

For a full sib pair, the IBD sharing at a locus is calculated:

$\pi = 0.5\pi^p + 0.5\pi^m$, and $E(\pi^p) = E(\pi^m) = 0.5$, $Var(\pi) = 1/4Var(\pi^p) + 1/4Var(\pi^m)$, and $Cov(\pi^p, \pi^m) = 0$ because the transmission of the paternal haploids is independent from the maternal ones.

For a chromosome, with k genotyped markers, $E(\pi) = E(0.5\frac{\sum \pi_i^p}{k} + 0.5\frac{\sum \pi_i^m}{k}) = 0.5$, and

$$Var(\frac{\sum \pi_i^p}{k} + \frac{\sum \pi_i^m}{k}) = \frac{1}{4k^2} \{ [\sum_i \sum_j Cov(\pi_i^p, \pi_j^p)] + [\sum_i \sum_j Cov(\pi_i^m, \pi_j^m)] \} \quad (1)$$

Note: all $2 \sum \sum Cov(\pi_i^p, \pi_j^m) = 0$ and thus eliminated; 2) As when $i = j$, $Cov(\pi_i^p, \pi_i^p) = Var(\pi_i^p)$, the equation

above included the variance terms. It should be noticed that, in W.G.Hill's paper (J Hered 84:212-213), the statement that "contains many more covariance than variance terms, so terms in $Var(Z_{i(n)})$ can be ignored" is incorrect because the variance terms are included in the expression.

Now, we consider the IBD transmitted from paternal origins. We know, $Cov(\pi_i^p, \pi_j^p) = E(\pi_i^p \pi_j^p) - E(\pi_i^p)E(\pi_j^p)$,

the term $E(\pi_i^p \pi_j^p)$ can be calculated below $P(\pi_i^p = \delta_i, \pi_j^p = \delta_j) = P(\pi_i^p = \delta_i)P(\pi_j^p | \pi_i^p = \delta_j)$, where $\delta = 1$ if the alleles are IBD or 0 otherwise.

		Locus j	
		$\pi_j^p = 1$	$\pi_j^p = 0$
Locus i	$\pi_i^p = 1$	$\frac{r_p^2 + (1 - r_p)^2}{2}$ State $(\pi_i^p = 1, \pi_j^p = 1)$	$r_p(1 - r_p)$ State $(\pi_i^p = 1, \pi_j^p = 0)$
	$\pi_i^p = 0$	$r_p(1 - r_p)$ State $(\pi_i^p = 0, \pi_j^p = 1)$	$\frac{r_p^2 + (1 - r_p)^2}{2}$ State $(\pi_i^p = 0, \pi_j^p = 0)$

$$\begin{aligned} Cov(\pi_i^p, \pi_j^p) &= 1 \cdot 1 \cdot p(\pi_i^p = 1, \pi_j^p = 1) + 1 \cdot 0 \cdot p(\pi_i^p = 1, \pi_j^p = 0) \\ &\quad + 0 \cdot 1 \cdot p(\pi_i^p = 0, \pi_j^p = 1) + 0 \cdot 0 \cdot p(\pi_i^p = 0, \pi_j^p = 0) \\ &\quad - 1/4 \\ &= \frac{r_p^2 + (1 - r_p)^2}{2} - 1/4 \\ &= \frac{(1 - 2r_p)^2}{4} \end{aligned} \quad (2)$$

Assuming the Haldane mapping function $r_p = 0.5[1 - \exp(-2d)]$, then $Cov(\pi_i^p, \pi_j^p) = \frac{\exp(-4d)}{4}$, where d is the genetic distance measured in Morgan.

For the maternal haploids, if the recombination fractions are different from that of paternal, the similar table should

be made for maternally raised IBD. $P(\pi_i^m = \delta, \pi_j^m = \delta) = P(\pi_i^m = \delta)P(\pi_j^m | \pi_i^m = \delta)$, and it is

		Locus j	
		$\pi_j^m = 1$	$\pi_j^m = 0$
Locus i	$\pi_i^m = 1$	$\frac{r_m^2 + (1 - r_m)^2}{2}$ State $(\pi_i^m = 1, \pi_j^m = 1)$	$r_m(1 - r_m)$ State $(\pi_i^m = 1, \pi_j^m = 0)$
	$\pi_i^m = 0$	$r_m(1 - r_m)$ State $(\pi_i^m = 0, \pi_j^m = 1)$	$\frac{r_m^2 + (1 - r_m)^2}{2}$ State $(\pi_i^m = 0, \pi_j^m = 0)$

Assume the difference between maternal and paternal recombination fractions is ϵ ,

$$Cov(\pi_i^m, \pi_j^m) = (1 - 2r_m)^2 = \exp(-4d) + \epsilon \quad (3)$$

Now

$$\begin{aligned} Var\left(\frac{\sum \pi_i^p}{k} + \frac{\sum \pi_i^m}{k}\right) &= \frac{1}{4k^2} \{ [\sum_i \sum_j Cov(\pi_i^p, \pi_j^p)] + [\sum_i \sum_j Cov(\pi_i^m, \pi_j^m)] \} \\ &= \frac{1}{4k^2} \{ \sum_i \sum_j \frac{\exp(-4|x_i - x_j|)}{4} + \sum_i \sum_j [\frac{\exp(-4|x_i - x_j|)}{4} + \epsilon_{ij}] \} \\ &= \frac{1}{4k^2} \{ \sum_i \sum_j \frac{\exp(-4|x_i - x_j|)}{4} + \sum_i \sum_j \frac{\exp(-4|x_i - x_j|)}{4} \} + \frac{1}{4k^2} \sum_i \sum_j \epsilon_{ij} \end{aligned}$$

When k is very large, it can be expressed as an integral, and the analytical solution is:

$$\begin{aligned} Var\left(\frac{\sum \pi_i^p}{k} + \frac{\sum \pi_i^m}{k}\right) &= \frac{1}{4l^2} \int_0^l \int_0^l \exp(-4|x_i - x_j|) dx_i dx_j + \frac{1}{4n^2} \int_0^l \int_0^l \epsilon_{ij} dx_i dx_j \\ &= \frac{1}{16l^2} [l - 0.5r_p(2l)] + \Omega \end{aligned}$$

Where Ω is the sum of the terms associated with ϵ .

The existence of Ω makes difference to what is in Peter Visscher's paper (PLoS Genet 2:e41) in which table 3 gives the theoretical sd when assuming the recombination fractions are same between the males and the females.

Dominance

For dominance code, it requires a full sib pair sharing the both paternal and maternal alleles are IBD for a given locus.

Let $\Delta_i=1$ if both $\pi_i^p = 1$ and $\pi_i^m = 1$ are IBD, or 0 otherwise.

$$p(\pi^p = 1, \pi^m = 1) = 1/4, p(\pi^p = 1, \pi^m = 0) = p(\pi^p = 0, \pi^m = 1) = p(\pi^p = 0, \pi^m = 0) = 1/4$$

$$E(\Delta_i) = 1/4, \text{ and } Var(\Delta_i) = 3/16$$

$$Var\left(\frac{\sum \Delta_i}{k}\right) = \frac{1}{k^2} \sum_i \sum_j Cov(\Delta_i, \Delta_j)$$

Now, let's look at the covariance term $Cov(\Delta_i, \Delta_j) = E(\Delta_i \Delta_j) - E(\Delta_i)E(\Delta_j)$, in order to get the form of $E(\Delta_i \Delta_j)$, we need a table

This table represents

$$\begin{aligned}
 p(\Delta_i = \eta_i, \Delta_j = \eta_j) &= p(\pi_i^p = \delta_i^p, \pi_j^p = \delta_j^p, \pi_i^m = \delta_i^m, \pi_j^m = \delta_j^m) \\
 &= p(\pi_i^p = \delta_i^p, \pi_j^p = \delta_j^p, \pi_i^m = \delta_i^m, \pi_j^m = \delta_j^m) \\
 &= p(\pi_i^p = \delta_i^p) p(\pi_i^p = \delta_i^p | \pi_j^p = \delta_j^p) p(\pi_i^m = \delta_i^m) p(\pi_i^m = \delta_i^m | \pi_j^m = \delta_j^m)
 \end{aligned}$$

		Locus j			
		$\Delta_j = 1$ state ($\delta_j^p = 1, \delta_j^m = 1$)	$\Delta_j = 0$ state ($\delta_j^p = 1, \delta_j^m = 0$)	$\Delta_j = 0$ state ($\delta_j^p = 0, \delta_j^m = 1$)	$\Delta_j = 0$ state ($\delta_j^p = 0, \delta_j^m = 0$)
Locus i	$\Delta_i = 1$ state ($\delta_i^p = 1, \delta_i^m = 1$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 1, \delta_j^p = 1$) ($\delta_i^m = 1, \delta_j^m = 1$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 1, \delta_j^p = 1$) ($\delta_i^m = 1, \delta_j^m = 0$)	$[r_p(1 - r_p)]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 1, \delta_j^p = 0$) ($\delta_i^m = 1, \delta_j^m = 1$)	$[r_p(1 - r_p)]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 1, \delta_j^p = 0$) ($\delta_i^m = 1, \delta_j^m = 0$)
	$\Delta_i = 0$ state ($\delta_i^p = 1, \delta_i^m = 0$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 1, \delta_j^p = 1$) ($\delta_i^m = 0, \delta_j^m = 1$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 1, \delta_j^p = 1$) ($\delta_i^m = 0, \delta_j^m = 0$)	$[r_p(1 - r_p)]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 1, \delta_j^p = 0$) ($\delta_i^m = 0, \delta_j^m = 1$)	$[r_p(1 - r_p)]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 1, \delta_j^p = 0$) ($\delta_i^m = 0, \delta_j^m = 0$)
	$\Delta_i = 0$ state ($\delta_i^p = 0, \delta_i^m = 1$)	$[r_p(1 - r_p)]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 0, \delta_j^p = 1$) ($\delta_i^m = 1, \delta_j^m = 1$)	$[r_p(1 - r_p)]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 0, \delta_j^p = 1$) ($\delta_i^m = 1, \delta_j^m = 0$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 0, \delta_j^p = 0$) ($\delta_i^m = 1, \delta_j^m = 1$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 0, \delta_j^p = 0$) ($\delta_i^m = 1, \delta_j^m = 0$)
	$\Delta_i = 1$ state ($\delta_i^p = 0, \delta_i^m = 0$)	$[r_p(1 - r_p)]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 0, \delta_j^p = 1$) ($\delta_i^m = 0, \delta_j^m = 1$)	$[r_p(1 - r_p)]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 0, \delta_j^p = 1$) ($\delta_i^m = 0, \delta_j^m = 0$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times [r_m(1 - r_m)]$ State ($\delta_i^p = 0, \delta_j^p = 0$) ($\delta_i^m = 0, \delta_j^m = 1$)	$\left[\frac{r_p^2 + (1 - r_p)^2}{2} \right]$ $\times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right]$ State ($\delta_i^p = 0, \delta_j^p = 0$) ($\delta_i^m = 0, \delta_j^m = 0$)

$$\begin{aligned}
 Cov(\Delta_i, \Delta_j) &= \left[\frac{r_p^2 + (1 - r_p)^2}{2} \right] \times \left[\frac{r_m^2 + (1 - r_m)^2}{2} \right] - 1/16 \\
 &= 1/4[(1 - r)^2 + r^2] - 1/16 \\
 &= 1/8[\exp(-2d) + 1]^2 - 1/16
 \end{aligned}$$

If we assume Haldane's mapping function.

Analogous to the IBD sharing, when k is large,

$$\begin{aligned} Var(\frac{\sum \Delta_i}{k}) &= \frac{1}{k^2} \sum_i \sum_j Cov(\Delta_i, \Delta_j) \\ &= \frac{1}{l^2} \int_0^l \int_0^l \{1/8[\exp(-2|x_i - x_j| + 1))^2 - 1/16\} dx_i dx_y \end{aligned}$$