Assignment 2

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Question 1

(a)

First, we need to define states. Since there are three types, we can define each type as a state. Therefore, we have three states: 1) $\boldsymbol{GG}$, 2) $\boldsymbol{Gg}$, 3) $\boldsymbol{gg}$.

Then the matrix is as following,

$$P = \begin{bmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{bmatrix}$$

(b)

We need to calculate $P^3_{Gg,GG}$

$$\begin{aligned} P^3_{Gg,GG} &= P_{Gg,GG}P^2_{GG,GG} + P_{Gg,Gg}P^2_{Gg,GG} + P_{Gg,gg}P^2_{gg,GG} \\ &= 0.25 \cdot P^2_{GG,GG} + 0.5 \cdot P^2_{Gg,GG} + 0.25 \cdot P^2_{gg,GG} \end{aligned}$$

$$\begin{aligned} P^2_{GG,GG} &= P_{GG,GG}P_{GG,GG} + P_{GG,Gg}P_{Gg,GG} + P_{GG,gg}P_{gg,GG} \\ &= 0.5 \times 0.5 + 0.5 \times 0.25 + 0 \\ &= 0.375 \end{aligned}$$

$$\begin{aligned} P^2_{Gg,GG} &= P_{Gg,GG}P_{GG,GG} + P_{Gg,Gg}P_{Gg,GG} + P_{Gg,gg}P_{gg,GG} \\ &= 0.25 \times 0.5 + 0.5 \times 0.25 + 0 \\ &= 0.25 \end{aligned}$$

$$\begin{aligned} P^2_{gg,GG} &= P_{gg,GG}P_{GG,GG} + P_{gg,Gg}P_{Gg,GG} + P_{gg,gg}P_{gg,GG} \\ &= 0 + 0.5 \times 0.25 + 0 \\ &= 0.125 \end{aligned}$$

Finally,

$$P^3_{Gg,GG} = 0.25 \times 0.375 + 0.5 \times 0.25 + 0.25 \times 0.125 = 0.25$$

(c)

Solve $\boldsymbol{\pi P} = \boldsymbol{\pi}$, $\sum \pi_i = 1$.

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

By solving the equations above, we get $\pi_0 = 0.25$, $\pi_1 = 0.5$, $\pi_2 = 0.25$, and π_2 is what we are looking for.

Question 2

(a)

The state is (x, a, b), where x is the current page and a and b are the pages in the cache. Thus, we have the following transition matrix. (The order is (1, 1, 2), (2, 1, 2), (1, 1, 3), (3, 1, 3), (2, 2, 3), (3, 2, 3).)

$$P = \begin{bmatrix} 0 & x & 0 & 0 & 0 & 1-x \\ y & 0 & 0 & 0 & 0 & 1-y \\ 0 & 0 & 0 & 1-x & x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & y & 0 & 0 & 1-y \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

To get the proportion of time that the cache contains page 1 and 2, we need to get $\pi_0 + \pi_1$.

Solve $\boldsymbol{\pi P} = \boldsymbol{\pi}$, $\sum \pi_i = 1$. We get

$$\begin{aligned} y\pi_1 &= \pi_0 \\ x\pi_0 &= \pi_1 \\ \cdots \end{aligned}$$

and we are given that $0 < x < y < \frac{1}{2}$, therefore, we must have $\pi_0 = \pi_1 = 0$.

Finally, the answer is $\pi_0 + \pi_1 = 0 + 0 = 0$

(b)

There are three scenarios when a cache miss happens.

1. cache contains 1 and 2, and is requesting 3.
2. cache contains 1 and 3, and is requesting 2.
3. cache contains 2 and 3, and is requesting 1.

$$\begin{aligned} P\{\text{cache miss}\} &= \pi_0 P\{1 \to 3\} + \pi_1 P\{2 \to 3\} + \pi_2 P\{1 \to 2\} + \pi_3 P\{3 \to 2\} + \pi_4 P\{2 \to 1\} + \pi_5 P\{3 \to 1\} \\ &= 0 \times (1-x) + 0 \times (1-y) + \pi_2 \times x + \pi_3 \times 1 + \pi_4 \times y + \pi_5 \times 0 \\ &= x\pi_2 + \pi_3 + y\pi_4 \end{aligned}$$

Question 3

(a)

The probability that up today and up tomorrow is $1 - \frac{1}{4} - \frac{1}{6} = \frac{7}{12}$, the answer is $(\frac{7}{12})^5$.

(b)

What we need to calculate is $P^5_{up,down}$

$$P^5_{up,down} = P^5_{up,soft} + P^5_{up,hard}$$

We are given the formula,

$$P^n_{ij} = \sum_{k=0}^{M-1} P_{ik} P^{n-1}_{kj}$$

I use the following code to solve this question:

```
def P(matrix, begin, end, step, total):
    if step==1:
        return matrix[begin][end]
    res = 0
    for i in range(total):
        res += P(matrix, begin, i, 1, total)*P(matrix, i, end, step-1, total)
    return res

if __name__ == '__main__':
    matrix = [[7/12, 1/4, 1/6],[3/4, 1/4, 0],[1, 0, 0]]
    print(P(matrix, 0, 1, 5, 3)+P(matrix, 0, 2, 5, 3))
```

This code gives me 0.3341.

(c)

Solve $\boldsymbol{\pi P} = \boldsymbol{\pi}$, $\sum \pi_i = 1$. We get $\pi_0 = \frac{2}{3}$, $\pi_1 = \frac{2}{9}$, $\pi_3 = \frac{1}{9}$, and π_0 is what we want.