Spații metrice

1. X – mult. nevidă

$d: X \times X \to \mathbb{R}$

$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ e o metrică.

i) $d(x,y) = d(y,x), (\forall) x, y \in X$

ii) $d(x,y) \geq 0$

$d(x,y) = 0 \Leftrightarrow x = y$

iii) $d(x,y) + d(y,z) \geq d(x,z), (\forall) x, y, z \in X$

iii) $\begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} + \begin{cases} 0, & y = z \\ 1, & y \neq z \end{cases} \geq \begin{cases} 0, & x = z \\ 1, & x \neq z \end{cases}$

2. $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right] \subset \mathbb{R}$ $\quad d(x,y) = |\sin(x-y)|$

$d: X \times X \to \mathbb{R}$

d – metrică, $d\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

i) $d(x,y) = d(y,x), (\forall) x, y \in X$ ✓

ii) $d(x,y) \geq 0$ ✓

$d(x,y) = 0 \Leftrightarrow x = y$ ✓

iii) $d(x,y) + d(y,z) \geq d(x,z), (\forall) x, y, z \in X$

iii) $* \; |\sin(x-y)| + |\sin(y-z)| \geq |\sin(x-z)|$

$|a| + |b| \geq |a+b|$

$* \geq |\sin(x-y) + \sin(y-z)| = \left|2 \sin \frac{x-y+y-z}{2} \cos \frac{x-y-y+z}{2}\right|$

$\geq |\sin(x-z)|$

$\left|2 \sin \frac{x-z}{2} \cos \frac{x-y-y+z}{2}\right| \geq \left|2 \sin \frac{x-z}{2}\right| \geq |\sin(x-z)|$

3. Fie $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $d(\bar{x}, \bar{y}) = \sum_{i=1}^{n} |x_i - y_i|$

a) $(\mathbb{R}^n, d)$ – sp. metric.

i) $d(x,y) = d(y,x)$

ii) $d(x,y) \geq 0$

$d(x,y) = 0 \Leftrightarrow x = y$

iii) $d(x,y) + d(y,z) \geq d(x,z)$

$\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| \geq d(x,z)$

$\geq |x_i - y_i + y_i - z_i| \geq |x_i - z_i| = d(x,z)$

b) $\overline{S}_d(x_0, R) = \{ x \in M \mid d(x, x_0) \leq r \}$

$n = 2$, $x_0 = (1, -1)$, $R = 2$

$M = \mathbb{R}^2$

$d(x, x_0) = d((x,y), (1,-1)) =$

$= |x-1| + |y+1|$

$\overline{S}_d(x_0, R) = \{ x \in \mathbb{R}^2 \mid |x-1| + |y+1| \leq 2 \}$

I. $x - x + y + 1 = 2$

II. $x - 1 - y - 1 = 2$

III. $1 - x + y + 1 = 2$

IV. $x - x - y + 1 = 2$

I. $x + y = 2$

II. $x - y = 4$

III. $x = y$

IV. $x + y = -2$

c) $\mathcal{C}(x_0, R = 2)$, $x_0 = (1, -1)$

$d_e: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$, $d_e(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$d_e(x, x_0) \leq \mathbb{R}$

$\sqrt{(x-1)^2 + (y+1)^2} \leq 2 \quad |^2$

$(x-1)^2 + (y+1)^2 \leq 4$

$(x-1)^2 + (y+1)^2 \leq 4$

$$|\sin(x-y)| + |\sin(y-z)| \geq |\sin(x-z)|$$

$$|\sin(x-y)| \leq |\sin(x-z+z-y)| =$$

$$= |\sin(x-z)\cos(z-y) + \cos(x-z)\sin(z-y)| =$$

$$= |\sin(x-z)||\cos(z-y)| + |\cos(x-z)||\sin(z-y)| =$$

$$= |\sin(x-z)| + |\sin(z-y)| =$$

$$= d(x,z) + d(z,y) \geq d(x,y) \quad \checkmark$$