Digital Control Systems

1 Introduction

The use of a digital computer as a compensator (controller) device has grown during the past two decades as the price and reliability of digital computers has improved. A block diagram of a single-loop digital control system is shown in Figure 1 [2].

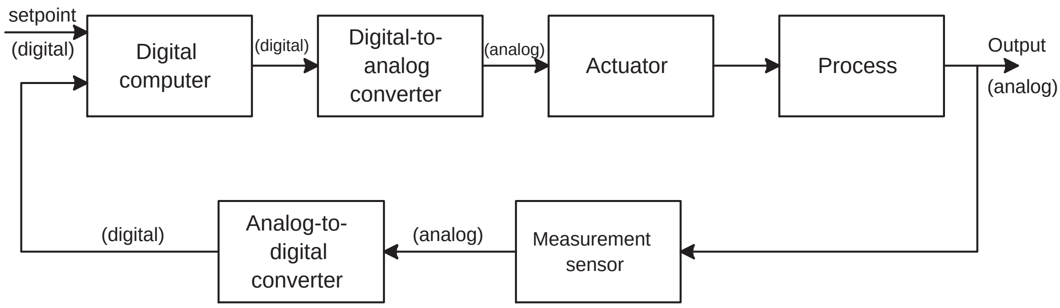

Figure 1: A block diagram of a computer control system

The digital computer in this system configuration receives the error in digital form and performs calculations in order to provide an output in digital form. The computer may be programmed to provide an output so that the performance of the process is near or equal to the desired performance.

Many computers are able to receive and manipulate several inputs, so a digital computer control system can often be a multivariable system.

A digital computer receives and operates on signals in digital (numerical) form as contrasted to continuous signals. A *digital control system* uses digital signals and a digital computer to control a process. The measurement data are converted from analog form to digital form by means of the analog-to-digital converter (A/D) shown in Figure 1. After processing the inputs, the digital computer provides an output in digital form. This output is then converted to analog form by the digital-to-analog converter (D/A) shown in Figure 1.

The advantages of using digital control include improved measurement sensitivity, the use of digitally coded signals, digital sensors and transducers, and microprocessors; reduced sensitivity to signal noise, and the capability to easily reconfigure the control algorithm in software. Improved sensitivity results from the low-energy signals required by digital sensors and devices. The use of digitally coded signals permits the wide application of digital devices and communications.

Digital sensors and transducers can effectively measure, transmit and couple signals and devices. In addition, many systems are inherently digital because they send out pulse signals.

2 Sampled-Data Systems

Computers used in control systems are interconnected to the actuator and the process by means of signal converters. We will assume that all the numbers that enter or leave the computer do so at the same fixed period T, called the **sampling period**. Thus, for example, the reference input shown in Figure 2 is a sequence of sample values $r(kT)$.

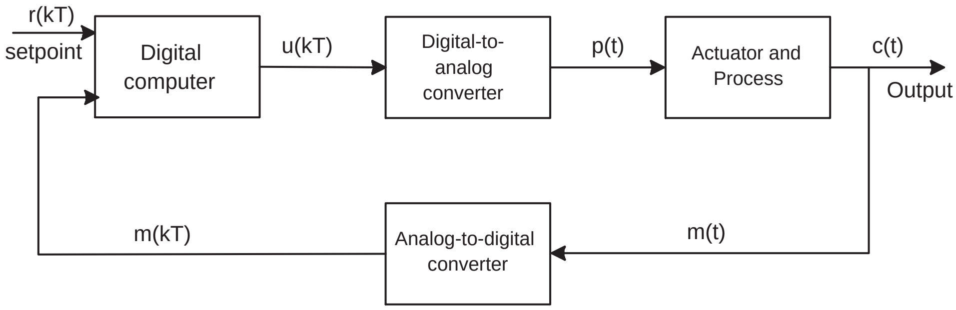

Figure 2: A digital control system

The variables $r(kT)$, $m(kT)$ and $u(kT)$ are discrete signals in contrast to $m(t)$, $c(t)$ which are continuous functions of time.

A typical configuration of a digital control system includes, [3]:

- A processing unit (digital computer or other kind of signal processors) to perform the necessary computations.
- Analog to digital converters to read the process signals into the computer. This is the process of *sampling.*
- Digital to analog converters to take the control signals from the computer and transform them into a form whereby they can be applied back to the physical process. This is a process of *signal reconstruction.*

2.1 Sampling

In choosing the sampling period T, or the sampling frequency $\omega_s = 2\pi/T$, most authors approaching this field refer to Shannon's sampling theorem. It states that:

'A function $f(t)$ that has a bandwidth ω_b is uniquely determined by a discrete set of sample values provided that the sampling frequency is greater than $\omega_s = 2\omega_b$. The sampling frequency $\omega_s = 2\omega_b$ is called the *Nyquist frequency*' [1].

It is rare in practical situations to work near the limit given by Shannon's theorem. *A useful rule in applications is to sample the signal at about 10 times higher than the highest frequency thought to be present [1].*

If a signal is sampled below Shannon's limit, then a lower frequency signal, called an *alias* may be constructed as shown in Figure 3(a).

The original signal in Figure 3(a) is $u(t) = sin(2t)$, thus the frequency is $\omega = 2$. The sampling frequency was chosen as $\omega_s = 1.2\omega$ (or the sampling period $T = 2\pi/\omega_s$). The same signal, sampled with a frequency $\omega_s = 8\omega$ (or a sampling period $T = 2\pi/\omega_s$) is well enough approximated by the discrete sequence of samples, as shown in Figure 3(b).

To ensure that aliasing does not take place, it is common practice to place an anti-aliasing filter before the A/D converter. This is an analog low-pass filter with a break frequency of $0.5\omega_s$ (and $omega_s > 10\omega_b$). The higher ω_s is in comparison to ω_b, the more closely the digital system resembles an analog one [1].

A **sampler** is basically a switch that closes every T seconds for one instant of time.

Consider an ideal sampler as shown in Figure 4.

The input is $r(t)$ and the output $r^*(t)$, where nT is the current sample time and the current value of $r^*(t)$ is $r(kT)$.

For a given time t, we have:

$$r^*(t) = r(kT) \cdot \delta(t - kT)$$

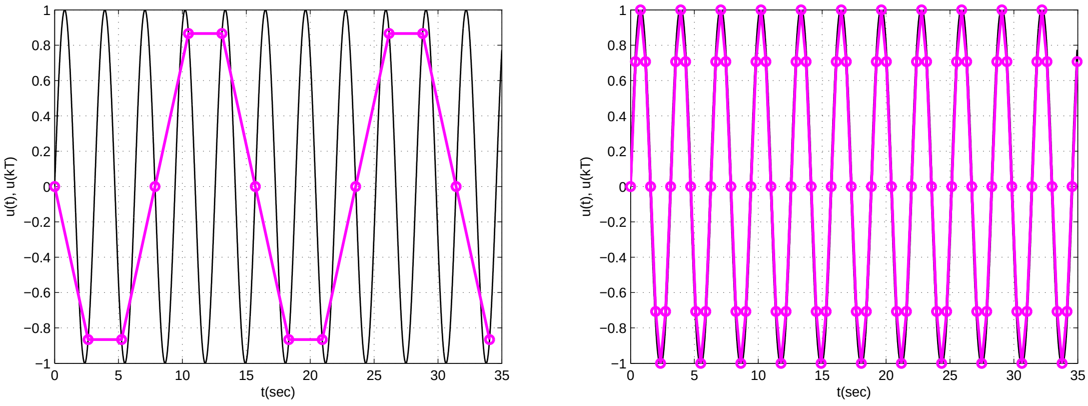

Figure 3: (a) Alias due to undersampling, (b) Sampled signal

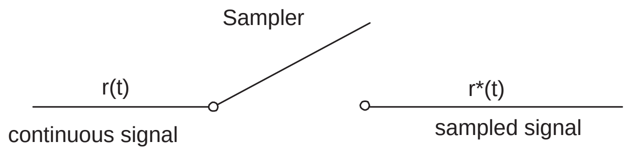

Figure 4: An ideal sampler with an input r(t)

where $\delta(t-kT)$ is the unit impulse function occurring at $t=kT$.

Then we portray the series for $r^*(t)$ as a string of impulses starting at $t=0$ spaced at T seconds and of amplitude $r^*(kT)$.

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t-kT)$$

For example see Figure 5.

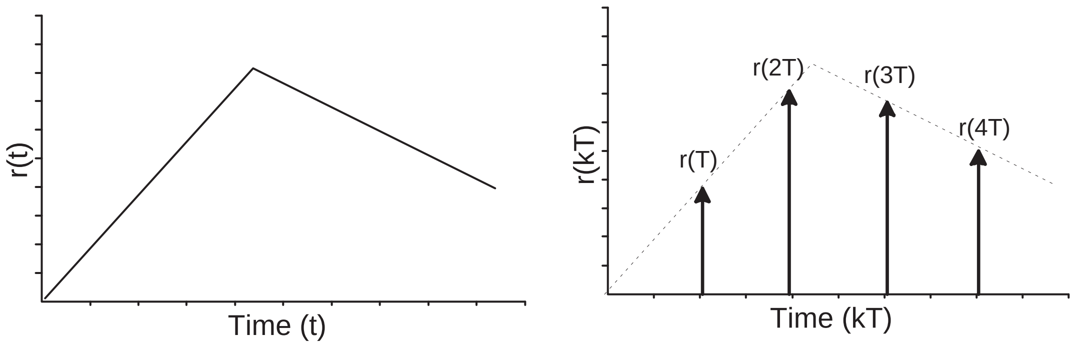

Figure 5: (a) An input r(t) and (b) the sampled signal

2.2 Signal reconstruction

A digital-to-analog (D/A) converter serves as a device that converts the sampled signal $r^*(t)$ to a continuous signal $p(t)$. The D/A converter can usually be represented by a zero-order hold (ZOH) circuit, as shown in Figure 6.

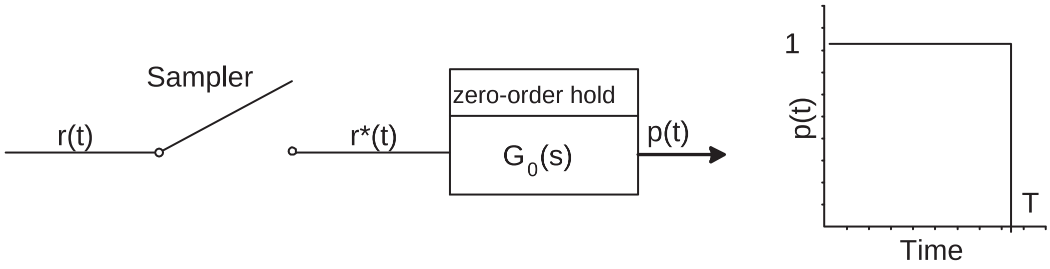

Figure 6: A sampler and zero-order circuit (left); the response of a zero-order hold to an impulse input r(kT), which equals 1 when k=0 and equals zero when k≠0 (right)

The ZOH takes the value $r(kT)$ and holds it constantly for $kT \leq t \leq (k+1)T$ as shown in Figure 6 for k=0. Thus we use $r(kT)$during the sampling period.

A sampler and ZOH can accurately follow the input signal if T is small compared to the transient changes in the signal. The response of a sampler and ZOH for a ramp input is shown in Figure 7.

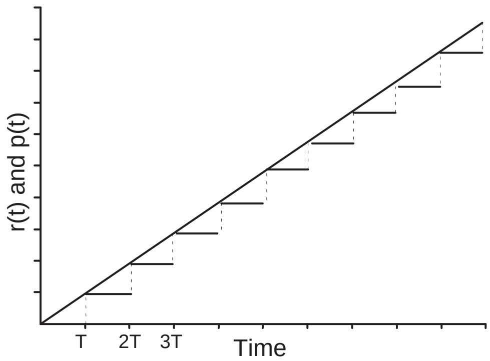

Figure 7: The response of a sampler and zero-order hold for a ramp input r(t)=t

A ZOH converts a series of impulses into a series of pulses of width T. As shown in Figure 8, a unit impulse, at time t, is converted into a pulse of width T, which may be created by a positive unit step at time t ($1(t)$) followed by a negative unit step ($1(t-T)$).

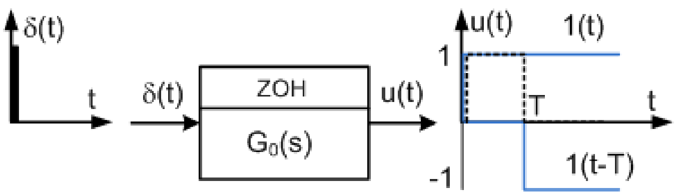

Figure 8: The impulse response of a ZOH

Then the transfer function of the zero-order hold is:

$$G_0(s) = \frac{\pounds\{u(t)\}}{\pounds\{\delta(t)\}} = \pounds\{1(t) - 1(t-T)\} = \frac{1}{s} - \frac{1}{s}e^{-sT} = \frac{1-e^{-sT}}{s} \tag{1}$$

3 The z-Transform

The z-transform is the main analytical tool for SISO discrete-time systems and it analogous to the Laplace transform for continuous systems. The symbol z can be associated with a discrete time shifting in a difference equation in the same way that s can be associated with differentiation in a differential equation.

Because the output of the ideal sampler, $r^*(t)$, is a series of impulses with values $r(kT)$ we have:

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT) \tag{2}$$

for a signal for t>0. Using the Laplace transform, we have:

$$\pounds[r^*(t)] = \sum_{k=0}^{\infty} r(kT)e^{-ksT} \tag{3}$$

We now have an infinite series that involves factors of e^{sT} and its powers. We define

$$z = e^{sT} \tag{4}$$

where this relationship involves a conformal mapping from the s-plane to the z-plane. We then define a new transform, called the **z-Transform**, so that:

$$Z[r(t)] = Z[r^*(t)] = R(z) = \sum_{k=0}^{\infty} r(kT)z^{-k} \tag{5}$$

As an example, let us determine the z-transform of the unit step function u(t)=1. We obtain:

$$Z[u(t)] = \sum_{k=0}^{\infty} u(kT)z^{-k} = \sum_{k=0}^{\infty} z^{-k} \tag{6}$$

since u(kT)=1, for k≥0. This series can be written in closed form as:

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \tag{7}$$

Obs. Recall that the infinite geometric series may be written: $(1 - bx)^{-1} = 1 + bx + (bx)^2 + ...$ *if* $(bx)^2 < 1$

Example. Transform of exponential

Let us determine the z-transform of $f(t) = e^{-at}$, $t \geq 0$

$$Z[e^{-at}] = F(z) = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = \sum_{k=0}^{\infty} (ze^{aT})^{-k} \tag{8}$$

Again, this series can be written in closed form as:

$$F(z) = \frac{1}{1 - (ze^{aT})^{-1}} = \frac{z}{z - e^{-aT}} \tag{9}$$

In general we may show that

$$Z[e^{-at}f(t)] = F(e^{aT}z) \tag{10}$$

Table 1. z-Transforms

x(t)	X(s)	X(z)
$\delta(t) = \{ \begin{array}{c} 1, t=0 \\ 0, t=kT, k \neq 0 \end{array}$	1	1
$\delta(t-kT) = \{ \begin{array}{c} 1, t=kT \\ 0, t \neq kT, \end{array}$	e^{-kTs}	z^{-k}
u(t)=1, unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$1-e^{-at}$	$\frac{1}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
$sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{zsin\omega T}{z^2-2zcos\omega T+1}$

Table2. Properties of z-transform

$x(t)$	$X(z)$
$kx(t)$	$kX(z)$
$x_1(t)+x_2(t)$	$X_1(z)+X_2(z)$
$x(t+T)$	$zX(z)-zx(0)$
$tx(t)$	$-Tz\frac{dX(z)}{dz}$
$e^{-at}x(t)$	$X(ze^{aT})$
x(0), initial value	$lim_{z\to\infty}X(z)$ if the limit exists
x(∞), final value	$lim_{z\to 1}(z-1)X(z)$ if the limit exists and the system is stable

4 The pulse transfer function

Consider the block diagram shown in Figure 9. The signal $U^*(s)$ is a sampled input to $G(s)$ which gives a continuous output $Y(s)$ which when sampled becomes $Y^*(s)$. Figure (b) shows the pulse transfer function where $U(z)$ is equivalent to $U^*(s)$ and $Y(z)$ is equivalent to $Y^*(s)$, [1].

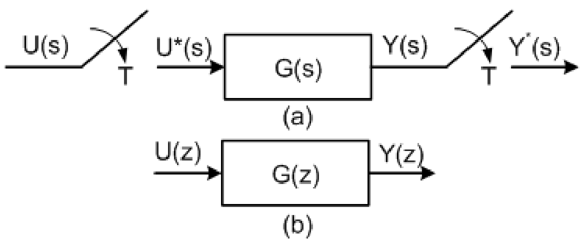

Figure 9: $G(s)$ and $G(z)$

The *pulse transfer function* is

$$G(z) = \frac{Y(z)}{U(z)} \tag{11}$$

4.1 Blocks in cascade

In Figure 10 there are samplers either side of the blocks $G_1(s)$ and $G_2(s)$. The pulse transfer function is:

$$\frac{Y(z)}{U(z)} = G_1(z)G_2(z) \tag{12}$$

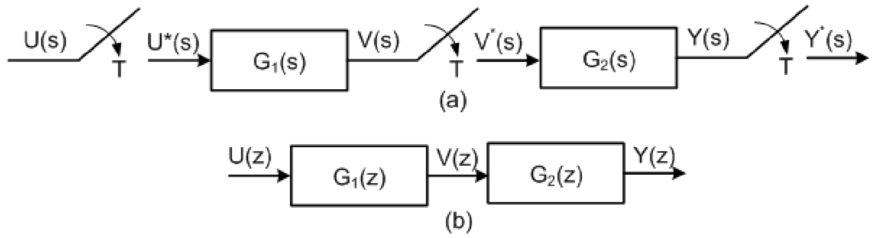

Figure 10: Blocks in cascade - all signals sampled

In figure 11 there is no sampler between $G_1(s)$ and $G_2(s)$ so they can be combined to give $G_1(s)G_2(s) = G_1G_2(s)$. The pulse transfer function is computed as:

$$\frac{Y(z)}{U(z)} = G_1G_2(z) = Z\{G_1G_2(s)\} \tag{13}$$

Note that $G_1(z)G_2(z) \neq G_1G_2(z)$.

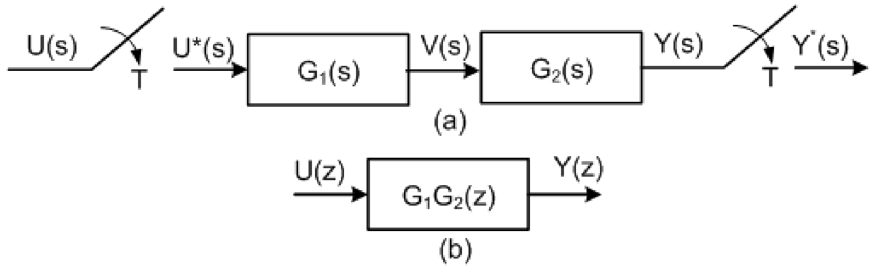

Figure 11: Blocks in cascade - input and output sampled

4.2 Inverse transformation

The discrete time response can be found using a number of methods.

4.2.1 Infinite power series method

Example. If the z-transform of a signal is obtained in the form:

$$X(z) = \frac{z^2}{z^2 - 1.36z + 0.36}, \tag{14}$$

by long division we obtain:

$$X(z) = z^2 \quad : \quad (z^2 - 1.36z + 0.36) = 1 + 1.36z^{-1} + 1.5z^{-2} + ... \tag{15}$$

From equation (5) we have:

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} = x(0)z^0 + x(T)z^{-1} + x(2T)z^{-2} + ... \tag{16}$$

Thus:

$$x(0) = 1, \quad x(T) = 1.36, \quad x(2T) = 1.5, \quad \text{etc} \tag{17}$$

4.2.2 Difference equation method

Consider a system of the form:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ...}{1 + a_1 z^{-1} + a_2 z^{-2} + ...} \tag{18}$$

By cross-multiplication we get:

$$(1 + a_1 z^{-1} + a_2 z^{-2} + ...)Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + ...)X(z) \tag{19}$$

or

$$Y(z) = -(a_1 z^{-1} + a_2 z^{-2} + ...)Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2} + ...)Y(z) \tag{20}$$

$$Y(z) = -a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + ... + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... \tag{21}$$

The equation above can be expressed as a difference equation of the form:

$$y(kT) = -a_1 y((k-1)T) + a_2 y((k-2)T) + ... + b_0 x(kT) + b_1 x((k-1)T) + b_2 x((k-2)T) + ... \tag{22}$$

or simply:

$$y_k = -a_1 y_{k-1} + a_2 y_{k-2} + ... + b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} + ... \tag{23}$$

Example. [1] Consider a transfer function:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - 0.368} \tag{24}$$

The equation (27) can be written as:

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.368 z^{-1}} \tag{25}$$

or

$$(1 - 0.368 z^{-1})Y(z) = X(z) \quad \text{or} \quad Y(z) = 0.368 z^{-1} Y(z) + X(z) \tag{26}$$

It can be expressed as a difference equation:

$$y_k = 0.368 y_{k-1} + x_k \tag{27}$$

If the first two samples are known, the values for y_k can be determined by an iterative procedure.

4.3 Digitizing analog transfer functions

4.3.1 ZOH

The conversion of an analog transfer function to a digital one is generally best done by one of the approximations of the z-transform. Just as the Laplace transform maps continuous time quantities into the complex frequency domain, thus allowing linear transfer function models to be written in terms of the variable s, so the z-transform maps linear transfer function models into discrete time, and allows transfer function models for discrete time systems to be written in terms of a variable z.

One approach in digitizing an analog transfer function is transformation into z-domain, as described in the previous subsections. Given an s-domain transfer function $G(s)$, the z-transform can be obtained from:

$$G(z) = Z\left(G_0(s)G(s)\right) = Z\left(\frac{1 - e^{-sT}}{s} G(s)\right) = (1 - z^{-1}) Z\left(\frac{G(s)}{s}\right) \tag{28}$$

where $G_0(s)$ is the transfer function of a zero-order hold, and T is the sampling time.

Example. Let us consider the system in Figure 12, for $T = 1$.

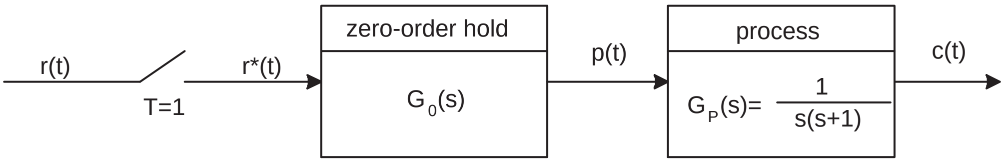

Figure 12: An open-loop sampled data system

The transfer function of a ZOH is

$$G_0(s) = \frac{1 - e^{-sT}}{s}$$

Therefore the transfer function C(s)/R*(s) is

$$\frac{C(s)}{R^*(s)} = G_0(s)G_P(s) = G(s) = \frac{1 - e^{-sT}}{s^2(s+1)} \tag{29}$$

Expanding into partial fractions, we have:

$$G(s) = (1 - e^{-sT}) \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right) \tag{30}$$

$$G(z) = Z[G(s)] = (1 - z^{-1}) Z \left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right] \tag{31}$$

Using the entries of Table 1 to obtain the z-transform of each term we have:

$$G(z) = Z[G(s)] = (1 - z^{-1}) \left(\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z - e^{-aT}} \right) = \frac{(ze^{-T} - z + Tz) + (1 - e^{-T} - Te^{-T})}{(z-1)(z - e^{-T})} \tag{32}$$

When T=1, we obtain:

$$G(z) = \frac{ze^{-1} + 1 - 2e^{-1}}{(z-1)(z - e^{-1})} = \frac{0.3678z + 0.2644}{(z-1)(z - 0.3678)} = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678} \tag{33}$$

The response of this system to unit impulse is obtained for R(z)=1 so that C(z)=G(z)·1. We may obtain C(z) by dividing the denominator into the numerator as:

$$C(z) = 0.3678z + 0.2644 : z^2 - 1.3678z + 0.3678 = 0.3678z^{-1} + 0.7675z^{-2} + 0.9145z^{-3} + ... \tag{34}$$

The calculation yields the response at the sampling instants and can be carried out asfar as is needed for C(z). From equation (5) we have:

$$C(z) = \sum_{k=0}^{\infty} c(kT) z^{-k} \tag{35}$$

In this case we have obtained c(kT) as follows: $c(0) = 0$, $c(T) = 0.3678$, $c(2T) = 0.7675$, $c(3T) = 0.9145...$

Note that $c(kT)$ provides the values of $c(t)$ at $t = kT$.

We have determined $C(z)$, the z-transform of the output sampled signal. The z-transform of the input signal is $R(z)$. The transfer function in the z-domain is:

$$\frac{C(z)}{R(z)} = G(z) \tag{36}$$

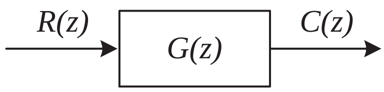

Figure 13: The z-transform transfer function in block diagram

4.3.2 Simple substitution

This variable z, strictly arises from a discrete time summation expression related to the definition of the Laplace transform. However, it is usually approximated by various substitutions for s in the linear transfer function model. These different possible derivations lead to different possible z-transform models of the same system, each having slightly different properties.

The variable z is first introduced in the simplest manner possible. That is, a multiplication by z^{-1} is used as a notation to represent a *delay* of one time step in a discrete-time signal.

In obtaining z-transfer functions from linear transfer functions by making substitutions for s, many of the generally used substitutions depend on the property that multiplying by s implies differentiation in the time domain. Multiplying the z-transform by z^{-1} implies an additional delay of one sampling interval. The argument for the simplest of the substitutions suggests that the gradient of the time response of, for example, a signal e(t) at the present time (instant k) is given by:

$$\frac{de}{dt}\mid_k = \frac{(latest\_sample\_of\_e) - (last\_sample\_of\_e)}{sampling\_interval} = \frac{e_k - e_{k-1}}{T} \tag{37}$$

In the z-domain, when e_k becomes E(z) (that is, the z-transform of e_k) this will convert to:

$$\frac{E(z) - E(z) \cdot z^{-1}}{T}$$

or

$$E(z)\frac{1 - z^{-1}}{T}$$

Multiplying E(z) by $\frac{1-z^{-1}}{T}$ is therefore equivalent to differentiation, that is, to multiplying E(s) by s. This means that substituting the following expressions for s in a linear transfer function:

$$s = \frac{1 - z^{-1}}{T} \tag{38}$$

will give an approximately equivalent discrete-time version (in z) for the analog linear transfer function. It is then easy to convert the z-transfer function to an algorithm from which a computer program can be written.

4.3.3 Tustin substitution

A more exact substitution for s (known as the Tustin transformation) is:

$$s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})} \tag{39}$$

This more complicated substitution often allows close approximation of the behavior of the analog controller using a longer sampling interval (T) than would be the case with the simpler substitution for s. The longer sampling interval is beneficial in that either a slower processor can be used to perform the calculation, or a given processor can cope with a faster sampling rate.

Example. Consider a continuous system having the transfer function:

$$G(s) = \frac{1}{s + 4} \tag{40}$$

First we shall determine the pulse transfer function using the ZOH method:

$$G_1(z) = Z\{G_0(s)G(s)\} = Z\{\frac{1-e^{-sT}}{s}\frac{1}{s+4}\} = (1-z^{-1})\frac{1}{4}Z\{\frac{1}{s} - \frac{1}{s+4}\} \tag{41}$$

Using Table 1 to determine the z-transforms of basic signals we obtain:

$$G_1(z) = \frac{z-1}{z}\frac{1}{4}(\frac{z}{z-1} - \frac{z}{z-e^{-4T}}) = \frac{1}{4}\frac{1-e^{-4T}}{z-e^{-4T}} \tag{42}$$

Using the simple substitution $s = \frac{1-z^{-1}}{T}$, the pulse transfer function is:

$$G_2(z) = \frac{1}{\frac{1-z^{-1}}{T}+4} = \frac{T}{1+4T-z^{-1}} \tag{43}$$

Using the Tustin substitution, the pulse transfer function result as:

$$G_3(z) = \frac{1}{\frac{2(1-z^{-1})}{T(1+z^{-1})}+4} = \frac{T(1+z^{-1})}{2+4T+(4T-2)z^{-1}} \tag{44}$$

For $T = 0.1$, the pulse transfer functions (written in z instead of z^{-1}) in all cases above are:

$$G_1(z) = \frac{0.08242}{z-0.6703}, \quad G_2(z) = \frac{0.1z}{1.4z-1}, \quad G_3(z) = \frac{0.04167z+0.04167}{z-0.6667} \tag{45}$$

The simulated unit step response in all three cases is shown in Figure 14

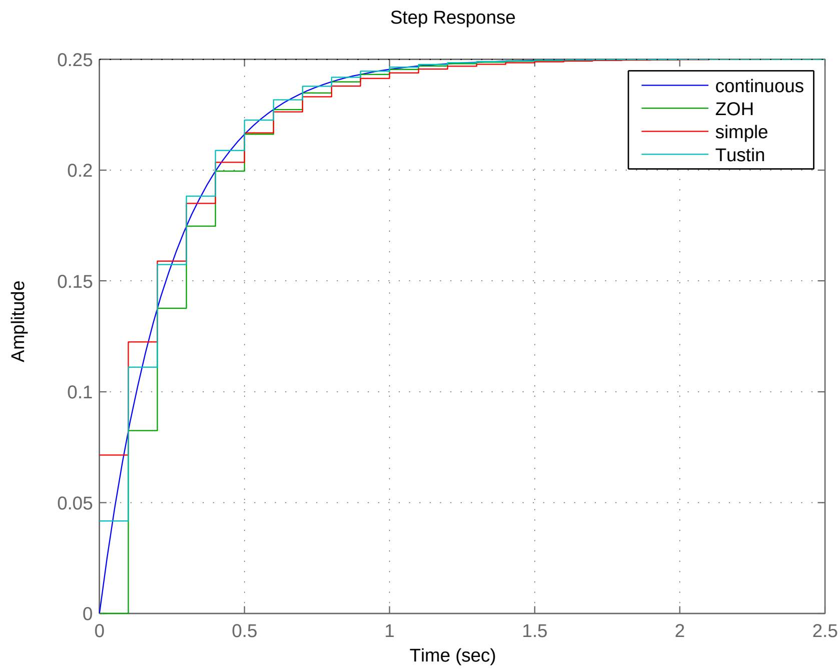

Figure 14: Step response

4.4 Closed-Loop Sampled Data Systems

Consider the sampled data z-transform model of a system with sampled output signal $C(z)$ shown in Figure 15. The closed-loop transfer function $T(z)$ (using block diagram reduction) is:

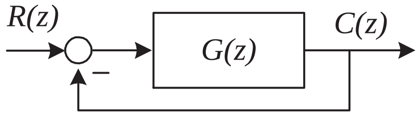

Figure 15: Feedback control system with unity feedback

$$\frac{C(z)}{R(z)} = T(z) = \frac{G(z)}{1+G(z)} \tag{46}$$

We assume that $G(z)$ is the z-transform of $G_0(s)G_p(s)$, where $G_0(s)$ is the zero-order hold and $G_p(s)$ is the plant transfer function.

A feedback control system with a digital controller is shown in Figure 16. The z-transform of the block diagram model is:

$$\frac{C(z)}{R(z)} = T(z) = \frac{G(z)D(z)}{1+G(z)D(z)} \tag{47}$$

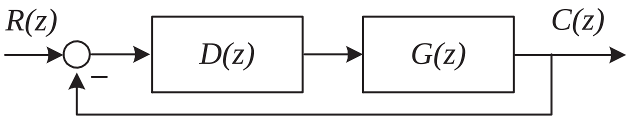

Figure 16: Feedback control system with digital controller

Example. Response of a closed-loop system

We consider a closed-loop sampled-data control system shown in Figure 17.

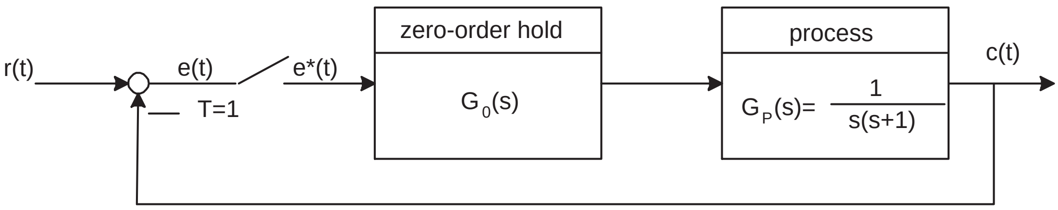

Figure 17: A closed-loop sampled data system

The z-transform of the open-loop system $G(z)$ was obtained in the previous Example (equation (33)). Therefore we have:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.3678z+0.2644}{z^2-z+0.6322} \tag{48}$$

Since the input is a unit step

$$R(z) = \frac{z}{z-1} \tag{49}$$

then

$$C(z) = \frac{z(0.3678z+0.2644)}{(z-1)(z^2-z+0.6322)} = \frac{0.3678z^2+0.2644z}{z^3-2z^2+1.6322z-0.6322} \tag{50}$$

Completing the division we have:

$$C(z) = 0.3678z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.147z^{-5} \tag{51}$$

The values of c(kT) are shown in Figure 18 (b). The complete response of the sampled-data closed-loop system is shown and contrasted to the response of a continuous system (where T=0). The overshoot of the sampled data system is 40% in contrast to 17% for a continuous system. Furthermore, the settling time is twice as long as of the continuous system.

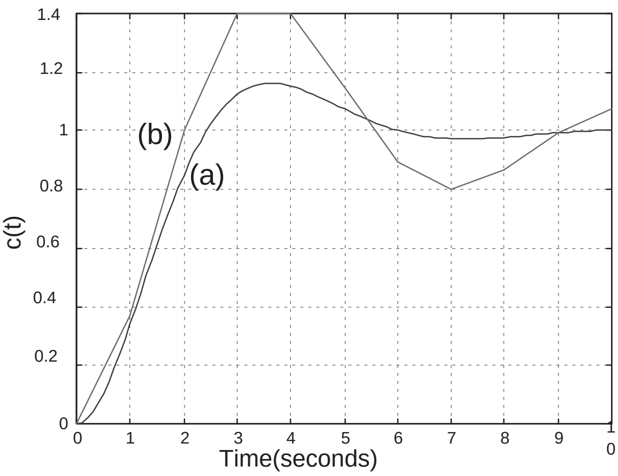

Figure 18: The response of a second-order system (a)continuous T=0 (b) sampled system T=1

5 Stability analysis in z-plane

A linear continuous feedback control system is stable if all the poles of the closed-loop transfer function $T(s)$ lie in the left half of the s-plane. The s-plane is related to the z-plane by the transformation:

$$z = e^{sT} = e^{(\sigma + j\omega)T} \tag{52}$$

We may also write this relationship as:

$$|z| = e^{\sigma T} \tag{53}$$

and

$$\angle z = \omega T \tag{54}$$

In the left-hand s-plane, the real part of s, $\sigma < 0$ and therefore the related magnitude of z varies between 0 and 1 ($0 < e^{\sigma T} < 1$). Therefore the imaginary axis of the s-plane corresponds to the unit circle in the z-plane, and the inside of the unit circle corresponds to the left half of the s-plane.

Therefore we can state that *a sampled data system is stable if all the poles of the closed-loop transfer function $T(z)$ lie within the unit circle of the z-plane.*

As shown in Figure 19 [4], each region of the s-plane can be mapped into a corresponding region on the z-plane. Points that have negative values of σ (left half s-plane, region A) map into the inside of the unit circle on the z-plane. Points on the $j\omega$ axis, region B, have zero values of σ and yield points on the unit circle on the z-plane. The points that have positive values of σ are in the right half of the s-plane, region C. The magnitudes of the mapped points are $e^{sigmaT} > 1$, thus they are mapped into points outside the circle on the z-plane.

Example. Consider a feedback control system as the one shown in Figure 15, where $G(z)$ is the z-transform of open-loop transfer function:

$$G(s) = \frac{k}{s(s+1)} \tag{55}$$

$$G(z) = Z\left(G_0(s)G(s)\right) = Z\left(\frac{1-e^{-sT}}{s}\frac{k}{s(s+1)}\right) = (1-z^{-1})Z\left(\frac{k}{s^2(s+1)}\right) \tag{56}$$

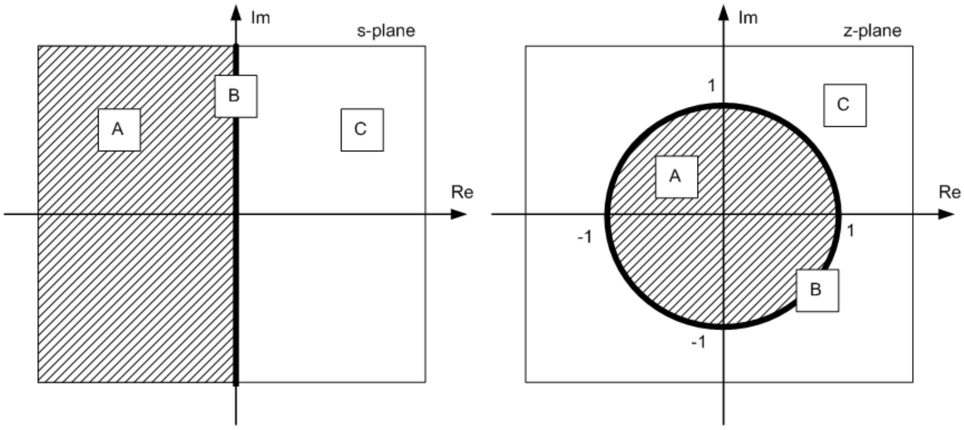

Figure 19: Mapping regions of the s-plane onto the z-plane

For $T = 1$, we have:

$$G(z) = \frac{k(0.367z + 0.264)}{z^2 - 1.367z + 0.367} \tag{57}$$

The closed-loop transfer function (Figure 15) is calculated as:

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{k(0.367z + 0.264)}{z^2 + (0.367k - 1.367)z + 0.264k + 0.367} \tag{58}$$

The poles of the closed-loop transfer function $T(z)$ are the roots of the equation $q(z) = 1 + G(z) = 0$ (the characteristic equation).

When $k = 1$, we have:

$$q(z) = z^2 + (0.367k - 1.367)z + 0.264k + 0.367 = z^2 - z + 0.631 \tag{59}$$

The roots of $q(z) = 0$ are: $z_1 = 0.5 + 0.6173j$ and $z_2 = 0.5 - 0.6173j$. The system is stable because the roots lie within the unit circle.

When $k = 10$ we have:

$$q(z) = z^2 + (0.367k - 1.367)z + 0.264k + 0.367 = z^2 + 2.31z + 3.01 \tag{60}$$

The roots of $q(z) = 0$ are: $z_1 = -1.1550 + 1.2946j$ and $z_2 = 1.1550 + 1.2946j$. The system is not stable because the roots lie outside the unit circle.

Example. A digital controller implemented following various different z-transform methods.

A plant modeled by the continuous transfer function:

$$H_P(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^3 + 7s^2 + 6s} \tag{61}$$

is to be controlled in the closed-loop with unity negative feedback, using a forward path lead compensator of transfer function

$$D(s) = \frac{1.5(s + 1)}{s + 3} \tag{62}$$

The controller is to be implemented in digital form. We shall investigate the performance of implementations having a sampling period of 0.1s and being converted into the z-domain by:

a) The simple method (substitution (38))

b) The Tustin method (substitution (39))

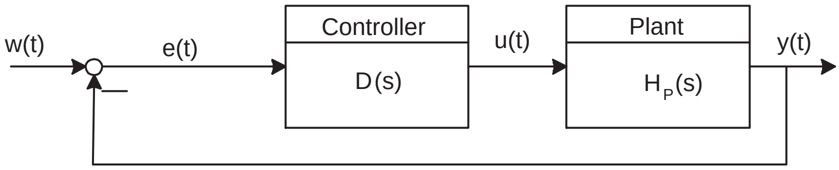

Figure 20: A closed-loop control system

c) The z-transform method

Evaluating the controller transfer functions:

a) Using equation (38):

$$s = \frac{1 - z^{-1}}{T}$$

with T=0.1s gives

$$s = 10(1 - z^{-1}) \tag{63}$$

and so

$$D(z) = \frac{1.5[10(1 - z^{-1}) + 1]}{10(1 - z^{-1}) + 3} = \frac{16.6 - 15z^{-1}}{13 - 10z^{-1}} = \frac{1.2692 - 1.1538z^{-1}}{1 - 0.7692z^{-1}} \tag{64}$$

b) Using Tustin substitution for s (39):

$$s = \frac{2(1 - z^{-1})}{T(1 + z^{-1})} \tag{65}$$

the transfer function converts to:

$$D(z) = \frac{1.5\left[\frac{2(1-z^{-1})}{T(1+z^{-1})} + 1\right]}{\left[\frac{2(1-z^{-1})}{T(1+z^{-1})} + 3\right]} = \frac{1.3696 - 1.2391z^{-1}}{1 - 0.7391z^{-1}} \tag{66}$$

c) Using the z-transform method

$$D(z) = Z\left(\frac{1 - e^{-sT}}{s}G(s)\right) = (1 - z^{-1})Z\left(\frac{1.5(s + 1)}{s(s + 3)}\right) = \frac{1.50 - 1.3704z^{-1}}{1 - 0.7408z^{-1}} \tag{67}$$

The step responses of the controlled systems may be compared by means of an appropriate computer package or by means of Matlab. Graphs of the step responses are shown in Figure 21 and the following table compares them for the maximum overshoot at the time at which it occurs.

Conversion type	*simple*	*Tustin*
overshoot %	3.6	2.9
peak time, sec	3.2	3.35

For comparison, the analog controller would give an overshoot of 1.6% at 3.5 seconds, so all the digital implementations produce some degree of performance degradation.

In order to produce a digital control system arrangement for one of the above transfer functions in z, it will be necessary to produce a suitable program for the processor. A first step in doing so is to convert the transfer function into a discrete time (difference) equation from which an algorithm or pseudo-code may be developed.

The compensator transfer function in z is, by definition, equal to the ratio $U(z)/E(z)$ of the z-transform of the controller action u and the error signal e, respectively, thus:

$$D(z) = \frac{U(z)}{E(z)} = \frac{1.2692 - 1.1538z^{-1}}{1 - 0.7692z^{-1}}$$

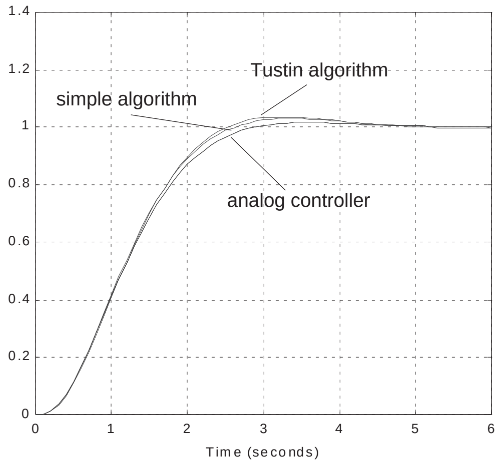

Figure 21: Closed-loop step responses

which will give

$$(1 - 0.7692z^{-1})U(z) = (1.2692 - 1.1538z^{-1})E(z)$$

Such equation can be returned easily to the time domain by recalling that multiplying by z^{-1} represents a delaying of the associated signal by one sampling interval. For example $E(z)$ represents the latest sample of the error signal and $z^{-1}E(z)$ represents the sample at the previous sampling interval. Returning to the time-domain notation, it can therefore be seen that

$$u(k) - 0.7692u(k-1) = 1.2692e(k) - 1.1538e(k-1)$$

or

$$u(k) = 0.7692u(k-1) + 1.2692e(k) - 1.1538e(k-1) \tag{68}$$

The outline algorithm could then be:

Initialize: set $u(k-1) = 0$, $e(k-1) = 0$ or to their actual values if they are available

Loop: Reset sampling interval timer

Input e (=$e(k)$)

Calculate $u(k)$ from equation (68)

Output $u(k)$

Set $u(k-1) = u(k)$

Set $e(k-1) = e(k)$

Wait for end of sampling interval

End Loop

The "Wait for end of sampling interval" is because the calculations are unlikely to take the same exact time and, even they do, that time is unlikely to be the sampling interval. The easiest way of ensuring that sampling does take place at equal intervals of time maybe to program the "Loop" routine as an interrupt routine which is called by an appropriate pulse train (often generated by a support chip having a software-settable timing signal generator)

Example. Implementation of Digital PID Controllers

We will consider the PID controller with a s-domain transfer function:

$$\frac{U(s)}{X(s)} = G_C(s) = k_1 + \frac{k_2}{s} + k_3 s \tag{69}$$

We can determine a digital implementation of this controller by using a discrete time approximation for the derivative and integration. For the time derivative we use the *backward difference rule*

$$u(kT) = \frac{dx}{dt}|_{t=kT} = \frac{1}{T}\left[x(kT) - x((k-1)T)\right] \tag{70}$$

The z-transform of equation (70) is then:

$$U(z) = \frac{1 - z^{-1}}{T} X(z) = \frac{z-1}{Tz} X(z) \tag{71}$$

The integration of x(t) can be represented by the *forward rectangular integration* at t=kT as:

$$u(kT) = u((k-1)T) + Tx(kT) \tag{72}$$

where $u(kT)$ is the output of the integrator at t=kT.

The z-transform of equation (72) is:

$$U(z) = z^{-1}U(z) + TX(z) \tag{73}$$

and the transfer function is then:

$$\frac{U(z)}{X(z)} = \frac{Tz}{z-1} \tag{74}$$

Hence, the z-domain transfer function of the PID controller is:

$$G_C(z) = k_1 + \frac{k_2 T z}{z-1} + k_3 \frac{z-1}{Tz} \tag{75}$$

The complete difference equation algorithm that provides PID controller is obtained by adding the three terms to obtain (we use x(kT)=x(k)):

$$u(k) = k_1 x(k) + k_2[u(k-1) + Tx(k)] + \frac{k_3}{T}[x(k) - x(k-1)] = [k_1 + k_2 T + \frac{k_3}{T}]x(k) + k_3 T x(k-1) + k_2 u(k-1) \tag{76}$$

Equation (76) can be implemented using a digital computer or microprocessor. Of course, we can obtain a PI or a PD controller by setting an appropriate gain equal to zero.

6 Exercises

6.1 Example 1: Pulse transfer function and system stability

Consider a first order system having the transfer function:

$$G(s) = \frac{1}{s+4} \tag{77}$$

Compute the pulse transfer function and analyze the system stability.

Solution. We shall determine the pulse transfer function $G(z)$ using two methods (ZOH and simple substitution), as shown in Chapter 6 (Digital control systems).

Using the ZOH method, $G(z)$ is obtained from:

$$G_1(z) = Z\{G_0(s)G(s)\} = Z\{\frac{1-e^{-sT}}{s}\frac{1}{s+4}\} = (1-z^{-1})\frac{1}{4}Z\{\frac{1}{s} - \frac{1}{s+4}\} \tag{78}$$

Using a table of the z-transforms of basic signals we obtain:

$$G_1(z) = \frac{z-1}{z}\frac{1}{4}(\frac{z}{z-1} - \frac{z}{z-e^{-4T}}) = \frac{1}{4}\frac{1-e^{-4T}}{z-e^{-4T}} \tag{79}$$

Using the simple substitution $s = \frac{1-z^{-1}}{T}$, the pulse transfer function is:

$$G_2(z) = \frac{1}{\frac{1-z^{-1}}{T}+4} = \frac{T}{1+4T-z^{-1}} = \frac{Tz}{(1+4T)z-1} \tag{80}$$

For $T = 0.1$, the pulse transfer functions (written in z instead of z^{-1}) in all cases above are:

$$G_1(z) = \frac{0.08242}{z-0.6703}, \quad G_2(z) = \frac{0.1z}{1.4z-1} \tag{81}$$

The continuous system (77) is clearly stable, since it has only one negative pole at -4.

The discrete system is described by any of the pulse transfer functions $G_1(z)$, $G_2(z)$. Each of them has one pole that is located inside the unit circle:

$$G_1(z): \quad z_1 = 0.6703 < 1, \quad G_2(z): \quad z_2 = \frac{1}{1.4} < 1 \tag{82}$$

thus the discrete system is stable, for both methods.

Moreover, if we analyze the general form of the pulse transfer functions (79) for ZOH and (80) for the simple substitution, the poles are smaller than 1 for any value of the sampling period T:

$$G_1(z): \quad z_1 = e^{-4T} < 1, \text{ and } G_2(z): \quad z_2 = \frac{1}{1+4T} < 1, \text{ for any } T > 0 \tag{83}$$

6.2 Example: Pulse transfer function and system stability

Consider a critically stable system having the transfer function:

$$G(s) = \frac{1}{s^2+1}$$

The system has two poles on the imaginary axis, $s_{1,2} = \pm j$, thus it is critically stable.

We shall determine the pulse transfer function $G(z)$ and analyze the stability of the discrete system.

Solution. Using the ZOH method, the pulse transfer function result as:

$$G_1(z) = Z\{G_0(s)G(s)\} = Z\{\frac{1-e^{-sT}}{s}\frac{1}{s^2+1}\} = (1-z^{-1})Z\{\frac{1}{s} - \frac{s}{s^2+1}\} \tag{84}$$

Using a table of the z-transforms of basic signals (a step and a cosine) we obtain:

$$G_1(z) = \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z(z-cosT)}{z^2-2zcosT+1}\right) \tag{85}$$

where the z-transform of the second term is obtained from:

$$£\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

and

$$Z\{\cos\omega t\} = \frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}, \text{ with } \omega = 1$$

Equation (85) can be written as:

$$G_1(z) = 1 - \frac{(z-1)(z-\cos T)}{z^2-2z\cos T+1} = \frac{z^2-2z\cos T+1-(z-1)(z-\cos T)}{z^2-2z\cos T+1} \tag{86}$$

The poles of the discrete system are the roots of the denominator:

$$z_{1,2} = \cos T \pm \sqrt{(\cos T)^2-1} = \cos T \pm j\sin T \tag{87}$$

The roots are complex and they are located exactly on the unit circle because the magnitude of the poles is unity: $|z_{1,2}| = \sqrt{(\cos T)^2+(\sin T)^2} = 1$.

We shall compare this result with the poles of the discrete system resulted by simple transformation $s = (1-z^{-1})/T = (z-1)/zT$:

$$G_2(z) = \frac{1}{\left(\frac{z-1}{zT}\right)^2+1} = \frac{z^2T^2}{(T^2+1)z^2-2z+1} \tag{88}$$

with the poles:

$$z_{1,2} = \frac{1\pm\sqrt{1-(T^2+1)}}{T^2+1} = \frac{1\pm Tj}{T^2+1} \tag{89}$$

It is clear that $z_{1,2}$ will have a magnitude of 1 only for $T=0$. For other values of the sampling time, the simple transformation is an inaccurate approximation. If, for example $T=0.1$:

$$z_{1,2} = \frac{1\pm 0.1j}{0.1^2+1} = 0.9901 \pm 0.0990j, \text{ and } |z_{1,2}| = 0.995 < 1$$

For $T=0.01$:

$$z_{1,2} = \frac{1\pm 0.01j}{0.01^2+1} = 0.9999 \pm 0.0100j, \text{ and } |z_{1,2}| = 0.99995 < 1$$

The approximation is better when the sampling period decreases towards 0.

6.3 Example: Pulse transfer function from a difference equation

Consider a system with input u and output y whose behavior can be described with a difference equation:

$$y(k) + 2y(k-1) + \frac{3}{4}y(k-2) = u(k-1) \tag{90}$$

1. Define the pulse transfer function of the system

2. Is the system stable? Justify your answer.

Obs. $y(k) = y(kT)$, $y(k-1) = y((k-1)T)$ where T is the sampling period

Solution. If the initial conditions are considered zero, the following property of the Z transform is useful:

$$\text{Shift right: } Z\{f(k-n)\} = z^{-n}F(z)$$

The z-transform of equation (93) is:

$$Y(z) + 2z^{-1}Y(z) + \frac{3}{4}z^{-2}Y(z) = z^{-1}U(z) \tag{91}$$

or

$$Y(z)(1 + 2z^{-1} + \frac{3}{4}z^{-2}) = z^{-1}U(z) \tag{92}$$

and the transfer function:

$$\frac{Y(z)}{U(z)} = \frac{z^{-1}}{1 + 2z^{-1} + \frac{3}{4}z^{-2}} = \frac{z}{z^2 + 2z + \frac{3}{4}} \tag{93}$$

The poles are:

$$z_{1,2} = -1 \pm \frac{1}{2}, \quad z_1 = -1.5, \quad z_2 = -0.5$$

One pole is located inside the unit circle while the other one is outside, therefore the system is unstable.

6.4 Example: Systems in cascade and the pulse transfer function

Determine the overall pulse transfer function for the systems shown in Figure 22. Compare cases (a) and (b).

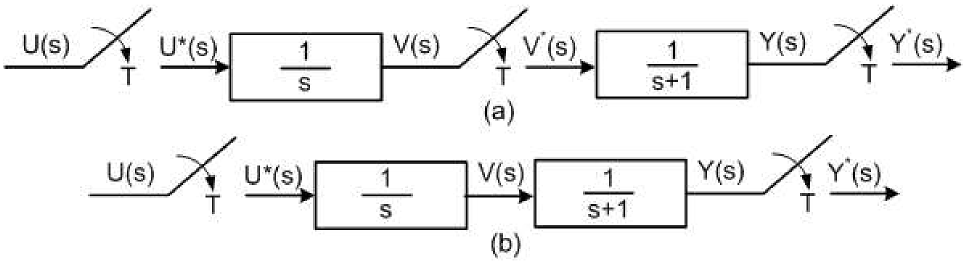

Figure 22: Systems in cascade

Solution. In case (a), all signals are sampled, therefore we can determine a pulse transfer function for each of the blocks shown in Figure 22. For case (b), an overall transfer function for the continuous systems must be computed and the pulse transfer function will be determined as explained in Chapter 6 (Section 4). Let

$$G_1(s) = \frac{1}{s}, \text{ and } G_2(s) = \frac{1}{s+1}$$

(a) In the first case:

$$G_1(z) = (1 - z^{-1})Z\left(\frac{G_1(s)}{s}\right) = (1 - z^{-1})Z\left(\frac{1}{s^2}\right) = (1 - z^{-1})\frac{Tz}{(z-1)^2} = \frac{T}{z-1}$$

$$G_2(z) = (1 - z^{-1})Z\left(\frac{G_2(s)}{s}\right) = (1 - z^{-1})Z\left(\frac{1}{s(s+1)}\right) = (1 - z^{-1})Z\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

or

$$G_2(z) = \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right) = \frac{1-e^{-T}}{z-e^{-T}}$$

The overall transfer function is

$$G_{0a}(z) = G_1(z)G_2(z) = \frac{T}{z-1}\frac{1-e^{-T}}{z-e^{-T}}$$

(b) In the second case, the continuous equivalent transfer function is:

$$G_1G_2(s) = \frac{1}{s(s+1)}$$

Then:

$$G_{0b}(z) = (1-z^{-1})Z\{\frac{G_1G_2(s)}{s}\} = (1-z^{-1})Z\{\frac{1}{s^2(s+1)}\} = (1-z^{-1})Z\{\frac{1}{s+1} + \frac{1}{s^2} - \frac{1}{s}\}$$

$$G_{0b}(z) = \frac{z-1}{z}\left(\frac{z}{z-e^{-T}} + \frac{Tz}{(z-1)^2} - \frac{z}{z-1}\right)$$

and:

$$G_{0b}(z) = \frac{z-1}{z-e^{-T}} + \frac{T}{z-1} - 1 = \frac{(z-1)^2 + (z-e^{-T} - (z-1)(z-e^{-T}))}{(z-1)(z-e^{-T})} = \frac{ze^{-T} + 1 - 2e^{-T}}{(z-1)(z-e^{-T})}$$

It is clear that $G_{0a}(z) \neq G_{0b}(z)$ for any positive value of T.

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