PS

tema 11

5. Se consideră vectorul aleator discret (X, Y) cu densitatea de probabilitate

$$P(x,y) = \begin{cases} \frac{1}{21} & \text{dacă } x = 0,1,2,3,4,5, \quad y = 0,1,\ldots,x \\ 0 & \text{dacă altfel} \end{cases}$$

Să se determine $cov(X, Y)$.

$x \in [0, 5] \qquad y \in [0, x]$

$x \backslash y$	0	1	2	3	4	5
0	$\frac{1}{21}$	0	0	0	0	0
1	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0	0
2	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0
3	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0
4	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0
5	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{2}{21} & \ldots & & & \frac{6}{21} \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 1 & 2 & \ldots & 5 \\ \frac{6}{21} & \ldots & & & \frac{1}{21} \end{pmatrix}$$

$$cov(X, Y) = M(X \cdot Y) - M(X) \cdot M(Y)$$

$$(X\,Y) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 12 & 15 & 16 & 20 & 25 \\ \frac{6}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{2}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} \end{pmatrix}$$

$$M(X) = \frac{70}{21} \qquad M(Y) = \frac{35}{21} \qquad M(XY) = \frac{140}{21}$$

$$cov(X, Y) = \frac{140}{21} - \frac{70 \cdot 35}{21^2} = \frac{490}{441}$$

6. Se consideră vectorul aleator continuu (X, Y) cu densitatea de probabilitate

$$f(x,y) = \begin{cases} \frac{x+y}{3} & \text{dacă } 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{dacă altfel} \end{cases}$$

Să se determine $cov(X, Y)$. $= M(xy) - M(x) \cdot M(y)$

$$f_x(x) = \int_0^2 \frac{x+y}{3}\, dy = \left(\frac{x}{3} y + \frac{1}{3} \frac{y^2}{2}\right)\Big|_0^2 = 2\frac{x}{3} + \frac{1}{3} \quad (\text{dom } y)$$

$$M(x) = \int_0^1 x \cdot \frac{2x}{3} + \frac{1}{3} x\, dx = \left(\frac{2}{3} \frac{x^3}{3} + \frac{1}{3} \frac{x^2}{2}\right)\Big|_0^1 = {}^{2)}\frac{2}{9} + {}^{3)}\frac{1}{6} = \frac{7}{18}$$

$$f_y(y) = \int_0^1 \frac{x+y}{3}\, dx = \left(\frac{y}{3} x + \frac{1}{3} \frac{x^2}{2}\right)\Big|_0^1 = \frac{y}{3} + \frac{1}{6} \quad (\text{dom } x)$$

$$M(y) = \int_0^2 \left(\frac{y}{3} + \frac{1}{6}\right) y\, dy = \left(\frac{1}{3} \frac{y^3}{3} + \frac{1}{6} \frac{y^2}{2}\right)\Big|_0^2 =$$

$$= {}^{4)}\frac{1}{9}\cdot 8 + {}^{3)}\frac{1}{12}\cdot 4 = \frac{32+12}{36} = \frac{44}{36} = \frac{22}{18} = \frac{11}{9}$$

$$M(X\cdot Y) = \int_0^1\int_0^2 xy\,\frac{x+y}{3}\,dy\,dx = \int_0^1\int_0^2 \frac{x^2y}{3} + \frac{xy^2}{3}\,dy\,dx$$

$$= \int_0^1 \left(\frac{x^2}{3}\cdot\frac{y^2}{2} + \frac{x}{3}\cdot\frac{y^3}{3}\right)\Bigg|_0^2 dx =$$

$$= \int_0^1 \frac{x^3}{3}\cdot 2 + \frac{x}{3}\cdot\frac{8}{3}\,dx =$$

$$= \left(\frac{\cancel{2}}{3}\cdot\frac{x^4}{\cancel{4}_2} + \frac{\cancel{8}^4}{9}\cdot\frac{x^2}{\cancel{2}}\right)\Bigg|_0^1 = {}^{3)}\frac{1}{6} + {}^{2)}\frac{4}{9} = \frac{3+8}{18} = \frac{11}{18}$$

$$\mathrm{cov}(X,Y) = {}^{9)}\frac{11}{18} - \frac{7}{18}\cdot\frac{11}{9} = \frac{99-77}{18\cdot 9} = \frac{22}{18\cdot 9} = \frac{11}{81}$$

7. Daca matricea de covarianta a vectorului (X,Y) este $\Sigma = \begin{pmatrix} 4 & -4 \\ -4 & 25 \end{pmatrix}$, sa se calculeze $\rho(X,Y)$ şi $\sigma^2(X+2Y)$.

$\sigma^2(X) = 4$ $\qquad$ $\mathrm{cov}(X,Y) = -4$

$\sigma^2(Y) = 25$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma(X)\,\sigma(Y)} = \frac{-4}{2\cdot 5} = -\frac{2}{5}$$

$$\sigma^2(X+2Y) = \sigma^2(X) + \sigma^2(2Y) + 2\,\mathrm{cov}(X, 2Y) =$$

$$= 4 + 4\cdot 25 + 2\cdot 2\,(-4) =$$

$$= 4 + 100 - 16 = 100 - 12 = 88$$

8. Fie variabilele aleatoare X şi Y înre care există relaţia $Y = X - 2$. Să se calculeze covarianţa şi coeficientul de corelaţie pentru variabilele X, Y, şiind că $\sigma^2(X) = 0.01$. Să se determine matricea de covarianţă asociată vectorului (X,Y).

10^{-2}

$Y = X - 2$

$\sigma^2(Y) = \sigma^2(X-2) = \sigma^2(X) = 0{,}01$

$\sqrt{10^{-2}} =$

$= 10^{-1}$

$Y = aX + b \qquad a = 1,\quad b = -2$

$a > 0 \Rightarrow \rho(X,Y) = 1$

$$\rho(x,y) = \frac{cov(x,y)}{\sigma(x) \cdot \sigma(y)} \Rightarrow$$

$$cov(x,y) = 1 \cdot \frac{1}{100} = \frac{1}{100} = 0{,}01$$

$$\Sigma = \begin{pmatrix} 0{,}01 & 0{,}01 \\ 0{,}01 & 0{,}01 \end{pmatrix}$$

9. Dacă X şi Y sunt două variabile aleatore astfel încât $\rho(X,Y) = 0$, atunci sunt X şi Y independente? Dar necorelate?

- $\rho(x,y) = 0 \Rightarrow x, y$ – necorelate

$\Rightarrow cov(x,y) = 0 \longrightarrow$ nu se poate preciza

- necorelate ✓ (teorie)

10. Se consideră vectorul aleator continuu (X,Y) cu densitatea de probabilitate

$$f(x,y) = \begin{cases} 4xy & \text{dacă } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{dacă altfel} \end{cases}$$

Să se determine $cov(X,Y)$.

$$f_x(x) = \int_0^1 4xy\,dy = \cancel{4}^{2}x \frac{y^2}{\cancel{2}}\Big|_0^1 = 2x$$

$$\Rightarrow M(x) = \int_0^1 2x^2\,dx = 2\frac{x^3}{3} = \frac{2}{3}$$

Analog $M(y) = \frac{2}{3}$

$$M(xy) = \int_0^1\int_0^1 4x^2y^2\,dy\,dx = \int_0^1 \frac{4}{3}x^2 y^3\Big|_0^1 dx =$$

$$= \frac{4}{3}\frac{x^3}{3}\Big|_0^1 = \frac{4}{9}$$

$$cov(x,y) = \frac{4}{9} - \frac{4}{9} = 0$$