# Part III - Modular Forms

Lectured by Jack Thorne

Artur Avameri

## Contents

# 1 Introduction

06 Oct 2022, Lecture 1

**Definition 1.1.** We define the following groups:

$$\begin{aligned}
\mathfrak{H} &= \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \\
GL_2(\mathbb{R})^+ &= \{g \in GL_2(\mathbb{R}) \mid \det(g) > 0\} \\
\Gamma(1) = SL_2(\mathbb{Z}) &= \{g \in M_2(\mathbb{Z}) \mid \det(g) = 1\}.
\end{aligned}$$

Note that $\Gamma(1)$ is a subgroup of $GL_2(\mathbb{R})^+$.

**Lemma 1.1.** $GL_2(\mathbb{R})^+$ acts transitively on $\mathfrak{H}$ by Möbius transformations.

*Proof.* Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{H}$. Then

$$\mathrm{Im}(g\tau) = \frac{1}{2i}\left(\frac{a\tau + b}{c\tau + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right) = \frac{1}{2i}\frac{(ad - bc)(\tau - \overline{\tau})}{|c\tau + d|^2} = \frac{\det(g)\mathrm{Im}(\tau)}{|c\tau + d|^2} > 0,$$

so $g\tau \in \mathfrak{H}$. This action is transitive since

$$x + iy \in \mathfrak{H} \implies \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = x + iy,$$

so everything in $\mathfrak{H}$ is conjugate to $i$. □

**Definition 1.2.** If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $\tau \in \mathfrak{H}$, then define

$$j(g, \tau) = c\tau + d.$$

This is called a **modular cocycle**. If $k \in \mathbb{Z}$ and $f : \mathfrak{H} \to \mathbb{C}$, then

$$f|_k[g] : \mathfrak{H} \to \mathbb{C}$$

is defined by

$$f|_k[g](\tau) = \det(g)^{k-1} f(g\tau) j(g, \tau)^{-k}.$$

This is the **weight $k$ action of $g$ on $f$**.

**Lemma 1.2.** This is a right action of $GL_2(\mathbb{R})^+$: if $g, h \in GL_2(\mathbb{R})^+$, then

$$f|_k[gh] = (f|_k[g])|_k[h].$$

*Proof.* We compute

$$\begin{aligned}
&(f|_k[g])|_k[h](\tau) = \det(h)^{k-1} f|_k[g](h\tau) j(h,\tau)^{-k} = \\
&\det(h)^{k-1}\det(g)^{k-1} f(gh\tau) j(g,h\tau)^{-k} j(h,\tau)^{-k} \stackrel{?}{=} \\
&\det(gh)^{k-1} f(gh\tau) j(gh,\tau)^{-k} = f|_k[gh](\tau).
\end{aligned}$$

Hence we need to check that $j(gh,\tau) = j(gh,\tau)j(h,\tau)$. Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$g\begin{pmatrix}\tau & 1\end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix}.$$

We now get

$$j(gh,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh\begin{pmatrix}\tau 1\end{pmatrix} = g\left(j(h,\tau)\begin{pmatrix} h\tau \\ 1\end{pmatrix}\right) = j(h,\tau)j(g,h\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix},$$

which finishes the computation and proof. □

**Formulae.** For $g \in GL_2(\mathbb{R})^+, \tau \in \mathfrak{H}$, we have

$$\operatorname{Im}(g\tau) = \det(g)\frac{\operatorname{Im}(\tau)}{|j(g,\tau)|^2} \text{ and } j(g,\tau)\begin{pmatrix} g\tau \\ 1\end{pmatrix} = g\begin{pmatrix}\tau \\ 1\end{pmatrix}.$$

**Definition 1.3.** Let $k \in \mathbb{Z}$ and $\gamma \leq \Gamma(1)$ of finite index[1]. A **weakly modular function of weight $k$ and level $\Gamma$** is a meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ which is invariant under the weight $k$ action of $\Gamma$, i.e. such that

$$\forall \tau \in \mathfrak{H}, \forall \gamma \in \Gamma, f|_k(\gamma) = f.$$

We will define modular forms next time: they are weakly modular functions which are holomorphic both in $\mathfrak{H}$ and at $\infty$.

It is a fact that modular forms of fixed weight and level live in finite-dimensional $\mathbb{C}$-vector spaces called $M_k(\Gamma)$. These form the main objects of study in this course.

**Motivation.** Why study modular forms?

(1) They are related to the theory of elliptic functions. Let $E/\mathbb{C}$ be an elliptic curve and $\omega$ a holomorphic non–zero 1–form. Then there exists a unique lattice[2] $\Lambda \in \mathbb{C}$ and isomorphism $\phi : \mathbb{C}/\Lambda \to E$ such that $\phi^*(\omega) = dz$. Then

[1] In other words, $\gamma$ is a (finite index) subgroup of $\Gamma(1)$.

[2] i.e. a discrete cocompact subgroup, or an abelian subgroup which is freely generated by two elements that are linearly independent over $\mathbb{R}$.

$E$ is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$ where if $k \in \mathbb{Z}$, then $G_k(\Lambda) = \sum_{\lambda \in \Lambda - \{0\}} \lambda^{-k}$. This converges absolutely for $k > 2$.

If $\tau \in \mathfrak{H}$, then $\Lambda\tau = \mathbb{Z}\tau \oplus \mathbb{Z} \subset \mathbb{C}$ is a lattice and $G_k(\tau) = G_k(\Lambda_\tau)$. This is a modular form of weight $k$ and level $\Gamma(1)$, called an Eisenstein series.

$\mathfrak{H}/SL_2(\mathbb{Z})$ can be identified with the set of (isomorphism classes of) elliptic curves over $\mathbb{C}$.

(2) Modular forms $f$ have Fourier expansions $\sum_{n \in \mathbb{Z}} a_n g^n$, $a_n \in \mathbb{C}$ and they often serve as a generating functions for arithmetically interesting sequences $a_n$.

For example, take $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. If $k \in 2\mathbb{N}$, then $\theta^k$ is a modular form with $q$–expansion $\theta^k = \sum_{n \in \mathbb{Z}} r_k(n) e^{\pi i n \tau}$, where $r_k(n)$ is the number of ways of writing $n$ as a sum of $k$ squares, i.e. $r_k(n) = |\{x \in \mathbb{Z}^k \mid \sum_{i=1}^k x_i^2 = n\}|$. By expressing $\theta^k$ in terms of other modular forms, we can prove formulae such as $r_4(n) = 8 \sum_{d|n, 4 \nmid d} d$.

(3) The Riemann zeta function $\zeta(s)$ is an important object of study. Its pleasant features include:

- The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.
- It has a meromorphic continuation to $\mathbb{C}$ and has a functional equation relating $\zeta(s)$ and $\zeta(1 - s)$.

A Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ which has similar properties (Euler product, meromorphic extension, some nice function equation) is called an $L$–function. Modular forms can be used to construct interesting examples of $L$–functions. In practice, we take $M_k(\Gamma)$ and decompose it under Hecke operators to get Hecke eigenforms, the nicest possible modular forms, which have the above properties.

(4) The Langlands program predicts a relation between modular forms and objects in arithmetic geometry. A special case of this is the modularity conjecture, which says that there is a bijective correspondence between elliptic curves $E/\mathbb{C}$ up to isogeny and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. Note that this is formulated in the language of Hecke operators and $L$–functions.

**Homework.** There is a handout on Moodle called "Reminder on Complex Analysis". Have a look at it before the next lecture.

**Warning.**