Limite + continuitate

1. ii) $f(x,y) = e^{-xy}$, în $(0,1)$

$$A = \{(x,y) \in \mathbb{R}^2 \mid y = mx+1,\ x \in \mathbb{R}\}$$

$$\lim_{\substack{(x,y)\to(0,1)\\(x,y)\in A}} f(x,y) = \lim_{\substack{x\to 0\\ y=mx+1}} e^{-x(mx+1)} = 1$$

$$l_{12} = \lim_{x\to 0}\left(\lim_{y\to 1} f(x,y)\right) = \lim_{x\to 0} e^{-x} = 1.$$

$$l_{21} = \lim_{y\to 1}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 1} 1 = 1.$$

$$\boxed{L = \lim_{(x,y)\to(0,1)} e^{-xy} = \lim_{(x,y)\to(0,1)} e^{-0\cdot 1} = 1.}$$

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{n+1}{n}\right) \to (0,1)$$

$$\lim_{n\to\infty} e^{-\frac{1}{n}\cdot\frac{n+1}{n}} = e^0 = 1.$$

iv) $\frac{\sin(xy)}{2x}$, în $(0,2)$

$$l_{12} = \lim_{x\to 0}\left(\lim_{y\to 2} \frac{\sin(xy)}{2x}\right) = \lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$$

$$l_{21} = \lim_{y\to 2}\left(\lim_{x\to 0} \frac{\sin(xy)}{2x}\right) = \lim_{y\to 2}\left(\lim_{x\to 0} \frac{\sin(xy)}{xy}\cdot \frac{\cancel{x}y}{2\cancel{x}}\right) = \lim_{y\to 2} \frac{y}{2} = 1.$$

$$\left|\frac{\sin(xy)}{2x} - 1\right| \le \frac{xy}{2x} - 1 \le \frac{y}{2} - 1 \xrightarrow[(x,y)\to(0,2)]{} \frac{2}{2} - 1 = 0.$$

$$g(x,y) = \frac{y}{2} - 1$$

$$\Rightarrow \lim_{(x,y)\to(0,2)} f(x,y) = 1.$$

vi) $\frac{x^4-y^4}{x^4+2x^2y^2+y^4}$ în $(0,0)$

$$l_{12}=\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^4-y^4}{x^4+2x^2y^2+y^4}\right)=\lim_{x\to 0}\frac{x^4}{x^4}=1.$$

$$l_{21}=\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4-y^4}{x^4+2x^2y^2+y^4}\right)=\lim_{y\to 0}\frac{-y^4}{y^4}=-1$$

Cum $l_{12}\neq l_{21}$ și ambele există $\Rightarrow (\not\exists)\lim\limits_{(x,y)\to(0,0)} f(x,y)$

2. i) $f(x,y)=\frac{y^2+2x}{y^2-3x}$ nu are l în $(0,0)$

$y^2\neq 3x$

$$(x_n,y_n)=\left(\frac{1}{n},\frac{1}{\sqrt{n}}\right)\to(0,0)$$

$$\lim_{n\to\infty} f(x_n,y_n)=\lim_{n\to\infty}\frac{\frac{1}{n}+\frac{2}{n}}{\frac{1}{n}-\frac{3}{n}}=\lim_{n\to\infty}\frac{\frac{3}{n}}{-\frac{2}{n}}=-\frac{3}{2}$$

$$(x_n',y_n')=\left(-\frac{1}{n},\frac{1}{\sqrt{n}}\right)\to(0,0)$$

$$\lim_{n\to\infty} f(x_n',y_n')=\lim_{n\to\infty}\frac{\frac{1}{n}-\frac{2}{n}}{\frac{1}{n}-\frac{3}{n}}=\lim_{n\to\infty}\frac{-\frac{1}{n}}{-\frac{2}{n}}=\frac{1}{2}$$

Cum cele 2 lim. sunt diferite $\xRightarrow{\text{T. Heine}} (\not\exists)\lim\limits_{(x,y)\to(0,0)} f(x,y)$

iii) $f(x,y)=\frac{y\,e^{-\frac{1}{x^2}}}{y^2+e^{-\frac{2}{x^2}}}$ în $(0,0)$

$$(x_n,y_n)=\left(\frac{1}{n},\frac{1}{e^{n^2}}\right)\to(0,0)$$

$$\lim_{n\to\infty}\frac{e^{-n^2}\cdot e^{-n^2}}{e^{-2n^2}+e^{-2n^2}}=\lim_{n\to\infty}\frac{\cancel{e^{-2n^2}}}{2\cdot \cancel{e^{-2n^2}}}=\frac{1}{2}$$

$$(x_n', y_n') = \left(-\frac{1}{\sqrt{n}}, \frac{1}{e^n}\right) \to (0,0)$$

$$\lim_{n\to\infty} \frac{e^{-n}\cdot e^{n}}{e^{-2n}-e^{2n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{e^{2n}}-e^{2n}} = \frac{1}{0-e^{\infty}} = \frac{1}{\infty} =$$

$$= 0.$$

3. iii) $f(x,y) = y\sin\frac{1}{x}\sin\frac{1}{y}$ $\qquad -y \le y\sin\frac{1}{y} \le y$

l_{12}, l_{21}, l în $(0,0)$ $\qquad \searrow \quad \downarrow \quad \swarrow$ $\quad 0$

$$l_{12} = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 0$$

$$l_{21} = \lim_{y\to 0}\left(\lim_{x\to 0} y\sin\frac{1}{x}\sin\frac{1}{y}\right)$$

$\lim\limits_{x\to 0} \sin\frac{1}{x}$, nu există $\quad g(x) = \sin\frac{1}{x}$

$$\left.\begin{aligned} x_n = \frac{1}{2n\pi} \quad & \lim_{n\to\infty} g(x_n) = \sin\frac{1}{\frac{1}{2n\pi}} = \sin 2n\pi = 0 \\ x_n' = \frac{2}{\pi} \quad & \lim_{n\to\infty} g(x_n') = \sin\frac{1}{\frac{2}{\pi}} = \sin\frac{\pi}{2} = 1 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow (\not\exists) \lim\limits_{x\to 0} \sin\frac{1}{x}$, $(\not\exists)\ l_{21}$

$$\left|y\sin\frac{1}{x}\sin\frac{1}{y} - 0\right| \le |y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

v) $f(x,y) = \dfrac{x^2-2y^2}{2x^2+y^2}$

$$\left.\begin{aligned} l_{12} &= \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2-2y^2}{2x^2+y^2}\right) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \\ l_{21} &= \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x^2-2y^2}{2x^2+y^2}\right) = \lim_{y\to 0} \frac{-2y^2}{y^2} = -2 \end{aligned}\right\} \Rightarrow l_{12} \ne l_{21} \text{ și ambele există}$$

$\Rightarrow (\not\exists)\ l$

$$f(x,y) = \frac{e^{x^2y}-1}{x^2y^2} = \frac{e^{x^2y}-1}{x^2y}\cdot\frac{1}{y}$$

$$x_n = \left(\frac{1}{\sqrt{n}}, \frac{1}{n}\right) \to (0,0)$$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{e^{\frac{1}{n^2}}-1}{\frac{1}{n^2}}\cdot n = \ln e\cdot\infty = \infty$$

3. iii) $\displaystyle\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{x+y}{x^2+y^2} = 0$

Pp. $\displaystyle l = \lim_{\substack{x\to\infty\\ y\to\infty}} f(x,y) = 0$

$$\left|\frac{x+y}{x^2+y^2}\right| \le \left|\frac{x}{x^2+y^2}\right| + \left|\frac{y}{x^2+y^2}\right| \le \frac{x}{2xy} + \frac{y}{2xy} = \frac{1}{2y} + \frac{1}{2x}$$

$$\left|\frac{x+y}{x^2+y^2}\right| \le \left|\frac{x+y}{2xy}\right| = \frac{1}{2y} + \frac{1}{2x} \xrightarrow[\substack{x\to\infty\\ y\to\infty}]{} 0$$

5. $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \sqrt{\frac{x^2+y^2-x}{2x-x^2-y^2}}$$

a) D ?!

$$\frac{x^2+y^2-x}{2x-x^2-y^2} \ge 0 \qquad x^2+y^2-x \ge 0$$

$$2x - x^2 - y^2 \ne 0$$

b) $$l_{12} = \lim_{x\to 0}\left(\lim_{y\to 0}\sqrt{\frac{x^2+y^2-x}{2x-x^2-y^2}}\right) = \lim_{x\to 0}\sqrt{\frac{x^2-x}{2x-x^2}} =$$

$$= \lim_{x\to 0}\sqrt{\frac{\cancel{x}(x-1)}{\cancel{x}(2-x)}} = \cancel{\dots (\nexists)}$$

$$l_{21} = \lim_{y\to 0}\left(\lim_{x\to 0}\sqrt{\frac{x^2+y^2-x}{2x-x^2-y^2}}\right) = \lim_{y\to 0}\sqrt{\frac{y^2}{-y^2}}$$

c) $A = \{(x,y)\in\mathbb{R}^2 \mid y = mx,\ x\in\mathbb{R}\}$

$$\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\in A}} f(x,y) = \lim_{\substack{x\to 0\\ y=mx}} \sqrt{\frac{x^2+(mx)^2-x}{2x-x^2-(mx)^2}} =$$

$$= \lim_{\substack{x \to 0 \\ y = mx}} \sqrt{\frac{\cancel{x}(x + m^2 x - 1)}{\cancel{x}(2 - x - m^2 x)}} = \sqrt{-\frac{1}{2}}$$

6. $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$x^2 + y^4 = 0$

$\Leftrightarrow x = 0$ și $y = 0$

$x^2 + (y^2)^2 = 0$

$D = \mathbb{R}^*$

i) f - cont parțial în $(0,0)$

$$\left.\begin{aligned} \lim_{x \to 0} f(x,0) = 0 \\ \lim_{y \to 0} f(0,y) = 0 \end{aligned}\right\} = f(0,0)$$

$\Leftrightarrow f$ - parțial cont

ii) f - discont.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

$$\left|\frac{xy^2}{x^2+y^4} - 0\right| \le \left|\frac{xy^2}{2xy^2}\right| \le \frac{1}{2} \underset{(x,y)\to(0,0)}{\not\to} 0$$

$\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 \Rightarrow f$ e discontinuă

iii) f e mărginită pe $\mathbb{R}^2$

$$\frac{xy^2}{x^2+y^4} \le \frac{xy^2}{2xy^2} = \frac{1}{2}$$

$$\frac{1}{x^2+y^4} \le -\frac{1}{2xy^2} \,\Big|\cdot xy^2$$

$$\frac{xy^2}{x^2+y^4} \le -\frac{xy^2}{2xy^2}$$

$$x^2 + y^4 \ge -2xy^2$$

$$\frac{xy^2}{x^2+y^4} \ge -\frac{xy^2}{2xy^2} = -\frac{1}{2}$$

$$x^2 + 2xy^2 + y^4 \ge 0$$

$$(x+y^2)^2 \ge 0$$

f e mărg. între $\left[-\frac{1}{2}; \frac{1}{2}\right]$

8. $f(x,y) = \begin{cases} \frac{\sin(x^3+y^3)}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$\sin x \le x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^3+y^3} \cdot \left(\frac{x^3+y^3}{x^2+y^2}\right) \to 0$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| = \frac{x^3}{2xy}$$

$$|x| \cdot \underbrace{\left|\frac{x^2}{x^2+y^2}\right|}_{\le 1} + |y| \cdot \underbrace{\left|\frac{y^2}{x^2+y^2}\right|}_{\le 1} = |x|+|y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow f$ e continuă

b) $$f(x,y) = \begin{cases} \dfrac{1-\cos(x^3y^3)}{x^2+2y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{1-\cos(x^3y^3)}{(x^3y^3)^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{x^6y^6}{x^2+2y^4}}_{\to 0}$$

$$\left|\frac{x^6y^6}{x^2+2y^4} - 0\right| \le \left|\frac{x^6y^6}{4xy^2}\right| = \frac{x^5y^4}{4} \xrightarrow[(x,y)\to(0,0)]{} 0.$$

c) $$f(x,y) = \begin{cases} \dfrac{e^{xy^2}-\cos(xy)}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy^2}-\cos(xy)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{e^{xy^2}-1+1-\cos(xy)}{x^2+y^2} =$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{e^{xy^2}-1}{xy^2} \cdot \frac{xy^2}{x^2+y^2} + \frac{1-\cos xy}{x^2y^2} \cdot \frac{x^2y^2}{x^2+y^2}\right) =$$

$$= 1 \cdot 0 + \frac{1}{2} \cdot 0 = 0$$

$$\left|\frac{xy^2}{x^2+y^2} - 0\right| \le \left|\frac{xy^2}{2xy}\right| = \frac{y}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| \le \left|\frac{x^2y^2}{2xy}\right| = \frac{xy}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$f(x,y) = \dfrac{e^{x^2y}-1}{x^2y^2}$ nu are lim. în $(0,0)$

$$\lim_{(x,y)\to(0,0)} \left(\underbrace{\frac{e^{\overset{\to 0}{x^2y}}-1}{x^2y}}_{\to 1} \cdot \frac{1}{y} \right) \quad ?!?$$