Integrale generalizate.

1. $$\int_{-1}^{7} \frac{dx}{\sqrt[3]{1+x}} = \lim_{u \to -1} \int_u^7 \frac{dx}{\sqrt[3]{1+x}} = \lim_{u\to -1} \frac{3}{2}(x+1)^{\frac{2}{3}} \Big|_u^7 \overset{(\#)}{=}$$

$t = 1+x$
$dt = dx$

$$J = \int \frac{1}{\sqrt[3]{t}}\, dt = \int t^{-\frac{1}{3}} dt = \frac{3}{2} t^{\frac{2}{3}}$$

$$\overset{(\#)}{=} \lim_{u \to -1} \frac{3}{2}\left(2^{3\cdot\frac{2}{3}} - (u+1)^{\frac{2}{3}}\right) = \frac{3}{2}\cdot 4 = 6 \in \mathbb{R}$$

$\Rightarrow J$ – conv.

ii) $$\int_1^3 \frac{x}{\sqrt{3-x}}\, dx = \lim_{u \to 3}\left(-6\sqrt{3-x} + \frac{2}{3}\cdot(3-x)\sqrt{3-x}\right)\Big|_1^u \overset{(\#)}{=}$$

$t = \sqrt{3-x}$
$t^2 = 3-x$
$x = 3-t^2$
$dx = -2t\,dt$

$$J = \int \frac{3-t^2}{\cancel{t}}\cdot(-2\cancel{t})\,dt =$$
$$= \int(-6+2t^2)\,dt =$$
$$= -6t + \frac{2t^3}{3}.$$

$$\overset{(\#)}{=} \lim_{u\to 3}\Big(\underbrace{-6\sqrt{3-u}}_{=0} + \underbrace{\frac{2}{3}(3-u)\sqrt{3-u}}_{=0} + 6\sqrt{3-1} - \frac{2}{3}(3-1)\sqrt{3-1}\Big) =$$

$$= 6\sqrt{2} - \frac{4}{3}\sqrt{2} = \left(6 - \frac{4}{3}\right)\sqrt{2} = \frac{14\sqrt{2}}{3} \Rightarrow J \text{ – conv.}$$

2. i) $$\int_0^\infty \frac{x}{\sqrt{x^4+1}}\, dx = \frac{1}{2}\int_0^\infty \frac{2x}{\sqrt{(x^2)^2+1^2}}\, dx = (\#)$$

$$I = \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$t = x^2$
$dt = 2x\,dx$

$$J = \int \frac{dt}{\sqrt{t^2+1^2}} = \ln\left(t + \sqrt{t^2+1^2}\right)$$

$$\overset{(\#)}{=} \lim_{v\to\infty} \frac{1}{2}\ln\left(x^2+\sqrt{x^2+1}\right)\Big|_0^v = \lim_{v\to\infty}\frac{1}{2}\ln\left(v^2+\sqrt{v^2+1}\right) =$$

$= \infty. \Rightarrow$ integrală divergentă

3. iii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^6-x^3+1}\,dx = \quad (\#)$

$$\frac{x^2}{(x^3)^2-x^3+1}$$

$t = x^3$

$dt = 3x^2 dx$

$\frac{1}{3}dt = x^2 dx$

$$J = \frac{1}{3}\int \frac{dt}{t^2-t+1}$$

$$J = \frac{1}{3}\int \frac{dt}{t^2-2t\cdot\frac{1}{2}+\frac{1}{4}+\frac{3}{4}} = \frac{1}{3}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\cdot \operatorname{arctg}\frac{\left(t-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$\overset{(\#)}{=} \int_{-\infty}^{0}\frac{x^2}{x^6-x^3+1}\,dx + \int_0^{\infty}\frac{x^2}{x^6-x^3+1}\,dx =$$

$$= \lim_{v\to-\infty}\frac{2}{3\sqrt{3}}\operatorname{arctg}\frac{x^3-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Big|_v^0 + \lim_{v\to\infty}\frac{2}{3\sqrt{3}}\operatorname{arctg}\frac{x^3-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Big|_0^v =$$

$$= \lim_{v\to-\infty}\left(\frac{2}{3\sqrt{3}}\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)-\frac{2}{3\sqrt{3}}\operatorname{arctg}\frac{v^3-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+$$

$$+\lim_{v\to\infty}\left(\frac{2}{3\sqrt{3}}\operatorname{arctg}\frac{v^3-\frac{1}{2}}{\frac{\sqrt{3}}{2}}-\frac{2}{3\sqrt{3}}\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)\right)=$$

$$= \cancel{-\frac{2}{3\sqrt{3}}\cdot\frac{\pi}{6}} + \frac{2}{3\sqrt{3}}\cdot\frac{\pi}{\cancel{2}} + \frac{\cancel{2}}{3\sqrt{3}}\cdot\frac{\pi}{\cancel{2}} + \cancel{\frac{2}{3\sqrt{3}}\cdot\frac{\pi}{6}} = \frac{2\pi}{3\sqrt{3}}$$

$\Rightarrow$ integrala e conv.

5. $I = \displaystyle\int_3^{\infty}\frac{dx}{x^2+x-2}$; $f(x) = \dfrac{1}{x^2+x-2}$

$$\lim_{x\to\infty} x^{\alpha} f(x) \overset{\alpha=2}{=} \lim_{x\to\infty} \frac{x^2}{x^2+x-2} = 1.$$

$\alpha > 1$ și $L < \infty \Rightarrow I = \text{conv.}$

$$I = \int_3^{\infty} \frac{dx}{\underbrace{x^2-1}_{x+2}+\underbrace{x-1}_{x-1}} = \int_3^{\infty} \frac{dx}{(x-1)(x+1)+(x-1)} = \int_3^{\infty} \frac{dx}{(x-1)(x+2)}$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x+2A-B}{(x-1)(x+2)} =$$

$$\begin{cases} A+B=0 \\ 2A-B=1 \end{cases} \Rightarrow B = -\frac{1}{3}$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$\frac{1}{(x-1)(x+2)} = \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right)$$

$$I = \int_3^{\infty} \frac{dx}{x^2+x-2} = \int_3^{\infty} \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right) = \frac{1}{3}\int_3 \frac{1}{x-1} - \frac{1}{3} \; (\#)$$

$$x^2+x-2=0$$

$$x_{1,2} = \frac{-1\pm 3}{2} \quad \begin{matrix} x_1 = -2 \\ x_2 = 1 \end{matrix}$$

$$\overset{(\#)}{=} -\frac{1}{3}\int_3^{\infty} \frac{1}{x+2} = \lim_{v\to\infty}\left(\frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x+2)\right)\Big|_3^{v} =$$

$$= \lim_{v\to\infty}\left(\frac{1}{3}\ln\frac{x-1}{x+2}\right)\Big|_3^{v} = \lim_{v\to\infty}\left(\frac{1}{3}\ln\frac{v-1}{v+2} - \frac{1}{3}\ln\frac{2}{5}\right) =$$

$$= \lim_{v\to\infty} \frac{1}{3}\ln\frac{2}{5}\cdot\frac{v-1}{v+2} = \frac{1}{3}\ln\frac{2}{5}.$$

6. ii) $I_2 = \int_0^5 \frac{1}{\sqrt{|x-2|}}\,dx = \int_0^2 \frac{1}{\sqrt{2-x}}\,dx + \int_2^5 \frac{1}{\sqrt{x-2}}\,dx =$

$$= \int_0^2 (2-x)^{-\frac{1}{2}}\,dx + \int_2^5 (x-2)^{-\frac{1}{2}}\,dx =$$

$$= \lim_{u \to 2} -\frac{(2-x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_0^u + \lim_{v \to 2} \frac{(x-2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_v^5 =$$

$$= \lim_{u \to 2} -2\sqrt{2-x}\Big|_0^u + \lim_{v \to 2} 2\sqrt{x-2}\Big|_v^5 =$$

$$= \lim_{u \to 2}\Big(\underbrace{-2\sqrt{2-u}}_{\to 0} + 2\sqrt{2}\Big) + \lim_{v \to 2}\Big(2\sqrt{3} - \underbrace{2\sqrt{v-2}}_{\to 0}\Big) =$$

$$= 2\sqrt{2} + 2\sqrt{3} = 2(\sqrt{3}+\sqrt{2})$$

v) $I = \int_{-\infty}^{\infty} \frac{1}{x^2+16}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+16} + \int_0^{\infty} \frac{1}{x^2+16} =$

$$= \lim_{u \to -\infty} \frac{1}{4}\operatorname{arctg}\frac{x}{4}\Big|_u^0 + \lim_{v \to \infty} \frac{1}{4}\operatorname{arctg}\frac{x}{4}\Big|_0^v =$$

$$= \lim_{u \to -\infty} -\frac{1}{4}\operatorname{arctg}\frac{u}{4} + \lim_{v \to \infty} \frac{1}{4}\operatorname{arctg}\frac{v}{4} =$$

$$= \frac{1}{4}\cdot\frac{\pi}{2} + \frac{1}{4}\cdot\frac{\pi}{2} = \frac{2\pi}{8} = \frac{\pi}{4}.$$

vi) $I = \int_2^{\infty} \frac{1}{x^2(x+2)}\,dx$

$$\lim_{x \to \infty} x^{\alpha} f(x) \overset{\alpha=3}{=} \lim_{x \to \infty} \frac{x^3}{x^3+2x^2} = 1$$

$L = 1 < \infty$, $\alpha = 3 > 1 \Rightarrow I$ - CONV

$$\frac{1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}.$$

$\boxed{\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1}\,dx}$ - CONV, $(\forall)\, t > 0$

$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ - CONV $(\forall)\, p, q > 0$

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\cdot\sqrt{\pi}, (\forall) n\in\mathbb{N}^*$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\Gamma(n+1)=n!, n\in\mathbb{N}$$

$$\Gamma(t+1)=t\,\Gamma(t)$$

$$\Gamma(1)=1$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\Gamma(p)\cdot\Gamma(1-p)=\frac{\pi}{\sin(p\cdot\pi)}$$

$$p\in(0,1)$$

iv) $$\int_1^\infty x\sqrt[3]{x^2-1}\,e^{-x^2}dx=\int_0^\infty \frac{1}{2}\cdot\sqrt[3]{t}\cdot\frac{1}{e}\cdot e^{-t}dt$$

$t=x^2-1$ $\qquad x=1 \Leftrightarrow t=0$

$dt=2x\,dx$ $\qquad x=\infty \Leftrightarrow t=\infty$

$\frac{1}{2}dt=x\,dx$ $\qquad e^{-(x^2-1)-1}=e^{-(x^2-1)}\cdot e^{-1}=e^{-t}\cdot\frac{1}{e}$

$$=\frac{1}{2e}\int_0^\infty t^{+\frac{1}{3}}\cdot e^{-t}dt=\frac{1}{2e}\int_0^\infty x^{+\frac{1}{3}}e^{-x}dt$$

$$\frac{1}{3}=p-1 \Rightarrow p=1+\frac{1}{3}=\frac{4}{3}$$

$$\Rightarrow I=\frac{1}{2e}\Gamma\left(\frac{4}{3}\right) - \text{conv.}$$

$$\Gamma\left(\frac{4}{3}\right)=\Gamma\left(1+\frac{1}{3}\right)=\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$I=\frac{1}{6e}\Gamma\left(\frac{1}{3}\right)$$

2. ii) $$\int_0^1 x^3\sqrt{1-\sqrt{x}}\,dx=\int_0^1 t^6\cdot\sqrt{1-t}\cdot 2t\,dt=$$

$$B(p,q)=\int_0^1 x^{p-1}\cdot(1-x)^{q-1}dx, \quad p,q>0$$

$t=\sqrt{x}$ $\qquad x=0 \Leftrightarrow t=0$

$t^2=x$ $\qquad x=1 \Leftrightarrow t=1$

$2t\,dt=dx$

$$=2\int_0^1 t^7(1-t)^{\frac{1}{2}}dt=2B\left(8,\frac{3}{2}\right)=$$

$p-1=7 \Rightarrow p=8$

$q-1=\frac{1}{2} \Rightarrow q=\frac{3}{2}$

$$= 2\cdot\frac{\Gamma(8)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(8+\frac{3}{2}\right)} = 2\cdot\frac{7!\cdot\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(9+\frac{1}{2}\right)} =$$

$$= \cancel{2}\cdot\frac{7!\cdot\frac{1!!\cdot\sqrt{\pi}}{\cancel{2}}}{\frac{17!!\cdot\sqrt{\pi}}{2^9}} = \frac{2^9\cdot 7!}{17!!}$$

$$\int_0^1 x(\ln x)^7 dx = \int_{-\infty}^0 e^t\cdot t^7\cdot e^t = \int_{-\infty}^0 t^7 e^{2t} dt$$

$t=\ln x$ $\qquad x=0 \Leftrightarrow t=-\infty$

$\qquad\qquad\quad x=1 \Leftrightarrow x=0$

$e^t=x$

$e^t dt = dx$

$y=-t$ $\qquad t=-\infty \Leftrightarrow y=\infty$

$dy=-dt$ $\qquad t=0 \Leftrightarrow y=0$

$-dy=dt$

$$= \int_\infty^0 (-y)^7\cdot e^{-2y}(-dy) = \int_0^\infty -y^7\cdot e^{-2y} dy =$$

$v=2y \Rightarrow y=\frac{v}{2}$ $\qquad y=0 \Leftrightarrow v=0$

$dv=2dy$ $\qquad y=\infty \Leftrightarrow v=\infty$

$\frac{1}{2}dv=dy$

$$= \int_0^\infty -\left(\frac{v}{2}\right)^7\cdot e^{-2v}\cdot\frac{1}{2}dv = -\frac{1}{2^8}\int_0^\infty v^7\cdot e^{-v} dv =$$

$\Rightarrow p-1=7 \Rightarrow p=8$

$$\Rightarrow I = -\frac{1}{2^8}\Gamma(8) = -\frac{7!}{2^8}$$

$$I_{17} = \int_0^\infty \frac{x}{(1+x^3)^2} dx$$

$x^3=\frac{t}{1-t}$

$$x^3(1-t) = t$$

$$x^3 - x^3 t = t$$

$$x^3 = t(1+x^3)$$

$$t = \frac{x^3}{1+x^3}$$

$$dt = \frac{3x^2(1+x^3) - x^3 \cdot 3x^2}{(1+x^3)^2} dx$$

$$dt = \frac{3x^2 + \cancel{3x^5} - \cancel{3x^5}}{(1+x^3)^2} dx$$

$$dt = \frac{3x^2}{(1+x^3)^2} dx$$

$$I_{18} = \int_0^\infty \frac{\sqrt{x}}{1+x^6} dx$$

$$= \int_0^\infty x^{-\frac{1}{2}} (1+x^6) dx$$