Integrale cu parametru

1. *Folosind teorema de derivare a integralelor cu parametru, să se calculeze următoarele integrale:*

1. *Să se calculeze integralele*

i) $I = \int\limits_0^2 \frac{1}{1+\alpha x}dx,\ \alpha > 0, \quad J = \int\limits_0^2 \frac{x}{(1+3x)^2}dx;$

ii) $I = \int\limits_0^3 \frac{1}{\alpha^2 + x^2}dx,\ \alpha > 0, \quad J = \int\limits_0^3 \frac{1}{(9+x^2)^2}dx.$

Soluţie. i) Prin calcul direct avem

$$I = \int\limits_0^2 \frac{1}{1+\alpha x}dx = \frac{1}{\alpha}\ln(1+\alpha x)\Bigg|_0^2 = \frac{1}{\alpha}\ln(1+2\alpha).$$

Dacă se consideră integrala I ca o integrală depinzând de parametrul α, din teorema de derivare a integralelor cu parametru avem

$$I'(\alpha) = \frac{d}{d\alpha}\int\limits_0^2 \frac{1}{1+\alpha x}dx = \int\limits_0^2 -\frac{x}{(1+\alpha x)^2}dx.$$

Pe de altă parte, cum $I(\alpha) = \frac{1}{\alpha}\ln(1+2\alpha)$, derivând acest rezultat avem

$$I'(\alpha) = -\frac{1}{\alpha^2}\ln(1+2\alpha) + \frac{1}{\alpha}\cdot\frac{2}{1+2\alpha}.$$

Prin urmare, se obţine

$$\int\limits_0^2 -\frac{x}{(1+\alpha x)^2}dx = -\frac{1}{\alpha^2}\ln(1+2\alpha) + \frac{1}{\alpha}\cdot\frac{2}{1+2\alpha}.$$

Pentru $\alpha = 3$ în relaţia de mai sus, rezultă

$$J = \int\limits_0^3 \frac{1}{(9+x^2)^2}dx = \frac{1}{9}\ln 7 - \frac{2}{21}.$$

ii) Integrala I se calculează direct:

$$I = \int_0^3 \frac{1}{\alpha^2 + x^2} dx = \frac{1}{\alpha} \arctan \frac{x}{\alpha} \Bigg|_0^3 = \frac{1}{\alpha} \arctan \frac{3}{\alpha}.$$

Dacă se consideră integrala I ca o integrală depinzând de parametrul α, din teorema de derivare a integralelor cu parametru avem

$$I'(\alpha) = \frac{d}{d\alpha} \int_0^3 \frac{1}{\alpha^2 + x^2} dx = \frac{d}{d\alpha} \int_0^3 -\frac{2\alpha}{(\alpha^2 + x^2)^2} dx.$$

Pe de altă parte avem

$$I'(\alpha) = -\frac{1}{\alpha^2} \arctan \frac{3}{\alpha} - \frac{3}{\alpha(\alpha^2 + 9)}.$$

Egalând cele două relaţii obţinute anterior, rezultă

$$-2\alpha \int_0^3 \frac{1}{(\alpha^2 + x^2)^2} dx = -\frac{1}{\alpha^2} \arctan \frac{3}{\alpha} - \frac{3}{\alpha(\alpha^2 + 9)},$$

sau echivalent

$$\int_0^3 \frac{1}{(\alpha^2 + x^2)^2} dx = \frac{1}{2\alpha^3} \arctan \frac{3}{\alpha} + \frac{3}{2\alpha^2(\alpha^2 + 9)}.$$

În ultima relaţie dând lui α valoarea 3 se obţine

$$J = \int_0^3 \frac{1}{(9 + x^2)^2} dx = \frac{\pi + 2}{216}.$$

2. Să se calculeze

$$I = \int_0^\infty \frac{1}{(4 + x^2)^6} dx$$

folosind integralele cu parametru.

Soluţie. Considerăm integrala cu parametru $I_1(a) = \int\limits_0^\infty \frac{1}{a^2+x^2}dx$. Se obţine uşor valoarea acestei integrale ca fiind $I_1(a) = \frac{\pi}{2a}$. Aplicând teorema de derivare a integralelor cu parametru se obţine $I_1'(a) = -\int\limits_0^\infty \frac{2a}{(a^2+x^2)^2}dx$. Pe de altă parte cum $I_1'(a) = \left(\frac{\pi}{2a}\right)' = -\frac{\pi}{2a^2}$, rezultă

$$I_2(a) = \int\limits_0^\infty \frac{1}{(a^2+x^2)^2}dx = \frac{\pi}{2^2a^3}.$$

Aplicând din nou, teorema de derivare sub semnul integrală avem $I_2'(a) = -\int\limits_0^\infty \frac{4a}{(a^2+x^2)^3}dx$. Totodată avem $I_2'(a) = \left(\frac{\pi}{2^2a^3}\right)' = -\frac{3\pi}{2^2a^4}$, deci

$$I_3(a) = \int\limits_0^\infty \frac{1}{(a^2+x^2)^3}dx = \frac{3\pi}{2^4a^5}.$$

Repetând procedeul se obţine

$$I_4(a) = \int\limits_0^\infty \frac{1}{(a^2+x^2)^4}dx = \frac{5\pi}{2^5a^7}, \quad I_5(a) = \int\limits_0^\infty \frac{1}{(a^2+x^2)^5}dx = \frac{5\cdot 7\pi}{2^8a^9},$$

$$I_6(a) = \int\limits_0^\infty \frac{1}{(a^2+x^2)^6}dx = \frac{63\pi}{2^9a^9}.$$

Se observă că $I = I_6(2) = \int\limits_0^\infty \frac{1}{(4+x^2)^6}dx = \frac{63\pi}{2^{18}}$.

3. Folosind posibilitatea permutării integralelor depinzând de un parametru, să se calculeze:

i) $I_1 = \int\limits_0^1 \frac{\sqrt{x} - \sqrt[3]{x}}{\ln x} dx$; **ii)** $I_2 = \int\limits_0^\infty e^{-ax} \frac{\sin bx - \sin cx}{x} dx, \quad a > 0.$

Soluţie. i) Se consideră integrala $I(\beta, \alpha) = \int\limits_0^1 \frac{x^\beta - x^\alpha}{\ln x} dx, \quad \alpha, \beta > -1$. Se observă că

$$\int\limits_\alpha^\beta x^t dt = \frac{x^t}{\ln x}\Bigg|_\alpha^\beta = \frac{x^\beta - x^\alpha}{\ln x}.$$

Din teorema lui Fubini, rezultă

$$\begin{aligned} I(\beta, \alpha) &= \int\limits_0^1 \frac{x^\beta - x^\alpha}{\ln x} dx = \int\limits_0^1 \left(\int\limits_\alpha^\beta x^t dt \right) dx = \int\limits_\alpha^\beta \left(\int\limits_0^1 x^t dx \right) dt \\ &= \int\limits_\alpha^\beta \frac{x^{t+1}}{t+1}\Bigg|_0^1 dt = \int\limits_\alpha^\beta \frac{1}{t+1} dt = \ln \frac{\beta + 1}{\alpha + 1}. \end{aligned}$$

Pentru $\beta = \frac{1}{2}$, $\alpha = \frac{1}{3}$ avem

$$I_1 = I\left(\frac{1}{2}, \frac{1}{3}\right) = \ln \frac{9}{8}.$$

ii) Deoarece $\int\limits_c^b e^{-ax} \cos tx dt = e^{-ax} \frac{\sin bx - \sin cx}{x}$, din teorema lui Fubini, rezultă

$$\begin{aligned} \int\limits_0^\infty e^{-ax} \frac{\sin bx - \sin cx}{x} dx &= \int\limits_0^\infty \left(\int\limits_c^b e^{-ax} \cos tx dt \right) dx \\ &= \int\limits_c^b \left(\int\limits_0^\infty e^{-ax} \cos tx dx \right) dt. \end{aligned}$$

Din problema 1 avem $\int\limits_0^\infty e^{-ax}\cos tx dx = C(a,t) = \dfrac{a}{a^2+t^2}$. Prin urmare, se obţine

$$\begin{aligned}\int\limits_0^\infty e^{-ax}\frac{\sin bx - \sin cx}{x}dx &= \int\limits_c^b \frac{a}{a^2+t^2}dt = \arctan\frac{t}{a}\Bigg|_c^b \\ &= \arctan\frac{b}{a} - \arctan\frac{c}{a} = \arctan\frac{a(b-c)}{a^2+bc}.\end{aligned}$$

Funcţiile lui Euler

1. *Să se calculeze următoarele integrale, folosind funcţia* Γ*:*
i) $\int_0^\infty x^3e^{-x}dx$, **ii)** $\int_0^\infty \sqrt{x}e^{-x}dx$, **iii)** $\int_0^\infty \dfrac{e^{-x}}{\sqrt{x}}dx$,
iv) $\int_1^\infty x\sqrt[3]{x^2-1}e^{-x^2}dx$, **v)** $\int_{-\infty}^0 x^4e^x dx$.

Soluţie. i) Integrala $\int_0^\infty x^3e^{-x}dx$ se poate calcula folosind funcţia Γ :

$$\int_0^\infty x^3e^{-x}dx = \Gamma(4) = 3! = 6.$$

ii) Integrala $\int_0^\infty \sqrt{x}e^{-x}dx$ nu mai poate fi calculată cu tehnicile învăţate la liceu, deci vom folosi funcţia Γ a lui Euler.

$$\int_0^\infty \sqrt{x}e^{-x}dx = \int_0^\infty x^{\frac{1}{2}}e^{-x}dx = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

iii) Se observă că

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx = \int_0^\infty x^{-\frac{1}{2}}e^{-x}dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

iv) Se face schimbarea de variabilă $x^2 - 1 = t$. Atunci $t \in [0, \infty]$, iar $2xdx = dt$. Cu acestea integrala devine

$$\int_1^\infty x\sqrt[3]{x^2-1}e^{-x^2}dx = \frac{1}{2}\int_0^\infty \sqrt[3]{t}e^{-t-1}dt = \frac{1}{2e}\Gamma\left(\frac{1}{3}+1\right) = \frac{1}{6e}\Gamma\left(\frac{1}{3}\right).$$

v) Prin simpla schimbare $x \mapsto -x$, avem

$$\int_{-\infty}^0 x^4 e^x dx = \int_0^\infty x^4 e^{-x} dx = \Gamma(5) = 4! = 24.$$

2. *Folosind proprietăţile funcţiilor* Γ *şi* β*, să se calculeze valoarea următoarelor integrale:*

i) $\displaystyle\int_0^1 x^2\sqrt[3]{1-x}dx$; **ii)** $\displaystyle\int_0^1 x^3\sqrt{1-\sqrt{x}}dx$; **iii)** $\displaystyle\int_0^1 x^{10}(1-x)^{15}dx$;
iv) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x}dx$, **v)** $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}dx$; **vi)** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sqrt[6]{\cos x}}dx$.

Soluţie. i) Integrală se poate calcula cu ajutorul funcţiile speciale ale lui Euler Γ şi β.

$$\int_0^1 x^2\sqrt[3]{1-x}dx = \beta\left(3, \frac{4}{3}\right) = \frac{\Gamma(3)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(3+\frac{4}{3}\right)} = \frac{2!\Gamma\left(1+\frac{1}{3}\right)}{\Gamma\left(1+\frac{10}{3}\right)} = \frac{2\cdot\frac{1}{3}\Gamma\left(\frac{4}{3}\right)}{\frac{10}{3}\Gamma\left(\frac{10}{3}\right)}$$
$$= \frac{1}{5}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(1+\frac{7}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)}{\frac{5\cdot 7}{3}\cdot\Gamma\left(\frac{7}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)}{\frac{5\cdot 7\cdot 4}{3^2}\cdot\Gamma\left(\frac{4}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)}{\frac{5\cdot 7\cdot 4}{3^3}\cdot\Gamma\left(\frac{1}{3}\right)} = \frac{27}{140}.$$

ii) Substituţia $x = t^2$ conduce la $dx = 2tdt$ şi exprimă integrala cu ajutorul funcţiei β.

$$\int_0^1 x^3\sqrt{1-\sqrt{x}}dx = \int_0^1 2t^7(1-t)^{\frac{1}{2}}dt = 2\beta\left(8, \frac{3}{2}\right)$$
$$= 2\frac{\Gamma(8)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(8+\frac{3}{2}\right)} = 2\frac{7!\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(9+\frac{1}{2}\right)} = \frac{2^9\cdot 7!}{17!!}.$$

iii) $\displaystyle\int_0^1 x^{10}(1-x)^{15}dx = \beta(11,16) = \frac{\Gamma(11)\Gamma(16)}{\Gamma(27)} = \frac{10!\cdot 15!}{26!}.$

iv) Cu substituţia $\sin x = \sqrt{t}$, se obţine

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x}dx = \int_0^1 \sqrt[3]{\sqrt{t}}\cdot\frac{1}{\sqrt{1-t}}\cdot\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\beta\left(\frac{2}{3},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{6}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{7}{6}\right)} = 3\sqrt{\pi}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}.$$

v) Schimbarea de variabilă $x^2 = \dfrac{t}{1-t}$ conduce la $x = \sqrt{\dfrac{t}{1-t}}$ de unde rezultă $dx = \dfrac{1}{2}\cdot\dfrac{\sqrt{1-t}}{\sqrt{t}}\cdot\dfrac{1}{(1-t)^2}dt$. Pentru $x = 0 \Rightarrow t = 0$, iar pentru $x = \infty \Rightarrow t = 1$. Prin urmare, integrala dvine

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}dx = \frac{1}{2}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}dt = \frac{1}{2}\beta\left(\frac{2}{3};\frac{1}{3}\right) =$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{2}} = \frac{\pi\sqrt{3}}{3}.$$

vi) Cu schimbarea de variabilă $x = \arcsin\sqrt{t}$, se obţine

$$\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sqrt[6]{\cos x}}dx = \frac{1}{2}\beta\left(3;\frac{5}{12}\right) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma\left(\frac{5}{12}\right)}{\Gamma\left(\frac{41}{12}\right)} = \frac{1728}{2465},$$

deoarece

$$\Gamma\left(\frac{41}{12}\right) = \Gamma\left(1+\frac{29}{12}\right) = \frac{29}{12}\Gamma\left(1+\frac{17}{12}\right) = \frac{29}{12}\cdot\frac{17}{12}\Gamma\left(1+\frac{5}{12}\right) = \frac{29\cdot 17\cdot 5}{12^3}\Gamma\left(\frac{5}{12}\right).$$

Probleme propuse

1. *Folosind funcţiile lui Euler, să se calculeze următoarele integrale:*

i) $I_1 = \int_0^\infty x^8 e^{-2x} dx$; $I_2 = \int_0^\infty x^{18} e^{-x^2} dx$; $I_3 = \int_0^\infty x^{28} e^{-x^3} dx$;

ii) $I_4 = \int_{-\infty}^{-5} (x+5)^8 e^{x+5} dx$; $I_5 = \int_{-\infty}^{2} (x-2)^{17} e^{x-3} dx$;

$I_6 = \int_{-\infty}^{-1} (x+1)^{23} e^{x+23} dx$; $I_7 = \int_0^1 x(\ln x)^7 dx$;

iii) $I_8 = \int_0^1 \sqrt{x^{16} - x^{17}} dx$; $I_9 = \int_0^1 x^{23} \sqrt{1-x} dx$;

$I_{10} = \int_{-1}^{0} x^{43} (1+x)^{34} dx$;

iv) $I_{11} = \int_0^7 x^2 \sqrt{49 - x^2} dx$; $I_{12} = \int_0^3 x^2 \sqrt{(9-x^2)^{11}} dx$;

v) $I_{13} = \int_0^{\frac{\pi}{2}} \sin^{43} x \cos^{34} x dx$; $I_{14} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sqrt[3]{\cos x}} dx$;

$I_{15} = \int_0^{\frac{\pi}{2}} \frac{\cos^{10} x}{\sqrt[8]{\sin x}} dx$; $I_{16} = \int_0^{\frac{\pi}{2}} \cos^{54} x dx$;

vi) $I_{17} = \int_0^\infty \frac{x}{(1+x^3)^2} dx$; $I_{18} = \int_0^\infty \frac{\sqrt{x}}{1+x^6} dx$; $I_{19} = \int_0^{\frac{\pi}{2}} \sqrt[3]{\tan x} dx$;

vii) $I_{20} = \int_0^\infty \frac{x^5 - \sqrt{x^5}}{e^x} dx$.

Indicaţii şi răspunsuri. i) $I_1 = \frac{8!}{2^9} = \frac{315}{2}$; $I_2 = \frac{17!!\sqrt{\pi}}{2^{10}}$; ii) I_4 se face schimbarea de variabilă $x+5=-t$; $I_5 = -\frac{17!}{e}$; iii) $I_9 = \frac{23! \cdot 2^{24}}{49!!}$; iv) Cu substituţia $x = 7u$ rezultă $I_{11} = \frac{2401\pi}{16}$; v) $I_{13} = \frac{21! \cdot 33!! \cdot 2^{21}}{77!!}$; $I_{16} = \frac{53!!\pi}{27! \cdot 2^{28}}$.

vi) La I_{17} se face schimbarea de variabilă $x^3 = \frac{t}{1-t}$.

Bibliografie *D. Păunescu, A. Juratoni*-Calcul integral avansat, Editura Orizonturi Universitare, Timişoara, 2015.