Quantum Computing

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2.75 Reduced Density

Define $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.

$$|\Phi_\pm\rangle \langle\Phi_\pm|_{AB} = \frac{1}{2}(|00\rangle \langle 00| \pm |00\rangle \langle 11| \pm |11\rangle \langle 00| + |11\rangle \langle 11|)$$

$$Tr_B(|\Phi_\pm\rangle \langle\Phi_\pm|_{AB}) = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{I}{2}$$

$$|\Psi_\pm\rangle \langle\Psi_\pm| = \frac{1}{2}(|01\rangle \langle 01| \pm |01\rangle \langle 10| \pm |10\rangle \langle 01| + |10\rangle \langle 10|)$$

$$Tr_B(|\Psi_\pm\rangle \langle\Psi_\pm|) = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{I}{2}$$

2.78 Schmidt Number

Proof. First Part

If $|\psi\rangle$ is product, then there exist a state $|\phi_A\rangle$ for system A, and a state $|\phi_B\rangle$ for system B such that $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$.

Obviously, this Schmidt number is 1.

Conversely, if Schmidt number is 1, the state is written as $|\psi\rangle = |\phi_A\rangle |\phi_B\rangle$. Hence this is a product state. □

Proof. Later part.

($\Rightarrow$) Proved by exercise 2.74.

($\Leftarrow$) Let a pure state be $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$. Then $\rho_A = Tr_B(|\psi\rangle \langle\psi|) = \sum_i \lambda_i^2 |i\rangle \langle i|$. If ρ_A is a pure state, then $\lambda_j = 1$ and otherwise 0 for some j. It follows that $|\psi_j\rangle = |j_A\rangle |j_B\rangle$. Thus $|\psi\rangle$ is a product state. $\square$

2.79 Schmidt Decomposition

Procedure of Schmidt decomposition.
Goal: $|\psi\rangle = \sum_i \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$

- Diagonalize reduced density matrix $\rho_A = \sum_i \lambda_i |i_A\rangle \langle i_A|$.
- Derive $|i_B\rangle$, $|i_B\rangle = \dfrac{(I \otimes \langle i_A|) |\psi\rangle}{\sqrt{\lambda_i}}$
- Construct $|\psi\rangle$.

(i)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ This is already decomposed.}$$

(ii)

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = |\psi\rangle |\psi\rangle \text{ where } |\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

(iii)

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$
$$\rho_{AB} = \psi_{AB}\rangle$$

$$\rho_A =_B (\rho_{AB}) = \frac{1}{3}(20 + 01 + 10 + 1) \tag{1}$$
$$\det(\rho_A - \lambda I) = \left(\frac{2}{3} - \lambda\right)\left(\frac{1}{3} - \lambda\right) - \frac{1}{9} = 0 \tag{2}$$
$$\lambda^2 - \lambda + \frac{1}{9} = 0 \tag{3}$$
$$\lambda = \frac{1 \pm \sqrt{5}/3}{2} = \frac{3 \pm \sqrt{5}}{6} \tag{4}$$

Eigenvector with eigenvalue $\lambda_0 \equiv \dfrac{3+\sqrt{5}}{6}$ is $|\lambda_0\rangle \equiv \dfrac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

Eigenvector with eigenvalue $\lambda_1 \equiv \frac{3-\sqrt{5}}{6}$ is $|\lambda_1\rangle \equiv \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.

$$\rho_A = \lambda_0 \lambda_0 + \lambda_1 \lambda_1.$$

$$|a_0\rangle \equiv \frac{(I \otimes \langle\lambda_0|)\,|\psi\rangle}{\sqrt{\lambda_0}}$$
$$|a_1\rangle \equiv \frac{(I \otimes \langle\lambda_1|)\,|\psi\rangle}{\sqrt{\lambda_1}}$$

Then

$$|\psi\rangle = \sum_{i=0}^{1} \sqrt{\lambda_i}\,|a_i\rangle\,|\lambda_i\rangle\,.$$

Calculate $|a_i\rangle$

2.80 Schmidt Coefficient

Let $|\psi\rangle = \sum_i \lambda_i\,|\psi_i\rangle_A\,|\psi_i\rangle_B$ and $|\varphi\rangle = \sum_i \lambda_i\,|\varphi_i\rangle_A\,|\varphi_i\rangle_B$.
Define $U = \sum_i |\psi_j\rangle\,\langle\varphi_j|_A$ and $V = \sum_j |\psi_j\rangle\,\langle\varphi_j|_B$.
Then

$$\begin{aligned}(U \otimes V)\,|\varphi\rangle &= \sum_i \lambda_i U\,|\varphi_i\rangle_A\,V\,|\varphi_i\rangle_B \\ &= \sum_i \lambda_i\,|\psi_i\rangle_A\,|\psi_i\rangle_B \\ &= |\psi\rangle\,.\end{aligned}$$

2.81 Purification

Let the Schmidt decomposition of $|AR_1\rangle$ be $|AR_1\rangle = \sum_i \sqrt{p_i}\,|\psi_i^A\rangle\,|\psi_i^R\rangle$ and let $|AR_2\rangle = \sum_i \sqrt{q_i}\,|\phi_i^A\rangle\,|\phi_i^R\rangle$.
Suppose ρ^A has orthonormal decomposition $\rho^A = \sum_i p_i\,|i\rangle\,\langle i|$.
Since $|AR_1\rangle$ and $|AR_2\rangle$ are purifications of the ρ^A, we have

$$\begin{gathered}Tr_R(|AR_1\rangle\,\langle AR_1|) = Tr_R(|AR_2\rangle\,\langle AR_2|) = \rho^A \\ \sum_i p_i\,|\psi_i^A\rangle\,\langle\psi_i^A| = \sum_i q_i\,|\phi_i^A\rangle\,\langle\phi_i^A| = \sum_i \lambda_i\,|i\rangle\,\langle i|\,.\end{gathered}$$

The $|i\rangle$, $|\psi_i^A\rangle$, and $|\psi_i^A\rangle$ are orthonormal bases and they are eigenvectors of ρ^A. Hence without loss of generality, we can consider

$$\lambda_i = p_i = q_i \text{ and } |i\rangle = |\psi_i^A\rangle = |\phi_i^A\rangle\,.$$

Then

$$|AR_1\rangle = \sum_i \lambda_i |i\rangle |\psi_i^R\rangle$$
$$|AR_2\rangle = \sum_i \lambda_i |i\rangle |\phi_i^R\rangle$$

Since $|AR_1\rangle$ and $|AR_2\rangle$ have same Schmidt numbers, there are two unitary operators U and V such that $|AR_1\rangle = (U \otimes V) |AR_2\rangle$ from exercise 2.80.

Suppose $U = I$ and $V = \sum_i |\psi_i^R\rangle \langle\phi_i^R|$. Then

$$\begin{aligned}\left(I \otimes \sum_j |\psi_j^R\rangle \langle\phi_j^R|\right) |AR_2\rangle &= \sum_i \lambda_i |i\rangle \left(\sum_j |\psi_j^R\rangle \langle\phi_j^R| |\phi_i^R\rangle\right)\\ &= \sum_i \lambda_i |i\rangle |\psi_i^R\rangle\\ &= |AR_1\rangle .\end{aligned}$$

Therefore there exists a unitary transformation U_R acting on system R such that $|AR_1\rangle = (I \otimes U_R) |AR_2\rangle$.

2.82

(1)
Let $|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$.

$$\begin{aligned}Tr_R(|\psi\rangle\langle\psi|) &= \sum_{i,j} \sqrt{p_i}\sqrt{p_j} |\psi_i\rangle\langle\psi_j| Tr_R(|i\rangle\langle j|)\\ &= \sum_{i,j} \sqrt{p_i}\sqrt{p_j} |\psi_i\rangle\langle\psi_j| \delta_{ij}\\ &= \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho.\end{aligned}$$

Thus $|\psi\rangle$ is a purification of ρ.

(2)
Define the projector P by $P = I \otimes |i\rangle\langle i|$. The probability we get the result i is

$$Tr\left[P|\psi\rangle\langle\psi|\right] = \langle\psi|P|\psi\rangle = \langle\psi|(I \otimes i)|\psi\rangle = p_i \langle\psi_i|\psi_i\rangle = p_i.$$

The post-measurement state is

$$\frac{P|\psi\rangle}{\sqrt{p_i}} = \frac{(I \otimes |i\rangle\langle i|)|\psi\rangle}{\sqrt{p_i}} = \frac{\sqrt{p_i}|\psi_i\rangle|i\rangle}{\sqrt{p_i}} = |\psi_i\rangle|i\rangle .$$

If we only focus on the state on system A,

$$Tr_R(|\psi_i\rangle |i\rangle) = |\psi_i\rangle .$$

(3)
($\{|\psi_i\rangle\}$ is not necessary an orthonormal basis.)

Suppose $|AR\rangle$ is a purification of ρ and its Schmidt decomposition is $|AR\rangle = \sum_i \sqrt{\lambda_i} |\phi_i^A\rangle |\phi_i^R\rangle$.

From assumption

$$Tr_R\left(|AR\rangle \langle AR|\right) = \sum_i \lambda_i |\phi_i^A\rangle \langle \phi_i^A| = \sum_i p_i |\psi_i\rangle \langle \psi_i| .$$

By theorem 2.6, there exits an unitary matrix u_{ij} such that $\sqrt{\lambda_i} |\phi_i^A\rangle = \sum_j u_{ij}\sqrt{p_j} |\psi_j\rangle$. Then

$$\begin{aligned}
|AR\rangle &= \sum_i \left(\sum_j u_{ij}\sqrt{p_j} |\psi_j\rangle \right) |\phi_i^R\rangle \\
&= \sum_j \sqrt{p_j} |\psi_j\rangle \otimes \left(\sum_i u_{ij} |\phi_i^R\rangle \right) \\
&= \sum_j \sqrt{p_j} |\psi_j\rangle |j\rangle \\
&= \sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle
\end{aligned}$$

where $|i\rangle = \sum_k u_{ki} |\phi_k^R\rangle$.

About $|i\rangle$,

$$\begin{aligned}
\langle k|l\rangle &= \sum_{m,n} u_{mk}^* u_{nl} \langle \phi_m^R|\phi_n^R\rangle \\
&= \sum_{m,n} u_{mk}^* u_{nl} \delta_{mn} \\
&= \sum_m u_{mk}^* u_{ml} \\
s &= \delta_{kl}, \quad (u_{ij} \text{ is unitary.})
\end{aligned}$$

which implies $|j\rangle$ is an orthonormal basis for system R.

Therefore if we measure system R w.r.t $|j\rangle$, we obtain j with probability p_j and post-measurement state for A is $|\psi_j\rangle$ from (2). Thus for any purification $|AR\rangle$, there exists an orthonormal basis $|i\rangle$ which satisfies the assertion.