Limite iterate

1. $f: \mathbb{R}^2 \to \mathbb{R}$, $l_{12} \neq l_{21}$ în $(0,0)$

Ce puteți spune despre limita în rap. cu ansamblul variabilelor?

$$\boxed{f = \frac{x-y}{x+y}}$$

$$A = \{(x,y) \in \mathbb{R}^2 \mid y = mx,\ m \in \mathbb{R}\}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A}} f(x,y) = \lim_{\substack{x\to 0\\ y=mx}} \frac{x-mx}{x+mx} = \lim_{x\to 0} \frac{\cancel{x}(1-m)}{\cancel{x}(1+m)} =$$

$$= \frac{1-m}{1+m}, \text{ deci } (\nexists)\ l$$

Metrică: o aplicație $d: X \times X \to \mathbb{R}$ cu prop:
- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0$

  $d(x,y) = 0 \iff x = y$
- $d(x,y) + d(y,z) \leq d(x,z)$

Ex: $d_e: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $d_e(\bar{x},\bar{y}) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}$

$d_e: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $d_e(\bar{x},\bar{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

2. l_{12}, $l_{21} (\nexists)$, are lim. în rap. cu ans. var.

$$\boxed{f(x,y) = y^2 \cos\frac{1}{x^2} \cos\frac{1}{y}}$$

$l_{12} = 0$, $l_{21} (\nexists)$, $l = 0$.

$$l_{12} = \lim_{x\to 0}\left(\lim_{y\to 0} y^2 \cos\frac{1}{x^2} \cos\frac{1}{y}\right) = 0.$$

$$\lim_{y\to 0} y^2 \cos\frac{1}{y} = 0$$

$$-y^2 \le y^2 \cos\frac{1}{y} \le y^2$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

$$l_{21} = \lim_{y\to 0}\left(\lim_{x\to 0} y^2 \cos\frac{1}{x^2}\cos\frac{1}{y}\right) = (\nexists)$$

$$\lim_{x\to 0} \cos\frac{1}{x^2} \quad (\nexists)$$

$$x_n = \frac{1}{\sqrt{2n\pi}} \longrightarrow 0$$

$$\lim_{n\to\infty} \cos\frac{1}{\frac{1}{2n\pi}} = \lim_{n\to\infty} \cos(2n\pi) = 1$$

$$y_n = \frac{1}{\sqrt{(2n+1)\pi}} \longrightarrow 0$$

$$\lim_{n\to\infty} \cos\frac{1}{\frac{1}{(2n+1)\pi}} = \lim_{n\to\infty} \cos(2n+1)\pi = -1$$

$$A = \{(x,y)\in\mathbb{R}^2 \mid y = mx \ ,\ m\in\mathbb{R}\}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A}} f(x,y) = \lim_{\substack{x\to 0\\ y=mx}} y^2\cos\frac{1}{x^2}\cos\frac{1}{y} =$$

$$= \lim_{x\to 0} (mx)^2 \underbrace{\cos\frac{1}{x^2}\cos}\frac{1}{mx} =$$

$$= \lim_{x\to 0} (mx)^2 \frac{1}{2}\left[\cos\left(\frac{1}{x^2}-\frac{1}{mx}\right)+\cos\left(\frac{1}{x^2}+\frac{1}{mx}\right)\right] =$$

$$= \lim_{x\to 0} \frac{(mx^2)}{2}\cos\left(\frac{1}{x^2}-\frac{1}{mx}\right) + \lim_{x\to 0}\frac{(mx)^2}{2}\cos\left(\frac{1}{x^2}+\frac{1}{mx}\right) =$$

$$-\frac{m^2x^2}{2} \le \frac{m^2x^2}{2}\cos\left(\frac{1}{x^2}\pm\frac{1}{mx}\right) \le \frac{m^2x^2}{2}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

$\Rightarrow \ell = 0$

Derivate parțiale mixte de ordinul 2 pt. o fct în 3 variabile în pct. (a,b,c)

$$\frac{\partial^2 f}{\partial x^i \partial y^j \partial z^k}(a,b,c), i+j+k=2.$$

3. $\ell_{12} = \ell_{21} = 0$.

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$\ell_{12} = \ell_{21} = 0$

$A = \{(x,y) \in \mathbb{R}^2 \mid y = mx, m \in \mathbb{R}\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=mx}} \frac{x \cdot mx}{x^2+m^2x^2} = \lim_{x\to 0} \frac{mx^2}{(m^2+1)x^2} =$$

$$= \frac{m}{m^2+1} \neq 0 \Rightarrow (\nexists)\ \ell$$

Teorema lui Heine

Fie $f: \mathbb{R}^2 \to \mathbb{R}$, $\ell = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

Fie $x_n = (x_{n_1}, x_{n_2}) \xrightarrow[n\to\infty]{} (x_0, y_0)$

$y_n = (y_{n_1}, y_{n_2}) \xrightarrow[n\to\infty]{} (x_0, y_0)$

Dacă $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ Atunci

$(\nexists)\ \ell = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$

4. l_{12} ($\exists$), l_{21} ($\nexists$), l ($\nexists$)

$$f(x,y)=\frac{x^4+y^3}{x^4+y^4}$$

$$l_{12}=\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=1.$$

$$l_{21}=\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)=\lim_{y\to 0}\frac{1}{y}, \ (\nexists)$$

$$l=\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A}} f(x,y)=\lim_{\substack{x\to 0\\ y=mx}}\frac{x^4+(mx)^3}{x^4+\underbrace{(mx)^4}_{x^3}}=$$

$$\frac{16}{3}-4=\frac{4}{3} \qquad y=mx^{\frac{4}{3}} \qquad \frac{x^4+\left[(mx)^{\frac{4}{3}}\right]^3}{x^4+\left[(mx)^{\frac{16}{3}}\right]^4}=\frac{x^4+m^4x^4}{x^4+m^{\frac{4}{3}}x^{\frac{16}{3}}}=$$

$$=\frac{\cancel{x^4}(1+m^4)}{\cancel{x^4}\left(1+m^{\frac{4}{3}}x^{\frac{4}{3}}\right)}\xrightarrow[x\to 0]{}\frac{1+m^4}{1+m^{\frac{4}{3}}\cdot 0}=1+m^4\neq 0.$$

T. lui Schwarz.

Dacă una dintre derivatele parțiale mixte e continuă într-un punct at.

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(x,y)=\frac{\partial^2 f}{\partial x_j\partial x_i}(x,y)$$

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{\sqrt{x^2+y^4}}, & (x,y)\neq(0,0)\\[2ex] 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x,y)=0.$$

$$\left|\frac{x^2y^3}{\sqrt{x^2+y^4}}-0\right|\leq\sqrt{\frac{x^4y^6}{x^2+y^4}}\leq\sqrt{\frac{x^4y^6}{2xy^2}}\leq\sqrt{\frac{x^3y^4}{2}}\to 0$$

(7) $w : \mathbb{R}^2 \to \mathbb{R}$, nulă și continuă în $(0,0)$

a.î. $f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) +$

$+ w(x,y)\sqrt{(x-0)^2+(y-0)^2}$

$f(0,0) = 0$

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$w(x,y) = \frac{\frac{x^2y^3}{\sqrt{x^2+y^4}}}{\sqrt{x^2+y^2}} = \frac{x^2y^3}{\sqrt{x^2+y^4}\cdot\sqrt{x^2+y^2}}$

$\lim_{(x,y)\to(0,0)} w(x,y) = 0$

$\left| \frac{x^2y^3}{\sqrt{x^2+y^4}\cdot\sqrt{x^2+y^2}} - 0 \right| \leq \frac{x^2y^3}{\sqrt{2xy^2}\cdot\sqrt{2xy}} \leq \frac{x^2y^3}{2xy\sqrt{y}} =$

$= \frac{xy^{\frac{3}{2}}}{2} \xrightarrow[(x,y)\to(0,0)]{} 0$