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Isserlis' theorem

In probability theory, **Isserlis' theorem** or **Wick's probability theorem** is a formula that allows one to compute higher-order moments of the multivariate normal distribution in terms of its covariance matrix. It is named after Leon Isserlis.

This theorem is also particularly important in particle physics, where it is known as Wick's theorem after the work of Wick (1950).[1] Other applications include the analysis of portfolio returns,[2] quantum field theory[3] and generation of colored noise.[4]

Contents

Statement

If $(X_1, \dots, X_n)$ is a zero-mean multivariate normal random vector, then

$$\mathrm{E}[\,X_1 X_2 \cdots X_n\,] = \sum_{p \in P_n^2} \prod_{\{i,j\} \in p} \mathrm{E}[\,X_i X_j\,] = \sum_{p \in P_n^2} \prod_{\{i,j\} \in p} \mathrm{Cov}(\,X_i, X_j\,),$$

where the sum is over all the pairings of $\{1, \dots, n\}$, i.e. all distinct ways of partitioning $\{1, \dots, n\}$ into pairs $\{i, j\}$, and the product is over the pairs contained in p.[5][6]

In his original paper,[7] Leon Isserlis proves this theorem by mathematical induction, generalizing the formula for the 4^{th} order moments,[8] which takes the appearance

$$\mathrm{E}[\,X_1 X_2 X_3 X_4\,] = \mathrm{E}[X_1 X_2]\,\mathrm{E}[X_3 X_4] + \mathrm{E}[X_1 X_3]\,\mathrm{E}[X_2 X_4] + \mathrm{E}[X_1 X_4]\,\mathrm{E}[X_2 X_3].$$

Odd case, $n \in 2\mathbb{N}+1$

If $n = 2m+1$ is odd, there does not exist any pairing of $\{1, \dots, 2m+1\}$. Under this hypothesis, Isserlis' theorem implies that:

$$\mathrm{E}[\,X_1 X_2 \cdots X_{2m+1}\,] = 0.$$

Even case, $n \in 2\mathbb{N}$

If $n = 2m$ is even, there exists $(2m)!/(2^m m!) = (2m-1)!!$ (see double factorial) pair partitions of $\{1, \ldots, 2m\}$: this yields $(2m)!/(2^m m!) = (2m-1)!!$ of terms in the sum. For example, for 4^{th} order moments (i.e. 4 random variables) there are three terms. For 6^{th}-order moments there are $3 \times 5 = 15$ terms, and for 8^{th}-order moments there are $3 \times 5 \times 7 = 105$ terms.

Generalizations

Gaussian integration by part

An equivalent formulation of the Wick's probability formula is the Gaussian integration by part. If $(X_1, \ldots X_{2n})$ is a zero-mean multivariate normal random vector, then

$$\mathrm{E}(X_1 f(X_1, \ldots, X_n)) = \sum_{i=1}^{n} \mathrm{Cov}(X_1 X_i)\, \mathrm{E}(\partial_{X_i} f(X_1, \ldots, X_n)).$$

The Wick's probability formula can be recovered by induction, considering the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by: $f(x_1, \ldots, x_n) = x_2 \ldots x_n$. Among other things, this formulation is important in Liouville Conformal Field Theory to obtain conformal Ward's identities, BPZ equations[9] and to prove Fyodorov-Bouchaud formula.[10]

Non-Gaussian random variables

For non-Gaussian random variables, the moment-cumulants formula replaces the Wick's probability formula. If $(X_1, \ldots X_n)$ is a vector of random variables, then

$$\mathrm{E}(X_1 \ldots X_n) = \sum_{p \in P_n} \prod_{b \in p} \kappa\big((X_i)_{i \in b}\big),$$

where the sum is over all the partitions of $\{1, \ldots, n\}$, the product is over the blocks of p and $\kappa\big((X_i)_{i \in b}\big)$ is the cumulants of $(X_i)_{i \in b}$.

See also

- Wick's theorem
- Cumulants
- Normal distribution

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Further reading

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This page was last edited on 7 December 2019, at 12:17 (UTC).

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