Homework 2

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Problem 2.1. Find MLE for the following parameters:
(a) Probability of success p in Bernoulli(p) model

Solution (a). Let X be a bernoulli random variable with parameter p. Let X_1,...,X_n be the random sample of X.
PDF for the bernoulli distribution with parameter p is

$$f(x) = p^x(1-p)^{1-x} \qquad\qquad x = 0,1$$

Likelihood function of the sample:

$$\begin{aligned}
&= \prod_{i=1}^{n} f(x_i, p) \\
&= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \\
&= p^{\sum\limits_{i=1}^{n} x_i}(1-p)^{n-\sum\limits_{i=1}^{n} x_i}
\end{aligned}$$

Log-likelihood function of the sample:

$$\begin{aligned}
&= \ln\left(p^{\sum\limits_{i=1}^{n} x_i}(1-p)^{n-\sum\limits_{i=1}^{n} x_i}\right) \\
&= \left(\sum_{i=1}^{n} x_i\right)\ln(p) + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-p)
\end{aligned}$$

Maximum value of this function can be found by taking the derivative of the above function with respect to p, and setting it to 0.

$$= \frac{\sum\limits_{i=1}^{n} x_i}{p} - \left(\frac{n - \sum\limits_{i=1}^{n} x_i}{1-p}\right) = 0$$

gives that $p = \frac{\sum\limits_{i=1}^{n} x_i}{n}$

(b) Probability of success p in Binomial(n,p) model

Solution (b). Let X be a binomail random variable with parameters n and p. Let X_1,...,X_m be the random sample of X.
PDF for the binomial distribution with parameters n and p is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, .., n.$$

Likelihood function of the sample:

$$\begin{aligned} &= \prod_{i=1}^{m} f(x_i, p) \\ &= \prod_{i=1}^{m} \binom{n}{x} p^{x_i} (1-p)^{n-x_i} \end{aligned}$$

Log-likelihood function of the sample:

$$\begin{aligned} &= \ln\left(\prod_{i=1}^{m} \binom{n}{x} p^{x_i} (1-p)^{n-x_i}\right) \\ &= \sum_{i=1}^{m} \ln\binom{n}{x_i} + \left(\sum_{i=1}^{n} x_i\right) \ln(p) + \left(mn - \sum_{i=1}^{m} x_i\right) \ln(1-p) \end{aligned}$$

Maximum value of this function can be found by taking the derivative of the above function with respect to p, and setting it to 0.

$$= 0 + \frac{\sum\limits_{i=1}^{m} x_i}{p} - \left(\frac{mn - \sum\limits_{i=1}^{m} x_i}{1-p}\right) = 0$$

gives that $p = \frac{\sum_{i=1}^{m} x_i}{mn}$

(c) Probability of success p in Geometric(p) model

Solution (c). Let X be a geometric random variable with parameter p. Let X_1,...,X_n be the random sample of X.
PDF for the geometric distribution with parameter p is

$$f(x) = p(1-p)^{x-1} \qquad x = 1, 2, ...$$

Likelihood function of the sample:

$$= \prod_{i=1}^{n} f(x_i, p)$$
$$= \prod_{i=1}^{n} p(1-p)^{x_i - 1}$$
$$= p^n (1-p)^{\sum\limits_{i=1}^{n} x_i - n}$$

Log-likelihood function of the sample:

$$= \ln\left(p^n (1-p)^{\sum\limits_{i=1}^{n} x_i - n}\right)$$
$$= n\ln(p) + \left(\sum_{i=1}^{m} x_i - n\right)\ln(1-p)$$

Maximum value of this function can be found by taking the derivative of the above function with respect to p, and setting it to 0.

$$= \frac{n}{p} + \left(\frac{n - \sum\limits_{i=1}^{m} x_i}{1-p}\right) = 0$$

gives that $p = \frac{n}{\sum\limits_{i=1}^{m} x_i}$

(d) Intensity λ in Poisson(λ) model

Solution (d). Let X be a Poisson random variable with parameter λ. Let X_1,...,X_n be the random sample of X.
PDF for the poisson distribution with parameter λ is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad\qquad x = 0, 1, ...$$

Likelihood function of the sample:

$$= \prod_{i=1}^{n} f(x_i, \lambda)$$
$$= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$
$$= e^{-\lambda n}.\lambda^{\sum\limits_{i=1}^{n} x_i} C \qquad\qquad \textit{where} \qquad C = \prod_{i=1}^{n} \frac{1}{x_i!}$$

Log-likelihood function of the sample:

$$= \ln\left(e^{-\lambda n}.\lambda^{\sum_{i=1}^{n} x_i} C\right)$$

$$= -n\lambda + \left(\sum_{i=1}^{m} x_i\right)\ln(\lambda) + \ln(C)$$

Maximum value of this function can be found by taking the derivative of the above function with respect to λ, and setting it to 0.

$$= -n + \left(\frac{\sum_{i=1}^{m} x_i}{\lambda}\right) + 0 = 0$$

gives that $\lambda = \frac{\sum_{i=1}^{m} x_i}{n}$

Problem 2.2. Find the Fisher information for the models of problem 2.1
(a) Probability of success p in Bernoulli(p) model

Solution (a). The Fisher Information for parameter p̂ can be written two ways: I(p̂) $=E[(\frac{\partial}{\partial p}\ln f(x,p))^2|p]$ or I(p̂) $=-E[\frac{\partial^2}{\partial p^2}\ln f(x,p)|p]$.

From problem 2.1 (a) we know

$$\frac{\partial}{\partial p}\ln f(x,p) = \frac{\partial}{\partial p}[xp + (1-x)\ln(1-p)]$$
$$= \frac{x}{p} - \frac{1-x}{1-p}$$

Recall that $E(X) = p$ where X is the bernoulli random variable.

Therefore,

$$= I(\hat{p}) = -E\Big[\frac{\partial^2}{\partial p^2}\ln f(x,p)\Big]$$
$$= -E\Big[-\frac{X}{p^2} - \left(\frac{1-X}{q^2}\right)\Big]$$
$$= \frac{E(X)}{p^2} + \frac{E(1-X)}{(1-p)^2}$$
$$= \frac{1}{p} + \frac{1}{1-p}$$

gives $\frac{1}{p(1-p)}$

(b) Probability of success p in Binomial(n,p) model

Solution (b). The Fisher Information for parameter p̂ can be written two ways: I(p̂) $=E\Big[(\frac{\partial}{\partial p}\ln f(x,p))^2|p\Big]$ or I(p̂) $=-E\Big[\frac{\partial^2}{\partial p^2}\ln f(x,p)|p\Big]$.

From problem 2.1 (b) we know

$$\frac{\partial}{\partial p}\ln f(x,p) = \frac{\partial}{\partial p}[\ln\binom{n}{x} + x\ln p + (n-x)\ln(1-p)]$$
$$= \frac{x}{p} - \frac{n-x}{1-q}$$
$$\left(\frac{x}{p} - \frac{n-x}{1-q}\right)^2 = x^2\frac{1}{p^2q^2} - 2x\frac{n}{q^2p} + \frac{n^2}{q^2} \qquad \textit{where} \qquad q = (1-p)$$

Thus,

$$E\Big[X^2\frac{1}{p^2q^2} - 2x\frac{n}{q^2p} + \frac{n^2}{q^2}\Big]$$

Recall that $E(X) = np$ and $E(X^2) = npq + n^2p^2$ where X is the binomial random variable.

Therefore,

$$= \frac{npq}{p^2q^2} + \frac{n^2p^2}{p^2q^2} - 2\frac{n^2p}{q^2p} + \frac{n^2}{q^2}$$
$$= \frac{n}{pq} + \frac{n^2}{q^2} - 2\frac{n^2}{q^2} + \frac{n^2}{q^2}$$

$\frac{n}{pq}$

(c) Probability of success p in Geometric(p) model

Solution (c). The Fisher Information for parameter p̂ can be written two ways: I(p̂) $=E\Big[(\frac{\partial}{\partial p}\ln f(x,p))^2|p\Big]$ or I(p̂) $=-E\Big[\frac{\partial^2}{\partial p^2}\ln f(x,p)|p\Big]$.

From problem 2.1 (c) we know

$$\frac{\partial}{\partial p}\ln f(x,p) = \frac{\partial}{\partial p}[\ln p + (x-1)\ln(1-p)]$$
$$= \frac{1}{p} - \left(\frac{x}{1-p}\right)$$

Recall that $E(X) = \frac{1}{p}$ where X is the geometric random variable.

Therefore,

$$I(\hat{p}) = -E\Big[\frac{\partial^2}{\partial p^2}\ln f(x,p)\Big]$$
$$= -E\Big[\frac{-1}{p^2} - \Big(\frac{X}{1-p^2}\Big)\Big]$$
$$= \frac{1}{p^2} + \frac{E(X)}{(1-p)^2}$$
$$= \frac{1}{p^2} + \frac{1}{p(1-p)^2}$$
$$= \frac{1-p}{p^2(1-p)^2}$$

gives $\frac{1}{p^2(1-p)}$

(d) Intensity λ in Poisson(λ) model

Solution (d). The Fisher Information for parameter $\hat{\lambda}$ can be written two ways: $I(\hat{\lambda}) = E\Big[(\frac{\partial}{\partial\lambda}\ln f(x,\lambda))^2|\lambda\Big]$ or $I(\hat{\lambda}) = -E\Big[\frac{\partial^2}{\partial\lambda^2}\ln f(x,\lambda)\Big]$.

From problem 2.1 (d) we know

$$\frac{\partial}{\partial p}\ln f(x,\lambda) = \frac{\partial}{\partial p}[-\lambda + xln\lambda + \ln\frac{x}{x!}]$$
$$= \frac{x}{\lambda} - 1$$
$$\frac{\partial^2}{\partial\lambda^2}\ln f(x,\lambda) = \frac{-x}{\lambda^2}$$

Recall that $E(X) = \lambda$ where X is the poisson random variable with parameter λ.

Therefore,

$$I(\lambda) = -E\Big[\frac{\partial^2}{\partial p^2}\ln f(x,p)\Big]$$
$$= -E(\frac{-X}{\lambda^2})$$
$$= \frac{1}{\lambda^2}E(X)$$
$$= \frac{\lambda}{\lambda^2}$$

gives $\frac{1}{\lambda}$

Problem 2.3. Assess the efficiency of the MLE estimators for the models of problem 2.1.
(a) Probability of success p in Bernoulli(p) model

Solution (a). According to Cramer-Rao inequality $Var(\hat{p}) \geq \frac{1}{nI(\hat{p})}$
It states that if there is an estimator that achieves this lower bound, then it is the efficient estimator for that parameter.

From problem 2.1 (a) we have that $\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$. So, the variance of this parameter is

$$\begin{aligned} var(\hat{p}) &= var\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \\ &= \frac{var \sum\limits_{i=1}^{n} x_i}{n^2} \\ &= \frac{np(1-p)}{n^2} \\ &= \frac{p(1-p)}{n} \end{aligned}$$

Also, from problem 2.2 (a), $I(\hat{p}) = \frac{1}{p(1-p)}$. Hence $\hat{p}$ is an efficient estimator.

(b) Probability of success p in Binomial(n, p) model

Solution (b). From problem 2.1 (b) we have that $\hat{p} = \frac{\sum_{i=1}^{m} x_i}{mn}$. So, the variance of this parameter is

$$\begin{aligned} var(\hat{p}) &= var\left(\frac{\sum\limits_{i=1}^{m} x_i}{mn}\right) \\ &= \frac{var \sum\limits_{i=1}^{m} x_i}{m^2n^2} \\ &= \frac{mnp(1-p)}{m^2n^2} \\ &= \frac{p(1-p)}{mn} \end{aligned}$$

Also, from problem 2.2 (b), $I(\hat{p}) = \frac{n}{p(1-p)}$. Hence $\hat{p}$ is an efficient estimator.

(c) Probability of success p in Geometric(p) model

Solution (c). From problem 2.1 (c) we have that $\hat{p} = \frac{n}{\sum_{i=1}^{m} x_i} = \frac{1}{\bar{X}}$. Variance of this estimator can be calculated using delta method.

Delta method says that:
if θ follows normal distribution asymptotically with mean $= \theta$ and variance $= \sigma^2$ then $f(\theta)$ follows normal distribution asymptotically with mean $f(\theta)$ and variance $f'(\theta)^2.\sigma^2$.

From CLT we know that $\bar{X}$ follows normal distribution if n is large. Hence $\bar{X}$ follows normal distribution asymptotically with mean $= \frac{1}{p}$ and variance $= \frac{1-p}{np^2}$. Here the function is $f(\theta) = \frac{1}{\bar{X}}$ and $f'(\theta) = \frac{-1}{\bar{X}^2}$. From the delta method, mean of $f(\theta)$ is p and variance of $f(\theta)$ is $p^4.\frac{1-p}{np^2} = \frac{(1-p)p^2}{n}$. So, the variance of this parameter is

$$var(\hat{p}) = \frac{(1-p)p^2}{n}$$

Also, from problem 2.2 (c), $I(\hat{p}) = \frac{1}{p^2(1-p)}$. Hence $\hat{p}$ is an efficient estimator.

Solution (d). From problem 2.1 (d) we have that $\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$. So, the variance of this parameter is

$$\begin{aligned} var(\hat{\lambda}) &= var\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \\ &= \frac{var\sum\limits_{i=1}^{n} x_i}{n^2} \\ &= \frac{n\lambda}{n^2} \\ &= \frac{\lambda}{n} \end{aligned}$$

Also, from problem 2.2 (d), $I(\hat{\lambda}) = \frac{1}{\lambda}$. Hence $\hat{\lambda}$ is an efficient estimator.