Part III - Algebraic Geometry

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0 Introduction

6 Oct 2022, Lecture 1

The course consists of four parts.

(1) Basics of sheaves on topological spaces.

(2) Definition of schemes and morphisms.

(3) Properties of schemes (e.g. the algebraic geometry notion of compactness and other properties).

(4) A rapid introduction to the cohomology of schemes.

The main reference for the course is Hartshorne's *Algebraic Geometry*.

1 Beyond algebraic varieties

08 Oct 2022, Lecture 2

1.1 Summary of classical algebraic geometry

We let $k = \overline{k}$ be a algebraically closed field and consider $\mathbb{A}^n_k = \mathbb{A}^n = k^n$ as a set.

Definition 1.1. An **affine variety** is a subset $V \subset \mathbb{A}^n$ of the form $\mathbb{V}(S)$ with $S \subset k[x_1, \ldots, x_n]$, where $\mathbb{V}$ is the common vanishing locus.

Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$ (the ideal generated by S). By Hilbert Basis Theorem (since $k[x_1, \ldots, x_n]$ is noetherian), $\mathbb{V}(I(S)) = \mathbb{V}(S')$ for some finite set $S \subset k[x_1, \ldots, x_n]$.

In fact, $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$, where

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}$$

is the **radical** of I. For example, in $k[x]$, if $I = (x^2)$, then $\sqrt{I} = (x)$.

Definition 1.2. Given varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$, a **morphism** is a (set-theoretic) map $\phi : V \to W \subset \mathbb{A}^m_k$ such that if $\phi = (f_1, \ldots, f_m)$, then each f_i is the restriction of a polynomial in $\{x_1, \ldots, x_n\}$.

An **isomorphism** is a morphism with a two–sided inverse.

Our basic correspondence is

$$\{\text{Affine varieties over } k\}/\text{up to isomorphism}$$
$$\leftrightarrow$$
$$\{\text{finitely generated } k\text{–algebras } A \text{ without nilpotent elements}\}$$

A finitely generated k–algebra is just a quotient of a polynomial ring in finitely many variables. A nilpotent element is such that some power of it is zero. For example, in $k[x]/(x^2)$, the element x is nilpotent.

How does this correspondence work? Given a variety V (representing an isomorphism class), we write $V = \mathbb{V}(I)$ for $I \subset k[x_1, \dots, x_n]$ a radical ideal[1], and map $V \mapsto k[x_1, \dots, x_n]/I$.

For the reverse, if A is a finitely generated nilpotent free algebra, then $A \cong k[y_1, \dots, y_m]/J$ where we can choose J to be radical (exercise: why?).

We have to check that this is independent of our choice on both sides (exercise: think through this, it should be clear).

Definition 1.3. The algebra associated to V is classically denoted $k[V]$ and called the **coordinate ring of** V.

We have the compatibility of morphisms with our basic correspondence: there is a bijection between

$$\text{Morphisms}(V, W) \leftrightarrow \text{Ring homomorphisms}_k(k[W], k[V])$$

(here RingHom_k means that our homomorphisms preserve k).

We can now make our set into a topological space:

Definition 1.4. Let $V = \mathbb{V}(I) \subset \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The **Zariski topology** on V is defined such that the closed sets are $\mathbb{V}(S)$, where $S \subset k[V]$.

If $V \cong W$, then the Zariski topological spaces are homeomorphic as varieties (exercise).

Theorem 1.1 (Nullstellensatz)**.** Fix V a variety and let $k[V]$ be its coordinate ring. Given $p \in V$, we can produce a homomorphism $\text{ev}_p : k[V] \to k$ by sending $f \mapsto f(p)$. Note that ev_p is surjective (since we have constant functions), hence $\ker(\text{ev}_p) = m_p$ is a maximal ideal, giving us a map

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}.$$

Nullstellensatz says that this is actually a bijection. For the converse map, given $m \subset k[V]$, we get a quotient $k[V] \to k[V]/m = k$ (Nullstellensatz says this extension is finite, hence must be k). So using/choosing a representation for V in $k[x_1, \dots, x_n]$ gives a surjective homomorphism onto k and specifies a bunch of points.

[1] A radical ideal is an ideal equal to its radical.

1.2 Limitations of classical algebraic geometry

Question. What is an abstract variety, i.e. "some "space" X such that locally as a cover $\{U_i\}$, each U_i is an affine variety, compatible with overlaps".

Example 1.1 (non–algebraically closed fields)**.** Take $I = (x^2+y^2+1) \subset \mathbb{R}[x, y]$. Then $\mathbb{V}(I) = \varnothing \subset \mathbb{R}^2$, but I is prime, so radical, so nullstellensatz fails.

Question. On what topological space is $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ "naturally" the set of functions? (or $\mathbb{Z}$, or $\mathbb{Z}[x]$).

Example 1.2 (Why restrict to radical ideals?)**.** Take $C = \mathbb{V}(y - x^2) \subset \mathbb{A}^2_k$ and $D = \mathbb{V}(x, y)$, so $C \cap D = \mathbb{V}(y, y - x^2) = \mathbb{V}(x, y) = \{(0, 0)\}$. This is a single point, but if $D_\delta = \mathbb{V}(y + \delta)$ for some $\delta \in k$, then $C \cap D_\delta = \{\pm\sqrt{\delta}\}$, which is 2 points for all $\delta \neq 0$. In other words, intersections of varieties don't want to be varieties.

1.3 The spectrum of a ring

11 Oct 2022, Lecture 3

Let A be a commutative ring with identity. We will define a topological space on which A is the ring of functions.

Definition 1.5. The **Zariski spectrum** of A is

$$\text{Spec } A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

A ring homomorphism $\phi : A \to B$ induces a map $\phi^{-1} : \text{Spec } B \to \text{Spec } A$ by $q \mapsto \phi^{-1}(q)$. In general, the preimage of a prime ideal is a prime ideal.

Warning. This would fail if we only considered maximal ideals, since the preimage of a maximal ideal need not be maximal.

Given $f \in A$ and $\mathfrak{p} \in \text{Spec}(A)$, we have an induced $\overline{f} \in A/\mathfrak{p}$ obtained via a quotient. Informally, we can evaluate any $f \in A$ at points $\mathfrak{p} \in \text{Spec}(A)$ with the caveat that the codomain of this evaluation depends on $\mathfrak{p}$.

Example 1.3. Take $A = \mathbb{Z}$. Then $\text{Spec } A = \text{Spec } (\mathbb{Z}) = \{p \mid p \text{ is prime}\} \cup \{(0)\}$. Let's pick an element in $\mathbb{Z}$, say $132 \in \mathbb{Z}$. Given a prime p, we can look at $132 \pmod p \in \mathbb{Z}/p$. The takeaway here is that

$$\begin{gathered}\text{Spec } \mathbb{Z} \to \text{Space}\\ 132 \in \mathbb{Z} \to \text{a function}\\ 132 \pmod p \to \text{value of that function at } p.\end{gathered}$$

Note that based on the value of p, our codomain changes from point to point.

Example 1.4. Take $A = \mathbb{R}[x]$, then $\text{Spec } \mathbb{R}[x] = \mathbb{C}/\text{complex conjugation} \cup \{(0)\}$.

Exercise. Draw Spec $\mathbb{Z}[x]$ and Spec $k[x]$ for k any field (i.e. describe all prime ideals and their containment). This is on example sheet 1.

Example 1.5. If $A = \mathbb{C}[x]$, then Spec $A = \mathbb{C} \cup \{(0)\}$, where given $a \in \mathbb{C}$, we send it to the maximal ideal $\langle z - a \rangle$.

1.4 A topology on Spec A

Fix $f \in A$. Then $\mathbb{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \equiv 0 \pmod{\mathfrak{p}}\} \subset \text{Spec } A$. (Note that $f \equiv 0 \pmod{\mathfrak{p}}$ is the same as $f \in \mathfrak{p}$).

Similarly for $J \subset A$ an ideal, $\mathbb{V}(J) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p} \ \forall f \in J\}$.

Proposition 1.2. The sets $\mathbb{V}(J) \subset \text{Spec } A$ ranging over all ideals J form the closed sets of a topology on Spec A. This topology is called the **Zariski topology**.

Proof. Easy fact: $\varnothing$ and Spec A are closed, since we have functions 1 (vanishing nowhere) and 0 (vanishing everywhere). Since $\mathbb{V}(\sum_\alpha I_\alpha) = \bigcap_\alpha \mathbb{V}(I_\alpha)$ (this is because $I_1 + I_2$ is the smallest ideal containing $I_1 \cup I_2$), arbitrary intersections are closed.

Finally, we claim $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$. The containment $\subset$ is clear: if a prime ideal contains I_1 or I_2, it contains $I_1 \cap I_2$. Conversely, $I_1 I_2 \subset I_1 \cap I_2$, so if $I_1 I_2 \subset I_1 \cap I_2 \subset \mathfrak{p}$, then by primality $I_1 \subset \mathfrak{p}$ or $I_2 \subset \mathfrak{p}$. ☐

Example 1.6. Let $k = \mathbb{C}$ and consider Spec $\mathbb{C}[x, y]$. We make a few observations:

- The point $(0) \in \text{Spec } \mathbb{C}[x, y]$ is dense in the Zariski topology, i.e. $\overline{\{(0)\}} = \text{Spec } \mathbb{C}[x, y]$ because every prime ideal contains (0) (because we are in an integral domain).
- Consider the prime ideal $(y^2 - x^3)$ (which is prime since the quotient is an integral domain). Consider a maximal ideal $\mathfrak{m}_{a,b} = (x - a, y - b)$. We can ask: when is $\mathfrak{m}_{a,b} \in \overline{\{(y^2 - x^3)\}}$? The answer: if and only if $b^2 = a^3$, e.g. $(1, 1)$ (see example sheet 1). The lesson here is that points are not closed in the Zariski topology.

1.5 Functions on opens

Definition 1.6. Let $f \in A$. Define the **distinguished open** corresponding to f to be

$$\mathcal{U}_f = (\text{Spec}(A))/\mathbb{V}(f).$$

Example 1.7.

- Let $A = \mathbb{C}[x]$, so Spec $A = \mathbb{C} \cup \{(0)\}$ (with the Zariski topology). Take $f = x$ and consider $\mathcal{U}_x$. Recall the bijection Spec $\mathbb{C} \leftrightarrow \mathbb{C} \cup \{(0)\}$ by $(x - a) \leftarrow\!\shortmid a \in \mathbb{C}$ and $(0) \leftarrow\!\shortmid (0)$. Then $\mathbb{V}(x) = \{\mathfrak{p} \in \text{Spec } A \mid x \in \mathfrak{p}\} = \{(x)\}$, so $\mathcal{U}_f = \text{Spec } A \setminus \{(x)\}$.

- More generally, suppose we fix $a_1, \dots, a_r \in \mathbb{C}$, then Spec $A \backslash \{(x-a_i)\}_{i=1}^r = \mathcal{U}$ and $\mathcal{U} = \mathcal{U}_f$, where $f = \prod_{i=1}^r (x-a_i)$.

Lemma 1.3. The distinguished opens $\mathcal{U}_f$ taken over all $f \in A$ form a basis for the Zariski topology on Spec A.

Proof. Left as an exercise on example sheet 1. □

A bit of commutative algebra:

Definition 1.7. Given $f \in A$, the **localization of** A **at** f is $A_f = A[x]/(xf-1)$, which we can informally think of as $A_f = A[\frac{1}{f}]$.

Lemma 1.4. The distinguished open $\mathcal{U}_f \subset$ Spec A is naturally homeomorphic to Spec A_f via the ring homomorphism $A \xrightarrow{j} A_f$, which produces the inverse j^{-1} : Spec $A_f \to$ Spec A.

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Proof. Primes in the ring A_f are in bijection with primes of A that miss f via j^{-1}. We exhibit this bijection:

- Given $q \subset A_f$ prime, take $j^{-1}(q) \subset A$, which is prime.
- Given $p \subset A$ a prime ideal, take $p_f = j(p)A_f$. We claim p_f is a prime exactly when $f \notin p$.
  - If $f \in p$, then p_f contains f, which is a unit, so $p_f = (1)$ is not prime.
  - If $f \notin p$, then $(A_f/p_f) \cong (A/p)_{\overline{f}}$, where $\overline{f}$ is $f + p$, a coset (exercise: check this formally). Hence $(A/p)_{\overline{f}} \subset FF(A/p)$ (FF stands for fraction field), so it is an integral domain, so p_f is prime.

Finally we need to check that these maps are inverses. This is left as an exercise. □

Facts about distinguished opens:

- $U_f \cap U_g = U_{fg}$ (easy fact).
- $U_{f^n} = U_f$ for all $n \geq 1$ (easy fact).
- The rings A_f and A_{f^n} for $n \geq 1$ are isomorphic. Why? Since $A_f = A[x]/(xf-1)$ and $A_{f^n} = A[y]/(yf^n-1)$, the isomorphism is given by $A_f \to A_{f^n}$ by $x \mapsto f^{n-1}y$ and $A_{f^n} \to A_f$ by $y \mapsto x^n$ (check these are inverses).
- Containment. $U_f \subset U_g \iff f^n$ is a multiple of g for some $n \geq 1$. To orient ourselves: if $f = gf'$, then $U_f \subset U_g$.

Proof. The ($\implies$) direction is clear by the orientation above. Conversely, suppose $U_f \subset U_g$, so $\mathbb{V}(f) \supset \mathbb{V}(g)$. The set $\mathbb{V}(f)$ is the set of all primes containing (f). We claim that $\sqrt{(f)} \subset \sqrt{(g)}$. But what is the radical of I? It is the intersection of all primes containing the ideal I. $\square$

Foreshadowing: fix A. We've made an assignment from distinguished opens in Spec A to rings by mapping $U_f \mapsto A_f$. The association is "functorial", i.e. if $U_{f_1} \subset U_{f_2}$, then we can assume that $f_1^n = f_2 f_3$, so $U_{f_1} = U_{f_1^n} = U_{f_2 f_3} \subset U_{f_2}$, so there is a homomorphism $A_{f_2} \to A_{f_1}$. This is the restriction map.

Question: can we extend this association to all open sets? See notes for the answer (yes).

2 Sheaves

2.1 Presheaves

Let X be a topological space.

Definition 2.1. A **presheaf $\mathcal{F}$ on X of abelian groups** is an association from the set of open sets in X to abelian groups given by $U \mapsto \mathcal{F}(U)$ and for $U \subset V$ opens, a homomorphism $\mathrm{res}_u^v : \mathcal{F}(V) \to \mathcal{F}(U)$ (a **restriction map**) such that $\mathrm{res}_u^u = \mathrm{id}$ and $\mathrm{res}_u^v \circ \mathrm{res}_v^w = \mathrm{res}_u^w$ when $U \subset V \subset W$ are opens.

Example 2.1. For any space X, take $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$ with the usual restriction.

Similarly we can get sheaves of rings, sets, modules over a fixed ring, etc.

Definition 2.2. A **morphism** $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves on X is, for each $U \subset X$ open, a homomorphism $\phi(u) : \mathcal{F}(u) \to \mathcal{G}(u)$ compatible with restriction maps, i.e. if $V \subset U$, then the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(u) \arrow[r, "\phi(u)"] \arrow[d, "\mathrm{res}_v^u"] & \mathcal{G}(u) \arrow[d, "\mathrm{res}_v^u"] \\
\mathcal{F}(v) \arrow[r, "\phi(v)"'] & \mathcal{G}(v)
\end{tikzcd}$$

Definition 2.3. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of preshaves is injective (surjective) if $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (surjective) for all $U \subset X$.

2.2 Sheaves

16 Oct 2022, Lecture 5

Definition 2.4. A **sheaf** is a presheaf $\mathcal{F}$ such that

(1) If $U \subset X$ is open and $\{U_i\}$ is an open cover of U, then for $s \in \mathcal{F}(U)$, if $s|_{U_i} = \mathrm{res}^U_{U_i}(s) = 0$ for all i, then $s = 0$.

(2) If U and $\{U_i\}$ are as in (1), then given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Remark. These axioms imply $\mathcal{F}(\varnothing) = 0$ (exercise).

A **morphism** of sheaves is a morphism of the underlying presheaves.

Example 2.2. If X is a topological space, $\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$, then f is a sheaf.

Non–example. Let $X = \mathbb{C}$ with the Euclidean topology and take $\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic and bounded}\}$. Then $\mathcal{F}$ is not a sheaf, since bounded functions may glue to unbounded functions. For example, take $U = \mathbb{C}$ and $U_i = D(0, i)$. Then $f(z) = z$ is bounded on each U_i, but not on U. In general, the characterization of elements of a sheaf should be purely local, and being bounded is not a local condition.

Non–example. Fix a group G and a set $\mathcal{F}(U) = G$ (the **constant presheaf**). If U_1, U_2 are disjoint, then $\mathcal{F}(U_1 \cup U_2) = G \times G$.

Example 2.3. Give G the discrete topology (every subset is open and closed) and define

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\} = \{f : U \to G \mid f \text{ is locally constant}\}.$$

This is the **constant sheaf**.

Example 2.4. If V is an irreducible variety, then

$$\mathcal{O}_V(v) = \{f \in k[V] \mid f \text{ is regular at } p \;\forall p \in U\}.$$

Here regular at p means that $f = \frac{g}{h}$ in a neighborhood of p with g, h polynomials and $h(p) \neq 0$. $\mathcal{O}_V$ is the **structure sheaf** of V.

This is a sheaf, since we have a local condition.

2.3 Basic constructions

Terminology. A **section** of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.

Construction of stalks. Fix $p \in X$ and $\mathcal{F}$ a presheaf on X. Then $\mathcal{F}_p$, the **stalk** of $\mathcal{F}$ at p, is defined to be

$$\mathcal{F}_p = \{(U, s) \mid s \in \mathcal{F}(U), p \in U\}/\sim$$

with $(U, s) \sim (V, s')$ if $\exists W \subset U \cap V$ with $p \in W$ such that $s|_W = s'|_W$.

The elements of $\mathcal{F}_p$ are called **germs**.

Example 2.5. Take $\mathbb{A}^1$, the affine line, then $\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \mid g(0) \neq 0 \right\} = k[t]_{(t)} \subset k(t)$.

Proposition 2.1. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X such that for all $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism, then f is an isomorphism.

Here $f_p((U, s)) = (U, f_U(s))$, which is well–defined.

Proof. We will show $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for each U, and we can then define f^{-1} by $(f^{-1})_U = (f_U)^{-1}$.

f_U is injective: suppose $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Since f_p is injective, $(U, s) = 0$ in $\mathcal{F}_p$ for every $p \in U$. Thus for every $p \in U$, there exists an open neighborhood U_p of p such that $s|_{U_p} = 0$. But $\{U_p \mid p \in U\}$ is a cover of U, so $s = 0$ in $\mathcal{F}(U)$ by the first condition of being a sheaf.

f_U is surjective: take $t \in \mathcal{G}(U)$. For each $p \in U$, we have $(U_p, s_p) \in \mathcal{F}_p$ with $f_p(U_p, s_p) = (U, t) \in \mathcal{G}_p$. By shrinking U_p if necessary, we can assume $f_{U_p}(s_p) = t|_{U_p}$. For points $p, p' \in U$,

$$f_{(U_p \cap U_{p'})} \left(s_p|_{U_p \cap U_{p'}} \setminus s_{p'}|_{U_p \cap U_{p'}} \right) = t|_{U_p \cap U_{p'}} - t|_{U_p \cap U_{p'}} = 0.$$

Thus $s_p|_{U_p \cap U_{p'}} - s_{p'}|_{U_p \cap U_{p'}} = 0$ by the injectivity of $f_{U_p \cap U_{p'}}$. Thus by the second sheaf axiom, $\exists s \in \mathcal{F}(U)$ with $s|_{U_p} = s_p$. Now $f_U(s)|_{U_p} = f_{U_p}(s|_{U_p}) = f_{U_p}(s_p) = t|_{U_p}$. Thus $f_U(s) = t$ by the first sheaf axiom. ☐

We emphasize that this proof is asymmetric in the sense that we need to first prove injectivity to be able to prove surjectivity. 18 Oct 2022, Lecture 6

Exercises.

(i) There is a map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ mapping $s \mapsto ((U, s))_{p \in U}$. The claim is that this map is injective (by sheaf axiom 1).

(ii) Given two maps $\phi, \psi : \mathcal{F} \to \mathcal{G}$ with $\phi_p = \psi_p \; \forall p \in X$, we have $\phi = \psi$.

Definition 2.5 (Sheafification)**.** If $\mathcal{F}$ is a presheaf on X, then a morphism $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ to the sheaf $\mathcal{F}^{\mathrm{sh}}$ is a **sheafification** if for any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ for $\mathcal{G}$ a sheaf there is a unique commutative diagram of the following form:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\phi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

Remark. Since this is a definition by universal property, $\mathcal{F}^{\text{sh}}$ and the map $\mathcal{F} \to \mathcal{F}^{\text{sh}}$ are unique (up to unique isomorphism).

A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism of sheaves $\mathcal{F}^{\text{sh}} \to \mathcal{G}^{\text{sh}}$.

Proposition 2.2. Sheafification exists.

Proof. Given a presheaf $\mathcal{F}$ on X, define

$$\mathcal{F}^{\text{sh}}(U) = \{f : U \to \bigsqcup_{p \in U} \mathcal{F}_p \mid f(p) \in \mathcal{F}_p \text{ and for all } p \in U, \text{ there exists an open neighborhood } V_p \subset U \text{ and } s \in \mathcal{F}(V_p) \text{ such that } (V_p, q) = f(q) \in \mathcal{F}_q \ \forall q \in V_p\}.$$

This is clearly a sheaf. Verifying the universal property is left as an exercise. □

Corollary 2.3. The stalks of $\mathcal{F}$ and $\mathcal{F}^{\text{sh}}$ coincide.

Proof. Easy exercise from the definitions. □

Exercise. Find a nonzero presheaf $\mathcal{F}$ with $\mathcal{F}^{\text{sh}} = 0$. (Comment by Dhruv: this is rather stupid).

2.4 Kernels, cokernels, etc.

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then we can define presheaves ker ϕ, coker ϕ, im ϕ by

$$\begin{aligned}
(\ker\, \phi)(u) &= \ker\, \phi_u : \mathcal{F}(U) \to \mathcal{G}(U) \\
(\operatorname{coker}\, \phi)(u) &= \operatorname{coker}\, \phi_u \\
(\operatorname{im}\, \phi)(u) &= \operatorname{im}\, \phi_u.
\end{aligned}$$

These are all presheaves.

Exercise. The presheaf kernel for a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is also a sheaf.

This is not true for coker ϕ in general!

Example 2.6. Take $X = \mathbb{C}$ with the Euclidean topology, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X (with addition as its group operation). Let $\mathcal{O}_X^*$ be the sheaf of nowhere vanishing holomorphic functions (with multiplication as its group operation).

We have a morphism of sheaves $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ by $f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^*(U)$. Thus $\ker(\exp) = 2\pi i\mathbb{Z}$ with $\mathbb{Z}$ the constant sheaf, but coker(exp) is not a sheaf: if we let $U_1 = \mathbb{C} \setminus [0, \infty)$, $U_2 = \mathbb{C} \setminus (-\infty, 0]$ and $U = U_1 \cup U_2 = \mathbb{C} \setminus \{\infty\}$ and we let $f(z) = z \in \mathcal{O}_X^*(U)$, then it is not in the image of $\exp : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ since $\log z$ is not single–valued on U. Thus f defines a nonzero section of (coker exp)(U). But $f|_{U_i}$ is in the image of $\exp_{U_i}$, since we just choose some branch of $\log z$. Thus $f|_{U_i} = 1$ in coker exp, so sheaf axiom 1 fails.

Definition 2.6. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define the **sheaf cokernel** and the **sheaf image** to be the sheafification of the presheaf cokernel and the presheaf image.

Remark. Crucial fact: there is an exact sequence of sheaves

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1.$$

In other words, $2\pi i\mathbb{Z} = \ker(\exp)$ and $\operatorname{coker}(\exp) = 1$ (the first of these we showed, the second of this we will show once we've developed the necessary theory).

Remark. $\ker\phi, \operatorname{coker}\phi$ satisfy the category theoretic definitions of kernels and cokernels, i.e. they are universal in the appropriate sense. For example, for the kernel, if $\ker\phi : \mathcal{F} \to \mathcal{G}$, then for any other sheaf $\mathcal{L}$ with a map ψ to $\mathcal{F}$ such that $\phi \circ \psi = 0$, this map factors uniquely through the kernel. This is easy to check and left as an exercise.

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, dashed, "\exists!"'] \arrow[d, "\psi"] \arrow[dr, "\phi\circ\psi=0"] & \\
\ker\phi \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{F} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

For the cokernel, reverse all the arrows and check that $\operatorname{coker}\phi$ satisfies the universal property (exercise).

Adjacent notions.

(i) **Subsheaves**. $\mathcal{F} \subset \mathcal{G}$ is there exist inclusions $\mathcal{F}(U) \subset \mathcal{G}(U)$ compatible with restrictions. For example, $\ker(\phi : \mathcal{F} \to \mathcal{G}) \subset \mathcal{F}$.

(ii) **Quotient sheaves**. To be added at a later date.

2.5 Moving between spaces

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Definition 2.7. Given $f : X \to Y$ continuous with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, the **presheaf pushforward** $f_\star\mathcal{F}$ is defined by

$$\mathcal{U} \mapsto \mathcal{F}(f^{-1}(\mathcal{U}))$$

for an open set $\mathcal{U} \subset \mathcal{Y}$.

Proposition 2.4. The presheaf pushforward of a sheaf is a sheaf.

Proof. Trivial. □

Definition 2.8. Given $f : X \to Y$ continuous with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, the **inverse image presheaf** $(f^{-1}\mathcal{G})^{\text{pre}}$ is defined by (for V open in X)

$$(f^{-1}\mathcal{G})^{\text{pre}}(V) = \{(s_U, U) \mid U \text{ is an open set containing } V, s_U \in \mathcal{G}(U)\}/\sim$$

where $\sim$ is an equivalence relation that identifies pairs that agree on a smaller open set containing V.

The **inverse image sheaf** is given by $f^{-1}\mathcal{G} = ((f^{-1}\mathcal{G})^{\text{pre}})^{\text{sh}}$.

Example 2.7. Take Y a topological space and set $X = Y \sqcup Y$. Take $\mathcal{G} = \mathbb{Z}$ to be the constant sheaf and $\mathcal{F} = (f^{-1}\mathcal{G})^{\text{pre}}$. Fix $U \subset Y$ open and $V = f^{-1}(U)$. Then $\mathcal{F}(V) = \mathcal{G}(U) = \mathbb{Z}$, constant (assuming U is connected). But $V = U \sqcup U$, so $\mathcal{F}^{\text{sh}}(V) = \mathcal{G}(U) \times \mathcal{G}(U) = \mathbb{Z}^2$. This happens because this isn't a local condition.

Example 2.8. Let $\mathcal{F}$ be a sheaf of X and $\pi : X \to \text{point}$. Then $\pi_\star\mathcal{F}$ is a sheaf on a point, i.e. an abelian group, specifically $\mathcal{F}(\pi^{-1}(\text{point})) = \mathcal{F}(X)$.

Notation. We write $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$, called the global section, and $\mathcal{F}(X) = H^0(X, \mathcal{F})$, the 0^{th} cohomology of coefficients in $\mathcal{F}$.

For $p \in X$, $i : \{p\} \to X$, $\mathcal{G}$ a sheaf on the point, i.e. an abelian group A, we can consider $i_\star\mathcal{G}$. This is the sheaf on X such that $i_\star(\mathcal{G})(U) = \begin{cases} 0 & p \notin U. \\ A & p \in U. \end{cases}$ This is called the skyscraper at p with value A.

3 Schemes

The summary: $\text{Spec}(A)$ has a sheaf $\mathcal{O}_{\text{Spec}(A)}$ such that the value on a distinguished open $\mathcal{U}_f = A_f$, and then globalize this to get a scheme. We now spell this out in detail.

Definition 3.1. Let A be a ring and $S \subset A$ a set that is closed under multiplication. The two examples we should keep in mind are $S = \{1, f, f^2, f^3, \ldots\}$ or $S = A \setminus \mathfrak{p}$ for $\mathfrak{p}$ a prime ideal. The **localization** of A at S is

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\}/\sim$$

where $(a, s) \sim (a', s') \iff \exists s'' \in S$ such that $s''(as' - a's) = 0 \in A$. We read $\frac{a}{s}$ for the equivalence class of (a, s).

Warning. The map $A \to S^{-1}A$ need not be injective, e.g. if S contains a zero divisor.

What's going to happen now? We will define a sheaf $\mathcal{O}_{\text{Spec}(A)}$ on the topological space $\text{Spec}(A)$ with two features:

- The stalk at a prime $\mathfrak{p}$ will be $(A \setminus \mathfrak{p})^{-1}A$.
- If $\mathcal{U}_f$ is a distinguished open, then $\mathcal{O}_{\mathrm{Spec}(A)}(\mathcal{U}_f) = A_f$.

A sheaf on a base. Fix a topological space X and $\mathcal{B}$ a basis for the topology. A **sheaf on the base** $\mathcal{B}$, $\mathfrak{F}$, consists of assignments $B_i \mapsto \mathfrak{F}(B_i)$ on abelian groups/rings/some objects with restriction maps $\mathfrak{F}(B_i) \to \mathfrak{F}(B_j)$ whenever $B_j \subset B_i$. These satisfy the usual commutativity and the identities when $B_i \subset B_j \subset B_k$ or $B_i = B_j$, as well as the following two conditions:

(1) If $B = \cup B_i$ with $B, B_i \in \mathcal{B}$ and $f, g \in \mathfrak{F}(B)$ such that $f|_{B_i} = g|_{B_i}$ $\forall i$, then $f = g$.

(2) If $B = \cup B_i$ as above with $f_i \in \mathfrak{F}(B_i)$ such that they agree where assigned (i.e. $\forall i, j$, if $B' = B_i \cap B_j$, then $f_i|_{B'} = f_j|_{B'}$), then $\exists f \in \mathfrak{F}(B)$ with $f|_{B_i} = f_i$.

Proposition 3.1. Let F be a sheaf on a base $\mathcal{B}$ of a topological space X. Then this uniquely (up to unique isomorphism) determines a sheaf $\mathfrak{F}$ by $\mathfrak{F}(B_i) = F(B_i)$ agreeing with restriction maps.

Proof. Define the stalks of $\mathfrak{F}$ first, i.e. $\mathfrak{F}_{\mathfrak{p}} = \{(S_B, B) \mid B \text{ is a basic open containing } \mathfrak{p}\}/\sim$. Now use the sheafification trick to define

$$\mathfrak{F}(U) = \{(f_p \in \mathfrak{F}_p)_{p \in U} \mid \forall p \in U, \exists \text{ basic open } B \text{ containing } p \text{ and } s \in F(B) \text{ with } s_q = f_q \text{ in } \mathcal{F}_q\ \forall q \in B\}.$$

Thirdly, the natural maps $F(B) \to \mathfrak{F}(B)$ are isomorphic by sheaf axioms. The final fact that this is unique (up to unique isomorphism) is left as an exercise. $\square$

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Setup so far: Spec A is a topological space with base $\{U_f\}$ for $U_f = \mathbb{V}(f)^C$ over $f \in A$. Recall also that $U_f = U_g$ if and only if $\sqrt{(f)} = \sqrt{(g)}$. Also, if $U_f = U_g$, then the localizations $A_f \cong A_g$ are isomorphic. Therefore, the assignment $U_f \mapsto A_f$ is well–defined.

Proposition 3.2. The assignment $U_f \to A_f$ defines a sheaf (of rings) on the base of the topology of Spec A given by distinguished opens.

As a consequence, Spec A inherits a sheaf of rings, denoted $\mathcal{O}_{\mathrm{Spec}\ A}$ and called **the structure sheaf**.

Prelude. Suppose $\{U_{f_i}\}_{i \in I}$ covers Spec A. Then there exists a finite subcover. In other words, Spec A is quasi–compact. Why? Since the U_{f_i} cover, there exists no prime ideal $\mathfrak{p} \subset A$ containing all (f_i) $\iff$ $\sum_{i \in I}(f_i) = (1)$. Hence $1 = \sum_i a_i f_i$, where all but finitely many $a_i = 0$. So if $J \subset I$ are the indices with nonzero coefficient, then $\{U_{f_i}\}_{i \in J}$ cover.

Proof. We need to check that axioms 1 and 2 of a sheaf hold. We will check these for a the basic open $B = \text{Spec } A$ itself (the general case is similar, restrict to a basic open and repeat the proof).

Axiom 1: Suppose $\text{Spec } A = \bigcup_{i=1}^n U_{f_i}$ (finite by the prelude). Given $s \in A$ such that $s|_{U_{f_i}} = 0\ \forall i$, by the definition of localization $f_i^m s = 0$ for some m large enough. But $(1) = (f_i^m)_{i=1}^n$ for any $m > 0$ because $\{U_{f_i}\}$ cover Spec A, so hence so do $\{U_{f_i^m}\}$. Hence $1 = (\sum r_i f_i^m)$ and multiplying by s on both sides gives us $s = 0$.

Axiom 2: Say $\text{Spec } A = \bigcup_{i \in I} U_{f_i}$ and choose elements in each A_{f_i} that agree in $A_{f_i f_j}$, i.e. if $s_i \in A_{f_i}$, then the images of s_i and s_j in $A_{f_i f_j}$ coincide. We need to build an element in A with these restrictions.

First suppose I is finite. On U_{f_i}, we've chosen an element $\frac{a_i}{f_i^{l_i}} \in A_{f_i}$. Write $g_i = f_i^{l_i}$, noting $U_{f_i} = U_{g_i}$. On overlaps, restrict to $A_{g_i g_j}$. The condition for the second axiom is $(g_i g_j)^{m_j}(a_i g_j - a_j g_i) = 0$. Rewriting this using algebra and $U_f = U_{f^k}\ \forall k \geq 1$, we may assume $m = m_{ij}$ by taking the largest. Write $b_i = a_i g_i^m$ and $h_i = g_i^{m+1}$, so on each U_{h_i} we've chosen an element $\frac{b_i}{h_i}$.[2] But U_{h_i} cover Spec A, so $1 = \sum_i r_i h_i$ for $r_i \in A$. Now we construct $r = \sum r_i b_i$ with r_i as above. This restricts correctly to $\frac{b_i}{h_i}$ on U_{h_i} (i.e. in the localization A_{h_i}).

When I is infinite, pick a finite subcover $(f_1, \ldots, f_n) = A$ such that U_{f_i} form a cover and use the above to build r. But given $(f_1, \ldots, f_n, f_\alpha) = A$, the same construction gives a "new" r'. But $r' = r$ by the first axiom. $\square$

Definition 3.2. The structure sheaf on Spec A is the sheaf associated to the sheaf on the base sending $U_f \mapsto A_f$, denoted $\mathcal{O}_{\text{Spec } A}$.

Observation. The stalk $\mathcal{O}_{\text{Spec } A, \mathfrak{p}} = A_\mathfrak{p}$.

Terminology. A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$. An isomorphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the combination of a homomorphism $\pi : X \to Y$ and an isomorphism of sheaves on Y, $\mathcal{O}_Y \xrightarrow{\sim} \pi_\star \mathcal{O}_X$.

Definition 3.3. An **affine scheme** is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic (as a ringed space) to $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

Definition 3.4. A **scheme** is a ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme.

Intuitively, every point $p \in X$ has a neighborhood U_p such that the ringed space $(U_p, \mathcal{O}_{U_p})$ is isomorphic to some affine scheme (possibly depending on p). Note that if $U \subset X$ is open, then U is naturally a ringed space with $\mathcal{O}_U(V) = \mathcal{O}_X(V)$.

[2] So far, this is just rewriting everything symbolically with no actual content.

Example 3.1. Spec A for various rings A.

Example 3.2. Take $X = \text{Spec } \mathbb{C}[x,y]$ and $U = \{(x,y)\}^C$. Then the scheme U is not an affine scheme.

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Example 3.3. Open subschemes. Let X be a scheme and $U \subset X$ be open.[3] Write $i : U \to X$ for the inclusion map. Take $\mathcal{O}_U = \mathcal{O}_X|_U = i^{-1}\mathcal{O}_X$ to be the structure sheaf of U.

Proposition 3.3. The ringed space $(U, \mathcal{O}_U)$ is a scheme.

Simple case: take $X = \text{Spec } A$, $U = U_f$ for $f \in A$. Then $(U, \mathcal{O}_U) \cong (\text{Spec } A_f, \mathcal{O}_{\text{Spec } A_f})$.

Proof. Let $p \in U \subset X$ be a point. Since X is a scheme, we can find some $(V_p, \mathcal{O}_X|_{V_p})$ inside X with $p \in V$ such that V_p is isomorphic to an affine scheme. Take $V_p \cap U \subset U$ with the structure sheaf via restriction. However, this may not be affine. But V_p is affine, say $V \cong \text{Spec } B$, and the distinguished opens in Spec B form a basis for the topology. Hence we've reduced to the "simple case" and we're done. ☐

Example 3.4. Define $\mathbb{A}^n_k = \text{Spec } k[x_1, \ldots, x_n]$. Take $U = \mathbb{A}^{n^2} - \{\det(x_{ij}) = 0\}$, i.e. "$U = GL_2(k)$".

Example 3.5. A non–affine scheme. Take $X = \mathbb{A}^2_k = \text{Spec } k[x,y]$ and $U = \mathbb{A}^2_k \setminus \{(x,y)\}$.[4] Then the claim is that U is not affine. We will calculate $\mathcal{O}_U(U)$. Write $U_x = \mathbb{V}(x)^C \subset \mathbb{A}^2$ and $U_y = \mathbb{V}(y)^C \subset \mathbb{A}^2$. Note that $U = U_x \cup U_y$ and $U_x \cap U_y = \mathbb{A}^2 \setminus \mathbb{V}(xy)$.

We have $\mathcal{O}_U(U_x) = k[x, x^{-1}, y]$, $\mathcal{O}_U(U_y) = k[x, y, y^{-1}]$, and $\mathcal{O}_U(U_x \cap U_y) = k[x, x^{-1}, y, y^{-1}]$. Also, the restriction maps $\mathcal{O}_U(U_x) \to \mathcal{O}_U(U_{xy})$ are the obvious ones.

By sheaf axioms, $\mathcal{O}_U(U) = k[x, x^{-1}, y] \cap k[x, y, y^{-1}]$ (inside $k[x, x^{-1}, y, y^{-1}]$). Hence $\mathcal{O}_U(U) = k[x,y]$. This is a contradiction. Why? One way: there exists (in $(U, \mathcal{O}_U)$) a maximal ideal in the global sections ring with empty vanishing locus, namely $(x,y) \subset k[x,y]$. On the other hand, there is no maximal ideal in Spec $k[x,y]$ with empty vanishing locus.

This is a bit of a hack and there is a better conceptual approach that we will discover soon.

A little more on $U = \mathbb{A}^2_k \setminus \{(x,y)\}$ not being affine (this was talked about at the beginning of the following lecture). Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. Fix $p \in X$. Then there's a well–defined stalk $\mathcal{O}_{X,p}$ of $\mathcal{O}_X$ at p. The

[3]From now on, whenever we say "let X be a scheme", we silently take that to mean $(X, \mathcal{O}_X)$.
[4]Illegally, we are allowed to think of this as $\mathbb{R}^2 \setminus \{(0,0)\}$.

stalk is of the form $A_\mathfrak{p}$, where A is a ring and $\mathfrak{p} \subset A$ is a prime ideal. In particular, $A_\mathfrak{p}$ has a unique maximal ideal, namely $\mathfrak{p}A_\mathfrak{p}$. Say f vanishes at $\mathfrak{p}$ if its image in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is 0 (i.e $f \in \mathfrak{p}A_\mathfrak{p}$). (Here we're using an isomorphism $p \ni V_p$ open to Spec A). For $f \in \Gamma(X, \mathcal{O}_X)$, $\mathbb{V}(f)$, the vanishing locus of $f \subset X$ is well–defined.

3.1 Interlude: gluing sheaves

Let X be a topological space with cover $\{U_\alpha\}$, sheaves $\{\mathcal{F}_\alpha\}$ on $\{U_\alpha\}$ and isomorphisms (of sheaves) $\phi_{\alpha,\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$ such that $\phi_{\alpha,\alpha} = \mathrm{id}$, $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$ and $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (the **cocycle condition**).

Construction. We will build a sheaf $\mathcal{F}$ on X. Given $V \subset X$ open, define

$$\mathcal{F}(V) = \{(S_\alpha)_\alpha, S_\alpha \in \mathcal{F}_\alpha(U_\alpha \cap V) \mid \phi_{\alpha,\beta}(S_\alpha|_{V \cap U_\alpha \cap U_\beta}) = S_\beta|_{V \cap U_\alpha \cap U_\beta}\}.$$

$\mathcal{F}$ is a presheaf, since given $(S_\alpha) \in \mathcal{F}(V)$ and $W \subset V$ open, we can take $(S_a)|_W = \left(\mathrm{res}^{V \cap U_\alpha}_{W \cap U_\alpha}(S_\alpha)\right)_\alpha$. This lies in $\mathcal{F}(W)$ by sheaf axioms.

Proposition 3.4. $\mathcal{F}$ is a sheaf and $\mathcal{F}|_{U_\alpha} = \mathcal{F}_\alpha$ on U_α.

Proof. It is a presheaf, and both sheaf axioms are clear (exercise: check this). But we need to check/build an isomorphism $\mathcal{F}|_{U_\gamma} \to \mathcal{F}_\gamma$. Given $V \subset U_\gamma$ and $S \in \mathcal{F}_\gamma(V)$, define its image in $\mathcal{F}_{U_\gamma}$ to be $(\phi_{\gamma,\alpha}(S|_{V \cap U_\alpha}))_\alpha$. We need to check that this lies in $\mathcal{F}|_{U_\gamma}(V) = \mathcal{F}(V)$, but this follows from the cocycle condition: $\phi_{\alpha,\beta} \circ \phi_{\gamma,\alpha}(S|_{V \cap U_\alpha \cap U_\beta}) = \phi_{\gamma,\beta}(S|_{V \cap U_\alpha \cap U_\beta})$. □

3.2 More schemes

Take schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ with opens $U \subset X, V \subset Y$ and an isomorphism $(U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_Y|_V)$. We can glue both the topological spaces and the schemes: $X \sqcup Y/(U \sim V)$ with the sheaf glued as in the previous construction.

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How to glue: Take $(X \sqcup Y)/(U \sim V)$. By definition of the quotient topology, the image of X, Y in S form an open cover and their intersection is the image of U (or V). Now glue the structure sheaves on these opens as in the previous lecture (to get $(S, \mathcal{O}_S)$). Note that there is no cocycle condition, since we only have the intersection of two and not three opens.

Example 3.6. The bug–eyed line, i.e. the line with two origins. Let k be a field and $U \subset X = \mathrm{Spec}\ k[t]$, $V \subset Y = \mathrm{Spec}\ k[u]$, $U = \mathrm{Spec}\ k[t, t^{-1}]$, $V = \mathrm{Spec}\ k[u, u^{-1}]$. We have the isomorphism $U \to V$ by $t \mapsto u$. (Really, this is an isomorphism of rings $k[u, u^{-1}] \to k[t, t^{-1}]$ with $u \mapsto t$ and now take Spec).

On the level of topological spaces, $X = \mathbb{A}^1_k$ and $Y = \mathbb{A}^1_k$ with $U = \mathbb{A}^1 \setminus \{(t)\}$ (i.e. "U minus a point", similarly for V). Hence $X \sqcup Y/\sim$ gives the line with two origins.

What are the types of opens in this scheme?

- W could be contained inside X or Y (inside S). There are nice, easy open sets.
- $W = S \setminus \{p_1, \dots, p_r\}$ where $p_i \in U$ or $p_i \in V$. The simplest of these is when $W = S$.

What is $\mathcal{O}_S(S)$? Use sheaf axioms to find that $\mathcal{O}_S(S) \cong k[t]$. Conclusion: S is not affine.

Example 3.7. $\mathbb{P}^1_k$. Same setup: $X = \text{Spec } k[t], Y = \text{Spec } k[s], U = \text{Spec } k[t, t^{-1}]$, $V = \text{Spec } k[s, s^{-1}]$. We glue via the isomorphism $s \mapsto t^{-1}$. Then $\mathbb{P}^1_k$ is the result of the gluing (we can consider this as a definition for now).

Proposition 3.5. $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong k$.

Proof. Important exercise: the only elements of $k[t, t^{-1}]$ that are both polynomials in t and t^{-1} are the constants. (Do this!). In particular, $\mathbb{P}^1$ is not affine. $\square$

Example 3.8. Similarly we can build $S =$ "$\mathbb{A}^2_k$ with doubled origin" – this has the interseting property that there exist affine open subschemes $U_1, U_2 \subset S$ such that $U_1 \cap U_2$ is not affine. We flag this example for later.

Gluing schemes. (Example sheet 1). Given schemes X_i for $i \in I$, open subschemes $X_{ij} \subset X_i$ with $X_{ii} = X_i$, isomorphisms $f_{ij} : X_{ij} \xrightarrow{\sim} X_{ji}$ with $f_{ii} = \text{id}_{X_i}$, and the cocycle condition $f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$, there is a unique scheme X with an open cover given by X_i, glued along $X_{ij} \cong X_{ji}$.

Example 3.9 (Key example)**.** Take A any ring, $X_i = \text{Spec } A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$, $X_{ij} = \mathbb{V}\left(\frac{x_j}{x_i}\right)^C \subset X_i$, and isomorphisms $X_{ij} \to X_{ji}$ by $\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j}\left(\frac{x_i}{x_j}\right)^{-1}$. The resulting glued scheme is called the **projective** n–**space**, denoted $\mathbb{P}^n_A$.

Exercise/calculation. $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

3.3 Proj construction

Definition 3.5. A $\mathbb{Z}$–**grading** on a ring A is a decomposition $A = \bigoplus_{i \in \mathbb{Z}} A_i$ as abelian groups such that $A_i A_j \subset A_{i+j}$.

Example 3.10. Take $A = k[x_0, \dots, x_n]$ and write $A_d = \{\text{degree } d \text{ homogeneous polynomials}\} \cup \{0\}$.

Also: Let $I \subset k[x_0, \dots, x_n]$ be a homogeneous ideal (i.e. generated by homogeneous elements of possibly different degree). Then $k[x_0, \dots, x_n]/I$ is also naturally graded. (Exercise: think about how).

Remark. A_0 is always a subring of A.

Assumption. From now on, we will assume that the degree 1 elements generate A as an algebra over A_0. 30 Oct 2022, Lecture 11

Another assumption. We assume $A_i = 0$ for all $i < 0$.

Terminology. $A_+ = \bigoplus_{i \geq 1} A_i \subset A$ is the subgroup of positive degree elements. It forms an ideal, called the **irrelevant ideal**.

A **homogeneous element** $f \in A$ is an element contained in some A_d.

An ideal $I \subset A$ is called **homogeneous** if it is generated by homogeneous elements (possibly of different degrees).

Definition 3.6. The set Proj A is the set of all homogeneous primes in A that do not contain the irrelevant ideal A_+.

If $I \subset A$ is homogeneous, then $\mathbb{V}(I) = \{\mathfrak{p} \in \text{Proj } A \mid \mathfrak{p} \text{ contains } I\}$. Given this, the Zariski topology on Proj A has closed sets $\mathbb{V}(I)$ for $I \subset A$ homogeneous.

Let $f \in A_i$ and $U_f = \text{Proj } A \setminus \mathbb{V}(f)$. Then observe that $\{U_f\}_{f \in A_1}$ covers Proj A. Also, the ring $A[\frac{1}{f}] = A_f$ is naturally $\mathbb{Z}$–graded by $\deg(f^{-1}) = -\deg(f)$.

Example 3.11. Let $A = k[x_0, x_1]$ and $f = x_0$, then $A[\frac{1}{f}] = k[x_0, x_1, x_0^{-1}]$. The degree 0 elements of this include $\lambda (\lambda \in k)$, $\frac{x_1}{x_0}$, $\frac{x_1^2 + x_1 x_0}{x_0^2}$, etc. Similarly, degree 1 elements include $\frac{x_1^2}{x_0}$, etc.

Proposition 3.6. There is a natural bijection between

$$\{\text{homogeneous primes in } A \text{ missing } f\} \leftrightarrow \{\text{primes in } (A_f)_{\text{degree } 0}\}.$$

(Equivalently, the LHS is the homogeneous primes in A_f).

Proof and construction: Primes in A missing f are naturally in bijection with homogeneous primes in A_f. Suppose $\mathfrak{q} \subset (A[\frac{1}{f}])_0$ is a prime. Then let $\Psi(\mathfrak{q})$ be generated by

$$\bigcup_{d \geq 0} \left\{ a \in A_d \mid \frac{a}{f^d} \in \mathfrak{q} \right\} \subset A.$$

Exercise/easy check: this is prime.

Given $\mathfrak{p} \subset A$ homogeneous missing f, take $\phi(\mathfrak{p}) = \left(\mathfrak{p} A[\frac{1}{f}] \cap (A[\frac{1}{f}]_0) \right)$.

We need to check two compositions. $\phi \circ \Psi = \text{id}$ is easy and left as an exercise. However, $\Psi \circ \phi$ is trickier. We will show $\mathfrak{p} = \Psi(\phi(p))$ by exhibiting both containments. Suppose $\mathfrak{p} \in U_f \subset \text{Proj } A$. Then if $a \in \mathfrak{p} \cap A_d$, then $\frac{a}{f^d} \subset \phi(p)$, so $a \in \Psi(\phi(\mathfrak{p}))$. Conversely, if $a \in \Psi(\phi(\mathfrak{p}))$, then $\frac{a}{f^d} \in \phi(\mathfrak{p})$ for some d. Hence there exists $b \in \mathfrak{p}$ such that $\frac{b}{f^e} = \frac{a}{f^d}$ in the localization after inverting f. For some $k \geq 0$, we have $f^k(f^d b - f^e a) = 0$, but $f^{e+k} \not\subset \mathfrak{p}$. Hence by primality, $a \in \mathfrak{p}$, giving the reverse containment. $\square$

Remark. The bijection we constructed is compatible with ideal containment, so it gives a homeomorphism from U_f to $\mathrm{Spec}(A_f)_0$.

Proj A is covered by open sets, each isomorphic to Spec $(A_f)_0$ for some f. If $f, g \in A_1$, then $U_f \cap U_g$ is naturally homeomorphic to $\mathrm{Spec}(A[\frac{1}{f}])_0[\frac{f}{g}] = \mathrm{Spec}(A[f^{-1}, g^{-1}])_0$. Call this property $(\star)$.

Take the open cover $\{U_f\}$ with structure sheaf $\mathcal{O}_{\mathrm{Spec}(A_f)_0}$ on each U_f with isomorphisms on $U_f \cap U_g$ by $(\star)$. The cocycle condition follows immediately from the formal properties of localization (exercise: check this).

Terminology. If $A = k[x_0, \ldots, x_n]$ with the standard grading, then Proj A is denoted as $\mathbb{P}^n_k$. This is the same as the projective n–space defined earlier, but we need further work to show this.

4 Morphisms

We have already seen a few "examples". For example, we have a morphism given by inclusion for $U \subset X$. Similarly, if $A \to B$ is a ring homomorphism, then Spec $B \to$ Spec A should be another morphism.

4.1 Morphisms of schemes and locally ringed spaces

Given a scheme $(X, \mathcal{O}_X)$, the stalks $\mathcal{O}_{X,p}$ are **local rings** (i.e. they have a unique maximal ideal). Given a function $f \in \mathcal{O}_X(U)$ with $p \in U$, we can ask whether f vanishes at p, i.e. is the image of f in $\mathcal{O}_{X,p}$ contained in the maximal ideal?

Definition 4.1. A morphism of **ringed spaces** (i.e. a topological space and a sheaf of rings) is $(f, f^\#)$ such that:

- $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$
- $f : X \to Y$ is continuous.
- $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$, a morphism of sheaves of rings on Y.

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Warning. It is possible to find $(f, f^\#)$ from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ of schemes such that there exists $U \subset Y$ open with $q \in U$ and $h \in \mathcal{O}_Y(U)$ such that h vanishes at q, but $f^\#(h) \in \mathcal{O}_X(f^{-1}(U))$ does not vanish at $p \in X$ such that $f(p) = q$.

Observation. Given $f : X \to Y$ a ringed space morphism and $p \in X$, there is an induced map $f^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$. To spell this out, given $s \in \mathcal{O}_{Y,f(p)}$, we can represent it by (S_U, U) for U open, $f(p) \in U$ and $S_U \in \mathcal{O}_Y(U)$. Therefore $f^\#(S_U) \in \mathcal{O}_X(f^{-1}(U))$, so the pair $(f^\#(S_U), f^{-1}(U))$ defines an element in $\mathcal{O}_{X,p}$.

Definition 4.2. $(X, \mathcal{O}_X)$, a ringed space, is called **locally ringed** if $\forall p \in X$, $\mathcal{O}_{X,p}$ is a local ring (i.e. has a unique maximal ideal). A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism as ringed spaces such that if $\mathfrak{m}_p$ denotes the maximal ideal in $\mathcal{O}_{X,p}$, then $f^\#(\mathfrak{m}_{f(p)}) \subset \mathfrak{m}_p$ (in the stalks).

Definition 4.3. A **morphism of schemes** $X \to Y$ is a morphism as locally ringed spaces.

Theorem 4.1. There is a natural bijection between

$\{$Scheme theoretic morphisms Spec $B \to$ Spec $A\} \leftrightarrow \{$Ring homomorphisms $A \to B\}$.

Prologue. Recall that section of a sheaf $\mathcal{F}$ on U, i.e. $s \in \mathcal{F}(U)$, is a coherent collection of elements $s(p) \in \mathcal{F}_p$ (the stalk) for all $p \in U$.

Proof. We will first show that every ring map $A \to B$ induces a scheme map (we will construct it), and then conversely we will show that every scheme map Spec $B \to$ Spec A arises via our construction.

Given $\phi : A \to B$, we can take ϕ^{-1} : Spec $B \to$ Spec A as the topological part (continuity is formal). Now we build $\phi^\# : \mathcal{O}_{\text{Spec }A} \to \phi_\star^{-1}\mathcal{O}_{\text{Spec }B}$. First, at the stalk level, take $A_{\phi^{-1}(p)} \to B_p$ sending $\frac{a}{s} \mapsto \frac{\phi(a)}{\phi(s)}$ induced by ϕ. This makes sense because if $s \notin \phi^{-1}(p)$, then $\phi(s) \notin p$ (we're treating p as a prime ideal here). Observe that this is automatically a local homomorphism of local rings.

Secondly, on the open set level, given $U \subset$ Spec A, we need to define $\phi^\#$: $\mathcal{O}_{\text{Spec }A}(U) \to \mathcal{O}_{\text{Spec }B}((\phi^{-1})^{-1}(U))$ (this just means take the preimage of U inside Spec B). An element $s \in \mathcal{O}_{\text{Spec }A}(U)$ is a collection of assignments $[p \mapsto s(p)]_{p\in U}$ with $p \in U$, $s(p) \in A_p$. We define $\phi^\#$ by sending $[p \mapsto s(p)]_{p\in U} \mapsto [q \mapsto \phi_q(s(\phi^{-1}(q)))]_{q\in(\phi^{-1})^{-1}(U)}$, where ϕ_q is the map on stalks at q. We can check that this glues (see official notes if we're still in disbelief).

Conversely, suppose we're given $(f, f^\#)$: Spec $B \to$ Spec A. Using $\mathcal{O}_{\text{Spec }A}(\text{Spec }A) \to \mathcal{O}_{\text{Spec }B}(\text{Spec }B)$, we get $g : A \to B$ a ring homomorphism. We need to check that g^{-1} gives the right topological map and that the construction from the first part gives the right map on sheaves.

For the first part, the maps on stalks are compatible with restriction. For instance, $\Gamma(\text{Spec }A, \mathcal{O}_{\text{Spec }A}) \to \Gamma(\text{Spec }B, \mathcal{O}_{\text{Spec }B})$ is compatible with restricting to the stalks: $\mathcal{O}_{\text{Spec }A,f(p)} \to \mathcal{O}_{\text{Spec }B,p}$ for all $p \in$ Spec B. Hence the following map commutes.

$$\begin{tikzcd}
\Gamma(\text{Spec }A, \mathcal{O}_{\text{Spec }A}) \arrow[r] \arrow[d] & \Gamma(\text{Spec }B, \mathcal{O}_{\text{Spec }B}) \arrow[d] \\
\mathcal{O}_{\text{Spec }A,f(p)} \arrow[r] & \mathcal{O}_{\text{Spec }B,p}
\end{tikzcd}$$

Equivalently, the following map commutes for all $p \in \text{Spec } B$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ A_{f(p)} \arrow[r, "f^\#"] & B_p \end{tikzcd}$$

Since the morphism is local: $(f^\#)^{-1} p B_p = f(p) A_{f(p)}$. By commutativity, $g^{-1} = f$ topologically. The structure sheaf maps agree at the stalk level by construction, so we're done. $\square$

Housekeeping.

Definition 4.4. Let X, Y be schemes. A morphism $f : X \to Y$ is an **open immersion** if f induces an isomorphism of X onto an open subscheme of Y (i.e. $(U, \mathcal{O}_{\mathcal{Y}}|_U)$ for $U \subset Y$ open).

$g : X \to Y$ is a **closed immersion** if the topological map is a homeomorphism onto a closed subset of Y and the map $g^\# : \mathcal{O}_Y \to g_\star \mathcal{O}_X$ is surjective.

Example 4.1. Take $k[t] \to k[t]/t^2$ and take Spec. This is a closed immersion.