$y_1 \; n_1\times 1$

$y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad K_1=\frac{\tilde{x}_1\tilde{x}_1^T}{m}$ $\quad I_{N_1\times N_1}$

$y_2 \; n_2\times 1$ $\quad K_A=\frac{\tilde{x}_1\tilde{x}_2^T}{m}$ $\quad [\,[I_{n_1\times n_2}]\; 0\,]$

① target function

$$Q_{\min_\Delta}=\left\| yy^T-\begin{pmatrix}\sigma_{g_1}^2 K_1 & r_g K_A \\ r_g K_A^T & \sigma_{g_2}^2 K_2\end{pmatrix}-\begin{pmatrix}\sigma_{e_1}^2 I_{N_1} & r_e C \\ r_e C & \sigma_{e_2}^2 I_{N_2}\end{pmatrix}\right\|_F^2 .$$

$K_2=\frac{\tilde{x}_2\tilde{x}_2^T}{m}$ $\quad I_{N_2\times N_2}$ $\quad \|A\|_F^2 = tr(A^2).$

② $\Delta=[\sigma_{g_1}^2,\ \sigma_{g_2}^2,\ \sigma_{e_1}^2,\ \sigma_{e_2}^2,\ r_g,\ r_e].$

求导

$$\frac{\partial Q}{\partial \sigma_{g_1}^2}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial \sigma_{g_1}^2},\quad \frac{\partial Q}{\partial \sigma_{g_2}^2}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial \sigma_{g_2}^2},\quad \frac{\partial Q}{\partial \sigma_{e_1}^2}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial \sigma_{e_1}^2},\quad \frac{\partial Q}{\partial \sigma_{e_2}^2}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial^2 \sigma_{e_2}}$$

$$\frac{\partial Q}{\partial r_g}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial r_g},\quad \frac{\partial Q}{\partial r_e}=\frac{\partial Q}{\partial \Delta}\frac{\partial \Delta}{\partial r_e}$$

2.1 其中 $\frac{\partial Q}{\partial \Delta}=2\left[yy^T-\begin{pmatrix}\sigma_{g_1}^2 K_1 & r_g K_A \\ r_g K_A & \sigma_{g_2}^2 K_2\end{pmatrix}-\begin{pmatrix}\sigma_{e_1}^2 I_{N_1} & r_e C \\ r_e C & \sigma_{e_2}^2 I_{N_2}\end{pmatrix}\right]$

2.2

$$\frac{\partial \Delta}{\partial \sigma_{g_1}^2}=\begin{pmatrix}K_1 & 0 \\ 0 & 0\end{pmatrix}\qquad \frac{\partial \Delta}{\partial \sigma_{e_1}^2}=\begin{pmatrix}I_{N_1} & 0 \\ 0 & 0\end{pmatrix}\qquad \frac{\partial \Delta}{\partial r_g}=\begin{pmatrix}0 & K_A \\ K_A & 0\end{pmatrix}$$

$$\frac{\partial \Delta}{\partial \sigma_{g_2}^2}=\begin{pmatrix}0 & 0 \\ 0 & K_2\end{pmatrix}\qquad \frac{\partial \Delta}{\partial \sigma_{e_2}^2}=\begin{pmatrix}0 & 0 \\ 0 & I_{N_2}\end{pmatrix}\qquad \frac{\partial \Delta}{\partial r_e}=\begin{pmatrix}0 & C \\ C & 0\end{pmatrix}$$

③

$$\frac{\partial Q}{\partial \sigma_{g_1}^2}=2tr\{y_1y_1^TK_1-\sigma_{g_1}^2K_1K_1^T-\sigma_{e_1}^2I_{N_1}K_1\}=0$$

$$\frac{\partial Q}{\partial \sigma_{e_1}^2}=2tr\{y_1y_1^TI_{N_1}-\sigma_{g_1}^2K_1I_{N_1}-\sigma_{e_1}^2I_{N_1}^2\}=0$$

$$\frac{\partial Q}{\partial \sigma_{g_2}^2}=2tr\{y_2y_2^TK_2-\sigma_{g_2}^2K_2K_2^T-\sigma_{e_2}^2I_{N_2}K_2\}=0$$

$$\frac{\partial Q}{\partial \sigma_{e_2}^2}=2tr\{y_2y_2^TI_{N_2}-\sigma_{g_2}^2K_2I_{N_2}-\sigma_{e_2}^2I_{N_2}^2\}=0$$

$$\frac{\partial Q}{\partial r_g}=2tr\{y_1y_2^TK_A^T+y_2y_1^TK_A-K_A\cdot K_A^Tr_g-K_A^T\cdot K_Ar_g-C\cdot K_A^Tr_e-C^TK_A\cdot r_e\}=0$$

$$\frac{\partial Q}{\partial r_e}=2tr\{y_1y_2^TC^T+y_2y_1^TC-K_A\cdot C^Tr_e-K_A^T\cdot C\,r_e-C\cdot C^Tr_e-C^TC\,r_e\}=0$$

因为 $K_A\cdot K_A^T$, $C\cdot K_A^T$, $y_1y_2^TK_A^T$ 是 square matrix

则 $tr(K_A\cdot K_A^T)=tr(K_A^T\cdot K_A)$, $tr(C\cdot K_A^T)=tr(K_A\cdot C^T)$, $tr(y_1y_2^TK_A^T)=tr(y_1^Ty_2K_A)$

$$\frac{\partial Q}{\partial r_g}=4tr\{y_1y_2^TK_A^T-K_AK_A^Tr_g-CK_A^Tr_e\}=0$$

因为 K_AC^T, CC^T, $y_1y_2^TC^T$ 是 square matrix

$$\frac{\partial Q}{\partial r_e}=4tr\{y_1y_2^TC^T-K_AC^Tr_e-CC^Tr_e\}=0$$

$$
\begin{bmatrix}
tr(K_1 \cdot K_1^T) & tr(I_{N_1} \cdot K_1^T) & 0 & 0 & 0 & 0 \\
tr(I_{N_1} K_1^T) & tr(I_{N_1}) & 0 & 0 & 0 & 0 \\
0 & 0 & tr(K_2 \cdot K_2^T) & tr(I_{N_2} \cdot K_2^T) & 0 & 0 \\
0 & 0 & tr(I_{N_2} K_2) & tr(I_{N_2}) & 0 & 0 \\
0 & 0 & 0 & 0 & tr(K_A \cdot K_A^T) & tr(C K_A^T) \\
0 & 0 & 0 & 0 & tr(K_A C^T) & tr(C C^T)
\end{bmatrix}
\begin{bmatrix}
\sigma_{g_1}^2 \\ \sigma_{e_1}^2 \\ \sigma_{g_2}^2 \\ \sigma_{e_2}^2 \\ r_g \\ r_e
\end{bmatrix}
$$

$$
= \begin{bmatrix}
tr(y_1 y_1^T K_1) \\
tr(y_1 y_1^T I_{N_1}) \\
tr(y_2 y_2^T K_2) \\
tr(y_2 y_2^T I_{N_2}) \\
tr(y_1 y_2^T K_A^T) \\
tr(y_1 y_2^T C^T)
\end{bmatrix}
= \begin{bmatrix}
tr(y_1^T K_1 y_1) \\
tr(y_1^T I_{N_1} y_1) \\
tr(y_2^T K_2 y_2) \\
tr(y_2^T I_{N_2} y_2) \\
tr(y_2^T K_A^T y_1) \\
tr(y_2^T C^T y_1)
\end{bmatrix}
= \begin{bmatrix}
y_1^T K_1 y_1 \\
y_1^T I_{N_1} y_1 \\
y_2^T K_2 y_2 \\
y_2^T I_{N_2} y_2 \\
y_2^T K_A^T y_1 \\
y_2^T C^T y_1
\end{bmatrix}
$$

Cycling property of trace.

$tr(ABC)$

$= tr(BCA)$

$= tr(CAB)$.

Property of Quatric form.

$tr(y_1^T K_1 y_1) = y_1^T K_1 y_1$

可将上述方程组重构为

$$
\begin{bmatrix}
tr(K_1 \cdot K_1^T) & tr(I_{N_1} K_1^T) \\
tr(I_{N_1} K_1^T) & tr(I_{N_1})
\end{bmatrix}
\begin{bmatrix}
\sigma_{g_1}^2 \\ \sigma_{e_1}^2
\end{bmatrix}
= \begin{bmatrix}
y_1^T K_1 y_1 \\ y_1^T I_{N_1} y_1
\end{bmatrix}
$$

$$
\begin{bmatrix}
tr(K_2 K_2^T) & tr(I_{N_2} K_2^T) \\
tr(I_{N_2} K_2) & tr(I_{N_2})
\end{bmatrix}
\begin{bmatrix}
\sigma_{g_2}^2 \\ \sigma_{e_2}^2
\end{bmatrix}
= \begin{bmatrix}
y_2^T K_2 y_2 \\ y_2^T K_2 y_2
\end{bmatrix}
$$

$$
\begin{bmatrix}
tr(K_A \cdot K_A^T) & tr(C K_A^T) \\
tr(K_A C^T) & tr(C C^T)
\end{bmatrix}
\begin{bmatrix}
r_g \\ r_e
\end{bmatrix}
= \begin{bmatrix}
y_2^T K_A^T y_1 \\ y_2^T C^T y_1
\end{bmatrix}
$$