

$$(1) \text{ GRM} = \{[G_1 - M_1] \cdot V_1\} \{[G_2 - M_2] \cdot V_2\}^T$$

$G_1 = n_1 \times 3^{n_1}$ 矩阵.

$M_1 = n_1 \times 3^{n_1}$ 矩阵. 每列是位点基因频率

$V_1 = 3^{n_1} \times 3^{n_1}$ 对称矩阵.

$$\begin{bmatrix} f_1 & f_1 & \dots & f_m \\ f_1 & f_2 & \dots & f_m \\ f_1 & f_2 & \dots & f_m \\ f_1 & f_2 & \dots & f_m \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2f_1(1-f_1)} \\ \sqrt{2f_2(1-f_2)} \\ \vdots \\ \sqrt{2f_m(1-f_m)} \end{bmatrix}$$

(2)

$$\text{记 } [G_1 - M_1] \cdot V_1 = G_{S_1}$$

则 G_{S_1} 可分解为 U_1 与 P_1 .

$$U_1 = \begin{bmatrix} S_{11} & S_{11} & S_{11} & S_{12} & S_{12} & S_{12} & S_{13} & S_{13} & S_{13} \\ S_{21} & S_{22} & S_{23} & S_{21} & S_{22} & S_{23} & S_{21} & S_{22} & S_{23} \end{bmatrix}$$

S_{11}, S_{12}, S_{13} 是位点 1 的三种可能性

$$S_{11} = \frac{0-2f_1}{\sqrt{2f_1(1-f_1)}}$$

$$S_{12} = \frac{1-2f_1}{\sqrt{2f_1(1-f_1)}}$$

$$S_{13} = \frac{2-2f_1}{\sqrt{2f_1(1-f_1)}}$$

同理.

$$S_{21} = \frac{0-2f_2}{\sqrt{2f_2(1-f_2)}}$$

$$S_{22} = \frac{1-2f_2}{\sqrt{2f_2(1-f_2)}}$$

$$S_{23} = \frac{2-2f_2}{\sqrt{2f_2(1-f_2)}}$$

①

$$G_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 2 & 2 \\ 2 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \quad M_1 = \begin{bmatrix} 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \\ 2t_1 & 2t_2 \end{bmatrix} \quad V_1 = \begin{bmatrix} \sqrt{2t_1(t_1+t_2)} & 0 \\ 0 & \sqrt{2t_2(t_1+t_2)} \end{bmatrix}$$

②

$$GS_1 = (G_1 - M_1) \cdot V_1$$

假定 $t_1 = 0.5$, $t_2 = 0.25$.

$$GS_1 = \begin{bmatrix} \frac{0-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{0-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{1-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{0-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{1-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{1-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{2-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{2-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{2-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{0-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{1-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{0-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{1-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{0-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{0-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{1-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \\ \frac{2-1}{\sqrt{2 \cdot 0.5 \cdot 0.5}} & \frac{1-0.5}{\sqrt{2 \cdot 0.25 \cdot 0.75}} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{0.5} & -0.5/\sqrt{0.375} \\ 0 & -0.5/\sqrt{0.375} \\ 0 & 0.5/\sqrt{0.375} \\ 1/\sqrt{0.5} & 1.5/\sqrt{0.375} \\ 1/\sqrt{0.5} & -0.5/\sqrt{0.375} \\ 0 & -0.5/\sqrt{0.375} \\ 0 & -0.5/\sqrt{0.375} \\ -1/\sqrt{0.5} & 0.5/\sqrt{0.375} \\ 1/\sqrt{0.5} & 0.5/\sqrt{0.375} \end{bmatrix}$$

$$GS_1^T = U \cdot P = \begin{bmatrix} \frac{-1}{\sqrt{0.5}} & \frac{-1}{\sqrt{0.5}} & \frac{-1}{\sqrt{0.5}} & 0 & 0 & 0 & \frac{1}{\sqrt{0.5}} & \frac{1}{\sqrt{0.5}} & \frac{1}{\sqrt{0.5}} \\ \frac{-0.5}{\sqrt{0.375}} & \frac{0.5}{\sqrt{0.375}} & \frac{1.5}{\sqrt{0.375}} & \frac{-0.5}{\sqrt{0.375}} & \frac{0.5}{\sqrt{0.375}} & \frac{1.5}{\sqrt{0.375}} & \frac{-0.5}{\sqrt{0.375}} & \frac{0.5}{\sqrt{0.375}} & \frac{1.5}{\sqrt{0.375}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

实例. $G_S \cdot X = P^T U^T \cdot X$

$$\tilde{X} = \begin{bmatrix} \frac{1}{\sqrt{0.5}} \\ 1.5/\sqrt{0.375} \end{bmatrix}$$

X在两个位点基因型为 [2, 2]

则标准化后有 \tilde{X}

$$U^T \cdot \tilde{X} = \begin{bmatrix} \frac{-1}{\sqrt{0.5}} & \frac{-0.5}{\sqrt{0.375}} \\ \frac{-1}{\sqrt{0.5}} & \frac{0.5}{\sqrt{0.375}} \\ \frac{-1}{\sqrt{0.5}} & \frac{1.5}{\sqrt{0.375}} \\ 0 & \frac{-0.5}{\sqrt{0.375}} \\ 0 & \frac{0.5}{\sqrt{0.375}} \\ 0 & \frac{1.5}{\sqrt{0.375}} \\ \frac{1}{\sqrt{0.5}} & \frac{-0.5}{\sqrt{0.375}} \\ \frac{1}{\sqrt{0.5}} & \frac{0.5}{\sqrt{0.375}} \\ \frac{1}{\sqrt{0.5}} & \frac{1.5}{\sqrt{0.375}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{0.5}} \\ \frac{1.5}{\sqrt{0.375}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{0.5} & \frac{-0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ \frac{-1}{0.5} & \frac{0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ \frac{-1}{0.5} & \frac{1.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ 0 & \frac{-0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ 0 & \frac{0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ 0 & \frac{1.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ \frac{1}{0.5} & \frac{-0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ \frac{1}{0.5} & \frac{0.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \\ \frac{1}{0.5} & \frac{1.5}{\sqrt{0.375}} \cdot \frac{1.5}{\sqrt{0.375}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{0.5} + \frac{-0.5 \cdot 1.5}{0.375} \\ \frac{-1}{0.5} + \frac{0.5 \cdot 1.5}{0.375} \\ \frac{-1}{0.5} + \frac{1.5 \cdot 1.5}{0.375} \\ \dots \\ 0 + \frac{-0.5 \cdot 1.5}{0.375} \\ 0 + \frac{0.5 \cdot 1.5}{0.375} \\ 0 + \frac{1.5 \cdot 1.5}{0.375} \\ \dots \\ \frac{1}{0.5} + \frac{-0.5 \cdot 1.5}{0.375} \\ \frac{1}{0.5} + \frac{0.5 \cdot 1.5}{0.375} \\ \frac{1}{0.5} + \frac{1.5 \cdot 1.5}{0.375} \end{bmatrix}$$

$$P^T (U^T X) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \tilde{X} = \begin{bmatrix} U^T X [1] \\ U^T X [2] \\ U^T X [3] \\ \vdots \\ U^T X [9] \\ U^T X [7] \\ U^T X [4] \\ U^T X [4] \\ U^T X [2] \\ U^T X [8] \end{bmatrix} = \begin{bmatrix} \frac{-1}{0.5} + \frac{-0.5 \cdot 1.5}{0.375} \\ 0 + \frac{-0.5 \cdot 1.5}{0.375} \\ 0 + \frac{0.5 \cdot 1.5}{0.375} \\ \vdots \\ \frac{1}{0.5} + \frac{1.5 \cdot 1.5}{0.375} \\ \frac{1}{0.5} + \frac{-0.5 \cdot 1.5}{0.375} \\ 0 + \frac{-0.5 \cdot 1.5}{0.375} \\ 0 + \frac{0.5 \cdot 1.5}{0.375} \\ \frac{-1}{0.5} + \frac{0.5 \cdot 1.5}{0.375} \\ \frac{1}{0.5} + \frac{0.5 \cdot 1.5}{0.375} \end{bmatrix}$$