

# Analiza sistemelor lineare in timp continuu. Sisteme cu timp mort (cu intarzieri). Regulatoare PID

Special thanks to:

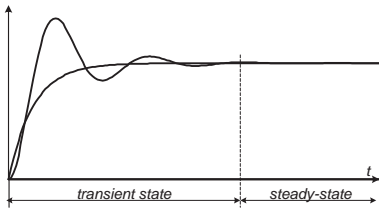
A. Bemporad, Automatic Control 1, Lecture Notes, University of Trento, Italy, 2011,

[http://cse.lab.imtlucca.it/~bemporad/automatic\\_control\\_course.html](http://cse.lab.imtlucca.it/~bemporad/automatic_control_course.html)

P. Raica, Systems Theory, Lecture Notes, Technical University of Cluj-Napoca, Cluj-Napoca,

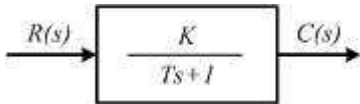
2012, [https://cs.utcluj.ro/files/educatie/licenta/2021-2022/22\\_CALCen\\_ST.pdf](https://cs.utcluj.ro/files/educatie/licenta/2021-2022/22_CALCen_ST.pdf)

- ▶ First step: derive a mathematical model
- ▶ Various methods are available for analysis
- ▶ Performance is analyzed based on various test signals
- ▶ The aim of analysis: study the system behavior in transient and steady-state when the system model and input are known.
- ▶ Typical test signals: step, ramp, impulse, sinusoidal



- ▶ Predict the dynamic behavior of the system from a knowledge of the system model.
- ▶ Important characteristics: absolute stability, transient response, steady-state error
  - ▶ A stable LTI system: the output comes back to its equilibrium state (system is subjected to a disturbance)
  - ▶ An unstable LTI system: either sustained oscillation of the output or the output diverges from equilibrium (system is subjected to a disturbance)
  - ▶ If the output of a system at steady state does not exactly agree with the input, the system is said to have *steady-state error*

- ▶ The system is broken down in simple elements of at most second order and the effects of each element are analyzed
- ▶ The behavior of simple elements can be studied using some characteristic parameters:
  - ▶ Time constants,  $T$
  - ▶ Time delay constant,  $T_m$
  - ▶ Damping factor  $\zeta$
  - ▶ Natural frequency  $\omega_n$
  - ▶ Gain constant,  $K$



The input-output relationship is given by:

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

Analyze the system responses to inputs as the unit step, unit ramp and unit impulse functions. The initial conditions are assumed to be zero.

$$r(t) = 1, \quad R(s) = \frac{1}{s}, \quad C(s) = \frac{K}{Ts + 1} \frac{1}{s}$$

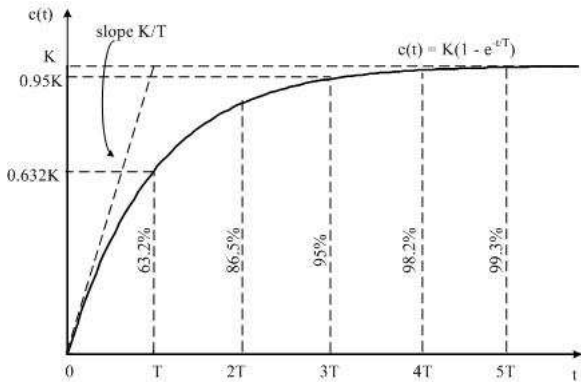
$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1} \left[ \frac{K}{s} - \frac{KT}{Ts + 1} \right] = K(1 - e^{-t/T}), \quad (t \geq 0)$$

*Property:* At  $t = T$  the value of  $c(t)$  is  $0.632K$ , or the response has reached 63.2% of its total change:

$$c(T) = K(1 - e^{-1}) = 0.632K$$

*Property:* The slope of the tangent at  $t = 0$  is  $1/T$ :

$$\frac{dc(t)}{dt} = \frac{K}{T} e^{-t/T} \Big|_{t=0} = \frac{K}{T}$$



For  $t \geq 4T$  the response remains within 2% of the final value. The response time is about 4 time constants.

$$r(t) = t, \quad R(s) = \frac{1}{s^2}, \quad C(s) = \frac{K}{Ts + 1} \frac{1}{s^2}$$

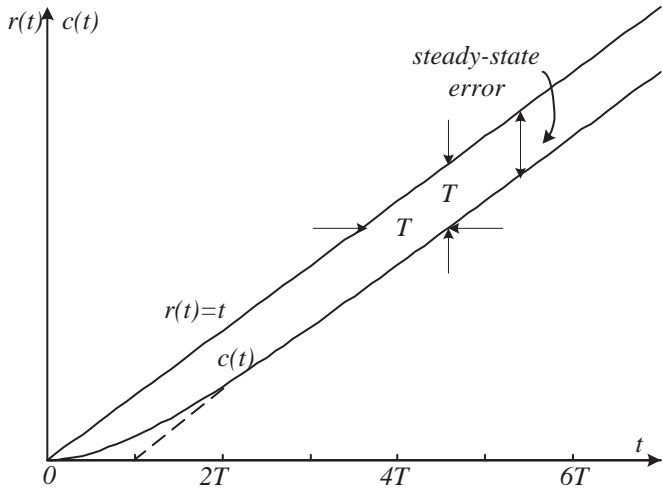
Expanding  $C(s)$  into partial fraction gives

$$\begin{aligned} c(t) &= \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1} \left[ K \left( \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \right) \right] \\ &= K(t - T + Te^{-t/T}), \quad (t \geq 0) \end{aligned}$$

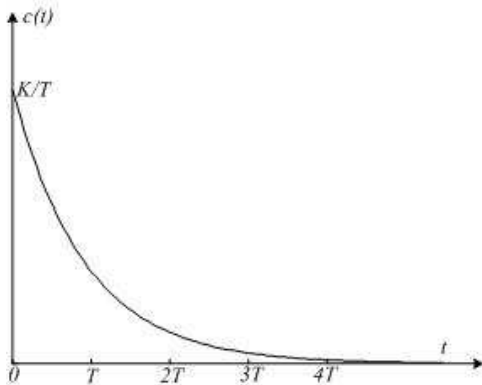
For  $K = 1$ , the error signal  $e(t)$ :

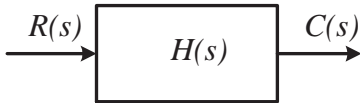
$$e(t) = r(t) - c(t) = T(1 - e^{-t/T}), \quad e(\infty) = T$$





$$r(t) = \delta(t), \quad R(s) = 1, \quad C(s) = \frac{K}{Ts + 1}, \quad c(t) = \frac{K}{T}e^{-t/T}, \quad (t \geq 0)$$





$$H(s) = \frac{C(s)}{R(s)} = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2}$$

$\omega_n$  - the natural frequency,  $\zeta$  - the damping factor

**Example.**

$$H(s) = \frac{1}{s^2 + s + 1}, \quad \frac{1}{\omega_n^2} = 1; \quad \frac{2\zeta}{\omega_n} = 1; \quad \Rightarrow \omega_n = 1; \quad \zeta = \frac{1}{2}$$

*The roots of the characteristic equation (System poles)*

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

*are:*

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

*Poles are complex for  $0 < \zeta < 1$  and real for  $\zeta \geq 1$ .*

$0 < \zeta < 1$  : the system is **underdamped** and the transient response is oscillatory.

$\zeta = 1$  : the system is **critically damped**

$\zeta > 1$  : system is **overdamped** (response does not oscillate)

$\zeta = 0$ : the transient response does not die out.

$$r(t) = 1, \quad R(s) = \frac{1}{s}, \quad C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

**Underdamped case ( $0 < \zeta < 1$ )**

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

where  $\omega_d = \omega_n\sqrt{1 - \zeta^2}$  - **damped natural frequency**.

$$\mathcal{L}^{-1}[C(s)] = c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \cdot \sin\left(\omega_d t + \arctan\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)$$

$\zeta = 0$ : the response becomes undamped and oscillations continue indefinitely.

$$c(t) = 1 - \cos \omega_n t, \quad (t \geq 0)$$

$\omega_n$  = the undamped natural frequency of the system = frequency at which the system would oscillate if the damping were decreased to zero.

**Critically damped case, ( $\zeta = 1$ ):** poles are equal and real

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}, \Rightarrow, c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad (t \geq 0)$$

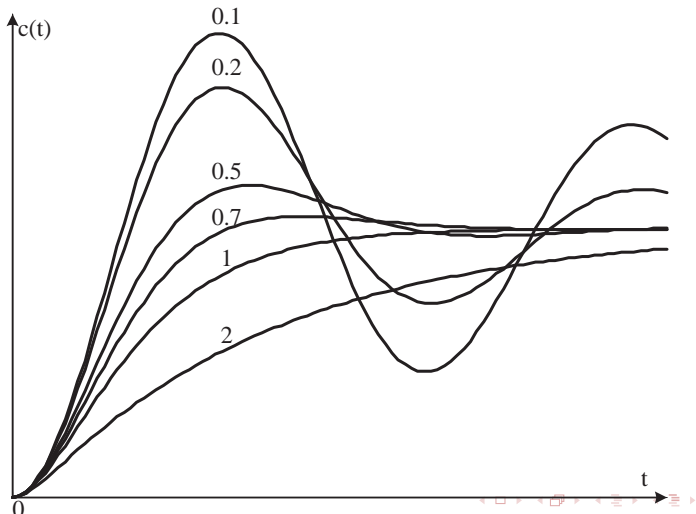
**Overdamped case, ( $\zeta > 1$ ):** poles are negative real

$$C(s) = \frac{\omega_n^2}{s(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right)$$

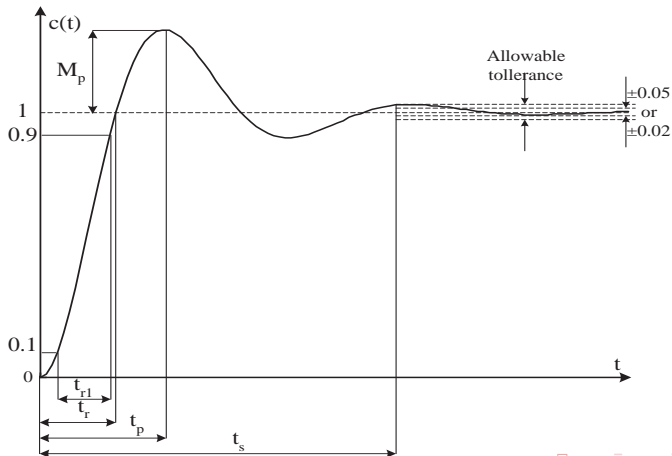
where  $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$  and  $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$  are the system poles.

The response  $c(t)$  includes two decaying exponential terms.





# Rise time, Peak time, Maximum overshoot, Settling time



- ▶ Rise time,  $t_r$  - the time required for the response to rise from 10% to 90%, 5% to 95% or 0% to 100% of its final value.
- ▶ Peak time,  $t_p$  - the time required for the response to reach the first peak of the overshoot.
- ▶ Maximum (percent) overshoot  $M_p$  - the maximum peak value of the response curve measured from the steady-state value of the response
- ▶ Settling time,  $t_s$  - the time required for the response curve to reach and stay within a range of 2% or 5% about the final value.

Obtain the rise time  $t_r$  by letting  $c(t_r) = 1$  or

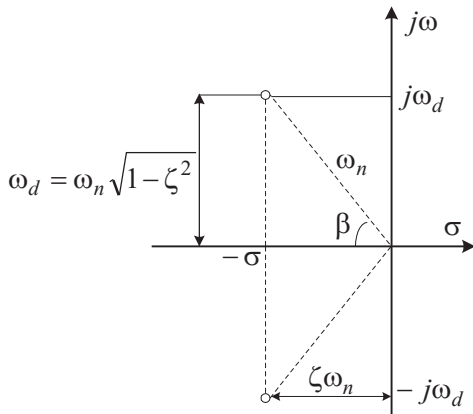
$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r} \left( \cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t_r \right)$$

$$\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t_r = 0, \quad \tan\omega_d t_r = -\frac{\zeta}{\sqrt{1-\zeta^2}} = -\frac{\omega_d}{\sigma}$$

$$t_r = \frac{1}{\omega_d} \cdot \arctan \left( \frac{\omega_d}{-\sigma} \right) = \frac{\pi - \beta}{\omega_d}$$

$\beta$  and  $\sigma$  are defined in next Figure.

Important picture !!!



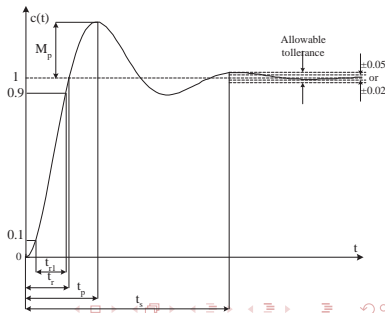
Obtain the peak time by differentiating  $c(t)$  with respect to time and letting the derivative equal zero:

$$\left. \frac{dc(t)}{dt} \right|_{t=t_p} = \sin(\omega_d t_p) \cdot \frac{\omega_n}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta \omega_n t_p} = 0$$

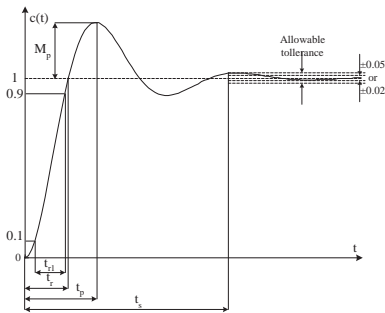
$$\sin(\omega_d t_p) = 0$$

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots,$$

$$t_p = \frac{\pi}{\omega_d}$$



$M_p$  occurs at the peak time or at  $t = t_p = \frac{\pi}{\omega_d}$ .



$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\pi/\omega_d} \left( \cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\pi \right) = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}}$$

$$c(t) = 1 - e^{-\zeta\omega_n t} \cdot \sin\left(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

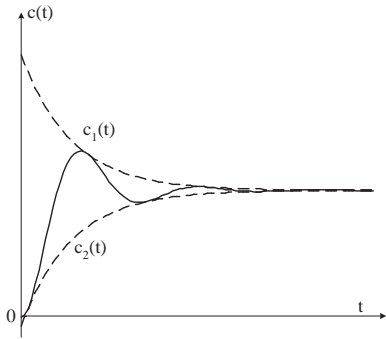
envelope curves:

$$c_{1,2}(t) = \pm e^{-\zeta\omega_n t} / \sqrt{1-\zeta^2}$$

$c_1, c_2, c(t)$  will reach 2% from the final value approximately when

$$e^{-\zeta\omega_n t_s} < 0.02, \text{ or } \zeta\omega_n t_s \cong 4$$

$$t_s = \frac{4}{\zeta\omega_n}$$



Consider the closed-loop system with the transfer function:

$$H(s) = \frac{25}{s^2 + 6s + 25} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

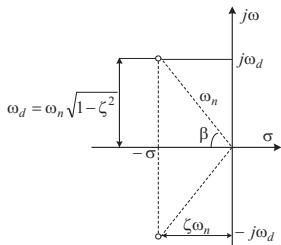
Calculate:

$$\omega_n = 5, \quad \zeta = 0.6, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} = 5\sqrt{1 - 0.6^2} = 4$$

The poles negative real part:

$$\sigma = -\zeta\omega_n = -3.$$





$$\beta = \arctan \frac{\omega_d}{\sigma} = 0.93$$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - 0.93}{4} = 0.55 \text{sec}$$

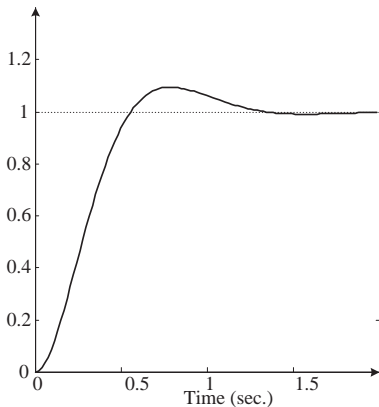
$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.78 \text{sec}$$

$$M_p = e^{-\pi \zeta / \sqrt{1 - \zeta^2}} = 0.095$$

$$M_p(\%) = 9.5\%$$

$$t_s = \frac{4}{-\sigma} = \frac{4}{\zeta \omega_n} = \frac{4}{3} = 1.33 \text{sec}$$

The step response of the system. The values of the system parameters can be seen from the plot.



Consider a 1st or 2nd order system with the transfer function

$$H_k(s) = \frac{k}{Ts + 1} = k \cdot H(s), \text{ or } H_k(s) = k \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = k \cdot H(s)$$

The system response when the input is a unit step  $R(s) = 1/s$  is:

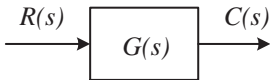
$$c(t) = \mathcal{L}^{-1}[H_k(s) \cdot R(s)] = \mathcal{L}^{-1}\left[\frac{k \cdot H(s)}{s}\right] = k \cdot \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$$

The time response is  $k \cdot c(t)$  where  $c(t)$  is the response of the system with a unity gain.

The **steady-state error** = the error between the input ( $r(t)$ ) and the output ( $c(t)$ ) at steady-state.

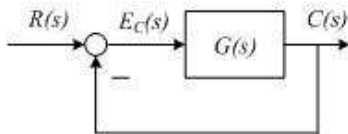
$$e_{ss} = \lim_{t \rightarrow \infty} (r(t) - c(t))$$

The open-loop system *error* :



$$E(s) = R(s) - C(s) = (1 - G(s))R(s)$$

For a unity feedback closed-loop system, the *error* is:



$$E(s) = R(s) - C(s) = R(s) - R(s) \frac{G(s)}{1 + G(s)} = \frac{1}{1 + G(s)} R(s)$$

Final value theorem:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

## For a unit step input

Open-loop system:

$$e_{ss} = \lim_{s \rightarrow 0} s(1 - G(s))\left(\frac{1}{s}\right) = \lim_{s \rightarrow 0} (1 - G(s)) = 1 - G(0)$$

Unity feedback closed-loop system:

$$e_{ss} = \lim_{s \rightarrow 0} s \left( \frac{1}{1 + G(s)} \right) \left( \frac{1}{s} \right) = \frac{1}{1 + G(0)}$$

Consider a first-order system:

$$G(s) = \frac{k}{Ts + 1}$$

Input:  $R(s) = 1/s$ . The steady-state error of the open-loop system:

$$e_{ss} = 1 - G(0) = 1 - k$$

If  $G(s)$  is the open-loop transfer function for a unity feedback closed-loop system:

$$e_{ss} = \frac{1}{1 + G(0)} = \frac{1}{1 + k}$$

Consider a system with the transfer function  $H(s)$ . The system step response:

$$y(t) = \mathcal{L}^{-1} [Y(s)] = \mathcal{L}^{-1} \left[ \frac{H(s)}{s} \right]$$

Add a zero at  $-a$  and divide the transfer function with  $a$  (the gain of the new system is unchanged):

$$H_z(s) = \frac{s+a}{a} H(s) = \frac{s}{a} H(s) + H(s)$$

The step response of the system  $H_z(s)$  result:

$$y_z(t) = \mathcal{L}^{-1} \left[ \frac{1}{s} \left( \frac{s}{a} H(s) + H(s) \right) \right] = \frac{1}{a} \dot{y}(t) + y(t)$$



If  $a$  is small (the zero is close to the imaginary axis)  $1/a$  is large  $\Rightarrow$   
The step response of  $H_z(s)$  will increase with the quantity  
 $1/a \cdot \dot{y}(t)$ .

The effect of addition of a zero is the increase of the overshoot.

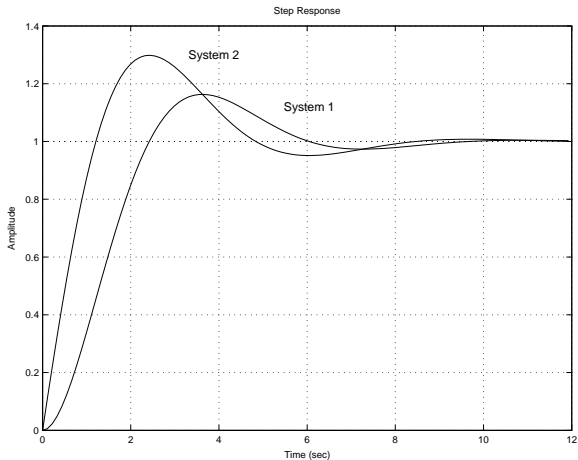
**Example.** Consider the system with the transfer function:

$$\text{System 1: } H_1(s) = \frac{1}{s^2 + s + 1}$$

We add a zero at  $-1$  and obtain:

$$\text{System 2: } H_2(s) = \frac{s + 1}{s^2 + s + 1}$$

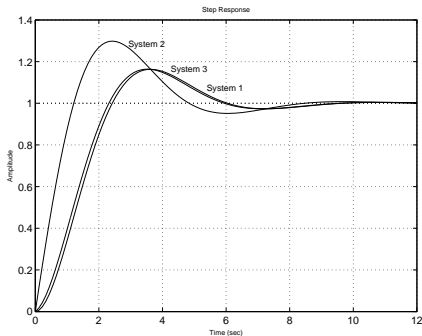
System1 : no zeros; System 2: a zero at -1



If the zero was added at -10 (the gain divided by 10)

$$\text{System 3: } H_3(s) = \frac{0.1(s + 10)}{s^2 + s + 1}$$

System 1; System 2: zero at -1; System 3: zero at -10



Consider a system  $H(s)$ , with unit step input  $R(s) = 1/s$  an output  $C(s)$

$$C(s) = H(s) \cdot R(s) = \frac{a^m s^m + \dots + a_1 s + a_0}{s(b^n s^n + \dots + b_1 s + b_0)}, \quad (m \leq n)$$

$$C(s) = \frac{K \prod_{i=1}^m (s + z_i)}{s \prod_{j=1}^q (s + p_j) \prod_{k=1}^r (s^2 + 2\zeta_k \omega_k s + \omega_k^2)}$$

where  $q + 2r = n$ .

$$C(s) = \frac{a}{s} + \sum_{j=1}^q \frac{a_j}{s + p_j} + \sum_{k=1}^r \frac{b_k(s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$$\begin{aligned}
 c(t) = & a + \sum_{j=1}^q a_j e^{-p_j t} + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \cos \omega_k \sqrt{1 - \zeta_k^2} t \\
 & + \sum_{k=1}^r b_k e^{-\zeta_k \omega_k t} \sin \omega_k \sqrt{1 - \zeta_k^2} t
 \end{aligned}$$

If the poles lie in the left-half s-plane:  $c(\infty) = a$ .

The poles located far from the  $j\omega$  (imaginary) axis have large negative real parts. The exponential terms that correspond to these poles decay very rapidly to zero.

The poles located nearest the  $j\omega$  axis correspond to transient response terms that decay slowly: **dominant poles**

## System approximation using the concept of dominant poles

Consider a system with a transfer function:

$$H(s) = \frac{k(s + a)}{(\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1)(Ts + 1)}$$

$$z_1 = -a, p_{1,2} = -\zeta\omega_n \pm j\sqrt{1 - \zeta^2}, p_3 = -1/T.$$

Consider the real pole far from the  $j\omega$  axis; complex poles are dominant. The system order can be reduced by *neglecting the real pole*.

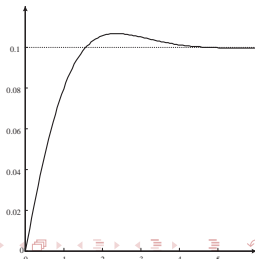
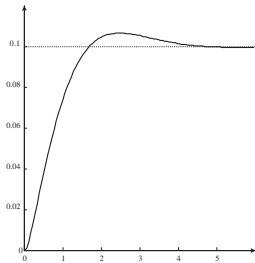
!!! The gain factor must be multiplied by the absolute value of the time constant or  $1/\text{pole}$  ( $\Rightarrow$  same steady-state value).

$$H_1(s) = \frac{s + 2}{(s^2 + 2s + 2)(s + 10)}$$

$p_{1,2} = -1 \pm j$ : dominant,  $p_3 = -10$ : can be neglected.

Divide the system gain by  $|p_3| = 10$  and obtain:

$$H_2(s) = \frac{0.1(s + 2)}{s^2 + 2s + 2}$$

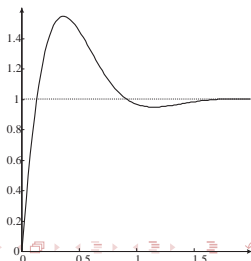
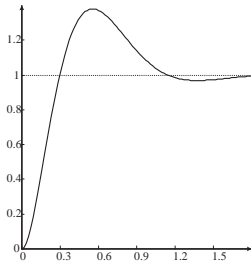


$$H_1(s) = \frac{62.5(s + 2.5)}{(s^2 + 6s + 25)(s + 6.25)}$$

$$p_{1,2} = -3 \pm 4 \cdot j, \quad p_3 = -6.25.$$

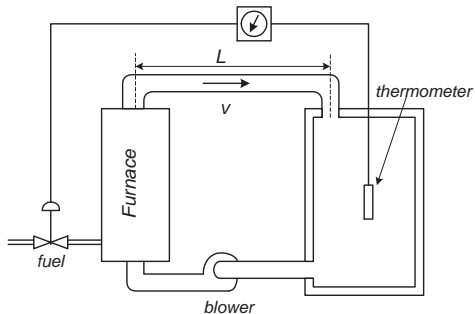
Neglect the real pole and obtain:

$$H_2(s) = \frac{10(s + 2.5)}{s^2 + 6s + 25}$$





Systems with transport lag (dead time)



A thermal system. Hot water is circulated to keep the temperature of a chamber constant.

$L$  - distance between the furnace and measure element,  
 $v$  - the air velocity,

$T = L/v$  **transport lag** or **dead time**

A **dead time** is the time interval between the start of an event at one point in a system and its resulting action at another point in the system.

The input  $x(t)$  and the output  $y(t)$  of a transport lag or dead time element are related by

$$y(t) = x(t - T), \text{ where } T \text{ is the dead time.}$$

The transfer function of transport lag or dead time is given by:

$$H(s) = \frac{\mathcal{L}[x(t - T)]}{\mathcal{L}[x(t)]} = \frac{X(s)e^{-sT}}{X(s)} = e^{-sT}$$

A linear system with dead time:

$$\sum_{j=0}^m a_j \frac{d^j x(t - T)}{dt^j} = \sum_{j=0}^n b_j \frac{d^j y(t)}{dt^j}$$

where  $x(t)$  is the input signal,  $y(t)$  is the output

The Laplace transform of the differential equation:

$$e^{-sT} \cdot \sum_{j=0}^m a_j s^j \mathcal{L}[x(t)] = \sum_{j=0}^n b_j s^j \mathcal{L}[y(t)]$$

and the transfer function is:

$$H(s) = e^{-sT} \frac{\sum_{j=0}^m a_j s^j}{\sum_{j=0}^n b_j s^j} = \frac{Y(s)}{X(s)}$$

If we use a Taylor series expansion for  $e^{-sT}$ :

$$e^{-sT} = 1 - Ts + \frac{1}{2!} T^2 s^2 - \frac{1}{3!} T^3 s^3 + \dots$$

and the truncated Taylor series expansion of a ratio of two polynomials:

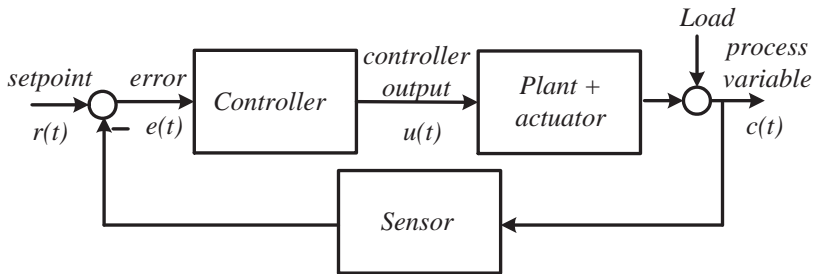
$$\frac{1 + \alpha Ts}{1 + \beta Ts} = 1 + (\alpha + \beta) Ts - \beta(\alpha - \beta) T^2 s^2 + \beta^2(\alpha - \beta) T^3 s^3,$$

$\Rightarrow$  *Pade approximation*:

$$a) e^{-sT} = \frac{1 - \frac{1}{2} Ts}{1 + \frac{1}{2} Ts}, \quad b) e^{-sT} = \frac{1 - \frac{1}{2} Ts + \frac{1}{12} T^2 s^2}{1 + \frac{1}{2} Ts + \frac{1}{12} T^2 s^2}$$

# PID controllers

A feedback controller is designed to generate an *output* that causes some corrective effort to be applied to a *process* so as to drive a measurable *process variable* towards a desired value known as the *setpoint*.



## Temperature control in a room:

- ▶ Process - the room,
- ▶ Process output - temperature
- ▶ Setpoint - desired room temperature,
- ▶ Transducer/Sensor - thermocouple
- ▶ Controller - thermostat
- ▶ Controller output - the activation signal to the air conditioner
- ▶ Actuator - the air conditioner
- ▶ Load(s) - random heat sources (such as sunshine and warm bodies)



- ▶ PID (Proportional-Integral-Derivative) is the control algorithm most often used in industrial control (95 %).

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt}$$

- ▶  $u(t)$  - the current controller output
- ▶  $e(t)$  - error = setpoint - measured output
- ▶  $K_P$  - *proportional tuning constant*
- ▶  $K_I$  - *integral tuning constant*
- ▶  $K_D$  - *derivative tuning constant*

Tuning = setting the  $K_P$ ,  $K_I$ , and  $K_D$  so that the weighted sum of the P, I and D terms produces a controller output that drives the process variable in the direction required to eliminate the error.

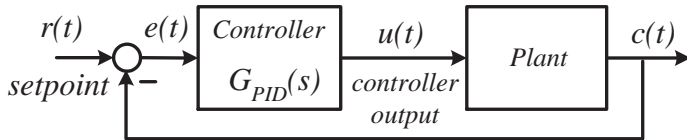
$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt}$$

Sluggish process:

- ▶ If an error is introduced abruptly, the controller's initial reaction will be determined primarily by the D term.
- ▶ The proportional term will then come in to play: keeps the controller's output going until the error is eliminated.
- ▶ The integral term will also begin to contribute to the controller's output as the error accumulates over time. The process variable may overshoot.

Three approaches:

1. 'Trial-and-error' (experienced engineers). Example: decrease of  $K_I \Rightarrow$  reduces the overshoot but slows the rate of change of the error
2. The analytical approach. If a mathematical model of the process is known, there are various methods for controller tuning.
3. A compromise between purely heuristic trial-and-error techniques and the analytical techniques. It was originally proposed in 1942 by John G. Ziegler and Nathaniel B. Nichols of Taylor Instruments.



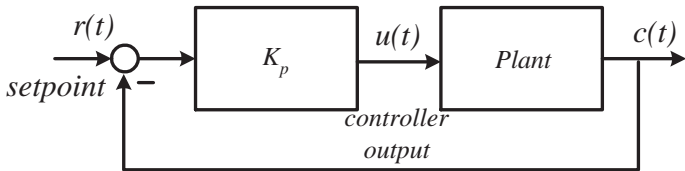
In practice the output of a PID controller is given by:

$$u(t) = K_p \left[ e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right]$$

$$G_{PID}(s) = \frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right)$$

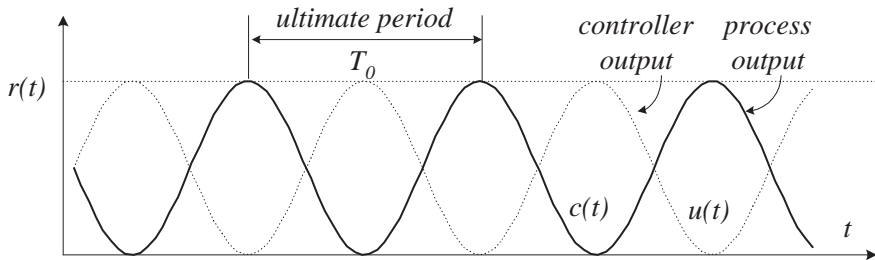
where  $K_p$  = proportional gain,  $T_i$  = integral time,  $T_d$  = derivative time

## The Ultimate Sensitivity Method



- ▶ Set  $T_i = \infty$ , and  $T_d = 0$ .
- ▶ Increase  $K_p$  from 0 up to a critical value  $K_0$  where the output,  $c(t)$ , first exhibits sustained oscillations.

# The Ultimate Sensitivity Method



Determine experimentally:

- ▶  $K_0$  - the ultimate gain
- ▶  $T_0$  - critical period.

## The Ultimate Sensitivity Method

Type of controller	$K_p$	$T_i$	$T_d$
P	$0.5K_0$	$\infty$	0
PI	$0.45K_0$	$1/1.2T_0$	0
PID	$0.6K_0$	$0.5T_0$	$0.125T_0$

The PID controller tuned by the ultimate sensitivity method of Ziegler-Nichols gives:

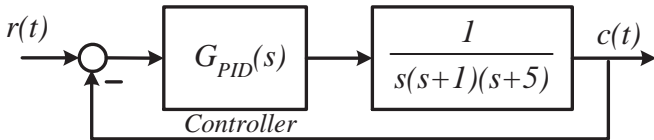
$$\begin{aligned} G_{PID}(s) &= K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = 0.6 K_0 \left( 1 + \frac{1}{0.5 T_0 s} + 0.125 T_0 s \right) \\ &= 0.075 K_0 T_0 \frac{(s + 4/T_0)^2}{s} \end{aligned}$$

Thus, the PID controller has a pole at the origin and double zeros at  $s = -4/T_0$ .

*Derivative cautions.* If the process output rate of change is caused by noise, the D term may cause over-corrections. (ex. in pressure and level control)



Ziegler-Nichols tuning rules can be applied to plants where the model is known.



$T_i = \infty$ ,  $T_d = 0$ , closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5) + K_p}$$

The value of  $K_p$  that makes the system marginally stable so that sustained oscillation occurs can be obtained by use of Routh's stability criterion.

The characteristic equation:

$$s^3 + 6s^2 + 5s + K_p = 0$$

$$\Rightarrow K_0 = 30$$

$$(s + 1)(s^2 + 5) = (s + 1)(s + \omega_n^2) = 0$$

$$\Rightarrow \omega_n = \sqrt{5},$$

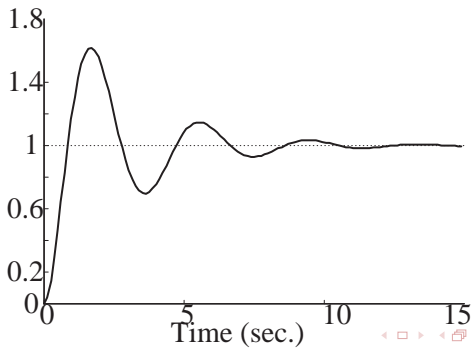
$$T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{5}} = 2.81$$

$$K_p = 0.6K_0 = 18, \quad T_i = 0.5T_0 = 1.405, \quad T_d = 0.125T_0 = 0.35$$

The transfer function of the PID controller is:

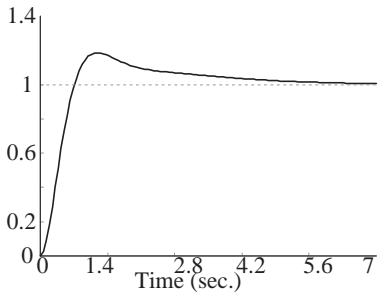
$$G_{PID}(s) = 18 \left( 1 + \frac{1}{1.405s} + 0.35s \right) = \frac{6.32(s + 1.42)^2}{s}$$

The unit step response of this system (computer simulation):

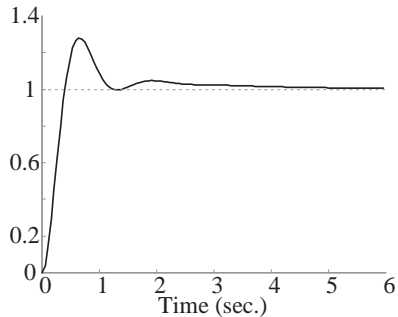


Increase  $K_p$  to 39.42:

$$G_{PID}(s) = \frac{13.84(s + 0.65)^2}{s}$$

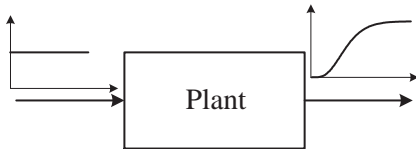


$$G_{PID}(s) = \frac{30.322(s + 0.65)^2}{s}$$



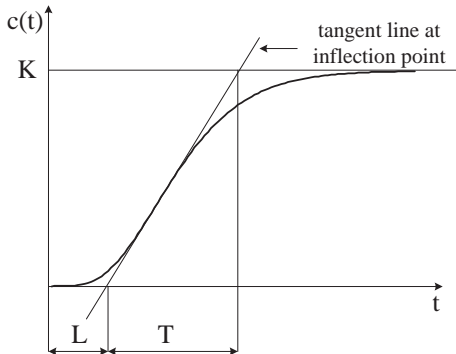
**Transient response method** or *reaction curve (open-loop) method*

Make a step change to the process and record the process output:



⇒ open-loop gain  $K$ , the loop apparent deadtime  $L$ , the loop time constant  $T$ .

## Transient response method



The transfer function of the plant = first order with time delay:

## Transient response method

Type of controller	$K_p$	$T_i$	$T_d$
P	$T/L$	$\infty$	0
PI	$0.9T/L$	$L/0.3$	0
PID	$1.2T/L$	$2L$	$0.5L$

PID controller:

$$G_{PID}(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = 0.6T \frac{(s + 1/L)^2}{s}$$

a pole at the origin and double zeros at  $s = -1/L$ .

Zeigler-Nichols tuning methods tend to produce systems whose transient response is rather oscillatory and so will need to be tuned further prior to putting the system into closed-loop operation.



Thank you very much for your attention!