Quadratic form (statistics)

In multivariate statistics, if ε is a vector of n random variables, and Λ is an n-dimensional symmetric matrix, then the scalar quantity $\varepsilon^T \Lambda \varepsilon$ is known as a **quadratic form** in ε.

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Expectation

It can be shown that[1]

$$\mathrm{E}\left[\varepsilon^T \Lambda \varepsilon\right] = \operatorname{tr}[\Lambda \Sigma] + \mu^T \Lambda \mu$$

where μ and Σ are the expected value and variance-covariance matrix of ε, respectively, and tr denotes the trace of a matrix. This result only depends on the existence of μ and Σ; in particular, normality of ε is *not* required.

A book treatment of the topic of quadratic forms in random variables is that of Mathai and Provost.[2]

Proof

Since the quadratic form is a scalar quantity, $\varepsilon^T \Lambda \varepsilon = \operatorname{tr}(\varepsilon^T \Lambda \varepsilon)$.

Next, by the cyclic property of the trace operator,

$$\mathrm{E}[\operatorname{tr}(\varepsilon^T \Lambda \varepsilon)] = \mathrm{E}[\operatorname{tr}(\Lambda \varepsilon \varepsilon^T)].$$

Since the trace operator is a linear combination of the components of the matrix, it therefore follows from the linearity of the expectation operator that

$$\mathrm{E}[\operatorname{tr}(\Lambda \varepsilon \varepsilon^T)] = \operatorname{tr}(\Lambda\, \mathrm{E}(\varepsilon \varepsilon^T)).$$

A standard property of variances then tells us that this is

$$\operatorname{tr}(\Lambda(\Sigma + \mu \mu^T)).$$

Applying the cyclic property of the trace operator again, we get

$$\operatorname{tr}(\Lambda\Sigma) + \operatorname{tr}(\Lambda\mu\mu^T) = \operatorname{tr}(\Lambda\Sigma) + \operatorname{tr}(\mu^T\Lambda\mu) = \operatorname{tr}(\Lambda\Sigma) + \mu^T\Lambda\mu.$$

Variance in the Gaussian case

In general, the variance of a quadratic form depends greatly on the distribution of ε. However, if ε *does* follow a multivariate normal distribution, the variance of the quadratic form becomes particularly tractable. Assume for the moment that Λ is a symmetric matrix. Then,

$$\operatorname{var}\left[\varepsilon^T\Lambda\varepsilon\right] = 2\operatorname{tr}[\Lambda\Sigma\Lambda\Sigma] + 4\mu^T\Lambda\Sigma\Lambda\mu$$ [3].

In fact, this can be generalized to find the covariance between two quadratic forms on the same ε (once again, Λ_1 and Λ_2 must both be symmetric):

$$\operatorname{cov}\left[\varepsilon^T\Lambda_1\varepsilon, \varepsilon^T\Lambda_2\varepsilon\right] = 2\operatorname{tr}[\Lambda_1\Sigma\Lambda_2\Sigma] + 4\mu^T\Lambda_1\Sigma\Lambda_2\mu.$$

Computing the variance in the non-symmetric case

Some texts incorrectly state that the above variance or covariance results hold without requiring Λ to be symmetric. The case for general Λ can be derived by noting that

$$\varepsilon^T\Lambda^T\varepsilon = \varepsilon^T\Lambda\varepsilon$$

so

$$\varepsilon^T\tilde{\Lambda}\varepsilon = \varepsilon^T\left(\Lambda + \Lambda^T\right)\varepsilon/2$$

is a quadratic form in the symmetric matrix $\tilde{\Lambda} = \left(\Lambda + \Lambda^T\right)/2$, so the mean and variance expressions are the same, provided Λ is replaced by $\tilde{\Lambda}$ therein.

Examples of quadratic forms

In the setting where one has a set of observations y and an operator matrix H, then the residual sum of squares can be written as a quadratic form in y:

$$\text{RSS} = y^T(I-H)^T(I-H)y.$$

For procedures where the matrix H is symmetric and idempotent, and the errors are Gaussian with covariance matrix $\sigma^2 I$, RSS/σ^2 has a chi-squared distribution with k degrees of freedom and noncentrality parameter λ, where

$$k = \operatorname{tr}\left[(I-H)^T(I-H)\right]$$
$$\lambda = \mu^T(I-H)^T(I-H)\mu/2$$

may be found by matching the first two central moments of a noncentral chi-squared random variable to the expressions given in the first two sections. If Hy estimates μ with no bias, then the noncentrality λ is zero and RSS/σ^2 follows a central chi-squared distribution.

See also

- Quadratic form
- Covariance matrix
- Matrix representation of conic sections

References

1. Bates, Douglas. "Quadratic Forms of Random Variables" (http://www.stat.wisc.edu/~st849-1/lectures/Ch02.pdf) (PDF). *STAT 849 lectures*. Retrieved August 21, 2011.
2. Mathai, A. M. & Provost, Serge B. (1992). *Quadratic Forms in Random Variables*. CRC Press. p. 424. ISBN 978-0824786915.
3. Rencher, Alvin C.; Schaalje, G. Bruce. (2008). *Linear models in statistics* (2nd ed.). Hoboken, N.J.: Wiley-Interscience. ISBN 9780471754985. OCLC 212120778 (https://www.worldcat.org/oclc/212120778).