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Mathematics of Games

Exam 1 - 2023 - Solution

Time: 120 minutes.
Total : $6 \times 10 = 60$ Points

1. Audi and BMW are producing cars in a Cournot market. The public is willing to pay a price of $p_A = 42 - Q$ for an Audi and $p_B = 51 - Q$ for a BMW, where $Q = q_A + q_B$ is the aggregate quantity produced by Audi and BMW. The production cost c_A per car for Audi is drawn uniformly at random from the interval $[0, 12]$, while the production cost c_B of BMW is drawn uniformly at random from the interval $[6, 18]$. The actual production cost of a company is private information of that company.

   Suppose, in an equilibrium market, a customer ends up paying 17 for an Audi. Determine $c_A + c_B$, the sum of the actually drawn production costs.

   Solution:

   From an outside point of view, the expected production costs for Audi and BMW respectively are

$$\mathbb{E}[c_A] = 6 \quad \wedge \quad \mathbb{E}[c_B] = 12$$

   The expected payoff for Audi is

$$u_A(q_A, q_B) = q_A(42 - q_A - q_B - c_A),$$

   which is maximized by differentiation for

$$q_A^* = \frac{1}{2}(42 - q_B - c_A)$$

   Analogously, we obtain

$$q_B^* = \frac{1}{2}(51 - q_A - c_B).$$

   Plugging this into one another, we obtain

$$\begin{aligned} q_A^* &= \frac{1}{2}\left(42 - \frac{1}{2}(51 - q_A - c_B) - c_A\right) \\ &= 21 - \frac{51}{4} + \frac{1}{4}c_B - \frac{1}{2}c_A + \frac{1}{4}q_A \\ \iff q_A^* &= \frac{4}{3}\left(21 - \frac{51}{4} + \frac{1}{4}\mathbb{E}[c_B] - \frac{1}{2}c_A\right) \\ &= \frac{1}{3}\left(84 - 51 + 12 - 2c_A\right) \\ &= 15 - \frac{2}{3}c_A \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
q_B^* &= \frac{1}{2}\left(51 - \frac{1}{2}(42 - q_B - c_A) - c_B\right) \\
&= \frac{51}{2} - \frac{42}{4} + \frac{1}{4}c_A - \frac{1}{2}c_B + \frac{1}{4}q_B \\
\iff \quad q_B^* &= \frac{4}{3}\left(\frac{51}{2} - \frac{42}{4} + \frac{1}{4}\mathbb{E}[c_A] - \frac{1}{2}c_B\right) \\
&= \frac{1}{3}\left(102 - 42 + 6 - 2c_B\right) \\
&= 22 - \frac{2}{3}c_B.
\end{aligned}$$

Hence, in equilibrium, the public pays a price of

$$\begin{aligned}
p_A &= 42 - \left(15 - \frac{2}{3}c_A\right) - \left(22 - \frac{2}{3}c_B\right) \\
&= 5 + \frac{2}{3}(c_A + c_B) \\
&= 17 \\
\iff \quad c_A + c_B &= \frac{3}{2}(17 - 5) \\
&= 18.
\end{aligned}$$

2. For the following normal form table game, give a value for x, such that there is a proper mixed strategy Nash equilibrium, where (T, L) is played with a probability of 20%.

	L	R
T	$2, 5$	$3, x$
B	$4, 3$	$1, 5$

Solution:

Suppose in a mixed strategy player 1 plays T with probability p and player 2 plays L with probability q. In a MSNE, each player has to be indifferent between either of his options, that means the respective payoff for either option has to be the same:

$$\begin{aligned} EPO_1[T] = 2q + 3(1-q) &= 4q + (1-q) = EPO_1[B] \\ \iff 2(1-q) &= 2q \\ \iff q &= \frac{1}{2} \end{aligned}$$

For player 2 we obtain relative to x

$$\begin{aligned} EPO_2[L] = 5p + 3(1-p) &= xp + 5(1-p) = EPO_2[R] \\ \iff (5-x)p &= 2(1-p) \\ \iff (7-x)p &= 2 \iff p = \frac{2}{7-x} \end{aligned}$$

Since (T, L) is played with probability 20%, we obtain that $p \cdot q = \frac{1}{5}$, which implies with $q = \frac{1}{2}$, that $p = \frac{2}{5}$ and hence

$$\frac{2}{5} = p = \frac{2}{7-x} \iff x = 2.$$

3. In the following signalling game, player 1 is of type P or Q with equal probability and can choose an action $\sigma_1 \in \{A, B\}$, which is observed by player 2, who can then react with a reaction $\sigma_2 \in \{X, Y, Z\}$. The payoffs are given in the following table. Give all perfect Bayesian equilibria and check them with the intuitive criterion. Give your equilibria in the form

$$\big((\text{Action type } P, \text{ Action type } Q), (\text{Reaction to } A, \text{ Reaction to } B), \mu(P|A), \mu(P|B)\big)$$

(σ_1, σ_2)	P	Q
(A,X)	9,8	6,3
(A,Y)	7,4	1,9
(A,Z)	6,3	3,6
(B,X)	8,2	5,3
(B,Y)	7,3	5,4
(B,Z)	7,5	7,2

Solution:

We first observe, that when A is played, Z is strictly dominated by Y and if B is played, then X is strictly dominated by Y. Hence, we can eliminate these options.

For the pooling equilibria, we first consider (A, A). Since $8 + 3 < 4 + 9$, player 2 reacts with Y. However, here Q gets payoff 1, while he gets at least 5 for any reaction to B, hence he would deviate and there is no equilibrium with (A, A),

For the second type of pooling equilibria with (B, B), we notice that player 2 is indifferent between reacting to B with Y or Z, since $3 + 4 = 5 + 2$. Hence, we first consider the reaction with Y to B. Here, player 2 needs to react to A with Y in order to prevent player 1 from deviating. This is a best response for player 2 if for $\mu = \mu(P|A)$ it holds that

$$8\mu + 3(1 - \mu) \leq 4\mu + 9(1 - \mu) \iff 4\mu \leq 6(1 - \mu) \iff 10\mu \leq 6 \iff \mu \leq \frac{3}{5}.$$

This yields the first pooling equilibrium

$$\left((B, B), (Y, Y), \mu(P|A) \leq \frac{3}{5}, \frac{1}{2}\right).$$

Here, both types could potentially make a better payoff by deviating to (A, X) and our belief is not completely unreasonable, making this equilibrium survive the intuitive criterion.

Secondly, we consider the pooling equilibrium with reaction Z to B. Here, player 2 also needs to react with Y to A in order to prevent type P from deviating. This is again a best response for player 2 if $\mu(P|A) \leq \frac{3}{5}$, yielding the last pooling equilibrium

$$\left((B, B), (Y, Z), \mu(P|A) \leq \frac{3}{5}, \frac{1}{2}\right).$$

Here, we see that type Q already obtains his unique highest possible payoff in the game and there should be no reason to belief it is type Q if A is observed, yet we assign positive probability to that event. Hence, this equilibrium does not survive the intuitive criterion.

For the separating equilibria, we first look at (A, B), where a best response is (X, Y), which in turn would make type Q deviate, hence, no such equilibrium.

Alternatively for (B, A), we see that a best response is (Z, Y), which again would make type Q deviate, which means there are no separating equilibria at all.

4. Consider the following infinite 2-player game with payoffs given below. Give a largest possible interval $I \subseteq (0,1)$, such that the following strategy is a subgame perfect Nash equilibrium for all values of the discount factor $\delta \in I$:

Play (B,X) in the first turn. In every turn after that play (B,X) if (B,X) was played in all previous turns. If a first deviation was done by player 1, always play (B,Y) afterwards and if a first deviation was done by player 2, always play (A,X) afterwards.

	X	Y
A	$8,1$	$1,0$
B	$4,6$	$2,7$

Solution:

We consider the payoffs for sticking to the strategy and compare them to the payoffs for deviating in the first turn. For player 1, sticking to the strategy is worth it if and only if

$$\begin{aligned} PO_1[\text{strat}] = \sum_{t=0}^{\infty} \delta^t 4 = \frac{4}{1-\delta} \geq PO_1[\text{dev}] = 8 + \sum_{t=1}^{\infty} \delta^t 2 = 8 + \frac{2\delta}{1-\delta} \\ \iff 4 \geq 8 - 8\delta + 2\delta \\ \iff 6\delta \geq 4 \\ \iff \delta \geq \frac{2}{3}. \end{aligned}$$

Similarly for player 2 we obtain that sticking to the strategy is worth it if and only if

$$\begin{aligned} PO_2[\text{strat}] = \sum_{t=0}^{\infty} \delta^t 6 = \frac{6}{1-\delta} \geq PO_2[\text{dev}] = 7 + \sum_{t=1}^{\infty} \delta^t 1 = 7 + \frac{\delta}{1-\delta} \\ \iff 6 \geq 7 - 7\delta + \delta \\ \iff 6\delta \geq 1 \\ \iff \delta \geq \frac{1}{6}. \end{aligned}$$

Since we need to keep both players from deviating and $\frac{2}{3} \geq \frac{1}{6}$, we obtain that the above described strategy is a subgame perfect Nash equilibrium if and only if $\frac{2}{3} \leq \delta < 1$, that is $I = \left[\frac{2}{3}, 1\right)$.

5. The following game is repeated as long as the outcome of every previous turn was (A, A) and ends after the first turn, where at least one player plays B. There is no discount factor. Give a symmetric, subgame perfect, mixed strategy Nash equilibrium for this stopping game, where each player plays A with probability p.

	A	B
A	$-1, -1$	$-1, 0$
B	$0, -1$	$-4, -4$

Solution:

In such an equilibrium in a stopping game, a player is indifferent between picking B this turn or the next turn. W.l.o.g. we consider this in the very first turn. His expected payoff for playing B in the first turn is $p \cdot 0 + (1-p) \cdot (-4)$, while his expected payoff for playing A in the first turn and then playing B in the second turn, in case the game continues, is $-1 + p(p \cdot 0 + (1-p)(-4))$. Setting these two payoffs equal, we obtain

$$\begin{aligned}
p \cdot 0 + (1-p) \cdot (-4) &= -1 + p(p \cdot 0 + (1-p)(-4)) \\
\iff 4p - 4 &= -1 + p(1-p)(-4) \\
\iff 4p &= 3 - 4p + 4p^2 \\
\iff 0 &= 4p^2 - 8p + 3 \\
\iff p &= \frac{8 \pm \sqrt{8^2 - 4 \cdot 4 \cdot 3}}{2 \cdot 4} \\
&= \frac{8 \pm \sqrt{16}}{8} \\
&= \frac{1}{2}.
\end{aligned}$$

6. Alice and Bob are bargaining over one dollar. They play a three-period bargaining game with alternating offers and discount factor $\delta = \frac{1}{2}$. In particular, Alice offers a split division in periods 0 and 2 (if the game does not end before $t = 2$), while Bob makes his proposal in period 1 (if the game does not end in stage 0).

   Compared to standard bargaining games, this one has the following additional feature: In each period, besides saying Yes or No, the respondent can choose an outside option, Out, that is worth a fraction α, where $(0 < \alpha < 1)$. If the respondent chooses the outside option, the proposer gets nothing and the game terminates. The parameter α is commonly known by both players and also affected by the discount factor. The game ends when a proposal is accepted, or the outside option is chosen, or at the end of period 2. For every $\alpha \in \left(0, \frac{1}{3}\right)$, find a subgame perfect equilibrium.

   Solution:
   In turn $t = 2$, Alice has to offer at least the Out amount, that is, she offers a split of $(1 - \alpha, \alpha)$ to Bob and he accepts. Note that since $\alpha < \frac{1}{3}$, Out is no option for Alice in turn $t = 2$.

   In turn $t = 1$, Bob knows that Alice can guarantee herself at least $\frac{1-\alpha}{4}$ in the next (and last) turn by saying No. Hence, he offers her just that, $\frac{1-\alpha}{4}$ and to himself $\frac{1}{2} - \frac{1-\alpha}{4} = \frac{1+\alpha}{4}$, that is a split of $\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right)$. If Alice were to choose the Out option instead, she would get a total of $\frac{\alpha}{2}$, which is less than $\frac{1-\alpha}{4}$ for $\alpha < \frac{1}{3}$.

   In turn $t = 0$, Bob knows he can propose for himself at least $\frac{1+\alpha}{4}$ in turn $t = 1$. His Out option would yield him α, which is again less than $\frac{1}{2} - \frac{1-\alpha}{4}$ for $\alpha < \frac{1}{3}$. Hence, in turn $t = 0$, Alice offers Bob $\frac{1+\alpha}{4}$ and for herself $\frac{3-\alpha}{4}$, which is also larger than her Out option, α for $\alpha < \frac{1}{3}$, hence, a split of $\left(\frac{3-\alpha}{4}, \frac{1+\alpha}{4}\right)$.