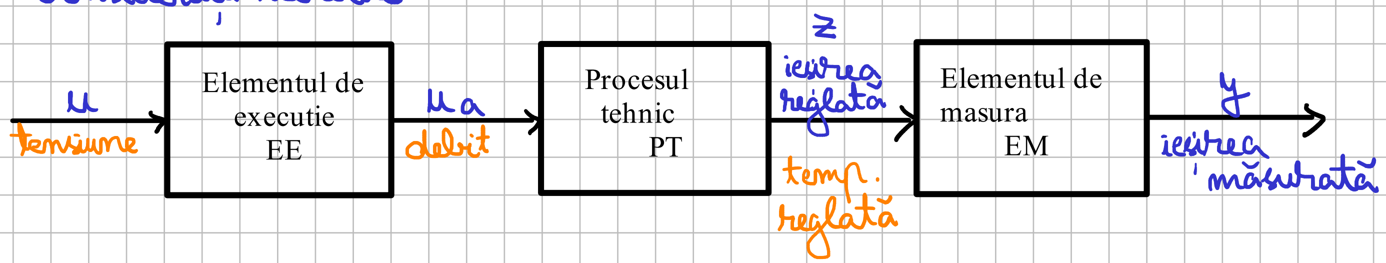
Teoria sistemelor

(laborator 1 - S1)

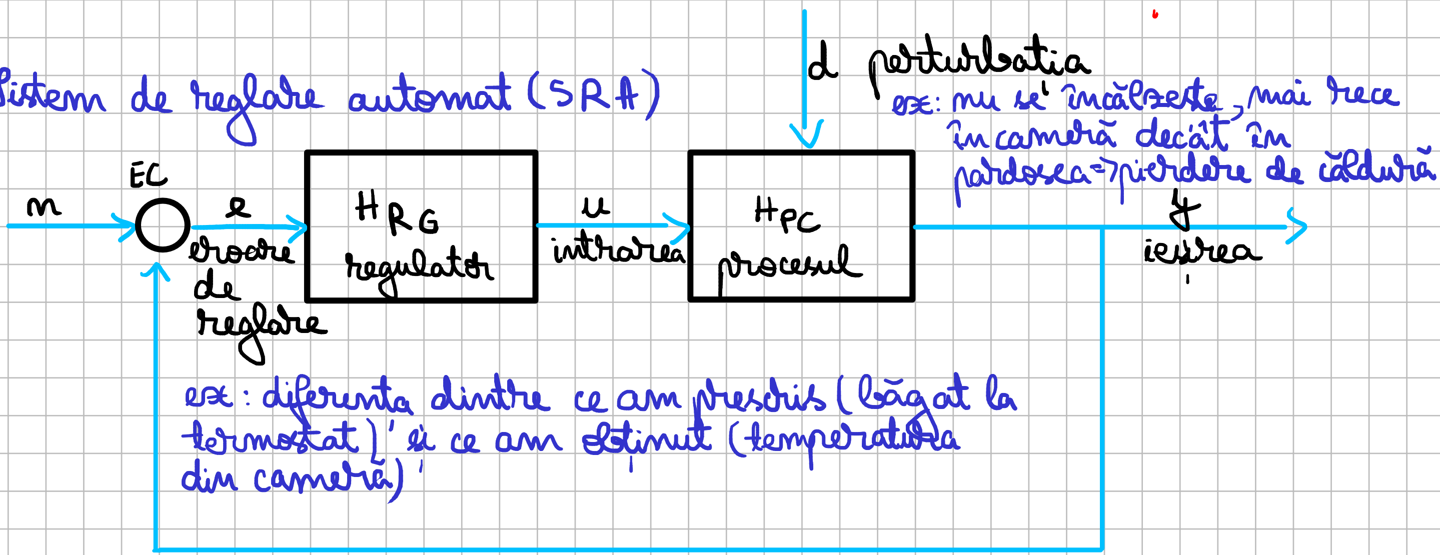
Considerații teoretice



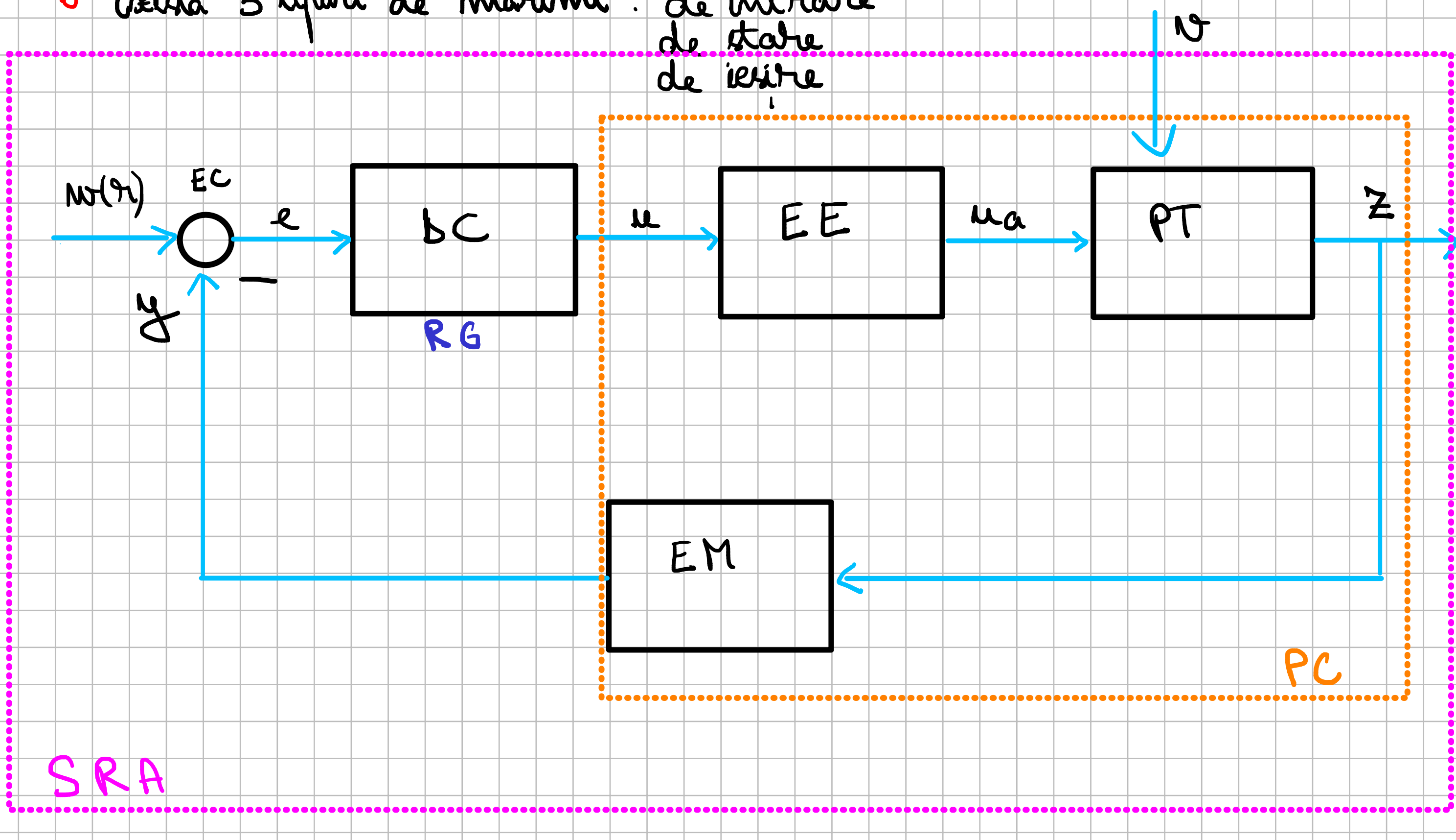
! între elemente există ecuații (relații primare)

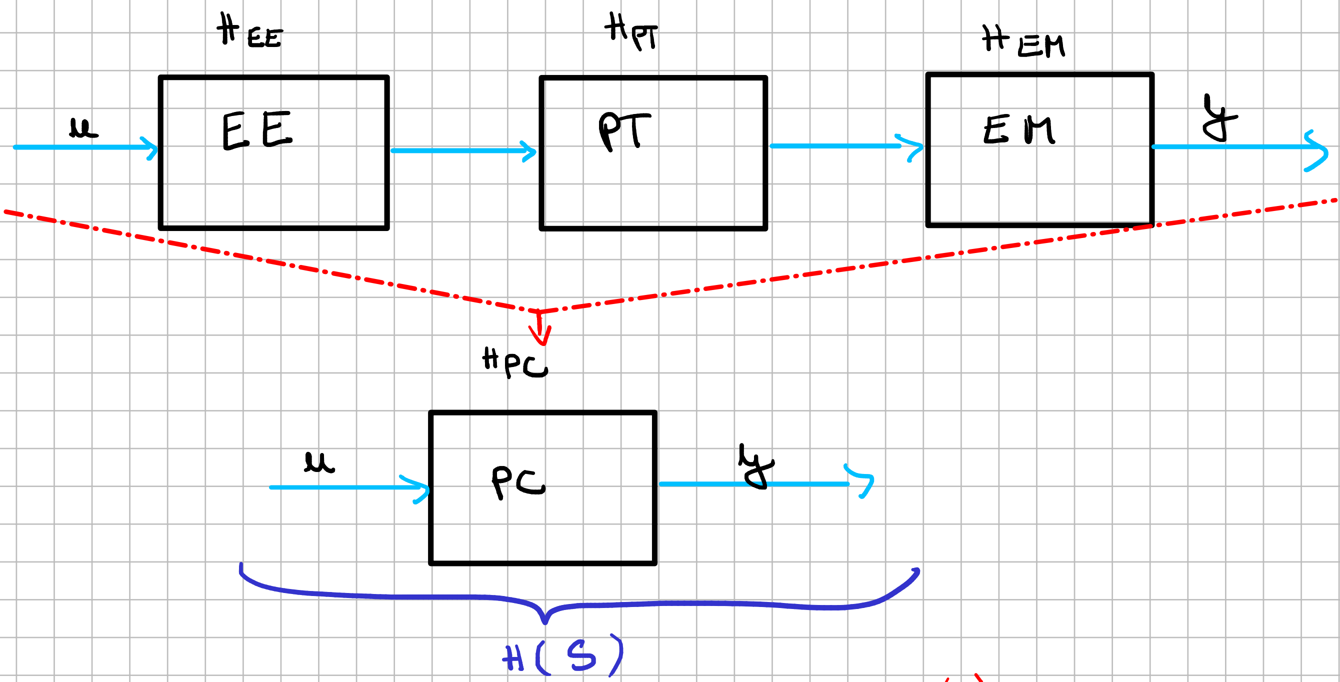
⇒ 2 modele matematice → intrare - ieșire
→ intrare - stare - ieșire

Sistem de reglare automat (SRA)



! Există 3 tipuri de mărimi : de intrare
de stare
de ieșire





$$y(s) = H(s) \cdot u(s) \Rightarrow H(s) = \frac{y(s)}{u(s)}$$

(demonstrație după funcția de transfer)

w sau r ⟶ referința

e ⟶ eroarea de reglare (nu trebuie să ∃)

DC ⟶ dispozitiv de conducere

RG ⟶ regulator

u ⟶ tensiunea de comandă (tensiune)

EE ⟶ element de execuție

u_a ⟶ tensiunea de acționare

v ⟶ perturbație (d)

PT ⟶ proces tehnic

z ⟶ ieșirea de reglare / reglată

EM ⟶ element de măsură

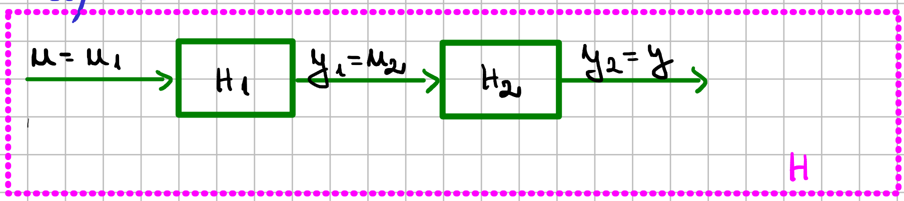
PC ⟶ proces condus

y ⟶ ieșire măsurată

SRA ⟶ sistem de reglare automat

Principalele conexiuni de sisteme

a) serie

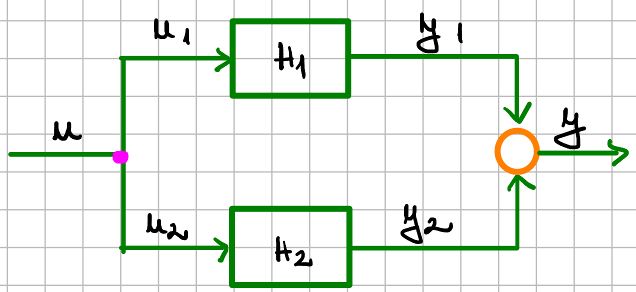


$H = H_1 \cdot H_2$

Demonstrație

$$\boxed{y} = y_2 = H_2 \cdot u_2 = H_2 \cdot y_1 = H_2 \cdot H_1 \cdot u_1 = \underline{H_2 \cdot H_1 \cdot u} = \boxed{H \cdot u}$$

b) paralel



• → $u = u_1 = u_2$ punct de ramificație

o → $y = y_1 + y_2$ sumator

(ieșirile se însumează)

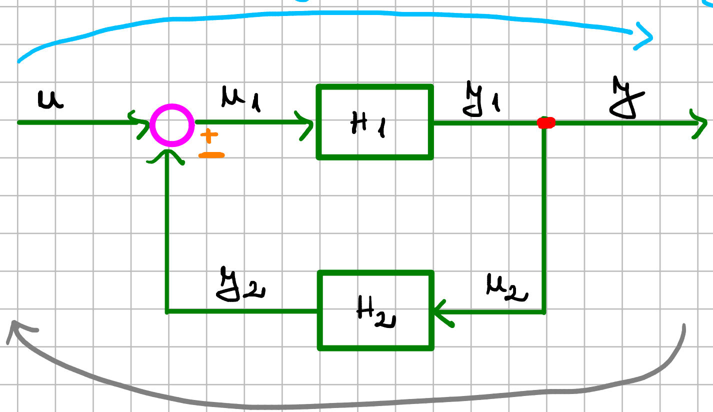
$H = H_1 + H_2$

Demonstrație

$$\boxed{y} = y_1 + y_2 = H_1 \cdot u_1 + H_2 \cdot u_2 = H_1 \cdot u + H_2 \cdot u = \underline{(H_1 + H_2) \cdot u} = \boxed{H \cdot u}$$

c) cu reacție

calea directă $u \to y$



o+ → reacție pozitivă

$u_1 = u + y_2$

o− → reacție negativă

$u_1 = u - y_2$

• $y_1 = y = u_2$

calea cu reacție $y \to u$

(observăm ce face semnul de intrare)

$$H = \frac{H_1}{1 \mp H_1 \cdot H_2}$$

$$y = y_1 = H_1 \cdot u_1 = H_1 (u \pm y_2) = H_1 \cdot (u \pm H_2 \cdot u_2) =$$

$$= H_1 \cdot u \pm H_1 \cdot H_2 \cdot u_2 = H_1 \cdot u \pm H_1 \cdot H_2 \cdot y$$

$$\Rightarrow y \mp H_1 \cdot H_2 \cdot y = H_1 \cdot u$$

$$\Leftrightarrow y (1 \mp H_1 \cdot H_2) = H_1 \cdot u$$

$$\Leftrightarrow y = \frac{H_1}{1 \mp H_1 \cdot H_2} \cdot u \; \Big| : u \qquad H = \frac{y}{u}$$

$$H = \frac{H_1}{1 \mp H_1 H_2}$$

Modele matematice intrare - ieșire

1) MM-II în timp continuu

$$a_n y^{(n)}(t) + a_{n-1} \cdot y^{(n-1)}(t) + \ldots + a_1 y^{(1)}(t) + a_0 y(t) =$$

$$= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \ldots + b_1 u^{(1)}(t) + b_0 u(t)$$

(Laplace)

$$y^{(n)}(t) = s^n y(s) \quad \Rightarrow$$

$$a_n s^n y(s) + a_{n-1} s^{n-1} y(s) + \ldots + a_1 s y(s) + a_0 y(s) =$$

$$= b_m s^m u(s) + b_{m-1} s^{m-1} u(s) + \ldots + b_1 s \cdot u(s) + b_0 u(s)$$

Dăm factor comun $y(s)$ și $u(s)$

$$y(s) \left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \right] = u(s) \left[b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0 \right]$$

$$y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \, u(s) \Big| : u(s) \quad H(s) = \frac{y(s)}{u(s)}$$

$$H(s)=\frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \qquad m < n$$

2) MM-II în timp discret

$$a_n\, y(k+n) + a_{n-1}\, y(k+n-1) + \dots + a_1\, y(k+1) + a_0\, y(k) =$$

$$= b_m\, u(k+m) + b_{m-1}\, u(k+m-1) + \dots + b_1\, u(k+1) + b_0\, u(k)$$

Se aplică transformata Z

$$a_n z^n y(z) + a_{n-1} z^{n-1} y(z) + \dots + a_1 z\, y(z) + a_0\, y(z) =$$

$$= b_m z^m u(z) + b_{m-1} z^{m-1} u(z) + \dots + b_1 z\, u(z) + b_0\, u(z)$$

$$y(z)\left[a_n z^n + \dots + a_1 z + a_0\right] = u(z)\left[b_m z^m + \dots + b_1 z + b_0\right]$$

$$H(z) = \frac{y(z)}{u(z)}$$

$$\Rightarrow \quad H(z) = \frac{b_m z^m + \dots + b_1 z + b_0}{a_n z^n + \dots + a_1 z + a_0}$$

Modele matematice intrare-stare-ieșire

1) MM-ISI în timp continuu

$$\begin{cases} \dot{x}(t) = A \cdot x(t) + b \cdot u(t) \\ y(t) = c^T \cdot x(t) \end{cases} \quad \Big| \text{ se aplică Laplace}$$

$$\Rightarrow \begin{cases} s \cdot x(s) = A \cdot x(s) + b \cdot u(s) \\ y(s) = c^T \cdot x(s) \end{cases} \Rightarrow s \cdot x(s) - A \cdot x(s) = b \cdot u(s)$$

$$\Rightarrow \begin{cases} s \cdot i \cdot x(s) - A \cdot x(s) = b \cdot u(s) \\ y(s) = c^T \cdot x(s) \end{cases}$$

$$\Rightarrow \begin{cases} x(s)\left[s \cdot i - A\right] = b \cdot u(s) \\ y(s) = c^T \cdot x(s) \end{cases}$$

$$\Rightarrow \begin{cases} x(s) = [s \cdot i - A]^{-1} \cdot b \cdot u(s) \\ y(s) = c^T \cdot x(s) \end{cases}$$

$$\Rightarrow y(s) = \underbrace{c^T \cdot [s \cdot i - A]^{-1} \cdot b}_{H_{PC}} \cdot u(s) \quad \Big| : u(s)$$

$$H(s) = \frac{y(s)}{u(s)} = c^T \cdot [s \cdot i - A]^{-1} \cdot b$$

$$H(s) = \underline{c}^T (s\underline{I} - \underline{A})^{-1} \underline{b},\ \underline{A} = \begin{bmatrix} 0 & 1 & 0 & .. & 0 & 0 \\ 0 & 0 & 1 & .. & 0 & 0 \\ 0 & 0 & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & .. & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix} \in \Re^{n \times n},$$

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .. \\ 0 \\ \frac{1}{a_n} \end{bmatrix} \in \Re^{n \times 1},\ \underline{c}^T = [b_0\ b_1\ \dots b_m\ 0\ \dots 0] \in \Re^{1 \times n}$$

2) MM - ISI în timp discret

$$\begin{cases} x(k+1) = A\, x(k) + b\, u(k) \\ y(k) = c^T \cdot x(k) \end{cases} \quad \Big| \text{ Aplicăm } Z$$

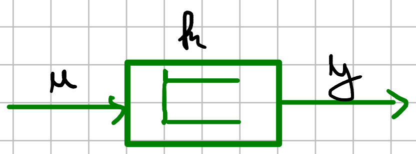
$$\Rightarrow \begin{cases} z\, x(z) = A\, x(z) + b\, u(z) \Rightarrow x(z)\, (z i - A)^{-1} = b\, u(z) \\ y(z) = c^T \cdot x(z) \end{cases}$$

$$\Rightarrow y(z) = c^T \cdot (z i - A)^{-1} \cdot b\, u(z)$$

$$\Rightarrow H(z) = \frac{y(z)}{u(z)} = c^T (z i - A)^{-1} \cdot b$$

Elemente de transfer (ET)

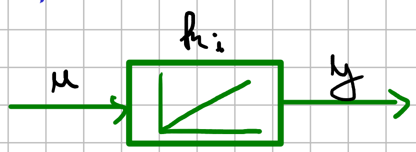
1) Elementul de transfer de tip proporțional (ET-P)



MM-ii: $y(t) = k \cdot u(t) \mid \mathcal{L}$

$y(s) = k\, u(s) \Rightarrow H(s) = k$

2) Elementul de transfer de tip integrator (ET-i)



MM-ii: $y(t) = k_i \int_0^t u(\tau)\, d\tau$

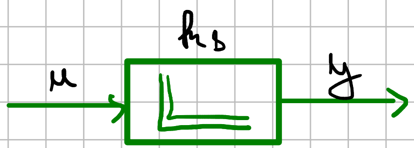
$\dot{y}(t) = k_i\, u(t) \mid \mathcal{L}$

$s\, y(s) = k_i\, u(s)$

$\Rightarrow y(s) = \frac{k_i}{s}\, u(s)$

$\Rightarrow H(s) = \frac{k_i}{s}$

3) Elementul de transfer de tip derivator (ET-d)

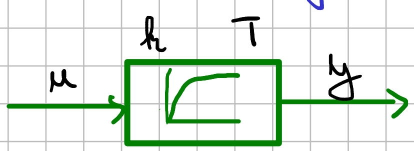


MM-ii: $y(t) = k_d \cdot \dot{u}(t) \mid \mathcal{L}$

$y(s) = k_d\, s\, u(s)$

$\Rightarrow H(s) = k_d \cdot s$

4) Elementul de transfer de tip proporțional cu temporizare de ordinul 1 (ET-PT1) $\longrightarrow$ filtru trece-jos



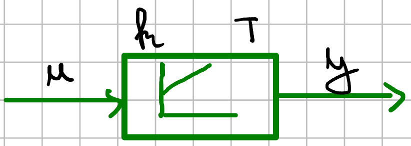
MM-ii: $T \cdot \dot{y}(t) + y(t) = k \cdot u(t) \mid \mathcal{L}$

$T \cdot s\, y(s) + y(s) = k \cdot u(s)$

$\Rightarrow y(s)[T \cdot s + 1] = k \cdot u(s)$

$\Rightarrow H(s) = \frac{k}{T \cdot s + 1}$

5) Elementul de transfer de tip proporțional - integrator (ET-PI)



$$MM\text{-ii}: y(t) = k\left[u(t) + \frac{1}{T}\int_0^t u(\tau)\,d\tau\right] \Rightarrow$$

$$y(t) = k\,u(t) + \frac{k}{T}\int_0^t u(\tau)\,d\tau \quad \Big|\ \text{derivăm}$$

$$\dot{y}(t) = k\,\dot{u}(t) + \frac{k}{T}\,u(t) \quad \Big|\ \mathcal{L}$$

$$\Rightarrow s\,y(s) = k\,s\,u(s) + \frac{k}{T}\,u(s)$$

$$\Rightarrow s\,y(s) = u(s)\left[k\,s + \frac{k}{T}\right] = u(s)\cdot\frac{k\,(sT+1)}{T}$$

$$\Rightarrow \quad H(s) = \frac{k}{sT}\left[sT+1\right]$$

- ET - element de transfer
- ET-P : proporțional
- ET-I : integrator
- ET-D : derivator
- ET-PT1 : proporțional cu temporizare de ordinul 1
- ET-PD-T1 : proporțional derivator cu temporizare de ordinul 1
- ET-DT1 : derivator cu temporizare de ordinul 1
- ET-PI : proporțional integrator
- ET-PD : proporțional derivator
- ET-PID : proporțional integrator derivator
- ET-PT2 : proporțional cu temporizare de ordinul 2

Transformata Z

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) \cdot z^{-k}$$

$\div$ $\quad S_n = \begin{cases} b_1 \cdot \dfrac{q^n - 1}{q - 1}, & q \neq 1 \\ n \cdot b_1, & q = 1 \end{cases} \qquad n \geq 1$

1) $f(k) = \sigma = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

$$\Rightarrow Z\{f(k)\} = \sum_{k=0}^{\infty} \sigma(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + \ldots + z^{-n} =$$

$$n \to \infty \qquad = 1 \cdot \frac{\overbrace{(z^{-1})^{n+1}}^{\to 0} - 1}{z^{-1} - 1} = \frac{-1}{z^{-1} - 1} = \frac{1}{1 - z^{-1}}$$

$$\Rightarrow Z\{f(k)\} = Z\{\sigma(k)\} = \frac{1}{1 - z^{-1}}$$

2) $f(k) = e^{-a \cdot k}$

$$Z\{f(k)\} = \sum_{k=0}^{\infty} e^{-ak} \cdot z^{-k} = \sum_{k=0}^{\infty} (e^{-a} \cdot z)^{k} =$$

$$= (e^{-0} \cdot z)^0 + (e^{-1} \cdot z)^1 + \ldots + (e^{-n} \cdot z)^n, \quad n \to \infty$$

$$= 1 \cdot \frac{(e^{-a} \cdot z)^n - 1}{e^{-a} \cdot z - 1} = \frac{1}{1 - e^{-a} \cdot z} = Z\{e^{-ak}\}$$

3) $$Z\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1) \cdot z^{-k} = \sum_{\tau=1}^{\infty} f(\tau) \cdot z^{-\tau+1} =$$

$$= \sum_{\tau=1}^{\infty} f(\tau) \cdot z^{-\tau} \cdot z$$

Notăm $k+1 = \tau$ $\quad\Big|\quad$ $k \to \infty \Rightarrow \tau \to \infty$

$k = \tau - 1$ $\quad\Big|\quad$ $k \to 0 \Rightarrow \tau \to 1$

$$= z \sum_{\tau=1}^{\infty} f(\tau) \cdot z^{-\tau} =$$

$$= z\left[f(1)\cdot z^{-1} + \dots + f(m)\, z^{-m} \right] \quad , m \to \infty$$

Adăugăm și scădem un $f(0)$

$$= z\left[F(z) - f(0)\right]$$

$Z\{f(k+2)\}$ poate fi demonstrat în același fel că este

$$= z^2\left[F(z) - f(0) - f(1)\, z^{-1}\right]$$

$$\Rightarrow Z\{f(k+m)\} = z^m\left[F(z) - f(0) - f(1)\, z^{-1} - \dots - f(m-1)\, z^{m-1}\right] =$$

$$= z^m\left[F(z) - \sum_{k=0}^{m-1} f(k)\cdot z^{-m}\right]$$

④ Rezolvați ecuația diferențială folosind transformata Z:

$$y(k+1) - a\,y(k) = 0, \quad y(0) = y_0$$

$$Z\{y(k+1) - a\,y\{k\}\} = Z\{0\} \Rightarrow Z\{y(k+1)\} - aZ\{y(k)\} = 0 \Rightarrow$$

$$z\left[Y(z) - y(0)\right] - a\,Y(z) = 0 \Rightarrow z\,Y(z) - z\,y_0 - a\,Y(z) = 0 \Rightarrow Y(z)(z-a) = z\,y_0 \Rightarrow$$

$$Y(z) = \frac{z\,y_0}{z-a} = y_0 \frac{1}{1-az^{-1}} \Rightarrow Z^{-1}\{Z\{y(k)\}\} = Z^{-1}\{Y(z)\} = y(k) \Rightarrow$$

$$y(k) = Z^{-1}\left\{y_0 \frac{1}{1-az^{-1}}\right\} = y_0\, Z^{-1}\left\{\frac{1}{1-az^{-1}}\right\} = y_0 a^k \Rightarrow \boxed{y(k) = y_0 a^k}$$

⑤ Rezolvați ecuația diferențială de ordinul 2 folosind transformata Z:

$$y(k+2) - 18\,y(k+1) + 32\,y(k) = 0, \quad y(0) = 0, \quad y(1) = 2$$

$$Z\{y(k+2) - 18y(k+1) + 32y(k)\} = Z\{0\} \Rightarrow Z\{y(k+2)\} - 18Z\{y(k+1)\} + 32Z\{y(k)\} = 0$$

$$\Rightarrow z^2\left[Y(z) - y(0) - y(1)z^{-1}\right] - 18z\left[Y(z) - y(0)\right] + 32\,Y(z) = 0 \Rightarrow$$

$$z^2\left[Y(z) - 2z^{-1}\right] - 18z\,Y(z) + 32\,Y(z) = 0 \Rightarrow z^2 Y(z) - 2z - 18z\,Y(z) + 32\,Y(z) = 0$$

$$Y(z)(z^2 - 18z + 32) = 2z \Rightarrow Y(z) = \frac{2z}{z^2 - 18z + 32} = \frac{2z}{(z-16)(z-2)} = \frac{2z}{z^2(1-16z^{-1})(1-2z^{-1})}$$

$$Y(z) = \frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})} \Rightarrow y(k) = Z^{-1}\{Y(z)\} = Z^{-1}\{Z\{y(k)\}\} \Rightarrow$$

$$y(k) = Z^{-1}\left\{\frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})}\right\}$$

$$\frac{2z^{-1}}{(1-2z^{-1})(1-16z^{-1})} = \frac{A}{1-2z^{-1}} + \frac{B}{1-16z^{-1}} \Rightarrow A(1-16z^{-1}) + B(1-2z^{-1}) = 2z^{-1}$$

$$A - 16Az^{-1} + B - 2Bz^{-1} = 2z^{-1} \Rightarrow$$

$$\begin{cases} A + B = 0 \\ -16A - 2B = 2 \end{cases} \Rightarrow \begin{array}{r} A + B = 0 \\ -8A - B = 1 \\ \hline -7A \quad / = 1 \end{array} \Rightarrow \boxed{A = -\frac{1}{7}} \Rightarrow \boxed{B = -A = \frac{1}{7}}$$

$$y(k) = Z^{-1}\left\{\frac{-1/7}{1-2z^{-1}}\right\} + Z^{-1}\left\{\frac{1/7}{1-16z^{-1}}\right\} = -\frac{1}{7} Z^{-1}\left\{\frac{1}{1-2z^{-1}}\right\} + \frac{1}{7} Z^{-1}\left\{\frac{1}{1-16z^{-1}}\right\} \Rightarrow$$

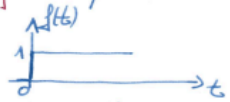
$$\boxed{y(k) = -\frac{1}{7} 2^k + \frac{1}{7} 16^k}$$

Definiție: În matematică, transformata Laplace este o integrală care convertește o funcție a unei variabile reale $\boxed{t}$ (de obicei t=timp) într-o funcție a unei variabile complexe $\boxed{s}$ (în domeniul frecvenței, cunoscut și sub numele de domeniul s sau planul s). Transformarea are multe aplicații în știință și inginerie, deoarece este un instrument pentru rezolvarea ecuațiilor diferențiale, în special transformă ecuațiile diferențiale în ecuații algebrice.

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt}$$

$\mathcal{L}$- operator liniar

1) $f(t)=\sigma(t) = \begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$



$$\mathcal{L}\{f(t)\}=\mathcal{L}\{\sigma(t)\}=F(s)=\int_0^\infty \sigma(t)e^{-st}dt=\int_0^\infty e^{-st}dt = -\frac{1}{s}\int_0^\infty -se^{-st}dt=$$

$(e^{-st})'=-se^{-st}$

$$=-\frac1s\int_0^\infty (e^{-st})'dt = -\frac1s e^{-st}\Big|_0^\infty$$

$$=-\frac1s\left(\cancel{\lim_{t\to\infty} e^{-st}}^{0} - \lim_{t\to0}e^{-st}\right)$$

$$=-\frac1s(0-1)=\frac1s \Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\}=\mathcal{L}\{\sigma(t)\}=F(s)=\frac1s}$$

2) $f(t)=e^{-at}\Rightarrow \mathcal{L}\{f(t)\}=F(s)=\int_0^\infty e^{-at}e^{-st}dt=\int_0^\infty e^{-(s+a)t}dt \Rightarrow$

$[e^{-(s+a)t}]'=-(s+a)e^{-(s+a)t}$

$$\mathcal{L}\{f(t)\}=F(s)=\int_0^\infty e^{-(s+a)t}dt = \frac{-1}{s+a}\int_0^\infty -(s+a)e^{-(s+a)t}dt = -\frac{1}{s+a}\int_0^\infty [e^{-(s+a)t}]'dt$$

$$=-\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{-1}{s+a}\left(\cancel{\lim_{t\to\infty}e^{-(s+a)t}}^{0} - \lim_{t\to0}e^{-(s+a)t}\right) = \frac{1}{s+a}\Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\}=\mathcal{L}\{e^{-at}\}=F(s)=\frac{1}{s+a}}$$

3) $f(t)=f(t-m)\Rightarrow \mathcal{L}\{f(t)\}=F(s)=\int_0^\infty f(t-m)e^{-st}dt=\int_{-m}^\infty f(\tau)e^{-s(\tau+m)}d\tau$

notăm: $t-m=\tau \Rightarrow t=\tau+m\Rightarrow dt=d\tau$

$t\to\infty\Rightarrow\tau\to\infty$

$t\to0\Rightarrow\tau\to-m$

$$\Rightarrow \mathcal{L}\{f(t)\}=e^{-sm}\int_{-m}^\infty f(\tau)e^{-s\tau}d\tau = e^{-sm}\left[\cancel{\int_{-m}^0 f(\tau)e^{-s\tau}d\tau}^{0} + \int_0^\infty f(\tau)e^{-s\tau}d\tau\right]=$$

$$=e^{-sm}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sm}\mathcal{L}\{f(\tau)\}=e^{-sm}F(s)\Rightarrow$$

$$\boxed{\mathcal{L}\{f(t)\}=\mathcal{L}\{f(t-m)\}=e^{-sm}F(s)}$$

4) $\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(e^{-st})'dt=$

$$=\underbrace{\lim_{t\to\infty}[f(t)e^{-st}]}_{\cong 0} - \underbrace{\lim_{t\to0}[f(t)e^{-st}]}_{f(0)} - \int_0^\infty f(t)(-se^{-st})dt = -f(0)+s\underbrace{\int_0^\infty f(t)e^{-st}dt}_{F(s)}$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\}=sF(s)-f(0)}$$

$$\mathcal{L}\{f''(t)\}=\int_0^\infty (f'(t))'e^{-st}dt = f'(t)e^{-st}\Big|_0^\infty - \int_0^\infty f'(t)(e^{-st})'dt=$$

$$=\cancel{\lim_{t\to\infty}[f'(t)e^{-st}]}^{0} - \underbrace{\lim_{t\to0}[f'(t)e^{-st}]}_{f'(0)} + s\underbrace{\int_0^\infty f'(t)e^{-st}dt}_{\mathcal{L}\{f'(t)\}} = s\mathcal{L}\{f'(t)\}-f'(0)=$$

$$=s[sF(s)-f(0)]-f'(0)=s^2F(s)-sf(0)-f'(0)\Rightarrow$$

$$\boxed{\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)}$$

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\dots-sf^{(n-2)}(0)-f^{(n-1)}(0)\Rightarrow$$

$$\boxed{\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)}$$

5) Rezolvați ecuația diferențială $y'(t)+by(t)=1$, $y(0)=y_0=0$

$\mathcal{L}\{y'(t)+by(t)\}=\mathcal{L}\{1\}$ => proprietatea de liniaritate

$\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}=aF(s)+bG(s)$ =>

$\mathcal{L}\{y'(t)\}+b\mathcal{L}\{y(t)\}=\mathcal{L}\{1\} \Rightarrow sY(s)-y(0)+bY(s)=\frac{1}{s} \Rightarrow Y(s)(s+b)=\frac{1}{s}+y(0)$

$\Rightarrow Y(s)(s+b)=\frac{1}{s} \Rightarrow Y(s)=\frac{1}{s(s+b)} \Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s(s+b)}\right\}=y(t)$

$\frac{1}{s(s+b)}=\frac{A}{s}+\frac{B}{s+b} \Rightarrow A(s+b)+Bs=1 \Rightarrow As+Ab+Bs=1 \Rightarrow$

$s(A+B)+Ab=1 \Rightarrow \begin{cases} A+B=0 \\ Ab=1 \end{cases} \Rightarrow \boxed{A=\frac{1}{b}} \Rightarrow \boxed{B=-A=-\frac{1}{b}}$

$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s(s+b)}\right\}=\mathcal{L}^{-1}\left\{\frac{1/b}{s}-\frac{1/b}{s+b}\right\} \Rightarrow$

$y(t)=\mathcal{L}^{-1}\left\{\frac{1/b}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1/b}{s+b}\right\}=\frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{b}\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}=\frac{1}{b}-\frac{1}{b}e^{-bt}$

$\Rightarrow \boxed{y(t)=\frac{1}{b}(1-e^{-bt})}$

6) Rezolvați ecuația diferențială $y''(t)-2y'(t)+y(t)=3e^t$, $y(0)=1$, $y'(0)=1$

$\mathcal{L}\{y''(t)-2y'(t)+y(t)\}=\mathcal{L}\{3e^t\} \Rightarrow$

$\mathcal{L}\{y''(t)\}-2\mathcal{L}\{y'(t)\}+\mathcal{L}\{y(t)\}=3\mathcal{L}\{e^t\} \Rightarrow$

$s^2Y(s)-sy(0)-y'(0)-2\left[sY(s)-y(0)\right]+Y(s)=\frac{3}{s-1} \Rightarrow$

$s^2Y(s)-s-1-2sY(s)+2+Y(s)=\frac{3}{s-1} \Rightarrow Y(s)(s^2-2s+1)-s+1=\frac{3}{s-1} \Rightarrow$

$Y(s)(s^2-2s+1)=\frac{3}{s-1}+(s-1)=\frac{3+(s-1)^2}{s-1}=\frac{3+s^2-2s+1}{s-1}=\frac{s^2-2s+4}{s-1} \Rightarrow$

$Y(s)=\frac{s^2-2s+4}{(s-1)^3} \Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s^2-2s+4}{(s-1)^3}\right\}=y(t)$

$\frac{s^2-2s+4}{(s-1)^3}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3} \Rightarrow A(s-1)^2+B(s-1)+C=s^2-2s+4$

$A(s^2-2s+1)+B(s-1)+C=s^2-2s+4$

$As^2-2As+A+Bs-B+C=s^2-2s+4 \Rightarrow s^2A+s(-2A+B)+A+C=s^2-2s+4 \Rightarrow$

$\begin{cases} A=1 \\ -2A+B=-2 \\ A+C=4 \end{cases} \Rightarrow \boxed{A=1}$ $\quad -2+B=-2 \Rightarrow \boxed{B=0}$

$C=4-A \Rightarrow \boxed{C=3}$

$y(t)=\mathcal{L}^{-1}\left\{\frac{s^2-2s+4}{(s-1)^3}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s-1}+\frac{3}{(s-1)^3}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}=$

$=e^t+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}=e^t+\frac{3}{2}t^2e^t \Rightarrow \boxed{y(t)=e^t+\frac{3}{2}t^2e^t}$

$\boxed{\text{Temă}}$: pag 9/3 b) Folosiți transformata Laplace pentru a rezolva următoarea ecuație diferențială:

$$y''(t)+4y(t)=\sin(2t) \qquad y(0)=1,\ y'(0)=0 \Rightarrow \mathcal{L}\{y''(t)+4y(t)\}=\mathcal{L}\{\sin(2t)\} \Rightarrow$$

$$\mathcal{L}\{y''(t)\}+4\mathcal{L}\{y(t)\}=\frac{2}{s^2+4} \Rightarrow s^2Y(s)-sy(0)-y'(0)+4Y(s)=\frac{2}{s^2+4}$$

$$Y(s)(s^2+4)-s=\frac{2}{s^2+4} \Rightarrow (s^2+4)Y(s)=\frac{2}{s^2+4}+s \Rightarrow (s^2+4)Y(s)=\frac{s^3+4s+2}{s^2+4}.$$

$$\Rightarrow Y(s)=\frac{s^3+4s+2}{(s^2+4)^2} \Rightarrow \mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\{\mathcal{L}\{y(t)\}\}=y(t)$$

$$\frac{s^3+4s+2}{(s^2+4)^2}=\frac{As+B}{s^2+4}+\frac{Cs+D}{(s^2+4)^2} \Rightarrow (As+B)(s^2+4)+Cs+D=s^3+4s+2$$

$$\Rightarrow As^3+4As+Bs^2+4B+Cs+D=s^3+4s+2 \Rightarrow As^3+Bs^2+s(4A+C)+4B+D=s^3+4s+2$$

$$\begin{cases} \boxed{A=1} \\ \boxed{B=0} \\ 4A+C=4 \Rightarrow 4+C=4 \Rightarrow \boxed{C=0} \\ 4B+D=2 \Rightarrow \boxed{D=2} \end{cases}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\}=\cos(2t)+\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} \Rightarrow$$

$$y(t)=\cos(2t)+\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2\cdot 2^3}{(s^2+2^2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\}=\sin(2t)-2t\cos(2t)$$

$$\Rightarrow y(t)=\cos(2t)+\frac{1}{8}\left[\sin(2t)-2t\cos(2t)\right]$$

$$y(t)=\cos(2t)-\frac{t}{4}\cos(2t)+\frac{1}{8}\sin(2t)$$

$$\boxed{y(t)=\left(1-\frac{t}{4}\right)\cos(2t)+\frac{1}{8}\sin(2t)}$$