = -1$ , whence  $\text{Re}(\tau) = \text{Re}(\gamma\tau) = -\frac{1}{2}$ , so  $\tau = \gamma\tau = \rho$ .

If  $a = 0$ , then  $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$ , which stabilizes  $i$ , and  $\langle -S \rangle = \langle S \rangle$ .

If  $a = -1$ , then  $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$ , which stabilizes  $\rho$ , and  $(ST)^3 = I$ , so  $\langle (ST)^2 \rangle = \langle ST \rangle$ .

- If  $c = 1, d = 1, \tau = \rho$ , then  $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ , so  $\rho = \gamma\rho = \frac{a\rho+b}{\rho+1}$ . We have  $\rho^2 + \rho + 1 = 0$ , so  $\rho^2 + \rho = -1$ , so  $a\rho + b = \rho^2 + \rho = -1$ . But  $a, b \in \mathbb{Z}$  and  $1, \rho$  are linearly independent over  $\mathbb{R}$ , so  $a = 0, b = -1$ , so  $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -ST$ , which stabilizes  $\rho$ .

- If  $c = -1$ , we can reduce this to the case  $c = 1$  by replacing  $\gamma$  with  $-\gamma$ .

We have now shown the first three parts of the proposition. It remains to show the last part, i.e.  $\Gamma(1) = \langle S, T \rangle$ . Since  $S^2 = -I$ , it is enough to show that  $H = G$ . Choose  $\tau \in \text{Int}(f)$ , so  $\text{Stab}_G(\tau) = \{I\}$ . Let  $g \in G$ . By our claim proving (i),  $\exists h \in H$  such that  $hg\tau \in \mathfrak{f}'$ . We must therefore have  $hg\tau = \tau$ , hence  $hg \in \text{Stab}_G(\tau) = \{I\}$ , so  $g = h^{-1} \in H$ .  $\square$

**Notation.** We write  $e_\tau = |\text{Stab}_G(\tau)|$ .

Let  $f$  be a nonzero modular function of weight  $k$ , level  $\Gamma(1)$ . If  $\tau \in \mathfrak{h}$ , then  $v_\tau(f)$  is the order of  $f$  at  $\tau$  (the unique  $n \in \mathbb{Z}$  such that  $f(z) = (z - \tau)^n g(z)$  for some meromorphic  $g$  that is holomorphic and non-vanishing at  $\tau$ ). We define  $v_\infty(f)$  to be the order of  $f$  at infinity, i.e.  $v_\infty(f) = v_0(\tilde{f})$  for  $\tilde{f}$  the meromorphic function in  $D(0, 1)$  with  $f(\tau) = \tilde{f}(e^{2\pi i \tau})$ .

**Proposition 2.4.** Let  $f$  be a nonzero modular function of weight  $k$ , level  $\Gamma(1)$ . Then

$$\sum_{\tau \in \Gamma(1) \backslash \mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.$$

*Proof.* We first check that the sum is well-defined:

- If  $\tau \in \mathfrak{h}$ , then  $e_\tau, v_\tau(f)$  only depend on the  $\Gamma(1)$ -orbit of  $\tau$ . This is because if  $\gamma \in \Gamma(1)$  and  $\tau \in \mathfrak{h}$ , then  $\text{Stab}_{\Gamma(1)}(\tau)$  and  $\text{Stab}_{\Gamma(1)}(\gamma\tau)$  are conjugate subgroups of  $\Gamma(1)$ , so  $e_\tau = e_{\gamma\tau}$ . On the other hand,  $f(\gamma\tau) = f(\tau)j(\gamma, \tau)^k$  and  $j(\gamma, \tau)$  is holomorphic and non-vanishing on  $\mathfrak{h}$ , so  $v_{\gamma\tau}(f) = v_\tau(f)$ .
- The sum only has a finite number of nonzero terms, since if  $f$  is a modular function and  $\tilde{f}$  is a meromorphic function on  $D(0, 1)$ , then  $\exists \delta > 0$  such that  $\tilde{f}$  is holomorphic and non-vanishing in  $D^*(0, \delta)$ . Thus  $\exists R > 0$  such that  $f$  is holomorphic and non-vanishing in  $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > R\}$ . Hence to show the sum is finite, it suffices to show that  $f$  only has a finite number of zeroes and poles in  $\mathfrak{f}$  (as  $\mathfrak{f}$  intersects every  $\Gamma(1)$ -orbit), for which it suffices to show that  $f$  has a finite number of zeroes and poles in  $\mathfrak{f} \cap \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \leq R\}$ , which is true as the set is compact (closed and bounded) and the zeroes and poles of  $f$  are discrete.

13 Oct 2022,  
Lecture 4

To prove the identity, we use contour integration. Setup: if  $U \subset \mathbb{C}$  is an open subset,  $f : U \rightarrow \mathbb{C}$  is holomorphic and  $\gamma : [0, 1] \rightarrow U$  is a path, then

$$\int_{\gamma} f(z) dz = \int_{t=0}^1 f(\gamma(t)) \gamma'(t) dt.$$

We have the pullback formula: if  $u : U \rightarrow V$  is a holomorphic map between open subsets of  $\mathbb{C}$ ,  $g : V \rightarrow \mathbb{C}$  is holomorphic and  $\gamma$  is a path in  $U$ , then

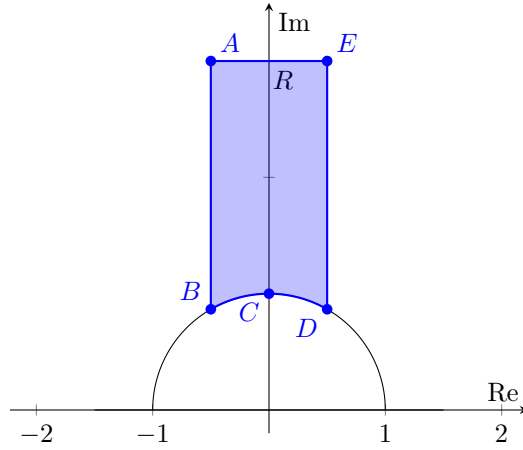
$$\int_{u \circ \gamma} g(z) dz = \int_{\gamma} u^*(g(z) dz) = \int_{\gamma} g(u(z)) u'(z) dz.$$

A particularly nice case: if  $g(z) = h'(z)/h(z)$ , then  $g(z) dz = d \log h$ , so  $\int_{u \circ \gamma} d \log h = \int_{\gamma} u^*(d \log h) = \int_{\gamma} d(\log h \circ u) = \int_{\gamma} \frac{(h \circ u)'(z)}{(h \circ u)(z)} dz$ .

We also have (Cauchy's) argument principle: if  $U \subset \mathbb{C}$  is a simply connected open subset,  $\gamma \subset U$  is a simple positively oriented closed path and  $g$  is a meromorphic function in  $U$  with no zeroes or poles on  $\gamma$ , then

$$\frac{1}{2\pi i} \oint_{\gamma} d \log g = \frac{1}{2\pi i} \oint_{\gamma} \frac{g'(z)}{g(z)} dz = \sum_{a \in \text{Int}(\gamma)} v_a(g).$$

We now apply this to our modular function  $f$ . Choose  $R > 0$  such that  $f$  has no zeroes or poles in  $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq R\}$ . We consider  $\frac{1}{2\pi i} \oint_{\gamma} d \log f$ , where  $\gamma$  is the contour  $ABCDE$ .



By choice of  $R$ , there are no zeroes or poles of  $f$  on  $AE$ . We first consider the case where  $f$  has no zeroes or poles at all on  $\gamma$ . Then the argument principle

gives

$$\frac{1}{2\pi i} \oint_{\gamma} d\log f = \frac{1}{2\pi i} \int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d\log f = \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_{\tau}} v_{\tau}(f)$$

(as  $v_{\tau}(f) \neq 0$ ,  $e_{\tau} = 1$  under our assumptions).

Apply the pullback formula with  $u(\tau) = \tau + 1$ . Then  $u(AB) = ED$ ,  $f \circ u = f$ , so

$$\int_{u(AB)} d\log f = \int_{AB} d\log f \circ u = \int_{AB} d\log f = \int_{ED} d\log f = - \int_{DE} d\log f.$$

Hence  $\int_{AB} + \int_{DE} d\log f = 0$ .

Now take  $q = e^{2\pi i \tau}$ , so  $f = \tilde{f} \circ q$  and  $q(AE)$  is a positively oriented circle around 0 in  $D(0, 1)$ . So

$$\frac{1}{2\pi i} \int_{q(AE)} d\log \tilde{f} = v_{\infty}(f) = \frac{1}{2\pi i} \int_{AE} d\log \tilde{f} \circ q = \frac{1}{2\pi i} \int_{AE} d\log f.$$

Now take  $v(\tau) = S(\tau) = -\frac{1}{\tau}$ . Then  $v(BC) = DC$  and we know  $f|_k[S](\tau) = f(-\frac{1}{\tau})\tau^{-k} = f(\tau)$ , so  $f \circ v = f(\tau)\tau^k$ . Hence

$$\begin{aligned} \int_{DC} d\log f &= \int_{v(BC)} d\log f = \int_{BC} d\log(f \circ v) = \int_{BC} d\log(f(\tau)\tau^k) \\ &= \int_{BC} d\log f + k d\log \tau = \int_{BC} d\log f + k(\log C - \log B) \end{aligned}$$

where here  $\log$  is any branch of the logarithm defined on  $BC$ . But  $B = \rho$ ,  $C = i$ , so  $\log B = i\frac{2\pi}{3}$  and  $\log C = i\frac{\pi}{2}$ . Hence

$$\int_{CD} d\log f = - \int_{DC} d\log f + k \left( \frac{2\pi i}{3} - \frac{2\pi i}{4} \right),$$

giving

$$\int_{BC} + \int_{CD} d\log f = 2\pi i k \frac{1}{12}.$$

We have

$$\begin{aligned} \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) &= \frac{1}{2\pi i} \left( \int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d \log f \right) \\ &= \frac{1}{2\pi i} \left( 0 + \frac{k}{12} + 0 - v_\infty(f) \right) \\ &\implies \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}. \end{aligned}$$

This finishes the proof in the case where there are no zeroes or poles. If there are zeroes or poles on  $\gamma$ , we need to modify the contour. For example, if there's a zero or a pole at a point  $P$  on  $AB$ , then consider the contour  $\gamma'$ , which is just  $\gamma$  but with a small semicircle around our (discrete) pole, which satisfies the property that  $f$  has no zeroes or poles on  $\gamma'$ . The trickiest case is when there is a zero or pole at  $B = \rho$  or  $C = i$ . This is Q3 on example sheet 1.  $\square$

16 Oct 2022,  
Lecture 5

**Example 2.1.** Take  $k = 4$ ,  $f = E_4 \in M_4(\Gamma(1))$ . Hence  $\forall \tau \in \mathfrak{h}, v_\tau(E_4) \geq 0$  (as it is holomorphic in  $\mathfrak{h}$ ). We know  $E_4(\tau) = 1 + \sum_{n \geq 1} a_n q^n$ , so  $E_4(\infty) \neq 0$  and  $v_\infty(E_4) = 0$ . Hence our formula gives

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v(E_4) = \frac{1}{3} v_\rho(E_4) + \frac{1}{2} v_i(E_4) + \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}, \tau \not\sim \rho, i} v_\tau(E_4) = \frac{1}{3}.$$

So we have  $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{3}$ , where  $a, b, c \in \mathbb{Z}_{\geq 0}$ , which gives the only solution  $a = 1, b = c = 0$ , so  $E_4(\rho) = 0$  and  $E_4(\tau) \neq 0$  if  $\tau \notin \Gamma(1)\rho$ .

If  $k = 6$ ,  $f = E_6$ , then we get

$$\frac{1}{3} v_\rho(E_6) + \frac{1}{2} v_i(E_6) + \sum_{\tau \not\sim \rho, i} v_\tau(E_6) = \frac{6}{12} = \frac{1}{2},$$

so this forces  $v_\rho(E_6) = 0$ ,  $v_i(E_6) = 1$ ,  $v_\tau(E_6) \neq 0$  if  $\tau \not\sim \rho$  and  $\tau \not\sim i$ , so  $E_6(i) = 0$ ,  $E_6(\tau) \neq 0$  if  $\tau \not\sim \rho, i$ .

Recall  $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$ . This is nonzero since  $\Delta(\rho) = \frac{E_4(\rho)^3 - E_6(\rho)^2}{1728} = -\frac{E_6(\rho)^2}{1728} \neq 0$ . We also have  $v_\infty(\Delta) \geq 1$  by construction, so plug in  $\Delta$  to our formula to get

$$\sum_{\tau} \frac{1}{e^\tau} v_\tau(\Delta) + v_\infty(\Delta) = 1,$$

so  $v_\infty(\Delta) = 1$ , so  $\Delta$  has a simple zero at  $\infty$  and  $\Delta$  is nonvanishing in  $\mathfrak{h}$ .

**Theorem 2.5.** Let  $k \in 2\mathbb{Z}$ . Then:

- (1) If  $k < 0$  or  $k = 2$ , then  $M_k(\Gamma(1)) = 0$ ; and  $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$ .
- (2) If  $4 \leq k \leq 10$ , then  $M_k(\Gamma(1)) = \mathbb{C} \cdot E_k$ .
- (3) For any  $k$ , multiplication by  $\Delta$  gives an isomorphism  $M_k(\Gamma(1)) \xrightarrow{\times \Delta} S_{k+12}(\Gamma(1))$ .

*Proof.* (1) Let  $f \in M_k(\Gamma(1))$  be nonzero. Then  $\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$ . Note the LHS is  $\geq 0$ , but for  $k < 0$ , the RHS is  $< 0$ . If  $k = 2$ , then we get the equation  $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{6}$  for  $a, b, c \in \mathbb{Z}_{\geq 0}$ , which has no solutions.

Suppose  $f \in M_0(\Gamma(1)) \setminus \mathbb{C} \cdot 1$ . Then  $f - f(\infty) \cdot 1 \in S_0(\Gamma(1))$  is a nonzero function (here 1 is the constant function 1). Then  $\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot 1) + \underbrace{v_\infty(f - f(\infty) \cdot 1)}_{\geq 1} = 0$ , a contradiction, so  $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$ .

- (2) Let  $4 \leq k \leq 10$  and  $f \in M_k(\Gamma(1))$ . Consider  $f - f(\infty) \cdot E_k \in S_k(\Gamma(1))$ . If this is nonzero, then

$$\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot E_k) + \underbrace{v_\infty(f - f(\infty) \cdot E_k)}_{\geq 1} = \frac{k}{12} < 1,$$

a contradiction. So  $f = f(\infty) \cdot E_k$ .

- (3) Our map  $\times \Delta : M_k(\Gamma(1)) \rightarrow S_{k+12}(\Gamma(1))$  is a well-defined  $\mathbb{C}$ -linear map. It is injective, since if  $\Delta f = 0$ , then  $f = 0$  (as  $\Delta$  is nonvanishing in  $\mathfrak{h}$ ). For surjectivity, if  $f \in S_{k+12}(\Gamma(1))$ , then  $\frac{f}{\Delta}$  is holomorphic in  $\mathfrak{h}$  and invariant under the weight  $k$  action of  $\Gamma(1)$ .

We need to show  $\frac{f}{\Delta}$  is holomorphic at  $\infty$ , as then  $\frac{f}{\Delta} \in M_k(\Gamma(1))$ , so  $f = \frac{f}{\Delta} \Delta \in \text{Im}(\times \Delta)$ . Hence we need  $v_\infty\left(\frac{f}{\Delta}\right) \geq 0$ . But  $v_\infty\left(\frac{f}{\Delta}\right) = \underbrace{v_\infty(f)}_{\geq 1} - \underbrace{v_\infty(\Delta)}_{=1} \geq 0$ , so we're done.

□

**Corollary 2.6.** If  $k \in 2\mathbb{Z}$ ,  $k \geq 0$ , then  $M_k(\Gamma(1))$  is finite-dimensional and

$$\dim_{\mathbb{C}} M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12}. \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}. \end{cases}$$

*Proof.* We proved this for  $0 \leq k \leq 10$ . In general, use induction on  $k$ : we need to show that for  $k \geq 0$ ,  $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = \dim_{\mathbb{C}} M_k(\Gamma(1)) + 1$ .

We know  $E_{k+12} \in M_{k+12}(\Gamma(1))$ , so  $M_{k+12}(\Gamma(1)) = \mathbb{C} E_{k+12} \oplus S_{k+12}(\Gamma(1))$ . But this equals  $\mathbb{C} E_{k+12} \oplus \Delta M_k(\Gamma(1))$ , so  $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$ .

□

**Example 2.2.** We have  $E_4^2 \in M_8(\Gamma(1)) = \mathbb{C}E_8$ . So there is a relation between  $E_4^2$  and  $E_8$  (in this case, one is a scalar multiple of the other), but we have  $E_8(\infty) = 1 = E_4(\infty)^2 \implies E_4^2 = E_8$ .

Similarly,  $E_4E_6 \in M_{10}(\Gamma(1)) = \mathbb{C}E_{10}$ , so we find  $E_4E_6 = E_{10}$ .

**Corollary 2.7.** If  $k \in 2\mathbb{N}$ , then  $M_k(\Gamma(1))$  is spanned as a  $\mathbb{C}$ -vector space by  $\{E_4^a E_6^b \mid a, b \in \mathbb{Z}_{\geq 0}, 4a + 6b = k\}$ . In other words, if  $\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma(1))$ , then  $\mathcal{M}$  is a graded  $\mathbb{C}$ -algebra generated by  $E_4$  and  $E_6$ .

*Proof.* We proved this for  $0 \leq k \leq 10$ . If  $k \geq 12$ , then

$$M_k(\Gamma(1)) = \mathbb{C}E_k \oplus \Delta M_{k-12}(\Gamma(1)) = \mathbb{C}f \oplus \Delta M_{k-12}(\Gamma(1))$$

for any  $f \in M_k(\Gamma(1))$  such that  $f(\infty) \neq 0$  by the same argument. We can always find some  $A, B \in \mathbb{Z}_{\geq 0}$  such that  $4A + 6B = k$ , so  $E_4^A E_6^B \in M_k(\Gamma(1))$  and  $(E_4^A E_6^B)(\infty) \neq 0$ . Now by induction,  $M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$ , so  $\Delta M_{k-12}(\Gamma(1)) = \langle \Delta E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$ . But  $\Delta \in \langle E_4^3, E_6^2 \rangle$ , so

$$\Delta M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$$

and  $E_4^A E_6^B \in \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$ , so  $M_k(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$ .  $\square$

**Theorem 2.8.** Let  $j(\tau) = \frac{E_4(\tau)^3}{\Delta}$ . Then  $j$  is a modular function of weight 0, level  $\Gamma(1)$  which is holomorphic on  $\mathfrak{h}$  and has a simple pole at  $\infty$ . It defines a bijection  $\Gamma(1) \setminus \mathfrak{h} \rightarrow \mathbb{C}$  given by  $\tau \rightarrow j(\tau)$ . Moreover, every modular function of weight 0, level  $\Gamma(1)$  is a rational function of  $j$ .<sup>6</sup>

The interpretation of this is that it is possible to define a Riemann surface structure on  $\Gamma(1) \setminus \mathfrak{h} \sqcup \{\infty\}$  such that we get a compact Riemann surface whose meromorphic functions are exactly the modular functions of weight 0. So the theorem says that this Riemann surface, called  $X(1)$ , is isomorphic to the Riemann sphere, and our formula says that if  $\mathcal{L}$  is an invertible sheaf on a compact Riemann surface and  $S$  is a meromorphic section, then  $\sum_a v_a(S) = \deg(\mathcal{L})$ . This is useful if we are also taking algebraic geometry.

18 Oct 2022,  
Lecture 6

*Proof.* We showed that  $\Delta$  is nonvanishing in  $\mathfrak{h}$  and has a simple zero at  $\infty$ . Hence  $j$  is holomorphic in  $\mathfrak{h}$  and  $v_\infty(j) = 3v_\infty(E_4) - v_\infty(\Delta) = -1$ . Note that if  $\gamma \in \Gamma(1)$ , then  $j|_0[\gamma](\tau) = j(\gamma\tau) = j(\tau)$  since the map is constant on  $\Gamma(1)$ -orbits. To show the map is a bijection, we need to show that  $\forall z \in \mathbb{C}$ , there exists a unique orbit  $\Gamma(1) \cdot \tau$  such that  $j(\tau) = z$ , i.e.  $v_\tau(j - z) > 0$ .

We know

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_\tau} \underbrace{v_\tau(j - z)}_{\geq 0, \text{ as } j - z \text{ is holomorphic in } \mathfrak{h}} = 1,$$

<sup>6</sup>Remember that  $\Gamma(1) \setminus \mathfrak{h}$  is the set of orbits of  $\Gamma(1)$  under  $\mathfrak{h}$ .



(since  $v_\infty(j-z) = -1$  and  $\frac{k}{12} = 0$ ) again giving  $\frac{a}{3} + \frac{b}{2} + c = 1$  for  $a, b, c \in \mathbb{Z}_{\geq 0}$ ,  $a = v_\rho(j-z), b = v_i(j-z), c = \sum_{\tau \neq \rho, i} v_\tau(j-z)$ . This gives the solutions

- $(a, b, c) = (0, 0, 1)$ , so  $j-z$  vanishes at a unique  $\Gamma(1) \cdot \tau$ .
- $(a, b, c) = (0, 2, 0)$ , so  $j-z$  vanishes at  $i$ .
- $(a, b, c) = (3, 0, 0)$ , so  $j-z$  vanishes at  $\rho$ .

Hence our map is bijective. Consider a nonzero modular function  $f$  of weight 0. To get rid of all the poles, we can consider a product  $f \cdot \prod_{i=0}^n (j(\tau) - j(a_i))^{b_i}$  for  $a_i \in \mathfrak{h}$ ,  $b_i \in \mathbb{Z}_{\geq 0}$ , where the  $a_i$  are among the poles of  $f$  in  $\mathfrak{h}$ . Hence to show  $f$  is a rational function of  $j$ , it is enough to consider the case where  $f$  is holomorphic in  $\mathfrak{h}$ . Then there exists  $m \geq 0$  such that  $\Delta^m f$  is holomorphic at  $\infty$ , so  $\Delta^m f$  is holomorphic in  $\mathfrak{h}$  and at  $\infty$ , so  $\Delta^m f \in M_{12m}(\Gamma(1))$ . We showed that  $M_{12m}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = 12m \rangle$ , so  $f$  is a linear combination of functions of the form  $\frac{E_4^a E_6^b}{\Delta^m}$ , where  $4a + 6b = 12m$ .

Hence it is enough to show that  $\frac{E_4^a E_6^b}{\Delta^m}$  is a rational function of  $j$  where  $4a + 6b = 12m$ ,  $a, b \in \mathbb{Z}_{\geq 0}$ . But then  $2a + 3b = 6m$ , which gives  $p, q \in \mathbb{Z}_{\geq 0}$  such that  $a = 3p, b = 2q$ , so  $p + q = m$ . Then

$$\frac{E_4^a E_6^b}{\Delta^m} = \left( \frac{E_4^3}{\Delta} \right)^p \left( \frac{E_6^2}{\Delta} \right)^q = j^p \left( \frac{E_6^2}{\Delta} \right)^q.$$

As  $E_4^3 - E_6^2 = 1728\Delta$ , we get  $j = \frac{E_6^2}{\Delta} + 1728$ . So this is a rational function of  $j$ .  $\square$

**Proposition 2.9.** Let  $k \geq 4$  be an even integer. Then

$$G_k(\tau) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where  $q = e^{2\pi i \tau}$  and  $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ .

*Proof.* We start from the identity

$$\pi \cot(\pi \tau) = \frac{1}{\tau} + \sum_{n \geq 1} \left( \frac{1}{\tau + n} + \frac{1}{\tau - n} \right).$$

This is true for  $\tau \in \mathfrak{h}$  and it is even locally uniformly convergent in  $\mathfrak{h}$ . We can write

$$\pi \cot(\pi \tau) = i\pi \frac{e^{\pi i \tau} + e^{-\pi i \tau}}{e^{\pi i \tau} - e^{-\pi i \tau}} = \pi i \frac{q + 1}{q - 1} = -\pi i (1+q)(1-q)^{-1} = -\pi i \left( 1 + 2 \sum_{n \geq 1} q^n \right).$$

Differentiate term-by-term  $k - 1$  times. The RHS of the bottom expression is

$$-2\pi i \left( \frac{d}{d\tau} \right)^{k-1} \left( \sum_{n \geq 1} q^n \right) = -(2\pi i)^k \sum_{n \geq 1} n^{k-1} q^n,$$

while the RHS of the top expression is

$$(-1)^{k-1} (k-1)! \left( \tau^{-k} + \sum_{n \geq 1} (\tau + n)^{-k} + (\tau - n)^{-k} \right) = (-1)^{k-1} (k-1)! \sum_{n \in \mathbb{Z}} (\tau + n)^{-k}.$$

Rearranging and using the fact that  $k$  is even (to make the sign go away) gives

$$\sum_{n \in \mathbb{Z}} (\tau + n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^n, \tau \in \mathfrak{h}.$$

Then

$$G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k} = 2\zeta(k) + \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus 0, \\ m \neq 0}} (m\tau + n)^{-k} = 2\zeta(k) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-k}.$$

Plug in our identity to get

$$G_k(\tau) = 2\zeta(k) + \sum_{m \geq 1} \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^{mn} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{N \geq 1} \underbrace{\left( \sum_{n|N} n^{k-1} \right)}_{=\sigma_{k-1}(N)} q^N.$$

□

**Corollary 2.10.**  $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n$  has all  $a_n \in \mathbb{Q}$ . Moreover, if  $k = 4$  or  $k = 6$ , then  $a_n \in \mathbb{Z}$ .

*Proof.* We have

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

Hence we need to show that  $\frac{\zeta(k)}{\pi^k}$  is rational. This is on example sheet 1 (when

$k$  is even). One can show that  $\zeta(4) = \frac{\pi^4}{90}$  and  $\zeta(6) = \frac{\pi^6}{945}$ , so

$$\begin{aligned} E_4(\tau) &= 1 + \frac{2^4 \pi^4 \cdot 90}{\pi^4 \cdot 6} \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \\ E_6(\tau) &= 1 - \frac{2^6 \pi^6 \cdot 3^3 \cdot 5 \cdot 7}{\pi^6 \cdot 5!} \sum_{n \geq 1} \sigma_5(n) q^n = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n. \end{aligned}$$

□

**Corollary 2.11.** If  $\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n$  is the  $q$ -expansion of  $\Delta$ , then  $\tau(1) = 1$  and  $\tau(n) \in \mathbb{Z} \forall n \geq 1$ .

*Proof.* Write  $E_4 = 1 + 240U$  and  $E_6 = 1 - 504V$  for  $U, V = q + \dots \in \mathbb{Z}[[q]]$ . Then

$$\begin{aligned} \Delta &= \frac{E_4^3 - E_6^2}{1728} = \frac{(1 + 240U)^3 - (1 - 504V)^2}{1728} \\ &= \frac{3 \cdot 240U + 3 \cdot 240^2 U^2 + 240^3 U^3 + 2 \cdot 504V - 504^2 V^2}{1728} \\ &= \frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} + R, \end{aligned}$$

where we claim  $R \in q^2 \mathbb{Z}[[q]]$ , but for this we just need to check that  $1728 \mid 3 \cdot 240^2, 1728 \mid 240^3, 1728 \mid 504^2$ , which is true.

We need to check that

$$\frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} = \frac{2^4 \cdot 3^2 \cdot 5 \cdot U + 2^4 \cdot 3^2 \cdot 7 \cdot V}{2^6 \cdot 3^3} \in \mathbb{Z}[[q]].$$

But this equals

$$\frac{5U + 7V}{12} = \frac{5(U - V)}{12} + V.$$

Hence we need to check that

$$\frac{5}{12}(\sigma_3(n) - \sigma_5(n)) \in \mathbb{Z} \forall n \geq 1,$$

i.e. we need to check that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod{12} \forall n \geq 1.$$

But this is true as  $d^3 \equiv d^5 \pmod{12} \forall d \in \mathbb{N}$ .

Finally, we compute  $\tau(1) = \frac{3 \cdot 240 + 2 \cdot 504}{1728} = 1$ . □

**Theorem 2.12.** Let  $k \geq 4$  be even and  $N = \dim_{\mathbb{C}} S_k(\Gamma(1))$ . Then there exists a unique basis  $f_0, \dots, f_N$  for  $M_k(\Gamma(1))$  as a  $\mathbb{C}$ -vector space such that

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(a)  $\forall 0 \leq i \leq N$ ,  $f_i = \sum_{n \geq 0} a_n(f_i) q^n$  for  $a_n(f_i) \in \mathbb{Z} \forall n \geq 0$ .

(b) If  $0 \leq i, n \leq N$ , then  $a_n(f_i) = \delta_{in}$ .

So in other words,  $f_i = q^i + O(q^{N+1})$ . This is important because  $M_k(\Gamma(1))$  has a  $\mathbb{Z}$ -structure, i.e. we can realize it as a tensor product  $M_k(\Gamma(1)) = M_k(\Gamma(1), \mathbb{Z}) \oplus \mathbb{C}$ , where  $M_k(\Gamma(1), \mathbb{Z}) = \{f \in M_k(\Gamma(1)) \mid \forall n \geq 0, a_n(f) \in \mathbb{Z}\}$ .

*Proof.* We first construct  $f_0, \dots, f_N \in M_k(\Gamma(1))$  with properties (a) and (b). Write  $k = 12a + d$ , for  $a, d \in \mathbb{Z}_{\geq 0}$  such that  $d = 14$  if  $k \equiv 2 \pmod{12}$ , or  $0 \leq d \leq 10$  if  $d \not\equiv 2 \pmod{12}$ .

Then

$$\left\lfloor \frac{k}{12} \right\rfloor = \begin{cases} a & k \not\equiv 2 \pmod{12} \\ a+1 & k \equiv 2 \pmod{12} \end{cases} \implies \lfloor a \rfloor = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & k \not\equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \pmod{12} \end{cases}.$$

We have  $\dim_{\mathbb{C}} M_k(\Gamma(1)) = N + 1 = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \pmod{12} \end{cases}$  so  $a = N$ ,  $k = 12N + d$ .

Now consider  $A, B \in \mathbb{Z}_{\geq 0}$  such that  $d = 4A + 6B$ . Consider the modular forms

$$g_i = \Delta^i E_4^A E_6^B E_6^{2(N-i)}$$

for  $0 \leq i \leq N$ . Each  $g_i$  has weight  $12i + 4A + 6B + 12(N-i) = 12N + d = k$ , so  $g_i \in M_k(\Gamma(1))$ . As  $E_4, E_6, \Delta$  have  $q$ -expansions in  $\mathbb{Z}[[q]]$ , so does  $g_i$ . The leading term of  $g_i$  is  $q^i$ , so the  $q$ -expansions look like

$$\begin{aligned} g_0 &= 1 + a_1(g_0)q + \dots + a_N(g_0)q^N + O(q^{N+1}) \\ &\vdots \\ g_{N-1} &= 0 + \dots + q_{N-1} + a_N(g_{N-1})q^N + O(q^{N+1}) \\ g_N &= 0 + \dots + 0 + q^N + O(q^{N+1}) \end{aligned}$$

We can now carry out row reduction on the  $g_i$  to obtain  $f_0, \dots, f_N$  satisfying (a) and (b). For uniqueness, consider the linear functionals

$$\begin{aligned} a_0, \dots, a_N : M_k(\Gamma(1)) &\rightarrow \mathbb{C} \\ f &\mapsto a_i(f), \quad f = \sum_{n \geq 0} a_n(f) q^n. \end{aligned}$$

Then  $a_i(f_j) = \delta_{ij}$ , which forces  $a_0, \dots, a_N$  to be linearly independent. Hence they form a basis of the dual vector space  $M_k(\Gamma(1))^*$ . So  $f_0, \dots, f_N$  is the dual basis of  $M_k(\Gamma(1))$ , and they form the unique basis with this property.  $\square$

### 3 Hecke operators

Hecke operators are just symmetries (linear endomorphisms) of spaces of modular forms. They can arise from either representation theory:  $\Gamma(1) \leq GL_2(\mathbb{Q})^+$ , which acts on  $\{f : \mathfrak{h} \rightarrow \mathbb{C}\}$  by  $f \mapsto f|_k[g]$ . But  $M_k(\Gamma(1)) \leq \{f : \mathfrak{h} \rightarrow \mathbb{C}\}^{\Gamma(1)}$ , and a general group theory fact says that under suitable conditions, there's an action by a big class of operators; or from geometry: we can think of modular forms as functions on the set of lattices  $\mathcal{L}$  in  $\mathbb{C}$ . In this course, we will follow the second point of view.

**Recall.** If  $V$  is a finite-dimensional  $\mathbb{R}$ -vector space, then a lattice  $\Lambda$  in  $V$  is a subgroup  $\Lambda \subset V$  which is discrete and cocompact (i.e.  $V/\Lambda$  is compact).

**Lemma 3.1.** A subgroup  $\Lambda \leq V$  is a lattice if and only if there exists a basis  $e_1, \dots, e_n$  for  $V$  as a  $\mathbb{R}$ -vector space such that  $\Lambda = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$ .

*Proof.* This is a question on example sheet 2.  $\square$

We study  $\mathcal{L} = \{\Lambda \leq \mathbb{C} \text{ a lattice}\}$  with its action by  $\mathbb{C}^\times$ , i.e.  $z\Lambda = \{z\lambda \mid \lambda \in \Lambda\}$  for  $z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}$ .

**Proposition 3.2.** The map  $\tau \mapsto \Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$  induces a bijection between

$$\Gamma(1) \backslash \mathfrak{h} \leftrightarrow \mathbb{C}^\times \backslash \mathcal{L}$$

(orbits of  $\Gamma(1)$  in  $\mathfrak{h}$  and the set of lattices in  $\mathbb{C}$  modulo scalar multiplication).

*Proof.* This map is well-defined, since if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ ,  $\tau \in \mathfrak{h}$ , then

$$\Lambda_{\gamma\tau} = \mathbb{Z} \left( \frac{a\tau + b}{c\tau + d} \right) \oplus \mathbb{Z} = (c\tau + d)^{-1} (\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)) = (c\tau + d)^{-1} \Lambda_\tau.$$

For surjectivity, if  $\Lambda$  is a lattice, then  $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$  with  $\text{Im} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \neq 0$ . Swapping  $e_1, e_2$  if necessary, we may assume that  $\text{Im} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} > 0$ . Then  $\Lambda = e_2(\mathbb{Z}e_1/e_2 \oplus \mathbb{Z}) = e_1\Lambda_\tau$  for  $\tau = \frac{e_1}{e_2}$ .

For injectivity, if  $\tau, \tau'$  have the same image, then  $\exists z \in \mathbb{C}^\times$  such that  $\mathbb{Z}\Lambda_\tau = \mathbb{Z}\Lambda_{\tau'}$ , i.e.  $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$  such that  $\tau' = az\tau + bz, 1 = cz\tau + dz$ . Then  $\tau' = \frac{az\tau + bz}{cz\tau + dz} = \frac{a\tau + b}{c\tau + d}$ . But  $\text{Im}(\tau') = \text{Im}(\gamma\tau) = \det(\gamma) \frac{\text{Im}(\tau)}{|c\tau + d|^2}$  and  $\text{Im}(\tau) > 0, \text{Im}(\tau') > 0$ , hence  $\det(\gamma) > 0$ , so  $\det(\gamma) = 1$  and so  $\gamma \in \Gamma(1)$ .  $\square$

**Definition 3.1.** If  $k \in \mathbb{Z}$ , say a function  $F : \mathcal{L} \rightarrow \mathbb{C}$  is **of weight  $k$**  if  $\forall z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}, F(z\Lambda) = z^{-k}F(\Lambda)$ .

**Proposition 3.3.** Let

$$\begin{aligned} V_k &= \{F : \mathcal{L} \rightarrow \mathbb{C} \text{ of weight } k\}. \\ W_k &= \{f : \mathfrak{h} \rightarrow \mathbb{C} \mid \forall \gamma \in \Gamma(1), f|_k[\gamma] = f\}. \end{aligned}$$

Then the map  $F \mapsto (f : \tau \mapsto F(\Lambda\tau))$  induces a  $\mathbb{C}$ -vector space isomorphism  $V_k \rightarrow W_k$ .

*Proof.* We first check that if  $F \in V_k$ ,  $f(\tau) = F(\Lambda\tau)$ , then  $f \in W_k$ . If  $\gamma \in \Gamma(1)$ ,

$$f|_k[g](\tau) = f(\gamma\tau)j(\gamma, \tau)^{-k} = F(\lambda\gamma\tau)j(\gamma, \tau)^{-k} = F(j(\gamma, \tau)\Lambda_{\gamma\tau}) = F(\Lambda\tau) = f(\tau),$$

so  $j(\gamma, \tau)\Lambda_{\gamma\tau} = \Lambda_\tau$ .

To show that the map is an isomorphism, we write down its inverse: define  $\alpha : W_k \rightarrow V_k$  by  $\alpha(f)(\Lambda) = e_2^{-k} f(e_1/e_2)$  if  $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$  with  $\text{Im}(e_1/e_2) > 0$ . This is well-defined, since if  $e'_1, e'_2$  is another basis with  $\text{Im}(e'_1/e'_2) > 0$ , then  $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$  such that  $e'_1 = ae_1 + be_2$ ,  $e'_2 = ce_1 + de_2$ . Then

$$\begin{aligned} e_2'^{-k} f(e'_1/e'_2) &= (ce_1 + de_2)^{-k} f\left(\frac{ae_1 + be_2}{ce_1 + de_2}\right) \\ &= e_2^{-k} (ce_1/e_2 + d)^{-k} f\left(\frac{ae_1/e_2 + b}{ce_1/e_2 + d}\right) = e_2^{-k} f\left(\frac{e_1}{e_2}\right). \end{aligned}$$

Exercise: check that the two maps are inverse to each other.  $\square$

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**Definition 3.2.** Let  $n \in \mathbb{N}$ . The  $n^{\text{th}}$  Hecke operator  $T_n : V_k \rightarrow V_k$  is defined by the formula

$$(T_n F)(\Lambda) = n^{k-1} \sum_{\substack{\Lambda' \leq \Lambda \\ n \mid \Lambda}} F(\Lambda').$$

Here  $\sum_{\Lambda' \leq \Lambda}^n$  means summing over all subgroups  $\Lambda'$  of  $\Lambda$  of index  $n$ .

We also write  $T_n : W_k \rightarrow W_k$  for the endomorphism arising from the isomorphism  $V_k \xrightarrow{\sim} W_k$ .

Why is  $T_n$  a well-defined endomorphism of  $V_k$ ? First of all, the sum is finite since there's a bijection

$$\begin{aligned} \{\Lambda' \leq \Lambda\} &\leftrightarrow \{H \leq \Lambda/n\Lambda \text{ of index } n\} \\ \Lambda' &\mapsto \Lambda'/n\Lambda \\ H + n\Lambda &\leftrightarrow H \end{aligned}$$

This is well-defined, since Lagrange's theorem implies that

$$\Lambda' \leq_n \Lambda \implies n(\Lambda/\Lambda') = 0 \implies n\Lambda \leq \Lambda'.$$

But  $\Lambda/n\Lambda \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  is finite, so it has finitely many subgroups of index  $n$ .

If  $\Lambda' \leq_n \Lambda$ , then  $n\Lambda \leq \Lambda' \leq \Lambda$ , so  $\Lambda'$  is also discrete and cocompact in  $\mathbb{C}$ .

We next check that  $T_n F$  is of weight  $k$ , i.e. that  $(T_n F)(z\Lambda) = z^{-k}(T_n F)(\Lambda)$ . We have an isomorphism  $\{\Lambda' \leq_n z\Lambda\} \leftrightarrow \{\Lambda' \leq_n \Lambda\}$  given by  $\Lambda' \mapsto z^{-1}\Lambda'$ , so

$$(T_n F)(z\Lambda) = n^{k-1} \sum_{\Lambda' \leq_n z\Lambda} F(\Lambda') = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} F(z\Lambda') = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} z^{-k} F(\Lambda') = z^{-k} (T_n F)(\Lambda).$$

**Proposition 3.4.** (1) If  $m, n \in \mathbb{N}$  with  $(m, n) = 1$ , then  $T_m T_n = T_{mn}$ .

(2) If  $p$  is a prime number and  $n \in \mathbb{N}$ , then  $T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{k-1}}$  (acting on  $V_k$ ).

*Proof.* Let  $m, n \in \mathbb{N}$ , not necessarily coprime. Then

$$\begin{aligned} (T_m(T_n F))(\Lambda) &= m^{k-1} \sum_{\Lambda' \leq_m \Lambda} (T_n F)(\Lambda') = (mn)^{k-1} \sum_{\Lambda' \leq_m \Lambda} \sum_{\Lambda'' \leq_n \Lambda'} F(\Lambda'') \\ &= (mn)^{k-1} \sum_{\Lambda'' \leq_{mn} \Lambda} a(\Lambda, \Lambda'') F(\Lambda''), \end{aligned}$$

where  $a(\Lambda, \Lambda'') = |\{\Lambda' \leq_m \Lambda'' \mid \Lambda' \leq_n \Lambda\}| = |H \leq \Lambda/\Lambda'' \mid |H| = n|$  is the number of ways to express  $\Lambda'$  as an intermediate subgroup. If  $(m, n) = 1$ , then  $a(\Lambda, \Lambda'') = 1$  for all  $\Lambda'' \leq \Lambda$  as any finite abelian group of order  $mn$  has a unique subgroup of order  $n$ .

(1) In this case, we find

$$T_m T_n F(\Lambda) = (mn)^{k-1} \sum_{\Lambda'' \leq_{mn} \Lambda} F(\Lambda'') = (T_{mn} F)(\Lambda) \implies T_m T_n = T_{mn}.$$

(2) The same computation gives (for  $p$  prime,  $n \in \mathbb{N}$ )

$$(T_{p^n}(T_p F))(\Lambda) = p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} \Lambda} a(\Lambda, \Lambda'') F(\Lambda''),$$

where  $a(\Lambda, \Lambda'') = |\{H \subset \Lambda/\Lambda'' \mid |H| = p\}|$ . But if  $\Lambda'' \leq_{p^{n+1}} \Lambda$ , then  $\Lambda/\Lambda''$  need not have a unique subgroup of order  $p$ , as  $\Lambda \cong \mathbb{Z}^2$ , so  $\Lambda/\Lambda''$  is a finite

abelian group of order  $p^{n+1}$  that can be generated by 2 elements. But any such group is isomorphic to  $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ , where  $a \geq b \geq 0$  are integers such that  $a + b = n + 1$ . We now split into two cases:

- $b = 0$ , so  $a = n + 1$  and  $\Lambda/\Lambda'' \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$ . This group is cyclic and has a unique subgroup of order  $p$ , so  $a(\Lambda, \Lambda'') = 1$ .
- $b > 0$ , so  $\Lambda/\Lambda'' \cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ . Let  $\Lambda/\Lambda''[p] = \{x \in \Lambda/\Lambda'' \mid px = 0\}$ . This is a subgroup of  $\Lambda/\Lambda''$ , and

$$\{H \leq \Lambda/\Lambda'' \mid |H| = p\} = \{H \leq \Lambda/\Lambda''[p] \mid |H| = p\}.$$

Hence  $\Lambda/\Lambda''[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  from our above isomorphism. So in this case,  $a(\Lambda, \Lambda'') = |\{H \leq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \mid |H| = p\}|$ . In other words,

$$a(\Lambda, \Lambda') = |\mathbb{P}^1(\mathbb{F}_p)| = |\mathbb{A}^1(\mathbb{F}_p) \cup \{\infty\}| = p + 1.$$

How do we distinguish between these two cases? We will show on example sheet 2 that if  $\Lambda'' \leq_{p^{n+1}} \Lambda$ , then there exists a  $\mathbb{Z}$ -basis  $e_1, e_2$  for  $\Lambda$  such that  $\Lambda'' = \mathbb{Z}p^a e_1 \oplus \mathbb{Z}p^b e_2$  for the same  $a, b$  satisfying  $a \geq b \geq 0, a + b = n + 1$  as before (this is a consequence of Smith normal form).

Hence we see that we are in case 2 if and only if  $\Lambda'' \leq p\Lambda$ . Thus we find

$$(T_{p^n}(T_p F)(\Lambda)) = p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} \Lambda} F(\Lambda'') + p^{(n+1)(k-1)} \sum_{\substack{\Lambda'' \leq p\Lambda \\ p^{n-1}}} pF(\Lambda'').$$

Here each  $\Lambda''$  in case 1 goes into the first sum and each  $\Lambda''$  in case 2 goes once into the first sum and  $p$  times into the second sum. We have

$$\begin{aligned} & p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} p\Lambda} pF(\Lambda'') = p^{(n-1)(k-1)} p^{2(k-1)} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} pF(p\Lambda'') \\ & = p^{(n-1)(k-1)} p^{2(k-1)} p^{1-k} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} F(\Lambda'') = p^{k-1} T_{p^{n+1}} F(\Lambda). \end{aligned}$$

$$\text{Hence } T_{p^n} T_p F(\Lambda) = T_{p^{n+1}} F(\Lambda) + p^{k-1} T_{p^{n-1}} F(\Lambda).$$

□

**Corollary 3.5.**  $\forall m, n \in \mathbb{N}, T_m T_n = T_n T_m$  as endomorphisms of  $V_k$ , i.e. all Hecke operators commute.

*Proof.* If we write  $m = \prod_{i=1}^r p_i^{a_i}$  for  $a_i \geq 1, p_i$  distinct, then  $T_m = T_{p_1^{a_1}} \dots T_{p_r^{a_r}}$ . We've shown that if  $p, q$  are distinct primes, then  $T_{p^a}, T_{q^b}$  commute  $\forall a, b \geq 1$ .



We need to show that if  $p$  is a prime and  $a, b \in \mathbb{N}$ , then  $T_{p^a}$  and  $T_{p^b}$  commute. But we have a stronger claim that  $\forall a \in \mathbb{N}$ ,  $T_{p^a}$  is a polynomial in  $T_p$ . We prove this by induction on  $a$ , the case  $a = 1$  being trivial.

In general,  $T_{p^{a+1}} = T_{p^a}T_p - p^{k-1}T_{p^{a-1}}$ , which proves the claim.  $\square$

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Lecture 9

**Lemma 3.6.** Let  $n \in \mathbb{N}$  and  $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \leq \mathbb{C}$  a lattice. Then  $\{\Lambda' \leq_n \Lambda\} = \{\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2 \mid a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, b < d\}$ , where this is isomorphic to the set  $\{a, b, d \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$ .

*Proof.* Consider the short exact sequence

$$0 \rightarrow \mathbb{Z}e_2 / \mathbb{Z}e_2 \cap \Lambda' \rightarrow \Lambda / \Lambda' \rightarrow \underbrace{\Lambda / \mathbb{Z}e_2 + \Lambda'}_{\cong \mathbb{Z}e_1 / \mathbb{Z}e_1 \cap (\mathbb{Z}e_2 + \Lambda)} \rightarrow 0.$$

Then  $|\Lambda / \Lambda'| = n$ . We let  $d = |\mathbb{Z}e_2 / \mathbb{Z}e_2 \cap \Lambda'| = \inf\{d \geq 1 \mid de_2 \in \Lambda'\}$  and  $a = |\Lambda / \mathbb{Z}e_2 + \Lambda'| = \inf\{a \geq 1 \mid \exists b \in \mathbb{Z} \text{ s.t. } ae_1 + be_2 \in \Lambda'\}$ . Then  $n = ad$  and there exists a unique  $0 \leq b < d$  such that  $ae_1 + be_2 \in \Lambda'$ .

We now claim that  $\Lambda' = \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$ . The inclusion  $\supseteq$  is clear. On the other hand, if  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z})$ ,  $\alpha\delta - \beta\gamma = N \in \mathbb{Z}$  is nonzero, then  $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = |N|$ . So  $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = n = [\Lambda : \Lambda']$ , so  $[\Lambda' : \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2] = 1$ , so they're equal.

We've defined a map  $\{\Lambda' \leq_n \Lambda\} \rightarrow \{(a, b, d) \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$ . This map has an inverse, given by  $(a, b, d) \mapsto \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$ , so it's a bijection.  $\square$

**Lemma 3.7.** Let  $f \in W_k$ . Then we have the two formulas

$$(T_n f)(\tau) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) = \sum_{\substack{ad=n \\ 0 \leq b < d}} f|_k \left[ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right].$$

*Proof.*  $f \leftrightarrow F \in V_k$  with  $f(\tau) = F(\Lambda_\tau)$ . By definition,

$$(T_n f)(\tau) = (T_n F)(\Lambda_\tau) = n^{k-1} \sum_{\Lambda' \leq_n \Lambda_\tau} F(\Lambda') = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} F(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}d).$$

This equals

$$n^{k-1} \sum_{a, b, d} F(d(\mathbb{Z}(\frac{a\tau + b}{d} \oplus \mathbb{Z}))) = n^{k-1} \sum_{a, b, d} d^{-k} F(\Lambda_{\frac{a\tau + b}{d}}) = n^{k-1} \sum_{a, b, d} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

For the second formula, recall that if  $g \in GL_2(\mathbb{R})^+$ , then  $f|_k[g] = \det(g)^{k-1} f(g\tau)j(g, \tau)^{k-1}$ , so

$$f|_k\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right](\tau) = n^{k-1} f\left(\frac{a\tau + b}{d}\right) d^{-k}.$$

$$\text{Hence } (T_n f)(\tau) = \sum_{a,b,d} f|_k\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]. \quad \square$$

**Corollary 3.8.** If  $f \in W_k$  and  $f$  is holomorphic, then  $T_n f$  is also holomorphic.

*Proof.* Look at the formula above:  $T_n f$  is a finite sum of holomorphic functions.  $\square$

**Proposition 3.9.** Let  $f \in W_k$  be holomorphic in  $\mathfrak{h}$  with  $q$ -expansion  $f(\tau) = \sum_{m \in \mathbb{Z}} b_m q^m$ . Then  $T_n f$  has  $q$ -expansion  $T_n f = \sum_{m \in \mathbb{Z}} c_m q^m$ , where

$$c_m = \sum_{\substack{a \in \mathbb{N} \\ a|(m,n)}} a^{k-1} b_{(mn/a^2)}.$$

*Proof.*

$$\begin{aligned} T_n f &= n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a \tau / d} e^{2\pi i m b \tau / d} \\ &= n^{k-1} \sum_{ad=n} d^{-k} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a \tau / d} \left( \sum_{0 \leq b < d} e^{2\pi i m b \tau / d} \right). \end{aligned}$$

Note that  $\sum_{0 \leq b < d} e^{2\pi i m b \tau / d} = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$ . Hence

$$T_n f = n^{k-1} \sum_{ad=n} d^{1-k} \sum_{m \in \mathbb{Z}} b_{dm} e^{2\pi i a m \tau}.$$

This gives

$$T_n f = \sum_{ad=n} \left(\frac{n}{d}\right)^{k-1} \sum_{m \in \mathbb{Z}} b_{dm} q^{am} = \sum_{a|n} a^{k-1} \sum_{m \in \mathbb{Z}} b_{nm/a} q^{am}.$$

This equals  $\sum_{N \in \mathbb{Z}} c_N q^N$ , where  $c_N = \sum_{\substack{a|m \\ a|n}} a^{k-1} b_{nN/a^2}$ .  $\square$

**Theorem 3.10.**  $T_n$  preserves the subspaces  $S_k(\Gamma(1)) \leq M_k(\Gamma(1)) \leq W_k \forall n \geq 1$ . Moreover, if  $f \in M_k(\Gamma(1))$ , then  $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$  and  $a_1(T_n f) = a_n(f)$ .

*Proof.* To show that  $T_n$  preserves  $M_k(\Gamma(1))$ , we need to show that if  $f \in M_k(\Gamma(1))$ , then  $T_n f$  is holomorphic in  $\mathfrak{h}$  (then we're done by the previous corollary) and at  $\infty$ , i.e.  $a_N(T_n f) = 0$  if  $N < 0$ .

But  $a_N(T_n f) = \sum_{a|(m,n)} a^{k-1} a_{Nn/a^2}(f)$ . Since  $Nn/a^2 < 0$  and  $f$  is holomorphic at  $\infty$ , all summands are 0, so  $T_n f$  is holomorphic at  $\infty$ .

We have  $a_0(T_n f) = \sum_{a|(n,0)} a^{k-1} a_{n \cdot 0/a^2}(f) = \sum_{a|n} a^{k-1} a_0(f) = \sigma_{k-1}(n) a_0(f)$ .

Also  $a_1(T_n f) = \sum_{a|(n,1)} a^{k-1} a_{n \cdot 1/a^2}(f) = a_n(f)$ .

Finally, if  $f \in S_k(\Gamma(1))$ , then  $a_0(f) = 0$ , and then  $T_n f \in M_k(\Gamma(1))$  and  $a_0(T_n f) = \sigma_{k-1}(n) a_0(f) = 0 \implies T_n f \in S_k(\Gamma(1))$ .  $\square$

Our next goal is to study the spectral decomposition of Hecke operators on  $M_k(\Gamma(1))$ , i.e. the decomposition of  $M_k(\Gamma(1))$  as a sum of (simultaneous) generalized eigenspaces for the  $T_n$ .

The simplest case is when  $M_k(\Gamma(1))$  or  $S_k(\Gamma(1))$  is 1-dimensional (as then every nonzero element is an eigenvector). For example,  $S_{12}(\Gamma(1))$  is 1-dimensional, spanned by  $\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n$ . So  $\Delta$  is a  $T_n$ -eigenvector for all  $n \geq 1$ . If  $T_n \Delta = \alpha_n \Delta$  for some  $\alpha_n \in \mathbb{C}$ , then  $a_1(T_n \Delta) = a_1(\alpha_n \Delta) = \alpha_n a_1(\Delta) = \alpha_n$  (as we proved  $a_1(\Delta) = 1$ ). But we also have  $a_1(T_n \Delta) = a_n(\Delta) = \tau(n)$ . Hence  $\alpha_n = \text{Hecke eigenvalue} = \tau(n) = \text{coefficient of } q^n$ .

Ramanujan conjectured in 1916 that  $\tau$  is multiplicative and  $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$  for  $p$  prime,  $n \in \mathbb{N}$ . These identities are true for Hecke operators (i.e.  $T_{mn} = T_m T_n$  and  $T_{p^{n+1}} = T_p T_{p^n} - p^{k-1} T_{p^{n-1}}$ ), hence also for the eigenvalues  $\alpha_n$ , hence for the numbers  $\tau(n)$ .

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Lecture 10

Our goal now is to study the spectral decomposition of  $M_k(\Gamma(1))$  and the arithmetic properties of Hecke eigenvalues.

**Definition 3.3.** If  $f \in M_k(\Gamma(1))$ , we say  $f$  is an **eigenform** if  $f$  is a  $T_n$ -eigenvector  $\forall n \geq 1$ .

We say  $f$  is a **normalized eigenform** if  $a_1(f) = 1$ .

**Lemma 3.11.** Suppose  $k > 0$ . Then any eigenform  $f \in M_k(\Gamma(1))$  is a scalar multiple of a unique normalized eigenform. Moreover, if  $f$  is normalized, then  $T_n(f) = a_n(f)f \ \forall n \geq 1$ . (In other words, the  $n^{\text{th}}$  Hecke eigenvalue = the  $n^{\text{th}}$   $q$ -expansion coefficient).

For example,  $\Delta$  is a normalized eigenform and  $\tau(n)\Delta = T_n \Delta$ .

*Proof.* We know  $a_1(T_n f) = a_n(f)$ . We need to show that if  $f$  is an eigenform, then  $a_1(f) \neq 0$  (as then  $f/a_1(f)$  is normalized). But if  $a_1 = 0$  and  $\alpha_n$  is the eigenvalue of  $T_n$  on  $f$ , then  $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = 0 \ \forall n \geq 1$ .

Then  $f = \sum_{n \geq 0} a_n(f)q^n = a_0(f)$ , which is a contradiction as constants are not modular forms of weights  $k > 0$ .

If  $f$  is normalized, then  $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = \alpha_n$ .  $\square$

**Proposition 3.12.** Let  $k \geq 4$  be even. Then  $G_k(\tau)$  is an eigenform.

*Proof.* We need to show that  $G_k$  is a  $T_n$ -eigenvector  $\forall n \geq 1$ . We know  $T_n$  is a polynomial in  $T_p$  for  $p$  ranging over  $p \mid n$  for  $p$  prime. Hence it is enough to show that  $G_k$  is a  $T_p$ -eigenvector  $\forall p$  prime.

$G_k(\tau) = G_k(\Lambda_\tau)$  for  $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ . Then

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\Lambda' \leq \frac{\Lambda}{p}} G_k(\Lambda') = p^{k-1} \sum_{\Lambda' \leq \frac{\Lambda}{p}} \sum_{\lambda \in \Lambda' \setminus 0} \lambda^{-k} = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} a(\Lambda, \lambda) \lambda^{-k}$$

where  $a(\Lambda, \lambda) = |\{\Lambda' \leq \frac{\Lambda}{p} \mid \lambda \in \Lambda'\}|$ . We know that if  $\Lambda' \leq \frac{\Lambda}{p}$ , then  $p\Lambda \leq \Lambda' \leq \Lambda$  and we have a bijection  $\{\Lambda' \leq \frac{\Lambda}{p}\} \leftrightarrow \{H \leq \Lambda/p\Lambda \mid |H| = p\}$ .

If  $\lambda \in p\Lambda$ , then  $\{\Lambda' \leq \frac{\Lambda}{p} \mid \lambda \in \Lambda'\} = \{\Lambda' \leq \frac{\Lambda}{p}\}$ , so  $a(\Lambda, \lambda) = p + 1$ .

If  $\lambda \notin p\Lambda$ , then  $\lambda \neq 0$  modulo  $p\Lambda$  and there exists a unique subgroup  $H \leq \Lambda/p\Lambda$  of order  $p$  such that  $\lambda \in H$ . Hence in this case,  $\{\Lambda' \leq \frac{\Lambda}{p}\} = \{\mathbb{Z}\lambda + p\Lambda\}$  and  $a(\Lambda, \lambda) = 1$ . Hence

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in p\Lambda \setminus 0} p \lambda^{-k}.$$

We get

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} p(p\lambda)^{-k} = p^{k-1} G_k(\Lambda) + G_k(\Lambda) = \sigma_{k-1}(p) G_k(\Lambda).$$

$\square$

We can compute the  $T_n$ -eigenvalues on  $G_k$  for all  $n$  now using  $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$ . So if  $f$  is an eigenform and  $a_0(f) \neq 0$ , then this forces the eigenvalue to be equal to  $\sigma_{k-1}(n)$ . So  $T_n G_k = \sigma_{k-1}(n) G_k \forall n \geq 1$ . The  $q$ -expansion of  $G_k$  is  $2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$  and we also defined  $E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ . Hence  $a_0(E_k) = 1$ , but  $E_k$  is not a normalized eigenform. Hence the associated normalized eigenform is

$$F_k(\tau) = \frac{\zeta(k)(k-1)!}{(2\pi i)^k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n = \frac{-B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

(here we gave multiple equivalent expressions).

We have a decomposition  $M_k(\Gamma(1)) = \mathbb{C}F_k \oplus S_k(\Gamma(1))$  (for  $k \geq 4$ ). Both summands are  $T_n$ -invariant, so it's enough to study the action of  $T_n$  on  $S_k$ .

**Remark.**  $T_n$  do not usually respect multiplication. In particular, the product of eigenforms is not usually an eigenform. For example,  $E_4^2 = E_8$ , but  $E_4^3 \in M_{12}(\Gamma(1)) = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$  requires both  $E_{12}$  and  $\mathbb{C}$  to be expressed and hence is not an eigenform.

**Proposition 3.13.** If  $f \in S_k(\Gamma(1))$  is a cuspidal eigenform, then all of the  $T_n$ -eigenvalues on  $f$  are algebraic integers. If  $f$  is normalized, then  $\mathbb{Q}(\{a_n(f)\}_{n=1}^\infty)$  has finite degree over  $\mathbb{Q}$  (i.e. it is a number field).

*Proof.* We will show that for all  $n \geq 1$ , all eigenvalues of  $T_n$  on  $S_k(\Gamma(1))$  are algebraic integers. We will do this by showing that the characteristic polynomial of  $T_n$  acting on  $S_k$  has integer coefficients (and it is of course monic).

We consider the basis  $f_1, \dots, f_N$  for  $S_k(SL_2(\mathbb{Z}))$  characterized by:

- $\forall 1 \leq i \leq N$  and  $\forall n \geq 1$ ,  $a_n(f_i) \in \mathbb{Z}$ .
- $\forall 1 \leq i, n \leq N$ ,  $a_n(f_i) = \delta_{in}$ .

Recall that this meant that  $f_1, \dots, f_N$  was the dual basis to the basis of functionals  $a_1, \dots, a_N$  of  $S_k(\Gamma(1))^*$ . Hence  $\forall f \in S_k(\Gamma(1))$ ,  $f = \sum_{i=1}^N a_i(f)f_i$  (this identity holds for any elements of a finite dimensional vector space with its basis and dual basis)

The claim is that if  $A$  denotes the matrix of  $T_n$  in the basis of  $f_1, \dots, f_N$ , then  $A$  has integer entries. As the characteristic polynomial of  $T_n$  is  $\det(X \cdot I - A)$ , this will show that the characteristic polynomial has coefficients in  $\mathbb{Z}$ .

By definition,  $T_n(j) = \sum_{i=1}^N A_{ij}f_i$ . Then for  $1 \leq m \leq N$ ,

$$a_m(T_n f_j) = \sum_{i=1}^N A_{ij} a_m(f_i) = \sum_{i=1}^N A_{ij} \delta_{im} = A_{mj}.$$

But  $a_m(T_n f_j) = \sum_{a|(m,n)} a^{k-1} a_{mn/a^2}(f_j)$  by the formula from the last lecture. Note that each  $a_{mn/a^2}(f_j)$  is in  $\mathbb{Z}$  by the definition of  $f_j$ , so  $\forall m, j$ ,  $A_{mj} \in \mathbb{Z}$ .

If  $f$  is a normalized eigenform,  $f = \sum_{i=1}^N a_i(f)f_i$ , then  $\forall n \geq 1$ ,  $a_n(f) = \sum_{i=1}^N a_i(f) \underbrace{a_n(f_i)}_{\in \mathbb{Z}}$ . Hence  $\mathbb{Q}(\{a_n(f)\}_{n \geq 1}) = \mathbb{Q}(\{a_n(f)\}_{n=1}^N)$  has finite degree over  $\mathbb{Q}$ . □