$$a_n = \frac{2}{T}\int_0^T f(x)\cos(wnx)dx, \quad n \geq 0$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(wnx)dx, \quad n \geq 1$$

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n(\cos wnx) + b_n \sin(wnx)\right)$$

1. a) $f: \mathbb{R} \to \mathbb{R}$, $T = \pi$

$f(x) = x+1$, $x \in [0, \pi)$

$$a_0 = \frac{2}{T}\int_0^T f(x)dx = \frac{2}{\pi}\int_0^{\pi}(x+1)dx =$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2} + x\right)\Big|_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2} - \frac{0^2}{2} + \pi - 0\right) =$$

$$= \boxed{\pi + 2}.$$

$$w = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\cos(2nx)dx =$$

$$= \frac{2}{\pi}\int_0^{\pi}(x+1)\left(\frac{\sin(2nx)}{2n}\right)'dx = \underbrace{\frac{2}{\pi}(x+1)\frac{\sin(2nx)}{2nx}\Big|_0^{\pi}}_{0} -$$

$$- \frac{2}{\pi}\int_0^{\pi}\frac{\sin(2nx)}{2n}dx =$$

$$= -\frac{1}{n\pi}\int_0^{\pi}\sin(2nx)dx = -\frac{1}{n\pi}\int_0^{\pi}\left(-\frac{\cos(2nx)}{2n}\right)'dx =$$

$$= \frac{\cos(2nx)}{2n^2\pi}\Big|_0^{\pi} = \frac{\cos(2n\pi) - \cos 0}{2n^2\pi}$$

$$= 0.$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(wnx)dx, \quad n \geq 1$$

$$= \frac{2}{\pi}\int_0^{\pi}(x+1)\sin(2nx)dx = \frac{2}{\pi}\int_0^{\pi}(x+1)\left(-\frac{\cos(2nx)}{2n}\right)'$$

$$= -\frac{2}{\pi}(x+1)\frac{\cos(2nx)}{2n}\Big|_0^{\pi} + \frac{2}{\pi}\int_0^{\pi} + \frac{\cos(2nx)}{2n}dx =$$

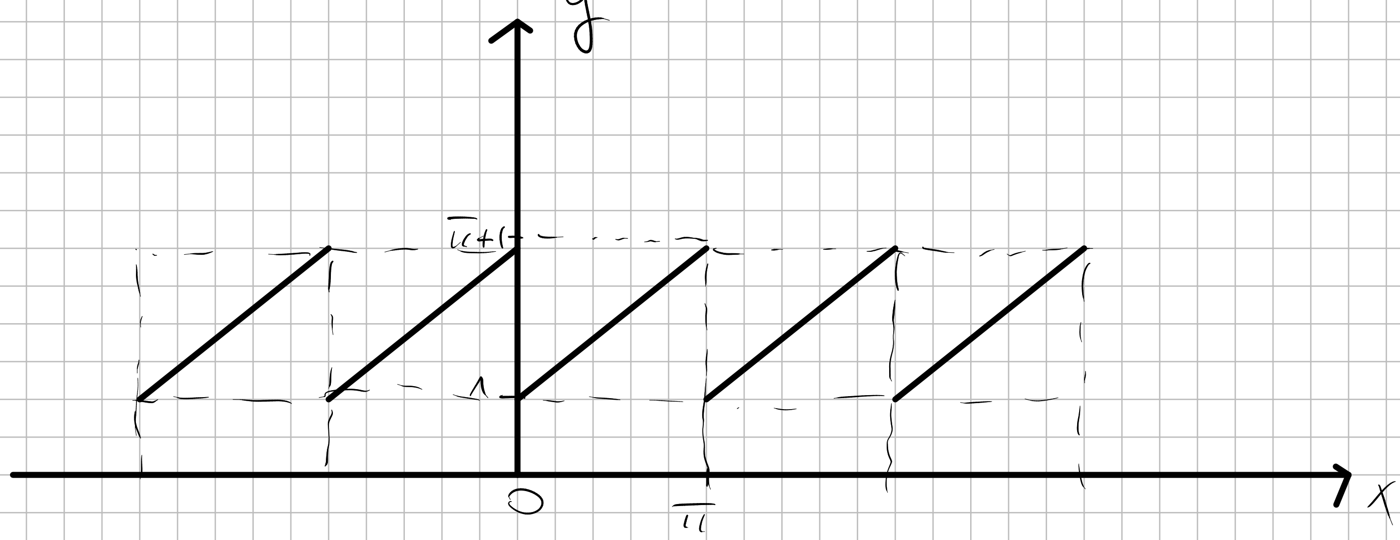
$$= -\frac{2}{\pi}(\pi+1)\frac{\cos(2n\pi)}{2n} + \frac{2}{\pi}\cdot\frac{\cos 0}{2n} + \frac{1}{n\pi}\int_0^{\pi}\cos(2nx) =$$

$$= -\frac{2}{\pi}(\pi+1)\cdot\frac{1}{2n} + \frac{1}{n\pi} + \frac{1}{n\pi}\int_0^{\pi}\left(\frac{\sin(2nx)}{2n}\right)' =$$

$$= -\frac{1}{n} - \cancel{\frac{1}{n\pi}} + \cancel{\frac{1}{n\pi}} + \frac{\sin(2nx)}{2n^2\pi}\Bigg|_0^{\pi} =$$

$$= \boxed{-\frac{1}{n}}$$

$f(x) = x+1,\ x \in [0, \pi)$



f e mărginită pe $[0, \pi)$

f admite un nr. finit de pct. de disc. de sp. I. $(x_k = k\pi,\ k \in \mathbb{Z})$

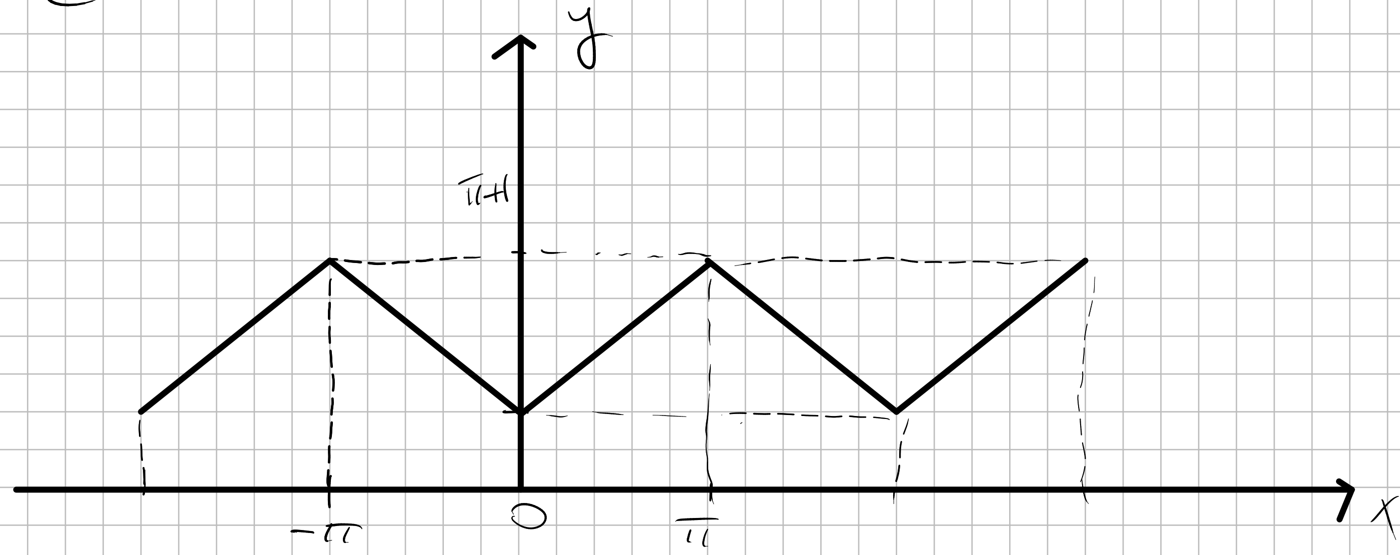
f e monotonă pe $[k\pi, (k+1)\pi)$

$$f(x) \to \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(wnx) + b_n\sin(wnx).$$

$$f(x) \to \frac{\pi+2}{2} + \sum_{n\geq 1} -\frac{1}{n}\sin(2nx) =$$

$$= \frac{\pi+2}{2} - \sum_{n\geq 1}\frac{1}{n}\sin(2nx) \stackrel{T.D.}{=}$$

$$= \begin{cases} \tilde{f}(x), & x \neq k\pi \\ \dfrac{f(0+0)+f(\pi-0)}{2} = \dfrac{1+\pi+1}{2} = 1+\dfrac{\pi}{2}, & x = k\pi \end{cases}$$



Deoarece f nu e pară, o prelung. prin paritate pe $[-\pi, 0)$ $\quad f_p(x): [-\pi;\pi] \to \mathbb{R}$

$$f_p(x) = \begin{cases} f(x), & x\in[0,\pi] \\ f(-x), & x\in[-\pi;0] \end{cases} = \begin{cases} x+1, & x\in[0,\pi] \\ -x+1, & x\in[-\pi,0] \end{cases}$$

$$\boxed{b_n = 0}$$

$$a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\,dx \qquad w = \frac{2\pi}{T} = 1$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos(wnx)\,dx \qquad T = 2\pi$$

$$a_0 = \frac{4}{2\pi}\int_0^{\pi}(x+1)\,dx = \frac{\cancel{4}^{\,2}}{2\pi}\left(\frac{x^2}{2}\Big|_0^{\pi} + x\Big|_0^{\pi}\right) =$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{2} + \pi\right) = \boxed{\pi + 2}$$

$$a_n = \frac{4}{2\pi}\int_0^{\pi}(x+1)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(x+1)\left(\frac{\sin(nx)}{n}\right)' =$$

$$= \underbrace{\frac{2}{\pi}(x+1)\frac{\sin(nx)}{n}\Big|_0^{\pi}}_{0} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin(nx)}{n}\,dx =$$

$$= 0 - \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx =$$

$$= -\frac{2}{n\pi}\int_0^{\pi}\left(-\frac{\cos(nx)}{n}\right)' = +\frac{2}{n\pi}\cdot\frac{\cos(nx)}{n}\Big|_0^{\pi} =$$

$$= \frac{2}{n^2\pi}\cdot\left(\cos(n\pi) - \cos\pi\right) = \boxed{\frac{2}{n^2\pi}\left((-1)^n - 1\right)}$$

$$(-1)^n - 1 = \begin{cases} 0, & n = 2k \\ -2, & n = 2k-1 \end{cases}$$

$$a_n = \begin{cases} 0, & n = 2k \\ -\frac{4}{n^2\pi}, & n = 2k-1 \end{cases}$$

$$f(x) = \frac{\pi+2}{2} + \sum_{n=1}^{\infty} a_n\cos(nx), (\forall)x\in[-\pi;\pi]$$

$$f(x) = \frac{\pi}{2}+1+\sum_{k=1}^{\infty} -\frac{4}{(2k-1)^2\pi}\cos(2k-1)x =$$

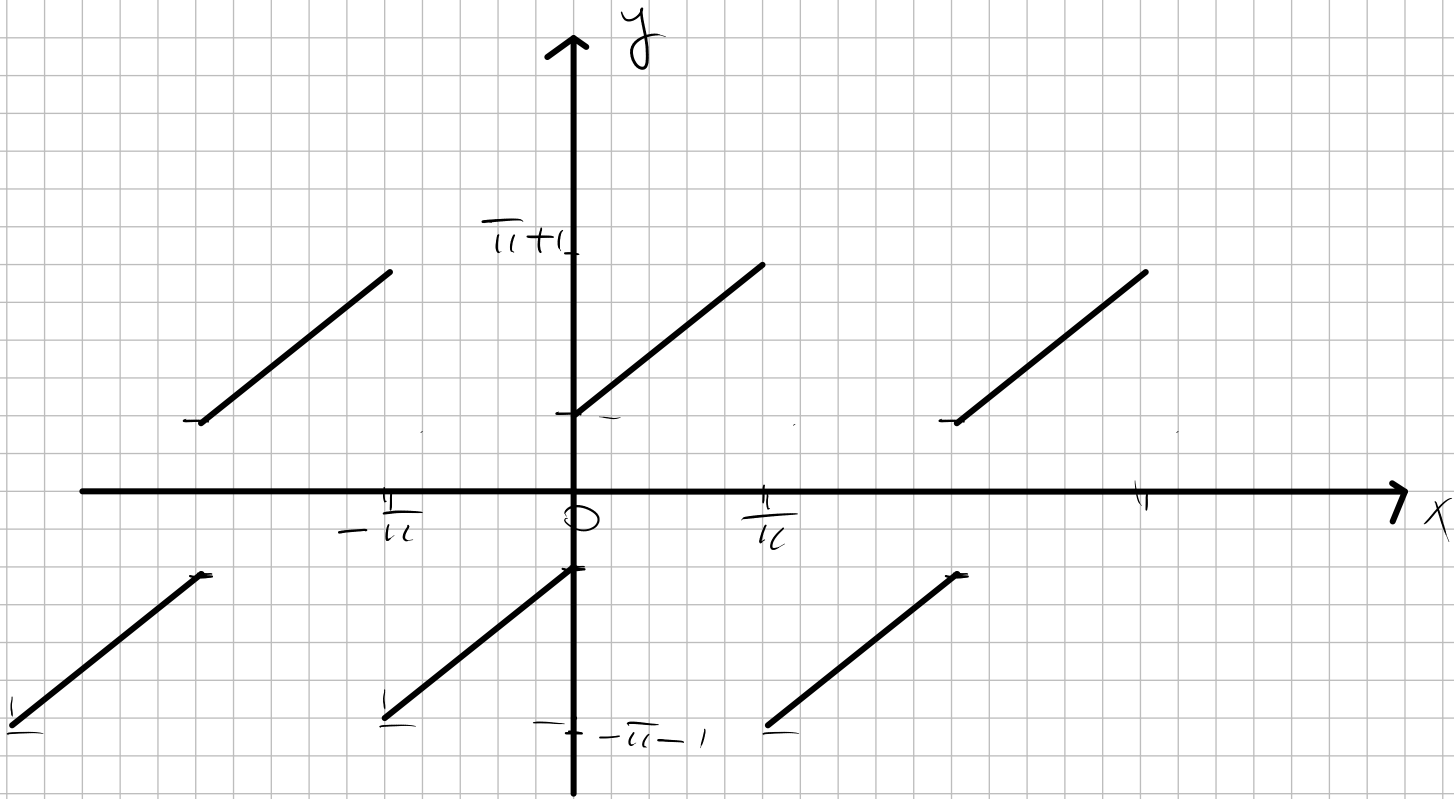
$$\boxed{= \frac{\pi}{2}+1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}}$$ – serie de cos.

Prelung. fct. prin imparitate.

$$f_i(x) = \begin{cases} f(x), & x\in[0,\pi) \\ -f(-x), & x\in(-\pi;0) \end{cases}$$

$$-f(-x) = \\ = -(-x+1) = \\ = x-1$$

$$f_i(x) = \begin{cases} x+1, & x\in[0,\pi) \\ x-1, & x\in(-\pi;0) \end{cases}$$



$$a_n = 0, \quad n \geq 0 \qquad T = 2\pi, \quad \omega = 1$$

$$b_n = \frac{4}{2\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin(nx)\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi}(x+1)\left(-\frac{\cos(nx)}{n}\right)' dx = -\frac{2}{\pi}(x+1)\frac{\cos nx}{n}\Big|_0^{\pi}$$

$$+ \frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}\,dx = -\frac{2}{\pi}(\pi+1)\frac{\cos(n\pi)}{n} + \frac{2}{\pi}\frac{\cos 0}{n} +$$

$$+ \frac{2}{n\pi}\underbrace{\int_0^{\pi}\left(\frac{\sin(nx)}{n}\right)' dx}_{0} = -2\cdot\frac{(-1)^n}{n} + \frac{2}{n\pi} - \frac{2\cdot(-1)^n}{n\pi}$$

$$b_{2n} = -\cancel{2}\cdot\frac{(-1)^{2n}}{\cancel{2}n} + \frac{2}{2n\pi} - 2\cdot\frac{(-1)^{2n}}{2n\pi} =$$

$$= -\frac{1}{n} + \cancel{\frac{1}{n\pi}} - \cancel{\frac{1}{n\pi}} = -\frac{1}{n}$$

$$b_{2n-1} = -2\cdot\frac{(-1)^{2n-1}}{2n-1} + \frac{2}{(2n-1)\pi} - 2\cdot\frac{(-1)^{2n-1}}{(2n-1)\pi} =$$

$$= -\frac{2}{2n+1} + \frac{2}{(2n-1)\pi} + \frac{2}{(2n+1)\pi} = -\frac{2}{2n-1} + \frac{4}{(2n-1)\pi}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\omega n x) = \sum_{k=1}^{\infty} -\frac{1}{2k}\cdot\sin(2kx) +$$

$$+\sum_{k=1}^{\infty} -\frac{2}{2k-1}\cdot \sin(2k-1)x + \frac{4}{(2k+1)\pi}\sin(2k-1)x =$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k} - 2\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{(2k-1)\pi} = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k} + \frac{4-2\pi}{\pi}\sum_{k=1}\frac{\sin(2k-1)x}{2k-1}$$

c) $\sum_{n\geq 1}\frac{1}{(2n-1)^2}$

$$f(x) = \frac{\pi}{2} + 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k+1)x}{(2k-1)^2}$$

$$f(0) = \frac{\pi}{2} + 1 - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2n+1)^2}$$

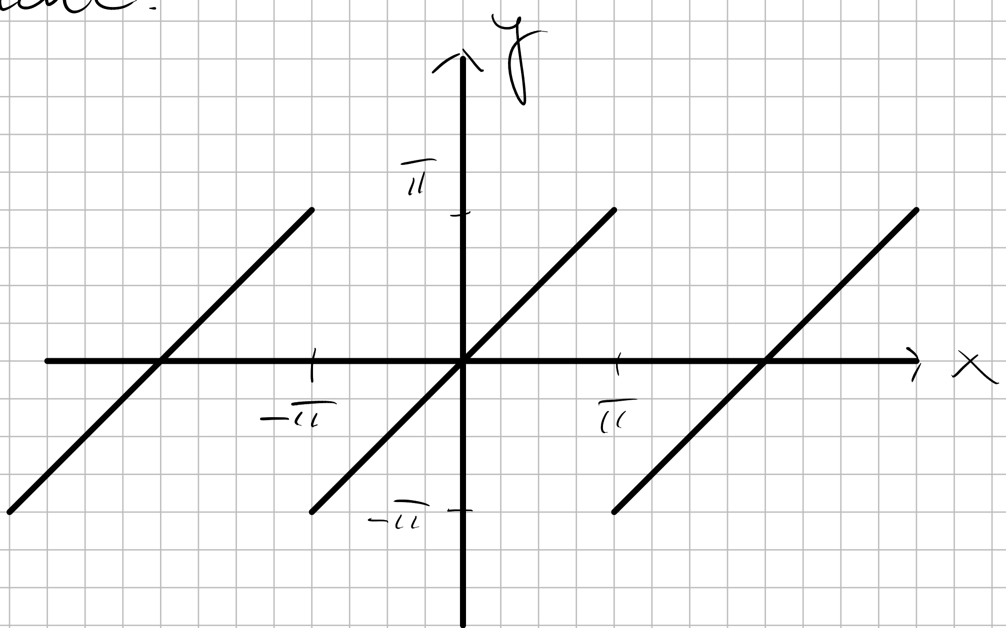
$$\cancel{1} = \frac{\pi}{2} + \cancel{1} - \frac{4}{\pi}S$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}S \implies S = -\frac{\pi}{2}\cdot\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{8}$$

$$\sum_{n\geq 1}\frac{1}{n^2}$$

3. Dezv. în serie Fourier, $T = 2\pi$

$f(x) = x$, $x \in (-\pi;\ \pi)$. Dom. de dezvoltabilitate.



T. D

$$\begin{cases} \circ f \text{ e mărginită pe } [-\pi;\pi] \\ \circ f \text{ admite un nr. finit de pct. de disc.} \\ \quad \text{de sp. I. } (x_k = k\pi) \\ \circ f \text{ e monotonă pe } [-\pi;\pi] \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} \left[a_n \cos(\omega n x) + b_n \sin(\omega n x) \right]$$

$$S(x) = \begin{cases} f(x), & x \in (-\pi;\pi) \\ \frac{f(-\pi+0)+f(\pi-0)}{2}, & x = \pm\pi \end{cases}$$

$$D = (-\pi;\pi)$$

$$S(\pm\pi) = \frac{f(-\pi+0)+f(\pi-0)}{2} = \frac{-\pi+\pi}{2} = 0 \neq f(\pm\pi) = \pm\pi$$