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Mathematics of Games

Exam 2 - 2023 - Solution

Time: 120 minutes.
Total : $6 \times 10 = 60$ Points

1. In the following signalling game, player 1 is of type P or Q with equal probability and can choose an action $\sigma_1 \in \{A, B, C\}$, which is observed by player 2, who can then react with a reaction $\sigma_2 \in \{X, Y\}$. The payoffs are given in the following table. Give all perfect Bayesian equilibria and check them with the intuitive criterion. Give your equilibria in the form

$$\big((\text{Action } P, \text{ Action } Q), (\text{Reaction } A, \text{ Reaction } B, \text{ Reaction } C),$$
$$\mu(P|A), \mu(P|B), \mu(P|C)\big).$$

(σ_1, σ_2)	P	Q
(A,X)	4,1	7,3
(A,Y)	9,7	2,4
(B,X)	2,6	8,6
(B,Y)	5,1	3,8
(C,X)	8,7	5,4
(C,Y)	6,4	4,5

Solution:

For Player 2, reacting to A with Y strictly dominates reacting to A with X. Hence, for type Q, A is strictly dominated by B and C. Similarly, for type P, playing C strictly dominates playing B. Hence, in no equilibrium, type P plays B or type Q plays A.

For our equilibria, this means we have to check the following combinations of (σ_1, σ_2):

- For (A, B), player 2 reacts to B with Y, yielding a payoff of 3 for Q. No matter how player 2 reacts to C, Q could improve his payoff. Hence, no such equilibrium
- For (A, C), player 2 reacts with (Y, Y, Y), in order to maximize his payoff for A and C and also in order to prevent Q from deviating to B. Here, neither P nor Q have an incentive to deviate. Reacting to B with Y is a best response for player 2 if it holds for $\mu = \mu(P|B)$, that

$$\begin{aligned} 6\mu + 6(1-\mu) &\leq 1\mu + 8(1-\mu) \\ \iff 6 &\leq 8 - 7\mu \\ \iff \mu &\leq \frac{2}{7}. \end{aligned}$$

$$((A,C),(Y,Y,Y),\mu(P|A)=1,\mu(P|B)\leq\frac{2}{7},\mu(P|C)=0)$$

The equilibrium survives the intuitive criterion, since it is certainly type Q if B is observed, because for type P, B is strictly dominated by C, implying $\mu(P|B)=0$, which is in accordance with our equilibrium.

- For (C,B), player 2 reacts to B with Y and to C with X, which would make Q want to deviate to C, so this constitutes no equilibrium
- For (C,C), player 2 would maximize his expected payoff by reacting to C with X. Now, if he reacts to A with X, Q would deviate and if he reacts to A with Y, P would deviate, hence also here no equilibrium.

By our above argumentation, this covers all possible actions by player 1 and thus all possible equilibria.

2. Airbus and Boeing are producing aircraft in a Cournot market, where each firm can choose their respective quantity to produce. The public is willing to pay a price of $x - Q$ for an aircraft from Airbus and a price of $12 - Q$ for an aircraft from Boeing, where $Q = q_A + q_B$ is the aggregate quantity produced by Airbus and Boeing. The production cost is $c_A = 3$ for Airbus and $c_B = 6$ for Boeing.

   Give a positive value for x, such that in an equilibrium market state, Airbus and Boeing make the same payoff.

   Solution:

   The payoff for Airbus is $u_A(q_A, q_B) = q_A \cdot (x - q_A - q_B - 3)$, which is maximized for $q_A = \frac{1}{2}(x - 3 - q_B)$. Similarly, the payoff for Boeing $u_B(q_A, q_B) = q_B \cdot (12 - q_A - q_B - 6)$ is maximized for $q_B = \frac{1}{2}(12 - 6 - q_A)$. Plugging this into one another, we obtain

$$\begin{aligned}
q_B &= \frac{1}{2}(6 - q_A) \\
&= \frac{1}{2}\left(6 - \frac{1}{2}(x - 3 - q_B)\right) \\
&= \frac{1}{2}\left(6 - \frac{1}{2}x + \frac{3}{2}\right) + \frac{1}{4}q_B \\
\iff q_B &= \frac{4}{3} \cdot \frac{1}{2} \cdot \left(\frac{15}{2} - \frac{1}{2}x\right) \\
&= 5 - \frac{1}{3}x, \text{ which implies} \\
\implies q_A &= \frac{1}{2}\left(x - 3 - 5 + \frac{x}{3}\right) \\
&= \frac{2}{3}x - 4.
\end{aligned}$$

   Plugging this into the respective payoffs, we obtain

$$\begin{aligned}
u_A(q_A, q_B) &= q_A \cdot (x - q_A - q_B - 3) \\
&= \left(\frac{2}{3}x - 4\right) \cdot \left(x - \frac{2}{3}x + 4 - 5 + \frac{1}{3}x - 3\right) \\
&= \left(\frac{2}{3}x - 4\right) \cdot \left(\frac{2}{3}x - 4\right) \\
&= \left(\frac{2}{3}x - 4\right)^2, \text{ and analogously} \\
u_B(q_A, q_B) &= q_B \cdot (12 - q_A - q_B - 6) \\
&= \left(5 - \frac{1}{3}x\right) \cdot \left(12 - \frac{2}{3}x + 4 - 5 + \frac{1}{3}x - 6\right) \\
&= \left(5 - \frac{1}{3}x\right)^2
\end{aligned}$$

which we can now set equal to obtain one of either

$$
\begin{aligned}
&\frac{2}{3}x - 4 = 5 - \frac{1}{3}x \\
&\iff x = 9, \text{ or} \\
&\frac{2}{3}x - 4 = -5 + \frac{1}{3}x \\
&\iff \frac{1}{3}x = -1 \\
&\iff x = -3, \text{ which is negative.}
\end{aligned}
$$

Hence, we obtain $x = 9$.

3. For the following normal form table game, determine all mixed strategy Nash equilibria.

	W	X	Y	Z
A	$6,6$	$4,3$	$7,4$	$5,8$
B	$1,7$	$4,7$	$6,2$	$7,9$
C	$4,3$	$3,5$	$8,1$	$6,4$
D	$7,1$	$7,4$	$5,3$	$3,2$

Solution:

We first apply the method of strict (expected) domination

- Z strictly dominates W
- **Option 1:**
  - $\frac{2}{3}X + \frac{1}{3}Z$ strictly dominates Y in expectation, since

$$\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 8 = \frac{14}{3} > 4$$
$$\frac{2}{3} \cdot 7 + \frac{1}{3} \cdot 9 = \frac{23}{3} > 2$$
$$\frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 4 = \frac{14}{3} > 1$$
$$\frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 2 = \frac{10}{3} > 3$$

  - B strictly dominates C
  - $\frac{2}{3}B + \frac{1}{3}D$ strictly dominates A in expectation, since

$$\frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 7 = \frac{15}{3} > 4$$
$$\frac{2}{3} \cdot 7 + \frac{1}{3} \cdot 3 = \frac{17}{3} > 5$$

- **Option 2:**
  - $\frac{7}{10}C + \frac{3}{10}D$ strictly dominates A in expectation, since

$$\frac{7}{10} \cdot 3 + \frac{3}{10} \cdot 7 = \frac{42}{10} > 4$$
$$\frac{7}{10} \cdot 8 + \frac{3}{10} \cdot 5 = \frac{71}{10} > 7$$
$$\frac{7}{10} \cdot 6 + \frac{3}{10} \cdot 3 = \frac{51}{10} > 5$$

  - X strictly dominates Y
  - B strictly dominates C

Either way, there are no further dominations to be found, so we can continue with the reduced table:

	X	Z
B	$4,7$	$7,9$
D	$7,4$	$3,2$

We may assume that player 1 plays B with probability b and player 2 plays X with probability x in any MSNE. Since each player needs to be indifferent between either of his options, this yields

$$\begin{aligned} 4x + 7(1-x) &= 7x + 3(1-x) \\ \iff -3x + 7 &= 4x + 3 \\ \iff 4 &= 7x \\ \iff x &= \frac{4}{7}, \text{ and similarly} \\ 7b + 4(1-b) &= 9b + 2(1-b) \\ \iff 3b + 4 &= 7b + 2 \\ \iff 2 &= 4b \\ \iff b &= \frac{1}{2} \end{aligned}$$

which implies that the only MSNE for our original game is

$$\left(\left(0, \frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{4}{7}, 0, \frac{3}{7}\right)\right).$$

4. For the following normal form table game, give a value for x, such that in a proper mixed strategy Nash equilibrium, player 1 (Options T, B) has twice the payoff of player 2 (Options L, R) in expectation.

	L	R
T	$x, 1$	$2, 2$
B	$2, 3$	$4, 0$

Solution:

Suppose in an equilibrium, player 1 plays T with probability p and player 2 plays L with probability q. In such a mixed equilibrium, each player is indifferent between either of his options and we see that

$$\begin{aligned} qx + 2(1-q) &= 2q + 4(1-q) \\ \iff qx &= 2q + 4 - 4q - 2 + 2q \\ \iff q &= \frac{2}{x} \end{aligned}$$

and similarly

$$\begin{aligned} 1p + 3(1-p) &= 2p \\ \iff 3 &= 4p \\ \iff p &= \frac{3}{4}. \end{aligned}$$

Since the expected payoff for each player is the same for each of his actions by requirement, we obtain that

$$\begin{aligned} EPO[P_1] &= x \cdot \frac{2}{x} + 2\left(1 - \frac{2}{x}\right) \\ &= 4 - \frac{4}{x}, \text{ and for player 2} \\ EPO[P_2] &= 2p + 0 \\ &= \frac{3}{2}. \end{aligned}$$

Seeing that the expected payoff of player 1 is twice that of player 2, we get the equation

$$\begin{aligned} 4 - \frac{4}{x} &= 2 \cdot \frac{3}{2} \\ \iff 1 &= \frac{4}{x} \\ \iff x &= 4. \end{aligned}$$

5. Consider the following infinite 2-player game with payoffs given below, where $x > y > 1$, and the following two strategies:

σ_1: *Play (B, C) in turn $t = 0$. In every turn after that play (B, C) if (B, C) was played in all previous turns. If a first deviation was done by player 1, always play (B, D) afterwards and if a first deviation was done by player 2, always play (A, C) afterwards.*

σ_2: *Starting in turn $t = 0$, in all even turns play (A, C) and in all odd turns, play (B, D).*

There is a discount factor $\delta \in (0, 1)$.

(a) For which values of δ is σ_1 a pure strategy subgame perfect Nash equilibrium?

(b) If σ_1 is a pure strategy subgame perfect Nash equilibrium, for which values of δ does σ_2 yield a higher payoff for player 1 than σ_1?

	C	D
A	$x, 1$	$0, 0$
B	y, y	$1, x$

Solution:

(a) σ_1 is a subgame perfect PSNE if and only if the payoff for sticking to the strategy is at least as high as the payoff for for deviating. We may assume, that a possible deviation takes place in the very first turn.

$$PO[\sigma_1, \text{strat.}] = y \sum_{t=0}^{\infty} \delta^t = \frac{y}{1-\delta} \geq x + \frac{\delta}{1-\delta} = x + y \sum_{t=1}^{\infty} \delta^t = PO[\sigma_2, \text{dev.}]$$

$$\begin{aligned} &\iff y \geq x - x\delta + \delta \\ &\iff y - x \geq \delta(1-x) \\ &\iff \frac{y-x}{1-x} \leq \delta \\ &\iff \delta \geq \frac{x-y}{x-1} \end{aligned}$$

(b) We have already seen that player 1's payoff for σ_1 when it is a s.p. PSNE is $\frac{y}{1-\delta}$. His payoff for σ_2 is

$$\begin{aligned} x \cdot \sum_{t=0}^{\infty} \delta^{2t} + 1 \cdot \sum_{t=0}^{\infty} \delta^{2t+1} &= x \cdot \frac{1}{1-\delta^2} + \frac{\delta}{1-\delta^2} \\ &> \frac{y}{1-\delta} = \frac{y(1+\delta)}{1-\delta^2} \\ &\iff x + \delta > y + y\delta \\ &\iff x - y > \delta(y-1) \\ &\iff \delta < \frac{x-y}{y-1} \end{aligned}$$

Hence, the condition is satisfied if and only if

$$\frac{x-y}{x-1} \leq \delta < \frac{x-y}{y-1}.$$

6. In the following stopping game, each of two players can either stop or pass in each round. If both pass, the game continues, if only one player stops, that player gets payoff 1 and the other 0. If both players stop, each of them gets payoff $-x$. The discount factor is $\frac{1}{2}$. Suppose there is a symmetric mixed strategy subgame perfect Nash equilibrium, where each player is equally likely to either stop or pass. Determine x.

Solution:

In an equilibrium, each player is indifferent between stopping in turn 0 or turn 1. Hence, the respective actions yield the same expected payoff:

$$\begin{aligned} \left(1-\frac{1}{2}\right)\cdot 1+\frac{1}{2}\cdot(-x) &= \frac{1}{2}\cdot 0+\left(1-\frac{1}{2}\right)\cdot\left(\left(1-\frac{1}{2}\right)\cdot 1+\frac{1}{2}\cdot(-x)\right)\cdot\frac{1}{2} \\ \iff \frac{1}{2}-\frac{x}{2} &= \frac{1}{2}\cdot\left(\frac{1}{2}-\frac{x}{2}\right)\cdot\frac{1}{2}. \end{aligned}$$

If $x \neq 1$, we can divide by $\left(\frac{1}{2}-\frac{x}{2}\right)$, to obtain $1=\frac{1}{4}$, which is false, hence we obtain $x=1$.