Derivate parțiale

1. $\frac{\partial f}{\partial x}\left(\frac{\pi}{4}, 1\right)$ $\qquad f(x,y) = \ln\left(\operatorname{tg}\frac{x}{y}\right)$

$\frac{\partial f}{\partial y}\left(\frac{\pi}{4}, 1\right)$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$$

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{4}, 1\right) = \lim_{x \to \frac{\pi}{4}} \frac{\ln(\operatorname{tg} x) - 0}{x - \frac{\pi}{4}} = (\#)$$

$f(x,1) = \ln(\operatorname{tg} x)$; $f\left(\frac{\pi}{4}, y\right) = \ln\left(\operatorname{tg}\frac{\pi}{4y}\right)$ $\qquad \frac{\ln(1+x)}{x} \xrightarrow[x \to 0]{} 1$

$f\left(\frac{\pi}{4}, 1\right) = \ln\left(\operatorname{tg}\frac{\pi}{4}\right) = \ln(1) = 0.$

$$\underline{\underline{(\#)}} \lim_{x \to \frac{\pi}{4}} \frac{\ln(\operatorname{tg} x)}{x - \frac{\pi}{4}} = \lim_{x \to \frac{\pi}{4}} \frac{\ln(1 + \operatorname{tg} x - 1)}{\operatorname{tg} x - 1} \cdot \frac{\operatorname{tg} x - 1}{x - \frac{\pi}{4}} =$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\operatorname{tg} x - 1}{x - \frac{\pi}{4}} = \lim_{y \to 0} \frac{\operatorname{tg}\left(y + \frac{\pi}{4}\right) - 1}{y} = (\#)$$

$x - \frac{\pi}{4} = y \qquad x \to \frac{\pi}{4} \Leftrightarrow y \to 0$

$x = y + \frac{\pi}{4} \qquad \operatorname{tg}(a+b) = \frac{\operatorname{tg} a + \operatorname{tg} b}{1 - \operatorname{tg} a \cdot \operatorname{tg} b}.$

$$\underline{\underline{(\#)}} \lim_{y \to 0} \frac{\frac{\operatorname{tg} y + 1}{1 - \operatorname{tg} y} - 1}{y} = \lim_{y \to 0} \frac{\operatorname{tg} y + \cancel{1} - \cancel{1} + \operatorname{tg} y}{1 - \operatorname{tg} y} \cdot \frac{1}{y} =$$

$$= \lim_{y \to 0} \frac{2 \operatorname{tg} y}{(1 - \operatorname{tg} y) \cdot y} = \lim_{y \to 0} 2 \cdot \frac{\operatorname{tg} y}{y} \cdot \frac{1}{1 - \operatorname{tg} y} = 2 \cdot 1 \cdot \frac{1}{1 - 0} = \boxed{2}$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{y \to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y - y_0}$$

$$\frac{\partial f}{\partial y}\left(\frac{\pi}{4}, 1\right) = \lim_{y \to 1} \frac{\ln\left(\operatorname{tg}\frac{\pi}{4y}\right)}{y - 1} = \lim_{y \to 1} \frac{1}{\operatorname{tg}\frac{\pi}{4y}} \cdot \frac{1}{\cos^2\frac{\pi}{4y}} \cdot \frac{\pi}{4}\left(-\frac{1}{y^2}\right)$$

$$= \frac{1}{tg\frac{\pi}{4}} \cdot \frac{1}{\frac{1}{2}} \cdot \frac{\pi}{4} \cdot \left(-\frac{1}{1^2}\right) = -\frac{\pi}{2}.$$

2. $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

are deriv. parțiale în $(0,0)$ dar nu e cont. în origine. $\lim\limits_{x\to 0} f(x,0) = \lim\limits_{y \to 0} f(0,y) = f(0,0) = 0$

$\Rightarrow$ deci f e cont. parțial în raport cu x și y.

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0.$$

$$|f(x,y)-0| = \left|\frac{xy}{x^2+y^2}\right| \leq \left|\frac{xy}{2xy}\right| = \frac{1}{2} \not\xrightarrow[(x,y)\to(0,0)]{} 0$$

3. i) $f(x,y) = \frac{xy}{x^2-y^2}$

$$xy^2 \frac{\partial f}{\partial x} + x^2 y \frac{\partial f}{\partial y} = (x^2+y^2)\cdot f$$

$$\frac{\partial f}{\partial x} = \left(\frac{xy}{x^2-y^2}\right)'_x = y \cdot \left(\frac{x}{x^2-y^2}\right)'_x = y \cdot \frac{1\cdot(x^2-y^2) - x\cdot(2x-0)}{(x^2-y^2)^2} =$$

$$= y \cdot \frac{x^2-y^2-2x^2}{(x^2-y^2)^2} = -y \cdot \frac{x^2+y^2}{(x^2-y^2)^2}.$$

$$\frac{\partial f}{\partial y} = \left(\frac{xy}{x^2-y^2}\right)'_y = x \cdot \left(\frac{y}{x^2-y^2}\right)'_y = x \cdot \frac{1\cdot(x^2-y^2) - y\cdot(0-2y)}{(x^2-y^2)^2}$$

$$= x \cdot \frac{x^2-y^2+2y^2}{(x^2-y^2)^2} = x \cdot \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$xy^2 \cdot \left(-y \cdot \frac{x^2+y^2}{(x^2-y^2)^2}\right) + x^2 \cdot y \cdot \left(x \cdot \frac{x^2+y^2}{(x^2-y^2)^2}\right) = (x^2+y^2)\frac{xy}{x^2-y^2}$$

$$(-xy^3+x^3y) \cdot \frac{x^2+y^2}{(x^2-y^2)^2} = (x^2+y^2)\frac{xy}{x^2-y^2}$$

$$xy\left(-\frac{y^2}{y}+x^2\right)\cdot\frac{x^2+y^2}{m(x^2-y^2)x}=(x^2+y^2)\frac{xy}{x^2-y^2}.$$

5. $g:\mathbb{R}^3\to\mathbb{R}^2$ $\qquad J[2][3]$

$$g(x,y,z)=(\underbrace{xy^2}_{g_1},\underbrace{y\ln z}_{g_2}),\ z>0$$

$\frac{\partial g_1}{\partial x}=y^2 \qquad \frac{\partial g_2}{\partial x}=0$

$\frac{\partial g_1}{\partial y}=2xy \qquad \frac{\partial g_2}{\partial y}=\ln z$

$\frac{\partial g_1}{\partial z}=0 \qquad \frac{\partial g_2}{\partial z}=\frac{y}{z}$

$$J_f=\begin{pmatrix} y^2 & 2xy & 0\\ 0 & \ln z & \frac{y}{z}\end{pmatrix}$$

6. $\frac{D(u,v,w)}{D(x,y,z)}$

ii) $u=xyz,\ v=xy-xyz,\ w=y-xy$

$J_f=?$

$\frac{\partial u}{\partial x}=yz \qquad \frac{\partial v}{\partial x}=y-yz \qquad \frac{\partial w}{\partial x}=-y$

$\frac{\partial u}{\partial y}=xz \qquad \frac{\partial v}{\partial y}=x-xz \qquad \frac{\partial w}{\partial y}=1-x$

$\frac{\partial u}{\partial z}=xy \qquad \frac{\partial v}{\partial z}=-xy \qquad \frac{\partial w}{\partial z}=0$

$$J_f=\begin{pmatrix} yz & xz & xy\\ y-yz & x-xz & -xy\\ -y & 1-x & 0\end{pmatrix}$$

$$\frac{D(u,v,w)}{D(x,y,z)}=\begin{vmatrix} yz & xz & xy\\ y(1-z) & x(1-z) & -xy\\ -y & 1-x & 0\end{vmatrix}=\begin{vmatrix} y & x & 0\\ y(1-z) & x(1-z) & -xy\\ -y & 1-x & 0\end{vmatrix}$$

$$= (-1)^{2+3} \cdot (-xy) \cdot \begin{vmatrix} y & x \\ -y & 1-x \end{vmatrix} = xy(y - \cancel{xy} + \cancel{xy}) = xy^2$$

7. $f(x,y,z) = \dfrac{yz}{x} + \dfrac{xz}{y} + \dfrac{xy}{z}$

$$f(tx,ty,tz) = \frac{t^2 yz}{tx} + \frac{t^2 xz}{ty} + \frac{t^2 xy}{tz} = t f(x,y,z)$$

$$\Rightarrow \boxed{p = 1}.$$

$$x \frac{\partial f}{\partial x}(x,y,z) + y \frac{\partial f}{\partial y}(x,y,z) + z \frac{\partial f}{\partial z}(x,y,z) = p f(x,y,z)$$

$$x^2 \frac{\partial^2 f}{\partial x^2}(x,y,z) + y^2 \frac{\partial^2 f}{\partial y^2}(x,y,z) + z^2 \frac{\partial^2 f}{\partial z^2}(x,y,z) +$$

$$2xy \frac{\partial^2 f}{\partial x \partial y} + 2yz \frac{\partial^2 f}{\partial y \partial z} + 2xz \frac{\partial^2 f}{\partial x \partial z} = p(p-1) \cdot f(x,y,z)$$

8. Fie $n \in \mathbb{N}$ şi funcţia $f_n : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $\quad f_n(x,y) = \dfrac{x^2y^3}{(x^4+y^4)^n}$.
Să se arate că are loc relaţia:

$$\frac{\partial^2 f_n}{\partial x^2}(1,1) + 2\frac{\partial^2 f_n}{\partial xy}(1,1) + \frac{\partial^2 f_n}{\partial y^2}(1,1) = \frac{4n^2 - 9n + 5}{2^{n-2}}.$$

$$f_n(tx,ty) = \frac{t^5 x^2 y^3}{(t^4x^4 + t^4y^4)^n} = \frac{t^5 x^2 y^3}{t^{4n}(x^4+y^4)^n} = t^{5-4n} f(x,y)$$

$$p = 5 - 4n, \quad n \in \mathbb{N}$$

$$\frac{\partial^2 f_n}{\partial x^2}(1,1) + 2 \cdot \frac{\partial^2 f_n}{\partial xy}(1,1) + \frac{\partial^2 f_n}{\partial y^2}(1,1) =$$

$$= p(p-1) \cdot f(1,1) = (5-4n)(4-4n) \cdot \frac{1}{2^n} = \frac{20 - 20n - 16n + 16n^2}{2^n}$$

$$= \frac{16n^2 - 36n + 20}{2^n} = \frac{4(4n^2 - 9n + 5)}{2^n} = \boxed{\frac{4n^2 - 9n + 5}{2^{n-2}}}$$

9. Să se calculeze derivatele parţiale de ordinul întai ale funcţiei compuse $F(x,y,z) = g(e^{xyz}, \sin(x+y)), g \in C^2(\mathbb{R}^2)$.

$u = e^{xyz}, \quad v = \sin(x+y)$

$$\frac{\partial u}{\partial x} = yz \cdot e^{xyz}, \quad \frac{\partial u}{\partial y} = xz e^{xyz}, \quad \frac{\partial u}{\partial z} = xy e^{xyz}$$

$$\frac{\partial v}{\partial x} = \cos(x+y); \quad \frac{\partial v}{\partial y} = \cos(x+y); \quad \frac{\partial v}{\partial z} = 0.$$

$$\frac{\partial F}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial F}{\partial z} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial z}$$

10. Să se calculeze derivatele parţiale de ordinul întai şi doi pentru funcţia compusă $F(x,y) = f(x^2+y^2, xy)$, $f \in C^2(\mathbb{R}^2)$.

$u = x^2 + y^2, \quad v = xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = x$$

$$\frac{\partial F}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} = 2x \frac{\partial g}{\partial u} + y \cdot \frac{\partial g}{\partial v}$$

$$\frac{\partial F}{\partial y} = 2y \frac{\partial g}{\partial u} + x \cdot \frac{\partial g}{\partial v}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x}\right) =$$

$$= \frac{\partial}{\partial x}\left(2x \frac{\partial g}{\partial u} + y \frac{\partial g}{\partial v}\right) =$$

$$= 2x\, \delta_x \frac{\partial g}{\partial u} + 2 \frac{\partial g}{\partial u} + y\, \delta_x \frac{\partial g}{\partial v} + 0 \cdot \frac{\partial g}{\partial v}$$

$$\widehat{\partial}_x = 2x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v}$$

$$= 2x\left(2x \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right) + y \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\right)\right) + 2\frac{\partial g}{\partial u} +$$

$$+ y \cdot \left(2x \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right) + y \cdot \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial v}\right)\right) =$$

$$= 4x^2 \frac{\partial^2 g}{\partial u^2} + 2xy \frac{\partial^2 g}{\partial v \partial u} + 2\frac{\partial g}{\partial u} + 2xy \frac{\partial^2 g}{\partial u \partial v} +$$

$$+ y^2 \frac{\partial^2 g}{\partial v^2} = 4x^2 \frac{\partial^2 g}{\partial u^2} + 4xy \frac{\partial^2 g}{\partial u \partial v} + 2\frac{\partial g}{\partial u} + y^2 \frac{\partial^2 g}{\partial v^2}$$

11. Folosind definiţia, să se arate că funcţia $f(x,y) = (x-1)^2 + y^2$ este diferenţiabilă în punctul $A(1,1)$.

f - dif. în $(1,1)$ $\overset{def.}{\Longleftrightarrow}$ $\frac{\partial f}{\partial x}(1,1) = \lim_{x \to 1} \frac{f(x,1) - f(1,1)}{x-1} =$

$f(1,1) = 1.$

$= \lim_{x\to 1} \frac{(x-1)^2 + \cancel{1^2} - \cancel{1^2}}{(x-1)} = 0.$ $\quad \frac{\partial f}{\partial x} = 2(x-1)$

$\frac{\partial f}{\partial y}(1,1) = \lim_{y \to 1} \frac{f(1,y) - f(1,1)}{y-1} = \lim_{y\to 1} \frac{y^2 - 1^2}{y-1} = 2.$

$\frac{\partial f}{\partial y} = 2y = 2$

$(\exists)\ \omega : \mathbb{R}^2 \to \mathbb{R}$, cont. și nulă în $(1,1)$

$f(x,y) = \underbrace{f(1,1)}_{=1} + \frac{\partial f}{\partial x}\underbrace{(1,1)}_{=0}(x-1) + \frac{\partial f}{\partial y}\underbrace{(1,1)}_{=2}\cdot(y-1)$

$+ \ \omega(x,y)\sqrt{(x-1)^2 + (y-1)^2}$

$f(x,y) = 1 + 2(y-1) + \omega(x,y)\sqrt{(x-1)^2 + (y-1)^2}$

$\omega(x,y) = \frac{(x-1)^2 + y^2 - 1 - 2(y-1)}{\sqrt{(x-1)^2+(y-1)^2}} = \frac{(x-1)^2 + (y-1)^2}{\sqrt{(x-1)^2+(y-1)^2}} =$

$= \sqrt{(x-1)^2 + (y-1)^2} \xrightarrow[(x,y)\to(1,1)]{} 0$

$\Rightarrow$ $\omega(x,y)$ - cont și nulă în $(1,1)$ $\Rightarrow$

$\Rightarrow$ f - diferențială.

13. Fie funcţia $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, $f(x,y) = \begin{cases} x^2 \sin \frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$.

(i) Studiaţi continuitatea derivatelor parţiale în (0,0);

(ii) Studiaţi diferenţiabilitatea funcţiei f în punctul (0,0);

$\frac{\partial f}{\partial x} = 2x \sin\frac{1}{y}$ $\qquad \frac{\partial f}{\partial x}(0,0)\ (\nexists)$

$\frac{\partial f}{\partial y} = x^2 \cos\frac{1}{y}\cdot\left(-\frac{1}{y^2}\right)$ $\qquad \frac{\partial f}{\partial y}(0,0)\ (\nexists)$ $\Bigg\} \Rightarrow$

$\Rightarrow$ derivatele nu sunt continue.

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0^2\cdot\sin\frac{1}{y}-0}{y}=0$$

$$\frac{\partial f}{\partial x}=2x\sin\frac{1}{y}$$

$$\frac{\partial f}{\partial x}=\begin{cases}2x\sin\frac{1}{y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)} 2x\sin\frac{1}{y}=0$$

$$\left|2x\sin\frac{1}{y}-0\right|\le|2x|\xrightarrow[(x,y)\to(0,0)]{}0 \Rightarrow \frac{\partial f}{\partial x} \text{ e cont}$$

$$\frac{\partial f}{\partial y}=x^2\cdot\cos\frac{1}{y}\cdot\left(-\frac{1}{y^2}\right)=-\frac{x^2}{y^2}\cdot\cos\frac{1}{y}$$

$$\frac{\partial f}{\partial x}=\begin{cases}-\frac{x^2}{y^2}\cos\frac{1}{y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)}-\frac{x^2}{y^2}\cos\frac{1}{y}=0$$

$$x_n=\left(\frac{1}{n};\frac{1}{2n\pi}\right)\to(0,0)$$

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}\frac{-\frac{1}{n^2}}{\frac{1}{4n^2\pi^2}}\cdot\cos\frac{1}{\frac{1}{2n\pi}}=-4\pi^2\cdot\cos(2n\pi)=-4\pi^2\cdot(-1)^{2n}=-4\pi^2.$$

$$x_n'=\left(\frac{1}{n},\frac{1}{(2n+1)\pi}\right)\to(0,0)$$

$$\lim_{n\to\infty}f(x_n')=\lim_{n\to\infty}\frac{-\frac{1}{n^2}}{\frac{1}{(2n+1)^2\pi^2}}\cdot\cos(2n+1)\pi=$$

$$=\lim_{n\to\infty}-\pi^2\cdot\frac{4n^2+4n+1}{n^2}\cdot(-1)^{2n+1}=4\pi^2$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y} \neq 0 \Rightarrow \frac{\partial f}{\partial y} \text{ nu e cont.}$$

$$f(x,y) = f(0,0) + \underbrace{\frac{\partial f}{\partial x}}_{=0}(x-0) + \underbrace{\frac{\partial f}{\partial y}}_{=0}(y-0) + w(x,y)\cdot\sqrt{x^2+y^2}$$

$$f(x,y) = w(x,y)\cdot\sqrt{x^2+y^2}$$

$$w(x,y) = \frac{x^2 \sin\frac{1}{y}}{\sqrt{x^2+y^2}} \qquad \sqrt{\frac{x^4}{x^2+y^2}} = \sqrt{x^2\cdot\frac{x^2}{x^2+y^2}}$$

$$\left|\frac{x^2\sin\frac{1}{y}}{\sqrt{x^2+y^2}} - 0\right| \leq \left|\frac{x^2}{\sqrt{x^2+y^2}}\right| = \left|\sqrt{\frac{x^4}{x^2+y^2}}\right| =$$

$$= \left|\sqrt{x^2\cdot\underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}}\right| \leq \left|\sqrt{x^2\cdot 1}\right| = |x| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow f$ - diferențiabilă în $(0,0)$

17) iv) $d^m f$, $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x) = e^{ax+by}$

$$d^m f = \frac{\partial^m f}{\partial x^m}(x,y)\,dx^m + \frac{\partial^m f}{\partial x^{m-1}\partial y} + \dots +$$

$$d^m f = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)^m = \sum_{k=0}^{m} C_m^k \cdot \left(\frac{\partial f}{\partial x}\right)^{m-k}\left(\frac{\partial f}{\partial y}\right)^k$$

$$d^m f = \sum_{k=0}^{m} C_m^k \cdot \frac{\partial^m f}{\partial x^{m-k}\partial y^k} = \sum_{k=0}^{m} C_m^k \cdot a^{m-k}\cdot b^k\cdot e^{ax+by}.$$

$$\sum_{k=0}^{m} C_m^k \cdot a^{m-k}\cdot b^k \cdot e^{ax+by}\cdot dx^{m-k}\cdot dy^k$$

18. Să se dezvolte polinomul $P(x,y) = 2x^3 + 4x^2y + y^2 - 1$ după puterile lui $x+1$ şi $y-1$.

$f(-1,1) = -2+4+1-1 = 2.$

$$f(x,y) = \underbrace{f(-1,1)}_{=2} + \frac{1}{1!}\left[\underbrace{\frac{\partial f}{\partial x}(-1,1)}_{=-2}(x+1) + \right.$$

$$+ \left.\underbrace{\frac{\partial f}{\partial y}(-1,1)}_{=6}(y-1)\right] + \frac{1}{2!}\left[\underbrace{\frac{\partial^2 f}{\partial x^2}(-1,1)}_{=-4}(x+1)^2 + \right.$$

$$2\underbrace{\frac{\partial^2 f}{\partial x\partial y}(-1,1)}_{=-8}(x+1)(y-1)+\underbrace{\frac{\partial^2 f}{\partial y^2}(-1,1)}_{=2}(y-1)^2\Big]+$$

$$+\frac{1}{3!}\Big[\underbrace{\frac{\partial^3 f}{\partial x^3}(-1,1)}_{=12}(x+1)^3+3\underbrace{\frac{\partial^3 f}{\partial x^2\partial y}(-1,1)}_{=8}(x+1)^2(y-1)+$$

$$+3\underbrace{\frac{\partial^3 f}{\partial x\partial y^2}(-1,1)}_{=0}(x+1)(y-1)^2+\underbrace{\frac{\partial^3 f}{\partial y^3}(-1,1)}_{=0}(y-1)^3\Big]$$

$$\frac{\partial f}{\partial x}=6x^2+8xy \qquad \frac{\partial f}{\partial x}(-1,1)=6-8=\boxed{-2}$$

$$\frac{\partial f}{\partial y}=4x^2+2y \qquad \frac{\partial f}{\partial y}(-1,1)=4+2=\boxed{6}$$

$$\frac{\partial^2 f}{\partial x^2}=12x+8y \qquad \frac{\partial^2 f}{\partial x^2}(-1,1)=-12+8=\boxed{-4}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial(4x^2+2y)}{\partial x}=8x \qquad \frac{\partial f}{\partial x\partial y}(-1,1)=\boxed{-8}$$

$$\frac{\partial^2 f}{\partial y^2}=\boxed{2}$$

$$\frac{\partial^3 f}{\partial x^3}=\boxed{12}$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2}{\partial x^2}(4x^2+2y)=\boxed{8}$$

$$\frac{\partial^3 f}{\partial x\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial}{\partial x}(2)=\boxed{0.}$$

$$\frac{\partial^3 f}{\partial y^3}=\boxed{0.}$$

(iv) $f:\mathbb{R}^3\to\mathbb{R},\ f(x,y,z)=x^2+y^2+4z^2-xy+xz+2yz;$

$$\begin{cases}\dfrac{\partial f}{\partial x}=0\\[2mm] \dfrac{\partial f}{\partial y}=0\\[2mm] \dfrac{\partial f}{\partial z}=0\end{cases}\qquad \begin{aligned}\frac{\partial f}{\partial x}&=2x-y+z=0\\ \frac{\partial f}{\partial y}&=2y-x+2z=0\\ \frac{\partial f}{\partial z}&=8z+x+2y=0\end{aligned}$$

$$\begin{cases} 2x - y + z = 0 \\ -x + 2y + 2z = 0 \\ x + 2y + 8z = 0 \end{cases} \Rightarrow M(0,0,0)$$

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \boxed{2} & -1 & 1 \\ -1 & 2 & 2 \\ 1 & 2 & 8 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0 \\ \Delta_3 = 32 - 2 - 2 - 2 - 8 - 8 = 8 > 0 \end{array}\right\} \overset{\text{Sylvester}}{\Longrightarrow}$$

$\Rightarrow M(0,0,0)$ pct. de minim.

(iii) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 6x^2 - 6y^2$;

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \quad \begin{array}{l} \frac{\partial f}{\partial x} = 3x^2 - 12x = 0 \mid :3 \\ \frac{\partial f}{\partial y} = 3y^2 - 12y = 0 \mid :3 \end{array} \quad \begin{cases} x^2 - 4x = 0 \\ y^2 - 4y = 0 \end{cases}$$

$$\begin{cases} x(x-4) = 0 \\ y(y-4) = 0 \end{cases} \quad M_1(0,0), M_2(0,4), M_3(4,0), M_4(4,4)$$

$$H_f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x-12 & 0 \\ 0 & 6y-12 \end{pmatrix} = \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix}$$

$\Delta_1 = -12 < 0$
$\Delta_2 = 144 > 0$ $\Big\}$ Sylvester $M_1(0,0)$ $\Rightarrow$ pct. de maxim

Ex: $f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0 \end{cases} \qquad \frac{\partial f}{\partial x} = 3x^2 + 12y \qquad \frac{\partial f}{\partial y} = 2y + 12x \qquad \frac{\partial f}{\partial z} = 2z + 2 \qquad \begin{cases} 3x^2 + 12y = 0 \\ 2y + 12x = 0 \\ 2z + 2 = 0 \Rightarrow z = -1 \end{cases}$$

$y = -6x$

$3x^2 - 72x = 0 \mid :3 \qquad x^2 - 24x = 0 \qquad x(x-24) = 0$

$\Rightarrow M_1(0,0,-1),\ M_2(24,-144,-1)$

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 6x & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$Hf(0,0,-1) = \begin{pmatrix} 0 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \Delta_1 = 0$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2}(0,0,-1)\,dx^2 + \frac{\partial^2 f}{\partial x \partial y}\,dx\,dy + \frac{\partial^2 f}{\partial x \partial z}\,dx\,dz + \frac{\partial^2 f}{\partial y^2}\,dy^2 + \frac{\partial^2 f}{\partial y \partial x}\,dy\,dx + \frac{\partial^2 f}{\partial y \partial z}\,dy\,dz$$
$$+ \frac{\partial^2 f}{\partial z^2}\,dz^2 + \frac{\partial^2 f}{\partial z \partial x}\,dz\,dx + \frac{\partial^2 f}{\partial z \partial y}\,dz\,dy =$$
$$= \frac{\partial^2 f}{\partial x^2}(0,0,-1)\,\underline{dx^2} + \frac{\partial^2 f}{\partial y^2}(0,0,-1)\,\underline{dy^2} + \frac{\partial^2 f}{\partial z^2}(0,0,-1)$$
$$\underline{dz^2} + \left(\frac{\partial^2 f}{\partial x \partial y}\,dx\,dy + \frac{\partial^2 f}{\partial x \partial z}\,dx\,dz + \frac{\partial^2 f}{\partial y \partial z}\,dy\,dz\right) \cdot 2 =$$

$= 2dy^2 + 2z^2 + 24dxdy$

$$\begin{pmatrix} 0 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 12 & 0 \\ 12 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(-1)^{3+3} \cdot (2-\lambda) \cdot \begin{vmatrix} -\lambda & 12 \\ 12 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(-2\lambda+\lambda^2-144)=0$

$(2-\lambda)(\lambda^2-2\lambda-144)=0$

$\Delta = 4+4\cdot 144 =$
$= 4+576 = 580$

$(2-\lambda)(\dots)=0$

$\lambda_1>0,\ \lambda_2<0,\ \lambda_3>0 \Rightarrow$ pct. nu e de extrem (M_1)

24. Determinaţi $\alpha, \beta, \gamma \in \mathbb{R}$ pentru care funcţia $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2x^2 + 2y^2 - 3xy + \alpha x + \beta y + \gamma$ admite un minim egal cu 0 în punctul $(2, -1)$.

$f(2,-1)=0$

$M(2,-1)$ punct de minim.

$f(2,-1) = 2\cdot 2^2 + 2\cdot(-1)^2 - 3\cdot 2\cdot(-1) + \alpha\cdot 2 + \beta\cdot(-1) + \gamma =$

$= 8+2+6+2\alpha-\beta+\gamma = 0$

$2\alpha - \beta + \gamma = -16$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(4x-3y+\alpha\right) =$$

$= 4$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(4y-3x+\beta\right) = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(4y-3x+\beta\right) = -3$$

$$H_f = \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix}$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 & \Rightarrow x = 2 \\ \frac{\partial f}{\partial y} = 0 & \Rightarrow y = -1 \end{cases}$$

$$\begin{cases} \frac{\partial f}{\partial x}(2,-1) = 0 \\ \frac{\partial f}{\partial y}(2,-1) = 0 \\ f(2,-1) = 0 \end{cases} \qquad \begin{cases} 8+3+\alpha = 0 \Rightarrow \boxed{\alpha = -11} \\ -4-6+\beta = 0 \Rightarrow \boxed{\beta = 10} \\ 2\alpha - \beta + \gamma = -16 \end{cases}$$

$$\frac{\partial f}{\partial x} = 4x-3y+\alpha \qquad -22-10+\gamma = -16$$

$$\boxed{\gamma = -16+32 = 16}$$

$$\frac{\partial f}{\partial y} = 4y-3x+\beta$$

$$S: \{(-11; 10; 16)\}$$

(i) $\dfrac{\partial^{13} f}{\partial x^6 \partial y^7}$ pentru funcţia $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = (x + y) \cdot \sin(x + y)$;

$\partial^{14} f$

$$\frac{\partial^{13} f}{\partial x^6 \partial y^7} = \frac{\partial^6}{\partial x^6}\left(\frac{\partial^7 f}{\partial y^7}\right)$$

$$\frac{\partial^7 f}{\partial y^7} = \Big[\underbrace{(x+y)}_{u}\underbrace{\sin(x+y)}_{v}\Big]^{(7)}_y = \sum_{k=0}^{7} C_7^k \cdot (x+y)^{(7-k)}_x \sin(x+y)^{(k)}_x$$

$$\frac{\partial u}{\partial y} = 1 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = \dots = \frac{\partial^7 u}{\partial y^7} = 0$$

$$= C_7^6 (x+y)^{(7-6)} \sin(x+y)^{(6)} + C_7^7 (x+y)^{(7-7)} \cdot \sin(x+y)^{(7)}$$

$$\sin(x+y)^{(n)} = \sin\left(x+y+\frac{n\pi}{2}\right)$$

$$= \frac{7!}{6!\cdot 1!}\cdot 1\cdot \sin\left(x+y+6\frac{\pi}{2}\right) + \frac{7!}{7!\cdot 0!}\cdot(x+y)\sin\left(x+y+\frac{7\pi}{2}\right)$$

$$= 7\sin(x+y+3\pi) + (x+y)\sin\left(x+y+3\pi+\frac{\pi}{2}\right)$$

$$\frac{\partial^6}{\partial x^6}\left(7\sin(x+y+3\pi) + \underbrace{(x+y)}_{u}\underbrace{\sin\left(x+y+3\pi+\frac{\pi}{2}\right)}_{v}\right) =$$

$$= \frac{\partial^6}{\partial x^6}\left(7\sin(x+y+3\pi)\right) + \frac{\partial^6}{\partial x^6}\left((x+y)\sin\left(x+y+\frac{7\pi}{2}\right)\right)$$

$$= 7\sin\left(x+y+3\pi+\frac{6\pi}{2}\right) + \sum_{k=0}^{6} C_6^k\cdot(x+y)_y^{(6-k)}\sin\left(x+y+\frac{7\pi}{2}\right)_y^{(k)}$$

$$\ln x = \left(\frac{1}{x}\right)^{(n-1)} = \frac{(-1)^{n-1}}{x^n}\cdot(n-1)!$$

$$= 7\sin(x+y+6\pi) + \frac{6!}{1!\cdot 5!}(x+y)_y^{(1)}\sin^{(5)}\left(x+y+\frac{7\pi}{2}\right)_y +$$
$$+ \frac{6!}{1!\cdot 6!}(x+y)\sin^{(6)}\left(x+y+\frac{7\pi}{2}\right) =$$

$$= 7\sin(x+y+6\pi) + 6\sin(x+y+6\pi) + (x+y)\sin$$
$$\left(x+y+\frac{7\pi}{2}+\frac{6\pi}{2}\right) = 13\sin(x+y+6\pi) + (x+y)\sin\left(x+y+\frac{13\pi}{2}\right)$$

(iii) $\dfrac{\partial^{m+n} f}{\partial x^m \partial y^n}$ pentru funcţia $f:\mathbf{R}^2\to\mathbf{R}$, $f(x,y) = x\cdot e^{5x+8y}$;

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} f_n \quad,\quad f:\mathbb{R}^2\to\mathbb{R},\ f(x,y) = x\cdot e^{5x+8y}$$

$$\frac{\partial^m}{\partial x^m}\left(\frac{\partial^n f}{\partial y^n}\right) = \frac{\partial^m}{\partial x^m}\left(\frac{\partial^n f}{\partial y^n}\right)$$

$$\frac{\partial^n}{\partial y^n}\left(\underbrace{x}_{u}\cdot\underbrace{e^{5x+8y}}_{v}\right) = \sum_{k=0}^{n} C_n^k\cdot u^{(n-k)}v^{(k)} = (\#)$$

$$u^{(0)} = x, \quad u^{(1)} = 1, \quad u^{(2)} = \dots = u^{(n)} = 0.$$

$$\overset{(\#)}{=} C_n^0 \cdot \underbrace{u^{(n)}}_{=0} v^{(0)} + C_n^1 \cdot \underbrace{u^{(n-1)}}_{=0} v^{(1)} \underbrace{+ \dots}_{=0} + C_n^{n-1} \cdot u^{(1)} v^{(n-1)}$$

$$+ C_n^n \cdot u^{(0)} \cdot v^{(n)} = \frac{n!}{1! \cdot (n-1)!} \cdot 1 \cdot 8^{n-1} e^{5x+8y} + x \cdot 8^n e^{5x+8y} \quad (\diamond)$$

$$v^{(0)} = e^{5x+8y}; \quad v^{(1)} = 8e^{5x+8y}; \quad v^{(2)} = 8^2 e^{5x+8y} \dots$$

$$\dots v^{(n)} = 8^n e^{5x+8y}$$

$$\overset{(\diamond)}{=} e^{5x+8y} \left(n 8^{n-1} + x \cdot 8^n \right) = 8^{n-1} e^{5x+8y} (n + 8x)$$

$$\frac{\partial^m}{\partial x^m} \left(8^{n-1} e^{5x+8y} (n + 8x) \right) = 8^{n-1} \frac{\partial^m}{\partial x^m} \Big(\underbrace{e^{5x+8y}}_{g_1}$$

$$\cdot \underbrace{(n+8x)}_{g_2} \Big) \quad (\#)$$

$$g_2^{(0)} = n + 8x; \quad g_2^{(1)} = 8; \quad g_2^{(2)} = \dots = g_2^{(m)} = 0$$

$$g_1^{(0)} = e^{5x+8y}; \quad g_1^{(1)} = 5e^{5x+8y} \Rightarrow g_1^{(m)} = 5^m e^{5x+8y}$$

$$\overset{(\#)}{=} 8^{n-1} \cdot \sum_{k=0}^{m} C_m^k \cdot \left(e^{5x+8y} \right)^{(m-k)} \cdot (n+8x)^{(k)} =$$

$$= 8^{n-1} \Big[C_m^0 \left(e^{5x+8y} \right)^{(m)} \cdot (n+8x)^{(0)} +$$

$$+ C_m^1 \left(e^{5x+8y} \right)^{(m-1)} \cdot (n+8x)^{(1)} \Big] =$$

$$= 8^{n-1} \cdot \Big[5^m e^{5x+8y} (n + 8x) + m \cdot 5^{m-1} \cdot e^{5x+8y}$$

$$\cdot 8 \Big] = 8^{n-1} \cdot 5^{m-1} \cdot e^{5x+8y} \left((n+8x) \cdot 5 + 8m \right) =$$

$$\boxed{= 8^{n-1} \cdot 5^{m-1} \cdot e^{5x+8y} (40x + 5n + 8m)}$$