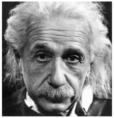
Modelare matematica. Identificarea sistemelor. Linearizarea modelelor neliniare. Functii de transfer in caracterizarea sistemelor liniare invariante in timp. Scheme bloc in modelare. Conexiuni de sisteme

Special thanks to:
A. Bemporad, Automatic Control 1, Lecture Notes, University of Trento, Italy, 2011, http://cse.lab.imtlucca.it/~bemporad/automatic_control_course.html
P. Raica, Systems Theory, Lecture Notes, Technical University of Cluj-Napoca, Cluj-Napoca, 2012, https://cs.utcluj.ro/files/educatie/licenta/2021-2022/22_CALCen_ST.pdf

Introduction

- Objective: Develop mathematical models of physical systems often encountered in practice
- Why? Mathematical models allow us to capture the main phenomena that take place in the system, in order to analyze, simulate, and control it
- We focus on *dynamical* models of *physical* (mechanical, electrical, thermal, hydraulic) systems

- Remember: A physical model for control design purposes should be
  - *Descriptive:* able to capture the main features of the system
  - *Simple:* the simpler the model, the simpler will be the synthesized control algorithm



Albert Einstein
(1879-1955)

"Make everything as simple as possible, but not simpler."
– Albert Einstein

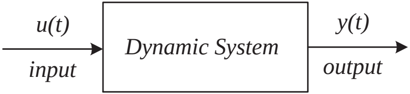
Today you will learn some basics of the art of modeling dynamical systems ...

A **mathematical model** is an equation or set of equations which adequately describes the behavior of a system.

Two approaches to finding the model:

- Lumped-parameter modeling: for each element a mathematical description is established from the physical laws.
- System identification: an experiment can be carried out and a mathematical model can be found from the results.

The important relationship is that between the manipulated inputs and measurable outputs.



The systems studied in this course are:

Linear - must obey the principle of superposition

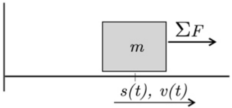
Stationary (or time invariant) - the parameters inside the element must not vary with time.

Deterministic - The outputs of the system at any time can be determined from a knowledge of the system's inputs up to that time.

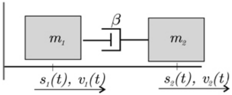
Examples.

- The resistor: $i(t) = \frac{1}{R}v(t)$
- The inductor: $i(t) = \frac{1}{L}\int v(t)dt$ or $v(t) = L\frac{di(t)}{dt}$
- The capacitor: $i(t) = C\frac{dv(t)}{dt}$

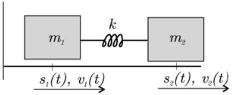
Mechanical systems – Linear motion



Newton's Law:
$$\sum F(t) = m\frac{dv(t)}{dt} = m\frac{d^2s(t)}{dt^2}$$

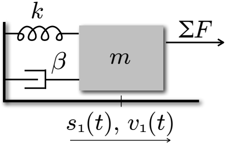


Viscous friction:
$$F_1(t) = \beta(v_2(t) - v_1(t)) = -F_2(t)$$



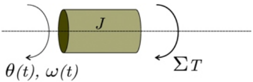
Elastic coupling:
$$F_1(t) = k(s_2(t) - s_1(t)) = -F_2(t)$$

- $s_i(t), v_i(t)$ = *position* and *velocity* of body i, with respect to a fixed (inertial) reference frame
- $F_i(t)$ = *force* acting on body i
- m, β, k = *mass*, *viscous friction coefficient*, *spring constant*



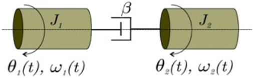
- Special case: $s_2(t) \equiv 0$, $v_2(t) \equiv 0$
  $F_1(t) = -ks_1(t)$, $F_1(t) = -\beta v_1(t)$

Mechanical systems – Rotational motion



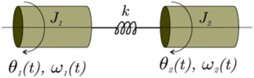
Newton's Law:

$$\sum \tau(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$



Viscous friction:

$$\tau_1(t) = \beta(\omega_2(t) - \omega_1(t)) = -\tau_2(t)$$



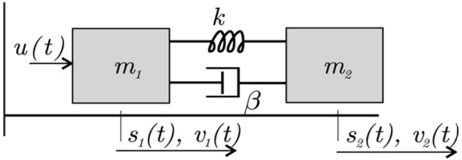
Elastic coupling:

$$\tau_1(t) = k(\theta_2(t) - \theta_1(t)) = -\tau_2(t)$$

- $\theta_i(t)$, $\omega_i(t)$ = *angular position* and *angular velocity* of body i, with respect to a fixed (inertial) reference frame
- $\tau_i(t)$: *torque* acting on body i
- J, β, k: *inertia*, *viscous friction coefficient*, *spring constant*

Example of mechanical system

Two masses connected by spring-damper (no dry friction with the surface)



Dynamics of mass m_1:

$$m_1 \frac{dv_1(t)}{dt} = u(t) + k(s_2(t) - s_1(t)) + \beta(v_2(t) - v_1(t))$$

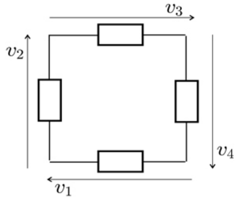
Dynamics of mass m_2:

$$m_2 \frac{dv_2(t)}{dt} = -k(s_2(t) - s_1(t)) - \beta(v_2(t) - v_1(t))$$

Note: viscous and elastic forces always oppose motion

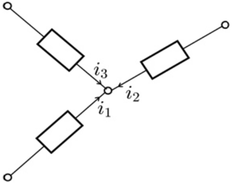
Electrical systems

Kirchhoff's voltage law: balance of voltages on a closed circuit



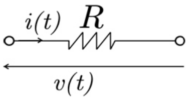
$$v_1 + v_2 + v_3 + v_4 = 0$$

Kirchhoff's current law: balance of the currents at a node

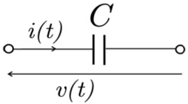


$$i_1 + i_2 + i_3 = 0$$

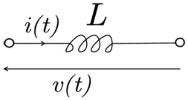
Electrical systems



Resistor: $v(t) = Ri(t)$



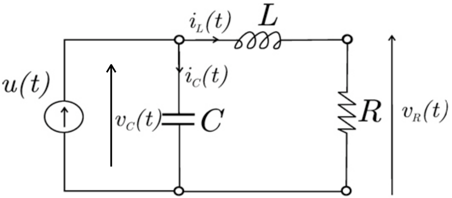
Capacitor: $i(t) = C\frac{dv(t)}{dt}$



Inductor: $v(t) = L\frac{di(t)}{dt}$

- $v(t)$: *voltage* across the component
- $i(t)$: *current* through the component
- R, C, L: *resistance*, *capacitance*, *inductance*

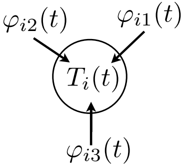
Example of electrical system



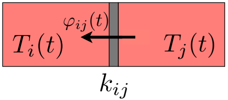
Kirchhoff's current law: $i_C(t) = C\frac{dv_C(t)}{dt} = u(t) - i_L(t)$

Kirchhoff's voltage law: $v_c(t) - L\frac{di_L(t)}{dt} - Ri_L(t) = 0$

Thermal systems



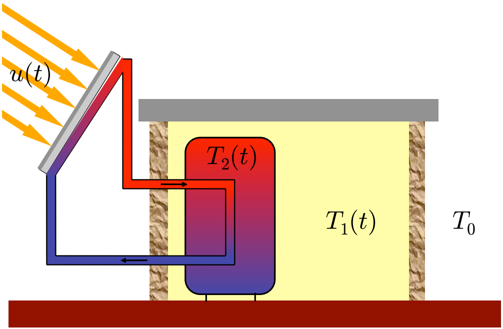
Heat transfer: energy balance
$\sum_j \varphi_{ij}(t) = C_i \frac{dT_i(t)}{dt}$



Conduction and/or convection
$\varphi_{ij}(t) = k_{ij}(T_j(t) - T_i(t))$

- $T_i(t)$, C_i = *temperature* and *heat capacity* of body i
- k_{ij} = *heat exchange coefficient* ($R_{ij} = 1/k_{ij}$ = *thermal resistance*)
- $\varphi_{ij}(t)$ = *thermal power* (=heat flow) from body j to body i

Example of thermal system



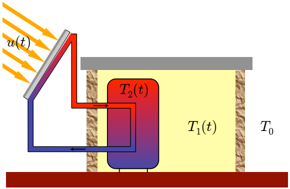
Heat transfer: energy balance

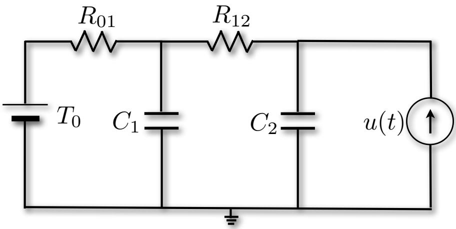
$$C_1\dot{T}_1(t) = -k_{01}(T_1(t) - T_0) + k_{12}(T_2(t) - T_1(t))$$
$$C_2\dot{T}_2(t) = -k_{12}(T_2(t) - T_1(t)) + u(t)$$

Electrical equivalent of thermal systems

thermal model	**electrical model**
reference temperature	ground
body	node
thermal capacitance	electrical capacitance connected to ground
thermal resistance	electrical resistance between nodes
thermal flow	current
temperature	voltage
thermal power input	current generator
constant temperature body	voltage generator

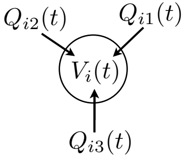
For the previous example:



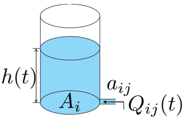


Hydraulic systems

Assumptions: the fluid is perfect (no shear stresses, no viscosity, no heat conduction), and subject only to gravity. Only one fluid is considered with constant density ρ (incompressible fluid). The orifices in the tanks are always at the bottom. The external pressure is constant (atmospheric pressure)



Mass (volume) balance
$$\sum_j Q_{ij}(t) = \frac{dV_i(t)}{dt} = A_i \frac{dh_i(t)}{dt}$$

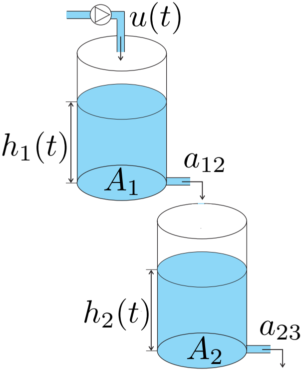


Torricelli's law

$$Q_{ij}(t) = -a_{ij}\sqrt{2gh_i(t)}$$

- A_i, $h_i(t)$ = *base area* and *fluid level* in tank i
- $Q_{ij}(t)$, a_{ij} = *volume flow* from tank j to tank i, area of orifice
- g: gravitational acceleration

Example of hydraulic system



Mass (volume) balance

$$A_1\dot{h}_1(t) = -a_{12}\sqrt{2gh_1(t)} + u(t)$$

$$A_2\dot{h}_2(t) = a_{12}\sqrt{2gh_1(t)} - a_{23}\sqrt{2gh_2(t)}$$

Choice of state variables

To obtain a state-space model one must choose state variables. How?

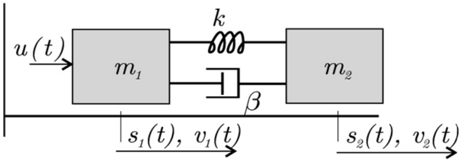
Rule of thumb: # state variables = # of energy storage elements

type	**element**	**energy**	**state**
mechanical	mass	kinetic energy: $\frac{1}{2}mv^2$	velocity
	spring	potential elastic energy: $\frac{1}{2}ks^2$	position
electrical	inductor	potential magnetic energy: $\frac{1}{2}Li^2$	current
	capacitor	potential electric energy: $\frac{1}{2}Cv^2$	voltage
thermal	body	internal energy: CT	temperature
hydraulic	tank	potential gravitational energy: ρgh	height

Choice of state variables also depends on selected output variables of interest ...

Example of mechanical system

Two masses connected by spring-damper (no dry friction on surface)



Dynamics of mass m_1:

$$m_1 \frac{dv_1(t)}{dt} = u(t) + k(s_2(t) - s_1(t)) + \beta(v_2(t) - v_1(t))$$

Dynamics of mass m_2:

$$m_2 \frac{dv_2(t)}{dt} = -k(s_2(t) - s_1(t)) - \beta(v_2(t) - v_1(t))$$

Example of mechanical system

Case 1. Output: $y = v_2$. Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_2 - s_1 \\ v_1 \\ v_2 \end{bmatrix}$$

$$y = x_3$$

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) - x_2(t) \\ \dot{x}_2(t) &= \frac{k}{m_1} x_1(t) + \frac{\beta}{m_1}(x_3(t) - x_2(t)) + \frac{1}{m_1} u(t) \\ \dot{x}_3(t) &= -\frac{k}{m_2} x_1(t) - \frac{\beta}{m_2}(x_3(t) - x_2(t)) \end{aligned}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 & 1 \\ \frac{k}{m_1} & -\frac{\beta}{m_1} & \frac{\beta}{m_1} \\ -\frac{k}{m_2} & \frac{\beta}{m_2} & -\frac{\beta}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t)$$

Example of mechanical system

Case 2. Output: $y = s_2$. Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ v_1 \\ s_2 \\ v_2 \end{bmatrix}$$

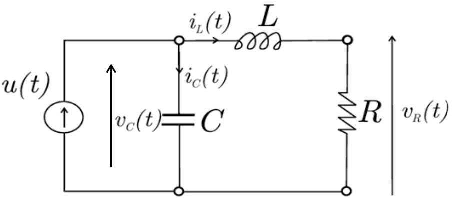
$$y = x_3$$

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{k}{m_1}(x_3(t) - x_1(t)) + \frac{\beta}{m_1}(x_4(t) - x_2(t)) + \frac{1}{m_1}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{m_2}(x_1(t) - x_3(t)) + \frac{\beta}{m_2}(x_2(t) - x_4(t))
\end{aligned}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{\beta}{m_1} & \frac{k}{m_1} & \frac{\beta}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{\beta}{m_2} & -\frac{k}{m_2} & -\frac{\beta}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

Example of electrical system



Kirchhoff's current law: $i_C = C\frac{dv_C}{dt} = u(t) - i_L$

Kirchhoff's voltage law: $L\frac{di_L}{dt} + Ri_L - v_c = 0$

Example of electrical system

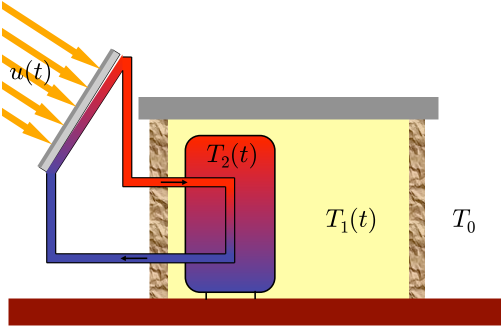
System output: $y = v_R = Ri_L$. Choice of state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$
$$y = Rx_2$$

$$\dot{x}_1(t) = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t)$$
$$\dot{x}_2(t) = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & R \end{bmatrix} x(t)$$

Example of thermal system



Heat transfer: energy balance

$$C_1\dot{T}_1(t) = -k_{01}(T_1(t) - T_0) + k_{12}(T_2(t) - T_1(t))$$
$$C_2\dot{T}_2(t) = -k_{12}(T_2(t) - T_1(t)) + u(t)$$

Example of thermal system

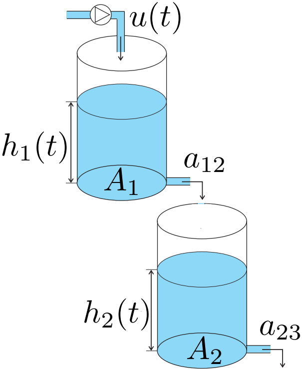
System output: $y = T_1 - T_0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1 - T_0 \\ T_2 - T_0 \end{bmatrix}$$
$$y = T_1 - T_0$$

$$\dot{x}_1(t) = \frac{1}{C_1}(-k_{12} - k_{01})x_1(t) + \frac{k_{12}}{C_1}x_2(t)$$
$$\dot{x}_2(t) = \frac{k_{12}}{C_2}x_1(t) - \frac{k_{12}}{C_2}x_2(t) + \frac{1}{C_2}u(t)$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{k_{12}+k_{01}}{C_1} & \frac{k_{12}}{C_1} \\ \frac{k_{12}}{C_2} & -\frac{k_{12}}{C_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{C_2} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Example of hydraulic system



Mass (volume) balance

$$A_1\dot{h}_1(t) = -a_{12}\sqrt{2gh_1(t)} + u(t)$$
$$A_2\dot{h}_2(t) = a_{12}\sqrt{2gh_1(t)} - a_{23}\sqrt{2gh_2(t)}$$

Example of hydraulic system

System output: h_2. Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$y = h_2$$

$$\dot{x}_1(t) = -\frac{a_{12}}{A_1}\sqrt{2gx_1(t)} + \frac{1}{A_1}u(t)$$

$$\dot{x}_2(t) = \frac{a_{12}}{A_2}\sqrt{2gx_1(t)} - \frac{a_{23}}{A_2}\sqrt{2gx_2(t)}$$

The model is nonlinear !

- We want to linearize the model around the equilibrium point (x_{1r}, x_{2r}), corresponding to the constant input u_r
- The linearized model will be useful to control the system near the equilibrium point

Example of hydraulic system

- Zero state derivatives

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), u(t)) = -\frac{a_{12}}{A_1}\sqrt{2gx_1(t)} + \frac{1}{A_1}u(t) = 0$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), u(t)) = \frac{a_{12}}{A_2}\sqrt{2gx_1(t)} - \frac{a_{23}}{A_2}\sqrt{2gx_2(t)} = 0$$

$$y(t) = \gamma(x_1(t), x_2(t), u(t)) = x_2(t)$$

- Substitute $u(t) = u_r$ and get $x_{1r} = \frac{u_r^2}{2ga_{12}^2}, x_{2r} = \frac{u_r^2}{2ga_{23}^2}, y_r = x_{2r}$
- Linearize

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \rightarrow \begin{array}{l} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow A = \begin{bmatrix} -\frac{a_{12}^2 g}{A_1 u_r} & 0 \\ \frac{a_{12}^2 g}{A_2 u_r} & -\frac{a_{23}^2 g}{A_2 u_r} \end{bmatrix}$$

Example of hydraulic system

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \rightarrow \begin{array}{l} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow B = \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix}$$

Note that here the input enters the state-update equation linearly, so there is no need to compute $\frac{\partial f_i}{\partial u}$ to get B

$$C = \begin{bmatrix} \frac{\partial \gamma}{\partial x_1} & \frac{\partial \gamma}{\partial x_2} \end{bmatrix} \rightarrow \begin{array}{l} \text{substitute } u = u_r, \\ x_1 = x_{1r},\ x_2 = x_{2r} \end{array} \rightarrow C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

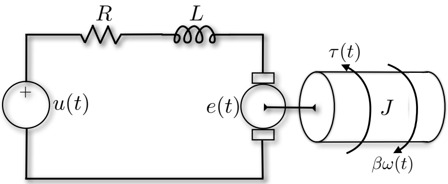
The output equation is also linear, and one can directly obtain C

The overall linearized system (with $\Delta x(t) = x(t) - x_r$, $\Delta u(t) = u(t) - u_r$, and $\Delta y(t) = y(t) - y_r$) is

$$\begin{aligned} \dot{\Delta x}(t) &= \begin{bmatrix} -\frac{a_{12}^2 g}{A_1 u_r} & 0 \\ \frac{a_{12}^2 g}{A_2 u_r} & -\frac{a_{23}^2 g}{A_2 u_r} \end{bmatrix} \Delta x(t) + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} \Delta u(t) \\ \Delta y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \Delta x(t) \end{aligned}$$

Electrical DC motor

Example of a (very common) system involving mechanical and electrical models



- Electrical part: $L\frac{di(t)}{dt} + Ri(t) + e(t) = u(t)$
  The back emf $e(t)$ is proportional to the motor speed: $e(t) = K\omega(t)$
- Mechanical part: $J\frac{d\omega(t)}{dt} + \beta\omega(t) = \tau(t)$
  The torque $\tau(t)$ is proportional to the armature current: $\tau(t) = Ki(t)$
- Overall model

$$L\frac{di(t)}{dt} = u(t) - Ri(t) - K\omega(t)$$
$$J\frac{d\omega(t)}{dt} = Ki(t) - \beta\omega(t)$$

Electrical DC motor

Case 1. System output: $y = \omega$

Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i \end{bmatrix}$$

$$y = \omega$$

$$\dot{x}_1(t) = \frac{K}{J} x_2(t) - \frac{\beta}{J} x_1(t)$$

$$\dot{x}_2(t) = \frac{1}{L} u(t) - \frac{R}{L} x_2(t) - \frac{K}{L} x_1(t)$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{\beta}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Electrical DC motor

Case 2. System output: $y = \theta$, angular position

Choice of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \omega \\ i \end{bmatrix}$$

$$y = \theta$$

$$\dot{x}_1(t) = x_2(t)$$

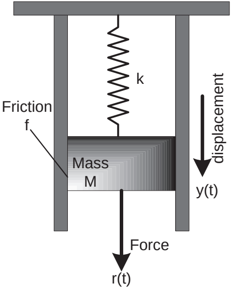
$$\dot{x}_2(t) = \frac{K}{J}x_3(t) - \frac{\beta}{J}x_2(t)$$

$$\dot{x}_3(t) = \frac{1}{L}u(t) - \frac{R}{L}x_3(t) - \frac{K}{L}x_2(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{\beta}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t)$$

Example



$$M\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + ky(t) = r(t)$$

where: f is the friction coefficient, M - the mass, k - the stiffness of the linear spring.

Linear approximation

A system is defined as linear in terms of the system excitation and response.

The principle of superposition

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) \\ x_2(t) &\rightarrow y_2(t) \\ x_1(t) + x_2(t) &\rightarrow y_1(t) + y_2(t) \end{aligned}$$

Homogeneity

$$\begin{aligned} x(t) &\rightarrow y(t) \\ mx(t) &\rightarrow my(t) \end{aligned}$$

Linear approximation

Nonlinear system

$$y = x^2$$

Nonlinear system

$$y = mx + b$$

Linear about an operating point x_0, y_0 for small changes Δx and Δy. When $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$:

$$y_0 + \Delta y = mx_0 + m\Delta x + b$$

and therefore

$$\Delta y = m\Delta x$$

Input $x(t)$ and a response $y(t)$: $y(t) = g(x(t))$
Taylor series expansion about the operating point x_0:

$$y = g(x) = g(x_0) + \frac{dg}{dx}|_{x=x_0} \frac{x - x_0}{1!} + \frac{d^2 g}{dx^2}|_{x=x_0} \frac{(x - x_0)^2}{2!} + \ldots$$

Linear approximation

The slope at the operating point,

$$m = \frac{dg}{dx}|_{x=x_0},$$

$$y = g(x_0) + \frac{dg}{dx}|_{x=x_0}(x - x_0) = y_0 + m(x - x_0),$$

Finally, this equation can be rewritten as the linear equation

$$(y - y_0) = m(x - x_0) \text{ or } \Delta y = m\Delta x$$

Linear approximation

If the dependent variable y depends upon several excitation variables $x_1, x_2, ..., x_n$:

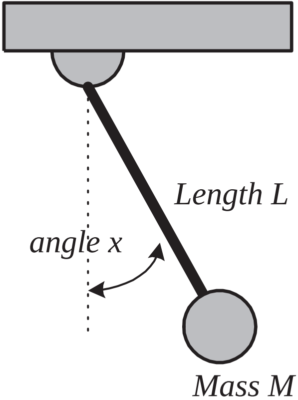
$$y = g(x_1, x_2, ..., x_n).$$

The Taylor series expansion about the operating point $x_{10}, x_{20}, ..., x_{n0}$ (the higher-order terms are neglected):

$$\begin{aligned} y &= g(x_{10}, x_{20}, ..., x_{n0}) + \frac{dg}{dx_1}|_{x=x_0}(x_1 - x_{10}) + \\ &+ \frac{dg}{dx_2}|_{x=x_0}(x_2 - x_{20}) + ... + \frac{dg}{dx_n}|_{x=x_0}(x_n - x_{n0}) \end{aligned}$$

where x_0 is the operating point.

Example - pendulum oscillator



The torque on the mass is:
$T = MgLsin(x)$
The equilibrium condition for the mass is $x_0 = 0^o$.

$$T - T_0 \cong MgL\frac{\partial sinx}{\partial x}|_{x=x_0}(x - x_0),$$

where $T_0 = 0$.

$$T = MgL(cos0^o)(x - 0^o) = MgLx$$

The approximation is reasonably accurate for $-\pi/4 \leq x \leq \pi/4$.

Laplace transform

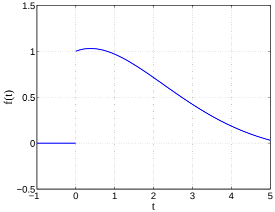
Consider a function $f(t)$, $f : \mathbb{R} \to \mathbb{R}$, $f(t) = 0$ for all $t < 0$.

Definition

The *Laplace transform* $\mathscr{L}[f]$ of f is the function $F : \mathbb{C} \to \mathbb{C}$ of complex variable $s \in \mathbb{C}$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

for all $s \in \mathbb{C}$ for which the integral exists



Pierre-Simon Laplace (1749-1827)

Once $F(s)$ is computed using the integral, it's extended to all $s \in \mathbb{C}$ for which $F(s)$ makes sense

Laplace transforms convert integral and differential equations into algebraic equations. We'll see how ...

Examples of Laplace transforms

- *Unit step*

$$f(t) = 1\!\!\mathrm{I}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \quad \Rightarrow \quad F(s) = \int_0^{+\infty} e^{-st} dt = -\frac{1}{s}\bigg|_0^{\infty} = \frac{1}{s}$$

- *Dirac's delta* (or *impulse function*[1])

$$f(t) = \delta(t) \triangleq \begin{cases} 0 & \text{if } t \neq 0 \\ +\infty & \text{if } t = 0 \end{cases} \quad \text{such that} \quad \int_{-\infty}^{\infty} \delta(t) = 1$$

$$F(s) = 1 = \int_0^{+\infty} \delta(t) e^{-st} dt = e^{-s0} = 1, \ \forall s \in \mathbb{C}$$

---

[1] The function $\delta(t)$ is can be considered as the limit of the sequence of functions $f_\epsilon(t)$ for $\epsilon \to 0$

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{se } 0 \leq t \leq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

To be mathematically correct, Dirac's δ is a *distribution*, not a function

Properties of Laplace transforms

- *Linearity*

$$\mathscr{L}[k_1 f_1(t) + k_2 f_2(t)] = k_1 \mathscr{L}[f_1(t)] + k_2 \mathscr{L}[f_2(t)]$$

Example: $f(t) = \delta(t) - 2\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = 1 - \frac{2}{s}$

- *Time delay*

$$\mathscr{L}[f(t-\tau)] = e^{-s\tau}\mathscr{L}[f(t)]$$

Example: $f(t) = 3\,\mathbb{1}(t-2) \Rightarrow \mathscr{L}[f] = \frac{3e^{-2s}}{s}$

- *Exponential scaling*

$$\mathscr{L}[e^{at}f(t)] = F(s-a), \text{ where } F(s) = \mathscr{L}[f(t)]$$

Example: $f(t) = e^{at}\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = \frac{1}{s-a}$

Example: $f(t) = \cos(\omega t)\,\mathbb{1}(t) = \frac{e^{j\omega t}+e^{-j\omega t}}{2}\,\mathbb{1}(t) \Rightarrow \mathscr{L}[f] = \frac{s}{s^2+\omega^2}$

Properties of Laplace transforms

- *Time derivative*[2]:

$$\mathscr{L}[\frac{d}{dt}f(t)] = s\mathscr{L}[f(t)] - f(0^+)$$

Example $\Longrightarrow f(t) = \sin(\omega t)\,\mathbb{1}(t) \Rightarrow L[f] = \frac{\omega}{s^2+\omega^2}$

- *Multiplication by t*

$$\mathscr{L}[tf(t)] = -\frac{d}{ds}\mathscr{L}[f(t)]$$

Example $\Longrightarrow f(t) = t\,\mathbb{1}(t) \Rightarrow L[f] = \frac{1}{s^2}$

[2] $f(0^+) = \lim_{t\to 0^+} f(t)$. If f is continuos in 0, $f(0^+) = f(0)$

Initial and final value theorems

Initial value theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Example: $f(t) = 1\!\!1(t) - t\,1\!\!1(t) \Rightarrow F(s) = \frac{1}{s} - \frac{1}{s^2}$
$f(0^+) = 1 = \lim_{s \to \infty} sF(s)$

Final value theorem

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} sF(s)$$

Example: $f(t) = 1\!\!1(t) - e^{-t}\,1\!\!1(t) \Rightarrow F(s) = \frac{1}{s} - \frac{1}{s+1}$
$f(+\infty) = 1 = \lim_{s \to 0} sF(s)$

Convolution

- The *convolution* $h = f * g$ of two signals f and g is the signal

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

- It's easy to see that $h = f * g = g * f$
- The Laplace transform of the convolution:

$$\mathscr{L}[f(t) * g(t)] = \mathscr{L}[f(t)]\mathscr{L}[g(t)]$$

- Laplace transforms turn convolution into multiplication !

Common Laplace transforms

1	$\dfrac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\dfrac{1}{s^2}$
$\dfrac{t^k}{k!},\ k \geq 0$	$\dfrac{1}{s^{k+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2} = \dfrac{1/2}{s-j\omega} + \dfrac{1/2}{s+j\omega}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2} = \dfrac{1/2j}{s-j\omega} - \dfrac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\dfrac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$
$e^{-at}\cos \omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin \omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$

In MATLAB use

`F = LAPLACE(f)`

MATLAB

```
» syms t
» f=exp(2*t)+t-t^2
» F=laplace(f)

F =

1/(s-2)+1/s^2-2/s^3
```

courtesy of S. Boyd, `http://www.stanford.edu/~boyd/ee102/`

Properties of Laplace transforms

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\, dt$
$f + g$	$F + G$
$\alpha f \ (\alpha \in \mathbf{R})$	αF
$\dfrac{df}{dt}$	$sF(s) - f(0)$
$\dfrac{d^k f}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2}\dfrac{df}{dt}(0) - \cdots - \dfrac{d^{k-1}f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\, d\tau$	$G(s) = \dfrac{F(s)}{s}$
$f(\alpha t),\ \alpha > 0$	$\dfrac{1}{\alpha} F(s/\alpha)$
$e^{at} f(t)$	$F(s-a)$
$tf(t)$	$-\dfrac{dF}{ds}$
$t^k f(t)$	$(-1)^k \dfrac{d^k F(s)}{ds^k}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(s)\, ds$
$g(t) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & t \ge T \end{cases},\ T \ge 0$	$G(s) = e^{-sT} F(s)$

courtesy of S. Boyd, `http://www.stanford.edu/~boyd/ee102/`

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

The Laplace transform

Table : Laplace transform operations

1	Linearity	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
2	Constant multiplication	af(t)	aF(s)
3	Complex shift theorem	$e^{\pm at}f(t)$	F(s±a)
4	Real shift theorem	f(t-T)	$e^{-Ts}F(s)$, $T \geq 0$
5	Scaling theorem	$f(\frac{t}{a})$	aF(as)
6	First derivative	$\frac{d}{dt}f(t)$	sF(s) - f(0)
7	First integral	$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$

The Laplace transform

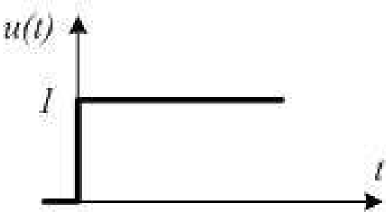
Table : Laplace transforms of common functions

	f(t)	F(s)
1	Unit impulse (Dirac) δ (t)	1
2	Unit step u(t)=1	$\frac{1}{s}$
3	Unit ramp v(t)=t	$\frac{1}{s^2}$
4	e^{at}	$\frac{1}{s-a}$
5	$cos\omega t$	$\frac{s}{s^2+\omega^2}$
6	$sin\omega t$	$\frac{\omega}{s^2+\omega^2}$

Signals

1. The unit step:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

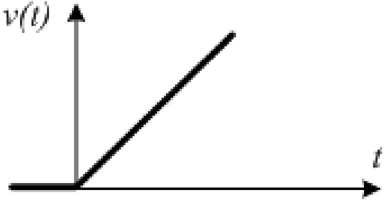


The Laplace transform of the step function:

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

2. The unit ramp

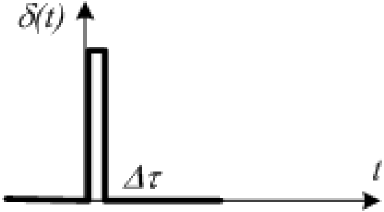
$$v(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$



The Laplace transform of the ramp signal is:

$$\mathcal{L}[v(t)] = \frac{1}{s^2}$$

3. The ideal impulse (Dirac)



$$\delta(t) = \begin{cases} 0, & t < 0 \;\; and \;\; t > \Delta\tau \\ A, & \tau \le t \le \tau + \Delta\tau \end{cases} \quad , \quad \lim_{\Delta\tau \to 0} \int_{\tau}^{\tau+\Delta\tau} \delta(t) dt = 1$$

The Laplace transform of the unit impulse is:

$$\mathcal{L}[\delta(t)] = 1$$

Transfer function models

- The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all the initial conditions assumed to be zero.

$$a_0 r(t)+a_1\frac{dr(t)}{dt}+...+a_m\frac{d^m r(t)}{dt^m} = b_0 y(t)+b_1\frac{dy(t)}{dt}+...+b_n\frac{d^n y(t)}{dt^n}$$

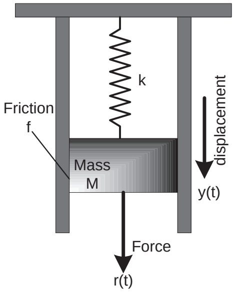
where $r(t)$ and $y(t)$ are the input and output variables.
Applying the Laplace transform for the initial conditions 0:

$$(a_0 + a_1 s + ... + a_m s^m)R(s) = (b_0 + b_1 s + ... + b_n s^n)Y(s)$$

and the transfer function will then be:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{a_0 + a_1 s + ... + a_m s^m}{b_0 + b_1 s + ... + b_n s^n}$$

Example: spring-mass-damper



$$M\frac{d^2y(t)}{dt} + f\frac{dy(t)}{dt} + ky(t) = r(t)$$

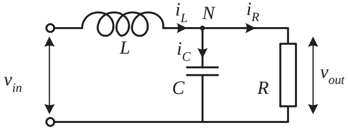
$$Ms^2Y(s) + fsY(s) + kY(s) = R(s)$$

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + fs + k}$$

"output = contents x input"

A transfer function $H(s)$ shows how the input is transferred to the output.

Example: electrical system



Inductor: $$\frac{di_L}{dt} = \frac{1}{L} v_L \qquad (1)$$

Capacitor: $$\frac{dv_C}{dt} = \frac{1}{C} i_C \qquad (2)$$

Resistor: $$v_R = R i_R \qquad (3)$$

Kirchhoff's current law:

$$i_L = i_C + i_R \qquad (4)$$

Kirchhoff's voltage law:

$$\begin{aligned} v_{in} &= v_L + v_C && (5) \\ v_C &= v_R = v_{out} && (6) \end{aligned}$$

Assume the initial conditions zero, apply the Laplace transform, eliminate everything except for input and output.

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{LCs^2 + \frac{L}{R}s + 1} = \frac{R}{RLCs^2 + Ls + R} \qquad (7)$$

Transfer function

For a **real physical** system the function $H(s)$ is a rational polynomial such that

$$H(s) = \frac{N(s)}{D(s)}$$

order of $D(s) \geq$ order of $N(s)$.

The characteristic equation: $D(s) = 0$

Roots of $D(s)$: **poles**.

Roots of $N(s)$: **zeros**

Highest degree of $D(s)$: **system order**

The poles and zeros of H(s) can be complex values, $s = \sigma + j\omega$.

Transfer function

$$H(s) = \frac{k(s-z_1)(s-z_2)...(s-z_m)}{s^r(s-p_1)(s-p_2)...(s-p_n)}$$

where $m \leq n$, p_i and z_i are the poles and zeros of the transfer function, r - the number of poles at the origin, n - the order of the system.

$$H(s) = \frac{k}{s^r}\frac{\prod_{j=1}^{m_1}(T_j s+1)\prod_{j=1}^{m_2}(\frac{1}{\omega_{nj}^2}s^2+\frac{2\zeta_j}{\omega_{nj}}s+1)}{\prod_{j=1}^{n_1}(T_j s+1)\prod_{j=1}^{n_2}(\frac{1}{\omega_{nj}^2}s^2+\frac{2\zeta_j}{\omega_{nj}}s+1)}$$

where k - the gain factor, ω_{nj} - the natural frequencies, T_j - the time constants, ζ_j - the damping factors.

Transfer function

- Let's apply the Laplace transform to continuous-time linear systems

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{array}\right.$$
$$x(0)=x_0$$

- Define $X(s)=\mathscr{L}[x(t)]$, $U(s)=\mathscr{L}[u(t)]$, $Y(s)=\mathscr{L}[y(t)]$
- Apply linearity and time-derivative rules

$$\left\{\begin{array}{rcl} sX(s)-x_0 & = & AX(s)+BU(s) \\ Y(s) & = & CX(s)+DU(s) \end{array}\right.$$

Transfer function

$$\begin{aligned} X(s) &= (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s) \\ Y(s) &= \underbrace{C(sI-A)^{-1}x_0}_{\text{Laplace transform of natural response}} + \underbrace{(C(sI-A)^{-1}B+D)U(s)}_{\text{Laplace transform of forced response}} \end{aligned}$$

Definition

The transfer function of a continuous-time linear system (A,B,C,D) is the ratio

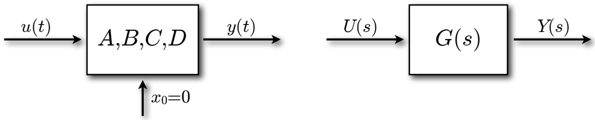
$$G(s) = C(sI-A)^{-1}B + D$$

between the Laplace transform $Y(s)$ of output and the Laplace transform $U(s)$ of the input signals *for the initial state* $x_0 = 0$

MATLAB

```
»sys=ss(A,B,C,D);
»G=tf(sys)
```

Transfer function



Example: The linear system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -10 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 2 \end{bmatrix} x(t) \end{cases}$$

has the transfer function

$$G(s) = \frac{2s+22}{s^2+11s+10}$$

Note: The transfer function does not depend on the input $u(t)$! It's only a property of the linear system.

MATLAB

```
»sys=ss([-10 1;
  0 -1],[0;1],[2 2],0);
»G=tf(sys)

Transfer function:
2 s + 22
-----------
s^2 + 11 s + 10
```

Transfer functions and linear ODEs

- Consider the n^{th}-order differential equation with input

$$\frac{dy^{(n)}(t)}{dt^n} + a_{n-1}\frac{dy^{(n-1)}(t)}{dt^{n-1}} + \cdots + a_1\dot{y}(t) + a_0 y(t) =$$
$$b_m\frac{du^{(m)}(t)}{dt^m} + b_{m-1}\frac{du^{(m-1)}(t)}{dt^{m-1}} + \cdots + b_1\dot{u}(t) + b_0 u(t)$$

- For initial conditions $y(0) = \dot{y}(0) = y^{(n-1)}(0)$, we obtain immediately the transfer function from u to y

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Example

$$\ddot{y} + 11\dot{y} + 10y = 2\dot{u} + 22u$$

$$G(s) = \frac{2s + 22}{s^2 + 11s + 10}$$

MATLAB

```
»G=tf([2 22],[1 11 10])

Transfer function:
2 s + 22
-----------
s^2 + 11 s + 10
```

Example

- Differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + y(t) = \dot{u}(t) + u(t)$$

- The transfer function is

$$G(s) = \frac{s+1}{s^2 + 3s + 1}$$

- The same transfer function $G(s)$ can be obtained through a state-space realization

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases}$$

  from which we compute

$$G(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+1}{s^2 + 3s + 1}$$

Some common transfer functions

- *Integrator*

$$\begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t) \end{cases} \qquad y(t) = \int_0^t u(\tau)d\tau \qquad \xrightarrow{u(t)} \boxed{\frac{1}{s}} \xrightarrow{y(t)}$$

- *Double integrator*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) \\ y(t) = x_1(t) \end{cases} \qquad y(t) = \int\int_0^t u(\tau)d\tau \qquad \xrightarrow{u(t)} \boxed{\frac{1}{s^2}} \xrightarrow{y(t)}$$

- *Damped oscillator* with frequency ω_0 rad/s and damping factor ζ

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases} \qquad \xrightarrow{u(t)} \boxed{\frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}} \xrightarrow{y(t)}$$

System response

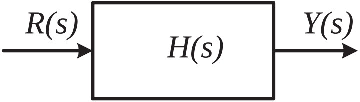


Figure : Block diagram of a system

From the definition of the transfer function:

$$Y(s) = H(s) \cdot R(s) \tag{8}$$

By applying the inverse Laplace transform we obtain: Zero initial conditions!

$$y(t) = \mathcal{L}^{-1}[H(s) \cdot R(s)]. \tag{9}$$

Example: spring-mass-damper

$$H(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + fs + k},\ R(s) = \mathcal{L}[\delta(t)],\ y(t) = \mathcal{L}^{-1}[H(s)\cdot 1]$$

$$M = 1, f = 3, k = 2$$

$$Y(s) = \frac{1}{(s+1)(s+2)}$$

$$y(t) = e^{-t} - e^{-2t}$$

$$M = 1, f = 1, k = 3$$

$$H(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

$$y(t) = \frac{2}{\sqrt{11}}e^{-t/2}sin(\frac{\sqrt{11}}{2}t)$$

Example: spring-mass-damper

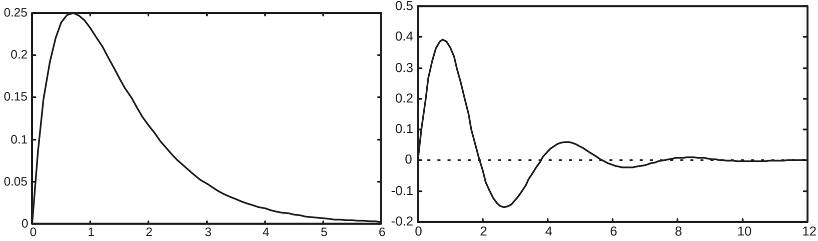


Figure : Spring-mass-damper response. Overdamped case (left). Underdamped case (right)

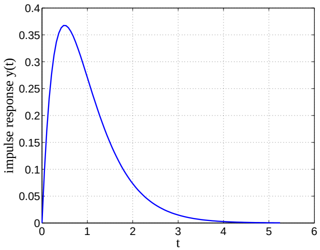
Impulse response

- Remember that an input signal $u(t)$ produces an output signal $y(t)$ whose Laplace transform $Y(s)$ is
$$Y(s) = G(s)U(s)$$
where $U(s) = \mathscr{L}[u]$, for initial state $x(0) = 0$
- Speciale case: impulsive input $u(t) = \delta(t)$, $U(s) = 1$. The corresponding output $y(t)$ is called the *impulse response*
- $G(s)$ is the Laplace transform of the impulse response $y(t)$
$$Y(s) = G(s) \cdot 1 = G(s)$$

Example:

$$G(s) = \frac{2}{s^2 + 3s + 1}$$

$$\mathscr{L}^{-1}[G(s)] = 2te^{-2t}$$



Inverse Laplace transform

- The impulse response $y(t)$ is therefore the *inverse Laplace transform* of the transfer function $G(s)$, $y(t) = \mathscr{L}^{-1}[G(s)]$
- The general formula for computing the inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

  where σ is large enough that $F(s)$ is defined for $\Re s \geq \sigma$
- This formula is not used very often

In MATLAB use

```
f = ILAPLACE(f)
```

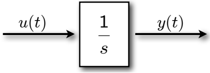
MATLAB

```
» syms s
» F=2*s/(s^2+1)
» f=ilaplace(F)

f = 2*cos(t)
```

Examples

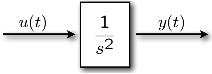
- *Integrator*

$$
\begin{aligned}
u(t) &= \delta(t) \\
y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s}] = 1\!\!1(t)
\end{aligned}
$$



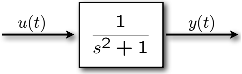
- *Double integrator*

$$
\begin{aligned}
u(t) &= \delta(t) \\
y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s^2}] = 1\!\!1(t)t
\end{aligned}
$$

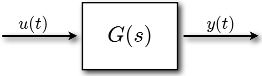


- *Undamped oscillator*

$$
\begin{aligned}
u(t) &= \delta(t) \\
y(t) &= \mathscr{L}^{-1}[\tfrac{1}{s^2+1}] = 1\!\!1(t)\sin t
\end{aligned}
$$



Poles and Zeros



- Rewrite the transfer function as the ratio of polynomials ($m < n$)

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

- The roots p_i of $D(s)$ are called the *poles* of the linear system $G(s)$
- The roots z_i of $N(s)$ are called the *zeros* of $G(s)$
- $G(s)$ is often written in zero/pole/gain form

$$G(s) = K\frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

In MATLAB use `ZPK` to transform to zero/pole/gain form

Examples

- Example 1

$$G(s) = \frac{s+2}{s^3+2s^2+3s+2} = \frac{s+2}{(s+1)(s^2+s+2)}$$

poles: $\{-1, -\frac{1}{2}+j\frac{\sqrt{7}}{2}, -\frac{1}{2}-j\frac{\sqrt{7}}{2}\}$, zeros: $\{-2\}$

- Example 2

$$G(s) = \frac{2s+22}{s^2+11s+10} = \frac{2(s+11)}{(s+10)(s+1)}$$

poles: $\{-10, -1\}$, zeros: $\{-11\}$

MATLAB

```
» G=tf([2 22],[1 11 10])
» zpk(G)

Zero/pole/gain:
2 (s+11)
---------
(s+10) (s+1)
```

Partial fraction decomposition

- The *partial fraction decomposition* of a rational function $G(s) = N(s)/D(s)$ is (assuming $p_i \neq p_j$)[3]

$$G(s) = \frac{\alpha_1}{s - p_1} + \cdots + \frac{\alpha_n}{s - p_n}$$

- α_i is called the *residue*[4] of $G(s)$ in $p_i \in \mathbb{C}$

$$\alpha_i = \lim_{s \to p_i} (s - p_i) G(s)$$

- The inverse Laplace transform of $G(s)$ is easily computed by inverting each term

$$\mathscr{L}^{-1}[G(s)] = \alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}$$

[3] For multiple poles p_i with multiplicity k we have the terms

$$\frac{\alpha_{i1}}{(s - p_i)} + \cdots + \frac{\alpha_{ik}}{(s - p_i)^k}, \ \alpha_{ij} = \frac{1}{(k-j)!} \lim_{s \to p_i} \frac{d^{(k-j)}}{ds^{(k-j)}} [(s - p_i)^k G(s)]$$

and the inverse Laplace transform is

$$\alpha_{i1} e^{p_i t} + \cdots + \alpha_{ik} \frac{t^{k-1}}{(k-1)!} e^{p_i t}$$

[4] Residues of conjugate poles are conjugate of each other: $p_i = \bar{p}_j \ \Rightarrow \ \alpha_i = \bar{\alpha}_j$

Linear algebra recalls

- The *inverse* of a matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

  is the matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$
- The inverse A^{-1} can be computed using the *adjugate* matrix $\mathrm{Adj}A$

$$A^{-1} = \frac{\mathrm{Adj}A}{\det A}$$

- The adjugate matrix is the transpose of the *cofactor matrix* C of A

$$\mathrm{Adj}A = C^T,\ C_{ij} = (-1)^{i+j}M_{ij}$$

  where M_{ij} is the (i,j) *cofactor* of A, that is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of A

Poles, eigenvalues, modes

- Linear system

$$\begin{cases} \dot{x}(t) &= Ax(t)+Bu(t) \\ y(t) &= Cx(t)+Du(t) \end{cases}$$
$$x(0)=0$$

$$G(s)=C(sI-A)^{-1}B+D \triangleq \frac{N_G(s)}{D_G(s)}$$

- Use the adjugate matrix to represent the inverse of $(sI-A)$

$$C(sI-A)^{-1}B+D=C\frac{C\,\mathrm{Adj}(sI-A)B}{\det(sI-A)}+D$$

- The denominator $D_G(s)=\det(sI-A)$!

The poles of $G(s)$ coincide with the eigenvalues of A

- Well, not always ...

Poles, eigenvalues, modes

- Some eigenvalues of A may not be poles of $G(s)$ in case of *pole/zero cancellations*
- Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$\det(sI - A) = (s-1)(s+1)$

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s+1}$$

- The pole $s=1$ has no influence on the input/output behavior of the system (but it has influence on the free response $x_1(t) = e^t x_{10}$)
- We'll better understand cancellations when investigating reachability and observability properties

Steady-state solution and DC gain

- Let A asymptotically stable. Natural response vanishes asymptotically
- Assume constant $u(t) \equiv u_r$. What is the asymptotic value $x_r = lim_{t\to\infty} x(t)$?

  Impose $0 = \dot{x}_r(t) = Ax_r + Bu_r$ and get $x_r = -A^{-1}Bu_r$

  The corresponding *steady-state* output $y_r = Cx_r + Du_r$ is

$$y_r = \underbrace{(-CA^{-1}B + D)}_{\text{DC gain}} u_r$$

- Cf. final value theorem:

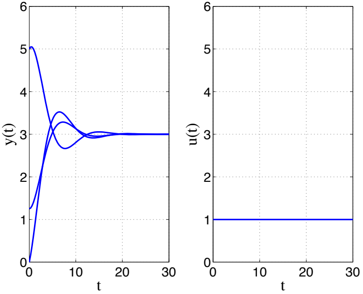
$$\begin{aligned} y_r &= \lim_{t\to+\infty} y(t) = \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} sG(s)U(s) = \lim_{s\to 0} sG(s)\frac{u_r}{s} \\ &= G(0)u_r = (-CA^{-1}B + D)u_r \end{aligned}$$

- $G(0)$ is called the *DC gain* of the system

DC gain - Example

$$\begin{cases} \dot{x}(t) & = & \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) & = & \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} x(t) \end{cases}$$

- DC gain: $-\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 3$
- Transfer function: $G(s) = \frac{2s+3}{4s^2+2s+1}$. G(0)=3



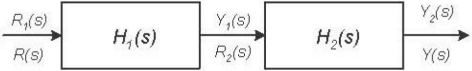
Output $y(t)$ for different initial conditions and input $u(t) \equiv 1$

MATLAB

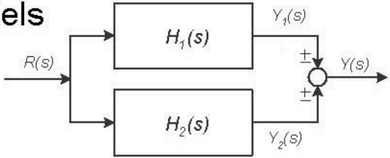
```
»sys=tf([2 3],[4 2 1]);
»dcgain(sys)

ans =

     3
```

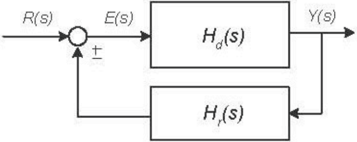
Block diagrams consist of *unidirectional*, operational blocks that represent transfer functions

Basic connections: series, parallel and feedback.



Block diagram models





Equivalent transfer functions

Series connection

$$H(s) = \frac{Y(s)}{R(s)} = \frac{Y_2(s)}{R_1(s)} = \frac{Y_2(s) \cdot Y_1(s)}{R_1(s) \cdot R_2(s)} = H_1(s) \cdot H_2(s)$$

Parallel connection

$$Y(s) = \pm Y_1(s) \pm Y_2(s), \quad H(s) = \frac{Y(s)}{R(s)} = \pm H_1(s) \pm H_2(s)$$

Feedback connection

$$H(s) = \frac{Y(s)}{R(s)} = \frac{H_d(s)}{1 \mp H_d(s) \cdot H_r(s)}$$

Diagram transformations

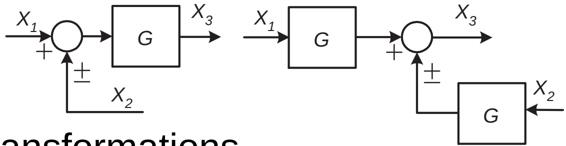


Figure : Moving a summing point behind a block

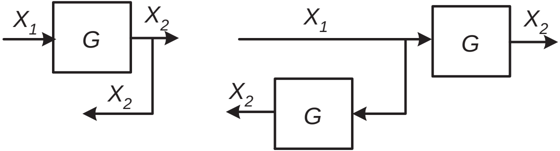


Figure : Moving a pickoff point ahead of a block

Diagram transformations

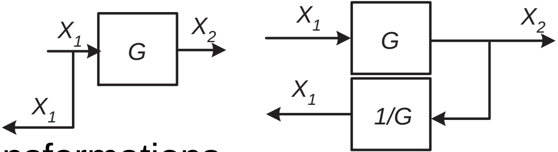


Figure : Moving a pickoff point behind a block

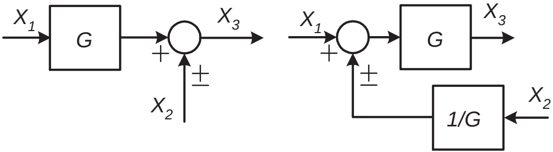
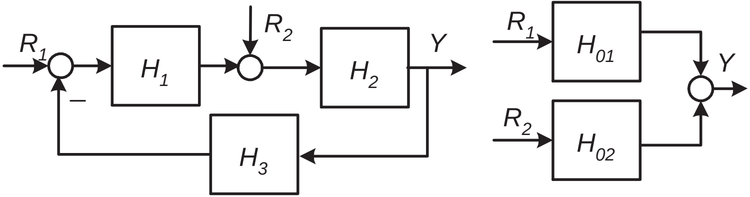


Figure : Moving a summing point ahead of a block

Overlap of signals



$$Y(s) = R_1(s) \cdot H_{01}(s)|_{R_2(s)=0} + R_2(s) \cdot H_{02}(s)|_{R_1(s)=0}$$

$$Y(s) = \frac{H_1 H_2}{1 + H_1 H_2 H_3} \cdot R_1(s) + \frac{H_2}{1 + H_1 H_2 H_3} \cdot R_2(s)$$

Transfer matrix (transfer function matrix)

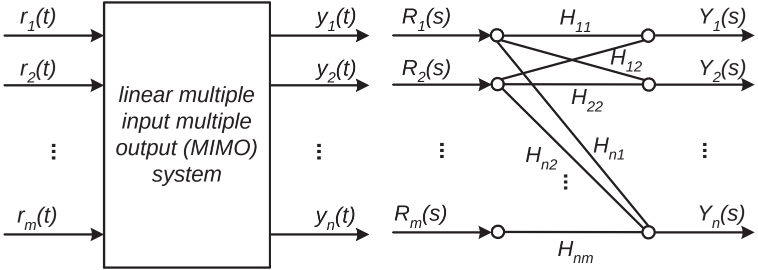


Figure : MIMO system

Transfer matrix (transfer function matrix)

$$
\begin{gathered}
Y_1 = H_{11}R_1 + H_{12}R_2 + \ldots H_{1m}R_m \\
Y_2 = H_{21}R_1 + H_{22}R_2 + \ldots H_{2m}R_m \\
\ldots \\
Y_n = H_{n1}R_1 + H_{n2}R_2 + \ldots H_{nm}R_m
\end{gathered}
$$

where the transfer function from the input k to the output j:

$$H_{jk} = \frac{Y_j}{R_k}$$

Transfer matrix (transfer function matrix)

Matrix form:

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{R}$$

Input and output vectors:

$$\mathbf{R} = [R_1(s)\ R_2(s)\ ...\ R_m(s)]^T, \ \ \mathbf{Y} = [Y_1(s)\ Y_2(s)\ ...\ Y_n(s)]^T$$

The transfer matrix:

$$\mathbf{H} = \left[\begin{array}{cccc} H_{11} & H_{12} & \dots & H_{1m} \\ H_{21} & H_{22} & \dots & H_{2m} \\ \dots & \dots & \dots & \dots \\ H_{n1} & H_{n2} & \dots & H_{nm} \end{array} \right]$$

Connections of MIMO systems

Series connection

$$\mathbf{H} = \mathbf{H}_2 \cdot \mathbf{H}_1,$$ for *n* systems $$\mathbf{H} = \prod_{j=n}^{1} \mathbf{H}_j$$

Parallel connection

$$\mathbf{H} = \pm\mathbf{H}_1 \pm \mathbf{H}_2$$

Feedback connection

$$\mathbf{H} = (\mathbf{I} \mp \mathbf{H}_d \cdot \mathbf{H}_r)^{-1} \cdot \mathbf{H}_d$$

Thank you very much for your attention!