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Analiză matematică

(seminar 1 - S1)

28.09.2023

$N.F. = \frac{2 \cdot N.E + N.S.}{3}$

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Recapitulare șiruri numerice

Criteriul rădăcinii/radicalului

Fie $(a_n)_{n\geq 1}$ un șir de numere reale pozitive. Dacă există $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \in [0,\infty)$, atunci $\lim_{n\to\infty} \sqrt[n]{a_n} = l$.

1. $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$

Fie $b_n = \sqrt[n]{\frac{n!}{n^n}} \Rightarrow a_n = \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \overset{1^\infty}{=}$$

$$\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}} = e^{-1} = \frac{1}{e}$$

$\overset{\text{crt. rad}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e}$

Lema Stolz - Cesàro

Fie $(x_n)_{n\geq 1}$ un șir de forma $x_n = \frac{a_n}{b_n}$; $a_n, b_n \in \mathbb{R}$, $b_n \neq 0$, $n \in \mathbb{N}$. Dacă a) (b_n) - șir strict monoton

b) $b_n \to \infty$ sau $a_n \to 0$, $b_n \to 0$

atunci $\exists \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = x \in \mathbb{R} \cup \{\pm\infty\} \Rightarrow \lim_{n\to\infty} \underset{\frac{a_n}{b_n}}{x_n} = x$

2. $\lim_{n\to\infty} \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} = \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}}{n+1-n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \overset{1}{=} \frac{1}{e}$

$a_n = \sqrt[n]{n!}$

$b_n = n \nearrow \infty$

Criteriul cleștelui

Fie (x_n) un șir de numere reale. Dacă $\exists$ două șiruri $(a_n)_{n\geq n_0}$ și $(b_n)_{n\geq n_0}$ care au aceeași limită, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = x$, a.î.

$a_n \leq x_n \leq b_n$, pt $n \geq n_0$, atunci șirul (x_n) are limită:

$$\lim_{n\to\infty} x_n = x$$

3. $$\lim_{n\to\infty} \underbrace{\left(\frac{1^2}{n^3+3}+\frac{2^2}{n^3+6}+\ldots+\frac{n^2}{n^3+3n}\right)}_{S_n} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{k^3+3k}$$

$$\sum_{k=1}^{n}\frac{k^2}{n^3+3} \geq S_n \geq \sum_{k=1}^{n}\frac{k^2}{n^3+3n}$$

$$\Leftrightarrow \frac{1}{n^3+3}\cdot\frac{\overset{2n^3}{n(n+1)(2n+1)}}{6} \geq S_n \geq \frac{1}{n^3+3n}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\frac{1}{n^3+3}\cdot\frac{n(n+1)(2n+1)}{6} \xrightarrow{n\to\infty} \frac{1}{3}, \quad S_n \to \frac{1}{3}, \quad \frac{1}{n^3+3n}\cdot\frac{n(n+1)(2n+1)}{6} \xrightarrow{n\to\infty} \frac{1}{3}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{3}$$

4. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{49k^2+7k-12}$$

$$49k^2+7k-12 = (7k-3)(7k+4)$$

$$\frac{1}{(7k-3)(7k+4)} = \frac{A}{7k-3}+\frac{B}{7k+4} \quad \Big|\cdot(7k-3)(7k+4)$$

$$1 = A\cdot(7k+4)+B(7k-3) = k\cdot 7(A+B)+4A-3B$$

$$\Rightarrow \begin{cases} A+B=0 \Rightarrow B=-A \\ 4A-3B=1 \end{cases} \Rightarrow 7A=1 \Rightarrow A=\frac{1}{7} \Rightarrow B=-\frac{1}{7}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{7}\sum_{k=1}^{n}\frac{1}{7k-3}-\frac{1}{7k+4} = \frac{1}{7}\lim_{n\to\infty}\left(\frac{1}{4}-\cancel{\frac{1}{11}}+\cancel{\frac{1}{11}}-\cancel{\frac{1}{18}}+\ldots+\cancel{\frac{1}{7n+3}}-\frac{1}{7n+4}\right)$$

$$= \frac{1}{7}\lim_{n\to\infty}\left(\frac{1}{4}-\underbrace{\frac{1}{7n+4}}_{\to 0}\right) = \frac{1}{7}\cdot\frac{1}{4} = \frac{1}{28}$$

(5.) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{2k+1}{2^k(k+1)!}$

$$\frac{2k+1}{2^k(k+1)!} = \frac{2k+2-1}{2^k(k+1)!} = \frac{2\cancel{(k+1)}}{2^k\cancel{(k+1)!}_{k!}} - \frac{1}{2^k(k+1)!} = \frac{1}{2^{k-1}k!} - \frac{1}{2^k(k+1)!}$$

$$\Rightarrow S = \frac{1}{1\cdot 1!} - \cancel{\frac{1}{2^1 2!}} + \cancel{\frac{1}{2^1\cdot 2!}} - \cancel{\frac{1}{2^2\cdot 3!}} + \ldots + \cancel{\frac{1}{2^{n-1}\cdot n!}} - \frac{1}{2^n(n+1)!} =$$

$$= 1 - \frac{1}{2^n(n+1)!} \xrightarrow{n\to\infty} 0$$

$\lim\limits_{n\to\infty} S = 1$

Criteriul majorării

Fie (x_n) un șir de numere reale.

(1) Dacă $\exists\, x \in \mathbb{R}$ și $\exists\, (y_n)_{n\geq n_0} \subset \mathbb{R}_+$ cu $y_n \to 0$ a.î.

$$|x_n - x| \leq y_n \text{ atunci } x_n \to 0 \ (\lim_{n\to\infty} x_n = x)$$

(2) Dacă $\exists\, (y_n)_{n\geq n_0}$ cu $y_n \to \infty$ a.î.

$$x_n \geq y_n \text{ atunci } x_n \to \infty$$

(3) Dacă $\exists\, (y_n)_{n\geq n_0}$ cu $y_n \to -\infty$ a.î.

$$x_n \leq y_n \text{ atunci } x_n \to -\infty$$

Criteriul raportului

Fie (x_n) un șir de nr reale pozitive a.î. $\exists \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = l \in [0,\infty)$

a) $l < 1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

b) $l > 1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

c) $l = 1$ limita nu poate fi determinată cu acest criteriu

Criteriul raportului generalizat

Fie (x_n) un șir de nr. reale pozitive a.î. $\exists$

$$\lim_{n\to\infty}\left[\left(\frac{x_{n+1}}{x_n}\right)^n\right] = l \in [0, \infty)$$

a) $l < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

b) $l > 1 \Rightarrow \lim_{n\to\infty} x_n = \infty$

Analiză matematică
(seminar 2 - S2)

05.10.2023

① Constanta Euler - Mascheroni

$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \quad , n \geq 2$ ⓘ convergent

$\left(1+\frac{1}{k}\right)^{k} < e < \left(1+\frac{1}{k}\right)^{k+1} \quad \Big| \ln$

$\Leftrightarrow k \ln\left(1+\frac{1}{k}\right) < 1 < (k+1)\ln\left(1+\frac{1}{k}\right)$

$\Leftrightarrow k < \frac{1}{\ln\left(1+\frac{1}{k}\right)} < k+1$

$\Leftrightarrow \underbrace{\frac{1}{k+1} < \ln\left(1+\frac{1}{k}\right) < \frac{1}{k}}_{(1)} \quad \Big| \sum_{k=1}^{n-1}$

$\Leftrightarrow \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \dots + \ln n - \cancel{\ln(n-1)} <$
$< 1 + \frac{1}{2} + \dots + \frac{1}{n-1}$

$\Leftrightarrow \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$

$\left.\begin{array}{l} \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n < 0 \;\; \big| +1 \Rightarrow \gamma_n < 1 \\ 0 < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} - \ln n \;\; \big| + \frac{1}{n} \Rightarrow 0 < \frac{1}{n} < \gamma_n \end{array}\right\} \Rightarrow \gamma_n \in (0,1)$

$\Rightarrow \gamma_n$ - mărginit (a)

$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} - \ln\left(1+\frac{1}{n}\right) \overset{(1)}{<} 0$

$\Rightarrow \gamma_n$ - strict descrescător (b)

din (a) și (b) $\Rightarrow \gamma_n$ - convergent

Obs: $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$

$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n$

Criteriul raportului ② $\lim\limits_{n\to\infty} \frac{(2n-1)!!}{2n!!}$

$\lim\limits_{n\to\infty} \frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{2n!!}{(2n-1)!!} = \lim\limits_{n\to\infty} \frac{2n+1}{2n+2} = 1 \Rightarrow$ încercăm cu

Criteriul raportului generalizat

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{-1}{2n+2}\right)^{-(2n+2)}\right]^{\frac{-n}{2n+2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} < 1$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

Şir Cauchy

(x_n) - şir fundamental (Cauchy) $\Leftrightarrow \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq n_\varepsilon$, $\forall p \in \mathbb{N}^*$ $|x_{n+p}-x_n| < \varepsilon$.

3. $x_n = \sum_{k=1}^{n} \frac{1}{k^4}$

$$\frac{1}{k^4} \leq \frac{1}{k^2} \leq \frac{1}{(k-1)k}$$

$$|x_{n+p}-x_n| = \left|\sum_{k=n+1}^{n+p} \frac{1}{k^4}\right| = \sum_{k=n+1}^{n+p}\frac{1}{k^4} \leq \sum_{k=n+1}^{n+p}\frac{1}{k^2} \leq \sum_{k=n+1}^{n+p}\frac{1}{k(k-1)} =$$

$$= \sum_{k=n+1}^{n+p} \frac{1}{k-1} - \frac{1}{k} = \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \dots + \cancel{\frac{1}{n+p-1}} - \frac{1}{n+p} =$$

$$= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} < \varepsilon$$

$$\Rightarrow n > \frac{1}{\varepsilon}$$

Fie $n_\varepsilon = \left[\frac{1}{\varepsilon}\right]+1 \Rightarrow x_n$ - şir fundamental

$\Rightarrow (x_n)$ - convergent

4. $y_n = \sum_{k=1}^{n} \frac{\cos(kx)}{2^k}$, $x \in \mathbb{R}$ $\qquad \cos^2 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$\cos x \in [-1,1]$ $\quad |\cos x| \leq 1$, $\forall x \in \mathbb{R}$

$$|x_{n+p}-x_n| = \left|\sum_{k=n+1}^{n+p}\frac{\cos kx}{2^k}\right| \leq \sum_{k=n+1}^{n+p}\left|\frac{\cos kx}{2^k}\right| \leq \sum_{k=n+1}^{n+p}\frac{1}{2^k} =$$

$$= \frac{1}{2^{n+1}}\left(1+\frac{1}{2}+\dots+\frac{1}{2^{p-1}}\right) = \frac{1}{2^{n+1}}\cdot 1 \cdot \frac{1-\left(\frac{1}{2}\right)^p}{1-\frac{1}{2}} = \frac{1}{2^n}\cdot\left(1-\left(\frac{1}{2}\right)^p\right) < \frac{1}{2^n} < \varepsilon$$

$$2^n > \frac{1}{\varepsilon} \;|\; \log_2$$

$$n > \log_2 \frac{1}{\varepsilon}$$

Fie $n_\varepsilon = \left[\log_2 \frac{1}{\varepsilon}\right]+1 \Rightarrow y_n$ - şir fundamental

$\Rightarrow (y_n)$ - convergent

5. Dem că șirul $x_n = \sum_{k=1}^{n} \frac{1}{k}$ nu e fundamental

$$|x_{n+p} - x_n| = \sum_{k=n+1}^{n+p} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} > \underbrace{\frac{1}{n+p} + \dots + \frac{1}{n+p}}_{\text{de } n \text{ ori}} = \frac{n}{n+p}$$

pt. $p = n \Rightarrow |x_{n+p} - x_n| > \frac{n}{2n} = \frac{1}{2} > \varepsilon$

$\Rightarrow$ șirul nu e fundamental

! $\exists\, \varepsilon > 0$ a.î. $\forall\, n_\varepsilon \in \mathbb{N}$ cu $n \geq n_\varepsilon$ și $\exists\, p \in \mathbb{N}^*$ a.î. $|x_{n+p} - x_n| > \varepsilon$

Analiză matematică 12.10.2023

(seminar 3 - S3)

Obs: $S_n = \sum_{k=1}^{n} a_k$

$$\lim_{n\to\infty} S_n = S \begin{cases} \exists \text{ și e finită} \Rightarrow \sum a_n \in C \\ \nexists \text{ sau e } \pm\infty \Rightarrow \sum a_n \in D \end{cases}$$

① $\sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) = \sum_{n\geq 2} \ln\left(\frac{n-1}{n}\right) = \sum_{n\geq 2}\left[\ln(n-1) - \ln n\right] =$

$= \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \ldots + \cancel{\ln(n-1)} - \ln n = -\ln n = S_n$

$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln \underbrace{\frac{1}{n}}_{\to 0} = -\infty \Rightarrow \sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) \in D$

② $\sum_{n\geq 1}^{n} \sin\frac{3}{2^{n+2}} \sin\frac{1}{2^{n+2}}$ $\qquad \sin a \sin b = \frac{1}{2}\left[\cos(a-b) \cos(a+b)\right]$

$\lim_{n\to\infty} \sin\frac{3}{2^3} \sin\frac{1}{2^3} + \sin\frac{3}{2^4}\cdot\sin\frac{1}{2^4} + \ldots + \sin\frac{3}{2^{n+2}}\sin\frac{1}{2^{n+2}} =$

$= \lim_{n\to\infty}\frac{1}{2}\left[\cos\frac{2}{2^3} - \cos\frac{2^{\cancel{2}}}{2^{\cancel{3}}} + \cos\frac{2}{2^4} - \cos\frac{2^2}{2^4} + \cos\frac{2}{2^5} - \cos\frac{2^2}{2^5} + \ldots + \cos\frac{2}{2^{n+2}} - \cos\frac{2^2}{2^{n+2}}\right] =$

$= \frac{1}{2}\lim_{n\to\infty}\left[\cancel{\cos\frac{1}{2^2}} - \cos\frac{1}{2} + \cancel{\cos\frac{1}{2^3}} - \cancel{\cos\frac{1}{2^2}} + \cancel{\cos\frac{1}{2^4}} - \cancel{\cos\frac{1}{2^3}} + \ldots + \cos\frac{1}{2^{n+1}} - \cancel{\cos\frac{1}{2^n}}\right] =$

$= \frac{1}{2}\lim_{n\to\infty} \underbrace{\cos \underbrace{\frac{1}{2^{n+1}}}_{\to 0}}_{\to 1} - \cos\frac{1}{2} = \frac{1}{2}\left[1 - \cos\frac{1}{2}\right] \Rightarrow \sum a_n \in C$

Să se studieze natura următoarelor serii folosind def. necesară de convergență a unei serii.

Condiția necesară de convergență a unei serii este ca termenul general să tindă la 0.

1) Dacă termenul general nu tinde la 0, seria este divergentă.

2) Dacă termenul general tinde la 0, nu putem afirma natura seriei (studiem fiecare caz în parte).

① $\sum_{n=1}^{\infty} n^2 \ln \cos \frac{2\pi}{3n}$

$\cos 2x = 1 - 2\sin^2 x$

$$\lim_{n\to\infty} n^2 \ln\left(1 - 2\sin^2 \frac{\pi}{3n}\right) = \lim_{n\to\infty} \cancel{n^2} \underbrace{\frac{\ln\left(1-2\sin^2\frac{\pi}{3n}\right)}{-2\sin^2\frac{\pi}{3n}}}_{\to 1} \cdot \frac{-2\overbrace{\sin^2\frac{\pi}{3n}}^{\to 1}}{\frac{\pi^2}{9n^2}} \cdot \frac{\pi^2}{9\cancel{n^2}}$$

$$= -2\frac{\pi^2}{9} \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ - divergentă}$$

Deoarece nu este satisfăcută condiția necesară de convergență a unei serii, atunci seria este divergentă.

② $\sum_{n=1}^{\infty} \left(1 + \sin\frac{\pi}{n}\right)^n$

$$1^\infty \quad \lim_{n\to\infty}\left[\left(1+\sin\frac{\pi}{n}\right)^{\frac{1}{\sin\frac{\pi}{n}}}\right]^{n\sin\frac{\pi}{n}} = e^{\lim_{n\to\infty} \cancel{n} \cdot \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \cdot \frac{\pi}{\cancel{n}}} = e^{\pi} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ - divergentă}$$

③ Folosind criteriul general de convergență a lui Cauchy, să se studieze natura seriei.

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+n}$$

Condiția necesară și suficientă ca o serie să fie convergentă este ca șirul sumelor parțiale să fie șir fundamental.

$$|S_{n+p} - S_n| = \left|\sum_{k=n+1}^{n+p} \frac{\sin kx}{k^2+k}\right| \leq \sum_{k=n+1}^{n+p} \left|\frac{\sin kx}{k(k+1)}\right| \leq \sum_{k=n+1}^{n+p}\left(\frac{1}{k} - \frac{1}{k+1}\right) =$$

$$= \frac{1}{n+1} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}} + \dots + \cancel{\frac{1}{n+p}} - \frac{1}{n+p+1} =$$

$$= \frac{1}{n+1} - \frac{1}{n+p+1} \leq \frac{1}{n+1} < \frac{1}{n} < \varepsilon \quad \text{pt } n > \frac{1}{\varepsilon}$$

Fie $n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1 \Rightarrow S_n$ - șir fundamental $\Rightarrow \sum a_n$ - conv.

Studiați natura urm. serii folosind criteriile de comparație

↓

rad. / raport

① $\sum_{n=1}^{\infty} 3^n \sin \frac{\pi}{5^n}$

$\sin x \leq x$

a) Dacă $\sum y_n$ - conv $\Rightarrow \sum x_n$ - conv.

$0 < x_n < y_n$

$\sum x_n$

$\sum y_n$

b) Dacă $\sum x_n$ - div. $\Rightarrow \sum y_n$ - div.

$$3^n \sin \frac{\pi}{5^n} \leq \frac{\pi}{5^n} \cdot 3^n = \pi \left(\frac{3}{5}\right)^n \overset{not}{=} y_n$$

$\sum_{n\geq 1} \pi \left(\frac{3}{5}\right)^n$ - convergentă (s. geometrică cu $q = \frac{3}{5} < 1$)

$\overset{crt.}{\underset{comp.}{\Rightarrow}} \sum x_n$ - conv.

$$\sum_{n=0}^{\infty} q^n \begin{cases} \text{conv} & |q| < 1 \\ \text{div} & |q| > 1 \end{cases}$$ seria geometrică

② $\sum_{n=1}^{\infty} \frac{3n+7}{5n^4+3n+7}$

Fie $y_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{3n^4+7n^3}{5n^4+3n+7} = \frac{3}{5} \in (0,\infty) \overset{crt.\ comp}{\underset{la\ lim}{\Longrightarrow}} \sum x_n \sim \sum y_n$$

$\sum \frac{1}{n^3}$ - conv (seria armonică generalizată cu $p=3$)

$\Rightarrow \sum x_n$ - conv

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{convergentă pt } p > 1 \\ \text{divergentă pt } p \leq 1 \end{cases}$$ seria armonică generalizată

③ $\sum_{n=1}^{\infty} \frac{\sqrt[5]{n^2} + 2\sqrt{n}}{3\sqrt[3]{n^2} + 4n}$ $\qquad x_n = \frac{n^{\frac{2}{5}} + 2n^{\frac{1}{2}}}{3n^{\frac{2}{3}} + 4n}$

$\frac{2}{5} + \frac{1}{2} = \frac{4+5}{2} = \frac{9}{2}$

$$y_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^{\frac{9}{2}} + 2n}{3n^{\frac{2}{3}} + 4n} = \frac{2}{4} = \frac{1}{2} \in (0,\infty)$$

$\overset{CCL}{\Longrightarrow} \sum x_n \sim \sum y_n$

$\sum y_n$ - div $\Rightarrow \sum x_n$ - div

Studiați natura urm. serii folosind criterii de conv. adecvate

(1) $\sum_{n=1}^{\infty} \left(\sqrt{n(n+a)} - n\right)^n \quad , a > 0$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{n(n+a)-n^2}{\sqrt{n(n+a)}+n} = \lim_{n\to\infty} \frac{\cancel{n}\cdot a}{\cancel{n}\left(\sqrt{1+\frac{a}{n}}+1\right)} = \frac{a}{2}$$

Dacă $\frac{a}{2} < 1 \Rightarrow \sum a_n$ - conv.

Dacă $\frac{a}{2} > 1 \Rightarrow \sum a_n$ - div.

Daca $\frac{a}{2} = 1 \Leftrightarrow a = 2 \Rightarrow$ criteriul rap. generalizat ?

În acest caz seria devine $\sum_{n=1}^{\infty} \left(\sqrt{n^2+2n} - n\right)^n$

$$\lim_{n\to\infty} \left(\sqrt{n^2+2n}-n\right)^n \overset{1^\infty}{=} \lim_{n\to\infty} \left(\frac{2n}{\sqrt{n^2+2n}+n}\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{n-\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}}\right)^{\text{inv}}\right]^{\frac{n\left(n-\sqrt{n^2+2n}\right)}{n+\sqrt{n^2+2n}}} =$$

$$= e^{\lim_{n\to\infty} \frac{n(\cancel{n^2}-\cancel{n^2}-2n)}{(n+\sqrt{n^2+2n})^2}} = e^{\lim_{n\to\infty} \frac{-2\cancel{n^2}}{\cancel{n^2}\left(1+2\sqrt{1+\frac{2}{n}}+1+\frac{2}{n}\right)}} = e^{-\frac{2}{4}} = e^{-\frac{1}{2}} \neq 0$$

$\Rightarrow \sum a_n$ - divergentă

(2) $\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!!}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+1)!!} \cdot \frac{(2n-1)!!}{n!} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$$

$\Rightarrow \sum x_n$ - conv.

(3) $\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\cdot\ldots\cdot(a+n)} \quad , a > 0$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{n!}}{\cancel{(a+1)\cdot\ldots\cdot(a+n)}} \cdot \frac{\cancel{(a+1)\cdot\ldots\cdot}(a+n+1)}{(n+1)\cancel{!}} - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{a+n+1}{n+1} - 1\right) = \lim_{n\to\infty} n\cdot\frac{a}{n+1} = a$$

Dacă $a < 1 \Rightarrow \sum_{n=1}^{\infty} x_n$ - divergentă

Dacă $a > 1 \Rightarrow \sum_{n=1}^{\infty} x_n$ - convergentă

Pt $a = 1$ seria devine $\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 3 \cdot \ldots (n+1)} = \sum_{n=1}^{\infty} \frac{n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n+1}$ - div

pt. că e serie armonică cu $p = 1$

19. 10. 2023

Analiza matematică
(seminar 4 - S4)

Să se studieze convergența absolută și convergența seriilor:

① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+n}$

$x_n = \frac{2n+1}{n^2+n} \qquad y_n = \frac{1}{n}$

$\lim_{x\to\infty} \frac{x_n}{y_n} = \lim_{x\to\infty} \frac{2n^2+n}{n^2+n} = 2 > 1$

$\xrightarrow[\text{la lim}]{\text{crt. comp}} \sum x_n \sim \sum y_n$

$\sum y_n$ - div $\Rightarrow \sum x_n$ - div.

$\Rightarrow \sum x_n$ - nu e absolut convergentă (1)

Crt. lui Leibniz $x_n \xrightarrow{n\to\infty} 0$

$x_{2n+1} - x_{2n} = \frac{2(2n+1)+1}{(2n+1)2(n+1)} - \frac{2\cdot 2n+1}{2n(2n+1)} =$

$= \frac{4n^2+2n+n-4n^2-4n-n-1}{2n(n+1)(2n+1)} = \frac{-\cancel{(2n+1)}}{2n(n+1)\cancel{(2n+1)}} = -\frac{1}{2n(n+1)} < 0$

$\Rightarrow x_n$ - descresc. $(*)$

$\lim_{n\to\infty} x_n = 0 \quad (**)$

$\xrightarrow{\text{crt. Leibniz}} \sum (-1)^{n+1} \frac{2n+1}{n(n+1)}$ este conv. (2)

din (1) și (2) $\Rightarrow \sum (-1)^{n+1} \frac{2n+1}{n(n+1)}$ semiconvergentă

② $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{\sqrt{n}+(-1)^{n+1}}$

③ $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n\sqrt{n}}$

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

serie armon cu $p>1$ $\Rightarrow$ $\sum |a_n|$ - conv. $\Rightarrow \sum (-1)^{n+1} \frac{1}{n\sqrt{n}}$ - absolut conv.

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$ - conv.

Să se studieze conv. cu ajutorul Crit. lui Leibniz

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2$$

$$\frac{x_{n+1}}{x_n} = \left[\frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{(2n)!!}{(2n-1)!!} \right]^2 = \left(\frac{2n+1}{2n+2} \right)^2 = \left(1 - \frac{1}{2n+2} \right)^2 < 1$$

$\Rightarrow x_{n+1} < x_n \Rightarrow (x_n) \searrow$ (1)

Pt. $\lim\limits_{n\to}$ putem face ca la ex 2 sau cu crit. cleștelui

$$x_n = \left[\frac{(2n-1)!!}{(2n)!!} \right]^2$$

$$y_n = \frac{(2n-1)!!}{(2n)!!}$$

$$0 < y_n = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)} = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n}{2n+1} \Big| \cdot y_n$$

$$0 < x_n < \frac{1}{2n+1}$$

$\to 0$ $\xRightarrow{\text{crit. cleștelui}} \lim\limits_{n\to\infty} x_n = 0$ (2)

din (1) și (2) $\xRightarrow{\text{crit. lui Leibniz}} \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{(2n-1)!!}{(2n)!!} \right]^2$ - convergentă

Studiați natura seriei

$$\sum_{n=2}^{\infty} \left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots + \frac{1}{\ln n}\right) \frac{\sin(n \cdot a)}{n-1}, \quad a \in \mathbb{R}$$

Fie $\alpha_n = \left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \dots + \frac{1}{\ln n}\right) \cdot \frac{1}{n-1}$? $\xrightarrow{n\to\infty} 0$

$u_n = \sin(n \cdot a)$? $|t_n| < M$

$$\alpha_{n+1} - \alpha_n = \left(\frac{1}{\ln 2} + \dots + \frac{1}{\ln n} + \frac{1}{\ln(n+1)}\right) \cdot \frac{1}{n} - \frac{1}{n-1}\left(\frac{1}{\ln 2} + \dots + \frac{1}{\ln n}\right) =$$

$$= \frac{1}{(n-1)n}\left(\frac{1}{\ln 2} - \frac{1}{\ln(n+1)} + \frac{1}{\ln 3} - \frac{1}{\ln(n+1)} + \dots + \frac{1}{n} - \frac{1}{\ln(n+1)}\right) =$$

$$= \underset{<0}{\frac{-1}{(n-1)n}} \underbrace{\left(\frac{\ln \frac{n+1}{2}}{\ln 2 \ln(n+1)} + \frac{\ln \frac{n+1}{3}}{\ln 3 \ln(n+1)} + \dots + \frac{\ln \frac{n+1}{n}}{\ln n \ln(n+1)}\right)}_{>0} < 0$$

$\ln$ - s.c.

$\Rightarrow \alpha_n \searrow$

$$\lim_{n\to\infty} \alpha_n \overset{S-C}{=} \lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)}}{n-(n-1)} = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$$

$\Rightarrow \alpha_n \xrightarrow{n\to\infty} 0 \quad (1)$

$$S_n = \sum_{k=2}^{\infty} u_n$$

$$|\sin 2a + \dots + \sin na| = \left|\frac{\cancel{2}\sin\left(a\frac{n+2}{2}\right)\sin\left(a\frac{n-1}{2}\right)}{\cancel{2}\sin\frac{n}{2}}\right| \leq \frac{1}{\left|\sin\frac{n}{2}\right|} = M$$

$\Rightarrow S_n$ - mărg. (2)

din (1) și (2) $\xRightarrow[\text{Dirichlet}]{\text{crit.}} \sum_{n=2}^{\infty} \alpha_n u_n$ - convergentă

Scanat cu CamScanner

Șiruri și serii de funcții

Să se studieze conv. simplă și uniformă pt. urm. șiruri de funcții

① $f_n : [0, \infty) \to \mathbb{R}$; $f_n(x) = \dfrac{x+n}{x+n+1}$

$\lim\limits_{n\to\infty} f_n = 1 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, $f(x) = 1$, $\forall x \in [0, \infty)$

$|f_n(x) - f(x)| = \left| \dfrac{-1}{x+n+1} \right| = \dfrac{1}{x+n+1} < \dfrac{1}{n+1} < \dfrac{1}{n} < \varepsilon$

$\Rightarrow n > \dfrac{1}{\varepsilon}$ Fie $n_\varepsilon = \left[\dfrac{1}{\varepsilon}\right] + 1$

$\Rightarrow \forall \varepsilon > 0, \forall x \in [0, \infty)$, $\exists\, n_\varepsilon \in \mathbb{N}$ a.î. $\forall n > n_\varepsilon$ $|f_n(x) - f(x)| < \varepsilon$

$\Rightarrow f_n \xrightarrow{u} f$

② $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \operatorname{arctg}(nx)$

$$\lim_{n\to\infty} \operatorname{arctg} x = \begin{cases} \dfrac{\pi}{2}, & x > 0 \\ 0, & x = 0 \\ -\dfrac{\pi}{2}, & x < 0 \end{cases}$$

$\Rightarrow f$ lim nu e cont $\Rightarrow f_n$ nu e u conv.

Utilizând proprietatea limită uniformă a unui șir de funcții continue este o funcție continuă.

(f - nu e cont $\Rightarrow f_n$ - nu e uniform convergentă)

③ $f_n : (0, \infty) \to \mathbb{R}$, $f_n(x) = \dfrac{n}{x+n}$

$\lim\limits_{n\to\infty} \dfrac{n}{x+n} = 1 \Rightarrow f_n \xrightarrow{s} f$, $f(x) = 1$

$|f_n(x) - f(x)| = \left| \dfrac{-x}{x+n} \right| \underset{x>0}{=} \dfrac{x}{x+n}$

Folosim prop. 2 $x_n = n$

$|f_n(x_n) - f(x_n)| = \dfrac{n}{2n} \xrightarrow[n\to\infty]{} \dfrac{1}{2} \overset{P_2}{\Rightarrow} f_n \overset{u}{\underset{n\to\infty}{\not\to}} f$

(4.) $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = \frac{x^2}{n^2+x^4}$

$\lim\limits_{n\to\infty} f_n = 0 \Rightarrow f_n \xrightarrow{s} f, \quad f(x) = 0$

$|f_n(x) - f(x)| = \left| \frac{x^2}{n^2+x^4} \right| = \frac{x^2}{n^2+x^4} \leq \left| \frac{x^2}{2nx^2} \right| = \frac{1}{2n} < \varepsilon$

$x^4 + n^2 \geq 2nx^2$

$(x^2 - n)^2 \geq 0$ ✓

$\Rightarrow n > \frac{1}{2\varepsilon}$

$\Rightarrow n_\varepsilon = \left[\frac{1}{2\varepsilon}\right] + 1$

$\Rightarrow \forall \varepsilon > 0, \forall x \in \mathbb{R}, \exists n_\varepsilon \in \mathbb{N}$ a.î. $n > n_\varepsilon \quad |f_n(x) - f(x)| < \varepsilon$

$\Rightarrow f_n \xrightarrow{u} f$

S8 joi
altă sală

Analiză matematică

(seminar 5 - S5)

26.10.2023

Șiruri și serii de funcții

Să se arate că urm. șiruri sunt uniform convergente

1. i) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{x^n}{e^{nx}}$

$\lim\limits_{n\to\infty} f_n(x) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, $f(x) = 0, \forall x \in [0,\infty)$

(exp > putere)

$\lim\limits_{n\to\infty} \sup\limits_{x\in[0,\infty)} |f_n(x) - f(x)| = 0$

$f_n'(x) = (x^n e^{-nx})' = n x^{n-1} e^{-nx} - x^n e^{-nx} \cdot n =$

$= n e^{-nx} x^{n-1} (1-x)$

$f_n'(x) = 0 \Leftrightarrow x = 0$ sau $x = 1$

x	0		1		∞
$f_n'(x)$	0	$+ \; +$	0	$- \; - \; - \; -$	
$f_n(x)$	0	$\nearrow$	$\frac{1}{e^n}$	$\searrow$	

$f_n(1) = \dfrac{1}{e^n}$

$\Rightarrow a = \sup\limits_{x\in[0,\infty)} |f_n(x) - f(x)| = \dfrac{1}{e^n}$

$\Rightarrow \lim\limits_{n\to\infty} a = \lim\limits_{n\to\infty} \dfrac{1}{e^n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f$

ii) $f_n(x) = \sqrt{1+nx} - \sqrt{nx}$

$$\lim_{n\to\infty} \sqrt{1+nx} - \sqrt{nx} = \lim_{n\to\infty} \frac{1+nx-nx}{\sqrt{1+nx}+\sqrt{nx}} = 0$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \ f(x) = 0, \forall x \in [1, \infty)$$

$$\lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0$$

$$|f_n(x) - f(x)| = |f_n(x)| = f_n(x)$$

$$f_n'(x) = \frac{1}{2\sqrt{1+nx}} \cdot n - \frac{1}{2\sqrt{nx}} \cdot n = -\frac{n}{2\sqrt{1+nx}\sqrt{nx}(\sqrt{nx}+\sqrt{1+nx})} < 0$$

x	1		∞
$f_n'(x)$	$-$	$-$ $-$ $-$ $-$	
$f_n(x)$	0	$\searrow$	0

$$f(1) = \sqrt{1+n} - \sqrt{n}$$

$$\Rightarrow a = \sup |f_n(x) - f(x)| = \sqrt{1+n} - \sqrt{n}$$

$$\Rightarrow \lim_{n\to\infty} a = \lim_{n\to\infty} \frac{1+n-n}{\sqrt{1+n}+\sqrt{n}} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f$$

② Să se calculeze $\lim_{n\to\infty} \int_2^{10} e^{-nx^2}\,dx$

$$f_n(x) = e^{-nx^2} = \frac{1}{e^{nx^2}}, \ x \in [2, 10]$$

$$\lim_{n\to\infty} f_n(x) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \ f(x) = 0$$

$$\lim_{n\to\infty} \sup_{x\in[2,10]} |f_n(x) - f(x)|$$

$$f_n'(x) = -2nx \cdot e^{-nx^2} \quad f_n'(x) = 0 \Leftrightarrow x = 0 \notin [2, 10]$$

x	2		10
$f_n'(x)$	$-$	$-$ $-$ $-$ $-$ $-$ $-$ $-$	$-$
$f_n(x)$		$\searrow$	

$$f_n(2) = \frac{1}{e^{4n}} = \sup_{x\in[2,10]} |f_n(x) - f(x)|$$

$$\Rightarrow \lim_{n\to\infty} f_n(2) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f \text{ pe } [2, 10]$$

Th. and integr.

$$\Rightarrow \lim_{n\to\infty} \left(\int_2^{10} e^{-nx^2}\,dx \right) = \int_2^{10} \left(\lim_{n\to\infty} e^{-nx^2} \right) dx = \int_2^{10} 0\,dx = 0$$

Studiați convergența uniformă a urm. șiruri de funcții

i) $f_n: \mathbb{R} \to \mathbb{R}$, $f_n(x) = e^{-nx^2} \sin(nx)$

$$\lim_{n\to\infty} f_n(x) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,\ f(x)=0$$

Fie $x_n = \frac{1}{n}$

$$|f_n(x_n) - f(x_n)| = \left|e^{-n\cdot\frac{1}{n^2}} \cdot \sin 1\right| = e^{-\frac{1}{n}} \sin 1 = \frac{\sin 1}{e^{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{\sin 1}{e^{\frac{1}{n}}} = \sin 1 \neq 0 \Rightarrow f_n \not\xrightarrow{u} f$$

ii) $f_n: [0,1] \to \mathbb{R}$, $f_n(x) = x^n$

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \quad \text{nu e cont}$$

$$\overset{T2.1.5}{\Longrightarrow} f_n \not\xrightarrow{u} f$$

Tema:

iii) $f_n: [0,1] \to \mathbb{R}$, $f_n(x) = x^{1+\frac{1}{n}}$, $n \geq 1$ $\boxed{\text{cu derivata}}$

$$\lim_{n\to\infty} f_n(x) = x^1 = x \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,\ f(x) = x$$

$f(x)$ - cont pe $[0,1]$

$$f'(x) = x \cdot x^{\frac{1}{n}} = x^{\frac{1}{n}} + \cancel{x} \frac{x^{\frac{1}{n}}}{n} = \frac{x^{\frac{1}{n}}(n+1)}{n} > 0$$

$$|f_n(x)| = |x \cdot x^{\frac{1}{n}}| = \underbrace{|x|}_{\in[0,1]} \cdot |x^{\frac{1}{n}}| \leq \underbrace{|x|}_{\in[0,1]} \leq 1 = g(x)$$

Fie $g(x) = 1$ $\quad \sum_{n=1}^{\infty} g(x)$

$$\left(\lim_{n\to\infty} f_n(x)\right)' = \lim_{n\to\infty} f'_n(x)$$

crit. Weierstrass $\Longrightarrow \sum_n f_n$ - u conv pe $[0,1]$

$x = 0$

x	0			1
$f'(x)$	+	+	+	+
$f(x)$	↗			1

$$\lim_{n\to\infty} |x^{1+\frac{1}{n}} - 1| = x \neq 0$$

$$\Rightarrow f_n \not\xrightarrow{u} f$$

Pr
(1

funcți
și are

$\lim_{x \to x_0}$

(

(4.) Fie seria de funcții:

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n^2+x^2+1}, x \in \mathbb{R} \quad u.c. \text{ și abs conv.}$$

$$\left| \frac{\cos nx}{n^2+x^2+1} \right| \leq \frac{1}{n^2+x^2+1} \leq \frac{1}{n^2+1} \overset{not}{=} a_n$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ - conv (serie arm)}$$

$$\xRightarrow{\text{Weierstrass}} \sum \frac{\cos nx}{n^2+x^2+1} \text{ - u.c.}$$

și $|f_n| \leq \frac{1}{n^2+1} \to$ abs. conv

(5.) $\sum_{n=0}^{\infty} \frac{1}{(x+n)(x+n+1)}$ este u.c. pe $(0, \infty)$

$$S_n(x) = f_0(x) + f_1(x) + \dots + f_n(x) =$$

$$= \frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \dots + \frac{1}{(x+n)(x+n+1)} =$$

$$= \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+1} - \frac{1}{x+2} + \dots + \frac{1}{x+n} - \frac{1}{x+n+1} =$$

$$= \frac{1}{x} - \frac{1}{x+n+1}$$

$$\lim_{n \to \infty} S_n(x) = \frac{1}{x} = S(x)$$

$$\Rightarrow S_n \xrightarrow[n \to \infty]{s} S, \quad S(x) = \frac{1}{x}$$

$$|S_n(x) - S(x)| = \left| \frac{1}{x+n+1} \right| < \frac{1}{n+1} < \varepsilon \quad \text{sau} \quad \frac{1}{n+1} = a_n \xrightarrow[n \to \infty]{} 0$$

$$\Rightarrow S_n \xrightarrow[n \to \infty]{u} S \overset{Def 2}{\Longrightarrow} \sum \frac{1}{(x+n)(x+n+1)} \text{ - u. conv.}$$

① $\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$, $C=(-1,1)$

② $\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$, $C=(-1,1)$ DE REȚINUT

③ $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$, $C=(-1,1)$

$f'(x) = \frac{1}{x^2+1}$ $f(x) = \operatorname{arctg} x$

Raza de convergență a seriei de puteri $R \in [0,\infty)$

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \quad \text{sau} \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Dacă $\exists \lim_{n\to\infty} \sqrt[n]{|a_n|}$ nu mai e nev de sup

Teorema I a lui Abel

-> oferă informații legate de modul de determinare a mulțimii C.

Fie $\sum_{n=0}^{\infty} c_n (x-x_0)^n$ și $R \in [0,\infty)$ Atunci:

(i) $R=0 \Rightarrow C=\{x_0\}$ seria e conv. doar în x_0

(ii) $R=\infty \Rightarrow C=\mathbb{R}$ seria e conv $\forall x \in \mathbb{R}$ -> $\sin x$

(iii) $R \in (0,\infty)$:

a) Seria e conv. $\forall x \in (x_0-R, x_0+R)$

b) Seria e div. $\forall x \in (-\infty, x_0-R) \cup (x_0+R, \infty)$

c) Seria e unif. conv. $\forall x \in [a,b] \subset (x_0-R, x_0+R)$

d) Dacă $x = x_0+R$ sau $x = x_0-R$ T.I. a lui Abel nu precizează natura seriei, dar seria se transformă într-o serie numerică

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \to R=1$$

$x_0=0$

a) $\Rightarrow$ seria este conv. pt. $x \in (-1,1)$

b) $\Rightarrow$ seria este div pt. $x \in (-\infty,-1) \cup (1,\infty)$

c) clar

d) $x=1 \to \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ - conv.

$x=-1$ $\sum_{n=0}^{\infty} \frac{1}{n}$ - div

$\Rightarrow C=(-1,1]$

(S6)

Det. mulțimea de conv. și suma urm. serii de puteri:

i) $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$ ii) $\sum_{n=0}^{\infty} n x^n$ iii) $\sum_{n=1}^{\infty} \frac{(n-1) x^{2n}}{n \cdot 3^n}$

iv) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ v) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2^n} x^{2n+1}$

vi) $\sum_{n=2}^{\infty} (-1)^n (n-1)(x-3)^n$

i) Dacă avem $(x-x_0)^{2n} \Rightarrow R^2 = \lim_{n\to} \left|\frac{a_n}{a_{n+1}}\right|$ (iii) a) abs. conv.

$a_n = \frac{1}{n \cdot 3^n} \Rightarrow R = \lim_{n\to\infty} \left| \frac{(n+1) \cancel{3^{n+1}}}{n \cdot \cancel{3^n}} \right| = 3$

$\Rightarrow C = (-3, 3)$ interval de conv.

T I Abel $\Rightarrow$ • seria este abs. conv. pt. $x \in C$

• seria este div. pt. $x \in (-\infty, -3) \cup (3, \infty)$

• $\forall [a, b] \subset C \Rightarrow$ seria este u.c.

• $x = -3 \Rightarrow \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ - conv (Leibniz)

• $x = 3 \Rightarrow \sum \frac{\cancel{3^n}}{n \cdot \cancel{3^n}} = \sum \frac{1}{n}$ - div. (serie arm.)

$\Rightarrow C = [-3, 3)$

Fie $S(x)$ = suma seriei

$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$, $x \in [-3, 3)$

T. de deriv. $S'(x) = \left(\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}\right)' = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n \cdot 3^n} = \frac{1}{3} \sum \left(\frac{x}{3}\right)^n$

(*) $\frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{1}{3-x}$

$\int S'(x)\,dx = \int \frac{1}{3-x}\,dx + C$

Scanat cu CamScanner

$S(x) = -\ln(3-x) + C$

$S(0) = 0$

$S(0) = -\ln 3 + C \Rightarrow C = \ln 3$

$\Rightarrow S(x) = -\ln(3-x) + \ln 3 = \ln \frac{3}{3-x}, \forall x \in [-3,3)$

ii) $\sum_{n=0}^{\infty} n x^n$

$a_n = n \Rightarrow R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$

$\Rightarrow C = (-1,1)$

Fie $S(x)$ - suma seriei, $S(x) = \sum_{n=0}^{\infty} n x^n$, $x \in (-1,1)$

T I, Abel
$$\begin{cases} \cdot \text{ abs. conv. } \forall x \in (-1,1) \\ \cdot \text{ div pt. } x \in (-\infty,-1) \cup (1,\infty) \\ \cdot \ \forall [a,b] \\ \cdot \ x = -1 \Rightarrow \sum_{n=1}^{\infty} n(-1)^n \text{ - div } (\nexists \lim) \\ \cdot \ x = 1 \Rightarrow \sum n 1^n = \sum n \text{ - div} \end{cases}$$

nu e necesară îndepl cond

$C = (-1,1)$

$(*) \left(\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \right)' = \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad | \cdot x$

$\Rightarrow S(x) = \frac{x}{(1-x)^2}, \forall x \in (-1,1)$

(iii) $\sum_{n=1}^{\infty} \frac{(n-1)x^{2n}}{n \cdot 3^n}$

$R^2 = \lim_{n\to\infty} \left| \frac{n-1}{n \cdot \cancel{3^n}} \cdot \frac{(n+1)3^{n+\cancel{1}}}{n} \right| = 3$

$\Rightarrow R = \sqrt{3}$

$I = (-\sqrt{3}, \sqrt{3})$

T. I. Abel. abs. conv. pt. $x \in (-\sqrt{3}, \sqrt{3})$

div. pt. $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

$\forall [a, b] \subset (-\sqrt{3}, \sqrt{3})$ - u. c.

$x = -\sqrt{3} \Rightarrow \sum \frac{(n-1)(-1)^{2n} \cdot \cancel{(\sqrt{3})^{2n}}}{n \cdot \cancel{3^n}} = \sum \underbrace{\frac{n-1}{n}}_{\not\to 0,\ n\to\infty}$ div

$x = \sqrt{3} \Rightarrow \sum \frac{(n-1)(\sqrt{3})^{2n}}{n \cdot 3^n} = \sum \frac{n-1}{n}$ div $\Rightarrow C = (-\sqrt{3}, \sqrt{3})$

Fie $S(x)$ - suma seriei, $S(x) = \sum \frac{(n-1)x^{2n}}{n \cdot 3^n}$, $\forall x \in I$

$S(x) = \underbrace{\sum \frac{n x^{2n}}{n \cdot 3^n}}_{S_1} - \underbrace{\sum \frac{x^{2n}}{n \cdot 3^n}}_{S_2}$ $\qquad \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$

$S_1(x) = \sum \frac{x^{2n}}{3^n} = \sum \left(\frac{x^2}{3}\right)^n \stackrel{*}{=} \frac{1}{1 + \frac{x^2}{3}} - 1 =$

$= \frac{3}{3 - x^2} - 1 = \frac{x^2}{3 - x^2}$

$S_2(x) = \sum \frac{x^{2n}}{n \cdot 3^n}$

$S_2'(x) = \sum \frac{2\cancel{n} x^{2n-1}}{\cancel{n} \cdot 3^n} = \frac{2x}{3} \sum_{n=1}^{\infty} \left(\frac{x^2}{3}\right)^{n-1} \stackrel{*}{=} \frac{2x}{3} \sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n =$

$= \frac{2x}{3} \cdot \frac{1}{1 - \frac{x^2}{3}} = \frac{2x}{3 - x^2}$

$\int \frac{2x}{3 - x^2} dx = -\ln(3 - x^2) + C$

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$$\left.\begin{array}{l} S_2(0) = -\ln 3 + C \\ S_2(0) = 0 \end{array}\right\} \Rightarrow C = \ln 3$$

$$\Rightarrow S_2(x) = -\ln(3-x^2) + \ln 3 = \ln \frac{3}{3-x^2}$$

$$S(x) = S_1(x) - S_2(x) = \frac{x^2}{3-x^2} - \ln\frac{3}{3-x^2}, \ x \in (-\sqrt{3}, \sqrt{3})$$

iv) $\sum\limits_{n=1}^{\infty} \frac{(x-5)^{2n}}{n}$ $\quad x_0 = 5$

$$a_n = \frac{1}{n} \Rightarrow R^2 = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1 \xRightarrow{R=1} J = (4, 6)$$

T I Abel:
- seria este abs conv pt $x \in (4, 6)$
- seria este div pt $x \in (-\infty, 4) \cup (6, \infty)$
- $\forall [a, b] \subset (4, 6)$ seria este u.c
- $x = 4 \Rightarrow \sum \frac{(-1)^{2n}}{n}$ - div.
- $x = 6 \Rightarrow \sum \frac{1}{n}$ - div

$\Rightarrow C = (4, 6)$

Fie $S(x) = \sum\limits_{n=1}^{\infty} \frac{(x-5)^{2n}}{n}, \ \forall x \in (4, 6)$

$$S'(x) = \sum_{n=1}^{\infty} \frac{2\cancel{n}(x-5)^{2n-1}}{\cancel{n}} = 2(x-5)\sum_{n=1}^{\infty} (x-5)^{2n-2} = 2(x-5) \cdot \frac{1}{1-(x-5)^2}$$

$$S(x) = \int S'(x)\,dx = \int \frac{2(x-5)}{1-(x-5)^2}\,dx = -\ln(1-(x-5)^2) + C$$

$$\left.\begin{array}{l} S(5) = C \\ S(5) = 0 \end{array}\right\} \Rightarrow C = 0 \Rightarrow S(x) = -\ln(1-(x-5)^2)$$

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v) $\sum_{m=0}^{\infty} (-1)^m \frac{m+1}{2^m} x^{2m+1}$ $a_6 = 0$

$a_m = (-1)^m \frac{m+1}{2^m}$ $R^2 = \lim_{m\to\infty} \left| \cancel{(-1)^m} \frac{m+1}{\cancel{2^m}} \cdot \frac{2^{\cancel{m+1}}}{m+2 \cancel{(-1)^{m+1}}} \right| = 2$

$D = (-\sqrt{2}, \sqrt{2})$

T I Abel abs conv pt $x \in (-\sqrt{2}, \sqrt{2})$

div pt. $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

$\forall [a,b] \subset (-\sqrt{2}, 2) \Rightarrow$ serie u c

$x = \sqrt{2} \Rightarrow \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{\cancel{2^m}} (-1)^{2m+1} \cancel{2^m} \cdot \sqrt{2}$ - div

$x = 2 \Rightarrow \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{\cancel{2^m}} (-1)^{2m+1} \cancel{2^m} \cdot \sqrt{2}$ - div

$C = (-\sqrt{2}, \sqrt{2})$

$S(x) = \sum_{m=0}^{\infty} (-1)^m \frac{m+1}{2^m} x^{2m+1}$, $\forall x \in (-\sqrt{2}, \sqrt{2})$

$S(x) = x \sum_{m=0}^{\infty} (m+1) \left(\frac{-x^2}{2}\right)^m$

$\int S(x)\, dx$

vi) $R = \lim_{n\to\infty} \left| (-1)^n (n-1) \cdot \frac{1}{n\cdot(-1)^{n+1}} \right| = 1$

$\Rightarrow J = (2,4)$

$x = 2$

$C = (2,4)$

$(-1)^n n (x-3)^n - (-1)^n (x-3)^n$

$n=0 = 1(x-3)^0 = 1$

$n=1 \quad -(x-3)^1 = -(x-3)$

$S(x) = \sum (-1)^n (n-1)(x-3)^n \qquad S_2(x)$

$S(x) = \sum_{n=2}^{\infty} (-1)^n n (x-3)^n - \underbrace{\sum_{n=2} (-1)^n (x-3)^n}_{}$ $\qquad S(x) = \left(\frac{x-3}{x-2}\right)^2$

$S_2(x) = \frac{1}{1+(x-3)} - 1 + x - 3 = \frac{1}{x-2} + x - 4 = \frac{1+x^2-2x-4x+8}{x-2} = \frac{x^2-6x+9}{x-2}$

$\left(\frac{1}{1+x}\right)' = \sum_{n=0}^{\infty} (-1)^n n (x)^{n-1} = \frac{-1}{(1+x)^2}$ $\qquad 1\cdot n\cdot(x-3)^{-1} = 0$

$n=0 \Rightarrow$

$S_1(x) = \sum_{n=2}^{\infty} (-1)^n n (x-3)^n = (x-3)\sum_{n=2}^{\infty} (-1)^n n (x-3)^{n-1} =$

$\hookrightarrow n=1 \Rightarrow n-1 = -1$

$= (x-3)\left[\frac{-1}{(1+x-3)^2} - 0 + 1\right] = (x-3)\left[\frac{-1}{(x-2)^2} + 1\right] =$

$= \frac{(x-3)\left[-1+x^2-4x+4\right]}{(x-2)^2} = \frac{(x-3)(x^2-4x+3)}{(x-2)} =$

$S_1(x) - S_2(x) = \frac{\cancel{x^3} - 4x^2 + 3x - 3x^2 + 12x - 9 - (\cancel{x^3} - 6x^2 + 9x - 2x^2 + 12x - 18)}{(x-2)^2} =$

$= \frac{x^2 - 6x + 9}{(x-2)^2} = \left(\frac{x-3}{x-2}\right)^2$

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57) 1) Să se studieze

i) $f(x) = sh\,x$, $f_1(x) = ch\,x$, $x \in \mathbb{R}$

ii) $f(x) = \cos^3 x$, $x \in \mathbb{R}$

iii) $f(x) = \frac{x^2}{e^x}$, $x \in \mathbb{R}$

i) $sh\,x = \frac{e^x - e^{-x}}{2}$

$ch\,x = \frac{e^x + e^{-x}}{2}$

Fie $g(x) = e^x$

$g(x) = T_n(x) + R_n(x)$

$T_n(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0) + \ldots + \frac{(x-x_0)^n}{n!} f^{(n)}(x)$

pol de grad n Taylor $x_0 = 0$ Mac Laurin

$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$, $c \in (x, x_0)$

Dacă $R_n(x) \xrightarrow[n\to\infty]{} 0$ at $\sum\limits_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$ s. n. seria Taylor

$(e^x)^{(n)} = e^x$, $\forall n \in \mathbb{N}$

$e^0 = 1$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$

$e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$ $\Rightarrow e^{-x} = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$, $x \in \mathbb{R}$

$sh\,x = \frac{1}{2} \sum \left[\frac{x^n}{n!} - (-1)^n \frac{x^n}{n!} \right] = \frac{1}{2} \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} (1 - (-1)^n) =$

$= \frac{1}{\cancel{2}} \sum\limits_{n=0}^{\infty} \cancel{2} \frac{x^{2n+1}}{(2n+1)!}$

$ch\,x = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

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ii) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\cos^3 x = \frac{\cos 3x - 3\cos x}{4}$

$\cos 3x = \sum (-1)^n \cdot \frac{3^{2n} \, x^{2n}}{(2n)!}$

$\cos^3 x = \frac{1}{4} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \left[3^{2n} - 3 \right] =$

$= \frac{3}{4} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \left[3^{2n-1} - 1 \right]$

iii) $f(x) = \frac{x^2}{e^x}$

$f(x) = \frac{x^2}{e^x} = x^2 \cdot e^{-x} = x^2 \sum (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n!}$

cântr

cân

Eb

f i

2) Să se dezvolte după puterile lui

i) $x, x-3, x+2: \quad f(x) = \frac{1}{2x+3}, \quad x \neq \frac{3}{2}$

ii) $x, x-3: \quad f(x) = \ln(1+x), \quad x > -1$

i) $f(x) = \frac{1}{3} \cdot \frac{1}{1+\frac{2}{3}x} = \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n \cdot x^n =$

$= \frac{1}{3} \sum \left(-\frac{2}{3}\right)^n x^n$

$\frac{2x}{3} \in (-1, 1) \Rightarrow x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

$f(x) = \frac{1}{2x+3} = \frac{1}{2(x+3)+3} = \frac{1}{2(x-3)+9} = \frac{1}{9} \cdot \frac{1}{1+\frac{2}{9}(x-3)} =$

$= \frac{1}{9}\sum_{n=0}^{\infty} (-1)^n \left[\frac{2(x-3)}{9}\right]^n, \quad \left|\frac{2(x-3)}{9}\right| < 1$

$\Rightarrow x \in \left(-\frac{3}{2}, \frac{15}{2}\right)$

$f(x) = \frac{1}{2(x+2-2)+3} = \frac{1}{2(x+2)-1} = -\frac{1}{1+(-2(x+2))} =$

$= \sum_{n=0}^{\infty} \underbrace{\left[2(x+2)\right]^n}$

$|\,| < 1$

$x \in \left(-\frac{5}{2}, -\frac{3}{2}\right)$

1

$dx =$

dx

$n \geq 1$

ii) $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$f(x) = \int \sum (-1)^n x^n dx = \sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum (-1)^n \frac{x^{n+1}}{n+1} + C$$

$x \in (-1, 1)$

$$\left.\begin{array}{l} f(0) = 0 \\ f(0) = C \end{array}\right\} \Rightarrow C = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, \ |x| < 1$$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1+(x-3+3)} = \frac{1}{4+(x-3)} = \frac{1}{4} \cdot \frac{1}{1+\frac{x-3}{4}} =$$

$$= \frac{1}{4} \sum (-1)^n \left(\frac{x-3}{4}\right)^n, \ \left|\frac{x-3}{4}\right| < 1 \Rightarrow x \in (-1, 7)$$

$$f(x) = \sum_{n=0}^{\infty} \int (-1)^n \frac{(x-3)^n}{4^{n+1}} dx = \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} (-1)^n \cdot \frac{(x-3)^{n+1}}{n+1} + C$$

$$\left.\begin{array}{l} f(3) = C \\ f(3) = \ln 4 \end{array}\right\} C = \ln 4$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1^{n+1}}{4^{n+1}} \frac{(x-3)^{n+1}}{n+1} + \ln 4$$

3. i) Folosind seria binomială să se dezvolte în serie de puteri ale lui x funcția

$f(x) = \frac{1}{\sqrt{1+x}}$ precizând și dom de conv. , $x > -1$

ii) Folosind rezultatul, să se det. suma seriei numerice

$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2n-1)!!}{(2n)!!}$

$(1+x)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)}{n!} \cdot x^n, \ |x|<1, \alpha \in \mathbb{R}\setminus\mathbb{N}$

seria binomială

i) pt. $\alpha = -\frac{1}{2} \Rightarrow 1 + \sum_{n=1}^{\infty} \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1+2n-2}{2}\right)}{n!} x^n$

$= 1 + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{(2n-1)!!}{2^n \cdot n!} \cdot x^n = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} x^n, \ |x|<$

pt. $x = -1$ seria devine $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} (-1)^n = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$

R - ~~..~~ - div.

$x = 1$ seria $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!}$ - conv. (Leibniz)

$\Rightarrow C = (-1, 1]$

ii) $x=1 \Rightarrow \frac{1}{\sqrt{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!}$

$\sum (-1)^n \frac{(2n-1)!!}{(2n)!!} = \frac{\sqrt{2}}{2} - 1 \ \Big| \ (-1)$

$\sum_{n=1} (-1)^{n+1} \frac{(2n-1)!!}{(2n)!!} = 1 - \frac{\sqrt{2}}{2}$

4) Folosind formula lui Taylor respectiv Mac-Laurin să se calculeze următoarele limite:

i) $\lim\limits_{x \to 1} \dfrac{24\ln x + 6x^4 - 32x^3 + 72x^2 - 96x + 50}{3(x-1)^5}$

Fie $f(x) = \ln x \simeq T_5(x, 1) = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f^{(3)}(1)$

$+ \frac{(x-1)^4}{4!} f^{(4)}(1) + \frac{(x-1)^5}{5!} f^{(5)}(1)$

$f'(x) = \frac{1}{x}$ $\quad f''(x) = -\frac{1}{x^2}$ $\quad f^{(3)}(x) = \frac{2}{x^3}$ $\quad f^{(4)}(x) = -\frac{2\cdot 3}{x^4}$

$f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5}$

$f'(1) = 1$ $\quad f''(1) = -1$ $\quad f^{3}(1) = 2$ $\quad f^{(4)}(1) = -3!$ $\quad f^{(5)}(1) = 4!$

$\ln x \simeq x - 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!} \cdot 2 + \frac{(x-3)^4}{4!} \cdot (-3!) + \frac{(x-3)^5}{5!} \cdot 4! =$

$\ln x \simeq x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$

fie $g(x) = 6x^4 - 32x^3 + 72x^2 - 96x + 50$

$g'(x) = 24x^3 - 96x^2 + 144x - 96$ $\qquad g'(1) = -24$

$\begin{array}{r} 168 - \\ 192 \\ \hline -24 \end{array}$

$g''(x) = 72x^2 - 192x + 144$ $\qquad g''(1) = 24$

$g'''(x) = 144x - 192$ $\qquad g'''(1) = -48$

$g^{IV}(x) = 144$ $\qquad g^{IV}(1) = 144$

$g(x) = -24 \cdot (x-1) + \frac{24}{2}(x-1)^2 - \frac{48}{3!}(x-1)^3 + \frac{144}{4!}(x-1)^4 =$

$= -24(x-1) + 12(x-1)^2 - 8(x-1)^3 + 6(x-1)^4$

$\lim\limits_{x \to 1} \dfrac{24(x-1) - 12(x-1)^2 + 8(x-1)^3 - 6(x-1)^4 + \frac{24}{5}(x-1)^5 - 24(x-1) + 12(x-1)^2 - 8(x-1)^3 + 6(x-1)^4}{3(x-1)^5}$

$= \frac{8}{5}$

ii) $\lim_{x \to 0} \frac{\ln(1+2x) - \sin 2x + 2x^2}{x^3} = 4$

$$f(x) = \ln(1+2x) \simeq T_3(x,0) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cancel{\frac{f^{IV}(c)}{(n+1)!}x^4}$$

$$f'(x) = \frac{1}{1+2x} \cdot 2 = \frac{2}{1+2x}$$

$$f''(x) = \frac{-2 \cdot 2}{(1+2x)^2} = -\frac{2^2}{(1+2x)^2}$$

$$f'''(x) = \frac{2^2 \cdot 2(1+2x) \cdot 2}{(1+2x)^4} = \frac{16}{(1+2x)^3}$$

$$\cancel{f^{IV}(x) = -\frac{2^3 \cdot 3(4x^2+8x+2)}{(1+2x)^6} = \frac{2^3 \cdot 3 \cdot 2(1+2x)^2}{(1+2x)^6} = -\frac{2^4 \cdot 3}{(1+2x)^4}}$$

$$f'(0) = 2 \qquad f''(0) = -4 \qquad f'''(0) = 16$$

$$\ln(1+2x) \simeq 0 + \frac{x}{1!} \cdot 2 + \frac{x^2}{2!} \cdot (-4) + \frac{x^3}{3!} \, 16 =$$

$$= 2x - 2x^2 + \frac{8}{3}x^3$$

$g(x) = \sin 2x \qquad g(0) = 0$

$g'(x) = 2\cos 2x \qquad g'(0) = 2$

$g''(x) = -4\sin 2x \qquad g''(0) = 0$

$g'''(x) = -8\cos 2x \qquad g'''(0) = -8$

$$g(x) = \sin 2x \simeq g(0) + \frac{x}{1!}g'(0) + \frac{x^2}{2!}g''(0) + \frac{x^3}{3!}g'''(0) =$$

$$= 0 + 2x + 0 - \frac{4}{3}x^3$$

$h(x) = 2x^2 \qquad h(0) = 0 \qquad 2x^2 \simeq h(0) + \frac{x}{1!}h'(0) + \frac{x^2}{2!}h''(0) =$

$h'(x) = 4x \qquad h'(0) = 0 \qquad = 2x^2$

$h''(x) = 4 \qquad h''(0) = 4$

$$\lim_{x \to 0} \frac{\frac{12}{3}x^3 +}{x^3} = 4$$

$$\lim_{x\to\infty} x\left[3-4x+6x^2-12x^3+12x^4\left(\ln(1+x)-\ln x\right)\right]=\frac{12}{5}$$

$x=\frac{1}{y}$

$x\to\infty \quad y\to 0 > 0$