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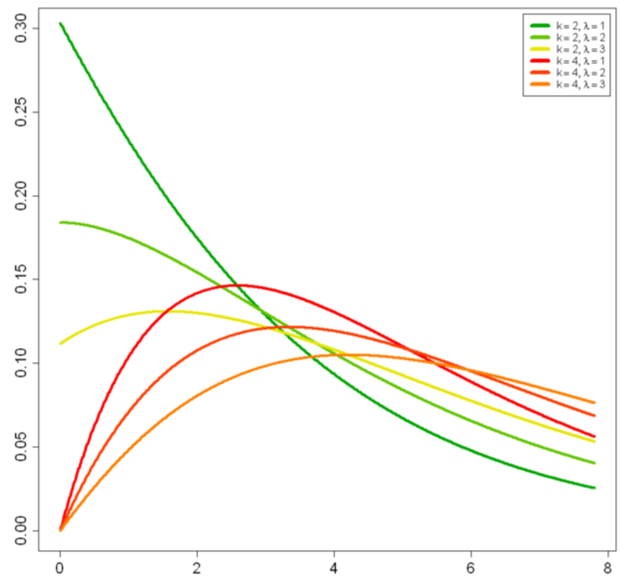
Noncentral chi-squared distribution

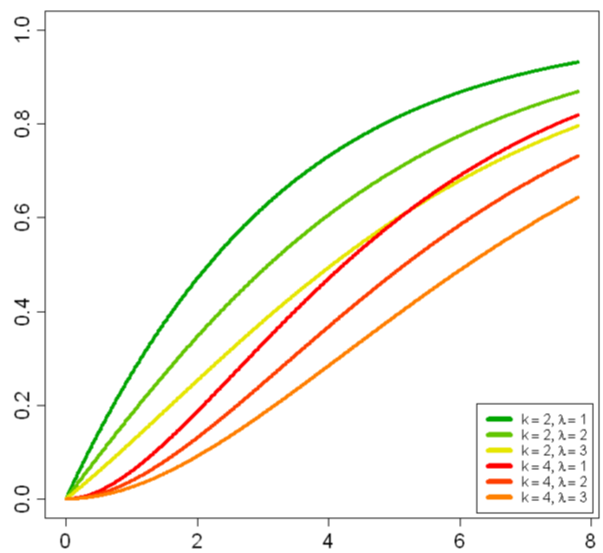
In probability theory and statistics, the **noncentral chi-square distribution** (or noncentral chi-squared distribution, **noncentral χ^2 distribution**) is a generalization of the chi-square distribution. It often arises in the power analysis of statistical tests in which the null distribution is (perhaps asymptotically) a chi-square distribution; important examples of such tests are the likelihood-ratio tests.

Noncentral chi-square

Probability density function



Cumulative distribution function



Parameters	$k > 0$ degrees of freedom $\lambda > 0$ non-centrality parameter
Support	$x \in [0; +\infty)$
PDF	$\frac{1}{2} e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{k/4-1/2} I_{k/2-1}(\sqrt{\lambda x})$
CDF	$1 - Q_{\frac{k}{2}}\left(\sqrt{\lambda}, \sqrt{x}\right)$ with Marcum Q-function $Q_M(a, b)$

Contents

Background

Let $(X_1, X_2, \ldots, X_i, \ldots, X_k)$ be k independent, normally distributed random variables with means μ_i and unit variances. Then the random variable

$$\sum_{i=1}^{k} X_i^2$$

is distributed according to the noncentral chi-square distribution. It has two parameters: k which specifies the number of degrees of freedom (i.e. the number of X_i), and λ which is related to the mean of the random variables X_i by:

$$\lambda = \sum_{i=1}^{k} \mu_i^2.$$

λ is sometimes called the noncentrality parameter. Note that some references define λ in other ways, such as half of the above sum, or its square root.

Mean	$k+\lambda$
Variance	$2(k+2\lambda)$
Skewness	$\frac{2^{3/2}(k+3\lambda)}{(k+2\lambda)^{3/2}}$
Ex. kurtosis	$\frac{12(k+4\lambda)}{(k+2\lambda)^2}$
MGF	$\frac{\exp\left(\frac{\lambda t}{1-2t}\right)}{(1-2t)^{k/2}}$ for $2t<1$
CF	$\frac{\exp\left(\frac{i\lambda t}{1-2it}\right)}{(1-2it)^{k/2}}$

This distribution arises in multivariate statistics as a derivative of the multivariate normal distribution. While the central chi-square distribution is the squared norm of a random vector with $N(0_k, I_k)$ distribution (i.e., the squared distance from the origin to a point taken at random from that distribution), the non-central χ^2 is the squared norm of a random vector with $N(\mu, I_k)$ distribution. Here 0_k is a zero vector of length k, $\mu = (\mu_1, \ldots, \mu_k)$ and I_k is the identity matrix of size k.

Definition

The probability density function (pdf) is given by

$$f_X(x; k, \lambda) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} f_{Y_{k+2i}}(x),$$

where Y_q is distributed as chi-square with q degrees of freedom.

From this representation, the noncentral chi-square distribution is seen to be a Poisson-weighted mixture of central chi-square distributions. Suppose that a random variable J has a Poisson distribution with mean $\lambda/2$, and the conditional distribution of Z given $J = i$ is chi-square with $k + 2i$ degrees of freedom. Then the unconditional distribution of Z is non-central chi-square with k degrees of freedom, and non-centrality parameter λ.

Alternatively, the pdf can be written as

$$f_X(x; k, \lambda) = \frac{1}{2} e^{-(x+\lambda)/2} \left(\frac{x}{\lambda}\right)^{k/4-1/2} I_{k/2-1}(\sqrt{\lambda x})$$

where $I_\nu(y)$ is a modified Bessel function of the first kind given by

$$I_\nu(y) = (y/2)^\nu \sum_{j=0}^{\infty} \frac{(y^2/4)^j}{j!\Gamma(\nu+j+1)}.$$

Using the relation between Bessel functions and hypergeometric functions, the pdf can also be written as:[1]

$$f_X(x; k, \lambda) = \mathrm{e}^{-\lambda/2} {}_0F_1(; k/2; \lambda x/4) \frac{1}{2^{k/2}\Gamma(k/2)} \mathrm{e}^{-x/2} x^{k/2-1}.$$

Siegel (1979) discusses the case $k = 0$ specifically (zero degrees of freedom), in which case the distribution has a discrete component at zero.

Properties

Moment generating function

The moment-generating function is given by

$$M(t;k,\lambda) = \frac{\exp\left(\frac{\lambda t}{1-2t}\right)}{(1-2t)^{k/2}}.$$

Moments

The first few raw moments are:

$$\begin{aligned}
\mu'_1 &= k+\lambda \\
\mu'_2 &= (k+\lambda)^2 + 2(k+2\lambda) \\
\mu'_3 &= (k+\lambda)^3 + 6(k+\lambda)(k+2\lambda) + 8(k+3\lambda) \\
\mu'_4 &= (k+\lambda)^4 + 12(k+\lambda)^2(k+2\lambda) + 4(11k^2 + 44k\lambda + 36\lambda^2) + 48(k+4\lambda)
\end{aligned}$$

The first few central moments are:

$$\begin{aligned}
\mu_2 &= 2(k+2\lambda) \\
\mu_3 &= 8(k+3\lambda) \\
\mu_4 &= 12(k+2\lambda)^2 + 48(k+4\lambda)
\end{aligned}$$

The nth cumulant is

$$K_n = 2^{n-1}(n-1)!(k+n\lambda).$$

Hence

$$\mu'_n = 2^{n-1}(n-1)!(k+n\lambda) + \sum_{j=1}^{n-1} \frac{(n-1)!2^{j-1}}{(n-j)!}(k+j\lambda)\mu'_{n-j}.$$

Cumulative distribution function

Again using the relation between the central and noncentral chi-square distributions, the cumulative distribution function (cdf) can be written as

$$P(x;k,\lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} Q(x;k+2j)$$

where $Q(x;k)$ is the cumulative distribution function of the central chi-square distribution with k degrees of freedom which is given by

$$Q(x;k) = \frac{\gamma(k/2, x/2)}{\Gamma(k/2)}$$

and where $\gamma(k, z)$ is the lower incomplete gamma function.

The Marcum Q-function $Q_M(a, b)$ can also be used to represent the cdf.[2]

$$P(x;k,\lambda) = 1 - Q_{\frac{k}{2}}\left(\sqrt{\lambda}, \sqrt{x}\right)$$

Approximation (including for quantiles)

Abdel-Aty [3] derives (as "first approx.") a non-central Wilson-Hilferty approximation:

$\left(\frac{\chi'^2}{k+\lambda}\right)^{\frac{1}{3}}$ is approximately normally distributed, $\sim \mathcal{N}\left(1 - \frac{2}{9f}, \frac{2}{9f}\right)$, i.e.,

$$P(x;k,\lambda) \approx \Phi\left\{\frac{\left(\frac{x}{k+\lambda}\right)^{1/3} - \left(1 - \frac{2}{9f}\right)}{\sqrt{\frac{2}{9f}}}\right\}, \text{where } f := \frac{(k+\lambda)^2}{k+2\lambda} = k + \frac{\lambda^2}{k+2\lambda},$$

which is quite accurate and well adapting to the noncentrality. Also, $f = f(k, \lambda)$ becomes $f = k$ for $\lambda = 0$, the (central) chi-squared case.

Sankaran [4] discusses a number of closed form approximations for the cumulative distribution function. In an earlier paper,[5] he derived and states the following approximation:

$$P(x;k,\lambda) \approx \Phi\left\{\frac{(\frac{x}{k+\lambda})^h - (1 + hp(h - 1 - 0.5(2-h)mp))}{h\sqrt{2p}(1 + 0.5mp)}\right\}$$

where

$\Phi\{\cdot\}$ denotes the cumulative distribution function of the standard normal distribution;
$h = 1 - \frac{2}{3}\frac{(k+\lambda)(k+3\lambda)}{(k+2\lambda)^2}$;
$p = \frac{k+2\lambda}{(k+\lambda)^2}$;
$m = (h-1)(1-3h)$.

This and other approximations are discussed in a later text book.[6]

For a given probability, these formulas are easily inverted to provide the corresponding approximation for x, to compute approximate quantiles.

Derivation of the pdf

The derivation of the probability density function is most easily done by performing the following steps:

1. Since $X_1, \ldots, X_k$ have unit variances, their joint distribution is spherically symmetric, up to a location shift.
2. The spherical symmetry then implies that the distribution of $X = X_1^2 + \cdots + X_k^2$ depends on the means only through the squared length, $\lambda = \mu_1^2 + \cdots + \mu_k^2$. Without loss of generality, we can therefore take $\mu_1 = \sqrt{\lambda}$ and $\mu_2 = \cdots = \mu_k = 0$.
3. Now derive the density of $X = X_1^2$ (i.e. the *k* = 1 case). Simple transformation of random variables shows that

$$\begin{aligned} f_X(x, 1, \lambda) &= \frac{1}{2\sqrt{x}} \left(\phi(\sqrt{x} - \sqrt{\lambda}) + \phi(\sqrt{x} + \sqrt{\lambda})\right) \\ &= \frac{1}{\sqrt{2\pi x}} e^{-(x+\lambda)/2} \cosh(\sqrt{\lambda x}), \end{aligned}$$

where $\phi(\cdot)$ is the standard normal density.

1. Expand the cosh term in a Taylor series. This gives the Poisson-weighted mixture representation of the density, still for *k* = 1. The indices on the chi-square random variables in the series above are 1 + 2*i* in this case.
2. Finally, for the general case. We've assumed, without loss of generality, that $X_2, \ldots, X_k$ are standard normal, and so $X_2^2 + \cdots + X_k^2$ has a *central* chi-square distribution with (*k* – 1) degrees of freedom, independent of X_1^2. Using the poisson-weighted mixture representation for X_1^2, and the fact that the sum of chi-square random variables is also a chi-square, completes the result. The indices in the series are (1 + 2*i*) + (*k* – 1) = *k* + 2*i* as required.

Related distributions

- If V is chi-square distributed $V \sim \chi_k^2$ then V is also non-central chi-square distributed: $V \sim {\chi'}_k^2(0)$
- A linear combination of noncentral chi-squared variables $\xi = \sum_i \lambda_i Y_i + c, \quad Y_i \sim \chi'^2(m_i, \delta_i^2)$, is generalized chi-square distributed.
- If $V_1 \sim {\chi'}_{k_1}^2(\lambda)$ and $V_2 \sim {\chi'}_{k_2}^2(0)$ and V_1 is independent of V_2 then a noncentral *F*-distributed variable is developed as $\frac{V_1/k_1}{V_2/k_2} \sim F'_{k_1,k_2}(\lambda)$
- If $J \sim \mathrm{Poisson}\left(\frac{1}{2}\lambda\right)$, then $\chi_{k+2J}^2 \sim {\chi'}_k^2(\lambda)$
- If $V \sim {\chi'}_2^2(\lambda)$, then $\sqrt{V}$ takes the Rice distribution with parameter $\sqrt{\lambda}$.
- Normal approximation:[7] if $V \sim {\chi'}_k^2(\lambda)$, then $\frac{V - (k + \lambda)}{\sqrt{2(k + 2\lambda)}} \to N(0, 1)$ in distribution as either $k \to \infty$ or $\lambda \to \infty$.
- If $V_1 \sim {\chi'}_{k_1}^2(\lambda_1)$and $V_2 \sim {\chi'}_{k_2}^2(\lambda_2)$, where V_1, V_2 are independent, then $W = (V_1 + V_2) \sim {\chi'}_k^2(\lambda_1 + \lambda_2)$ where $k = k_1 + k_2$.

- In general, for a finite set of $V_i \sim {\chi'}^2_{k_i}(\lambda_i), i \in \{1..N\}$, the sum of these non-central chi-square distributed random variables $Y = \sum_{i=1}^N V_i$ has the distribution $Y \sim {\chi'}^2_{k_y}(\lambda_y)$ where $k_y = \sum_{i=1}^N k_i, \lambda_y = \sum_{i=1}^N \lambda_i$. This can be seen using moment generating functions as follows: $M_Y(t) = M_{\sum_{i=1}^N V_i}(t) = \prod_{i=1}^N M_{V_i}(t)$ by the independence of the V_i random variables. It remains to plug in the MGF for the non-central chi square distributions into the product and compute the new MGF - this is left as an exercise. Alternatively it can be seen via the interpretation in the background section above as sums of squares of independent normally distributed random variables with variances of 1 and the specified means.
- The Complex noncentral Chi squared distribution has applications in radio communication and radar systems. Let $(z_1, \ldots, z_k)$ be independent scalar complex random variables with circular symmetry, means of μ_i and unit variances: $\operatorname{E}|z_i - \mu_i|^2 = 1$. Then the real random variable $S = \sum_{i=1}^k |z_i|^2$ is distributed according to the complex noncentral chi-square distribution

$$f_S(S) = \left(\frac{S}{\lambda}\right)^{(k-1)/2} e^{-(S+\lambda)} I_{k-1}(2\sqrt{S\lambda})$$

where $\lambda = \sum_{i=1}^k |\mu_i|^2$.

Transformations

Sankaran (1963) discusses the transformations of the form $z = [(X-b)/(k+\lambda)]^{1/2}$. He analyzes the expansions of the cumulants of z up to the term $O((k+\lambda)^{-4})$ and shows that the following choices of b produce reasonable results:

- $b = (k-1)/2$ makes the second cumulant of z approximately independent of λ
- $b = (k-1)/3$ makes the third cumulant of z approximately independent of λ
- $b = (k-1)/4$ makes the fourth cumulant of z approximately independent of λ

Also, a simpler transformation $z_1 = (X-(k-1)/2)^{1/2}$ can be used as a variance stabilizing transformation that produces a random variable with mean $(\lambda + (k-1)/2)^{1/2}$ and variance $O((k+\lambda)^{-2})$.

Usability of these transformations may be hampered by the need to take the square roots of negative numbers.

Various chi and chi-square distributions

Name	Statistic
chi-square distribution	$\sum_1^k \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2$
noncentral chi-square distribution	$\sum_1^k \left(\frac{X_i}{\sigma_i}\right)^2$
chi distribution	$\sqrt{\sum_1^k \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2}$
noncentral chi distribution	$\sqrt{\sum_1^k \left(\frac{X_i}{\sigma_i}\right)^2}$

Occurrences

Use in tolerance intervals

Two-sided normal regression tolerance intervals can be obtained based on the noncentral chi-square distribution.[8] This enables the calculation of a statistical interval within which, with some confidence level, a specified proportion of a sampled population falls.

Notes

1. Muirhead (2005) Theorem 1.3.4
2. Nuttall, Albert H. (1975): *Some Integrals Involving the Q_M Function (http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=1055327)*, *IEEE Transactions on Information Theory*, 21(1), 95–96, ISSN 0018-9448 (https://www.worldcat.org/search?fq=x0:jrnl&q=n2:0018-9448)
3. Abdel-Aty, S. (1954). Approximate Formulae for the Percentage Points and the Probability Integral of the Non-Central χ2 Distribution (https://www.jstor.org/stable/2332731) *Biometrika* 41, 538–540. doi:10.2307/2332731
4. Sankaran , M. (1963). Approximations to the non-central chi-squared distribution (http://biomet.oxfordjournals.org/cgi/content/citation/50/1-2/199) *Biometrika*, 50(1-2), 199–204
5. Sankaran , M. (1959). "On the non-central chi-squared distribution", *Biometrika* 46, 235–237
6. Johnson et al. (1995) *Continuous Univariate Distributions* Section 29.8
7. Muirhead (2005) pages 22–24 and problem 1.18.
8. Derek S. Young (August 2010). "tolerance: An R Package for Estimating Tolerance Intervals" (http://www.jstatsoft.org/v36/i05). *Journal of Statistical Software*. **36** (5): 1–39. ISSN 1548-7660 (https://www.worldcat.org/issn/1548-7660). Retrieved 19 February 2013., p.32

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