

Part III - Modular Forms

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Contents

1	Introduction	2
2	Modular Forms on $\Gamma(1)$	5
3	Hecke operators	21
4	L-functions	36

1 Introduction

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Lecture 1

Definition 1.1. We define the following groups:

$$\begin{aligned}\mathfrak{h} &= \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\} \\ GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\ \Gamma(1) &= SL_2(\mathbb{Z}) = \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.\end{aligned}$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

Lemma 1.1. $GL_2(\mathbb{R})^+$ acts transitively on \mathfrak{h} by Möbius transformations.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$. Then

$$\operatorname{Im}(g\tau) = \frac{1}{2i} \left(\frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) = \frac{1}{2i} \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\det(g)\operatorname{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{h}$. This action is transitive since

$$x + iy \in \mathfrak{h} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in \mathfrak{h} is conjugate to i . □

Definition 1.2. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{h}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{h} \rightarrow \mathbb{C}$, then

$$f|_k[g] : \mathfrak{h} \rightarrow \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight k action of g on f** .

Lemma 1.2. This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

Proof. We compute

$$\begin{aligned} (f|_k[g])|_k[h](\tau) &= \det(h)^{k-1} f|_k[g](h\tau) j(h, \tau)^{-k} = \\ &= \det(h)^{k-1} \det(g)^{k-1} f(gh\tau) j(g, h\tau)^{-k} j(h, \tau)^{-k} \stackrel{?}{=} \\ &= \det(gh)^{k-1} f(gh\tau) j(gh, \tau)^{-k} = f|_k[gh](\tau). \end{aligned}$$

Hence we need to check that $j(gh, \tau) = j(gh, \tau)j(h, \tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh, \tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g \left(j(h, \tau) \begin{pmatrix} h\tau \\ 1 \end{pmatrix} \right) = j(h, \tau) j(g, h\tau) \begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. \square

Formulae. For $g \in GL_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$, we have

$$\operatorname{Im}(g\tau) = \det(g) \frac{\operatorname{Im}(\tau)}{|j(g, \tau)|^2} \text{ and } j(g, \tau) \begin{pmatrix} g\tau \\ 1 \end{pmatrix} = g \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Definition 1.3. Let $k \in \mathbb{Z}$ and $\Gamma \leq \Gamma(1)$ of finite index¹. A **weakly modular function of weight k and level Γ** is a meromorphic function $f : \mathfrak{h} \rightarrow \mathbb{C}$ which is invariant under the weight k action of Γ , i.e. such that

$$\forall \tau \in \mathfrak{h}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in \mathfrak{h} and at ∞ .

It is a fact that modular forms of fixed weight and level live in finite-dimensional \mathbb{C} -vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

Motivation. Why study modular forms?

- (1) They are related to the theory of elliptic functions. Let E/\mathbb{C} be an elliptic curve and ω a holomorphic non-zero 1-form. Then there exists a unique lattice² $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \rightarrow E$ such that $\phi^*(\omega) = dz$. Then

¹In other words, Γ is a (finite index) subgroup of $\Gamma(1)$.

²i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over \mathbb{R} .

E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{h}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight k and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{h}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over \mathbb{C} .

- (2) Modular forms f have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences a_n .

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then θ^k is a modular form with q -expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing n as a sum of k squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing θ^k in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$.

- (3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to \mathbb{C} and has a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an L -function. Modular forms can be used to construct interesting examples of L -functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

- (4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves E/\mathbb{C} up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and L -functions.

Homework. There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

2 Modular Forms on $\Gamma(1)$

09 Oct 2022,
Lecture 2

Reminder. A **meromorphic** function in an open subset $U \subset \mathbb{C}$ is a closed subset $A \subset U$ and a holomorphic function $f : U \setminus A \rightarrow \mathbb{C}$ such that $\forall a \in A$, $\exists \delta > 0$ such that $D^*(a, \delta) \subset U \setminus A$ and $\exists n \geq 0$ such that $(z - a)^n f(z)$ extends to a holomorphic function in $D(a, \delta)$.

f then has a Laurent expansion $\sum_{m \in \mathbb{Z}} a_m (z - a)^m$ valid on $D^*(a, \delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a meromorphic function \tilde{f} in $D^*(0, 1)$ (the "q-disk") such that

$$f(\tau) = \tilde{f}(e^{2\pi i \tau}).$$

Proof. f is meromorphic in \mathfrak{h} by assumption. Take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then $f|_h[\gamma](\tau) = f(\gamma\tau) = f(\tau)$, as f is invariant under the weight k action of γ . But also $f(\gamma\tau) = f(\tau + 1)$, so f is periodic.

Now map a strip of \mathfrak{h} of width 1 to $D^*(0, 1)$ by $\tau \mapsto e^{2\pi i \tau}$. Let $a \in D^*(0, 1)$ and $\delta > 0$ be such that $D(a, \delta) \subset D^*(0, 1)$. Define \tilde{f} on $D(a, \delta)$ by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i} \log q\right),$$

for any branch of \log defined in $D(a, \delta)$. This is meromorphic and independent of the choice of the branch of \log , as f is periodic with period 1. This defines \tilde{f} in $D^*(0, 1)$. Finally, \tilde{f} is unique since $\tau \mapsto e^{2\pi i \tau}$ is surjective. \square

If \tilde{f} extends to a meromorphic function³ in $D(0, 1)$, then $\exists \delta > 0$ such that \tilde{f} has a Laurent expansion $\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ valid in $D^*(0, \delta)$.

In the region $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > \frac{1}{2\pi} \log \delta\}$, we have

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q = e^{2\pi i \tau}$. This is called the **q-expansion** of the weakly modular function f .

Definition 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is **meromorphic at ∞** if \tilde{f} extends to a meromorphic function in $D(0, 1)$.

We say f is **holomorphic at ∞** if \tilde{f} is meromorphic at ∞ and has a

³This might not be the case if the set of poles has a limit inside the disk.

removable singularity at $q = 0$. In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau).$$

We say f **vanishes at ∞** if f is holomorphic at ∞ and $f(\infty) = 0$.

Definition 2.2. A **modular function** (of weight k and level $\Gamma(1)$) is a weakly modular function (of weight k and level $\Gamma(1)$) which is meromorphic at ∞ .

A **modular form** is a weakly modular function which is holomorphic in \mathfrak{h} and holomorphic at ∞ .

A **cuspidal modular form** is a modular form that vanishes at ∞ .

Remark. We let $M_k(\Gamma(1))$ denote the set of modular forms of weight k and level $\Gamma(1)$. We write $S_k(\Gamma(1))$ for the set of cuspidal modular forms of weight k , level $\Gamma(1)$. Note $S_k(\Gamma(1)) \subset M_k(\Gamma(1))$. These are \mathbb{C} -vector spaces. If k is odd, then these both only contain the zero function, since taking $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ gives $f|_k[\gamma](\tau) = f(\tau)(-1)^k = f(\tau)$.

We now consider even weights only. If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k},$$

where $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ for any $\tau \in \mathfrak{h}$.

If $\gamma \in \Gamma(1)$, then formally we have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau)j(\gamma, \tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k}j(\gamma, \tau)^{-k},$$

but $\Lambda_{\gamma\tau} = \mathbb{Z} \frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z} = (c\tau+d)^{-1} (\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d)) = (c\tau+d)^{-1} \Lambda_\tau$.
Hence

$$\begin{aligned} G_k|_k[g](\tau) &= \sum_{\lambda \in (c\tau+d)^{-1} \Lambda_\tau \setminus 0} \lambda^{-k} (c\tau+d)^{-k} \\ &= \sum_{\lambda \in \Lambda_\tau \setminus 0} ((c\tau+d)^{-1} \lambda)^{-k} (c\tau+d)^{-k} = G_k(\tau). \end{aligned}$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely. Hence:

Proposition 2.2. Let $k > 2$ be an even integer. Then $G_k(\tau)$ converges absolutely and defines a modular form of weight k and level $\Gamma(1)$ which has

$G_k(\infty) = 2\zeta(k)$. G_k is the **weight k Eisenstein series**.

We will later see that $M_2(\Gamma(1)) = 0$.

Proof. We want to show absolute and locally uniform convergence in \mathfrak{h} . This will show that G_k is holomorphic by complex analysis. Let $A \geq 2$ and define $\Omega_A = \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq \frac{1}{A}, \text{Re}(\tau) \in [-A, A]\}$. We show uniform convergence in Ω_A . If $\tau \in \Omega_A, x \in \mathbb{R}$, then $|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A. \end{cases}$ Hence

$$|\tau + x| \stackrel{(\dagger)}{\geq} \sup \left(\frac{1}{A}, \frac{|x|}{2A^2} \right) \geq \sup \left(\frac{1}{2A^2}, \frac{|x|}{2A^2} \right) = \frac{1}{2A^2} \sup(1, |x|).$$

(\dagger) follows by drawing a diagram with the lines $y = \frac{1}{A}$ and $y = \frac{x}{2A^2}$ and marking the point $(2A, \frac{1}{A})$ on it, then noticing that our supremum always lies above the supremum of these two lines. If $(m, n) \in \mathbb{Z}^2, m \neq 0$, then

$$|m\tau + n| = |m| \left| \tau + \frac{n}{m} \right| \geq |m| \frac{1}{2A^2} \sup \left(1, \left| \frac{n}{m} \right| \right) = \frac{1}{2A^2} \sup(|m|, |n|).$$

This is also valid when $m = 0$ by inspection. If $\tau \in \Omega_A$, then

$$\begin{aligned} & \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} |m\tau + n|^{-k} \\ & \leq \left(\frac{1}{2A^2} \right)^{-k} \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \sup(|m|, |n|)^{-k} \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} \cdot |\{(m, n) \in \mathbb{Z}^2 \mid \sup(|m|, |n|) = d\}| \\ & = (2A^2)^k \sum_{d \in \mathbb{N}} d^{-k} 8d = 8(2A^2)^k \sum_{d \in \mathbb{N}} d^{1-k} \\ & < \infty \end{aligned}$$

whenever $k - 1 > 1$, i.e. $k > 2$. This shows absolute convergence, and uniform convergence in Ω_A by the Weierstrass M-test⁴. Hence G_k is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$. It remains to show that G_k is holomorphic at ∞ with $G_k(\infty) = 2\zeta(k)$. For this, it suffices to check that

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = 2\zeta(k).$$

⁴If we have a sequence of functions $f_n : \Omega \rightarrow \mathbb{C}$ and values $M_n > 0$ with $|f_n(x)| < M_n$ and $\sum M_n < \infty$, then $\sum f_n$ converges absolutely and uniformly on Ω . Here, replace n with d and sum d over $\sum_{(m,n) \in \mathbb{Z}^2 \setminus 0, \sup(|m|, |n|) = d} |m\tau + n|^{-k}$.

This follows from uniform convergence in Ω_A : we get

$$\lim_{\text{Im}(\tau) \rightarrow \infty} G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \lim_{\text{Im}(\tau) \rightarrow \infty} (m\tau + n)^{-k} = \sum_{n \in \mathbb{Z} \setminus 0} n^{-k} = 2 \sum_{n \geq 1} n^{-k} = 2\zeta(k).$$

□

11 Oct 2022,
Lecture 3

Recap. We defined what it means for a function $f : \mathfrak{h} \rightarrow \mathbb{C}$ to be a modular form of weight k and level $\Gamma(1)$. $M_k(\Gamma(1))$ is the \mathbb{C} -vector space of such forms. If $f \in M_k(\Gamma(1))$, then there exists a holomorphic $\tilde{f} : D(0, 1) \rightarrow \mathbb{C}$ (here we call $D(0, 1)$ the q -disk) such that $\forall \tau \in \mathfrak{h}$, $f(\tau) = \tilde{f}(e^{2\pi i \tau})$. The Taylor expansion of \tilde{f} gives the q -expansion

$$f(\tau) = \sum_{n \geq 0} a_n q^n, \quad q = e^{2\pi i \tau}.$$

We have $f(\infty) = \tilde{f}(0) = a_0$. If $k > 2$ is even, then $G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k}$ converges absolutely and defines an element of $M_k(\Gamma(1))$ with $G_k(\infty) = 2\zeta(k)$.

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n.$$

We will soon show that we have $a_n \in \mathbb{Q} \forall n \geq 1$.

We can construct more modular forms: if $f \in M_k(\Gamma(1))$ and $g \in M_l(\Gamma(1))$, then $fg \in M_{k+l}(\Gamma(1))$. To check this is a modular form, we need to check that:

- fg is holomorphic, which is true as f, g are holomorphic.
- fg is invariant under the weight $k + l$ action of $\Gamma(1)$, which is true as f, g are invariant under the weight k and l actions of $\Gamma(1)$ – this is just a computation.
- fg is holomorphic at ∞ . This is true as the q -expansions multiply, so since f, g have no negative terms, the same is true for fg .

Hence we get e.g. $E_4^3, E_6^2 \in M_{12}(\Gamma(1))$ and $\frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$ (i.e. it is cuspidal since zero at infinity). This difference is Ramanujan's Δ -function. We will show it is nonzero later.

We now want to show that $M_k(\Gamma(1))$ is finite-dimensional. We first study $\Gamma(1)/\mathfrak{h}$. For this, introduce a fundamental set $\mathfrak{f}' \subset \mathfrak{h}$ for the $\Gamma(1)$ -action. We define⁵ a fundamental set to be a set that intersects each $\Gamma(1)$ -orbit in exactly

⁵Definitions in literature may vary, so we omit a formal definition.

one element. Define

$$\mathfrak{f} = \left\{ \tau \in \mathfrak{h} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right], |\tau| \geq 1 \right\}.$$

$$\mathfrak{f}' = \left\{ \tau \in \mathfrak{f} \mid \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, \frac{1}{2} \right), |\tau| = 1 \implies \operatorname{Re}(\tau) \in \left[-\frac{1}{2}, 0 \right] \right\}.$$

Introduce $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\Gamma(1)$. We observe that every element of \mathfrak{f} is conjugate under S or T^{-1} to an element of \mathfrak{f}' , which is true since $T(\tau) = \tau + 1$ and $S(\tau) = -\frac{1}{\tau}$.



Proposition 2.3. Let $G = \Gamma(1)/\{\pm I\}$. Then

- (i) $\forall \tau \in \mathfrak{h}, \tau$ is $\Gamma(1)$ -conjugate to an element of \mathfrak{f}' .
- (ii) If $\tau, \tau' \in \mathfrak{f}'$ are $\Gamma(1)$ -conjugate, then $\tau = \tau'$.
- (iii) If $\tau \in \mathfrak{f}'$, then $\operatorname{Stab}_G(\tau)$ is trivial, except in the two cases $\operatorname{Stab}_G(i) = \langle S \rangle$ and $\operatorname{Stab}_G(\rho) = \langle ST \rangle$, where $\rho = e^{2\pi i/3}$.
- (iv) $\Gamma(1)$ is generated by S and T .

Proof. Let H be the subgroup of G generated by S and T .

Claim. Every $\tau \in \mathfrak{h}$ is H -conjugate to an element of \mathfrak{f}' .

Proof. By our above observation and since $S, T \in H$, it suffices to prove that every $\tau \in \mathfrak{h}$ is H -conjugate to \mathfrak{f} . Take $\tau \in \mathfrak{h}$. Recall that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2}$.

In particular, $\forall R \geq 0$, the intersection $H\tau \cap \{\text{Im}(\tau') > R\}$ is finite, since $\text{Im}(\gamma\tau) > R \iff |c\tau + d|^2 < \frac{\text{Im}(\tau)}{R}$, but $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ is a lattice, so the set $\{(c, d) \in \mathbb{Z}^2 \mid |c\tau + d| < R'\}$ is finite.

So there exists $h \in H$ such that $\text{Im}(h\tau) \geq \text{Im}(h'\tau) \forall h' \in H$. After replacing τ by $h\tau$, we can assume $\text{Im}(\tau) \geq \text{Im}(h\tau) \forall h \in H$. Since acting by T does not change $\text{Im}(\tau)$, we can also assume $\text{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]$. We have $\text{Im}(\tau) \geq \text{Im}(S\tau) = \frac{\text{Im}(\tau)}{|\tau|^2} \implies |\tau| \geq 1$, proving the claim and (i). \square

Now take $\tau, \tau' \in \mathfrak{f}'$ and suppose $\gamma\tau = \tau'$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. We want to show that either $\gamma = \pm I$ or $\tau = i, \rho$.

WLOG assume $\text{Im}(\tau') = \text{Im}(\gamma\tau) \geq \text{Im}(\tau)$, i.e. $\text{Im}(\gamma\tau) = \frac{\text{Im}(\tau)}{|c\tau + d|^2} \geq \text{Im}(\tau)$, so $|c\tau + d| \leq 1$. However, if $\tau \in \mathfrak{f}'$, then $\text{Im}(\tau) \geq \frac{\sqrt{3}}{2}$ with equality if and only if $\tau = \rho$. Hence $|c\tau + d| \geq |c|\text{Im}(\tau) \geq |c|\frac{\sqrt{3}}{2} \implies |c| \leq \frac{2}{\sqrt{3}} \implies |c| = 0, 1 \implies c = 0$ or $c = \pm 1$.

- If $c = 0$, then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, so $ad = 1 \implies a = d = \pm 1$, so $\gamma = \pm T^m$ for $m \in \mathbb{Z}$. However, T acts on \mathfrak{f}' by shifting the real part, so it can only stay in \mathfrak{f}' if $m = 0$ (as $\text{Re}(\mathfrak{f}') \in [-\frac{1}{2}, \frac{1}{2}]$), so $\gamma = \pm I$ and $\tau' = \tau$.
- If $c = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$ and $|\tau + d| \leq 1$. By drawing another picture, we see that the only circles centered at integers of radius 1 which intersect \mathfrak{f}' are centered at $-d = 0, -d = -1$. Hence either $d = 0$, whence $|\tau| = 1$, or $d = 1$, whence $\tau = \rho$.

– If $c = 1, d = 0, |\tau| = 1$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ since the determinant must be 1. Then $\gamma\tau = \frac{a\tau - 1}{\tau} = a - \frac{1}{\tau} = a - \bar{\tau}$, so $\text{Re}(\gamma\tau) = a - \text{Re}(\tau) \in \text{Re}(\mathfrak{f}' \cap \{|\tau| = 1\}) = [-\frac{1}{2}, 0]$. However, we also have $\text{Re}(\gamma\tau) \in a - [-\frac{1}{2}, 0] = a + [0, \frac{1}{2}]$.

The intersection $[-\frac{1}{2}, 0] \cap (a + [0, \frac{1}{2}])$ can be nonempty only if either $a = 0$, whence $\text{Re}(\gamma\tau) = \text{Re}(\tau) = 0$, so $\tau = \gamma\tau = i$, or $a = -1$, whence $\text{Re}(\tau) = \text{Re}(\gamma\tau) = -\frac{1}{2}$, so $\tau = \gamma\tau = \rho$.

If $a = 0$, then $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$, which stabilizes i , and $\langle -S \rangle = \langle S \rangle$.

If $a = -1$, then $\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (ST)^2$, which stabilizes ρ , and $(ST)^3 = I$, so $\langle (ST)^2 \rangle = \langle ST \rangle$.

- If $c = 1, d = 1, \tau = \rho$, then $\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$, so $\rho = \gamma\rho = \frac{a\rho+b}{\rho+1}$. We have $\rho^2 + \rho + 1 = 0$, so $\rho^2 + \rho = -1$, so $a\rho + b = \rho^2 + \rho = -1$. But $a, b \in \mathbb{Z}$ and $1, \rho$ are linearly independent over \mathbb{R} , so $a = 0, b = -1$, so $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -ST$, which stabilizes ρ .

- If $c = -1$, we can reduce this to the case $c = 1$ by replacing γ with $-\gamma$.

We have now shown the first three parts of the proposition. It remains to show the last part, i.e. $\Gamma(1) = \langle S, T \rangle$. Since $S^2 = -I$, it is enough to show that $H = G$. Choose $\tau \in \text{Int}(f)$, so $\text{Stab}_G(\tau) = \{I\}$. Let $g \in G$. By our claim proving (i), $\exists h \in H$ such that $hg\tau \in \mathfrak{f}'$. We must therefore have $hg\tau = \tau$, hence $hg \in \text{Stab}_G(\tau) = \{I\}$, so $g = h^{-1} \in H$. \square

Notation. We write $e_\tau = |\text{Stab}_G(\tau)|$.

Let f be a nonzero modular function of weight k , level $\Gamma(1)$. If $\tau \in \mathfrak{h}$, then $v_\tau(f)$ is the order of f at τ (the unique $n \in \mathbb{Z}$ such that $f(z) = (z - \tau)^n g(z)$ for some meromorphic g that is holomorphic and non-vanishing at τ). We define $v_\infty(f)$ to be the order of f at infinity, i.e. $v_\infty(f) = v_0(\tilde{f})$ for \tilde{f} the meromorphic function in $D(0, 1)$ with $f(\tau) = \tilde{f}(e^{2\pi i \tau})$.

Proposition 2.4. Let f be a nonzero modular function of weight k , level $\Gamma(1)$. Then

$$\sum_{\tau \in \Gamma(1) \backslash \mathfrak{h}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}.$$

Proof. We first check that the sum is well-defined:

- If $\tau \in \mathfrak{h}$, then $e_\tau, v_\tau(f)$ only depend on the $\Gamma(1)$ -orbit of τ . This is because if $\gamma \in \Gamma(1)$ and $\tau \in \mathfrak{h}$, then $\text{Stab}_{\Gamma(1)}(\tau)$ and $\text{Stab}_{\Gamma(1)}(\gamma\tau)$ are conjugate subgroups of $\Gamma(1)$, so $e_\tau = e_{\gamma\tau}$. On the other hand, $f(\gamma\tau) = f(\tau)j(\gamma, \tau)^k$ and $j(\gamma, \tau)$ is holomorphic and non-vanishing on \mathfrak{h} , so $v_{\gamma\tau}(f) = v_\tau(f)$.
- The sum only has a finite number of nonzero terms, since if f is a modular function and \tilde{f} is a meromorphic function on $D(0, 1)$, then $\exists \delta > 0$ such that \tilde{f} is holomorphic and non-vanishing in $D^*(0, \delta)$. Thus $\exists R > 0$ such that f is holomorphic and non-vanishing in $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) > R\}$. Hence to show the sum is finite, it suffices to show that f only has a finite number of zeroes and poles in \mathfrak{f} (as \mathfrak{f} intersects every $\Gamma(1)$ -orbit), for which it suffices to show that f has a finite number of zeroes and poles in $\mathfrak{f} \cap \{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \leq R\}$, which is true as the set is compact (closed and bounded) and the zeroes and poles of f are discrete.

13 Oct 2022,
Lecture 4

To prove the identity, we use contour integration. Setup: if $U \subset \mathbb{C}$ is an open subset, $f : U \rightarrow \mathbb{C}$ is holomorphic and $\gamma : [0, 1] \rightarrow U$ is a path, then

$$\int_{\gamma} f(z) dz = \int_{t=0}^1 f(\gamma(t)) \gamma'(t) dt.$$

We have the pullback formula: if $u : U \rightarrow V$ is a holomorphic map between open subsets of \mathbb{C} , $g : V \rightarrow \mathbb{C}$ is holomorphic and γ is a path in U , then

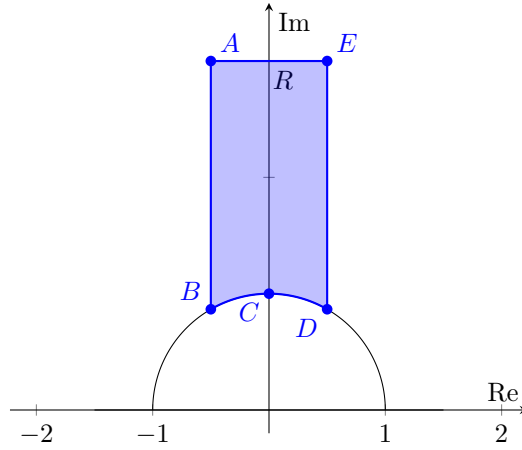
$$\int_{u \circ \gamma} g(z) dz = \int_{\gamma} u^*(g(z) dz) = \int_{\gamma} g(u(z)) u'(z) dz.$$

A particularly nice case: if $g(z) = h'(z)/h(z)$, then $g(z) dz = d \log h$, so $\int_{u \circ \gamma} d \log h = \int_{\gamma} u^*(d \log h) = \int_{\gamma} d(\log h \circ u) = \int_{\gamma} \frac{(h \circ u)'(z)}{(h \circ u)(z)} dz$.

We also have (Cauchy's) argument principle: if $U \subset \mathbb{C}$ is a simply connected open subset, $\gamma \subset U$ is a simple positively oriented closed path and g is a meromorphic function in U with no zeroes or poles on γ , then

$$\frac{1}{2\pi i} \oint_{\gamma} d \log g = \frac{1}{2\pi i} \oint_{\gamma} \frac{g'(z)}{g(z)} dz = \sum_{a \in \text{Int}(\gamma)} v_a(g).$$

We now apply this to our modular function f . Choose $R > 0$ such that f has no zeroes or poles in $\{\tau \in \mathfrak{h} \mid \text{Im}(\tau) \geq R\}$. We consider $\frac{1}{2\pi i} \oint_{\gamma} d \log f$, where γ is the contour $ABCDE$.



By choice of R , there are no zeroes or poles of f on AE . We first consider the case where f has no zeroes or poles at all on γ . Then the argument principle

gives

$$\frac{1}{2\pi i} \oint_{\gamma} d\log f = \frac{1}{2\pi i} \int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d\log f = \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_{\tau}} v_{\tau}(f)$$

(as $v_{\tau}(f) \neq 0$, $e_{\tau} = 1$ under our assumptions).

Apply the pullback formula with $u(\tau) = \tau + 1$. Then $u(AB) = ED$, $f \circ u = f$, so

$$\int_{u(AB)} d\log f = \int_{AB} d\log f \circ u = \int_{AB} d\log f = \int_{ED} d\log f = - \int_{DE} d\log f.$$

Hence $\int_{AB} + \int_{DE} d\log f = 0$.

Now take $q = e^{2\pi i \tau}$, so $f = \tilde{f} \circ q$ and $q(AE)$ is a positively oriented circle around 0 in $D(0, 1)$. So

$$\frac{1}{2\pi i} \int_{q(AE)} d\log \tilde{f} = v_{\infty}(f) = \frac{1}{2\pi i} \int_{AE} d\log \tilde{f} \circ q = \frac{1}{2\pi i} \int_{AE} d\log f.$$

Now take $v(\tau) = S(\tau) = -\frac{1}{\tau}$. Then $v(BC) = DC$ and we know $f|_k[S](\tau) = f(-\frac{1}{\tau})\tau^{-k} = f(\tau)$, so $f \circ v = f(\tau)\tau^k$. Hence

$$\begin{aligned} \int_{DC} d\log f &= \int_{v(BC)} d\log f = \int_{BC} d\log(f \circ v) = \int_{BC} d\log(f(\tau)\tau^k) \\ &= \int_{BC} d\log f + k d\log \tau = \int_{BC} d\log f + k(\log C - \log B) \end{aligned}$$

where here \log is any branch of the logarithm defined on BC . But $B = \rho$, $C = i$, so $\log B = i\frac{2\pi}{3}$ and $\log C = i\frac{\pi}{2}$. Hence

$$\int_{CD} d\log f = - \int_{DC} d\log f + k \left(\frac{2\pi i}{3} - \frac{2\pi i}{4} \right),$$

giving

$$\int_{BC} + \int_{CD} d\log f = 2\pi i k \frac{1}{12}.$$

We have

$$\begin{aligned} \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) &= \frac{1}{2\pi i} \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} d \log f \right) \\ &= \frac{1}{2\pi i} \left(0 + \frac{k}{12} + 0 - v_\infty(f) \right) \\ &\implies \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}. \end{aligned}$$

This finishes the proof in the case where there are no zeroes or poles. If there are zeroes or poles on γ , we need to modify the contour. For example, if there's a zero or a pole at a point P on AB , then consider the contour γ' , which is just γ but with a small semicircle around our (discrete) pole, which satisfies the property that f has no zeroes or poles on γ' . The trickiest case is when there is a zero or pole at $B = \rho$ or $C = i$. This is Q3 on example sheet 1. \square

16 Oct 2022,
Lecture 5

Example 2.1. Take $k = 4$, $f = E_4 \in M_4(\Gamma(1))$. Hence $\forall \tau \in \mathfrak{h}, v_\tau(E_4) \geq 0$ (as it is holomorphic in \mathfrak{h}). We know $E_4(\tau) = 1 + \sum_{n \geq 1} a_n q^n$, so $E_4(\infty) \neq 0$ and $v_\infty(E_4) = 0$. Hence our formula gives

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e^\tau} v_\tau(E_4) = \frac{1}{3} v_\rho(E_4) + \frac{1}{2} v_i(E_4) + \sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}, \tau \not\sim \rho, i} v_\tau(E_4) = \frac{1}{3}.$$

So we have $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{3}$, where $a, b, c \in \mathbb{Z}_{\geq 0}$, which gives the only solution $a = 1, b = c = 0$, so $E_4(\rho) = 0$ and $E_4(\tau) \neq 0$ if $\tau \notin \Gamma(1)\rho$.

If $k = 6$, $f = E_6$, then we get

$$\frac{1}{3} v_\rho(E_6) + \frac{1}{2} v_i(E_6) + \sum_{\tau \not\sim \rho, i} v_\tau(E_6) = \frac{6}{12} = \frac{1}{2},$$

so this forces $v_\rho(E_6) = 0$, $v_i(E_6) = 1$, $v_\tau(E_6) \neq 0$ if $\tau \not\sim \rho$ and $\tau \not\sim i$, so $E_6(i) = 0$, $E_6(\tau) \neq 0$ if $\tau \not\sim \rho, i$.

Recall $\Delta = \frac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma(1))$. This is nonzero since $\Delta(\rho) = \frac{E_4(\rho)^3 - E_6(\rho)^2}{1728} = -\frac{E_6(\rho)^2}{1728} \neq 0$. We also have $v_\infty(\Delta) \geq 1$ by construction, so plug in Δ to our formula to get

$$\sum_{\tau} \frac{1}{e^\tau} v_\tau(\Delta) + v_\infty(\Delta) = 1,$$

so $v_\infty(\Delta) = 1$, so Δ has a simple zero at ∞ and Δ is nonvanishing in \mathfrak{h} .

Theorem 2.5. Let $k \in 2\mathbb{Z}$. Then:

- (1) If $k < 0$ or $k = 2$, then $M_k(\Gamma(1)) = 0$; and $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.
- (2) If $4 \leq k \leq 10$, then $M_k(\Gamma(1)) = \mathbb{C} \cdot E_k$.
- (3) For any k , multiplication by Δ gives an isomorphism $M_k(\Gamma(1)) \xrightarrow{\times \Delta} S_{k+12}(\Gamma(1))$.

Proof. (1) Let $f \in M_k(\Gamma(1))$ be nonzero. Then $\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$. Note the LHS is ≥ 0 , but for $k < 0$, the RHS is < 0 . If $k = 2$, then we get the equation $\frac{a}{3} + \frac{b}{2} + c = \frac{1}{6}$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, which has no solutions.

Suppose $f \in M_0(\Gamma(1)) \setminus \mathbb{C} \cdot 1$. Then $f - f(\infty) \cdot 1 \in S_0(\Gamma(1))$ is a nonzero function (here 1 is the constant function 1). Then $\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot 1) + \underbrace{v_\infty(f - f(\infty) \cdot 1)}_{\geq 1} = 0$, a contradiction, so $M_0(\Gamma(1)) = \mathbb{C} \cdot 1$.

- (2) Let $4 \leq k \leq 10$ and $f \in M_k(\Gamma(1))$. Consider $f - f(\infty) \cdot E_k \in S_k(\Gamma(1))$. If this is nonzero, then

$$\sum_{e_\tau} \frac{1}{e_\tau} v_\tau(f - f(\infty) \cdot E_k) + \underbrace{v_\infty(f - f(\infty) \cdot E_k)}_{\geq 1} = \frac{k}{12} < 1,$$

a contradiction. So $f = f(\infty) \cdot E_k$.

- (3) Our map $\times \Delta : M_k(\Gamma(1)) \rightarrow S_{k+12}(\Gamma(1))$ is a well-defined \mathbb{C} -linear map. It is injective, since if $\Delta f = 0$, then $f = 0$ (as Δ is nonvanishing in \mathfrak{h}). For surjectivity, if $f \in S_{k+12}(\Gamma(1))$, then $\frac{f}{\Delta}$ is holomorphic in \mathfrak{h} and invariant under the weight k action of $\Gamma(1)$.

We need to show $\frac{f}{\Delta}$ is holomorphic at ∞ , as then $\frac{f}{\Delta} \in M_k(\Gamma(1))$, so $f = \frac{f}{\Delta} \Delta \in \text{Im}(\times \Delta)$. Hence we need $v_\infty\left(\frac{f}{\Delta}\right) \geq 0$. But $v_\infty\left(\frac{f}{\Delta}\right) = \underbrace{v_\infty(f)}_{\geq 1} - \underbrace{v_\infty(\Delta)}_{=1} \geq 0$, so we're done.

□

Corollary 2.6. If $k \in 2\mathbb{Z}$, $k \geq 0$, then $M_k(\Gamma(1))$ is finite-dimensional and

$$\dim_{\mathbb{C}} M_k(\Gamma(1)) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12}. \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}. \end{cases}$$

Proof. We proved this for $0 \leq k \leq 10$. In general, use induction on k : we need to show that for $k \geq 0$, $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = \dim_{\mathbb{C}} M_k(\Gamma(1)) + 1$.

We know $E_{k+12} \in M_{k+12}(\Gamma(1))$, so $M_{k+12}(\Gamma(1)) = \mathbb{C} E_{k+12} \oplus S_{k+12}(\Gamma(1))$. But this equals $\mathbb{C} E_{k+12} \oplus \Delta M_k(\Gamma(1))$, so $\dim_{\mathbb{C}} M_{k+12}(\Gamma(1)) = 1 + \dim_{\mathbb{C}} M_k(\Gamma(1))$.

□

Example 2.2. We have $E_4^2 \in M_8(\Gamma(1)) = \mathbb{C}E_8$. So there is a relation between E_4^2 and E_8 (in this case, one is a scalar multiple of the other), but we have $E_8(\infty) = 1 = E_4(\infty)^2 \implies E_4^2 = E_8$.

Similarly, $E_4E_6 \in M_{10}(\Gamma(1)) = \mathbb{C}E_{10}$, so we find $E_4E_6 = E_{10}$.

Corollary 2.7. If $k \in 2\mathbb{N}$, then $M_k(\Gamma(1))$ is spanned as a \mathbb{C} -vector space by $\{E_4^a E_6^b \mid a, b \in \mathbb{Z}_{\geq 0}, 4a + 6b = k\}$. In other words, if $\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma(1))$, then \mathcal{M} is a graded \mathbb{C} -algebra generated by E_4 and E_6 .

Proof. We proved this for $0 \leq k \leq 10$. If $k \geq 12$, then

$$M_k(\Gamma(1)) = \mathbb{C}E_k \oplus \Delta M_{k-12}(\Gamma(1)) = \mathbb{C}f \oplus \Delta M_{k-12}(\Gamma(1))$$

for any $f \in M_k(\Gamma(1))$ such that $f(\infty) \neq 0$ by the same argument. We can always find some $A, B \in \mathbb{Z}_{\geq 0}$ such that $4A + 6B = k$, so $E_4^A E_6^B \in M_k(\Gamma(1))$ and $(E_4^A E_6^B)(\infty) \neq 0$. Now by induction, $M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$, so $\Delta M_{k-12}(\Gamma(1)) = \langle \Delta E_4^a E_6^b \mid 4a + 6b = k - 12 \rangle$. But $\Delta \in \langle E_4^3, E_6^2 \rangle$, so

$$\Delta M_{k-12}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$$

and $E_4^A E_6^B \in \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$, so $M_k(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = k \rangle$. \square

Theorem 2.8. Let $j(\tau) = \frac{E_4(\tau)^3}{\Delta}$. Then j is a modular function of weight 0, level $\Gamma(1)$ which is holomorphic on \mathfrak{h} and has a simple pole at ∞ . It defines a bijection $\Gamma(1) \setminus \mathfrak{h} \rightarrow \mathbb{C}$ given by $\tau \rightarrow j(\tau)$. Moreover, every modular function of weight 0, level $\Gamma(1)$ is a rational function of j .⁶

The interpretation of this is that it is possible to define a Riemann surface structure on $\Gamma(1) \setminus \mathfrak{h} \sqcup \{\infty\}$ such that we get a compact Riemann surface whose meromorphic functions are exactly the modular functions of weight 0. So the theorem says that this Riemann surface, called $X(1)$, is isomorphic to the Riemann sphere, and our formula says that if \mathcal{L} is an invertible sheaf on a compact Riemann surface and S is a meromorphic section, then $\sum_a v_a(S) = \deg(\mathcal{L})$. This is useful if we are also taking algebraic geometry.

18 Oct 2022,
Lecture 6

Proof. We showed that Δ is nonvanishing in \mathfrak{h} and has a simple zero at ∞ . Hence j is holomorphic in \mathfrak{h} and $v_\infty(j) = 3v_\infty(E_4) - v_\infty(\Delta) = -1$. Note that if $\gamma \in \Gamma(1)$, then $j|_0[\gamma](\tau) = j(\gamma\tau) = j(\tau)$ since the map is constant on $\Gamma(1)$ -orbits. To show the map is a bijection, we need to show that $\forall z \in \mathbb{C}$, there exists a unique orbit $\Gamma(1) \cdot \tau$ such that $j(\tau) = z$, i.e. $v_\tau(j - z) > 0$.

We know

$$\sum_{\tau \in \Gamma(1) \setminus \mathfrak{h}} \frac{1}{e_\tau} \underbrace{v_\tau(j - z)}_{\geq 0, \text{ as } j - z \text{ is holomorphic in } \mathfrak{h}} = 1,$$

⁶Remember that $\Gamma(1) \setminus \mathfrak{h}$ is the set of orbits of $\Gamma(1)$ under \mathfrak{h} .

(since $v_\infty(j-z) = -1$ and $\frac{k}{12} = 0$) again giving $\frac{a}{3} + \frac{b}{2} + c = 1$ for $a, b, c \in \mathbb{Z}_{\geq 0}$, $a = v_\rho(j-z), b = v_i(j-z), c = \sum_{\tau \neq \rho, i} v_\tau(j-z)$. This gives the solutions

- $(a, b, c) = (0, 0, 1)$, so $j-z$ vanishes at a unique $\Gamma(1) \cdot \tau$.
- $(a, b, c) = (0, 2, 0)$, so $j-z$ vanishes at i .
- $(a, b, c) = (3, 0, 0)$, so $j-z$ vanishes at ρ .

Hence our map is bijective. Consider a nonzero modular function f of weight 0. To get rid of all the poles, we can consider a product $f \cdot \prod_{i=0}^n (j(\tau) - j(a_i))^{b_i}$ for $a_i \in \mathfrak{h}$, $b_i \in \mathbb{Z}_{\geq 0}$, where the a_i are among the poles of f in \mathfrak{h} . Hence to show f is a rational function of j , it is enough to consider the case where f is holomorphic in \mathfrak{h} . Then there exists $m \geq 0$ such that $\Delta^m f$ is holomorphic at ∞ , so $\Delta^m f$ is holomorphic in \mathfrak{h} and at ∞ , so $\Delta^m f \in M_{12m}(\Gamma(1))$. We showed that $M_{12m}(\Gamma(1)) = \langle E_4^a E_6^b \mid 4a + 6b = 12m \rangle$, so f is a linear combination of functions of the form $\frac{E_4^a E_6^b}{\Delta^m}$, where $4a + 6b = 12m$.

Hence it is enough to show that $\frac{E_4^a E_6^b}{\Delta^m}$ is a rational function of j where $4a + 6b = 12m$, $a, b \in \mathbb{Z}_{\geq 0}$. But then $2a + 3b = 6m$, which gives $p, q \in \mathbb{Z}_{\geq 0}$ such that $a = 3p, b = 2q$, so $p + q = m$. Then

$$\frac{E_4^a E_6^b}{\Delta^m} = \left(\frac{E_4^3}{\Delta} \right)^p \left(\frac{E_6^2}{\Delta} \right)^q = j^p \left(\frac{E_6^2}{\Delta} \right)^q.$$

As $E_4^3 - E_6^2 = 1728\Delta$, we get $j = \frac{E_6^2}{\Delta} + 1728$. So this is a rational function of j . \square

Proposition 2.9. Let $k \geq 4$ be an even integer. Then

$$G_k(\tau) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

where $q = e^{2\pi i \tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Proof. We start from the identity

$$\pi \cot(\pi \tau) = \frac{1}{\tau} + \sum_{n \geq 1} \left(\frac{1}{\tau + n} + \frac{1}{\tau - n} \right).$$

This is true for $\tau \in \mathfrak{h}$ and it is even locally uniformly convergent in \mathfrak{h} . We can write

$$\pi \cot(\pi \tau) = i\pi \frac{e^{\pi i \tau} + e^{-\pi i \tau}}{e^{\pi i \tau} - e^{-\pi i \tau}} = \pi i \frac{q + 1}{q - 1} = -\pi i (1+q)(1-q)^{-1} = -\pi i \left(1 + 2 \sum_{n \geq 1} q^n \right).$$

Differentiate term-by-term $k - 1$ times. The RHS of the bottom expression is

$$-2\pi i \left(\frac{d}{d\tau} \right)^{k-1} \left(\sum_{n \geq 1} q^n \right) = -(2\pi i)^k \sum_{n \geq 1} n^{k-1} q^n,$$

while the RHS of the top expression is

$$(-1)^{k-1} (k-1)! \left(\tau^{-k} + \sum_{n \geq 1} (\tau + n)^{-k} + (\tau - n)^{-k} \right) = (-1)^{k-1} (k-1)! \sum_{n \in \mathbb{Z}} (\tau + n)^{-k}.$$

Rearranging and using the fact that k is even (to make the sign go away) gives

$$\sum_{n \in \mathbb{Z}} (\tau + n)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^n, \tau \in \mathfrak{h}.$$

Then

$$G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k} = 2\zeta(k) + \sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus 0, \\ m \neq 0}} (m\tau + n)^{-k} = 2\zeta(k) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-k}.$$

Plug in our identity to get

$$G_k(\tau) = 2\zeta(k) + \sum_{m \geq 1} \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} q^{mn} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{N \geq 1} \underbrace{\left(\sum_{n|N} n^{k-1} \right)}_{=\sigma_{k-1}(N)} q^N.$$

□

Corollary 2.10. $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n \geq 1} a_n q^n$ has all $a_n \in \mathbb{Q}$. Moreover, if $k = 4$ or $k = 6$, then $a_n \in \mathbb{Z}$.

Proof. We have

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

Hence we need to show that $\frac{\zeta(k)}{\pi^k}$ is rational. This is on example sheet 1 (when

k is even). One can show that $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$, so

$$\begin{aligned} E_4(\tau) &= 1 + \frac{2^4 \pi^4 \cdot 90}{\pi^4 \cdot 6} \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \\ E_6(\tau) &= 1 - \frac{2^6 \pi^6 \cdot 3^3 \cdot 5 \cdot 7}{\pi^6 \cdot 5!} \sum_{n \geq 1} \sigma_5(n) q^n = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n. \end{aligned}$$

□

Corollary 2.11. If $\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n$ is the q -expansion of Δ , then $\tau(1) = 1$ and $\tau(n) \in \mathbb{Z} \forall n \geq 1$.

Proof. Write $E_4 = 1 + 240U$ and $E_6 = 1 - 504V$ for $U, V = q + \dots \in \mathbb{Z}[[q]]$. Then

$$\begin{aligned} \Delta &= \frac{E_4^3 - E_6^2}{1728} = \frac{(1 + 240U)^3 - (1 - 504V)^2}{1728} \\ &= \frac{3 \cdot 240U + 3 \cdot 240^2 U^2 + 240^3 U^3 + 2 \cdot 504V - 504^2 V^2}{1728} \\ &= \frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} + R, \end{aligned}$$

where we claim $R \in q^2 \mathbb{Z}[[q]]$, but for this we just need to check that $1728 \mid 3 \cdot 240^2, 1728 \mid 240^3, 1728 \mid 504^2$, which is true.

We need to check that

$$\frac{(3 \cdot 240U + 2 \cdot 504V)}{1728} = \frac{2^4 \cdot 3^2 \cdot 5 \cdot U + 2^4 \cdot 3^2 \cdot 7 \cdot V}{2^6 \cdot 3^3} \in \mathbb{Z}[[q]].$$

But this equals

$$\frac{5U + 7V}{12} = \frac{5(U - V)}{12} + V.$$

Hence we need to check that

$$\frac{5}{12}(\sigma_3(n) - \sigma_5(n)) \in \mathbb{Z} \forall n \geq 1,$$

i.e. we need to check that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod{12} \forall n \geq 1.$$

But this is true as $d^3 \equiv d^5 \pmod{12} \forall d \in \mathbb{N}$.

Finally, we compute $\tau(1) = \frac{3 \cdot 240 + 2 \cdot 504}{1728} = 1$. □

Theorem 2.12. Let $k \geq 4$ be even and $N = \dim_{\mathbb{C}} S_k(\Gamma(1))$. Then there exists a unique basis f_0, \dots, f_N for $M_k(\Gamma(1))$ as a \mathbb{C} -vector space such that

20 Oct 2022,
Lecture 7

(a) $\forall 0 \leq i \leq N$, $f_i = \sum_{n \geq 0} a_n(f_i) q^n$ for $a_n(f_i) \in \mathbb{Z} \forall n \geq 0$.

(b) If $0 \leq i, n \leq N$, then $a_n(f_i) = \delta_{in}$.

So in other words, $f_i = q^i + O(q^{N+1})$. This is important because $M_k(\Gamma(1))$ has a \mathbb{Z} -structure, i.e. we can realize it as a tensor product $M_k(\Gamma(1)) = M_k(\Gamma(1), \mathbb{Z}) \oplus \mathbb{C}$, where $M_k(\Gamma(1), \mathbb{Z}) = \{f \in M_k(\Gamma(1)) \mid \forall n \geq 0, a_n(f) \in \mathbb{Z}\}$.

Proof. We first construct $f_0, \dots, f_N \in M_k(\Gamma(1))$ with properties (a) and (b). Write $k = 12a + d$, for $a, d \in \mathbb{Z}_{\geq 0}$ such that $d = 14$ if $k \equiv 2 \pmod{12}$, or $0 \leq d \leq 10$ if $d \not\equiv 2 \pmod{12}$.

Then

$$\left\lfloor \frac{k}{12} \right\rfloor = \begin{cases} a & k \not\equiv 2 \pmod{12} \\ a+1 & k \equiv 2 \pmod{12} \end{cases} \implies \lfloor a \rfloor = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & k \not\equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \pmod{12} \end{cases}.$$

We have $\dim_{\mathbb{C}} M_k(\Gamma(1)) = N + 1 = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor + 1 & k \not\equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor & k \equiv 2 \pmod{12} \end{cases}$, so $a = N$, $k = 12N + d$.

Now consider $A, B \in \mathbb{Z}_{\geq 0}$ such that $d = 4A + 6B$. Consider the modular forms

$$g_i = \Delta^i E_4^A E_6^B E_6^{2(N-i)}$$

for $0 \leq i \leq N$. Each g_i has weight $12i + 4A + 6B + 12(N-i) = 12N + d = k$, so $g_i \in M_k(\Gamma(1))$. As E_4, E_6, Δ have q -expansions in $\mathbb{Z}[[q]]$, so does g_i . The leading term of g_i is q^i , so the q -expansions look like

$$\begin{aligned} g_0 &= 1 + a_1(g_0)q + \dots + a_N(g_0)q^N + O(q^{N+1}) \\ &\vdots \\ g_{N-1} &= 0 + \dots + q_{N-1} + a_N(g_{N-1})q^N + O(q^{N+1}) \\ g_N &= 0 + \dots + 0 + q^N + O(q^{N+1}) \end{aligned}$$

We can now carry out row reduction on the g_i to obtain f_0, \dots, f_N satisfying (a) and (b). For uniqueness, consider the linear functionals

$$\begin{aligned} a_0, \dots, a_N : M_k(\Gamma(1)) &\rightarrow \mathbb{C} \\ f &\mapsto a_i(f), \quad f = \sum_{n \geq 0} a_n(f) q^n. \end{aligned}$$

Then $a_i(f_j) = \delta_{ij}$, which forces a_0, \dots, a_N to be linearly independent. Hence they form a basis of the dual vector space $M_k(\Gamma(1))^*$. So f_0, \dots, f_N is the dual basis of $M_k(\Gamma(1))$, and they form the unique basis with this property. \square

3 Hecke operators

Hecke operators are just symmetries (linear endomorphisms) of spaces of modular forms. They can arise from either representation theory: $\Gamma(1) \leq GL_2(\mathbb{Q})^+$, which acts on $\{f : \mathfrak{h} \rightarrow \mathbb{C}\}$ by $f \mapsto f|_k[g]$. But $M_k(\Gamma(1)) \leq \{f : \mathfrak{h} \rightarrow \mathbb{C}\}^{\Gamma(1)}$, and a general group theory fact says that under suitable conditions, there's an action by a big class of operators; or from geometry: we can think of modular forms as functions on the set of lattices \mathcal{L} in \mathbb{C} . In this course, we will follow the second point of view.

Recall. If V is a finite-dimensional \mathbb{R} -vector space, then a lattice Λ in V is a subgroup $\Lambda \subset V$ which is discrete and cocompact (i.e. V/Λ is compact).

Lemma 3.1. A subgroup $\Lambda \leq V$ is a lattice if and only if there exists a basis e_1, \dots, e_n for V as a \mathbb{R} -vector space such that $\Lambda = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$.

Proof. This is a question on example sheet 2. \square

We study $\mathcal{L} = \{\Lambda \leq \mathbb{C} \text{ a lattice}\}$ with its action by \mathbb{C}^\times , i.e. $z\Lambda = \{z\lambda \mid \lambda \in \Lambda\}$ for $z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}$.

Proposition 3.2. The map $\tau \mapsto \Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$ induces a bijection between

$$\Gamma(1) \backslash \mathfrak{h} \leftrightarrow \mathbb{C}^\times \backslash \mathcal{L}$$

(orbits of $\Gamma(1)$ in \mathfrak{h} and the set of lattices in \mathbb{C} modulo scalar multiplication).

Proof. This map is well-defined, since if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, $\tau \in \mathfrak{h}$, then

$$\Lambda_{\gamma\tau} = \mathbb{Z} \left(\frac{a\tau + b}{c\tau + d} \right) \oplus \mathbb{Z} = (c\tau + d)^{-1} (\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)) = (c\tau + d)^{-1} \Lambda_\tau.$$

For surjectivity, if Λ is a lattice, then $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\text{Im} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \neq 0$. Swapping e_1, e_2 if necessary, we may assume that $\text{Im} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} > 0$. Then $\Lambda = e_2(\mathbb{Z}e_1/e_2 \oplus \mathbb{Z}) = e_1\Lambda_\tau$ for $\tau = \frac{e_1}{e_2}$.

For injectivity, if τ, τ' have the same image, then $\exists z \in \mathbb{C}^\times$ such that $\mathbb{Z}\Lambda_\tau = \mathbb{Z}\Lambda_{\tau'}$, i.e. $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $\tau' = az\tau + bz, 1 = cz\tau + dz$. Then $\tau' = \frac{az\tau + bz}{cz\tau + dz} = \frac{a\tau + b}{c\tau + d}$. But $\text{Im}(\tau') = \text{Im}(\gamma\tau) = \det(\gamma) \frac{\text{Im}(\tau)}{|c\tau + d|^2}$ and $\text{Im}(\tau) > 0, \text{Im}(\tau') > 0$, hence $\det(\gamma) > 0$, so $\det(\gamma) = 1$ and so $\gamma \in \Gamma(1)$. \square

Definition 3.1. If $k \in \mathbb{Z}$, say a function $F : \mathcal{L} \rightarrow \mathbb{C}$ is **of weight k** if $\forall z \in \mathbb{C}^\times, \Lambda \in \mathcal{L}, F(z\Lambda) = z^{-k}F(\Lambda)$.

Proposition 3.3. Let

$$\begin{aligned} V_k &= \{F : \mathcal{L} \rightarrow \mathbb{C} \text{ of weight } k\}. \\ W_k &= \{f : \mathfrak{h} \rightarrow \mathbb{C} \mid \forall \gamma \in \Gamma(1), f|_k[\gamma] = f\}. \end{aligned}$$

Then the map $F \mapsto (f : \tau \mapsto F(\Lambda\tau))$ induces a \mathbb{C} -vector space isomorphism $V_k \rightarrow W_k$.

Proof. We first check that if $F \in V_k$, $f(\tau) = F(\Lambda\tau)$, then $f \in W_k$. If $\gamma \in \Gamma(1)$,

$$f|_k[g](\tau) = f(\gamma\tau)j(\gamma, \tau)^{-k} = F(\lambda\gamma\tau)j(\gamma, \tau)^{-k} = F(j(\gamma, \tau)\Lambda_{\gamma\tau}) = F(\Lambda\tau) = f(\tau),$$

so $j(\gamma, \tau)\Lambda_{\gamma\tau} = \Lambda_\tau$.

To show that the map is an isomorphism, we write down its inverse: define $\alpha : W_k \rightarrow V_k$ by $\alpha(f)(\Lambda) = e_2^{-k} f(e_1/e_2)$ if $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\text{Im}(e_1/e_2) > 0$. This is well-defined, since if e'_1, e'_2 is another basis with $\text{Im}(e'_1/e'_2) > 0$, then $\exists \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ such that $e'_1 = ae_1 + be_2$, $e'_2 = ce_1 + de_2$. Then

$$\begin{aligned} e_2'^{-k} f(e'_1/e'_2) &= (ce_1 + de_2)^{-k} f\left(\frac{ae_1 + be_2}{ce_1 + de_2}\right) \\ &= e_2^{-k} (ce_1/e_2 + d)^{-k} f\left(\frac{ae_1/e_2 + b}{ce_1/e_2 + d}\right) = e_2^{-k} f\left(\frac{e_1}{e_2}\right). \end{aligned}$$

Exercise: check that the two maps are inverse to each other. \square

23 Oct 2022,
Lecture 8

Definition 3.2. Let $n \in \mathbb{N}$. The n^{th} Hecke operator $T_n : V_k \rightarrow V_k$ is defined by the formula

$$(T_n F)(\Lambda) = n^{k-1} \sum_{\substack{\Lambda' \leq \Lambda \\ n \mid \Lambda}} F(\Lambda').$$

Here $\sum_{\Lambda' \leq \Lambda}$ means summing over all subgroups Λ' of Λ of index n .

We also write $T_n : W_k \rightarrow W_k$ for the endomorphism arising from the isomorphism $V_k \xrightarrow{\sim} W_k$.

Why is T_n a well-defined endomorphism of V_k ? First of all, the sum is finite since there's a bijection

$$\begin{aligned} \{\Lambda' \leq \Lambda\} &\leftrightarrow \{H \leq \Lambda/n\Lambda \text{ of index } n\} \\ \Lambda' &\mapsto \Lambda'/n\Lambda \\ H + n\Lambda &\leftrightarrow H \end{aligned}$$

This is well-defined, since Lagrange's theorem implies that

$$\Lambda' \leq_n \Lambda \implies n(\Lambda/\Lambda') = 0 \implies n\Lambda \leq \Lambda'.$$

But $\Lambda/n\Lambda \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is finite, so it has finitely many subgroups of index n .

If $\Lambda' \leq_n \Lambda$, then $n\Lambda \leq \Lambda' \leq \Lambda$, so Λ' is also discrete and cocompact in \mathbb{C} .

We next check that $T_n F$ is of weight k , i.e. that $(T_n F)(z\Lambda) = z^{-k}(T_n F)(\Lambda)$. We have an isomorphism $\{\Lambda' \leq_n z\Lambda\} \leftrightarrow \{\Lambda' \leq_n \Lambda\}$ given by $\Lambda' \mapsto z^{-1}\Lambda'$, so

$$(T_n F)(z\Lambda) = n^{k-1} \sum_{\Lambda' \leq_n z\Lambda} F(\Lambda') = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} F(z\Lambda') = n^{k-1} \sum_{\Lambda' \leq_n \Lambda} z^{-k} F(\Lambda') = z^{-k} (T_n F)(\Lambda).$$

Proposition 3.4. (1) If $m, n \in \mathbb{N}$ with $(m, n) = 1$, then $T_m T_n = T_{mn}$.

(2) If p is a prime number and $n \in \mathbb{N}$, then $T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{k-1}}$ (acting on V_k).

Proof. Let $m, n \in \mathbb{N}$, not necessarily coprime. Then

$$\begin{aligned} (T_m(T_n F))(\Lambda) &= m^{k-1} \sum_{\Lambda' \leq_m \Lambda} (T_n F)(\Lambda') = (mn)^{k-1} \sum_{\Lambda' \leq_m \Lambda} \sum_{\Lambda'' \leq_n \Lambda'} F(\Lambda'') \\ &= (mn)^{k-1} \sum_{\Lambda'' \leq_{mn} \Lambda} a(\Lambda, \Lambda'') F(\Lambda''), \end{aligned}$$

where $a(\Lambda, \Lambda'') = |\{\Lambda' \leq_m \Lambda'' \mid \Lambda' \leq_n \Lambda\}| = |H \leq \Lambda/\Lambda'' \mid |H| = n|$ is the number of ways to express Λ' as an intermediate subgroup. If $(m, n) = 1$, then $a(\Lambda, \Lambda'') = 1$ for all $\Lambda'' \leq \Lambda$ as any finite abelian group of order mn has a unique subgroup of order n .

(1) In this case, we find

$$T_m T_n F(\Lambda) = (mn)^{k-1} \sum_{\Lambda'' \leq_{mn} \Lambda} F(\Lambda'') = (T_{mn} F)(\Lambda) \implies T_m T_n = T_{mn}.$$

(2) The same computation gives (for p prime, $n \in \mathbb{N}$)

$$(T_{p^n}(T_p F))(\Lambda) = p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} \Lambda} a(\Lambda, \Lambda'') F(\Lambda''),$$

where $a(\Lambda, \Lambda'') = |\{H \subset \Lambda/\Lambda'' \mid |H| = p\}|$. But if $\Lambda'' \leq_{p^{n+1}} \Lambda$, then Λ/Λ'' need not have a unique subgroup of order p , as $\Lambda \cong \mathbb{Z}^2$, so Λ/Λ'' is a finite

abelian group of order p^{n+1} that can be generated by 2 elements. But any such group is isomorphic to $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$, where $a \geq b \geq 0$ are integers such that $a + b = n + 1$. We now split into two cases:

- $b = 0$, so $a = n + 1$ and $\Lambda/\Lambda'' \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$. This group is cyclic and has a unique subgroup of order p , so $a(\Lambda, \Lambda'') = 1$.
- $b > 0$, so $\Lambda/\Lambda'' \cong \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$. Let $\Lambda/\Lambda''[p] = \{x \in \Lambda/\Lambda'' \mid px = 0\}$. This is a subgroup of Λ/Λ'' , and

$$\{H \leq \Lambda/\Lambda'' \mid |H| = p\} = \{H \leq \Lambda/\Lambda''[p] \mid |H| = p\}.$$

Hence $\Lambda/\Lambda''[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ from our above isomorphism. So in this case, $a(\Lambda, \Lambda'') = |\{H \leq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \mid |H| = p\}|$. In other words,

$$a(\Lambda, \Lambda') = |\mathbb{P}^1(\mathbb{F}_p)| = |\mathbb{A}^1(\mathbb{F}_p) \cup \{\infty\}| = p + 1.$$

How do we distinguish between these two cases? We will show on example sheet 2 that if $\Lambda'' \leq_{p^{n+1}} \Lambda$, then there exists a \mathbb{Z} -basis e_1, e_2 for Λ such that $\Lambda'' = \mathbb{Z}p^a e_1 \oplus \mathbb{Z}p^b e_2$ for the same a, b satisfying $a \geq b \geq 0, a + b = n + 1$ as before (this is a consequence of Smith normal form).

Hence we see that we are in case 2 if and only if $\Lambda'' \leq p\Lambda$. Thus we find

$$(T_{p^n}(T_p F)(\Lambda)) = p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} \Lambda} F(\Lambda'') + p^{(n+1)(k-1)} \sum_{\substack{\Lambda'' \leq p\Lambda \\ p^{n-1}}} pF(\Lambda'').$$

Here each Λ'' in case 1 goes into the first sum and each Λ'' in case 2 goes once into the first sum and p times into the second sum. We have

$$\begin{aligned} & p^{(n+1)(k-1)} \sum_{\Lambda'' \leq_{p^{n+1}} p\Lambda} pF(\Lambda'') = p^{(n-1)(k-1)} p^{2(k-1)} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} pF(p\Lambda'') \\ & = p^{(n-1)(k-1)} p^{2(k-1)} p^{1-k} \sum_{\Lambda'' \leq_{p^{n-1}} \Lambda} F(\Lambda'') = p^{k-1} T_{p^{n+1}} F(\Lambda). \end{aligned}$$

$$\text{Hence } T_{p^n} T_p F(\Lambda) = T_{p^{n+1}} F(\Lambda) + p^{k-1} T_{p^{n-1}} F(\Lambda).$$

□

Corollary 3.5. $\forall m, n \in \mathbb{N}, T_m T_n = T_n T_m$ as endomorphisms of V_k , i.e. all Hecke operators commute.

Proof. If we write $m = \prod_{i=1}^r p_i^{a_i}$ for $a_i \geq 1, p_i$ distinct, then $T_m = T_{p_1^{a_1}} \dots T_{p_r^{a_r}}$. We've shown that if p, q are distinct primes, then T_{p^a}, T_{q^b} commute $\forall a, b \geq 1$.

We need to show that if p is a prime and $a, b \in \mathbb{N}$, then T_{p^a} and T_{p^b} commute. But we have a stronger claim that $\forall a \in \mathbb{N}$, T_{p^a} is a polynomial in T_p . We prove this by induction on a , the case $a = 1$ being trivial.

In general, $T_{p^{a+1}} = T_{p^a}T_p - p^{k-1}T_{p^{a-1}}$, which proves the claim. \square

25 Oct 2022,
Lecture 9

Lemma 3.6. Let $n \in \mathbb{N}$ and $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \leq \mathbb{C}$ a lattice. Then $\{\Lambda' \leq_n \Lambda\} = \{\mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2 \mid a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, b < d\}$, where this is isomorphic to the set $\{a, b, d \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$.

Proof. Consider the short exact sequence

$$0 \rightarrow \mathbb{Z}e_2 / \mathbb{Z}e_2 \cap \Lambda' \rightarrow \Lambda / \Lambda' \rightarrow \underbrace{\Lambda / \mathbb{Z}e_2 + \Lambda'}_{\cong \mathbb{Z}e_1 / \mathbb{Z}e_1 \cap (\mathbb{Z}e_2 + \Lambda)} \rightarrow 0.$$

Then $|\Lambda / \Lambda'| = n$. We let $d = |\mathbb{Z}e_2 / \mathbb{Z}e_2 \cap \Lambda'| = \inf\{d \geq 1 \mid de_2 \in \Lambda'\}$ and $a = |\Lambda / \mathbb{Z}e_2 + \Lambda'| = \inf\{a \geq 1 \mid \exists b \in \mathbb{Z} \text{ s.t. } ae_1 + be_2 \in \Lambda'\}$. Then $n = ad$ and there exists a unique $0 \leq b < d$ such that $ae_1 + be_2 \in \Lambda'$.

We now claim that $\Lambda' = \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$. The inclusion \supseteq is clear. On the other hand, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z})$, $\alpha\delta - \beta\gamma = N \in \mathbb{Z}$ is nonzero, then $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = |N|$. So $[\Lambda : \mathbb{Z}(\alpha e_1 + \beta e_2) \oplus \mathbb{Z}(\gamma e_1 + \delta e_2)] = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = n = [\Lambda : \Lambda']$, so $[\Lambda' : \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2] = 1$, so they're equal.

We've defined a map $\{\Lambda' \leq_n \Lambda\} \rightarrow \{(a, b, d) \in \mathbb{Z}_{\geq 0} \mid ad = n, 0 \leq b < d\}$. This map has an inverse, given by $(a, b, d) \mapsto \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}de_2$, so it's a bijection. \square

Lemma 3.7. Let $f \in W_k$. Then we have the two formulas

$$(T_n f)(\tau) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) = \sum_{\substack{ad=n \\ 0 \leq b < d}} f|_k \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right].$$

Proof. $f \leftrightarrow F \in V_k$ with $f(\tau) = F(\Lambda_\tau)$. By definition,

$$(T_n f)(\tau) = (T_n F)(\Lambda_\tau) = n^{k-1} \sum_{\Lambda' \leq_n \Lambda_\tau} F(\Lambda') = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} F(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}d).$$

This equals

$$n^{k-1} \sum_{a,b,d} F(d(\mathbb{Z}(\frac{a\tau + b}{d} \oplus \mathbb{Z}))) = n^{k-1} \sum_{a,b,d} d^{-k} F(\Lambda_{\frac{a\tau + b}{d}}) = n^{k-1} \sum_{a,b,d} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

For the second formula, recall that if $g \in GL_2(\mathbb{R})^+$, then $f|_k[g] = \det(g)^{k-1} f(g\tau)j(g, \tau)^{k-1}$, so

$$f|_k\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right](\tau) = n^{k-1} f\left(\frac{a\tau + b}{d}\right) d^{-k}.$$

$$\text{Hence } (T_n f)(\tau) = \sum_{a,b,d} f|_k\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]. \quad \square$$

Corollary 3.8. If $f \in W_k$ and f is holomorphic, then $T_n f$ is also holomorphic.

Proof. Look at the formula above: $T_n f$ is a finite sum of holomorphic functions. \square

Proposition 3.9. Let $f \in W_k$ be holomorphic in \mathfrak{h} with q -expansion $f(\tau) = \sum_{m \in \mathbb{Z}} b_m q^m$. Then $T_n f$ has q -expansion $T_n f = \sum_{m \in \mathbb{Z}} c_m q^m$, where

$$c_m = \sum_{\substack{a \in \mathbb{N} \\ a|(m,n)}} a^{k-1} b_{(mn/a^2)}.$$

Proof.

$$\begin{aligned} T_n f &= n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) = n^{k-1} \sum_{\substack{ad=n \\ 0 \leq b < d}} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a \tau / d} e^{2\pi i m b \tau / d} \\ &= n^{k-1} \sum_{ad=n} d^{-k} \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m a \tau / d} \left(\sum_{0 \leq b < d} e^{2\pi i m b \tau / d} \right). \end{aligned}$$

Note that $\sum_{0 \leq b < d} e^{2\pi i m b \tau / d} = \begin{cases} d & d \mid m \\ 0 & \text{otherwise} \end{cases}$. Hence

$$T_n f = n^{k-1} \sum_{ad=n} d^{1-k} \sum_{m \in \mathbb{Z}} b_{dm} e^{2\pi i a m \tau}.$$

This gives

$$T_n f = \sum_{ad=n} \left(\frac{n}{d}\right)^{k-1} \sum_{m \in \mathbb{Z}} b_{dm} q^{am} = \sum_{a|n} a^{k-1} \sum_{m \in \mathbb{Z}} b_{nm/a} q^{am}.$$

This equals $\sum_{N \in \mathbb{Z}} c_N q^N$, where $c_N = \sum_{\substack{a|m \\ a|n}} a^{k-1} b_{nN/a^2}$. \square

Theorem 3.10. T_n preserves the subspaces $S_k(\Gamma(1)) \leq M_k(\Gamma(1)) \leq W_k \forall n \geq 1$. Moreover, if $f \in M_k(\Gamma(1))$, then $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$ and $a_1(T_n f) = a_n(f)$.

Proof. To show that T_n preserves $M_k(\Gamma(1))$, we need to show that if $f \in M_k(\Gamma(1))$, then $T_n f$ is holomorphic in \mathfrak{h} (then we're done by the previous corollary) and at ∞ , i.e. $a_N(T_n f) = 0$ if $N < 0$.

But $a_N(T_n f) = \sum_{a|(m,n)} a^{k-1} a_{Nn/a^2}(f)$. Since $Nn/a^2 < 0$ and f is holomorphic at ∞ , all summands are 0, so $T_n f$ is holomorphic at ∞ .

We have $a_0(T_n f) = \sum_{a|(n,0)} a^{k-1} a_{n \cdot 0/a^2}(f) = \sum_{a|n} a^{k-1} a_0(f) = \sigma_{k-1}(n) a_0(f)$.

Also $a_1(T_n f) = \sum_{a|(n,1)} a^{k-1} a_{n \cdot 1/a^2}(f) = a_n(f)$.

Finally, if $f \in S_k(\Gamma(1))$, then $a_0(f) = 0$, and then $T_n f \in M_k(\Gamma(1))$ and $a_0(T_n f) = \sigma_{k-1}(n) a_0(f) = 0 \implies T_n f \in S_k(\Gamma(1))$. \square

Our next goal is to study the spectral decomposition of Hecke operators on $M_k(\Gamma(1))$, i.e. the decomposition of $M_k(\Gamma(1))$ as a sum of (simultaneous) generalized eigenspaces for the T_n .

The simplest case is when $M_k(\Gamma(1))$ or $S_k(\Gamma(1))$ is 1-dimensional (as then every nonzero element is an eigenvector). For example, $S_{12}(\Gamma(1))$ is 1-dimensional, spanned by $\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n$. So Δ is a T_n -eigenvector for all $n \geq 1$. If $T_n \Delta = \alpha_n \Delta$ for some $\alpha_n \in \mathbb{C}$, then $a_1(T_n \Delta) = a_1(\alpha_n \Delta) = \alpha_n a_1(\Delta) = \alpha_n$ (as we proved $a_1(\Delta) = 1$). But we also have $a_1(T_n \Delta) = a_n(\Delta) = \tau(n)$. Hence $\alpha_n = \text{Hecke eigenvalue} = \tau(n) = \text{coefficient of } q^n$.

Ramanujan conjectured in 1916 that τ is multiplicative and $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$ for p prime, $n \in \mathbb{N}$. These identities are true for Hecke operators (i.e. $T_{mn} = T_m T_n$ and $T_{p^{n+1}} = T_p T_{p^n} - p^{k-1} T_{p^{n-1}}$), hence also for the eigenvalues α_n , hence for the numbers $\tau(n)$.

27 Oct 2022,
Lecture 10

Our goal now is to study the spectral decomposition of $M_k(\Gamma(1))$ and the arithmetic properties of Hecke eigenvalues.

Definition 3.3. If $f \in M_k(\Gamma(1))$, we say f is an **eigenform** if f is a T_n -eigenvector $\forall n \geq 1$.

We say f is a **normalized eigenform** if $a_1(f) = 1$.

Lemma 3.11. Suppose $k > 0$. Then any eigenform $f \in M_k(\Gamma(1))$ is a scalar multiple of a unique normalized eigenform. Moreover, if f is normalized, then $T_n(f) = a_n(f)f \ \forall n \geq 1$. (In other words, the n^{th} Hecke eigenvalue = the n^{th} q -expansion coefficient).

For example, Δ is a normalized eigenform and $\tau(n)\Delta = T_n \Delta$.

Proof. We know $a_1(T_n f) = a_n(f)$. We need to show that if f is an eigenform, then $a_1(f) \neq 0$ (as then $f/a_1(f)$ is normalized). But if $a_1 = 0$ and α_n is the eigenvalue of T_n on f , then $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = 0 \ \forall n \geq 1$.

Then $f = \sum_{n \geq 0} a_n(f)q^n = a_0(f)$, which is a contradiction as constants are not modular forms of weights $k > 0$.

If f is normalized, then $a_n(f) = a_1(T_n f) = a_1(\alpha_n f) = \alpha_n a_1(f) = \alpha_n$. \square

Proposition 3.12. Let $k \geq 4$ be even. Then $G_k(\tau)$ is an eigenform.

Proof. We need to show that G_k is a T_n -eigenvector $\forall n \geq 1$. We know T_n is a polynomial in T_p for p ranging over $p \mid n$ for p prime. Hence it is enough to show that G_k is a T_p -eigenvector $\forall p$ prime.

$G_k(\tau) = G_k(\Lambda_\tau)$ for $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$. Then

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\Lambda' \leq \frac{\Lambda}{p}} G_k(\Lambda') = p^{k-1} \sum_{\Lambda' \leq \frac{\Lambda}{p}} \sum_{\lambda \in \Lambda' \setminus 0} \lambda^{-k} = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} a(\Lambda, \lambda) \lambda^{-k}$$

where $a(\Lambda, \lambda) = |\{\Lambda' \leq \frac{\Lambda}{p} \mid \lambda \in \Lambda'\}|$. We know that if $\Lambda' \leq \frac{\Lambda}{p}$, then $p\Lambda \leq \Lambda' \leq \Lambda$ and we have a bijection $\{\Lambda' \leq \frac{\Lambda}{p}\} \leftrightarrow \{H \leq \Lambda/p\Lambda \mid |H| = p\}$.

If $\lambda \in p\Lambda$, then $\{\Lambda' \leq \frac{\Lambda}{p} \mid \lambda \in \Lambda'\} = \{\Lambda' \leq \frac{\Lambda}{p}\}$, so $a(\Lambda, \lambda) = p + 1$.

If $\lambda \notin p\Lambda$, then $\lambda \neq 0$ modulo $p\Lambda$ and there exists a unique subgroup $H \leq \Lambda/p\Lambda$ of order p such that $\lambda \in H$. Hence in this case, $\{\Lambda' \leq \frac{\Lambda}{p}\} = \{\mathbb{Z}\lambda + p\Lambda\}$ and $a(\Lambda, \lambda) = 1$. Hence

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in p\Lambda \setminus 0} p \lambda^{-k}.$$

We get

$$(T_p G_k)(\Lambda) = p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k} + p^{k-1} \sum_{\lambda \in \Lambda \setminus 0} p(p\lambda)^{-k} = p^{k-1} G_k(\Lambda) + G_k(\Lambda) = \sigma_{k-1}(p) G_k(\Lambda).$$

\square

We can compute the T_n -eigenvalues on G_k for all n now using $a_0(T_n f) = \sigma_{k-1}(n) a_0(f)$. So if f is an eigenform and $a_0(f) \neq 0$, then this forces the eigenvalue to be equal to $\sigma_{k-1}(n)$. So $T_n G_k = \sigma_{k-1}(n) G_k \forall n \geq 1$. The q -expansion of G_k is $2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ and we also defined $E_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$. Hence $a_0(E_k) = 1$, but E_k is not a normalized eigenform. Hence the associated normalized eigenform is

$$F_k(\tau) = \frac{\zeta(k)(k-1)!}{(2\pi i)^k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n = \frac{-B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n = \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

(here we gave multiple equivalent expressions).

We have a decomposition $M_k(\Gamma(1)) = \mathbb{C}F_k \oplus S_k(\Gamma(1))$ (for $k \geq 4$). Both summands are T_n -invariant, so it's enough to study the action of T_n on S_k .

Remark. T_n do not usually respect multiplication. In particular, the product of eigenforms is not usually an eigenform. For example, $E_4^2 = E_8$, but $E_4^3 \in M_{12}(\Gamma(1)) = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$ requires both E_{12} and \mathbb{C} to be expressed and hence is not an eigenform.

Proposition 3.13. If $f \in S_k(\Gamma(1))$ is a cuspidal eigenform, then all of the T_n -eigenvalues on f are algebraic integers. If f is normalized, then $\mathbb{Q}(\{a_n(f)\}_{n=1}^\infty)$ has finite degree over \mathbb{Q} (i.e. it is a number field).

Proof. We will show that for all $n \geq 1$, all eigenvalues of T_n on $S_k(\Gamma(1))$ are algebraic integers. We will do this by showing that the characteristic polynomial of T_n acting on S_k has integer coefficients (and it is of course monic).

We consider the basis f_1, \dots, f_N for $S_k(SL_2(\mathbb{Z}))$ characterized by:

- $\forall 1 \leq i \leq N$ and $\forall n \geq 1$, $a_n(f_i) \in \mathbb{Z}$.
- $\forall 1 \leq i, n \leq N$, $a_n(f_i) = \delta_{in}$.

Recall that this meant that f_1, \dots, f_N was the dual basis to the basis of functionals a_1, \dots, a_N of $S_k(\Gamma(1))^*$. Hence $\forall f \in S_k(\Gamma(1))$, $f = \sum_{i=1}^N a_i(f)f_i$ (this identity holds for any elements of a finite dimensional vector space with its basis and dual basis)

The claim is that if A denotes the matrix of T_n in the basis of f_1, \dots, f_N , then A has integer entries. As the characteristic polynomial of T_n is $\det(X \cdot I - A)$, this will show that the characteristic polynomial has coefficients in \mathbb{Z} .

By definition, $T_n(f) = \sum_{i=1}^N A_{ij}f_i$. Then for $1 \leq m \leq N$,

$$a_m(T_n f_j) = \sum_{i=1}^N A_{ij}a_m(f_j) = \sum_{i=1}^N A_{ij}\delta_{im} = A_{mj}.$$

But $a_m(T_n f_j) = \sum_{a|(m,n)} a^{k-1} a_{mn/a^2}(f_j)$ by the formula from the last lecture. Note that each $a_{mn/a^2}(f_j)$ is in \mathbb{Z} by the definition of f_j , so $\forall m, j$, $A_{mj} \in \mathbb{Z}$.

If f is a normalized eigenform, $f = \sum_{i=1}^N a_i(f)f_i$, then $\forall n \geq 1$, $a_n(f) = \sum_{i=1}^N a_i(f) \underbrace{a_n(f_i)}_{\in \mathbb{Z}}$. Hence $\mathbb{Q}(\{a_n(f)\}_{n \geq 1}) = \mathbb{Q}(\{a_n(f)\}_{n=1}^N)$ has finite degree over \mathbb{Q} . □

We can use this argument to compute Hecke eigenvalues.

Example 3.1. Take $k = 24$. We will compute the eigenvalues of T_{24} acting on $S_{24}(\Gamma(1))$. $S_{24}(\Gamma(1))$ has a unique basis f_1, f_2 with $f_1 = q + O(q^3)$ and $f_2 = q^2 +$

30 Oct 2022,
Lecture 11

$O(q^3)$. For any $f \in S_{24}(\Gamma(1))$, we have $f = a_1(f)f_1 + a_2(f)f_2$. So in particular, $T_2f_1 = a_1(T_2f_1)f_1 + a_2(T_2f_1)f_2$. We know $a_m(T_nf) = \sum_{a|(m,n)} a^{k-1}a_{mn/a^2}(f)$, so

$$T_2f_1 = a_1(T_2f_1)f_1 + a_2(T_2f_1)f_2 = a_2(f_1)f_1 + (a_4(f_1) + 2^{23}a_1(f_2))f_2 = (a_4(f_1) + 2^{23})f_2.$$

Similarly we get

$$T_2f_2 = a_2(f_2)f_1 + (a_4(f_2) + 2^{23}a_1(f_2))f_2 = f_1 + a_4(f_2)f_2.$$

In fact,

$$\begin{aligned} f_1 &= \Delta E_6^2 + 1032\Delta^2 = q + 195660q^3 + 12080128q^4 + \dots \\ f_2 &= \Delta^2 = q^2 - 48q^3 + 1080q^4 + \dots \end{aligned}$$

So the matrix of f_2 is

$$\begin{pmatrix} 0 & 1 \\ 20468736 & 1080 \end{pmatrix},$$

so the eigenvalues of T_2 on $S_{24}(\Gamma(1))$ are $12(45 \pm \sqrt{144169})$. Hence $S_{24}(\Gamma(1))$ has a basis of normalized eigenforms g_1, g_2 with q -expansion coefficients in $K_{g_i} = \mathbb{Q}(\sqrt{144169})$ (sidenote: this is a prime number).

Definition 3.4. Let $f : \mathfrak{h} \rightarrow \mathbb{C}$ be a continuous function that is invariant under the weight 0 action of $\Gamma(1)$, i.e. $f(\gamma\tau) = f(\tau) \forall \gamma \in \Gamma(1)$. We define

$$\int_{\Gamma(1) \backslash \mathfrak{h}} f(\tau) \frac{dx dy}{y^2} = \int_{\mathfrak{f}'} f(\tau) \frac{dx dy}{y^2}$$

(where $\tau = x + iy$).

The motivation for this is that the area form $\frac{dx \wedge dy}{y^2}$ on \mathfrak{h} is invariant under $GL_2(\mathbb{R})^+$ (i.e. $g^*(\omega) = \omega \forall g \in GL_2(\mathbb{R})^+$). We'd like to say that $\Gamma(1) \backslash \mathfrak{h} \cong \mathbb{C}$ is a manifold where $\omega = \frac{dx dy}{y^2}$ descends to $\Gamma(1) \backslash \mathfrak{h}$, so we could use integration on manifolds. This has the following problems:

- We don't assume any knowledge of differential geometry. (In general, if we have a manifold (M, ω) , then we have a volume form $\int_M \omega$).
- ω does not descend to $\Gamma(1) \backslash \mathfrak{h}$, because $\Gamma(1)/\{\pm I\}$ has fixed points in \mathfrak{h} . The solution for this is to choose a finite order subgroup $\Gamma \leq \Gamma(1)$ with no nontrivial elements of finite order. Then ω will descend to ω_Γ on $\Gamma \backslash \mathfrak{h}$ and $\frac{1}{[\Gamma(1):\Gamma]} \int_{\Gamma \backslash \mathfrak{h}} f \omega_\Gamma$ will be independent of the choice of Γ .

Lemma 3.14. Let $f, g \in S_k(\Gamma(1))$. Then the function $f(\tau)\overline{g(\tau)}\text{Im}(\tau)^k$ is

invariant under the weight 0 action of $\Gamma(1)$ and the integral

$$\int_{\Gamma(1) \backslash \mathfrak{h}} f(\tau) \overline{g(\tau)} \operatorname{Im}(\tau)^k \frac{dx dy}{y^2}$$

converges absolutely.

Proof. If $\gamma \in \Gamma(1)$, $f(\gamma\tau) = f(\tau)j(\gamma, \tau)^k$ and $\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}(\tau)}{|j(\gamma, \tau)|^2}$. So

$$f(\gamma\tau) \overline{g(\gamma\tau)} \operatorname{Im}(\gamma\tau)^k = f(\tau) \overline{g(\tau)} j(\gamma, \tau)^k \overline{j(\gamma, \tau)^k} \operatorname{Im}(\tau)^k \frac{1}{|j(\gamma, \tau)|^{2k}} = f(\tau) \overline{g(\tau)} \operatorname{Im}(\tau)^k.$$

If $f(\tau) = \tilde{f}(q)$ for $\tilde{f} : D(0, 1) \rightarrow \mathbb{C}$ holomorphic and vanishing at 0, then $\tilde{f}(q) = qh(q)$ for $h : D(0, 1) \rightarrow \mathbb{C}$ holomorphic. Hence $\forall \delta \in (0, 1)$, $\exists C_\delta > 0$ such that $|h(q)| \leq C_\delta$ if $0 \leq |q| \leq \delta$. Hence $|\tilde{f}(q)| \leq |q|C_\delta$ if $0 \leq |q| \leq \delta$.

So $\forall R \geq 0$, $\exists C_{f,R} > 0$ such that $\forall \tau \in \mathfrak{h}$ such that $\operatorname{Im}(\tau) \geq R$, $|f(\tau)| \leq |q|C_{f,R} = e^{-2\pi \operatorname{Im}(\tau)} C_{f,R}$. So

$$\int_{\Gamma(1) \backslash \mathfrak{h}} |f(\tau) \overline{g(\tau)} \operatorname{Im}(\tau)^k| \frac{dx dy}{y^2} \leq \int_{\mathfrak{f}'} C_{f, \frac{\sqrt{3}}{2}} C_{g, \frac{\sqrt{3}}{2}} e^{-2\pi y} e^{-2\pi y} y^k \frac{dx dy}{y^2}.$$

Furthermore, $\mathfrak{f}' \subset \left\{ x + iy \mid x \in \left[-\frac{1}{2}, \frac{1}{2}\right], y \in \left[\frac{\sqrt{3}}{2}, \infty\right) \right\}$. Hence our integral is

$$\leq \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=\frac{\sqrt{3}}{2}}^{\infty} e^{-4\pi y} y^{k-2} dx dy = \int_{y=\frac{\sqrt{3}}{2}}^{\infty} e^{-4\pi y} y^{k-2} dy < \infty.$$

□

Remark. The second part of the lemma does not hold if f, g are not assumed to be cuspidal.

Definition 3.5. The **Peterson inner product** on $S_k(\Gamma(1))$ is given by the formula

$$\langle f, g \rangle = \int_{\Gamma(1) \backslash \mathfrak{h}} f(\tau) \overline{g(\tau)} \operatorname{Im}(\tau)^k \frac{dx dy}{y^2}.$$

This is an inner product as $\langle f, f \rangle = \int_{\Gamma(1) \backslash \mathfrak{h}} |f(\tau)|^2 \operatorname{Im}(\tau)^k \frac{dx dy}{y^2}$. So if $\langle f, f \rangle = 0$, then $|f|^2 y^k = 0$, hence $f = 0$.

Theorem 3.15. For all $n \geq 1$, T_n is Hermitian with respect to the Peterson inner product, i.e. $\forall f, g \in S_k(\Gamma(1))$, $\langle T_n f, g \rangle = \langle f, T_n g \rangle$.

We will give a sketch proof of this next time.

Theorem 3.16. For all $k \geq 12$ even, there exists a basis f_1, \dots, f_N of normalized eigenforms for $S_k(\Gamma(1))$, unique up to reordering, with the following property:

$\forall 1 \leq i \leq N$, $K_{f_i} = \mathbb{Q}(\{a_n(f_i)\}_{n \geq 1})$ is a number field, contained in \mathbb{R} , and $\forall n \geq 1$, $a_n(f_i) \in \mathcal{O}_{K_{f_i}}$ (the algebraic integers in K_{f_i}).

Proof. We know from linear algebra that if $(V, (\cdot, \cdot))$ is an inner product space over \mathbb{C} , and $T : V \rightarrow V$ is a Hermitian endomorphism, then all eigenvalues of T are real and T is diagonalizable. We also know that if A_1, A_2, A_3, \dots is an infinite family of commuting Hermitian endomorphisms, then they can be diagonalized simultaneously. So in our case, we find a basis f_1, \dots, f_N of $S_k(\Gamma(1))$ of eigenforms, which we may assume are normalized. We only need to show that this basis is unique up to reordering, i.e. that all simultaneous eigenspaces are 1-dimensional. But if $f, g \in S_k(\Gamma(1))$ are normalized eigenforms with the same T_n -eigenvalues $\forall n \geq 1$, then $a_n(f) = a_n(g) \forall n \geq 1 \implies f = g$. \square

01 Nov 2022,
Lecture 12

These sequences $(a_1(f), a_2(f), \dots)_{n \geq 1}$ of eigenvalues of Hecke operators on normalized eigenforms f are among the most interesting objects in number theory. One reason for this is that the sequences $(a_p(f))_{p \text{ prime}}$ are exactly what we need in order to formulate the main conjectures of the Langlands program.

Ramanujan made conjectures concerning $\tau(n) = a_n(\Delta)$. One of them was $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$ and another was $\tau(p)\tau(p^n) = t(p^{n+1}) + p^{11}\tau(p^{n-1})$. These properties follow from basic properties of Hecke operators (and these properties also hold for general $a_n(f)$) for f a normalized eigenform.

While these two conjectures were proved the year after Ramanujan stated them, there is also a third conjecture that was only proved in the 1970s and Deligne won a Fields medal for it. To motivate this, let us prove:

Lemma 3.17. If p is prime, then

$$\sum_{n \geq 0} \tau(p^n) X^n = \frac{1}{(1 - \tau(p)X + p^{11}X^2)}.$$

Proof. We compute

$$\begin{aligned} & (1 - \tau(p)X + p^{11}X^2) \sum_{n \geq 0} \tau(p^n) X^n \\ &= 1 + \sum_{n \geq 2} (\tau(p^n)X^n - \tau(p)X\tau(p^{n-1})X^{n-1} + p^{11}X^2\tau(p^{n-2})X^{n-2}) \\ &= 1 + \sum_{n \geq 2} (\tau(p^n) - \tau(p)\tau(p^{n-1}) + p^{11}\tau(p^{n-2})) X^n = 1. \end{aligned}$$

\square

Let us factor $1 - \tau(p)X + p^{11}X^2 = (1 - \alpha_p X)(1 - \beta_p X)$ for $\alpha_p, \beta_p \in \mathbb{C}$. There are two possibilities:

- If $\tau(p)^2 - 4p^{11} > 0$, then α_p, β_p are distinct real numbers which hence have distinct absolute values.
- If $\tau(p)^2 - 4p^{11} \leq 0$, then α_p, β_p are conjugate complex numbers of the same absolute value $\sqrt{p^{11}}$.

Ramanujan conjectured that we always have the second case, i.e. $|\tau(p)| \leq 2p^{11/2}$ for any prime number p . The general form of this conjecture is:

Conjecture. Let $f \in S_k(\Gamma(1))$ be a normalized eigenform. Then $\forall p$ prime,

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}.$$

This is what Deligne proved in the 1970s.

Ramanujan proved the formula (for all p an odd prime)

$$r_{24}(p) = \left| \left\{ (x_1, \dots, x_{24}) \in \mathbb{Z}^{24} \mid \sum_{i=1}^{24} x_i^2 = p \right\} \right| = \frac{16}{691}(1 + p^{11}) + \frac{33152}{691}\tau(p).$$

A consequence of the Ramanujan conjecture is that

$$r_{24}(p) = \frac{16}{691}p^{11} + O(p^{11/2}).$$

We will now present a **non-examinable** sketch proof of Theorem 3.15. In particular, everything from now until the end of the lecture is non-examinable.

Sketch of proof, non-examinable. Recall $\langle f, g \rangle = \int_{\Gamma(1) \backslash \mathfrak{h}} f(\tau) \overline{g(\tau)} \text{Im}(\tau)^k \frac{dx dy}{y^2}$. Hence we want to show that

$$\int_{\Gamma(1) \backslash \mathfrak{h}} (T_n f) \overline{g} \text{Im}(\tau)^k \frac{dx dy}{y^2} = \int_{\Gamma(1) \backslash \mathfrak{h}} f(\overline{T_n g}) \text{Im}(\tau)^k \frac{dx dy}{y^2}$$

Initial reduction: it is enough to prove the theorem for $n = p$ a prime, since any T_n is a polynomial in T_p for $p \mid n$ with coefficients in \mathbb{Z} . We proved last time that the function $f(\tau) \overline{g(\tau)} \text{Im}(\tau)^k$ is invariant under the weight 0 action of $\Gamma(1)$, so it therefore corresponds to a function $\mathcal{L} \rightarrow \mathbb{C}$ invariant under \mathbb{C}^\times . We claim that this function is $\Lambda \mapsto F(\Lambda) \overline{G(\Lambda)} \text{covol}(\Lambda)^k$, where $F(\Lambda_\tau) = f(\tau)$, $G(\Lambda_\tau) = g(\tau)$, and $\text{covol}(\Lambda) = \int_{\mathbb{C}/\Lambda} dx dy = \left| \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \right|$ where $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $e_j = x_j + iy_j$.

Indeed, we can check that $\Lambda_\tau \mapsto F(\Lambda_\tau) \overline{G(\Lambda_\tau)} \text{covol}(\Lambda_\tau)^k$ and $\Lambda_\tau = \mathbb{Z}_\tau \oplus \mathbb{Z}$, so $\text{covol} \Lambda_\tau = y = \text{Im}(\tau)$. Now, if $A : \mathbb{C}^\times / \mathcal{L} \rightarrow \mathbb{C}$ is a continuous function, we

define $\int_{\mathbb{C}^\times/\mathcal{L}} A(\Lambda) d\Lambda = \int_{\Gamma(1)/\mathfrak{h}} a(\tau) \frac{dx dy}{y^2}$, where $a(\tau) = a(\Lambda\tau)$. Hence

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathbb{C}^\times/\mathcal{L}} F(\Lambda) \overline{G(\Lambda)} \text{covol}(\Lambda)^k d\Lambda \\ \langle T_p f, g \rangle &= p^{k-1} \int_{\mathbb{C}^\times/\mathcal{L}} \sum_{\Lambda' \leq_{\frac{1}{p}} \Lambda} F(\Lambda') \overline{G(\Lambda)} \text{covol}(\Lambda)^k d\Lambda \\ &\stackrel{?}{=} p^{k-1} \int_{\mathbb{C}^\times/\mathcal{L}} \sum_{\Lambda' \leq_{\frac{1}{p}} \Lambda} F(\Lambda) \overline{G(\Lambda')} \text{covol}(\Lambda)^k d\Lambda = \langle f, T_p g \rangle. \end{aligned}$$

Define $\mathcal{L}_p = \{(\Lambda', \Lambda) \mid \Lambda \in \mathcal{L}, \Lambda' \leq_{\frac{1}{p}} \Lambda\} \rightarrow \mathcal{L}$ by $(\Lambda', \Lambda) \mapsto \Lambda$.

Fact. There is a bijection $\mathfrak{h}/\Gamma_0(p) \rightarrow \mathbb{C}^\times/\mathcal{L}_p$ where $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod{p} \right\}$ given by $\tau \mapsto (\mathbb{Z}p\tau \oplus \mathbb{Z} \leq_{\frac{1}{p}} \mathbb{Z}\tau \oplus \mathbb{Z})$. (proving this fact is left as an exercise for the especially motivated).

If $A : \mathbb{C}^\times/\mathcal{L}_p \rightarrow \mathbb{C}$ is a continuous function, then we define

$$\int_{\mathbb{C}^\times/\mathcal{L}_p} A(\Lambda', \Lambda) d(\Lambda', \Lambda) = \int_{\Gamma_0(p)/\mathfrak{h}} a(\tau) \frac{dx dy}{y^2}.$$

We can rewrite

$$\begin{aligned} \langle T_p f, g \rangle &= p^{k-1} \int_{\mathbb{C}^\times/\mathcal{L}} \sum_{\Lambda' \leq_{\frac{1}{p}} \Lambda} F(\Lambda') \overline{G(\Lambda)} \text{covol}(\Lambda)^k d\Lambda \\ &= p^{k-1} \int_{\mathbb{C}^\times/\mathcal{L}_p} F(\Lambda') \overline{G(\Lambda)} \text{covol}(\Lambda)^k d(\Lambda', \Lambda) \\ &\stackrel{?}{=} p^{k-1} \int_{\mathbb{C}^\times/\mathcal{L}_p} F(\Lambda) \overline{G(\Lambda')} \text{covol}(\Lambda)^k d(\Lambda', \Lambda) = \langle f, T_p g \rangle. \end{aligned}$$

Observe that if $\Lambda' \leq_{\frac{1}{p}} \Lambda$, then $p\Lambda' \leq \Lambda$, so there's a map $\iota : \mathcal{L}_p \rightarrow \mathcal{L}_p$ by $(\Lambda', \Lambda) \mapsto (p\Lambda', \Lambda)$, so $\iota^2(\Lambda', \Lambda) = (p\Lambda', p\Lambda)$, so ι descends to a map $\bar{\iota} : \mathbb{C}^\times/\mathcal{L}_p \rightarrow \mathbb{C}^\times/\mathcal{L}_p$. The key point is that this map $\bar{\iota}$ is measure-preserving and transforms $\langle T_p f, g \rangle$ into $\langle f, T_p g \rangle$ (exercise).

Why is $\bar{\iota}$ measure-preserving? Under the bijection $\mathbb{C}^\times/\mathcal{L}_p \xrightarrow{\sim} \Gamma_0(p)/\mathfrak{h}$, it corresponds to the action of $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \in GL_2(\mathbb{Q})^+$. We defined integration on \mathfrak{h} using $\omega = \frac{dx dy}{y^2}$, which is invariant even in $GL_2(\mathbb{R})^+$. \square

Proposition 3.18. Let $f : \mathfrak{h} \rightarrow \mathbb{C}$ be continuous and Γ_∞ -invariant, where

03 Nov 2022,
Lecture 13

$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c = 0 \right\}$, i.e. $f(\tau) = f(\tau + 1)$. Suppose that $\forall \tau \in \mathfrak{h}$,

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} |f(\gamma\tau)| < \infty.$$

Also suppose that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} |f(x+iy)| \frac{dx dy}{y^2} < \infty.$$

Then $\sum_{\gamma \in \Gamma_0 \setminus \Gamma(1)} f(\gamma\tau)$ is a measurable function, $\Gamma(1)$ -invariant and

$$\int_{\Gamma(1) \setminus \mathfrak{h}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau) \frac{dx dy}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} |f(x+iy)| \frac{dx dy}{y^2}.$$

An application of this proposition is called "unfolding". This is because $\{\tau \in \mathfrak{h} \mid \operatorname{Re}(\tau) \in [-\frac{1}{2}, \frac{1}{2}]\}$ is a fundamental set for $\Gamma_\infty \setminus \{\pm I\}$, so

$$\int_{\Gamma(1) \setminus \mathfrak{h}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau) \frac{dx dy}{y^2} = \int_{\Gamma_\infty \setminus \mathfrak{h}} f(\tau) \frac{dx dy}{y^2} = \text{RHS of the prop above.}$$

Proof. We want to show that $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau)$ is measurable on f and the equality

$$\int_{\mathfrak{f}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau) \frac{dx dy}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} f(x+iy) \frac{dx dy}{y^2}.$$

Fubini's theorem says: Suppose $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\mathfrak{f}} |f(\gamma\tau)| \frac{dx dy}{y^2} < \infty$. Then $\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau)$ is measurable and absolutely integrable in \mathfrak{f} and

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\mathfrak{f}} f(\gamma\tau) \frac{dx dy}{y^2} = \int_{\mathfrak{f}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} f(\gamma\tau) \frac{dx dy}{y^2}.$$

We'll be done if we can show

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\mathfrak{f}} f(\gamma\tau) \frac{dx dy}{y^2} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=0}^{\infty} f(x+iy) \frac{dx dy}{y^2}.$$

But the LHS is equal to

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\gamma\mathfrak{f}} f(\tau) \frac{dx dy}{y^2}$$

by using the change of variable $f \mapsto \gamma f$ and invariance of $\frac{dx dy}{y^2}$ under the pullback by γ . Recall from the first example sheet that if $f^0 = \text{Int}(f)$, then $\forall \gamma \in \Gamma(1)$, $\gamma f^0 \cap \{\tau \in \mathfrak{h} \mid \text{Re}(\tau) \in \frac{1}{2} + \mathbb{Z}\} = \emptyset$. Hence for $\gamma \in \Gamma(1)$, γf^0 is contained in $\{\tau \in \mathfrak{h} \mid \text{Re}(\tau) \in (-\frac{1}{2}, \frac{1}{2}) + a\}$ for some $a \in \mathbb{Z}$. Also, there's a unique $\delta \in \Gamma_\infty \setminus \{\pm I\}$ such that $\delta \gamma f^0 \subset \{\tau \in \mathfrak{h} \mid \text{Re}(\tau) \in (-\frac{1}{2}, \frac{1}{2})\} = U$. Hence the set $\{\gamma \in \Gamma(1) \setminus \{\pm I\} \mid \gamma f^0 \subset U\}$ is a set of coset representatives for $\Gamma_\infty \setminus \Gamma(1)$. Thus

$$\sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\gamma f} f(\tau) \frac{dx dy}{y^2} = \sum_{\substack{\gamma \in \Gamma(1) \setminus \{\pm I\} \\ \gamma f^0 \subset U}} \int_{\gamma f} f(\tau) \frac{dx dy}{y^2} \stackrel{?}{=} \int_U f(\tau) \frac{dx dy}{y^2}.$$

But we know that $\mathfrak{h} = \left(\bigsqcup_{\gamma \in \Gamma(1) \setminus \{\pm I\}} \gamma f^0 \right) \sqcup W$ for W of measure zero, e.g. the union of all the $\Gamma(1)$ -translates of the vertical line $\text{Re}(\tau) = \frac{1}{2}$. Hence

$$U = \bigsqcup_{\gamma \in \Gamma(1) \setminus \{\pm I\}} (\gamma f^0 \cap U) \sqcup (W \cap U) = \bigsqcup_{\substack{\gamma \in \Gamma(1) \setminus \{\pm I\} \\ \gamma f^0 \subset U}} (\gamma f^0) \sqcup (W \cap U)$$

for $W \cap U$ of measure zero. Hence

$$\int_U f(\tau) \frac{dx dy}{y^2} = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma(1)} \int_{\gamma f^0} f(\tau) \frac{dx dy}{y^2}$$

which concludes the proof. \square

4 *L*-functions

Normalized eigenforms can be used to construct *L*-functions. What is an *L*-function? Motivation: the Riemann zeta function, $\zeta(s) = \sum_{n \geq 1} n^{-s}$. This converges absolutely in $\{s \mid \text{Re}(s) > 1\}$ and defines a holomorphic function in that region. Key properties:

- The Euler product: $\zeta(s) = \prod_p \text{prime} (1 - p^{-s})^{-1}$ (converges absolutely when $\text{Re}(s) > 1$).
- Meromorphic continuation: $\zeta(s)$ extends to a meromorphic function on \mathbb{C} with a simple pole at $s = 1$ and no other pole.
- Functional equation: Define $\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. Then $\xi(s) = \xi(1-s)$.
- Special values of $\zeta(s)$ at $s \in \mathbb{Z}$ shall have arithmetic meaning.

Other examples of functions of similar properties:

- Dirichlet *L*-functions $L(\chi, s) = \sum_{\substack{n \in \mathbb{N} \\ (n, N) = 1}} \chi(n \pmod{N}) n^{-s}$ associated to $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$.

- If E/\mathbb{Q} is an elliptic curve, then the Hasse–Weil L -function $L(E, s) = \sum_{n \geq 1} a_n n^{-s}$.

In general, an L -function is a Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$, $a_n \in \mathbb{C}$ which either provably has or is expected to have properties analogous to $\zeta(s)$.

Definition 4.1. Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma(1))$. Then its associated Dirichlet series is

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}.$$

We will consider separately the case of Eisenstein series and the case of cuspidal modular forms.

Let $F_k(\tau)$ be the normalized eigenform associated to G_k (for $k \geq 4$ even). Then

$$\begin{aligned} L(F_k, s) &= \sum_{n \geq 1} \sigma_{k-1}(n) n^{-s} = \sum_{n \geq 1} \sum_{d|n} d^{k-1} n^{-s} = \sum_{n \geq 1} \sum_{d|n} d^{k-1} d^{-s} \left(\frac{n}{d}\right)^{-s} \\ &= \sum_{a, d \geq 1} d^{k-1-s} a^{-s} = \zeta(s) \zeta(s+1-k). \end{aligned}$$

Lemma 4.1. Let $f \in S_k(\Gamma(1))$. Then $L(f, s)$ converges absolutely in the region $\{\operatorname{Re}(s) > 1 + \frac{k}{2}\}$ and defines a holomorphic function there.

Proof. We use a fact from the second example sheet: $\exists C_f > 0$ such that for all $n \geq 1$, $|a_n| \leq C_f n^{k/2}$. We then claim that $\forall \delta > 0$, $\sum_{n \geq 1} a_n n^{-s}$ converges absolutely and uniformly in $\{\operatorname{Re}(s) > 1 + \frac{k}{2} + \delta\}$. To prove this, we use the Weierstrass M -test.

Write $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$. Then $n^{-s} = \exp(-s \log n) \implies |n^{-s}| = \exp(-\sigma \log n) = n^{-\sigma}$. If $\sigma > 1 + \frac{k}{2} + \delta$, then

$$\sum_{n \geq 1} |a_n n^{-s}| \leq \sum_{n \geq 1} C_f n^{k/2} n^{-(1+k/2+\delta)} = \sum_{n \geq 1} C_f n^{-(1+\delta)} < \infty.$$

□

Remark. If we assume the Ramanujan–Petersson conjecture, we can get absolute convergence when $\operatorname{Re}(s) > \frac{1+k}{2}$.

06 Nov 2022,
Lecture 14

Theorem 4.2. Let $f \in S_k(\Gamma(1))$ be a cuspidal modular form. Then:

- (1) $L(f, s)$ admits an analytic continuation to \mathbb{C} .
- (2) If $\Lambda(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s)$, then $\Lambda(f, s) = i^k \Lambda(f, k-s)$.

To warm up, we consider the gamma function

$$\Gamma(s) = \int_{y=0}^{\infty} e^{-y} y^s \frac{dy}{y}$$

(when the integral converges absolutely).

Proposition 4.3. (i) $\Gamma(s)$ converges absolutely when $\operatorname{Re}(s) > 0$ and is a holomorphic function in $\{\operatorname{Re}(s) > 0\}$.

(ii) $\Gamma(s)$ admits a meromorphic continuation to \mathbb{C} with simple poles at $s = 0, -1, -2, \dots$, and no other poles.

Proof. $\Gamma(s)$ converges absolutely when $\operatorname{Re}(s) > 0$, i.e. $\int_{y=0}^{\infty} |e^{-y} y^s| \frac{dy}{y} < \infty$. Checking this is left as an easy exercise.

Next we show $\Gamma(s)$ is continuous in $\{\operatorname{Re}(s) > 0\}$. If $N > 1$, then define $\Gamma_N(s) = \int_{y=\frac{1}{N}}^N e^{-y} y^s \frac{dy}{y}$. We claim that $\Gamma_N(s)$ is continuous in $\{\operatorname{Re}(s) > 0\}$. But if $\operatorname{Re}(s) \geq 0$ and $\epsilon > 0$, then $\exists \delta > 0$ such that if $s' \in \mathbb{C}$ with $|s - s'| < \delta$ and $y \in [\frac{1}{N}, N]$, then $|y^s - y^{s'}| < \epsilon$ (since $(y, s) \mapsto y^s : [\frac{1}{N}, N] \times \mathbb{C} \rightarrow \mathbb{C}$ is continuous and $[\frac{1}{N}, N]$ is compact). Then

$$|\Gamma_N(s) - \Gamma_N(s')| \leq \int_{y=\frac{1}{N}}^N e^{-y} |y^s - y^{s'}| \frac{dy}{y} \leq \epsilon \int_{y=\frac{1}{N}}^N e^{-y} \frac{dy}{y} = C_N \epsilon,$$

so $\Gamma_N(s)$ is continuous.

To show $\Gamma(s)$ is holomorphic, we recall Morera's theorem: If $U \subset \mathbb{C}$ is open, $f : U \rightarrow \mathbb{C}$ is continuous and $\oint_{\gamma} f(z) dz = 0$ for all closed continuous paths γ in U , then f is holomorphic. We have

$$\oint_{\gamma} \Gamma_N(s) ds = \oint_{\gamma} \int_{y=\frac{1}{N}}^N e^{-y} y^s \frac{dy}{y} ds = \int_{y=\frac{1}{N}}^N e^{-y} \underbrace{\oint_{\gamma} y^s ds}_{0, \text{ as } y^s \text{ is holomorphic}} \frac{dy}{y} = 0.$$

Hence $\Gamma_N(s)$ is holomorphic by Morera's theorem.

To show Γ is holomorphic, we show $\Gamma_N \rightarrow \Gamma$ locally uniformly. In fact, we show uniform convergence in $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \in [\sigma_0, \sigma_1]\} \forall 0 < \sigma_0 < \sigma_1$, i.e. "uniform convergence in vertical strips". If s lies in this set, then

$$\begin{aligned} |\Gamma(s) - \Gamma_N(s)| &\leq \int_{y=0}^{\frac{1}{N}} |y^s e^{-y}| \frac{dy}{y} + \int_{y=N}^{\infty} |y^s e^{-y}| \frac{dy}{y} \\ &\leq \int_{y=0}^{\frac{1}{N}} y^{\sigma_0-1} e^{-y} dy + \int_{y=N}^{\infty} y_1^{\sigma_1-1} e^{-y} dy \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

at a rate independent of s , giving us uniform convergence and showing (i).

To prove (ii), we use the equation $s\Gamma(s) = \Gamma(s+1)$ (which we can prove from the definition by integrating by parts). This can be used to extend $\Gamma(s)$ into a meromorphic function on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > -k\}$ for any $k \in \mathbb{N}$ by induction on k , and the description of the poles also follows. \square

Proof of Theorem 4.2. Consider

$$F(s) = \int_{y=0}^{\infty} f(iy)y^s \frac{dy}{y},$$

called the Mellin transform of $f(iy)$. We claim that $F(s)$ converges for any $s \in \mathbb{C}$ and defines a holomorphic function. For absolute convergence, write

$$\int_{y=0}^{\infty} f(iy)y^s \frac{dy}{y} = \int_{y=0}^1 f(iy)y^s \frac{dy}{y} + \int_{y=1}^{\infty} f(iy)y^s \frac{dy}{y}.$$

We know $|f(\tau)| \leq C_f |e^{2\pi i\tau}|$, so $|f(iy)| \leq C_f e^{-2\pi y}$, so $\int_{y=1}^{\infty} |f(iy)y^s| \frac{dy}{y} < \infty$ for any $s \in \mathbb{C}$. For the first integral we have

$$\int_{y=0}^1 f(iy)y^s \frac{dy}{y} = \int_{y=1}^{\infty} f\left(\frac{1}{y}\right) y^{-s} \frac{dy}{y}.$$

But also $f(\tau) = f(-1/\tau)t^{-k}$, so $f(iy) = f\left(\frac{i}{y}\right) (iy)^{-k}$. Hence

$$\int_{y=1}^{\infty} \left| f\left(\frac{i}{y}\right) y^{-s} \right| \frac{dy}{y} = \int_{y=1}^{\infty} |f(iy)| y^{k-s} \frac{dy}{y} < \infty$$

for any $s \in \mathbb{C}$, since $f(iy)$ decays exponentially as $y \rightarrow \infty$.

The fact that F is holomorphic is left as an exercise: it is similar to the proof of holomorphicity of $\Gamma(s)$ above, but easier, since we don't have to worry about blowing up anywhere.

What is $F(s)$? We have

$$F(s) = \int_{y=0}^{\infty} \sum_{n=1}^{\infty} a_n e^{-2\pi n y} y^s \frac{dy}{y} \stackrel{(\star)}{=} \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-2\pi n y} y^s \frac{dy}{y}.$$

(\star) is justified by Fubini's theorem provided that

$$\sum_{n=1}^{\infty} |a_n| \int_{y=0}^{\infty} |e^{-2\pi n y} y^s| \frac{dy}{y} < \infty.$$

If we assume this holds, then we get

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-2\pi n y} y^s \frac{dy}{y} &= \sum_{n=1}^{\infty} a_n \int_{y=0}^{\infty} e^{-y} y^s (2\pi n)^{-s} \frac{dy}{y} \\ &= \sum_{n=1}^{\infty} (2\pi)^{-s} a_n n^{-s} \int_{y=0}^{\infty} e^{-y} y^s \frac{dy}{y} = \Lambda(f, s). \end{aligned}$$

To justify (\star) , we have

$$\sum_{n=1}^{\infty} |a_n| \int_{y=0}^{\infty} |e^{-2\pi n y} y^s| \frac{dy}{y} = (2\pi)^{-\sigma} \Gamma(\sigma) \sum_{n=1}^{\infty} |a_n| n^{-\sigma},$$

where $\sigma = \operatorname{Re}(s)$ (so $|y|^s = |y|^\sigma$, $|n^{-s}| = n^{-\sigma}$), i.e. whenever $L(f, s)$ is absolutely convergent.

We conclude that $F(s)$ is holomorphic in \mathbb{C} and equals $\Lambda(f, s)$ when $\operatorname{Re}(s) > 1 + \frac{k}{2}$, i.e. $\Lambda(f, s)$ has an analytic continuation to \mathbb{C} . We can write $L(f, s) = \frac{\Lambda(f, s)}{(2\pi)^{-s} \Gamma(s)}$, which is also analytic in \mathbb{C} , since $\frac{1}{\Gamma(s)}$ is entire.

For the last part, we have

$$\Lambda(f, s) = \int_{y=0}^{\infty} f(iy) y^s \frac{dy}{y} = \int_{y=1}^{\infty} f\left(\frac{i}{y}\right) y^{-s} \frac{dy}{y} + \int_{y=1}^{\infty} f(iy) y^s \frac{dy}{y}.$$

Use $f(i/y) = f(iy)(iy)^k$ to find that

$$\Lambda(f, s) = \int_{y=1}^{\infty} f(iy) (i^k y^{k-s} + y^s) \frac{dy}{y}$$

If $f \neq 0$, then k is even, so $i^k \in \{\pm 1\}$. Hence

$$\Lambda(f, k-s) = i^k \Lambda(f, s).$$

□

Theorem 4.4. Let $f \in S_k(\Gamma(1))$ be a normalized eigenform. Then

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p \text{ prime}} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

We can interpret this as either an equality of formal Dirichlet series or as an equality of complex numbers when $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent.

Proof. By an exercise on the third example sheet, it is enough to consider the formal identity. But we know $a_{mn} = a_m a_n$ if $(m, n) = 1$ (a property of Hecke

operators inherited by their eigenvalues). Hence

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_{p \text{ prime}} (1 + a_p p^{-s} + a_{p^2} p^{-2s} + a_{p^3} p^{-3s} + \dots).$$

So we need to show

$$1 + a_p p^{-s} + a_{p^2} p^{-2s} + a_{p^3} p^{-3s} + \dots = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

But this is equivalent to $a_{p^{n+1}} = a_p a_{p^n} - p^{k-1} a_{p^{n-1}}$, which we showed for Hecke operators. \square

08 Nov 2022,
Lecture 15

We will now use the following theorem as a black box result. For a proof, see Lang's *Algebraic Number Theory*.

Theorem 4.5 (Wiener–Ikehara Tauberian theorem). Consider a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s} = f(s)$, absolutely convergent when $\operatorname{Re}(s) > 1$ (so f is holomorphic in this region). Suppose f admits a meromorphic continuation to an open neighborhood of $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 1\}$ which is holomorphic on the line $\operatorname{Re}(s) = 1$, except possibly for a simple pole at $s = 1$ of residue α . Then

$$\sum_{1 \leq n \leq X} a_n = \alpha X + o(X) \text{ as } X \rightarrow \infty.$$

Here $o(X)$ denotes any function $g(X)$ such that $g(X)/X \rightarrow 0$ as $X \rightarrow \infty$, and $O(X)$ denotes any function $h(X)$ such that $h(X)/X$ is bounded as $X \rightarrow \infty$.

As an illustration, we prove:

Proposition 4.6. Suppose that the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ admits a meromorphic continuation to an open neighborhood of $\{\operatorname{Re}(s) \geq 1\}$ which is holomorphic and non-vanishing on the line $\operatorname{Re}(s) = 1$ except for a simple pole at $s = 1$. Then the Prime Number Theorem holds, i.e.

$$\pi(X) = \sum_{\substack{p \text{ prime} \\ p \leq X}} 1 = \frac{X}{\log X} + o\left(\frac{X}{\log X}\right)$$

as $X \rightarrow \infty$.

Proof. Note that we have the Taylor series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ for $-\log(1-z)$ valid for $|z| < 1$. A branch of $\log \zeta(s) = \log \prod_p (1 - p^{-s})^{-1}$ is given by

$$\sum_p \log(1 - p^{-s})^{-1} = \sum_p \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}.$$

This Dirichlet series is absolutely convergent when $\operatorname{Re}(s) > 1$, hence locally uniformly convergent, so we can compute the derivative term-by-term to find

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \sum_{k=1}^{\infty} \frac{d}{ds} \left(\frac{p^{-ks}}{k} \right) = \sum_p \sum_{k=1}^{\infty} (\log p) p^{-ks} \\ \implies -\frac{\zeta'(s)}{\zeta(s)} &= \sum_p (\log p) p^{-s} + \sum_p \sum_{k \geq 2} (\log p) p^{-ks}. \end{aligned}$$

Note the second term is absolutely convergent when $\operatorname{Re}(s) > \frac{1}{2}$. But $-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s)$, so if $\zeta(s)$ has a zero or a pole of order k at s_0 , then $-\frac{\zeta'(s)}{\zeta(s)}$ will have a simple pole at s_0 of residue $-k$.

We are assuming that $\zeta(s)$ has a meromorphic continuation which is holomorphic and non-vanishing on $\{\operatorname{Re}(s) = 1\}$ except for a simple pole at $s = 1$. Hence $-\frac{\zeta'(s)}{\zeta(s)}$ has a meromorphic continuation defined where ζ is defined, holomorphic on $\{\operatorname{Re}(s) = 1\}$ except for a simple pole at $s = 1$ of residue 1.

We conclude that $\sum_p (\log p) p^{-s}$ has a meromorphic continuation to a neighborhood of $\{\operatorname{Re}(s) \geq 1\}$, holomorphic on the line $\{\operatorname{Re}(s) = 1\}$ except for a simple pole at $s = 1$ of residue 1. Hence applying the Wiener-Ikehara Tauberian theorem to $\sum_p (\log p) p^{-s}$ gives

$$\theta(X) = \sum_{p \leq X} \log p = X + o(X).$$

To get back to $\pi(x)$, we use Lemma 4.7 (partial summation, to be proved after this proof). We take $a_n = \begin{cases} 0 & n \text{ not prime} \\ \log p & n = p \text{ prime} \end{cases}$ and $f(t) = \frac{1}{\log t}$. By partial summation,

$$\begin{aligned} \pi(X) &= 1 + \sum_{e < n \leq X} 1_{n \text{ is prime}} \\ &= 1 + \sum_{e < n \leq X} a_n f(n) = A(X) f(X) - A(e) f(e) + \int_{t=e}^X \frac{A(t)}{t(\log t)^2} dt. \end{aligned}$$

Note that $A(x) = \sum_{p \leq x} \log p = \theta(X) = X + o(X)$, so the above is

$$\begin{aligned} &= \frac{\theta(X)}{\log X} - A(e) f(e) + \int_{t=e}^X \frac{\theta(t)}{t(\log t)^2} dt \\ &= \frac{X}{\log X} + o\left(\frac{X}{\log X}\right) + \int_{t=e}^X \frac{\theta(t)}{t(\log t)^2} dt. \end{aligned}$$

To finish, we need to show that the last term can be absorbed into the error term. But $\theta(X) = X + o(X)$, so $\theta(X) = O(X)$, so $\exists C > 0$ such that $\theta(t) \leq Ct \forall t > 0$, so our integral is

$$\begin{aligned} &\leq C \int_{t=e}^X \frac{1}{(\log t)^2} dt = C \int_{t=e}^{\sqrt{X}} \frac{1}{(\log t)^2} dt + C \int_{t=\sqrt{X}}^X \frac{1}{(\log t)^2} dt \\ &\leq C\sqrt{X} + C \frac{X}{(\log \sqrt{X})^2} \\ &= C\sqrt{X} + \frac{4CX}{(\log X)^2} = o\left(\frac{X}{\log X}\right) \end{aligned}$$

as desired. □

Lemma 4.7 (Partial summation). Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers. Let $0 < X < Y$ be real numbers and let $f : [X, Y] \rightarrow \mathbb{R}$ be a continuously differentiable function. Let $A(t) = \sum_{0 \leq n \leq t} a_n$. Then

$$\sum_{X < n \leq Y} a_n f(n) = A(Y)f(Y) - A(X)f(X) - \int_{t=X}^Y A(t)f'(t)dt.$$

Proof. Elementary exercise. □

We will establish all required properties of $\zeta(s)$ for the proof later in the course using modular forms.

Theorem 4.8. Fix $n \geq 1$. Suppose we're given for all primes p a matrix $\Phi_p \in M_n(\mathbb{C})$ which is either zero or whose eigenvalues have value 1. Define

$$L(\{\Phi_p\}, s) = \prod_p \det(1_n - p^{-s}\Phi_p)^{-1}.$$

Then $L(\{\Phi_p\}, s)$ is absolutely convergent when $\operatorname{Re}(s) > 1$. Furthermore, suppose $L(\{\Phi_p\}, s)$ admits a meromorphic continuation to an open neighborhood of $\{\operatorname{Re}(s) \geq 1\}$ and is holomorphic and nonvanishing on the line $\{\operatorname{Re}(s) = 1\}$ except possibly for a pole at $s = 1$ of order δ . Then

$$\sum_{\substack{p \leq X \\ p \text{ prime}}} \operatorname{tr} \Phi_p = \frac{\delta X}{\log X} + o\left(\frac{X}{\log X}\right).$$

Proof. Left as an exercise on the third example sheet. This is just a generalization of the case $n = 1, \Phi_p = 1, L = \zeta$ that we just did. □

Example 4.1. Dirichlet's theorem on primes in arithmetic progressions. Fix $N \in \mathbb{N}, N \geq 1$. For any homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$, consider

$$L(\chi, s) = \sum_{\substack{(n, N)=1 \\ n \in \mathbb{N}}} \chi(n \bmod N) n^{-s} = \prod_{p \nmid N} (1 - \chi(p \bmod N) p^{-s})^{-1}.$$

These are called the Dirichlet L -functions. It is a fact that the hypotheses of Theorem 4.8 apply to $L(\chi, s)$, so we conclude that for any χ ,

$$\sum_{p \leq X} \chi(p \bmod N) = \text{ord}_{s=1} L(\chi, s) \cdot \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$

One can show that $\text{ord}_{s=1} L(\chi, s) = \begin{cases} -1 & \chi \text{ trivial.} \\ 0 & \chi \text{ nontrivial.} \end{cases}$

If $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, then $1_{a \bmod N}(g) = \frac{1}{\phi(N)} = \sum_{\chi} \overline{\chi(a)} \chi(g)$ for any $g \in (\mathbb{Z}/N\mathbb{Z})^\times$. Hence

$$\sum_{\substack{p \leq x \\ p \nmid N}} 1_{a \bmod N}(p) = \frac{1}{\phi(N)} \sum_{\chi} \overline{\chi(a)} \sum_{p \leq X} \chi(p \bmod N) = \frac{1}{\phi(N)} \frac{X}{\log X} + o\left(\frac{X}{\log X}\right).$$