AM(c)

Șiruri numerice

$x : \mathbb{N} \to \mathbb{R}$

$(x_n)_{n \in \mathbb{N}}, \; x_n = \dfrac{n}{n^2+2}$

$x : \mathbb{N}_0 \to \mathbb{R}$

$\mathbb{N}_0 \subset \mathbb{N}$

Șir de funcții

ex: $f_n(x) = \dfrac{nx}{n^2x+2}$, $f_n(x) : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$

$f(n, x)$

$x = 1 \Rightarrow f_n(1) = \dfrac{n}{n^2+2}$

Def 1. șirul (x_n) - mărg. dacă $\begin{cases} \exists M \in \mathbb{R}_+^* \text{ a.î. } |x_n| \leq M, \forall n \\ \exists a, b \in \mathbb{R} \text{ a.î. } a \leq x_n \leq b, \forall n \end{cases}$

Def 2. șirul (x_n) - nemărg. dacă $\forall M \in \mathbb{R}_+^*, \exists n \in \mathbb{N}$ a.î. $|x_n| > M$.

ex: $x_n = \cos(n)$, $|\cancel{\cos}\, x_n| \leq 1 \leq \ldots$ (orice nr. > 1)

↳ mărginit

! cel mai mic majorant → supremum

!! cel mai mare minorant → infimum

ex: $y_n = \dfrac{n^2+2}{n}$

↳ nemărginit

Fie $M = 5 \Rightarrow \exists n = 100$ (de ex.) a.î. $y_{100} > 5$

• Șirul (x_n) – monoton ↗ dacă $\underline{\underline{x_n \le x_{n+1}}}, \forall n$
– monoton ↘ dacă $x_n \ge x_{n+1}, \forall n$

$\dfrac{x_{n+1}}{x_n} \ge 1$

ex $x_n = 2^n < x_{n+1} = 2^{n+1}$ ↗

ex $y_n = e^{-n}, n \in \mathbb{N}$

$\quad \dfrac{1}{e^n} = \ldots$ ↘

ex $z_n = \dfrac{(-1)^n}{n}, n \in \mathbb{N}^*$

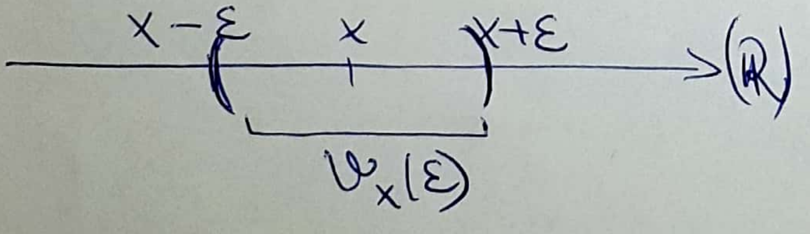
$$\left\{\begin{array}{l} z_{2n} = \dfrac{1^{2n}}{2n}, n \in \mathbb{N}^* \\ z_{2n+1} = -\dfrac{1^{2n+1}}{2n+1}, n \in \mathbb{N} \end{array}\right. \Rightarrow \text{sunt SUBȘIRURI ale șirului } (z_n)_{n \ge 0}$$

Def.2: Fie $(x_n)_{n} \in \mathbb{N}$ – șir NUMERIC și șirul de nr. nat
$0 \le k_0 \le k_1 \le \ldots \le k_n \le$ ↗.

⇒ Se num. SUBȘIR al lui (x_n) șirul $(y_n) = (x_{k_n})$.

<u>Def</u> Șirul (x_n) are limită (este convergent) dacă $\exists\, x \in \mathbb{R}$
a.î. $\exists\, n_0(\varepsilon) \in \mathbb{N}$ pt. care $\underline{\underline{|x_n - x|}} < \varepsilon, \forall n \ge n_0(\varepsilon)$

⇕



$-\varepsilon < x_n - x < \varepsilon \;|+x$

$-\varepsilon + x < x_n < \varepsilon + x$

$x - \varepsilon < x_n < x + \varepsilon$

$\lim\limits_{n \to \infty} x_n = x \Leftrightarrow x_n \xrightarrow{n \to \infty} x$

) $x_n = \frac{n}{2n+1} \xrightarrow{n \to \infty} \frac{1}{2} = 0,5$

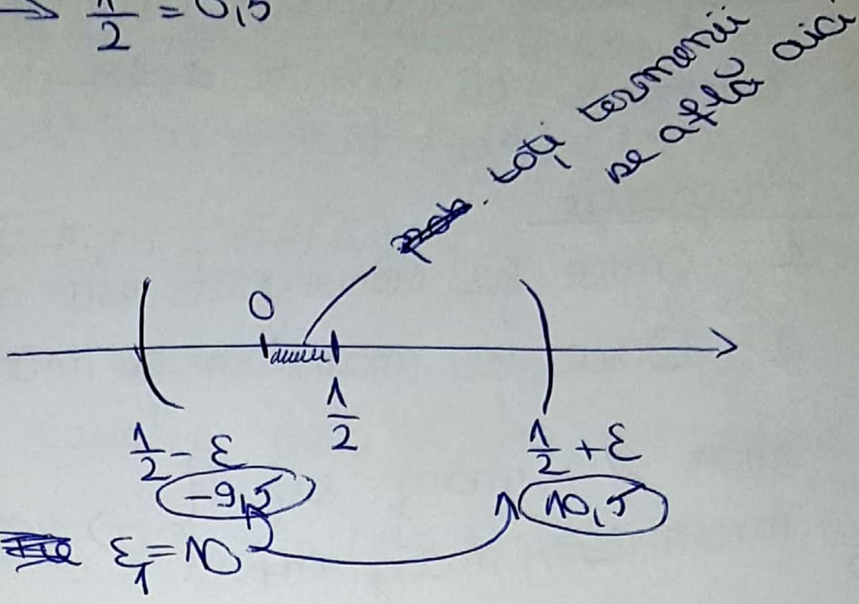
$x_0 = 0$

$x_1 = \frac{1}{3} \approx 0,33$

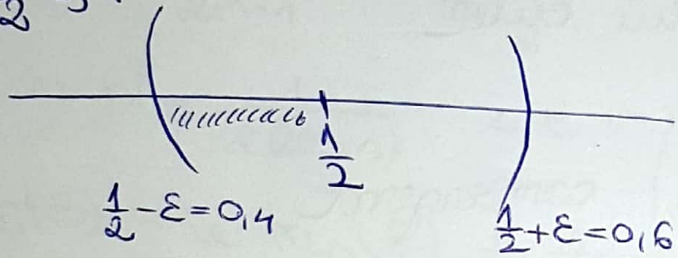
$x_2 = \frac{2}{5} \approx 0,4$

$x_3 = \frac{3}{7} \approx 0,42$

$x_4 = \frac{4}{9} \approx 0,44$



$\varepsilon_2 = 0,1$



$\varepsilon_3 = 0,01 \Rightarrow \mathcal{V}_{\frac{1}{2}}(\varepsilon_3)$

Def. Șirul (x_n) are limita ∞ $\left(\lim\limits_{n\to\infty} x_n = \infty\right)$ dacă $\forall \varepsilon > 0$ $\exists\, n_0(\varepsilon) \in \mathbb{N}$ a.î. $x_n > \varepsilon$, $\forall\, n \geq n_0(\varepsilon)$.

ex $x_n = n^2 \xrightarrow{n\to\infty} \infty$

$\varepsilon_1 = 100$; $n_0 = 11 \Rightarrow x_{n_0} = 121 > \varepsilon_1$

Convergent $\Rightarrow$ limită finită

Divergent $\Rightarrow$ limită infinită sau $\nexists$

Def (x_n) - divergent dacă (x_n) are limită infinită sau NU are limită.

ex $y_n = (-1)^n$ NU are limită

„Easy-way" de a dem. divergența este de a găsi 2 subșiruri cu limite ~~defer~~ diferite.

Propoziție

1. Orice șir convergent este mărginit.
2. Orice șir monoton și mărginit este convergent.

$$\left.\begin{array}{l}\text{mon.} \nearrow + \text{mărg. sup.} \\ \text{mon.} \searrow + \text{mărg. inf.}\end{array}\right\} \rightarrow \text{convergent}$$

$x_n = \left(1+\frac{1}{n}\right)^n \rightarrow$ Șirul lui Euler

$$\left.\begin{array}{l}\cdot\ 2 < x_n < 3 \\ \cdot\ x_n < x_{n+1}\end{array}\right\} \rightarrow (x_n) - \text{convergent}$$

ex $y_n = \frac{(-1)^n}{n} \xrightarrow{n\to\infty} 0$ (convergent)

$\hookrightarrow$ NU e monoton

Criteriul majorării

Șirul $(x_n)_{n\geq 0}$ converge la $x \in \mathbb{R}$ dacă $\exists (y_n)_{n\geq 0}$ a.î.

$$\begin{cases} |x_n - x| \leq y_n \\ \lim\limits_{n\to\infty} y_n = 0 \end{cases}$$

5.

Def Șirul $(x_n)_{n\geq 0}$ se num. ~~șir~~ ȘIR CAUCHY sau ȘIR FUNDAMENTAL dacă $\forall \varepsilon>0$ ~~a.î.~~ $\exists\, n_0(\varepsilon)\in\mathbb{N}$ a.î.
$|x_n - x_m| < \varepsilon,\ \forall\, n,m \geq n_0(\varepsilon)$.

$\Leftrightarrow$

$(x_n)_{n\geq 0}$ - șir Cauchy dacă $\forall\, \varepsilon>0, \exists\, n_0(\varepsilon)\in\mathbb{N}$ a.î. $|x_n - x_{n+p}| < \varepsilon,\ \forall\, n\geq n_0(\varepsilon), \forall\, p\in\mathbb{N}^*$.

ex $x_n = \dfrac{n}{2n+1} \xrightarrow{n\to\infty} \dfrac{1}{2}$

$|x_n - x| = \dfrac{1}{2(2n+1)} < \varepsilon \Leftrightarrow 2n+2 > \dfrac{1}{2\varepsilon} \Leftrightarrow \boxed{n > \left(\dfrac{1}{2\varepsilon}-1\right)\dfrac{1}{2}}$

~~$n_0(\varepsilon)$~~ Fie $n_0(\varepsilon) = \left[\left(\dfrac{1}{2\varepsilon}-1\right)\cdot\dfrac{1}{2}\right]+1 \in\mathbb{N}$

$\Rightarrow n\geq n_0(\varepsilon) = \left[\left(\dfrac{1}{2\varepsilon}-1\right)\cdot\dfrac{1}{2}\right]+1 > \dfrac{1}{2}\left(\dfrac{1}{2\varepsilon}-1\right)$

Teoremă Criteriul general de convergență al lui CAUCHY

$\hookrightarrow (x_n)_{n\geq 0}$ - convergent d.d. $(x_n)_{n\geq 0}$ - șir cauchy (fundamental)

AM(c)

$\cancel{L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}$

$\lim\limits_{n\to\infty} \frac{a^{x_n}-1}{x_n} = \ln a$ | $\lim\limits_{n\to\infty} \frac{e^{x_n}-1}{e^{x_n}} = \ln e = 1$

pt. $a>0$, $x_n \xrightarrow{n\to\infty} 0$ | pt. $x_n \xrightarrow{n\to\infty} 0$

$$\frac{e^{\ln \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}} - 1}{\ln \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}}$$

s-a folosit $e^{\ln B} = B$

$\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n \frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$ din ex. anterior

Dacă $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = x \in \mathbb{R}_+$

$\Rightarrow \sqrt[n]{x_n} \xrightarrow{n\to\infty} x$

$\frac{\sqrt[n+1]{(n+1)!^n}}{n!} \xrightarrow{?} e$

$\sqrt[n+1]{\frac{((n+1)!)^n}{(n!)^{n+1}}} = x_{n+1} \xrightarrow[?]{n\to\infty} e$

$\left.\begin{array}{l} |x_n - x| \le a_n \\ a_n \xrightarrow{n\to\infty} 0 \end{array}\right\} \Rightarrow x_n \xrightarrow{n\to\infty} x$

<u>Prop</u>. Șirul $(x_n)_{n\ge0}$ este CAUCHY dacă $\exists$ șirul $(a_n)_{n\ge0}$ a.î.

- $|x_{n+p} - x_n| \le a_n, \forall p \ge 1, \forall n \ge n_0, n_0 \in \mathbb{N}$
- $a_n \xrightarrow{n\to\infty} 0$

Ex. 14 Convergență

$$x_n = \frac{\sin x}{2} + \frac{\sin(2x)}{2^2} + \dots + \frac{\sin(nx)}{2^n}, \ n \geq 1$$

$$|x_{n+p} - x_n| = \left| \frac{\sin(n+1)x}{2^{n+1}} + \frac{\sin(n+2)x}{2^{n+2}} + \dots + \frac{\sin(n+p)x}{2^{n+p}} \right|$$

$$\leq \frac{|\sin(n+1)x|}{2^{n+1}} + \frac{|\sin(n+2)x|}{2^{n+2}} + \dots + \frac{|\sin(n+p)x|}{2^{n+p}} \leq$$

$\Rightarrow$ cea mai mare val. pe care $|\sin x|$ o poate lua e 1

$$\leq \underbrace{\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+p}}}_{\text{prog. geom.}} = \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \dots + \frac{1}{p}\right)$$

$$= \frac{1}{2^{n+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} = \frac{1}{2^n}\left(1 - \frac{1}{2^p}\right) \leq \frac{1}{2^n} \to 0$$

$$\Rightarrow \begin{cases} |x_{n+p} - x_n| \leq \frac{1}{2^n} \\ \frac{1}{2^n} \to 0 \end{cases} \Rightarrow (x_n)_{n \geq 0} \text{ - șir CAUCHY}$$

SERII DE NUMERE REALE

Fie $(x_n)_{n\geq 0} \in \mathbb{R}$.

Construim $\boxed{S_n := x_0 + x_1 + \dots + x_n}$

Șirul sumelor parțiale

$\sum_{n=0}^{\infty} x_n = x_0 + x_1 + \dots + x_n + \dots$ → Sumă INFINITĂ

x_n → termenul general al seriei

Ex. 1 Fie $(x_n)_{n\geq 2}$, $x_n = \frac{1}{n(n-1)}$

$$\sum_{n=2}^{\infty} x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n-1)} + \dots$$

$$\sum_{n\geq 2} x_n =$$

Def. 1 Seria $\sum_{n=0}^{\infty} x_n$ este convergentă dacă șirul sumelor parțiale S_n este convergent.

În plus, limita $S = \lim_{n\to\infty} S_n$ se numește SUMA INFINITĂ ($\in \mathbb{R}$) A SERIEI not. $\sum_{n=0}^{\infty} x_n$.

$$S_n = x_2 + x_3 + \dots + x_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \frac{1}{n-1} - \frac{1}{n}$$

$$= 1 - \frac{1}{n} = \frac{n-1}{n} \xrightarrow{n\to\infty} 1 \overset{not}{=} S$$

S_n - convergent (la 1) $\overset{Def}{\Rightarrow} \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ = conv. și are suma $S = 1$.

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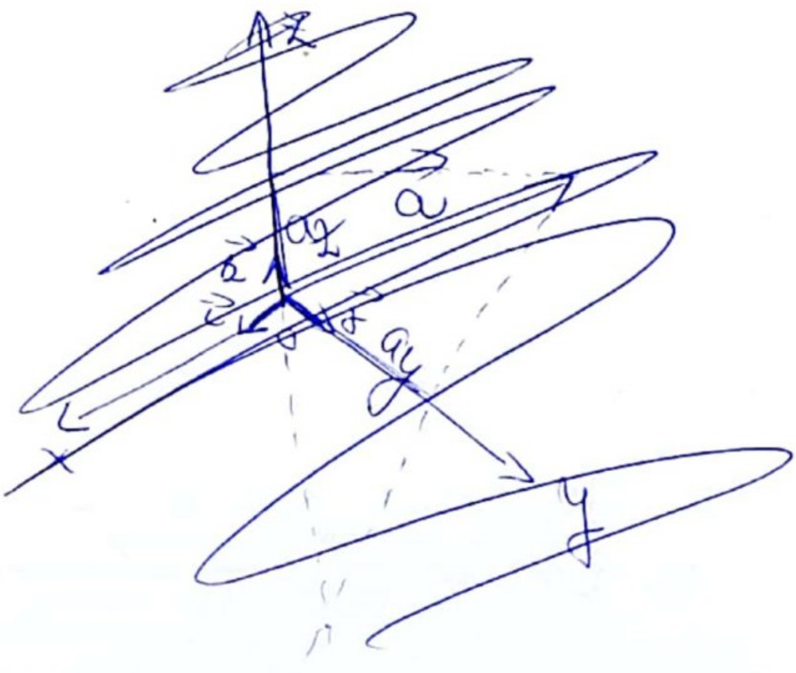
3. ~~Proiecția unui vector după trei direcții date~~



Def O serie este DIVERGENTĂ dacă S_n este divergent.

Ex. 2

$$\sum_{n=1}^{\infty} n = \text{serie DIV.} \left(\text{pt. că } S_n \xrightarrow{n\to\infty} \infty \right) \Rightarrow S_n \text{- div.})$$

Ex. 3 $p \in \mathbb{R}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergentă}, \ \forall\, p>1 \\ \text{divergentă}, \ \forall\, p\leq 1 \end{cases}}$$

↓

serie armonică generalizată

De reținut !

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = \text{DIV.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{CONV.}$$

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$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$S_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

Pp. prin R. A. că $S_n \xrightarrow{n\to\infty} \infty = S$

$\gamma_n = \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n}}_{\downarrow n\to\infty \; S} - \ln n \xrightarrow{n\to\infty} \gamma \in (0,1)$ $\Big\}$ contradicție

<u>Prop.</u> Dacă $\sum_{n=0}^{\infty} x_n = conv \Rightarrow \lim_{n\to\infty} x_n = 0$

Dem.

$\sum_{n=0}^{\infty} x_n - conv \Rightarrow (S_n)_{n\geq 0} = conv \Rightarrow \exists\, S = \lim_{n\to\infty} S_n$

și $x_n = S_n - S_{n-1} \xrightarrow{n\to\infty} S - S = 0$

<u>Prop. 2</u> (Criteriul de divergență)

Dacă $\lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} x_n = DIV$.

<u>Ex. 4</u> Seria geometrică

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots + r^n + \dots$$
$r \in \mathbb{R}$

① Dacă $r=1 \Rightarrow \sum_{n=0}^{\infty} 1^n = 1+1+\dots+1+\dots$

pt. că $x_n = 1 \not\xrightarrow{n\to\infty} 0$

SAU

$$S_n = \underbrace{1+1+\dots+1}_{n+1} = n+1 \xrightarrow{n\to\infty} \infty$$

$\Downarrow$
DIV

② Dacă $r\neq 1 \Rightarrow S_n = 1+r+\dots+r^n = \dfrac{r^{n+1}-1}{r-1}\ \cancel{\xrightarrow{n\to\infty}\frac{1}{1-r}}$

$$\frac{r^{n+1}-1}{r-1} \xrightarrow{n\to\infty} \begin{cases} \dfrac{1}{1-r}, & r\in(-1,1)\\ \infty, & r>1 \\ \nexists, & r\le -1\end{cases}$$

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Tot Serii

Crit. gen. de conv. al lui Cauchy

$\sum_{n\geq 0} x_n$ - conv. dacă $\forall\, \varepsilon>0, \exists\, n_0(\varepsilon)\in\mathbb{N}$ a.î.

$$|x_{n+1}+x_{n+2}+\ldots+x_{n+p}| < \varepsilon,\ \forall\, n\geq n_0(\varepsilon), \forall\, p\in\mathbb{N}^*$$

Dem

$\sum_{n\geq 0} x_n$ - conv $\Leftrightarrow (S_n)_{n\geq 0}$ - conv. $\Rightarrow (S_n)$ - şir Cauchy $\Leftrightarrow$

$\Leftrightarrow \forall\, \varepsilon>0, \exists\, n_0(\varepsilon)\in\mathbb{N}$ a.î. $\underbrace{|S_{n+p}-S_n|}_{|x_{n+1}+\ldots+x_{n+p}|} < \varepsilon, \forall\, n\geq n_0(\varepsilon)$, $\forall\, p\in\mathbb{N}^*$

Criterii de convergență pt. serii cu termeni pozitivi

Fie $\sum_{n\geq 0} x_n$, $x_n\geq 0, \forall\, n\in\mathbb{N}$.

Obs. $S_{n+1}-S_n = x_{n+1}\geq 0 \Rightarrow (S_n)_{n\geq 0} \nearrow$

Dacă (S_n) - mărg. sup. $\Bigg\} \Rightarrow$

$\Rightarrow (S_n)$ - conv. $\overset{Def}{\Rightarrow} \sum_{n\geq 0} x_n$ - conv.

Teorema 1 (Crit. COMPARAȚIEI)

Fie 2 serii $\sum_{n\geq 0} x_n$ și $\sum_{n\geq 0} y_n$, cu $x_n, y_n\geq 0, \forall n$

a.î. $\exists\, c>0$ cu prop. $\boxed{x_n \leq c\cdot y_n}$, $\forall\, n\geq n_0$.

At:

1) Dacă $\sum_{n\geq 0} y_n$ = conv $\Rightarrow \sum_{n\geq 0} x_n$ = conv.

2) Dacă $\sum_{n\geq 0} x_n$ = div. $\Rightarrow \sum_{n\geq 0} y_n$ = div.

$\boxed{c=1}$ Caz particular

① $\sum_{n=1}^{\infty} \underbrace{\frac{2n^2}{3n^6+1}}_{x_n}$. Fie $\sum_{n\geq 1} \frac{1}{n^4}$ = conv. $= y_n$

$x_n = \frac{2n^2}{3n^6+1} \leq \frac{1}{n^4} = y_n$

Cum $\sum_{n\geq 1} \frac{1}{n^4}$ - conv. ($p=4$ în seria armonică gen.) $\left.\right\} \xRightarrow[\text{Comp.}]{\text{Crit.}} \sum_{n\geq 1} \frac{2n^2}{3n^6+1}$ = conv.

~~Crit. COMPARAȚIEI~~

Teorema 2 (Crit. COMPARAȚIEI - forma la limită)

Fie 2 serii $\sum_{n\geq 0} x_n$ și $\sum_{n\geq 0} y_n$, cu $x_n, y_n > 0$.

Calculăm $\boxed{l = \lim_{n\to\infty} \frac{x_n}{y_n}} \in [0,\infty]$.

1) Dacă $l \in (0,\infty)$, at. $\sum_{n\geq 0} x_n \overset{\text{asemenea}}{\sim} \sum_{n\geq 0} y_n$ (au aceeași natură!).

2) Dacă $l=0$ și $\sum_{n\geq 0} y_n$ = conv. $\Rightarrow \sum_{n\geq 0} x_n$ = conv.

3) Dacă $l=\infty$ și $\sum_{n\geq 0} y_n$ = div. $\Rightarrow \sum_{n\geq 0} x_n$ = div.

Ex: $\sum\limits_{n\geq 0} \underbrace{\dfrac{n^2+\sqrt{n}}{n^2\cdot\sqrt{n}+1}}_{x_n}$

$$x_n = \frac{n^2+\sqrt{n}}{n^2\cdot\sqrt{n}+1} = \frac{n^2+n^{\frac{1}{2}}}{n^2\cdot n^{\frac{1}{2}}+1} = \frac{n^2+n^{\frac{1}{2}}}{n^{\frac{5}{2}}+1}$$

Fie $y_n = \dfrac{1}{n^{\frac{1}{2}}}$; $\sum\limits_{n\geq 1} \dfrac{1}{\sqrt{n}}$ = div.

$$l = \lim_{n\to\infty} \frac{x_n}{y_n} = \cancel{\lim_{n\to\infty} \frac{n^2+\sqrt{n}}{n^3+\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{n}+n}{n^2\sqrt{n}+1} = 1 \xRightarrow[\text{la lim.}]{\text{Crit. comp.}} \sum_{n\geq 1}\frac{1}{\sqrt{n}} \sim \sum_{n\geq 0}\frac{n^2+\sqrt{n}}{n^2\sqrt{n}+1}$$

$$\Rightarrow \sum_{n\geq 0}\frac{n^2+\sqrt{n}}{n^2\sqrt{n}+1} = \text{div.}$$

Ex: $\sum\limits_{n=2}^{\infty} \underbrace{(\sqrt[n]{n}-1)}_{x_n}$, $\sum\limits_{n\geq 2} y_n = ?$

$$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{\ln n}{n}}$$

$$x_n = \sqrt[n]{n}-1 = e^{\frac{\ln n}{n}}-1$$

$$\boxed{\begin{array}{l} \bullet\ \dfrac{a^x-1}{x} \xrightarrow{x\to 0} \ln a,\ a>0 \\ \bullet\ \dfrac{e^x-1}{x} \xrightarrow{x\to 0} \ln e = 1 \end{array}}$$

• Fie $y_n = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{e^{\frac{\ln n}{n}}-1}{\frac{\ln n}{n}} = 1 = l \Rightarrow \sum_{n\geq 2} x_n \sim \sum_{n\geq 2} y_n$$

$$\sum_{n\geq 2} \frac{\ln n}{n} = ?$$

Fie $\sum_{n\geq 2} z_n = \sum_{n\geq 2} \frac{1}{n}$ = div.

$l_2 = \lim_{n\to\infty} \frac{y_n}{z_n} = \lim_{n\to\infty} \ln n = \infty \xrightarrow[\text{(3)}]{\text{Crit. Comp.}} \sum_{n\geq 2} \frac{\ln n}{n}$ = div

$\left.\begin{array}{l} \sum_{n\geq 2} x_n \sim \sum_{n\geq 2} y_n \\ \sum_{n\geq 2} y_n = \text{div.} \end{array}\right\} \Rightarrow \sum_{n\geq 2} x_n = \text{div.}$

Teorema 3 (Criteriul RAPORTULUI) d'Alambert

Fie $\sum_{n\geq 0} x_n$, $x_n > 0$. Calc. $\boxed{l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}}$

1) Dacă $l < 1 \Rightarrow \sum_{n\geq 0} x_n$ = conv.

2) Dacă $l > 1 \Rightarrow \sum_{n\geq 0} x_n$ = div.

3) Dacă $l = 1 \Rightarrow$ natura $\sum_{n\geq 0} x_n$ (seriei) nu se poate stabili cu acest crit.

Teorema 4 (Crit. lui Raabe-Duhamel)

Fie $\sum_{n\geq 0} x_n$, $x_n>0$. Calc. $\boxed{L=\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)}$

1) Dacă $L<1 \Rightarrow \sum_{n\geq 0} x_n =$ div.

2) Dacă $L>1 \Rightarrow \sum_{n\geq 0} x_n =$ conv.

3) Dacă $L=1 \Rightarrow \sum_{n\geq 0} x_n$: natura seriei nu poate fi stabilită cu acest crit.

Ex:

$$\sum_{n\geq 1} \underbrace{\frac{1}{n^2\cdot 2^n}}_{= x_n}$$

$$l=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2\cdot 2^{n+1}}}{\frac{1}{n^2\cdot 2^n}}=$$

$$=\lim_{n\to\infty}\frac{n^2\cdot 2^n}{(n+1)^2\cdot 2^{n+1}}=\lim_{n\to\infty}\frac{n^2}{2(n+1)^2}=\frac{1}{2}<1$$

$$\xrightarrow[\text{rap. (1)}]{\text{Crit.}} \sum_{n\geq 1}\frac{1}{n^2\cdot 2^n}=\text{conv.}$$

Ex: $\sum_{n\geq 1} \underbrace{\frac{1}{n^2 \cdot a^n}}_{x_n}$, $a>0$

$\Rightarrow l = \frac{1}{a} \Rightarrow$

$l = \frac{1}{a} < 1 \to a > 1 \Rightarrow \sum_{n\geq 1} x_n =$ conv

$l = \frac{1}{a} < 1 \to a < 1 \to \sum_{n\geq 1} x_n =$ div.

$l = \frac{1}{a} = 1 \to a = 1 \Rightarrow \sum_{n\geq 1} x_n \overset{\text{deci me}}{=} \sum_{n\geq 1} \frac{1}{n^2}$

$=$ conv.

Ex: $\sum_{n\geq 1} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} = \sum_{n\geq 1} \frac{(2n-1)!!}{(2n)!!}$

$l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1 \overset{\text{Crit.}}{\underset{\text{rap.}}{\Rightarrow}}$ nu poate fi det. nat. serie

$L = \lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{2n+2}{2n+1} - 1\right)$

$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1 \overset{\text{Crit.}}{\underset{\text{R.D.}}{\Rightarrow}} \sum_{n\geq 1} x_n =$ div.

Teorema 5 (Criteriul RĂDĂCINII)

Fie $\sum_{n\geq 0} x_n$, $x_n > 0$. Calc. $\boxed{\ell = \lim_{n\to\infty} \sqrt[n]{x_n}}$

1) $\ell < 1 \Rightarrow \sum_{n\geq 0} x_n =$ conv.

2) $\ell > 1 \rightarrow \sum_{n\geq 0} x_n =$ div.

3) $\ell = 1 \Rightarrow$ nu se poate stab. nat. seriei cu acest criteriu.

Ex:

$$\sum_{n\geq 1} \left(1+\frac{1}{n}\right)^{n^2}$$

$$\ell = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e > 1 \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} x_n = \text{div.}$$

Teorema 6 (Criteriul de CONDENSARE a lui Cauchy)

Fie $(x_n)_{n\geq 1}$ șir ↓ de nr. poz.

At. $\sum_{n\geq 1} x_n \sim \sum_{n\geq 1} 2^n \cdot x_{2^n}$.

$\xlongequal{not} y_n$

Ex: $\sum_{n\geq 2} \underbrace{\frac{\ln n}{n}}_{=x_n} \stackrel{?}{=}$ div.

$\sum_{n\geq 2} 2^n \cdot x_{2^n} = \sum_{n\geq 2} 2^n \cdot \frac{\ln(2^n)}{2^n} = \ln 2 \sum_{n\geq 2} n =$ div.

Cum $x_n = \frac{\ln n}{n} \downarrow 0 \xRightarrow[\text{Cond.}]{\text{Crit.}} \sum_{n\geq 2} x_n \sim \sum_{n\geq 2} 2^n \cdot x_{2^n}$

$\sum_{n\geq 1} \underbrace{\frac{1}{n^p}}_{x_n} = \begin{cases} \text{conv}, & p>1 \\ \text{div}, & p\leq 1 \end{cases}$

1) Dacă $p \leq 0 \to x_n = \frac{1}{n^p} \overset{n\to\infty}{\not\to} 0$ → nu tinde la 0

$\xRightarrow[\text{div.}]{\text{Crit.}} \sum_{n\geq 1} x_n =$ div.

2) Dacă $p>0 \Rightarrow \begin{cases} x_n = \frac{1}{n^p} \xrightarrow{n\to\infty} 0 \\ (x_n) \downarrow \end{cases}$

$\sum_{n\geq 1} 2^n \cdot x_{2^n} = \sum_{n\geq 1} 2^n \cdot \frac{1}{2^{np}} = \sum_{n\geq 1} \frac{1}{2^{n(p-1)}} =$

$= \sum_{n\geq 1} \left(\frac{1}{2^{p-1}}\right)^n$

$$\boxed{\sum_{n\geq 0} q^{n} = \text{conv. dacă } q\in(-1,1)}$$

$$\Rightarrow \sum_{n\geq 1}\left(\frac{1}{2^{p-1}}\right)^{n} = \text{conv. pt. } q=\frac{1}{2^{p-1}}<1$$

$$\Leftrightarrow 2^{p-1}>1=2^{0} \Leftrightarrow p-1>0 \rightarrow p>1$$

$$\Rightarrow \sum_{n\geq 1}\left(\frac{1}{2^{p-1}}\right)^{n} = \text{conv.}$$

AM(c)

$\sum_{n=2}^{\infty} a^{\ln n}$, $a>0$ este $\to$ conv. pt. $a<\frac{1}{e}$

$\to$ div. pt. $a \geq \frac{1}{e}$

Crit. rap.: $l=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{a^{\ln(n+1)}}{a^{\ln n}}=$

$=\lim_{n\to\infty} a^{\ln\left(\frac{n+1}{n}\right)}=a^{\ln 1}=a^0=1\to$ nu decide natura Σ

Crit. Raabe-Duhamel: $L=\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=$

$=\lim_{n\to\infty} n\cdot\frac{a^{\ln\left(\frac{n}{n+1}\right)}-1}{\ln\left(\frac{n}{n+1}\right)}\cdot\ln\left(\frac{n}{n+1}\right)=$

$=\ln a\cdot\lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n=\ln a\cdot\lim_{n\to\infty}\ln\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)=$

$=\ln a\cdot\ln\frac{1}{e}=-\ln a$

$$\boxed{\frac{a^{\varepsilon(n)}-1}{\varepsilon(n)}\xrightarrow{\varepsilon(n)\to 0}\ln a,\ a>0}$$

- $L>1 \Leftrightarrow -\ln a>1=\ln e \Leftrightarrow \ln a<\ln e^{-1}\Leftrightarrow a<e^{-1}=\frac{1}{e}$ conv
- $L<1 \Leftrightarrow -\ln a<1=\ln e \Leftrightarrow \ln a>\ln e^{-1}\Leftrightarrow a>\frac{1}{e}$ div
- $L=1\to$ nu decide natura Σ

$\sum_{n=2}^{\infty}\left(\frac{1}{e}\right)^{\ln n}=\sum_{n\geq 2}\frac{1}{e^{\ln n}}=\sum_{n\geq 2}\frac{1}{n}=$ div.

$$\sum_{n\geq 2} \underbrace{a^{\ln n}}_{=x_n}, \quad a>0$$

$$\sum_{n\geq 2} \underbrace{a^{1+\frac{1}{2}+\dots+\frac{1}{n}}}_{=y_n}, \quad a>0$$

Crit. comp. la limită: $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = \lim\limits_{n\to\infty} \frac{a^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{a^{\ln n}} =$

$= \lim\limits_{n\to\infty} a^{1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n} = a^{c}$ (c = conv $\in(0,\infty)$) $\Rightarrow \sum\limits_{n\geq 2} x_n \sim \sum\limits_{n\geq 2} y_n \Rightarrow$

$\Rightarrow \sum\limits_{n\geq 2} a^{1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n} - \begin{cases} \text{conv. pt. } a<\frac{1}{e} \\ \text{div. pt. } a\geq\frac{1}{e} \end{cases}$

~~Serii~~

Criterii de convergență pentru serii cu termeni oarecare.

① Criteriul lui Leibniz

$\sum\limits_{n=0}^{\infty} (-1)^n \cdot a_n$ e CONV. dacă $(a_n)_{n\geq 0} \xrightarrow{n\to\infty} 0$

Ex $\sum\limits_{n\geq 1} \frac{(-1)^n}{n}$ = conv. pt. că $a_n = \frac{1}{n} \xrightarrow{n\to\infty} 0$

② Criteriul lui Dirichlet

$\sum\limits_{n=0}^{\infty} a_n \cdot u_n$. Dacă:

(i) $a_n \xrightarrow{n\to\infty} 0$

(ii) $(u_n)_{n\geq 0}$ are prop. că șirul sumelor parțiale asociat este mărginit $\Leftrightarrow \exists M>0$ a.î. $|\underbrace{u_0+u_1+\dots+u_n}_{\overset{\text{not}}{=} t_n}| \leq M$

$\Rightarrow \sum\limits_{n\geq 0} a_n \cdot u_n$ = CONV

b) $\sum_{n\geq 1} \frac{\sin(nx)}{n}, x\in\mathbb{R}$

alegem $a_n = \frac{1}{n} \searrow_{0}^{n\to\infty}$ şi $u_n = \sin(nx), x\in\mathbb{R}$.

$|t_n| = |\sin(x) + \sin(2x) + \dots + \sin(nx)| \leq (?)$, $\forall\, n\geq 1$

trb. să nu depindă de n

$S_1 = \sin(x) + \sin(2x) + \dots + \sin(nx) = \frac{\sin\left(\frac{nx}{2}\right) \cdot \sin\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}$

$S_2 = \cos(x) + \cos(2x) + \dots + \cos(nx) =$

$$= \frac{\sin\left(\frac{nx}{2}\right) \cdot \cos\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}$$

$|t_n| = \left|\frac{\sin\left(\frac{nx}{2}\right) \cdot \sin\left(\frac{(n+1)x}{2}\right)}{\sin\frac{x}{2}}\right| \leq \frac{1}{\sin\frac{x}{2}}$; $x\neq 2k\pi$ $\overset{not}{=} M$

$\xrightarrow[\text{Dir}]{\text{crit}} \sum_{n\geq 1} \frac{\sin(nx)}{n} = \text{CONV}.$

$S_2 + i\cdot S_1 = (\cos x + i\sin x) + \underbrace{(\cos(2x) + i\cdot\sin(2x))}_{(\cos x + i\sin x)^2} + \dots +$

$+ \underbrace{(\cos(nx) + i\cdot\sin(nx))}_{(\cos x + i\cdot\sin x)^n}$ Formula lui MOIVRE

$= \frac{(\cos x + i\sin x)((\cos x + i\sin x)^n - 1)}{\cos x + i\sin x - 1} =$

$= \frac{(\cos x + i\sin x)(\cos(nx) + i\sin(nx) - 1)}{2i\sin\frac{x}{2}\cos\frac{x}{2} - \underbrace{(1-\cos x)}_{2\sin^2\left(\frac{x}{2}\right)}}$

$\sin(2t) = 2\sin t \cos t$

$\cos(2t) = 2\cos^2 t - 1 = 1 - 2\sin^2 t$

$$= \frac{(\cos x + i\sin x)\overbrace{\left[2i\sin\frac{mx}{2}\cos\frac{mx}{2} + 2i^2\sin^2\frac{mx}{2}\right]}^{2i\sin\left(\frac{mx}{2}\right)\left(\cos\frac{mx}{2} + i\sin\frac{mx}{2}\right)}}{2i\cdot\sin\frac{x}{2}\cdot\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)} =$$

$$= \frac{\sin\left(\frac{mx}{2}\right)\cdot(\cos x + i\cdot\sin x)\left(\cos\frac{mx}{2} + i\cdot\sin\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)\cdot\left(\cos\left(\frac{x}{2}\right) + i\cdot\sin\left(\frac{x}{2}\right)\right)} =$$

$$(\cos a + i\sin a)(\cos b + i\sin b) = \cos(a+b) + i\cdot\sin(a+b)$$

$$= \frac{\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\left(\cos\left(\frac{(m+1)x}{2}\right) + i\cdot\sin\left(\frac{(m+1)x}{2}\right)\right) =$$

$$= \frac{\cos\frac{(m+1)x}{2}\cdot\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)} + i\cdot\frac{\sin\left(\frac{(m+1)x}{2}\right)\cdot\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$\Rightarrow S_2 + i\cdot S_1 = \frac{\cos\left(\frac{(m+1)x}{2}\right)\cdot\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)} + i\cdot\frac{\sin\left(\frac{(m+1)x}{2}\right)\cdot\sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

$$\Rightarrow \begin{cases} S_1 = \dfrac{\sin\left(\frac{mx}{2}\right)\cdot\sin\left(\frac{(m+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \\[2ex] S_2 = \dfrac{\sin\left(\frac{mx}{2}\right)\cdot\cos\left(\frac{(m+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \end{cases}$$

Criteriul lui ABEL

Fie $\sum_{n\geq 0} a_n \cdot u_n$. Dacă

(i) $(a_n)_{n\geq 0}$ - șir monoton și mărginit

(ii) $\sum_{n\geq 0} u_n$ = CONV

$\Rightarrow \sum_{n\geq 0} a_n \cdot u_n$ = CONV

Def: $\sum_{n\geq 0} x_n$ se num. ABSOLUT CONVERGENTĂ dacă $\sum_{n\geq 0} |x_n|$ = CONV.

Ex: $\sum_{n\geq 1} \frac{(-1)^{n+2}}{n^2}$ = ABS CONV pt. că $\sum_{n\geq 1} |x_n| = \sum_{n\geq 1} \frac{1}{n^2}$ = CONV

"x_n"

$\sum_{n\geq 1} \frac{(-1)^n}{n}$ = SEMICONV. pt. că $\sum_{n\geq 1} \frac{(-1)^n}{n}$ = CONV. și $\sum_{n\geq 1} |x_n| = \sum_{n\geq 1} \frac{1}{n}$ = DIV

Prop: Orice serie ABS CONV. este CONV.

Reciproca nu e ADV!!

Dem: $\sum_{n\geq 0} x_n$ = ABS CONV $\rightarrow \sum_{n\geq 0} |x_n|$ = CONV $\Longrightarrow$

$\xrightarrow[\text{conv. Cauchy}]{\text{Crit. gen.}} \forall \varepsilon > 0, \exists n(\varepsilon)$ a.î. $\big||x_n| + |x_{n+1}| + \dots + |x_{n+p}|\big| < \varepsilon$, $\forall n \geq n(\varepsilon)$, $\forall p \geq 1$

Dar $|x_n + x_{n+1} + \dots + x_{n+p}| \leq |x_n| + |x_{n+1}| + \dots + |x_{n+p}| < \varepsilon$, $\forall n \geq n_0(\varepsilon), \forall p \geq 1$

$\xrightarrow[\text{conv.}]{\text{Crit. gen.}} \sum_{n\geq 0} x_n$ = CONV.

ȘIRURI și SERII de FUNCȚII

$f_n : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, $\cancel{f(x)}$ $f(n,x) \overset{not}{=} f_n(x)$

Ex 1 $f_n(x) = x^n$, $f_n : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$

x_n = șir numeric	$f_n(x)$ = șir de funcții
$\sum_{n \geq 0} x_n$	$\sum_{n \geq 0} f_n(x)$

$n \to \infty$: $\boxed{x \in ?}$ a.î. $f_n(x) \xrightarrow{n\to\infty} f(x)$

D = domeniu de conv. = ?

- $x \in (-1,1) \Rightarrow x^n \xrightarrow{n\to\infty} 0$
- $x = 1 \Rightarrow x^n \equiv 1 \xrightarrow{n\to\infty} 1$
- $x \in \mathbb{R} \setminus (-1,1] \Rightarrow x^n$ nu e CONV.

$\Rightarrow f_n(x) \xrightarrow[n\to\infty]{s} f_n(x) = \begin{cases} 0, & x \in (-1,1) \\ 1, & x = 1 \end{cases}$

„f_n CONV SIMPLU la f"

$f_s(1) \neq f_d(1)$

f nu e cont.

Ex 2 $f_n(x) = \dfrac{nx^2}{n+x^2} \xrightarrow{n\to\infty} \dfrac{x^2}{1} = x^2 = f(x)$

$f_n : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$

Def 1 $f_n \xrightarrow[n\to\infty]{s} f$ d.d. $\forall \varepsilon > 0$, $\forall x \in D$, $\exists n_0(\varepsilon, x) \in \mathbb{N}$ a.î. $|f_n(x) - f(x)| < \varepsilon$, $\forall n > n_0$.

$\to$ converge SIMPLU

Def 2 $\to$ converge UNIFORM

$f_n \xrightarrow[n\to\infty]{u} f$ pe mulțimea D d.d. $\forall \varepsilon > 0$, $\forall x \in D$, $\exists n_0(\varepsilon) \in \mathbb{N}$ a.î. $|f_n(x) - f(x)| < \varepsilon$, $\forall n > n_0$.

Prop Dacă $f_n \xrightarrow[n\to\infty]{u} f \Rightarrow f_n \xrightarrow[n\to\infty]{s} f$ pe mulțimea D

Ex3 $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

(i) $f_n \xrightarrow[n\to\infty]{s} ?$

$f_n(x) \xrightarrow[n\to\infty]{s} f(x) = 0, \ \forall x \in \mathbb{R} = D$

(ii) $f_n \xrightarrow[n\to\infty]{u} ?$

$(\exists)$ $\underbrace{x_n}_{\text{şir numeric}} \in D$ a.î. $|f_n(x_n) - f(x_n)| \geq \varepsilon$, ε - găsit $\Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f$

∇ criteriu pt. a stabili că $f_n \xrightarrow{u} f$.

$f_n(x) = \dfrac{nx}{1+n^2x^2}$, $f(x) \equiv 0$; $\left|f_n(x_n) - \underbrace{f(x_n)}_{0}\right| = \left|\dfrac{n \cdot x_n}{1+n^2x_n^2}\right| =$

$x_n = \dfrac{1}{n}$ $\quad \dfrac{1}{2} \geq \dfrac{1}{4} = \varepsilon \Rightarrow f_n(x) \not\xrightarrow[n\to\infty]{u} f(x)$

Prop. 2 $\boxed{\text{Criteriul majorării}}$

$f_n \xrightarrow[n\to\infty]{u} f$ d.d. $\exists$ şirul numeric $(a_n)_{n\geq 0}$ a.î. $\begin{cases} a_n \xrightarrow{n\to\infty} 0 \\ |f_n(x) - f(x)| \leq a_n, \ \forall x \in D, \ \forall n \in \mathbb{N} \end{cases}$

Ex4

$f_n(x) = \dfrac{x}{1+n^2x^2}$, $n \in \mathbb{N}$, $x \in \mathbb{R}$

- $f_n \xrightarrow[n\to\infty]{s} 0 \equiv f(x)$
- $\left|f_n(x) - \underbrace{f(x)}_{0}\right| = |f_n(x)| = \left|\dfrac{x}{1+n^2x^2}\right| \leq \left|\dfrac{x}{2nx}\right| = \dfrac{1}{2n} \overset{\text{not}}{=} a_n$

$a^2 + b^2 \geq 2|a| \cdot |b|$

$a^2 + b^2 - 2ab \geq 0 \Leftrightarrow (a-b)^2 \geq 0 \Leftrightarrow \dfrac{1}{a^2+b^2} \leq \dfrac{1}{2|a|\cdot|b|}$

$a_n \xrightarrow{n\to\infty} 0 \Rightarrow f_n \xrightarrow{u} f$

Proprietățile șirurilor de funcții

1) TRANSFER DE CONTINUITATE

Dacă $f_n \xrightarrow[n\to\infty]{u} f$ pe mulțimea D și f_n = continuă pe D, $\forall n$, atunci f - continuă pe D.

2) TRANSFER DE DERIVABILITATE

Dacă $f_n \xrightarrow[n\to\infty]{u} f$ pe intervalul J și $\exists\, g: J \to \mathbb{R}$ a.î.

i) f_n - derivabilă pe J, $\forall n$

ii) $(f_n)' \xrightarrow[n\to\infty]{u} g$ pe J, at. f - derivabilă pe J și $\boxed{f' = g.}$

$$\boxed{\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} (f_n)'}$$

3) TRANSFER DE INTEGRABILITATE

Dacă $f_n \xrightarrow[n\to\infty]{u} f$ pe $[a,b]$ și f_n - integrabilă pe $[a,b]$ atunci f este integrabilă pe $[a,b]$ și are loc:

$$\boxed{\int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx = \lim_{n\to\infty}\left(\int_a^b f_n(x)\, dx\right)}$$

Ex) $f_n(x) = x^n \xrightarrow[n\to\infty]{s} f(x) = \begin{cases} 0, & x \in (-1,1) \\ 1, & x = 1 \end{cases}$

$f_n(x) \xrightarrow[n\to\infty]{u} f$??

$\hookrightarrow f_n \not\xrightarrow{u} f$ pt. că

R.A.: Dacă $f_n \xrightarrow[n\to\infty]{u} f \to f$ - cont., pt. că f_n - cont. pe $(-1,1]$ (transf. cont.)

ABSURD, pt. că f - discont. în $x_0 = 1$

$\xrightarrow{R.A.}$ ~~Crit. R.A~~ $f_n \xrightarrow{s} f$ și $f_n \not\xrightarrow{u} f$.

SERII DE FUNCȚII

Fie $(f_n)_{n\geq 0}$ - șir de funcții def. pe D

seria $\sum_{n\geq 0} f_n(x)$, $x\in D$
|| ← serie de șiruri de funcții
|| termenul general

$f_0(x)+f_1(x)+\dots+f_n(x)+\dots$

$S_n(x)=f_0(x)+f_1(x)+\dots+f_n(x)$ - șirul sumelor parțiale
← șir de funcții

Def. 1 $\sum_{n\geq 0} f_n(x)$ este SIMPLU CONV. dacă $S_n(x)$ este un șir de funcții simplu conv. pe D.

Def. 2 $\sum_{n\geq 0} f_n(x)$ este UNIFORM CONV ~~d~~ d.d. $S_n(x)$ este un șir de funcții uniform conv. pe D.

$$\boxed{\lim_{n\to\infty} S_n(x) \overset{not}{=} S(x) \Rightarrow \text{suma seriei de funcții}}$$

$\sum_{n\geq 0} x^n = 1+x+x^2+\dots+x^n+\dots$
↳ seria geometrică

$S_n(x)=1+x+\dots+x^n=\frac{x^{n+1}-1}{x-1} \xrightarrow{n\to\infty} \frac{1}{1-x} \overset{not}{=} S(x)$, $x\in(-1,1)$
$(x\neq 1)$

$\Rightarrow$ seria geometrică e SIMPLU CONV. $\forall\, x\in(-1,1)$ și are suma $S(x)=\frac{1}{1-x}$

Serii de șiruri de funcții

$$\sum_{n\geq 0} f_n(x) = f_0(x) + f_1(x) + \dots + f_n(x) + \dots$$

$$\underbrace{S_n(x)}_{\text{șir de funcții}} = f_0(x) + f_1(x) + \dots + f_n(x) \xrightarrow[n\to\infty]{s/u} S(x),\ x \in D = \text{dom. de CONV.}$$

$$\sum_{n\geq 0} \underbrace{x^n}_{f_n(x)} = \frac{1}{1-x},\ x \in (-1,1)$$

$$x^n \xrightarrow[x\in(-1,1)]{n\to\infty} 0$$

Prop. 1 Dacă $\sum_{n\geq 0} f_n(x)$ este U.C. pe mulțimea D, atunci

$$\boxed{f_n \xrightarrow[D]{u} 0}$$

Prop. 2 Dacă $f_n \overset{u}{\underset{D}{\not\to}} 0$, atunci $\sum_{n\geq 0} f_n(x)$ NU e U.C.

$$\Longleftrightarrow \boxed{\sup_{x\in D} |f_n(x)| \overset{n\to\infty}{\not\to} 0}$$

Teoremă (Criteriul lui Cauchy)

Fie $(f_n)_{n\geq 0}$ pe mulțimea de conv. D. Seria $\sum_{n\geq 0} f_n(x)$ este U.C. $\Longleftrightarrow \forall\ \varepsilon>0,\ \exists\ N(\varepsilon) \in \mathbb{N}$ a.î. $\underbrace{|f_{n+1}(x) + \dots + f_{n+p}(x)|}_{|S_{n+p}(x) - S_n(x)|} < \varepsilon$, $\forall\ n \geq N(\varepsilon),\ \forall\ p \in \mathbb{N}^*,\ \forall\ x \in D$.

Teoremă (Criteriul lui Weierstrass)

Fie $(f_n)_{n\geq 0}$ un şir de funcţii pe D şi $(a_n)_{n\geq 0}$ un şir numeric. Dacă:

i) $|f_n(x)| \leq a_n, \forall n \in \mathbb{N}, \forall x \in D$

ii) $\sum_{n\geq 0} a_n = \text{CONV.}$

$\Rightarrow$ atunci $\sum_{n\geq 0} f_n(x) = \text{U.C.}$

Ex $\sum_{n\geq 1} \frac{\sin(nx)}{n^3+1} = f_n(x)$

$$|f_n(x)| = \left|\frac{\sin(nx)}{n^3+1}\right| \leq \underbrace{\frac{1}{n^3+1}}_{a_n}, \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$$

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} \frac{1}{n^3+1} = \text{CONV.} \Rightarrow$$

$$\xrightarrow{\text{weierstrass}} \sum_{n\geq 1} f_n(x) = \cancel{\text{CONV}}. \sum_{n\geq 1} \frac{\sin(nx)}{n^3+1} = \text{U.C.}$$

$\boxed{\operatorname{arctg} x \leq x}$

Ex $\sum_{n\geq 1} \operatorname{arctg}\left(\frac{2x}{x^2+n^4}\right) \stackrel{?}{=} \text{U.C.}$

$$|f_n(x)| = \left|\operatorname{arctg}\left(\frac{2x}{x^2+n^4}\right)\right| \leq \frac{2x}{x^2+n^4} \leq \frac{2x}{2xn^2} = \frac{1}{n^2} \stackrel{not}{=} a_n$$

$$\sum_{n\geq 0} a_n = \sum_{n\geq 0} \frac{1}{n^2} = \text{CONV.} \Rightarrow$$

$$\xrightarrow{\text{weier.}} \sum_{n\geq 0} \operatorname{arctg}\left(\frac{2x}{x^2+n^4}\right) = \text{U.C.}$$

$\boxed{\frac{1}{a^2+b^2} \leq \frac{1}{2|a|\cdot|b|}}$

...oprietăți ale seriilor de funcții U.C

1) Transfer de trecere la limită

Fie $(f_n)_{n\geq 0}$ un șir de funcții pe D și $x_0 \in D'$ = mulțimea punctelor de acumulare

Dacă:

i) $\exists \lim\limits_{x \to x_0} \overline{f_n(x)} \in \mathbb{R}$ pt. $\forall\, n \geq 0$ (șir numeric)

ii) $\underbrace{\sum\limits_{n\geq 0} f_n(x)}_{\text{serie numerică}}$ este U.C. pe D și are suma f

$\Rightarrow$ atunci

$\sum\limits_{n\geq 0}\left(\lim\limits_{x\to x_0} f_n(x)\right)$ = CONV și are suma $\overline{\lim\limits_{x\to x_0} f(x)}$ = nr. real, adică

$$\boxed{\lim\left(\sum_{n\geq 0} f_n(x)\right) = \sum_{n\geq 0}\left(\lim f(x)\right)}$$

2) Transfer de derivabilitate

Fie $(f_n)_{n\geq 0}$ șir de funcții derivabile pe intervalul I. Dacă:

i) $\sum\limits_{n\geq 0} f_n(x)$ este U.C. pe I și are suma $f(x)$

ii) $\sum\limits_{n\geq 0} f'_n(x)$ este U.C. pe I și are suma $g(x)$

$\Rightarrow$ atunci

funcția f este derivabilă pe I și are loc $f' = g$, adică

$$\boxed{\left(\sum_{n\geq 0} f_n(x)\right)' = \sum_{n\geq 0} f'_n(x)}$$

3) Transfer de integrabilitate

Fie $f_n : [a,b] \to \mathbb{R}$ un șir de funcții integrabile pe $[a,b]$. D...

i) $\sum\limits_{n\geq 0} f_n(x)$ este U.C. pe $[a,b]$ și are suma $f(x)$ $\Rightarrow$ atunc...

f - integrabilă pe $[a,b]$ și $\boxed{\int_a^b\left(\sum\limits_{n\geq 0} f_n(x)\right)dx = \sum\limits_{n\geq 0}\left(\int_a^b f_n(x)\,d\ldots\right.}$

SERII DE PUTERI

$$\sum_{n\geq 0} \underbrace{a_n\cdot(x-x_0)^n}_{f_n(x)} = a_0 + a_1(x-x_0)+\dots+a_n(x-x_0)^n+\dots, x\in$$

$(a_n)_{n\geq 0}$ = şir numeric, $x_0 \in \mathbb{R}$
↳ fixat

↳ serie de puteri centrată în pct. x_0

$x_0=0$: $\sum_{n\geq 0} a_n \cdot x^n \Rightarrow$ seria de puteri centrată în $x_0=0$

Ex $\sum_{n\geq 0} \frac{x^n}{n!}$, $x\in\mathbb{R}$

$a_n = \frac{1}{n!}$

Ex $\sum_{n\geq 0} \underbrace{\frac{(-1)^n}{(2n+1)!}}_{=a_{2n+1}} \cdot x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$a_{2n+1} = \frac{(-1)^n}{(2n+1)!} \Rightarrow \begin{cases} \text{termeni impari} \text{ (fără } x^{2n+1}) \\ a_{2n}=0 \end{cases}$

Ex (Serii de tip geometric) → DE ŞTIUT

$$\boxed{\sum_{n\geq 0} x^n = \underbrace{\frac{1}{1-x}}_{S(x)}, x\in(-1,1)}$$

$$\boxed{\sum_{n\geq 0} (-1)^n\cdot x^n = \frac{1}{1+x}, x\in(-1,1)}$$

$$\boxed{\sum_{n\geq 0} x^{2n} = 1+x^2+x^4+\dots+x^{2n}+\dots = \frac{1}{1-x^2}, x\in(-1,1)}$$

$$\boxed{\sum_{n\geq 0} (-1)^n\cdot x^{2n} = \frac{1}{1+x^2}, x\in(-1,1)}$$

Raza de convergență $\sum_{n\geq 0} a_n(x-x_0)^n$

$R \in [0,\infty]$

1) $R^1 = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$

2) $R^1 = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$ $\Rightarrow$

$$\sum_{n\geq 0} \frac{3^n}{n+1} \cdot x^n$$

$$R = \lim_{n\to\infty} \frac{\left|\frac{3^n}{n+1}\right|}{\left|\frac{3^{n+1}}{n+2}\right|} = \frac{1}{3}$$

$\sum_{n\geq 0} a_n \cdot (x-x_0)^{\alpha \cdot n+\beta}$, $\alpha \in \mathbb{N}^*$, $\beta \in \{0, 1, \dots, \alpha-1\}$

$$\boxed{R^\alpha = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}}$$ ← cel mai des folosită

$\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$, $R^2 = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

Teorema I a lui ABEL

Fie $\sum_{n\geq 0} a_n \cdot (x-x_0)^n$ serie de puteri, R = raza de CONV.

i) Dacă $R=0$, atunci mulțimea de CONV. $C=\{x_0\} \overset{d.d.}{\Leftrightarrow}$ seria e CONV. în pct. x_0

ii) Dacă $R=\infty$, atunci $C=\mathbb{R}$ d.d. seria e CONV. în orice pct. din $\mathbb{R}$

iii) Dacă $R\in(0,\infty)$, atunci:

1. Seria este ABS CONV, $\forall x \in (x_0-R, x_0+R)$ și este DIV. $\forall x \in (-\infty, x_0-R) \cup (x_0+R, \infty)$.
2. Seria este U.C. pe orice interval compact $[\alpha,\beta]$, $[\alpha,\beta] \subset (x_0-R, x_0+R)$

!! PT. $x = x_0-R$ SAU $x = x_0+R$, CRIT. LUI ABEL NU OFERĂ INFO DESPRE CONV. SERIEI.

! conv. în $x_0 \pm R$ se face folosind seriile numerice.

(Ex) $\sum\limits_{n\geq 0} \frac{3^n}{n+1} \cdot x^n$; $R = \frac{1}{3}$

$x_0 = 0$

$\xrightarrow{\text{Th ABEL}}$ (i) $\sum\limits_{n\geq 0} \frac{3^n}{n+1} \cdot x^n$ = ABS CONV, $\forall x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

DIV, $\forall x \in \left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

(ii) $\sum\limits_{n\geq 0} \frac{3^n}{n+1} \cdot x^n$ = U.C., $\forall x \in [\alpha, \beta] \subset \left(-\frac{1}{3}, \frac{1}{3}\right)$

(iii) $x = \frac{1}{3} \Rightarrow \sum\limits_{n\geq 0} \frac{3^n}{n+1} \cdot \frac{1}{3^n} = \sum\limits_{n\geq 0} \frac{1}{n+1}$ = DIV

$x = -\frac{1}{3} \Rightarrow \sum\limits_{n\geq 0} \frac{3^n}{n+1} \cdot \frac{(-1)^n}{3^n} = \underbrace{\sum\limits_{n\geq 0} \frac{(-1)^n}{n+1}}_{\text{crit. Leibniz}}$ = CONV

$\Rightarrow$ mulțimea de conv: $C = \left[-\frac{1}{3}, \frac{1}{3}\right)$

AM (consultație)

① $\sum_{n=1}^{\infty} \frac{a^n \cdot n!}{n^n}$, $a>0$

x_n

$\cancel{\frac{x_{n+1}}{x}}$ Aplicăm Crit. rap.

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a \cdot a^n \cdot (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{a^n \cdot n!} =$

$= \lim_{n\to\infty} \frac{a \cdot (n+1) \cdot n^n}{(n+1)^{n+1}} = a \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$

$\cancel{= a \cdot \lim_{n\to\infty} 1^{\infty}} = a \cdot \frac{1}{e}$ $\left(\frac{1}{\frac{n+1}{n}}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

i) $l = \frac{a}{e} < 1 \to a < e \to$ CONV.

ii) $l = \frac{a}{e} > 1 \to a > e \to$ DIV

iii) $l = \frac{a}{e} = 1 \to a = e \to$ nu decide natura $\sum$

$a = e \to \sum_{n=1}^{\infty} \frac{e^n \cdot n!}{n^n}$.

Crit. Raabe-Duhamel

$L = \lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left(\frac{(n+1)^n}{n^n \cdot e} - 1\right)$

$= \lim_{n\to\infty} \frac{\left(\frac{1}{e} \cdot \left(1+\frac{1}{n}\right)^n - 1\right)}{\frac{1}{n}}$

$\lim_{x\to 0} \frac{\frac{1}{e} \cdot (1+x)^{\frac{1}{x}} - 1}{x} = \frac{1}{e} \cdot \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} =$

$$\overset{L'H}{=} \frac{1}{e} \cdot \lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2} \cdot \ln(x+1) + \frac{1}{x} \cdot \frac{1}{x+1}\right)}{1} =$$

$$(f^g)' = \left(e^{g \cdot \ln f}\right)' = e^{g \cdot \ln f} \cdot \left(g' \ln f + g \cdot \frac{f'}{f}\right) =$$

$$= f^g \left(g' \cdot \ln f + g \cdot \frac{f'}{f}\right)$$

$$= 1 \cdot \lim_{x\to 0} \frac{-(x+1) \cdot \ln(x+1) + (x)}{x^2 \cdot (x+1)} \overset{L'H}{=}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\ln(x+1) - (x+1)/(x+1) + 1}{3x^2 + 2x} =$$

$$= \lim_{x\to 0} \frac{-\ln(x+1)}{3x^2+2x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{-1}{x+1}}{6x+2} = -\frac{1}{2}$$

$$\Rightarrow L = -\frac{1}{2} < 1 \xRightarrow{R.D.} \Sigma = \text{Div}$$

$$\sum_{n\geq 0} \frac{\ln(1+a^n)}{n^2}$$

$$\frac{\ln(a^n)}{n^2} < \frac{\ln(1+a^n)}{n^2} \quad \checkmark$$

$$a^n < 1+a^n,\ a>0 \ \big|\cdot \ln \Rightarrow \ln(a^n) < \ln(1+a^n),\ a>0$$

$$y_n = \frac{\ln a}{n} < x_n = \frac{\ln(1+a^n)}{n^2}$$

$$\ln a \cdot \sum_{n\geq 1} \frac{1}{n} - \text{DIV}$$

$$\xrightarrow[\text{comp.}]{\text{Crit.}} \sum x_n = \text{DIV}$$

$$\cos(2x) = 1 - 2\sin^2 x$$

③ $x_1 = \sin^2\left(\pi\sqrt{n^2+3n+4}\right) \rightarrow \sin^2 x = \frac{1-\cos(2x)}{2}$

$$= \frac{1}{2}\left(\underset{\cos 0}{\underset{\parallel}{1}} - \cos\left(2\pi\sqrt{n^2+3n+4}\right)\right) =$$

$$= \cancel{\frac{1}{2}\left(-2\cdot\sin\sqrt{\pi\sqrt{n^2+3n+4}}\right)}\cdot\sin$$

$$= \frac{1}{2}\cdot(-2)\cdot\sin\frac{2\pi\left(1-\sqrt{n^2+3n+4}\right)}{2}\cdot\sin\frac{2\pi\left(1+\sqrt{n^2+3n+4}\right)}{2}$$

$$= -\sin\pi\left(\frac{\cancel{1+n^2}-(n^2+3n+4)}{1+\sqrt{n^2+3n+4}}\right)\cdot\sin\pi\left(1+\sqrt{n^3+3n+4}\right)$$

$$= -\sin\left(\frac{\cancel{1}-(3n+4)\pi}{n\left(1+\sqrt{1+\frac{3}{n}+\frac{4}{n^2}}\right)}\right)\cdot\sin\left(\pi\left(n+\sqrt{n^2+3n+4}\right)\right)$$

AM (consultații)

① $f_n : (0,\infty) \to \mathbb{R},\ f_n(x) = \dfrac{n}{e^{nx^2}}$

i) $f_n \xrightarrow{S} f \equiv 0$

ii) $f_n \not\xrightarrow{u} f \equiv 0$

$\dfrac{n}{e^{4n}} \xrightarrow{\frac{\infty}{\infty}} 0$

$\lim\limits_{y\to\infty} \dfrac{y}{e^{4y}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim\limits_{y\to\infty} \dfrac{1}{4e^{4y}} = \dfrac{1}{\infty} = 0$

i) $f_n(x) = \dfrac{n}{e^{nx^2}} \xrightarrow{n\to\infty} 0$

$\lim\limits_{y\to\infty} \dfrac{y}{e^{yx^2}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim\limits_{y\to\infty} \dfrac{1}{x^2 \cdot e^{yx^2}} = \dfrac{1}{x^2} \cdot \lim\limits_{y\to\infty} \dfrac{1}{e^{yx^2}} = \dfrac{1}{x^2} \cdot \dfrac{1}{\infty} = 0$

pt. $x \in (0,\infty)$, fixat

Am arătat că $f_n \xrightarrow[n\to\infty]{S} f \equiv 0$

ii) $|f_n(x_n) - f(x_n)| > \varepsilon$

Dacă $\exists\, x_n$ a.î. Σ, at. $f_n(x) \not\xrightarrow{u} f$

$x_n = \dfrac{1}{\sqrt{n}}$; $\left|f_n\left(\dfrac{1}{\sqrt{n}}\right) - \underbrace{f\left(\dfrac{1}{\sqrt{n}}\right)}_{=0}\right| = \dfrac{n}{e^{n\cdot\frac{1}{n}}} = \dfrac{n}{e} \xrightarrow{n\to\infty} \infty$

$|\underbrace{f_n(x_n)}_{=\frac{n}{e}}| > \varepsilon = \dfrac{1}{2e}$

Am arătat că $\exists\, \varepsilon = \dfrac{1}{2e}$ și $\exists\, x_n = \dfrac{1}{\sqrt{n}}$ a.î.

$|f_n(x_n) - f(x_n)| > \varepsilon \Rightarrow f_n \not\xrightarrow{u} f$.

② $f_n : \mathbb{R} \to \mathbb{R}, \ f_n(x) = \dfrac{x^2}{\sqrt{x^2 + \frac{1}{n}}}, \ n \in \mathbb{N}^*$

i) $f_n \xrightarrow[n\to\infty]{w} f = |x|, x \in \mathbb{R}$

• $\lim\limits_{n\to\infty} \dfrac{x^2}{\sqrt{x^2+\frac{1}{n}}} = \left(\dfrac{x^2}{\sqrt{x^2}} = \dfrac{x^2}{|x|} = |x| \right)$

$= \lim\limits_{n\to\infty} \dfrac{x^2}{|x| \sqrt{1 + \frac{1}{n} \cdot \frac{1}{x^2}}} = \left(\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}\cdot\frac{1}{x^2}}} \right) \cdot |x| =$

$= |x|, \ x \in \mathbb{R}^*$

$x = 0 \Rightarrow f_n(0) = \dfrac{0}{\sqrt{0+\frac{1}{n}}} = 0 \xrightarrow[n\to 0]{} f(0) = 0$

$x \neq 0 \Rightarrow f_n(x) = |x| \Rightarrow f_n(x) \xrightarrow[n\to\infty]{s} f(x) = |x|$

• $|f_n(x) - f(x)| \ \underline{< \varepsilon}$ sau $\underline{< a_n}$, $a_n \xrightarrow{n\to\infty} 0$

$|f_n(x) - f(x)| = \left| \dfrac{x^2}{\sqrt{x^2+\frac{1}{n}}} - |x| \right| =$

$= \left| \dfrac{|x|}{\sqrt{1+\frac{1}{n}\cdot\frac{1}{x^2}}} - |x| \right| = \left| |x| \left(\dfrac{1}{\sqrt{1+\frac{1}{n}\cdot\frac{1}{x^2}}} - 1 \right) \right| =$

$= |x| \cdot \left| \dfrac{1}{\sqrt{1+\frac{1}{n}\cdot\frac{1}{x^2}}} - \dfrac{\sqrt{\ }}{1} \right| = |x| \cdot \dfrac{\overset{\sqrt{\ }=1+\sqrt{\ }}{\left|1 - \sqrt{1+\frac{1}{nx^2}}\right|}}{\sqrt{1+\frac{1}{n}\cdot\frac{1}{x^2}}} =$

$= |x| \cdot \dfrac{\left|\cancel{1} - \cancel{1} - \frac{1}{nx^2}\right|}{\sqrt{1+\frac{1}{nx^2}}\left(1+\sqrt{1+\frac{1}{nx^2}}\right)} = |x| \cdot \dfrac{\left|-\frac{1}{nx^2}\right|}{\sqrt{1+\frac{1}{nx^2}}\left(1+\sqrt{1+\frac{1}{nx^2}}\right)} =$

$= \cancel{|x|} \cdot \dfrac{1}{n \cdot \cancel{|x|} \dfrac{\sqrt{nx^2+1}}{\sqrt{n}} \cdot \left(1+\sqrt{1+\frac{1}{nx^2}}\right)} = \dfrac{1}{\sqrt{n}\cdot\sqrt{nx^2+1}\left(1+\frac{\sqrt{nx^2+1}}{\sqrt{n}\cdot|x|}\right)}$

$\sqrt{nx^2} = \sqrt{n}\cdot|x|$

$$\frac{1}{\cancel{\sqrt{n}}\sqrt{nx^2+1}\cdot\frac{\sqrt{n}|x|+\sqrt{nx^2+1}}{\cancel{\sqrt{n}}\cdot|x|}} = \frac{|x|}{\sqrt{nx^2+1}\cdot(\sqrt{n}\cdot|x|+\sqrt{nx^2+1})}$$

$$\le \frac{|x|}{2\sqrt{n}\cdot\sqrt{x}\,(\sqrt{n}\cdot|x|+2\sqrt{n}\cdot\sqrt{x})} = \frac{1}{2\sqrt{n}(\sqrt{n}\cdot\sqrt{x}+2\sqrt{n})} =$$

$$= \frac{1}{2n}\cdot\underbrace{\frac{1}{\sqrt{x}+2}}^{\le\frac{1}{2}} \le \frac{1}{4n}$$

$$\Rightarrow \left|f_n(x)-f(x)\right| \le a_n = \frac{1}{4n}$$

$$\lim_{n\to\infty} a_n = 0 \quad \Rightarrow$$

$$\Rightarrow f_n \xrightarrow{u} f$$

③ $\sum_{n\ge 0}\frac{(n+1)^2}{3^n}$

$$l = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+2)^2}{3^{n+1}}\cdot\frac{3^n}{(n+1)^2} = \frac{1}{3}\in(0,\infty)$$

$\xrightarrow[\text{raportului}]{\text{Crit.}} \sum a_n = \text{CONV}$

$$S_n = a_0+a_1+\dots+a_n = \sum_{k=0}^{n}\frac{(k+1)^2}{3^k} = \sum_{k=0}^{n}\frac{k^2+2k+1}{3^k}$$

!! nu se poate calcula

④ $\sum_{n\geq 1} \frac{(2n-1)!!}{(2n+2)!!}$

Met. 1

$$a_n = \frac{(2n-1)!!\,(-1)}{(2n+2)!!} = \frac{(2n-1)!!\,\left((2n+1)-2n-2\right)}{(2n+2)!!}(-1)$$

$$=(-1)\frac{(2n+1)!!-(2n-1)!!\cdot(2n+2)}{(2n+2)!!} = -\frac{(2n+1)!!}{(2n+2)!!} + \frac{(2n-1)!!}{(2n)!!}$$

$$S_n = a_0 + a_1 + \dots + a_n = \dots = \frac{1!!}{2!!}$$

$$= \frac{1}{2} - \boxed{\frac{(2n+1)!!}{(2n+2)!!}} \xrightarrow{n\to\infty} \frac{1}{2}$$

$$y_n$$

$$L = \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{(2n+3)!!}{(2n+1)!!} \cdot \frac{(2n+2)!!}{(2n+4)!!} =$$

$$= \lim_{n\to\infty} \frac{2n+3}{2n+4} = 1 \Rightarrow \text{Crit. rap. nu decide}$$

$$L = \lim_{n\to\infty} \left(\frac{y_{n+1}}{y_n}\right)^n = \lim_{n\to\infty} \left(\frac{2n+3}{2n+4}\right)^n =$$

$$= \cancel{\lim_{n\to\infty}\left(\frac{2n+3}{2n+4}\right)} \lim_{n\to\infty} \left(1 + \frac{-1}{2n+4}\right)^n =$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{-1}{2n+4}\right)^{\frac{2n+4}{-1}}\right)^{\frac{-1}{2n+4}\cdot n} = e^{-\frac{1}{2}} \overset{?}{\lesseqgtr} 1$$

et .2

$$0 < y_m = \frac{(2m+1)!!}{(2m+2)!!} = \frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2m+1}{2m+2} < \cancel{\frac{4}{5}\cdot\frac{6}{7}}$$

$$< \frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{2m+2}{2m+3} = \frac{(2m+2)!!}{(2m+3)!!}$$

$$\cancel{\Rightarrow \frac{(2m+1)!!}{(2m+2)!!} < \frac{(2m+2)!!}{(2m+3)!!}}$$

$$\cancel{\Rightarrow \frac{(2m+1)!!}{(2m+2)!!}\cdot\frac{(2m+3)!!}{(2m+2)!!} < M}$$

$$0 < y_m^2 < \left(\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2m+1}{2m+2}\right)\left(\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{2m+2}{2m+3}\right)$$

$$= \frac{3}{2m+3} \Rightarrow y_m^2 \xrightarrow{m\to\infty} 0$$

Teorema a II-a a lui Abel

$\sum_{n\geq 0} a_n \underbrace{(x-x_0)}_{\text{not } y}^n$ - o serie de puteri cu raza de CONV

$R\in(0,\infty)$.

Dacă seria = CONV în $x=x_0\pm R$, atunci suma seriei $S(x)=\sum_{n\geq 0} a_n\cdot(x-x_0)^n$, $x\in C$ este o funcție continuă în punctele $x_0\pm R$ și

mulțimea de CONV.

$$\begin{cases} S(x_0-R) = \lim\limits_{\substack{x\to x_0-R\\ x>x_0-R}} S(x). \\ S(x_0+R) = \lim\limits_{\substack{x\to x_0+R\\ x<x_0+R}} S(x). \end{cases}$$

Ex. 1 Det. mulțimea de conv. și suma seriei

$\sum_{n\geq 1}\frac{x^n}{n\cdot 2^n}$, $a_n=\frac{1}{n\cdot 2^n}$

• Det. mulț. de conv.

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n}=2\cdot\lim_{n\to\infty}\frac{n+1}{n}=2$$

Aplicăm teorema I a lui Abel $\Rightarrow$

$$\Rightarrow\begin{cases}\text{seria e ABS. CONV. pt. } \forall\, x\in(-2,2) \text{ și e DIV. } \forall\, x\in(-\infty,-2)\cup(2,\infty)\\ \text{seria e U.C. pt. } \forall\, x\in[a,b]\subset(-2,2)\end{cases}$$

$* x=2 \Rightarrow \sum_{n\geq 1} \frac{2^n}{n\cdot 2^n} = \sum_{n\geq 1} \frac{1}{n} = $ DIV.

$* x=-2 \Rightarrow \sum_{n\geq 1} \frac{(-2)^n}{n\cdot 2^n} = \sum_{n\geq 1} \frac{(-1)^n\cdot \cancel{2^n}}{n\cdot \cancel{2^n}} = \sum_{n\geq 1} \frac{(-1)^n}{n} =$

$=$ CONV (crit. lui Leibniz)

$\Rightarrow$ mulțimea de conv. este $C=[-2,2)$.

• Det. suma seriei

$S(x)=\sum_{n\geq 1} \frac{x^n}{n\cdot 2^n}$

Met. 1

$\sum_{n\geq 1} \frac{y^n}{n} =$ CONV, $\forall\, y \in C=[-1,1)$

$(R=1)$

$S(y)=\sum_{n\geq 1} \frac{y^n}{n} \Big|' \Rightarrow S'(y)=\sum_{n\geq 1} \cancel{n}\cdot \frac{y^{n-1}}{\cancel{n}} =$

$\hookrightarrow$ asta nu se derivează pt. că va fi constantă dacă desfacem suma

$=\sum_{n\geq 1} y^{n-1} =$

$= \underbrace{1+y+\dots+y^{n-1}+y^n+\dots}_{\sum_{n\geq 0} y^n} = \frac{1}{1-y},\ y\in(-1,1)$

$\sum_{n\geq 0} y^n =$ seria ~~suma~~ geometrică

$S(y)=\int S'(y)\,dy=\int \frac{1}{1-y}\,dy = -\ln(1-y)+C,\ \forall y\in(-1,1)$

$\overset{x=0}{\Rightarrow} \overset{0}{S(0)} = -\ln 1 + C = C \Rightarrow C=0 \Rightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{y^n}{n} = -\ln(1-y),\ \forall\, y\in(-1,1)$

$\underline{x=-1} \Rightarrow S(-1) = \lim_{\substack{x\to -1 \\ x>-1}} S(x) = -\ln 2$

pt. $\underline{x=1} \Rightarrow S(1) = \lim_{\substack{x\to 1 \\ x<1}} S(x) = \infty$?

$\Rightarrow \sum_{n\geq 1} \frac{y^n}{n}$ are suma $S(y) = -\ln(1-y)$, $\forall y \in C$, $C = [-1,1)$.

$$S(x) = \sum_{n\geq 1} \frac{x^n}{2^n \cdot n} = \sum_{n\geq 1} \frac{1}{n}\cdot\left(\frac{x}{2}\right)^n = \sum_{n\geq 1} \frac{\left(\frac{x}{2}\right)^n}{n} =$$

$$= -\ln\left(1-\frac{x}{2}\right) = -\ln\left(\frac{2-x}{2}\right), \quad x \in [-2,2) = C.$$

Met. 2

$$S'(x) = \sum_{n\geq 1} \frac{1}{\cancel{n}}\cdot\frac{1}{2^n}\cdot \cancel{n}\cdot x^{n-1} = \frac{1}{2}\cdot\sum_{n\geq 1}\left(\frac{x}{2}\right)^{n-1} =$$

$$= \frac{1}{2}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\dots+\left(\frac{x}{2}\right)^n+\dots\right) =$$

$$= \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}, \ \cancel{\frac{x}{2}\in(-1,1)} = \frac{1}{2-x}, \quad \frac{x}{2}\in(-1,1)$$

Ex. 2

$$\sum_{n\geq 1} (-1)^{n-1}\cdot n \cdot x^{2n+1} \Rightarrow R^2 = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}\cdot n}{(-1)^n (n+1)}\right| = 1$$

$$\xRightarrow[R>1]{} R = 1$$

$\Rightarrow$ Începem mereu cu R și C.

$$R = \lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}\cdot n\cdot x^{2n+1}}{(-1)^n\cdot(n+1)\cdot x^{2n+3}}\right| =$$

$$= \lim_{n\to\infty}\left|-\frac{n}{(n+1)x^2}\right| = 1 \xrightarrow[\text{Abel}]{T.} \begin{cases} \Sigma = \text{CONV}, \ \forall x\in(-1,1) \\ \Sigma = \text{DIV}, \ \forall x\in(-\infty,-1)\cup(1,\infty) \\ \Sigma = \text{UC}, \ \forall x\in[\alpha,\beta]\subset(-1,1) \end{cases}$$

pt. $x=1 \Rightarrow \sum_{n\geq 1} (-1)^{n-1} \cdot n$ = DIV

pt. $x=-1 \Rightarrow \sum_{n\geq 1} (-1)^{n-1} \cdot n \cdot (-1)^{2n+1} = \sum_{n\geq 1} (-1)^{3n} \cdot n =$ = DIV

Mulțimea de conv. este $C=(-1,1)$.

• $S(x) = ?$

$$\sum_{n\geq 1} (-1)^{n-1} \cdot n \cdot x^{2n+1} = S(x), \ x\in(-1,1)$$

$$= x^2 \cdot \underbrace{\sum_{n\geq 1} (-1)^{n-1} \cdot n \cdot x^{2n-1}}_{P(x)}$$

$$\int P(x)\,dx = \int \left(\sum_{n\geq 1} (-1)^{n-1} \cdot n \cdot x^{2n-1}\right) dx =$$

$$= \sum_{n\geq 1} (-1)^{n-1} \cdot \cancel{n} \cdot \frac{x^{2n}}{2\cancel{n}} = \sum_{n\geq 1} (-1)^{n-1} \cdot \frac{x^{2n}}{2} =$$

$$= \frac{1}{2} \sum_{n\geq 1} (-1)^{n-1} \cdot x^{2n} = \frac{1}{2}\left(x^2 - x^4 + x^6 - x^8 + \dots\right) =$$

$$= \frac{x^2}{2} \underbrace{\left(1 - x^2 + x^4 - x^6 + \dots\right)}_{\sum_{n\geq 0} (-1)^n \cdot x^{2n} = \frac{1}{1+x^2}} = \frac{x^2}{2} \cdot \frac{1}{1+x^2}, \ \forall x \in (-1,1)$$

$$\int P(x)\,dx = \frac{x^2}{2(1+x^2)} \Rightarrow P(x) = \left(\frac{x^2}{2(1+x^2)}\right)' =$$

$$= \frac{1}{2} \cdot \frac{2x(1+x^2) - x^2 \cdot 2x}{(1+x^2)^2} = \frac{x}{(1+x^2)^2}$$

$f''(x) = \frac{x}{(1+x^2)^2}$

$x^2 \cdot P(x) = x^2 \cdot \frac{x}{(1+x^2)^2} = \frac{x^3}{(x^2+1)^2}, \forall x \in (-1,1)$

<u>Obs</u> SERIA BINOMIALĂ

$$\underbrace{1 + \frac{\alpha}{1!}x^1 + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-m+1)}{m!}x^m + \dots}_{} = \text{CONV}, \forall x \in (-1,1)$$

$$\| \quad S(x) = (1+x)^{\alpha}, \ \alpha \in \mathbb{R} \setminus \mathbb{N}$$

$$1 + \sum_{m \geq 1} \frac{\alpha(\alpha-1)(\alpha-2)\cdot \dots \cdot (\alpha-m+1)}{m!} \cdot x^m, x \in (-1,1)$$

$\alpha = \frac{1}{2} \Rightarrow ?$

$f(x) = \sqrt{1+x}, \quad \sqrt{2} = ?$

$\sqrt{17} = ?$ (16)

$$1 + \sum_{m \geq 1} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots}{m!} \cdot x^m$$

$x = 16$

SERII TAYLOR

Fie $f: J \to \mathbb{R}$, J = interval
derivabilă de n ori în x_0

• $(T_n f)(x) \overset{def}{=} f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$

$+ \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n$

→ polinomul Taylor de ordinul n asociat lui f în punctul x_0

• Funcția $R_n(x) = f(x) - (T_n f)(x)$ se num.
RESTUL TAYLOR DE ORDINUL N asociat lui f în x_0.

Obs Se poate demonstra:

$$\lim_{n \to \infty} (R_n f)(x) = 0$$

$$\boxed{f(x) = (T_n f)(x) + (R_n f)(x)}$$ FORMULA LUI TAYLOR

Obs

$$(R_n f)(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}, \ c \in (x_0, x)$$

→ Rest de tip Lagrange

...ază particular

Pt. $x_0 = 0 \longrightarrow f(x) = \underbrace{f(0) + \frac{x}{1!}\cdot f'(0) + \frac{x^2}{2!}\cdot f''(0) + \dots +}$

$$\underbrace{+ \frac{x^n}{n!}\cdot f^{(n)}(0)} + \underbrace{\frac{x^{n+1}}{(n+1)!}\cdot f^{(n+1)}(c)}_{\to 0},\quad c \in (0, x)$$

FORMULA LUI MAC-LAURIN

$$\underbrace{\sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}}_{a_n}(x-x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) +$$

$$+ \frac{f''(x_0)}{2!}\cdot(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n + \dots$$

$\hookrightarrow$ SERIA TAYLOR centrată în x_0, asociată funcției f

$\hookrightarrow \sum_{n\geq 0} a_n\cdot(x-x_0)^n$

Obs.

Dacă $\boxed{x_0 = 0}$, obținem seria $\boxed{\text{MAC-LAURIN}}$ asociată funcției f.

$$\Rightarrow \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}\cdot x^n = f(0) + \frac{f'(0)}{1!}\cdot x + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$S(x) \overset{\text{not}}{=} \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n$$

$\hookrightarrow$ suma seriei Taylor, $x \in C$ – mulț. de conv. a Σ

Cum se determină $S(x)=?$

Funcția $f: \underset{\mathbb{R}}{\overset{\cap}{J}} \to \mathbb{R}$ este DEZVOLTABILĂ în serie Taylor în punctul $x_0 \in J$ pe $C \subset J$ d.d. șirul de funcții $(R_n f)(x)$ converge simplu la 0 pe C, adică $f(x) = \sum\limits_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$\sum\limits_{n \geq 0} a_n \cdot x^n = S(x) = ?$

$f(x) = \sin x = \sum ?$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \ldots$

$\simeq 1$

AM(c)

$\sum_{n\geq 0} a_n \cdot x^n \to R$, D - dom. CONV

$S(x) = ?$ - suma seriei

$\Rightarrow S(x) = \sum_{n\geq 0} a_n \cdot x^n, \forall x \in D$

$f(x)$ - se dă

$f: I \subset \mathbb{R} \to \mathbb{R}$

$f(x) = \sum_{n\geq 0} ?$

$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} \cdot x^n$ = seria Mac-Laurin

Ex 1 de reținut

$$f(x) = e^x = \sum_{n\geq 0} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

$\sqrt{e} = ?$ $\to$ $x = \frac{1}{2}$

Ex 2 de reținut

$$g(x) = \sin x = \sum_{n\geq 0} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + \dots$$

Ex 3 de reținut

$$h(x) = \cos x = \sum_{n\geq 0} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

de reținut

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right) \qquad (\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$$

Ex 4 $f:[-1,1]\to\mathbb{R}$

$f(x)=\arcsin x \Rightarrow \sum_{m\geq 0}$?

$f^{(m)}(0)=?$

$f'(x)=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}$

$$(1+y)^{\alpha}=1+\sum_{m\geq 1}\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-m+1)}{m!}\cdot y^m$$

SERIA BINOMIALĂ $y\in(-1,1)$

$$\Rightarrow f'(x)=1+\sum_{m\geq 1}\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\ldots\cdot\left(-\frac{2m+1}{2}\right)}{m!}\cdot(-x^2)^m=$$

$$=1+\sum_{m\geq 1}\frac{(2m+1)!!}{2^m\cdot m!}\cdot x^{2m}=1+\sum_{m\geq 1}\frac{(2m+1)!!}{(2m)!!}\cdot x^{2m},$$

$x\in(-1,1)$

$$\Rightarrow f(x)=\int f'(x)\,dx$$

SERII FOURIER TRIGONOMETRICE

$$S_m(x)=\frac{a_0}{2}+\sum_{m\geq 1}\left(a_m\cdot\cos(\omega m x)+b_m\cdot\sin(\omega m x)\right),\ \omega=\text{pulsația}$$

$x\in\mathbb{R}$

$a_m, b_m \xrightarrow{\text{numiți}}$ coeficienții Fourier

$m\in\mathbb{N}$, $m\in\mathbb{N}^*$

→ se num. SERIE TRIGONOMETRICĂ

pt. $f(x)$ - dată → ∃ pos. de a fi dezvoltată în serie Fourier ??

$f:\mathbb{R}\to\mathbb{R}$, f - periodică de per. $T>0$ dacă $f(x)=f(x+T), \forall x\in\mathbb{R}$

per. princip.

Teorema lui DIRICHLET

Fie $f: \mathbb{R} \to \mathbb{R}$, f - periodică cu perioada T. Dacă:

i) f - mărginită pe $[a, a+T]$

ii) f - continuă pe $[a, a+T)$ sau admite un nr finit de puncte de discontinuitate de speța I pe $[a, a+T)$

iii) f - monotonă pe porțiuni pe $[a, a+T)$

$\Rightarrow$ atunci se poate construi seria FOURIER TRIGONOMETRICĂ

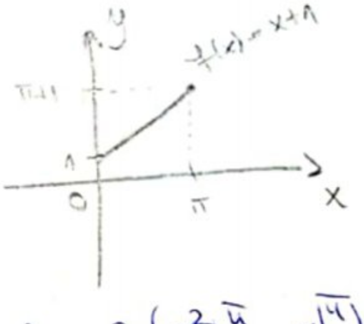
$$\boxed{S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\omega n x) + b_n \cdot \sin(\omega n x)\right)}, x \in [a, a+T],$$

$$\boxed{\omega = \frac{2\pi}{T}} \text{ și } \boxed{a_n = \frac{2}{T} \cdot \int_a^{a+T} f(x) \cdot \cos(\omega n x)\, dx}, \forall n \in \mathbb{N} \text{ și}$$

$$\boxed{b_n = \frac{2}{T} \int_a^{a+T} f(x) \cdot \sin(\omega n x)\, dx, \forall n \in \mathbb{N}^*}$$

În plus, are loc
- $S(x) = f(x)$ dacă x = punct de continuitate
- $S(x) = \frac{1}{2}\left(f(x-0) + f(x+0)\right)$ pt. x = punct de discontinuitate

Fie $\underline{f(x) = x+1}$, $f: \mathbb{R} \to \mathbb{R}$, f - periodică, $T = \pi$.
pt. $x \in [0, \pi)$



Dezvoltați f în serie Fourier trigo.

Se obs. că f satisface condițiile din T. Dir.

- $a_0 = \frac{2}{\pi} \int_0^{\pi} (x+1) \cos\left(\frac{\omega}{2} \cdot 0 \cdot x\right) dx = \frac{2}{\pi} \int_0^{\pi} (x+1)\, dx = \frac{2}{\pi}\left(\frac{x^2}{2}\Big|_0^{\pi} + x\Big|_0^{\pi}\right) =$

$= \frac{2}{\pi}\left(\frac{\pi^2}{2} + \pi\right) = \pi + 2$

- $a_n = \frac{2}{\pi} \int_0^{\pi} (x+1) \cdot \cos(2nx)\, dx = \frac{2}{\pi}\left((x+1) \cdot \frac{\sin(2nx)}{2n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(2nx)}{2n}\, dx\right) =$

$f'(x) = \cos(2nx) \to f(x) = \frac{\sin(2nx)}{2n}$

$g(x) = (x+1) \to g'(x) = 1$

$$= \frac{2}{\pi}\left(\frac{(\pi+1)\cdot\sin(2n\pi)-\sin 0}{2n} + \frac{1}{2n}\cdot\frac{\cos(2nx)}{2n}\Bigg|_0^{\pi}\right) =$$

$$= \frac{2}{\pi}\cdot\frac{1}{2n}\cdot\frac{\cos(2n\pi)-\cos 0}{4n^2} = 0$$

$$\bullet\ b_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\cdot\underbrace{\sin(2nx)}_{\left(-\frac{\cos(2nx)}{2n}\right)'}dx = \frac{2}{\pi}\left(\frac{-(x+1)\cdot\cos(2nx)}{2n}\Bigg|_0^{\pi} +\right.$$

$$\left.+\int_0^{\pi} 1\cdot\frac{\cos(2nx)}{2n}dx\right) = \frac{2}{\pi}\left(\frac{-(\pi+1)\cdot 1+1}{2n} + \frac{1}{2n}\cdot\frac{\sin(2nx)}{2n}\Bigg|_0^{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{2n} + \frac{1}{2n}\cdot\frac{1}{2n}\cdot 0\right) = -\frac{1}{n}$$

$$\Rightarrow f(x) = \frac{\pi}{2}+1+\sum_{n\geq 1}\left(-\frac{1}{n}\right)\cdot\sin(2nx) = \frac{\pi}{2}+1-\sum\frac{\sin(2nx)}{n},$$

$$x\in(0,\pi).$$

CAZURI PARTICULARE

1. Dacă $f:\mathbb{R}\to\mathbb{R}$ este o funcție $\boxed{\text{PARĂ}}$ pe $\left[-\frac{T}{2},\frac{T}{2}\right]$, atunci toți $\boxed{b_n=0}$, $\forall n\in\mathbb{N}^*$ și $\boxed{a_n=\frac{4}{T}\int_0^{\frac{T}{2}}f(x)\cdot\cos(\omega nx)dx}$, $\forall n\in\mathbb{N}$. În plus, seria devine $\boxed{S(x)=\frac{a_0}{2}+\sum_{n\geq 1}a_n\cos(\omega nx)}$

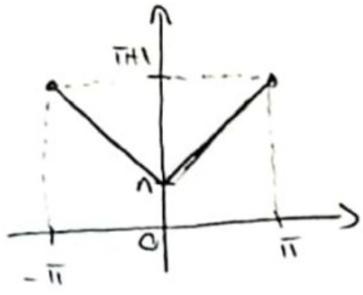
SERIE FOURIER TRIGO. DE COSINUSURI

2. Dacă $f:\mathbb{R}\to\mathbb{R}$ este o funcție $\boxed{\text{IMPARĂ}}$ pe $\left[-\frac{T}{2},\frac{T}{2}\right]$, atunci toți $\boxed{a_n=0}$, $\forall n\in\mathbb{N}$ și $\boxed{b_n=\frac{4}{T}\int_0^{\frac{T}{2}}f(x)\cdot\sin(\omega nx)dx}$, $\forall n\in\mathbb{N}^*$. În plus, seria devine $\boxed{S(x)=\sum_{n\geq 1}b_n\cdot\sin(\omega nx)}$

SERIE FOURIER TRIGO. DE SINUSURI

$) = x+1,\ T=\pi$

Dezvoltați f în SFT de cos



$$f_p(x) = \begin{cases} x+1, & x \in [0,\pi) \\ -x+1, & x \in (-\pi, 0) \end{cases}$$

$\Rightarrow T = 2\pi,\ \omega = \frac{2\pi}{T} = 1$

$\Rightarrow$ fct. pară $\Rightarrow$ se poate dez. în SFT. de cos.

- $b_n = 0,\ \forall\, n \geq 1$

- $$\underset{n\geq 1}{a_n} = \frac{4}{2\pi}\int_0^{\pi}(x+1)\cdot\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(x+1)\cdot\left(\frac{\sin(nx)}{n}\right)'dx =$$

$$= \frac{2}{\pi}\left(\frac{(x+1)\cdot\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\frac{\sin(nx)}{n}\,dx\right) =$$

$$= \frac{2}{\pi}\left(\frac{(\pi+1)\cdot\overbrace{\sin(n\pi)}^{0}}{n} + \frac{1}{n}\cdot\frac{\overbrace{\cos(nx)}^{(-1)^n}}{n}\Big|_0^{\pi}\right) =$$

$$= \frac{2}{\pi}\cdot\frac{\cos(n\pi)-\cos 0}{n^2} = \frac{2\left((-1)^n-1\right)}{\pi\cdot n^2},\ n\in\mathbb{N}^* = \begin{cases} 0, & n=2m \\ \frac{-4}{\pi(2m-1)^2}, & n=2m+1 \end{cases}$$

- $$a_0 = \frac{2}{\pi}\int_0^{\pi}(x+1)\,dx = \pi+2$$

$$\Rightarrow f_p(x) = \frac{\pi}{2}+1+2\sum_{n\geq 1}\frac{(-1)^n-1}{\pi n^2}\cdot\cos(nx) = \cancel{\frac{\pi}{2}+1-\frac{4}{\pi}\sum_{m\geq 1}}$$

$$= \frac{\pi}{2}+1-\frac{4}{\pi}\sum_{m\geq 1}\frac{\cos(2m-1)x}{(2m-1)^2},\ \forall\, x\in(-\pi,\pi)$$

$$x=0 \Rightarrow \underbrace{f_p(0)}_{0} = \frac{\pi}{2}+1-\frac{4}{\pi}\sum_{m\geq 1}\frac{\cos 0}{(2m-1)^2}$$

$$\Rightarrow \sum_{m\geq 1}\frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

$$\sum_{m\geq 1}\frac{1}{m^2};\ \sum_{m\geq 1}\frac{1}{(2m)^2}$$

Temă: Dezv. f în SFT de sin

a) $f(x) = x+1 \xrightarrow[SFT]{\cos} f(x) = \frac{\pi}{2} + 1 - \frac{4}{\pi} \sum_{n\geq 1} \frac{\cos(2n-1)x}{(2n-1)^2}$

$x=0 \Rightarrow \frac{\pi}{2} = \frac{4}{\pi} \sum_{n\geq 1} \frac{1}{(2n-1)^2} \Rightarrow \boxed{\sum_{n\geq 1} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$

pentru termeni impari

$S = \sum \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots =$

$= \underbrace{\left[1 + \frac{1}{3^2} + \dots + \frac{1}{(2n-1)^2}\right]}_{\frac{\pi^2}{8}} + \underbrace{\left[\frac{1}{2^2} + \frac{1}{4^2} + \dots + \frac{1}{2n^2}\right]}_{\frac{1}{4}\underbrace{\left(1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right)}_{S}} + \dots$

$\Rightarrow S = \frac{\pi^2}{8} + \frac{1}{4} S \Rightarrow S = \frac{\pi^2}{6} \Rightarrow \boxed{\sum \frac{1}{n^2} = \frac{\pi^2}{6}}$ toți termenii

$\Rightarrow S = \sum \frac{1}{(2n)^2} = \sum \frac{1}{4n^2} = \frac{S}{4} \Rightarrow \boxed{\sum \frac{1}{(2n)^2} = \frac{\pi^2}{24}}$ termeni pari

b) $\sum_{n\geq 1} \frac{(-1)^{n-1}}{(2n-1)^3} = ?$

Integrăm $f(x) \Rightarrow \int (x+1)\,dx = \left(\frac{\pi}{2}+1\right)x - \frac{4}{\pi} \cdot \sum_{n\geq 1} \frac{\sin(2n-1)x}{(2n-1)^3} + C$

$\Rightarrow \frac{x^2}{2} + x = \frac{\pi}{2}x + x - \frac{4}{\pi} \sum_{n\geq 1} \frac{\sin(2n-1)x}{(2n-1)^3} + C$

$x=0 \Rightarrow C=0$

$\Rightarrow \frac{4}{\pi} \sum_{n\geq 1} \frac{\sin(2n-1)x}{(2n-1)^3} = \frac{\pi}{2}x - \frac{x^2}{2}, \; x \in (-\pi, \pi)$

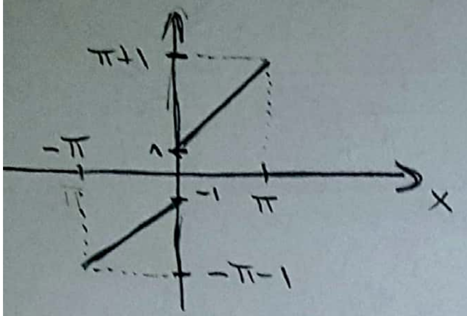
$\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$

$x = \frac{\pi}{2} \Rightarrow \frac{4}{\pi} \sum_{n\geq 1} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^2}{4} - \frac{\pi^2}{8} = \frac{\pi^2}{8} \,\Big|\cdot \frac{\pi}{4}$

$\Rightarrow \sum_{n\geq 1} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$

c) $f(x) = x+1$

$\rightarrow$ SFT de sinusuri $\Rightarrow f_i(x)$ impară



$T = 2\pi$

$\omega = 1$

$a_0 = 0$

$$b_n = \frac{4}{2\pi}\int_0^{\pi} (x+1)\cdot \sin(nx)\,dx$$

$$f_i(x) = \begin{cases} x+1, & x \in (0,\pi) \\ x-1, & x \in (-\pi, 0) \end{cases}$$

SPAȚII METRICE. NORME.

Fie X – mulțime nevidă.

$d: X \times X \to \mathbb{R}$

Def

Funcția d se numește METRICĂ / DISTANȚĂ pe X dacă:

i) $d(x,y) \geq 0, \ \forall x,y \in X$

ii) $d(x,y) = 0 \Leftrightarrow x = y$

iii) $d(x,y) = d(y,x), \ \forall x,y \in X$

iv) $\underbrace{d(x,y) \leq d(x,z) + d(z,y)}_{\text{inegalitatea } \Delta}, \ \forall x,y,z \in X$

METRICA EUCLIDIANĂ

$$d(\bar{x},\bar{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = (x_1-y_1)^2 + \dots + (x_n - y_n)^2$$

pt. $n=2 \rightarrow d(\bar{x},\bar{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

Exp. 3 $\boxed{\text{distanța dintre 2 funcții}}$

f, g - continue pe $[a, b]$

$$\boxed{d(f, g) = \int_a^b |f(x) - g(x)|\, dx}$$

$d: C([a,b]) \times C([a,b]) \to \mathbb{R}$

perechea (X, d) = spațiu metric

Exp. 4 $\boxed{\text{distanța Hamming}}$ → folosită în șirurile liniare

Fie X - spațiu liniar peste corpul $\mathbb{K}$ $(\mathbb{R}, \mathbb{C})$

$(X, +, \cdot) \to$ $x + y$ operație liniară

$\alpha \cdot x$ operație externă, $x \in X$ ~~pe~~ și $\alpha \in \mathbb{K}$.

$$\boxed{\|\cdot\| : X \to \mathbb{R}}$$

Def

Funcția $\|\cdot\|$ se numește NORMĂ pe X dacă:

i) $\|x\| \geq 0, \forall x \in X$

ii) $\|x\| = 0 \Leftrightarrow x = 0$

iii) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|, \forall \alpha \in K, \forall x \in X$

iv) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$

Exp. 1 modulul: $|\cdot|$

$(\mathbb{R}, |\cdot|)$ = spațiu normat

Exp. 2

$X = \mathbb{R}^n$, $\|\bar{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$

lungimea vectorului $\bar{x}$: $\|\bar{x}\|$.

Fie (X,d) spațiu metric.

Def

se numește SFERA DESCHISĂ / ÎNCHISĂ centrată în punctul x_0 și de rază $r>0$ în spațiul metric (X,d) mulțimea:

$$\boxed{S(x_0,r)=\{x\in X \mid d(x,x_0)<r\}} \text{ sfera DESCHISĂ}$$

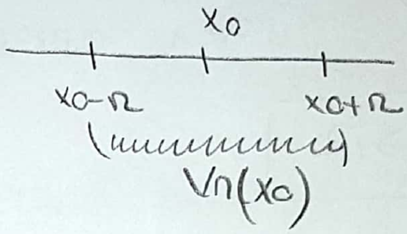
$$\boxed{\bar{S}(x_0,r)=\{x\in X \mid d(x,x_0)\le r\}} \text{ sfera ÎNCHISĂ}$$

Exp. 1

$X=\mathbb{R},\ x_0\in\mathbb{R}$

$d(x,y)=|x-y|$

Hai să construim sfera!



$S(x_0,r)=\{x\in\mathbb{R} \mid |x-x_0|<r\} \Leftrightarrow -r<x-x_0<r \ |+x_0$

$\Leftrightarrow x_0-r<x<x_0+r \Rightarrow \boxed{S(x_0,r)=(x_0-r,x_0+r)}$

$$\boxed{\bar{S}(x_0,r)=[x_0-r,x_0+r]}$$

Exp. 2

$X=\mathbb{R}^2,\ (x_0,y_0)\in\mathbb{R}^2,\ r>0$

$\left.\begin{array}{l}\bar{a}=(x,y)\\ \bar{b}=(u,v)\end{array}\right\}\ d(\bar{a},\bar{b})=\sqrt{(x-u)^2+(y-v)^2}$

$\Rightarrow S((x_0,y_0),r)=\{\bar{a}\in\mathbb{R}^2 \mid d(\bar{a},\bar{x_0})<r\}$

$d(\bar{a},\bar{x_0})<r \Leftrightarrow (x-x_0)^2+(y-y_0)^2<r$

$(x-x_0)^2+(y-y_0)^2<r^2$ ← sfera deschisă

$\Rightarrow$ interiorul unui cerc (fără contur)

$(x-x_0)^2+(y-y_0)^2 \leq r^2 \Leftarrow$ sfera închisă

$\Rightarrow$ interiorul unui cerc / disc + conturul

Exp. 3

$X=\mathbb{R}^3$

sfera deschisă : bila fără înveliș

sfera închisă : bila (interiorul unei sfere) cu înveliș

Ex

$$d(\bar{x},\bar{y})=\sum_{i=1}^{n}|x_i-y_i| \,,\; X=\mathbb{R}^n$$

$$n=2 \Rightarrow d(\bar{x},\bar{y})=|x_1-y_1|+|x_2-y_2|$$

$\bar{x}=(x_1,x_2)$

$\bar{y}=(y_1,y_2)$

$$\bar{S}(\bar{x_0},\bar{x})=\{\bar{x}\in\mathbb{R}^2 \mid |x_1-x_1^0|+|x_2-x_2^0|\leq r\}$$

$\bar{x_0} = (x_1^0,x_2^0)$

$$\left.\begin{array}{l} |x_1-x_1^0|+|x_2-x_2^0| \leq r \\ (x_1^0,x_2^0)=(0,0) \\ r=1 \end{array}\right\} \Rightarrow |x_1|+|x_2|\leq r=1$$

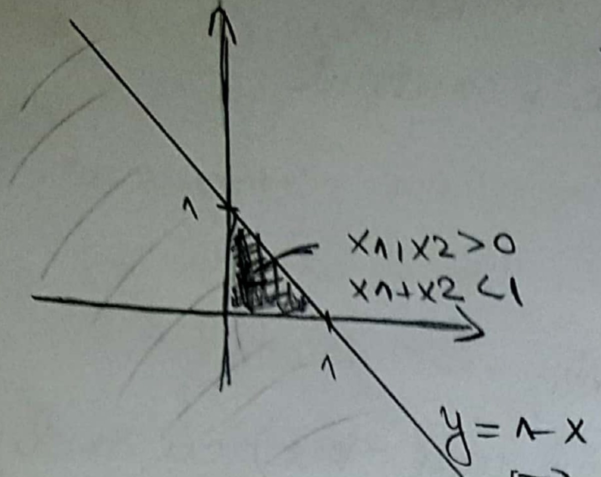
$$\Rightarrow \bar{S}=\{\bar{x}\in\mathbb{R}^2 \mid |x_1|+|x_2|\leq 1\}$$

$\hookrightarrow$ sfera închisă, centrată în $(0,0)$ de rază 1

~~pentru cazul~~ : $\cancel{x_1, x_2 \geq 0}$

$\cancel{x_1+x_2 \geq 0}$

$\cancel{\Leftrightarrow x_1+x_2 \leq 1}$

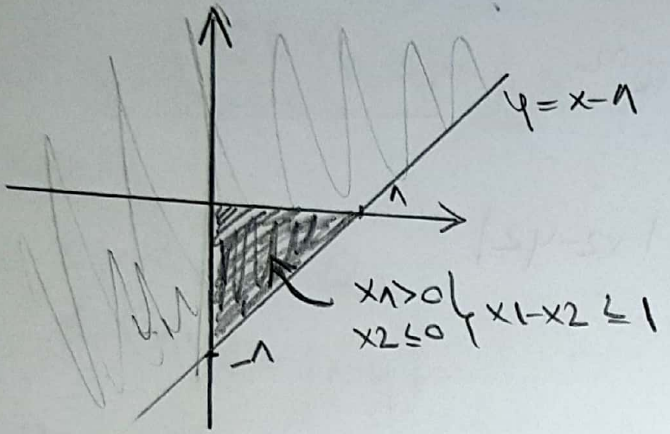


pentru cazul:

$\begin{cases} x_1, x_2 \geq 0 \\ x_1 + x_2 \geq 0 \end{cases}$

$\Rightarrow x_1 + x_2 \leq 1$

$\Rightarrow y_1 + y_2 = 1$



pentru cazul:

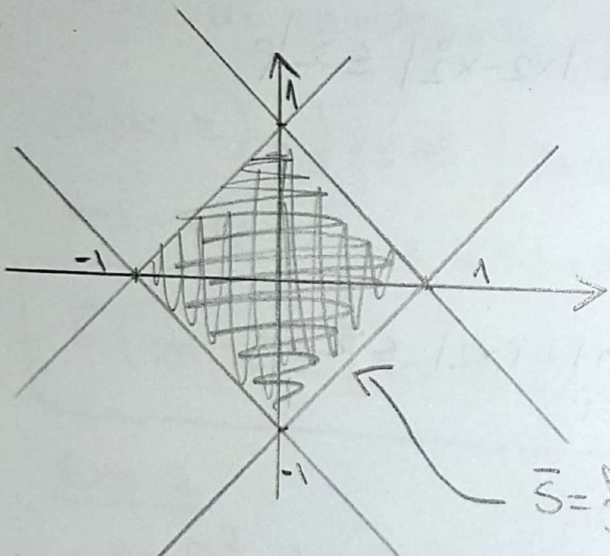
$\begin{cases} x_1 \geq 0 \\ x_2 \leq 0 \end{cases}$

$\Leftrightarrow x_1 - x_2 \leq 1$

Mai sunt încă 2 cazuri

$x_1 \leq 0,\ x_2 > 0$

$x_1 < 0,\ x_2 < 0$ | obținem



$\bar{S} = \{ \bar{x} \in \mathbb{R}^2 \mid |x_1| + |x_2| \leq 1 \}$

Un băiat și o fată se întâlnesc în intervalul 7-8 PM.

b → 15 min, apoi pleacă

f → 15 min, apoi pleacă

Probabilitatea să se întâlnească în [7, 8] PM:

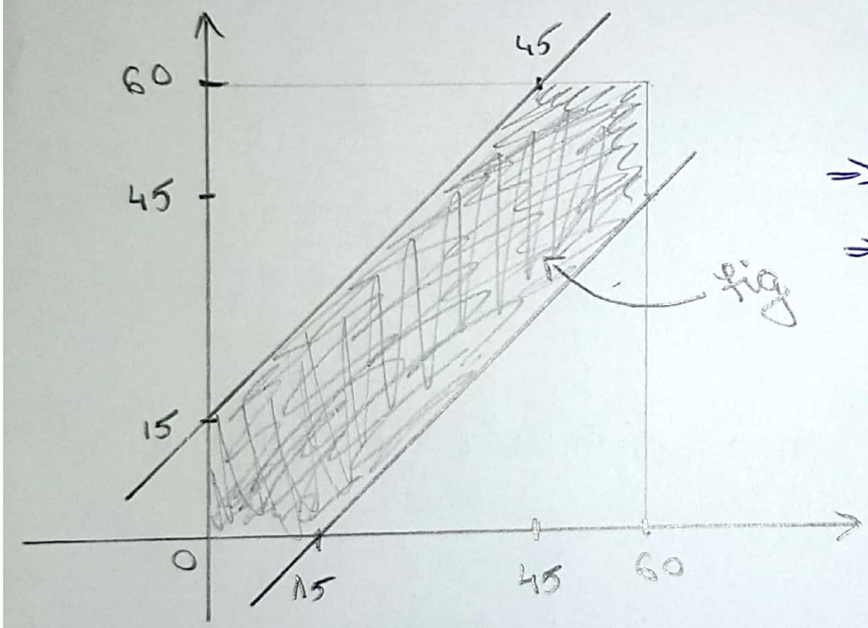
Pe Ox: x = min (b)

Pe Oy: y = min (f), cond. de întâlnire $|x-y| \le 15$

$x, y \in [0, 60]$

$\Rightarrow$ fig

$$P = \frac{A_{fig}}{A_{[60 \times 60]}}$$



$|x-y| \le 15$

$x-y = 15$

$\Rightarrow y = \cancel{A} \; x-15$

$\Rightarrow -15 \le x-y \le 15$

Funcții de mai multe variabile

$f: \mathbb{R}^n \to \mathbb{R}^m$

$f(\bar{x}) = f(x_1, x_2, \dots, x_n)$

$= (f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$

multifuncție

Exp. 1

$$f(x,y) = (\underbrace{x-y}_{f_1}, \underbrace{\sqrt{x^2+y^2}}_{f_2}, \underbrace{\ln(\sin(x-y))}_{f_3})$$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

$f'(x,y)$ ilegal!!

$f'_x(x,y) \leftarrow$ derivata în raport cu x

$$\frac{\partial f}{\partial x}(x,y)$$

$f: A \subset \mathbb{R} \to \mathbb{R}$, A' - mulțimea punctelor de acumulare pt. f

exp: $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$; $A' = [0,\infty)$

Are sens să calculezi limita în punctele de acumulare.

Def punctului de acumulare

$x \in A' \Leftrightarrow \forall\, U \in \mathcal{U}_x,\ U \cap (A \setminus \{x\}) \neq \emptyset$

Punctele de acumulare există și la multifuncții.

$\boxed{1}$ Mulțimea de convergență și suma seriei:

$$\sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (x-2)^n$$

$$a_n = (-1)^n \cdot n^2$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{\cancel{n}} \cdot n^2}{(-1)^n \cdot (-1) \cdot (n+1)^2}\right| =$$

$$= \lim_{n\to\infty} |-1| \cdot \left|\frac{n^2}{n^2+2n+1}\right| = 1 \cdot 1 = 1$$

$$J = (-1, 1)$$

$$x = -1 \Rightarrow \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (-1-2)^n =$$

$$= \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (-3)^n = \sum_{n\geq 1} (-1)^n \cdot (-1)^n \cdot n^2 \cdot 3^n =$$

$$= \sum_{n\geq 1} 3^n \cdot n^2$$

$$l = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot 3 \cdot (n+1)^2}{\cancel{3^n} \cdot n^2} = \lim_{n\to\infty} \left(3 \cdot \frac{n^2+2n+1}{n^2}\right) =$$

$$= 3 > 1$$

$$l > 1 \xrightarrow[\text{raportului}]{\text{crit.}} \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (-3)^n = DIV$$

$$x = 1 \Rightarrow \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (1-2)^n =$$

$$= \sum_{n\geq 1} (-1)^n \cdot (-1)^n \cdot n^2 = \sum_{n\geq 1} n^2 =$$

$T_1 f(x) = -\ln 2 + 2 \cdot \left(x + \frac{1}{2}\right)$

9. $\sum_{n \geq 1} \ln\left(1 + \frac{3}{n(n+4)}\right)$

$a_n = \ln\left(1 + \frac{3}{n(n+4)}\right) =$

$= \ln\left(1 + \frac{3}{n^2+4n}\right) =$

$= \ln\left(\frac{n^2+4n+3}{n^2+4n}\right) =$

$= \ln\left(\frac{(n+1)(n+3)}{n(n+4)}\right) =$

$= \ln(n+1) + \ln(n+3) - \ln n - \ln(n+4) =$

$= -\ln n + \ln(n+1) + \ln(n+3) - \ln(n+4)$

$\Rightarrow S_n = \sum_{k=1}^{n} -\ln k + \ln(k+1) + \ln(k+3) -$
$- \ln(k+4) =$

$= -\overset{0}{\ln 1} + \cancel{\ln 2} + \ln 4 - \ln 5 +$
$+ \cancel{\ln 2} + \ln 3 + \ln 5 - \ln 6 +$
$+ \dots - \ln n + \ln(n+1) + \ln(n+3) -$
$- \ln(n+4)$

$$\sum_{n\geq 1} \frac{1}{n^{-2}} = \text{div} \text{ (seria ~~g~~ armonică generalizată)}$$

$$\rightarrow \boxed{C = (-1, 1)}$$

Fie $S(x) \rightarrow$ suma seriei.

$$S(x) = \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (x-2)^n =$$

$$\cancel{= \sum_{n\geq 0} (-1)^{n+1} (n+1)^2 \cdot (x-2)^{n+1}}$$

$$= \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (x-2)^{n-1} \cdot (x-2) =$$

$$= (x-2) \cdot \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (x-2)^{n-1}$$

$$\rightarrow \frac{S(x)}{x-2} = \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot (x-2)^{n-1} \quad \Big| \int$$

$$\int \frac{S(x)}{x-2} dx = \sum_{n\geq 1} (-1)^n \cdot n^2 \cdot \frac{(x-2)^n}{\cancel{n}} =$$

$$= \sum_{n\geq 1} (-1)^n \cdot n \cdot (x-2)^n \quad \cancel{(\ast)}$$

$$\cancel{\frac{S(x)}{x-2} = \sum_{n\geq 1} (-1)^n \cdot n \cdot}$$

$$\int \frac{S(x)}{x-2}\,dx = \sum_{n\geq 1} (-1)^n \cdot n \cdot (x-2)^n$$

$$\sum_{n\geq 0} (-1)^n \cdot (x-2)^n = \frac{1}{1+(x-2)} = \frac{1}{x-1} \quad \Big|\,(\,)'$$

$$\sum_{n\geq 0} (-1)^n \cdot n\,(x-2)^{n-1} = \frac{-1}{(x-1)^2} \quad \Big|\cdot(x-2)$$

$$\sum_{n\geq 0} (-1)^n \cdot n \cdot (x-2)^n = \frac{-(x-2)}{(x-1)^2}$$

$$\Leftrightarrow 1 + \sum_{n\geq 1} (-1)^n \cdot n \cdot (x-2)^n = \frac{-(x-2)}{(x-1)^2}$$

$$\Rightarrow \sum_{n\geq 1} (-1)^n \cdot n \cdot (x-2)^n = \frac{-x+2}{(x-1)^2} - 1$$

$$\Rightarrow \int \frac{S(x)}{x-2}\,dx = \frac{-x+2}{(x-1)^2} - 1 \quad \Big|\,(\,)'$$

$$\frac{S(x)}{x-2} = \frac{-(x-1)^2 - (-x+2)\cdot 2\,(x-1)}{(x-1)^4}$$

$$\frac{S(x)}{x-2} = \frac{\cancel{(x-1)}\left[-x+1+2x-4\right]}{(x-1)^{\cancel{4}\,3}} \Leftrightarrow$$

$$\Leftrightarrow \frac{S(x)}{x-2} = \frac{x-3}{(x-1)^3} \Rightarrow \boxed{S(x) = \frac{(x-2)(x-3)}{(x-1)^3}}$$

$T = 2\pi$

$f : [0, \pi] \to \mathbb{R}$, $f(x) = |x|$

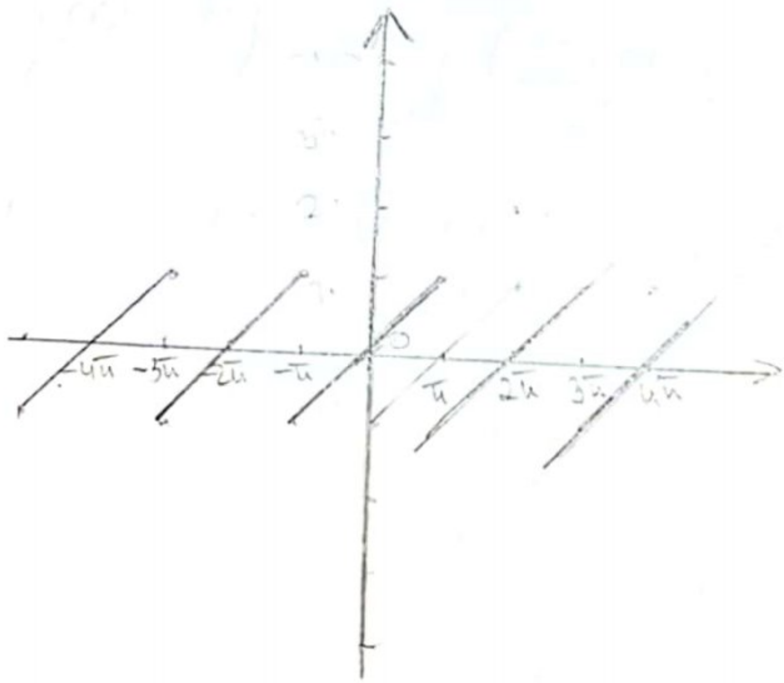
SFC, repr. grafică f + prelungire

1. Prelungim f prin paritate pe $[-\pi, \pi]$ și apoi prin periodicitate pe $\mathbb{R}$.

$$f_p(x) = \begin{cases} f(x), & x \in [0, \pi] \\ f(-x), & x \in [-\pi, 0) \end{cases} = \begin{cases} |x|, & x \in [0, \pi] \\ |-x|, & x \in [-\pi, 0) \end{cases} =$$

$$= \begin{cases} x, & x \in [0, \pi] \\ x, & x \in [-\pi, 0) \end{cases} = x, \ x \in [-\pi, \pi]$$

2. Graficul



3. Coeficienții Fourier

$b_n = 0, \forall n \geq 1$; $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$a_0 = \frac{4}{T} \cdot \int_0^{\frac{T}{2}} f(x)\,dx = \frac{4}{2\pi} \cdot \int_0^{\pi} x\,dx =$$

$$= \cancel{4}\,\frac{2}{\pi} \cdot \frac{x^2}{\cancel{2}}\Big|_0^{\pi} = \frac{1}{\pi}(\pi^2 - 0) = \pi$$

$$a_n = \frac{4}{T} \cdot \int_0^{\frac{T}{2}} f(x) \cdot \cos(\omega n x)\,dx =$$

$$= \frac{2}{\pi} \cdot \int_0^{\pi} x \cdot \cos(nx)\,dx =$$

$f(x) = x \quad \Rightarrow f'(x) = 1$

$g'(x) = \cos(nx) \Rightarrow g(x) = \frac{\sin(nx)}{n}$

$$= \frac{2}{\pi}\left(x \cdot \frac{\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx \right) =$$

$$= \frac{2}{\pi}\left(\frac{1}{n}\big(\pi \cdot \underbrace{\sin(n\pi)}_{0} - 0\big) - \frac{1}{n} \cdot \left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi} \right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\left(-\frac{1}{n}\right) \cdot \big(\cos(n\pi) - \cos 0\big)\right) =$$

$$= \frac{2}{\pi} \cdot \frac{1}{n^2}\left((-1)^n - 1\right) =$$

$$= \frac{2\left((-1)^n - 1\right)}{n^2 \cdot \pi}$$

$$\boxed{a_0 = \pi}$$

$$\boxed{a_n = \frac{2\left((-1)^n - 1\right)}{n^2 \cdot \pi}}$$

Seria Fourier de cosinusuri

$$S(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cdot \cos(\omega n x) =$$

$$= \frac{\pi}{2} + \sum_{n\geq 1} \frac{2((-1)^n - 1)}{n^2 \cdot \pi} \cdot \cos(nx) =$$

$$= \frac{\pi}{2} + \frac{2}{\pi} \cdot \sum_{n\geq 1} \frac{(-1)^n - 1}{n^2} \cdot \cos(nx)$$

$$\boxed{S(x) = \frac{\pi}{2} + \frac{2}{\pi} \cdot \sum_{n\geq 1} \frac{(-1)^n - 1}{n^2} \cdot \cos(nx)}$$

AM (c)

$\bar{a}, \bar{x}$ – vectori

Limite de funcții

Fie $f: A \subset \mathbb{R}^p \to \mathbb{R}$, $p \in \mathbb{N}^*$ $\qquad g = (f_1, f_2, \dots, f_q)$

$f(x_1, x_2, \dots, x_p) = \square \in \mathbb{R}$ $\qquad g: \mathbb{R}^p \to \mathbb{R}^q$

Def

Funcția f are limita $l \in \mathbb{R}$ în punctul $\bar{a} = (a_1, \dots, a_p) \in A'$ dacă $\forall \varepsilon > 0, \exists \eta(\varepsilon)$ a.î. $\forall \bar{x} \in A \setminus \{\bar{a}\}$ pt. care $|x_k - a_k| < \eta$ $(k = \overline{1,p})$, avem $|f(x) - l| < \varepsilon$.

Obs

Def. de mai sus exprimă faptul că f are limită în raport cu ansamblul variabilei.

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$, $\bar{a} = (a_1, a_2)$

$$\lim_{\bar{x} \to \vec{a}} f(x) = \cancel{\lim_{\bar{x}\to\bar{a}}} \lim_{(x_1,x_2) \to (a_1,a_2)} f(x_1, x_2) = \lim_{\substack{x_1 \to a_1 \\ x_2 \to a_2}} f(x_1, x_2)$$

Teorema lui Heine

Funcția $f: A \subset \mathbb{R}^p \to \mathbb{R}$ are limită l în punctul $\bar{a} \in A'$ d.d. pt. orice șir $(\overline{x_m})_{m \geq 0} \in A \setminus \{\bar{a}\}$ cu $\lim_{m\to\infty} \overline{x_m} = \bar{a}$, avem $\lim_{m \to \infty} f(\overline{x_m}) = l$.

Criteriul majorării („cleștelui")

Fie $f: A \subset \mathbb{R}^p \to \mathbb{R}$, $\bar{a} \in A'$ – mulț. punctelor de acumulare. Dacă $\exists g: A \subset \mathbb{R}^p \to \mathbb{R}_+$ a.î. $\lim_{\bar{x} \to \bar{a}} g(x) = 0$ și $|f(\bar{x}) - l| \leq g(x)$, $\forall x \in A$, atunci $\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = l$.

• se fol. at. când vrem să arătăm că fct. are lim

Obs

Dacă $h: A \subset \mathbb{R}^p \to \mathbb{R}^q$, $p, q \in \mathbb{N}^*$, $h = (h_1, h_2, \dots, h_q)$, $\bar{a} \in A'$, atunci: ~~limită~~ $\lim_{\vec{x} \to \bar{a}} h(\vec{x}) = \underbrace{\left(\lim_{\vec{x} \to \bar{a}} h_1(\vec{x}), \lim_{\vec{x} \to \bar{a}} h_2(\vec{x}), \dots, \lim_{\vec{x} \to \bar{a}} h_q(\vec{x})\right)}_{\text{dacă 1 dintre componente NU are limită, atunci } h(\vec{x}) \text{ NU are lim}} \in \mathbb{R}^q$.

$k = \overline{1, q}$, $h_k: A \subset \mathbb{R}^p \to \mathbb{R}$

Ex

1) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$\lim_{(x,y) \to (2,1)} f(x, y) = ?$; $\lim_{(x,y) \to (2,0)} f(x, y) = ?$; $\lim_{(x,y) \to (0,0)} f(x, y) = ?$

i) $\lim_{(x,y) \to (2,1)} f(x, y) = \frac{2 \cdot 1}{4 + 1} = \frac{2}{5}$

ii) $\lim_{(x,y) \to (2,0)} f(x, y) = \frac{2 \cdot 0}{4 + 0} = 0$

iii) $\lim_{(x,y) \to (0,0)} f(x, y) = „\frac{0}{0}"$

→ Arătăm că f nu are limită în $(0, 0)$.

• considerăm $(x_n^1, y_n^1) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow{n \to \infty} (0, 0)$

$f(x_n^1, y_n^1) = \frac{\left(\frac{1}{n}\right)^2}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{1}{2} \xrightarrow{n \to \infty} \frac{1}{2}$

• considerăm $(x_n^2, y_n^2) = \left(\frac{2}{n}, \frac{1}{n}\right) \xrightarrow{n \to \infty} (0, 0)$

$f(x_n^2, y_n^2) = \frac{\frac{2}{n^2}}{\frac{4}{n^2} + \frac{1}{n^2}} = \frac{2}{5} \xrightarrow{n \to \infty} \frac{2}{5} \neq \frac{1}{2} \longrightarrow$

$\xRightarrow{\text{T. Heine}} \nexists \lim_{(x,y) \to (0,0)} f(x, y)$

3.4° - Dacă ar exista, atunci orice pereche care tinde la (0,0) ~~și are același rezultat~~ $\{(a,a)\}$ ar trebui să ~~aibă~~ dea aceeași valoare pt. $f(x,y)$.

② $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (x+y)\cdot\cos\left(\frac{x}{y}\right), & y\neq 0 \\ 0, & y=0 \end{cases}$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

$|f(x,y)-0| = \left|(x+y)\cdot\cos\left(\frac{x}{y}\right)\right| = |x+y|\cdot\underbrace{\left|\cos\left(\frac{x}{y}\right)\right|}_{\leq 1} \leq |x+y| \overset{not}{=} g(x,y)$

$\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 \xrightarrow[maj.]{crit.} \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Limite iterate

$f: A\subset\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -$

$\bar{a} = (a_1, a_2) \in A'$

$$\boxed{\begin{aligned} l_{12} &= \lim_{x\to a_1}\left(\lim_{y\to a_2} f(x,y)\right) \\ l_{21} &= \lim_{y\to a_2}\left(\lim_{x\to a_1} f(x,y)\right) \end{aligned}}$$

Pt. $f(x,y,z) = -$, vom avea 3! limite iterate

$l_{123} = \lim\limits_{x\to a_1}\left(\lim\limits_{y\to a_2}\left(\lim\limits_{z\to a_3} f(x,y,z)\right)\right)$

③ $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$l_{12} = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right) = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{xy}{x^2+y^2}\right) =$

$= \lim\limits_{x\to 0} \frac{x\cdot 0}{x^2+0} = \lim\limits_{x\to 0} \frac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

$l_{21} = 0$

$\Rightarrow f(x,y)$ NU are l, dar are l_{12}, l_{21}.

④ $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (x+y)\cdot\cos\left(\frac{x}{y}\right), & \cancel{(x+y)\neq 0}\ y\neq 0 \\ 0, & y = 0 \end{cases}$

$l_{12} = \lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} (x+y)\cdot\cos\left(\frac{x}{y}\right)\right) =$

$= \lim\limits_{x\to 0}\left(\underbrace{\lim\limits_{y\to 0} x\cdot\cos\left(\frac{x}{y}\right)}_{x\cdot\lim\limits_{y\to 0}\cos\left(\frac{x}{y}\right)} + \underbrace{\lim\limits_{y\to 0} y\cdot\cos\left(\frac{x}{y}\right)}_{\to 0 \text{ (prin majorare)}}\right) \Rightarrow$

*

$\boxed{\nexists \lim\limits_{x\to\infty} \cos x}$!

($\nexists$, pe cos, cos mi ll cos x, care nu există)

$\to l_{12} =$ ceva ce ~~nu~~ $\nexists + 0 \Rightarrow l_{12} \nexists$

$\cos\left(\frac{x}{y}\right)$

$y_n^1 = \frac{x}{2n\pi} \xrightarrow{n\to\infty} 0 \to \cos\left(\frac{x}{y_n^1}\right) = \cos(2n\pi) = 1 \xrightarrow{n\to\infty} 1$

$y_n^2 = \frac{x}{(2n+1)\frac{\pi}{2}} \xrightarrow{n\to\infty} 0 \Rightarrow \cos\left(\frac{x}{y_n^2}\right) = \cos\left((2n+1)\frac{\pi}{2}\right) = 0 \xrightarrow{n\to\infty} 0$

$\xRightarrow{\text{T. Heine}} \nexists \lim\limits_{\cancel{(x,y)\to\infty}} \cos\left(\frac{x}{y}\right) \Rightarrow \nexists\ l_{12}$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)}$

$$l_{21} = \lim_{y\to 0}\left(\lim_{x\to 0}(x+y)\cdot\cos\frac{x}{y}\right) = \lim_{y\to 0}\left(\lim_{x\to 0} x\cdot\underbrace{\cos\frac{x}{y}}_{\to 0} +\right.$$

$$\left.+\lim_{x\to 0} y\cdot\cos\frac{x}{y}\right) = \lim_{y\to 0}\left(0 + y\cdot\underbrace{\lim_{x\to 0}\cos\frac{x}{y}}_{\cos 0 = 1}\right) =$$

$$= \lim_{y\to 0} y = 0$$

$\Rightarrow$ pt. $f(x,y)$: $\exists\, l$, $\exists\, l_{21}$, $\nexists\, l_{12}$

Legăturile dintre l și l_{12} / l_{21}

Prop

Dacă pt. o funcție f de mai multe variabile $\exists$ una din limitele iterate și $\exists$ limita l în punctul $\bar{a}$, atunci cele 2 limite existente sunt egale.

La ce folosește?

E util când vrem să arătăm că o funcție nu are lim. într-un anumit punct.

Ex $f(x,y) = \begin{cases} \dfrac{y^2-2x}{x^2-2y}, & (x,y) \neq (2,2) \\ 13, & (x,y) = (2,2) \end{cases}$

$l_{12}, l_{21} = ?$ în $\bar{a} = (2,2)$

$$l_{12} = \lim_{x\to 2}\left(\lim_{y\to 2}\frac{y^2-2x}{x^2-2y}\right) = \lim_{x\to 2}\frac{4-2x}{x^2-4} =$$

$$= \lim_{x\to 2}\frac{2\cancel{(2-x)}}{\cancel{(x-2)}(x+2)} = -\lim_{x\to 2}\frac{2}{x+2} = -\frac{1}{2}$$

$$l_{21} = \lim_{y\to 2}\left(\lim_{x\to 2}\frac{y^2-2x}{x^2-2y}\right) = \lim_{y\to 2}\frac{y^2-4}{4-2y} =$$

$$= \lim_{y\to 2}\frac{\cancel{(y-2)}(y+2)}{2\cancel{(2-y)}} = -\lim_{y\to 2}\frac{y+2}{2} = -2$$

$\Rightarrow l_{12} \neq l_{21}$ în $(2,2) \Rightarrow \nexists \lim\limits_{(x,y)\to(2,2)} f(x,y)$

! Limitele iterate au aceeași proprietate ca și limitele laterale (dacă sunt diferite, limita mare $\nexists$)

Def

Fie $f: A \subset \mathbb{R}^p \to \mathbb{R}$, $\bar{a} = (a_1, a_2, \dots, a_p) \in A$.

① Spunem că f - CONTINUĂ în punctul $\bar{a}$ dacă $\forall \varepsilon > 0$, $\exists \eta = \eta(\varepsilon) > 0$ a.î. $\forall \bar{x} \in A$ pt. care $|x_k - a_k| < \eta$ $(k = \overline{1,p})$, avem că $|f(\bar{x}) - f(\bar{a})| < \varepsilon$.

② Spunem că f - CONTINUĂ în punctul $\bar{a}$ d.d. $\forall (\bar{x}_n)_{n\in\mathbb{N}} \in A$ cu $\lim\limits_{n\to\infty} \bar{x}_n = \bar{a}$, avem că $\lim\limits_{n\to\infty} f(\bar{x}_n) = f(\bar{a})$.

③ Spunem că f - CONTINUĂ în punctul $\bar{a}$, dacă $\lim\limits_{\bar{x}\to\bar{a}} f(\bar{x}) = f(\bar{a})$.

Întrebare: f - continuă în pct. $(0,0)$

$f(0,0) \overset{?}{=} \lim\limits_{(x,y)\to(0,0)} f(x,y)$ | f NU e cont. în $(0,0)$

AM (c)

- $f(x_0,y_0) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \Rightarrow f$ - cont. în (x_0,y_0)

- $f: \mathbb{R}^p \to \mathbb{R}^q$, $f(\bar{x}) = (f_1(\bar{x}), \dots, f_q(\bar{x}))$

Spunem că f - cont. în $\bar{x_0}$, dacă $f_1, f_2, \dots, f_q$ sunt cont. în $\bar{x_0}$.

<u>Ex.</u> $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \begin{cases} \left(\dfrac{\ln(1+x^4+y^4)}{x^2+y^2}, x+y \sin\left(\dfrac{1}{x^2+y^2}\right)\right), & (x,y) \neq (0,0) \\ (0,0), & (x,y) = (0,0) \end{cases}$

~~$(x,y) \neq (0,0)$~~

Studiați cont. lui f pe $\mathbb{R}^2$.

- Pt. $(x,y) \neq (0,0)$ fct. $f_1(x,y) = \dfrac{\ln(1+x^4+y^4)}{x^2+y^2}$ și $f_2(x,y) = x+y \cdot \sin\left(\dfrac{1}{x^2+y^2}\right)$ sunt cont. (compuse de fct. elem.)

- Pt. $(x,y) = (0,0)$:

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} \frac{\ln(1+x^4+y^4)}{x^2+y^2} \overset{\frac{0}{0}}{=}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\ln(1+(x^4+y^4))}{x^4+y^4} \cdot \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = \text{„}\frac{0}{0}\text{"}$$

Pt a arăta că f ARE limită

Dacă $|f(x,y)-\ell| \le \underset{\longrightarrow (a,b)}{g(x,y)} \; 0$

COORDONATE POLARE

$$\left\{\begin{array}{l} x=\rho\cdot\cos\theta+x_0 \\ y=\rho\cdot\sin\theta+y_0 \end{array}\right. \text{ coord polare}$$

$$\mathcal{C}: (x-x_0)^2+(y-y_0)^2=\rho^2$$

$$\Delta: (x-x_0)^2+(y-y_0)^2\le\rho^2$$

UTIL

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} \xlongequal[y=\rho\cdot\sin\theta]{x=\rho\cdot\cos\theta} \lim_{\rho\to 0} \frac{\rho^4(\cos^4\theta+\sin^4\theta)}{\rho^2(\sin^2\theta+\cos^2\theta)}$$

$$=\lim_{\rho\to 0}\rho^2(\cos^4\theta+\sin^4\theta)=0=f_1(0,0)\Rightarrow$$

$\Rightarrow f_1$ - cont. în $(0,0)$

$$\lim_{(x,y)\to(0,0)} f_2(x,y)=\lim_{(x,y)\to(0,0)}\left(x+y\cdot\sin\frac{1}{x^2+y^2}\right)=$$

$$=\lim_{(x,y)\to(0,0)}\left(x+y\cdot\frac{\sin\frac{1}{x^2+y^2}}{\frac{1}{x^2+y^2}}\cdot\frac{1}{x^2+y^2}\right)=$$

$$=\lim_{(x,y)\to(0,0)}\left(x+\frac{y}{x^2+y^2}\right)$$

$|f_2(x,y) - 0| = \left|x + y \cdot \sin\left(\frac{1}{x^2+y^2}\right)\right| \leq |x| + |y \cdot \sin \frac{1}{x^2+y^2}|$

$\leq |x| + |y| \xrightarrow{(x,y)\to(0,0)} 0$

Crit. maj. $\Rightarrow f_2(x,y) \xrightarrow{(x,y)\to(0,0)} 0 = f_2(0,0) \to f_2$ - cont. în $(0,0)$

Derivabilitate și diferențiabilitate pt. funcții de mai multe variabile

Fie $f: \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$.

$$\boxed{\frac{\partial f}{\partial x}(x_0,y_0) \overset{def}{=} \lim_{x\to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x-x_0}}$$

$= f'_x(x_0,y_0)$

DERIVATA PARȚIALĂ A LUI F ÎN RAP. CU VARIABILA X ÎN PUNCTUL (x_0, y_0)

$$\boxed{\frac{\partial f}{\partial y}(x_0,y_0) \overset{def}{=} \lim_{y\to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y-y_0}}$$

$= f'_y(x_0,y_0)$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 \ln(y-x)$; $y - x > 0$

$\frac{\partial f}{\partial x}(1,2) = \lim_{x\to 1} \frac{f(x,2) - f(1,2)}{x-1} =$

$= \lim_{x\to 1} \frac{x^2 \ln(2-x) - \ln(1)}{x-1} = \lim_{x\to 1} \frac{x^2 \ln(2-x)}{x-1} =$

$\overset{0/0}{=} \lim_{x\to 1} \frac{x^2}{x-1} \cdot \lim_{x\to 1} \frac{\ln(1+1-x)}{1-x} \cdot 1 - x = -1$

$$\frac{\partial f}{\partial x} = 2x \ln(y-x) + x^2 \cdot \frac{-1}{y-x}$$

$$\frac{\partial f}{\partial x}(1,2) = 2\cdot 1 \ln(2-1) + 1 \cdot \frac{-1}{2-1} = -1$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 \cdot \frac{1}{y-x}$$

$$\frac{\partial f}{\partial y}(1,2) = 1 \cdot \frac{1}{2-1} = +1$$

$f : \mathbb{R}^p \to \mathbb{R}^q$, $f(\bar{x}) = f(x_1, \dots, x_p) = (f_1(\bar{x}), \dots, f_q(\bar{x}))$

$\bar{x} = x_1, \dots, x_p$

$$\boxed{\frac{\partial f}{\partial x_k}(\bar{x_0}) = \left(\frac{\partial f_1}{\partial x_k}(\bar{x_0}), \frac{\partial f_2}{\partial x_k}(\bar{x_0}), \dots, \frac{\partial f_q}{\partial x_k}(\bar{x_0})\right) \in \mathbb{R}^q \quad k = \overline{1,p}}$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x_0}) & \frac{\partial f_1}{\partial x_2}(\bar{x_0}) & \dots & \frac{\partial f_1}{\partial x_p}(\bar{x_0}) \\ \frac{\partial f_2}{\partial x_1}(\bar{x_0}) & \frac{\partial f_2}{\partial x_2}(\bar{x_0}) & \dots & \frac{\partial f_2}{\partial x_p}(\bar{x_0}) \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_q}{\partial x_1}(\bar{x_0}) & \frac{\partial f_q}{\partial x_2}(\bar{x_0}) & \dots & \frac{\partial f_q}{\partial x_p}(\bar{x_0}) \end{pmatrix} \overset{\text{not}}{=} J_f(\bar{x_0})$$

MATRICEA JACOBIANĂ

$$= \left(\frac{\partial f_i}{\partial x_j}(\bar{x_0})\right) \quad i = \overline{1,q}, \; j = \overline{1,p}$$

Scanned with CamScanner

Dacă $p = q \Rightarrow$ matricea se numește JACOBIAN
$\det(J_f(\overline{x_0}))$ = determinant = funcțional

$$= \frac{D(f_1, f_2, \dots, f_p)}{D(x_1, x_2, \dots, x_p)}(\overline{x_0}).$$

DERIVATE DE ORDIN SUPERIOR

$f: \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$

- $$\underbrace{\frac{\partial^2 f}{\partial x^2}(x_0, y_0)}_{\text{DERIVATĂ PARȚIALĂ DE ORDINUL 2 ÎN RAPORT CU } x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \lim_{x \to x_0} \frac{\frac{\partial f}{\partial x}(x, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)}{x - x_0}$$

- $$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = \lim_{y \to y_0} \frac{\frac{\partial f}{\partial y}(x_0, y) - \frac{\partial f}{\partial y}(x_0, y_0)}{y - y_0}$$

- $$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = \lim_{x \to x_0} \frac{\frac{\partial f}{\partial y}(x, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{x - x_0}$$

- $$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \lim_{y \to y_0} \frac{\frac{\partial f}{\partial x}(x_0, y) - \frac{\partial f}{\partial x}(x_0, y_0)}{y - y_0}$$

DERIVATE PARȚIALE MIXTE DE ORDINUL 2 (!! de obicei sunt egale)

Teorema lui Schwarz

- Dacă $f: \mathbb{R}^2 \to \mathbb{R}$ are (admite) derivate parțiale mixte de ordinul 2 într-o vecinătate a pct. (x_0, y_0) și dacă aceste derivate parțiale sunt continue în (x_0, y_0), atunci $\boxed{\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)}$ - nu se ind. deobicei când fct e def pe ramuri.

Ex $f(x, y) = x^4 y - \sin(x-y)$

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 y - \cos(x-y)$$

$$\frac{\partial f}{\partial y}(x,y) = x^4 + \cos(x-y)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(4x^3 y - \cos(x-y)\right) =$$

$$= 4x^3 - \sin(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^4 + \cos(x-y)\right) =$$

$$= 4x^3 - \sin(x-y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

Ex. $f: \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$

$$\boxed{\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(x_0, y_0) \overset{def}{=} \frac{\partial^m}{\partial x^m}\left(\frac{\partial^n f}{\partial y^n}\right)(x_0, y_0)}$$

$m, n \in \mathbb{N}^*$

$f: \mathbb{R}^p \to \mathbb{R}$, $\overline{x_0} = (x_1^0, x_2^0, \dots, x_p^0)$

$$\boxed{\frac{\partial^2 f}{\partial x_i \partial x_j}(\overline{x_0}) \overset{def}{=} \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(\overline{x_0})}$$

$i = \overline{1,p}$, $j = \overline{1,p}$

$$\frac{\partial^3 f}{\partial x^1 \partial y^1 \partial z^1}(x_0, y_0, z_0) \leftarrow$$ suma puterilor de sus trb. să fie egală cu suma puterilor de jos

$$\frac{\partial^3 f}{\partial x^2 \partial z^1}(x_0, y_0, z_0)$$

Ex. $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x, y) = \frac{x+y}{x-y}$, $x \neq y$ TEMĂ

$\underbrace{(x+y)}_{f} \cdot \underbrace{\frac{1}{x-y}}_{g}$

$$\frac{\partial^{100} f}{\partial x^{80} \partial y^{20}} = ?$$

Teorema lui Leibniz pt (cls. 11)

$$(f \cdot g)^{(n)}(x) = C_n^0 f^{(n)}(x) \cdot g(x) + C_n^1 \cdot f^{(n-1)}(x) \cdot g^{(1)}(x) +$$
$$+ \dots + C_n^n \cdot f(x) \cdot g^{(n)}(x)$$

FORMULA LUI LEIBNIZ

AM (c)

Ex. 1 $\frac{\partial^{11} f}{\partial x^6 \partial y^5}\left(\frac{\pi}{2}, 0\right) = ?$ $f(x,y) = (3y^2 - x) \cdot \sin(x-y)$

$$\frac{\partial^{11} f}{\partial x^6 \partial y^5}(x,y) = \frac{\partial^6}{\partial x^6}\left(\frac{\partial^5 f}{\partial y^5}\right)(x,y) =$$

$$(g \cdot h)^{(m)} = C_m^0 \, g^{(m)} \cdot h + C_m^1 \, g^{(m-1)} \cdot h' + \dots +$$
$$+ C_m^m \, g \cdot h^{(m)}$$

$$\frac{\partial^5 f}{\partial y^5} = \frac{\partial^5}{\partial y^5}\left((3y^2 - x)\sin(x-y)\right) \overset{m=5}{=} C_5^0 \cdot \underbrace{(3y^2-x)^{V}_{y}}_{0} \cdot \sin_y (x-y) +$$
$$+ C_5^1 \cdot \underbrace{(3y^2-x)^{IV}_{y}}_{0} \cdot \sin'(x-y)_y + C_5^2 \underbrace{(3y^2-x)^{III}_{y}}_{0} \sin''(x-y)_y +$$
$$+ C_5^3 (3y^2-x)^{II}_{y} \cdot \sin'''(x-y)_y + C_5^4 \cdot (3y^2-x)'_y \cdot \sin^{IV}(x-y)_y$$
$$+ C_5^5 (3y^2-x)_y \cdot \sin^{V}(x-y)_y =$$
$$= 0 + 0 + 0 + C_5^3 \cdot 6 \cdot \cos(x-y) + C_5^4 \cdot 6y \cdot \sin(x-y)$$
$$-(3y^2-x)\cos(x-y) =$$
$$= 60\cos(x-y) + 30y\sin(x-y) - (3y^2-x)\cos(x-y)$$

$$\Rightarrow \frac{\partial^{11} f}{\partial x^6 \partial y^5}(x,y) = \frac{\partial^6}{\partial x^6}\Big(60\cos(x-y) + 30y\sin(x-y) -$$
$$-(3y^2-x)\cos(x-y)\Big) =$$
$$= 60 \cdot (\cos(x-y))^{(6)}_x + 30y \cdot \sin(x-y)^{(6)}_x - \frac{\partial^6}{\partial x^6}\left((3y^2-x) \cdot \cos(x-y)\right)$$

FORMULĂ: $\boxed{\left(\frac{1}{x+a}\right)^{(m)} = \frac{m!\,(-1)^m}{(x+a)^{m+1}}}$

Elemente de calcul diferențial

Funcții diferențiabile

Fie $f: A \subset \mathbb{R}^m \to \mathbb{R}$. Asociem fiecărui vector $x \in \mathbb{R}^m$ o funcție liniară $d_{\bar{x}} f: \mathbb{R}^m \to \mathbb{R}$, numită DIFERENȚIALA LUI F ÎN PUNCTUL $\bar{x}$: $\boxed{\bar{x} \to (d_{\bar{x}} f)(\bar{h})}$

$\bar{h} \in \mathbb{R}^m$

Proprietăți

$$\boxed{\begin{aligned} &\cdot\ d_x(f+g) = d_x f + d_x g \\ &\cdot\ d_x(\alpha \cdot f) = \alpha \cdot d_x f \end{aligned}}$$

Fie $\bar{x} = (x_1, x_2, \dots, x_m) \Rightarrow$ $\boxed{d^1_{\bar{x}} f = \frac{\partial f}{\partial x_1}(\bar{x}) \cdot dx_1 + \dots + \frac{\partial f}{\partial x_m}(\bar{x})\, dx_m}$

Ex: $d^1_{(x,y)} f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ aplicăm o ridicare formală la $()^2$

$$d^2_{(x,y)} f = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$$

Pt. $f: \mathbb{R}^3 \to \mathbb{R}$

$$d^1_{(x,y,z)} f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Obs.

Pentru $d^2_{(x,y,z)}f$, „modelăm" formula

$$(A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC \Rightarrow$$

$$\Rightarrow d^2_{(x,y,z)}f = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial^2 f}{\partial z^2}dz^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + 2\frac{\partial^2 f}{\partial x \partial z}dx\,dz + 2\frac{\partial^2 f}{\partial y \partial z}\cdot dy\,dz$$

Def.

Funcția $f: D \subset \mathbb{R}^m \to \mathbb{R}$ este DIFERENȚIALĂ în punctul $\bar{a} = (a_1, \dots, a_m)$, dacă $\exists\, w_{\bar{a}}: D \subset \mathbb{R}^m \to \mathbb{R}$, continuă și nulă în $\bar{a}$, a.î. pt. $\forall\, \bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$:

$$\boxed{f(\bar{x}) = f(\bar{a}) = \sum_{k=1}^{m} \frac{\partial f}{\partial x_m} \cdot \bar{a}(x_k - a_k) + w_{\bar{a}}(\bar{x})\,\|\bar{x}-\bar{a}\|}$$

w_a - continuă și nulă în a

$$\Rightarrow \boxed{w_{\bar{a}}(\bar{a}) = \lim_{\bar{x} \to \bar{a}} w_{\bar{a}}(\bar{x})}$$

Caz particular

$f : A \subset \mathbb{R}^3 \to \mathbb{R}$, $\bar{a} = (a_1, a_2, a_3)$

f - diferențiabilă în $\bar{a}$ d.d.

$$f(x_1,x_2,x_3) = f(a_1,a_2,a_3) + \frac{\partial f}{\partial x_1}(\bar{a})(x_1-a_1) + \frac{\partial f}{\partial x_2}(\bar{a})(x_2-a_2) + \frac{\partial f}{\partial x_3}(\bar{a})(x_3-a_3) + \omega(x_1,x_2,x_3)\cdot\sqrt{(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2}$$

De aici:

$\bar{a} = (0,0,0)$

Criteriul de diferențiabilitate

Fie $f : A \subset \mathbb{R}^m \to \mathbb{R}$. Atunci, f - DIFERENȚIABILĂ în punctul $\bar{a}$, dacă TOATE derivatele sale parțiale $\frac{\partial f}{\partial x_k}$, $k=\overline{1,m}$ sunt continue în $\bar{a}$.

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} x^2 \sin\frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$

a) stud. cont. der. parțiale

b) stud. diferențiabilitatea lui f în $(0,0)$.

$\cdot)$ $\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x \cdot \sin\left(\frac{1}{y}\right), & y \neq 0 \\ 0, & y = 0 \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = \lim\limits_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x \to 0} \frac{0-0}{x} = 0$

• Studiem cont. în $(0,0)$ a lui $\frac{\partial f}{\partial x}$

↳ din def: Dacă $\frac{\partial f}{\partial x}(0,0) = \lim\limits_{(x,y) \to (0,0)} \frac{\partial f}{\partial x}(x,y) \Rightarrow$

$\Rightarrow$ f - cont. în $(0,0)$

$\lim\limits_{(x,y) \to (0,0)} \frac{\partial f}{\partial x}(0,0)$ cu crit. maj.

$\left| \frac{\partial f}{\partial x}(x,y) - 0 \right| = \left| 2x \cdot \sin\left(\frac{1}{y}\right) \right| \leq 2|x| \cdot \left|\sin \frac{1}{y}\right| \leq$

$\leq 2|x| \xrightarrow{x \to 0} 0 \Rightarrow \boxed{\frac{\partial f}{\partial x} \text{ continuă în } (0,0)}$

$\frac{\partial f}{\partial y}(x,y) = \begin{cases} -\frac{x^2}{y^2} \cos \frac{1}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$

$\frac{\partial f}{\partial y}(0,0) = \lim\limits_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y \to 0} \frac{0-0}{y} = 0$

• Studiem cont. în $\frac{\partial f}{\partial y}(0,0)$

↳ Vrem să arătăm că $\nexists$ lim

↳ cu Heine

$$(x_m, y_m) = \left(\frac{1}{2m\pi}, \frac{1}{2m\pi}\right) \xrightarrow{m\to\infty} (0,0)$$

$$\frac{\partial f}{\partial y}(x_m, y_m) = -\cos(2m\pi) = -1 \xrightarrow{m\to\infty} -1$$

$$(x_m^2, y_m^2) = \left(\frac{2}{(2m+1)\pi}, \frac{2}{(2m+1)\pi}\right) \xrightarrow{m\to\infty} (0,0)$$

$$\frac{\partial f}{\partial y}(x_m^2, y_m^2) = -\cos\left(\frac{(2m+1)\pi}{2}\right) = 0 \xrightarrow{m\to\infty} 0$$

$\Rightarrow$

$\Rightarrow -1 \neq 0 \xrightarrow{\text{Heine}} \nexists \lim \Rightarrow$ $\boxed{\frac{\partial f}{\partial y} \text{ nu e cont. în } (0,0)}$

ii) f - dif. în $(0,0)$, dacă $\exists\, w: \mathbb{R}^2 \to \mathbb{R}$,
$w(0,0) = \lim\limits_{(x,y)\to(0,0)} w(x,y) = 0$ a.î.

$$f(x,y) = \overset{0}{f(0,0)} + \overset{0}{\frac{\partial f}{\partial x}(0,0)}(x-0) + \overset{0}{\frac{\partial f}{\partial y}(0,0)}(y-0) + $$
$$+ w(x,y)\, \overbrace{\|(x,y)-(0,0)\|}^{\sqrt{x^2+y^2}} \Rightarrow$$

$\Rightarrow f(x,y) = w(x,y)\cdot\sqrt{x^2+y^2} \Rightarrow$

$$\Rightarrow w(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}} = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}}\cdot \sin\frac{1}{y}, & y\neq 0 \\ 0, & y = 0 \end{cases}$$

. verificăm $\omega(0,0)=\lim\limits_{(x,y)\to(0,0)} \omega(x,y)=0$

$\lim\limits_{(x,y)\to(0,0)} \omega(x,y)=\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}}\cdot \sin\frac{1}{y}$

$\left|\omega(x,y)-0\right|=\left|\frac{x^2}{\sqrt{x^2+y^2}}\cdot \sin\frac{1}{y}\right|=\frac{x^2}{\sqrt{x^2+y^2}}\cdot\left|\sin\frac{1}{y}\right|$

$\leq \frac{x^2}{\sqrt{x^2+y^2}} \underset{y^2>0}{<} \frac{x^2}{\sqrt{x^2}}=\sqrt{x^2}\xrightarrow{x\to 0} 0 \longrightarrow$

$\underset{\text{maj.}}{\overset{\text{crit.}}{\Longrightarrow}} \lim\limits_{(x,y)\to(0,0)} \omega(x,y)=0=\omega(0,0)\Rightarrow$

$\Rightarrow$ ω există și e continuă și nulă în

$(0,0)\overset{\text{def}}{\Longrightarrow} f(x,y)$ – dif. în $(0,0)$

Funcții omogene. Euler

Def: $f: \mathbb{R}^p \to \mathbb{R}$, f este OMOGENĂ de grad m ($m \in \mathbb{R}$) d.d. $\boxed{f(t\bar{x}) = t^m \cdot f(x)}$, $\forall x \in \mathbb{R}^p$, $\forall t > 0$.

Ex. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = \left(\operatorname{arctg}\sqrt{\frac{y}{x}} + e^{\ln \frac{y+z}{x}}\right) \cdot x$

$\frac{y}{x} \geq 0$, $\frac{y+z}{x} \geq 0$. Calculați gradul de omogenizare

$f(tx, ty, tz) = \dots = t^2 f(x,y,z)$.

TEOREMA LUI EULER

Dacă $f: A \subset \mathbb{R}^p \to \mathbb{R}$ este omogenă de grad m și are derivate parțiale continue în $\bar{x} = (x_1, x_2, x_0) \in A$, atunci f verifică relațiile lui Euler:

1) $x_1 \cdot \frac{\partial f}{\partial x_1}(\bar{x}) + x_2 \cdot \frac{\partial f}{\partial x_2}(\bar{x}) + \dots + x_p \cdot \frac{\partial f}{\partial x_p}(\bar{x}) = m \cdot f(\bar{x})$

2) $\left(x_1 \cdot \frac{\partial}{\partial x_1} + \dots + x_p \cdot \frac{\partial}{\partial x_p}\right)^{(2)}(f) = m(m-1) \cdot f$

$\Rightarrow x_1^2 \cdot \frac{\partial^2 f}{\partial x_1^2}(\bar{x}) + \dots + x_p^2 \cdot \frac{\partial^2 f}{\partial x_p^2}(\bar{x}) + 2 \cdot \frac{\partial^2 f}{\partial x_1 x_2}(\bar{x}) + \dots +$

$+ 2 \cdot \frac{\partial^2 f}{\partial x_{p-1} x_p}(\bar{x}) = m(m-1) \cdot f$

3) $\left(x_1 \cdot \frac{\partial}{\partial x_1} + \dots + x_p \cdot \frac{\partial}{\partial x_p}\right)^{(3)}(f) = m(m-1)(m-2) \cdot f$

k) $\left(x_1 \cdot \frac{\partial}{\partial x_1} + \dots + x_p \cdot \frac{\partial}{\partial x_p}\right)^{(k)}(f) = m(m-1) \cdot \ldots \cdot (m-k+1)\, f$

Ex: Fie $z(x,y) = g(\sqrt{x^2+y^3}, x\sqrt{y}), g \in C^3(\mathbb{R}^2)$

g- funcție omogenă de grad $n = \frac{4}{3}$

Calculați $E = x^3 \cdot \frac{\partial^3 z}{\partial x^3} + 3x^2 y \frac{\partial^3 z}{\partial x^2 \partial y} + 3xy^2 \cdot \frac{\partial^3 z}{\partial x \partial y^2} +$

$+ y^3 \cdot \frac{\partial^3 z}{\partial y^3}$.

$$E = \left(x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y}\right)^{(3)} z = m(m-1)(m-2) z(x,y)$$

$$z(tx, ty) = g(\sqrt{t^3x^3 + t^3y^3}, tx\sqrt{ty}) = g\left(t^{\frac{3}{2}}\sqrt{x^3+y^3}, t^{\frac{3}{2}} x\sqrt{y}\right)$$

$$= t^{\frac{3}{2} \cdot \frac{4}{3}} \cdot g(\sqrt{x^3+y^3}, x\sqrt{y}) = t^2 \cdot z(x,y) \Rightarrow$$

$\Rightarrow$ z- funcție omogenă de grad 2 $\Rightarrow$

$\Rightarrow m(m-1)(m-2) = 0 \Rightarrow E = 0$

Formula lui Taylor pentru multifuncții

Fie $f: A \subset \mathbb{R}^p \to \mathbb{R}, p \in \mathbb{N}^*, p > 1, \bar{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$

$$T_m(\bar{x}) = f(\bar{a}) + \frac{1}{1!} \cdot d^1_{\bar{a}} f(\bar{x} - \bar{a}) + \frac{1}{2!} \cdot d^2_{\bar{a}} f(\bar{x} - a) + \dots +$$
$$+ \frac{1}{m!} \cdot d^m_{\bar{a}} f(\bar{x} - \bar{a})$$

caz particular

$f: A \subset \mathbb{R}^2 \to \mathbb{R}, \bar{a} = (a,b) \in A \subset \mathbb{R}^2$

$$T_3(x,y) = f(a,b) + \frac{1}{1!} \cdot \left[\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-a)\right] +$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial^2 x}(a,b)(x-a)^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial^2 y}(a,b)(y-b)^2\right]$$

$+\frac{1}{3!}\left[\frac{\partial^3 f}{\partial x^3}(a,b)(x-a)^3+3\frac{\partial^3 f}{\partial x^2\partial y}(a,b)(x-a)^2(y-b)+\right.$

$\left.+3\cdot\frac{\partial^3 f}{\partial x\partial y^2}(a,b)(x-a)(y-b)^2+\frac{\partial^3 f}{\partial y^3}(a,b)(y-b)^3\right]$

Fie $f: A\subset\mathbb{R}^p\to\mathbb{R}$, $p>1$, f - diferențiabilă de $(m+1)$ ori pe mulțimea deschisă A $\Rightarrow$ Formula lui Taylor de ordinul $m\in\mathbb{N}^*$ asociată funcției f în punctul $\bar{a}$: $\boxed{f(\bar{x})=(T_m f)(x)+(R_m f)(x)}$

$$\boxed{(R_m f)(x)=\left(\frac{1}{(m+1)!}\cdot d^{m+1} f\right)(\bar{x}-\bar{a}),\quad \theta\in(0,1)}$$

Dacă $(R_m f)(\bar{x})\xrightarrow{m\to\infty}0$, atunci $f(\bar{x})\approx(T_m f)(\bar{x})$.

Obs

$a=0\Rightarrow$ formula lui Taylor $\equiv$ formula MacLaurin

Puncte de extrem pentru funcții de mai multe variabile

Def 1: Punctul $\bar{a}\in\mathbb{R}^p$ este punct de EXTREM LOCAL pt. funcția $A\subset\mathbb{R}^p\to\mathbb{R}$, $p\in\mathbb{N}$, $p>1$, dacă $\exists$ o vecinătate $V\in\mathcal{V}_{\bar{a}}$ a.î. $\forall x\in V\cap A$, expresia $\boxed{E=f(\bar{x})-f(\bar{a})}$ păstrează semn constant:

→ pentru $E>0$, $\bar{a}$ se num. punct de MINIM local

→ pentru $E<0$, $\bar{a}$ se num. punct de MAXIM local

Def. 2: Spunem că $\bar{a} \in \mathbb{R}^p$ este punct STAȚIONAR pentru funcția f, dacă f este diferențiabilă în A și $\boxed{\frac{\partial f}{\partial x_k}(\bar{a})=0}$, $\forall\, k=\overline{1,p}$.

Ex: pt. $f: A \subset \mathbb{R}^3 \to \mathbb{R}$, punctele staționare ~~pt.~~ se det. rezolvând sistemul
$$\begin{cases} \frac{\partial f}{\partial x}=0 \\ \frac{\partial f}{\partial y}=0 \\ \frac{\partial f}{\partial z}=0 \end{cases}$$

Teorema lui FERMAT

Dacă funcția $f: A \subset \mathbb{R}^p \to \mathbb{R}$ este diferențiabilă în punctul de extrem local $\bar{a} \in A$, atunci $\boxed{\frac{\partial f}{\partial x_k}(\bar{a})=0}$

$k=\overline{1,p}$ → pentru o funcție diferențiabilă, punctele de extrem se găsesc printre punctele staționare

Prop. (condiție suficientă de extrem)

Fie ~~A~~ $\bar{a} \in A$ punct staționar pentru f. Dacă $d^2_a f$ este
- → pozitiv definită ⇒ $\bar{a}$ punct de MINIM local
- → negativ definită ⇒ $\bar{a}$ punct de MAXIM local

Matricea HESSIANĂ (Hess) $\boxed{H^{p \times p}}$

$$d_{\bar{a}}^2 f \to H(\bar{a}) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(\bar{a})\right) \quad i,j = \overline{1,p}$$

$$H(\bar{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\bar{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p}(\bar{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\bar{a}) & \frac{\partial^2 f}{\partial x_2^2}(\bar{a}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_p}(\bar{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1}(\bar{a}) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_p^2}(\bar{a}) \end{pmatrix}$$

$$\Delta_1 = \frac{\partial^2 f}{\partial x_1^2}(\bar{a})$$

$$\Delta_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\bar{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\bar{a}) & \frac{\partial^2 f}{\partial x_2^2}(\bar{a}) \end{vmatrix}$$

$\vdots$

$$\Delta_p = \det(\text{Hess})$$

Criteriul lui SYLVESTER

i) $\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_p > 0 \Rightarrow \bar{a}$ – pct. de MINIM local

ii) $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 > 0, \ldots \Rightarrow \bar{a}$ – pct. de MAXIM local

iii) $\left.\begin{array}{l} \Delta_1 \geq 0, \Delta_2 \geq 0, \ldots, \Delta_p \geq 0 \\ \Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \ldots \end{array}\right\} \Rightarrow$ NU se poate stabili natura punctului $\bar{a}$. Dacă cel puțin un det. este NUL

iv) orice situație diferită $\Rightarrow \bar{a}$ nu este punct de extrem

AM (C)
~suplimentar~

1 Det. punctelor de extrem pt. $f: \mathbb{R}^3 \to \mathbb{R}$,

$$f(x,y,z) = x+y+z+\frac{1}{xyz}, \quad xyz \neq 0$$

Etapa 1 Se det. punctele staționare ale lui f, rezolvând sistemul $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0 \end{cases}$.

$$\Leftrightarrow \begin{cases} 1 - \frac{yz}{(xyz)^2} = 0 \\ 1 - \frac{xz}{(xyz)^2} = 0 \\ 1 - \frac{xy}{(xyz)^2} = 0 \end{cases} \Leftrightarrow \begin{cases} 1 - \frac{1}{yz} \cdot \frac{1}{x^2} = 0 \\ 1 - \frac{1}{xz} \cdot \frac{1}{y^2} = 0 \\ 1 - \frac{1}{xy} \cdot \frac{1}{z^2} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 = \frac{1}{x^2yz} \\ 1 = \frac{1}{xy^2z} \\ 1 = \frac{1}{xyz^2} \end{cases} \Leftrightarrow \begin{cases} x^2yz = 1 \\ xy^2z = 1 \\ xyz^2 = 1 \end{cases} \Big| : \Leftrightarrow \begin{cases} \frac{x}{y} = 1 \\ \frac{y}{z} = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{x=y=z} \Rightarrow x^4 - 1 = 0 \Leftrightarrow (x-1)(x+1)(x^2+1) = 0$$

$$\xRightarrow{x \in \mathbb{R}} x_1 = 1 \text{ sau } x_2 = -1 \Rightarrow \text{pct. staționare}$$

$A(-1,-1,-1)$, $B(1,1,1)$

Etapa 2 Construim matricea Hessiană

$$H(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \boxed{\frac{\partial^2 f}{\partial x \partial z}} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \boxed{\frac{\partial^2 f}{\partial z \partial x}} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} \Rightarrow$$

$$\Rightarrow H(x,y,z) = \begin{pmatrix} \frac{2}{x^3yz} & \frac{1}{x^2y^2z} & \frac{1}{x^2yz^2} \\ \frac{1}{x^2y^2z} & \frac{2}{xy^3z} & \frac{1}{xy^2z^2} \\ \frac{1}{x^2yz^2} & \frac{1}{xy^2z^2} & \frac{1}{xyz^3} \end{pmatrix}$$

$$\text{pt. } A(-1,-1,-1) \Rightarrow H(A) = \begin{pmatrix} \boxed{-2} & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

$\Delta_1 = -2 \quad < 0$

$\Delta_2 = \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 4-1 = 3 \quad > 0$

$\Delta_3 = \det(H(A)) = \begin{vmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{vmatrix} = -8-1-1+2+2+2 =$
$= -4 \quad < 0$

$\Rightarrow A$ - punct de MAXIM

pt. $B(1,1,1) \Rightarrow H(B) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\Delta_1 = 2 > 0$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3 > 0$

$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 8+1+1-2-2-2 = 4 > 0$

$\Rightarrow$ B– punct de MINIM

$\boxed{2}$. Fie $f: \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$, $f(x,y) = \sin x + \cos y + \cos(x-y)$

Etapa 1

$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} \cos x - \sin(x-y) = 0 \\ -\sin y + \sin(x-y) = 0 \end{cases} \Big| + \quad \to$

$\Rightarrow \cos x - \sin y = 0 \to \cos x = \sin y$, $x, y \in \left(0, \frac{\pi}{2}\right)$ $\Big\} \Rightarrow \boxed{x = y = \frac{\pi}{4}}$ $\boxed{x = \frac{\pi}{3}, y = \frac{\pi}{6}}$ $\boxed{x = \frac{\pi}{6}, y = \frac{\pi}{3}}$

$\sin y = \cos x \Leftrightarrow \sin y - \cos x = 0 \Leftrightarrow$

$\Leftrightarrow \sin y - \sin\left(\frac{\pi}{2} - x\right) = 0 \Rightarrow \ldots$

$\to \underbrace{A_1\left(\frac{\pi}{4}, \frac{\pi}{4}\right), A_2\left(\frac{\pi}{3}, \frac{\pi}{6}\right), A_3\left(\frac{\pi}{6}, \frac{\pi}{3}\right)}_{\text{trb. verificate !}}$

$\to$ nu convine $\quad\quad \to$ nu convine

Etapa 2

$H(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\sin x - \cos(x-y) & \cos(x-y) \\ \cos(x-y) & -\cos y - \cos(x-y) \end{pmatrix}$

~~Pt. $A_1\left(\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow H(A_1) = \begin{pmatrix} -\frac{\sqrt{2}}{2}-1 & 1 \\ 1 & -\frac{\sqrt{2}}{2}-1 \end{pmatrix}$~~

Pt. $A_2\left(\frac{\pi}{3}, \frac{\pi}{6}\right) \Rightarrow H(A_2) = \begin{pmatrix} -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \end{pmatrix} =$

$= \begin{pmatrix} \frac{-\sqrt{3}-\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}-\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$

$\Delta_1 = -\sqrt{3} < 0$

$\Delta_2 = \begin{vmatrix} -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\sqrt{3} \end{vmatrix} = 3 - \frac{3}{4} = \frac{9}{4} > 0$ $\Bigg\} \Rightarrow A_2$ - punct de MAXIM

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