COMP SCI/SFWR ENG 4/6E03 — Assignment 2 Solutions

1. (a) Let the states be 0 - GG, 1 - Gg, 2 - gg. Then the probability transition matrix is

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

   (b) $P^3(1,0) = 0.25$

   (c) Solve $\pi P = \pi$, $\sum \pi_i = 1$ to yield $\pi_2 = 0.25$

2. (a) One possible choice of state is three-valued – the first is the current page, the next two are the pages in the cache. This gives six possible states: (1,1,2), (2,1,2), (1,1,3), (3,1,3), (3,2,3), and (2,2,3). With the states in that order, the probability transition matrix is

$$P = \begin{bmatrix} 0 & x & 0 & 0 & 1-x & 0 \\ y & 0 & 0 & 0 & 1-y & 0 \\ 0 & 0 & 0 & 1-x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & y & 0 & 1-y & 0 \end{bmatrix}$$

I solved the following system of equations to get the stationary distribution:

$$\begin{aligned}
\pi_{(3,2,3)} &= \pi_{(2,2,3)}(1-y) + \pi_{(1,1,2)}(1-x) + \pi_{(2,1,2)}(1-y) \\
\pi_{(2,2,3)} &= \pi_{(3,2,3)} + \pi_{(3,1,3)} + x\pi_{(1,1,3)} \\
\pi_{(1,1,3)} &= y\pi_{(2,2,3)} \\
\pi_{(3,1,3)} &= (1-x)\pi_{(1,1,3)} \\
\pi_{(2,1,2)} &= x\pi_{(1,1,2)} \\
1 &= \pi_{(3,2,3)} + \pi_{(2,2,3)} + \pi_{(1,1,3)} + \pi_{(3,1,3)} + \pi_{(2,1,2)} + \pi_{(1,1,2)}
\end{aligned}$$

Solving yields

$$\begin{aligned}
\pi_{(2,1,2)} &= 0 \\
\pi_{(1,1,2)} &= 0 \\
\pi_{(2,2,3)} &= \frac{1}{2+y-xy} \\
\pi_{(3,1,3)} &= \frac{(1-x)y}{2+y-xy} \\
\pi_{(3,2,3)} &= \frac{1-y}{2+y-xy} \\
\pi_{(1,1,3)} &= \frac{y}{2+y-xy}
\end{aligned}$$

Note that you can see that the first two probabilities are zero as there are no transitions from the last four states to the first two. So, the required probability is 0.

(b) In general, the expression would be:

$$\pi_{(1,1,2)}P(1 \to 3)+\pi_{(2,1,2)}P(2 \to 3)+\pi_{(1,1,3)}P(1 \to 2)+\pi_{(3,1,3)}P(3 \to 2)+\pi_{(3,2,3)}P(3 \to 1)+\pi_{(2,2,3)}P(2 \to 1)$$

This simplifies to

$$x\pi_{(1,1,3)} + \pi_{(3,1,3)} + y\pi_{(2,2,3)}$$

3. (a) The probability that the server remains up the next day is $(1 - 1/6 - 1/4) = 7/12$. So, the required probability is $(7/12)^5$.

(b) Let the states be 0 - server up, 1 - server down due to hardware failure, 2 - server down due to software failure. Then the probability transition matrix is

$$P = \begin{bmatrix} 7/12 & 2/12 & 3/12 \\ 1 & 0 & 0 \\ 3/4 & 0 & 1/4 \end{bmatrix}$$

So, the required probability is $P^5(0,1) + P^5(0,2) = 0.3341$.

(c) Solve $\pi P = \pi$, $\sum \pi_i = 1$ to give $\pi_0 = 2/3$.