Time Series Analysis

Lecture **X**: Summary
Questions and Answers

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Course topics

- Time series, time series regression and exploratory analysis
  - Autocovariance, ACF
  - Sample ACF
  - Stationarity, detrending, differencing,
  - transformation and smoothing
- ARIMA models
  - AR, MA, ARMA, ARIMA, seasonal ARIMA
  - PACF
  - Model selection
  - Estimation
  - Forecasting
- State space models
  - Linear and Gaussian state space models
  - Kalman filtering, Kalman smoothing and Forecasting
  - Maximum likelihood estimate of the state space models
  - Stochastic volatility
- Recurrent Neural Networks (RNNs)

Stationarity

- Time series x_t is weakly stationary (stationary) if
  - $Ex_t = const$
  - $\gamma(s,t) = \gamma(|s-t|)$
  - $\text{var}(x_t) < \infty$
- $\gamma(t, t+h) = \gamma(|t+h-t|) = \gamma(h)$
  - Autocovariance depends on lag only!
- Autocovariance for stationary process $\gamma(h) = \text{cov}(x_t, x_{t+h})$
- ACF for stationary process $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$

Time domain: The Big Picture

Time Series data

Exploratory data analysis

Make data stationary

Suggest a model M(ϕ, θ)

$\phi = (\phi_1, ..., \phi_p)$

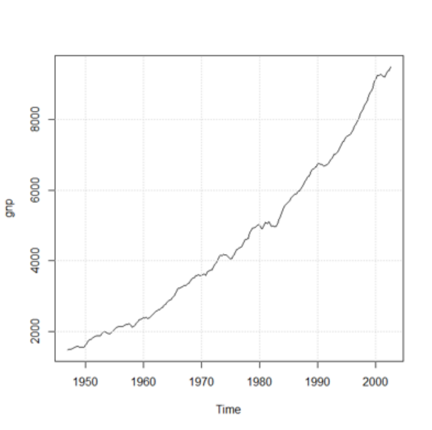
$\theta = (\theta_1, ..., \theta_q)$

Prediction

Mod

- A
  ar

- Estimate $\phi's$, $\theta's$, ...



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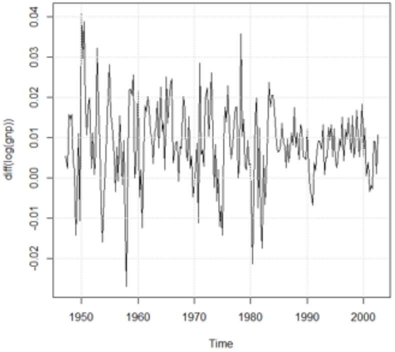
Estimate $\phi's$, $\theta's$, ...

Prediction

$Y_t = \nabla(\log(X_t))$



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Make data stationary

Suggest a model $M(\phi, \theta)$

$\phi = (\phi_1, ..., \phi_p)$

$\theta = (\theta_1, ..., \theta_q)$

Model Estimation

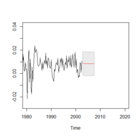
- Estimate $\phi's$, $\theta's$, ...

Model diagnostics and Model selection

- Are there better models with other p and q or even different kind?

no

Prediction



Time domain: The Big Picture

Time Series data

Exploratory data analysis

Make data stationary

Suggest a model $\mathrm{M}(\phi, \theta)$

$\phi = (\phi_1, ..., \phi_p)$

$\theta = (\theta_1, ..., \theta_q)$

Model Estimation

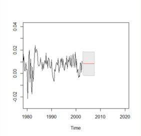
- Estimate $\phi' s$, $\theta' s$, ...

Model diagnostics and Model selection

- Are there better models with other p and q or even different kind?

no

Prediction



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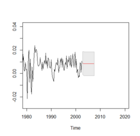

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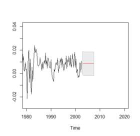


Time domain: The Big Picture

Time Series data

Exploratory data analysis

Make data stationary

Suggest a model $M(\phi, \theta)$

$$\phi = (\phi_1, ..., \phi_p)$$

$$\theta = (\theta_1, ..., \theta_q)$$

Model Estimation

- Estimate $\phi's$, $\theta's$, ...

Model diagnostics and Model selection

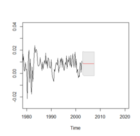
- Are there better models with other p and q or even different kind?

yes

Revise the model

no

Prediction



ARIMA modelling

- ARIMA models
  - AR, MA, ARMA, ARIMA, seasonal ARIMA
  - PACF
  - Model selection
  - Estimation
  - Forecasting

ARIMA models

Time series models so far

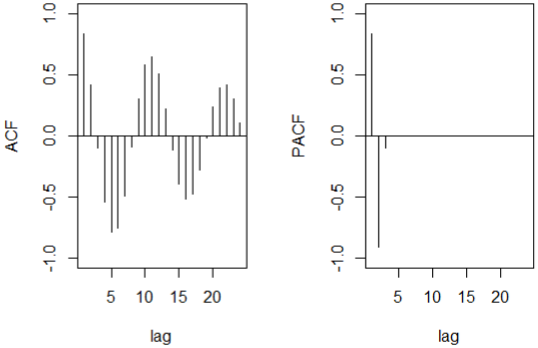
$$\phi^p(B)x_t = \theta^q(B)w_t$$

Model	Concise form
AR(p)	$\phi^p(B)x_t = w_t$
MA(q)	$x_t = \theta^q(B)w_t$
ARMA(p,q)	$\phi^p(B)x_t = \theta^q(B)w_t$
ARIMA(p,d,q)	$\phi^p(B)(1-B)^d x_t = \theta^q(B)w_t$
ARMA$(P,Q)_s$	$\Phi^P(B^s)x_t = \Theta^Q(s)w_t$
ARIMA$(P,D,Q)_s$	$\Phi^P(B^s)(1-B^s)^D x_t = \Theta^Q(B^s)w_t$
ARMA$(p,q)\times(P,Q)_s$	$\Phi^P(B^s)\phi^p(B)x_t = \Theta^Q(B^s)\theta^q(B)w_t$
ARIMA$(p,d,q)\times(P,D,Q)_s$	$\Phi^P(B^s)\phi^p(B)(1-B^s)^D(1-B)^d x_t = \Theta^Q(B^s)\theta^q(B)w_t$

* The notation used in this slide deviates from the notation used in the course literature so far.

PACF for AR(p)

- Example: AR(3) $\phi_1 = 1.5$, $\phi_2 = -0.75, \phi_3 = -0.1$



Seasonal?

ACF and PACF

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

How to differentiate between ARMA(p,q)?

Empirical ACF (EACF)

Idea:

- ARMA(p,q): $x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + \sum_{j=1}^{q} \theta_j w_{t-j} + w_t$
- If we can estimate $\phi_j \rightarrow x'_t = x_t - \sum_{j=1}^{p} \phi_j x_{t-j}$ is linear function in $w_t, ..., w_{t-q}$
- If we run regression x'_t against $w_t ... w_{t-j}$:
  - Residuals are white noise, $j \geq q \rightarrow$ ACFs not significant
    - ★ Some of the coefficients will be 0
  - Residuals are not white noise, $j < q \rightarrow$ ACFs significant
  - Note: $w_t s$ substituted by lagged residuals from a series of regressions
- If $x'_t = x_t - \sum_{j=1}^{k} \phi_j x_{t-j}, k < p \rightarrow$ white noise will never be achieved $\rightarrow$ ACFs are not zero

Empirical ACF (EACF)

- $k > p$ General result: ACFs are 0 for $j > q + (k - p)$
  - Example: ARMA(0,1)
- General conclusion for AR,MA =(k,j):
  - This is theoretical one! $\rightarrow$ not exactly the same for the samples

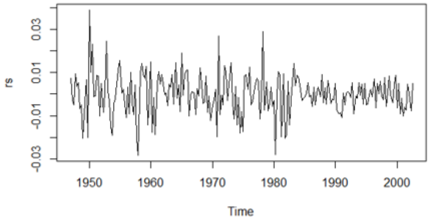
AR/MA	0	1	2	...	...	...	...
0	X	X	X	X	X	X	X
1	X	X	X	X	X	X	X
2	X	X	X	X	X	X	X
...	X	X	X	X	X	X	X
...	X	X	X	X	X	X	X
...	X	X	0	0	0	0	0
...	X	X	X	0	0	0	0
...	X	X	X	X	0	0	0
...	X	X	X	X	X	0	0

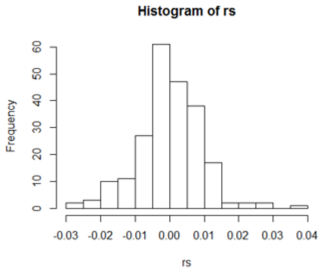
Residual analysis

- Residuals $r_t = x_t - \hat{x}_t^{t-1}$?they are innovations
  - Note: computed from one-step-ahead predictions!
  - Measures predictive quality of the model (compare OLS)
- Residual analysis
  - Visual inspection: stationary? Patterns?
  - Histograms, Q-Q plots
  - ACF, PACF
  - Runs test
  - Box-Ljung test

Residual analysis - Visual inspection

Histogram and visual inspection

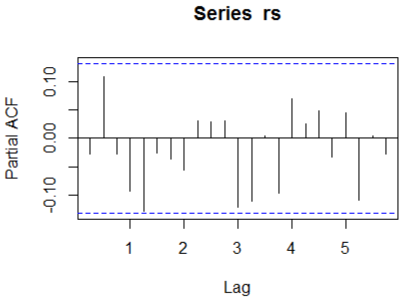


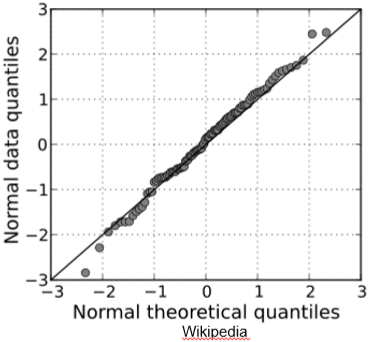


If looks white is good

If looks Normal is good

Residual analysis - ACF /PACF Q-Q plots





If between the blue lines good

If along the diagonal line GOOD

Statistical tests

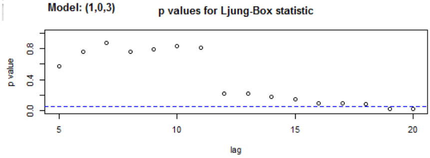
Tests are used to test independence

Runs test

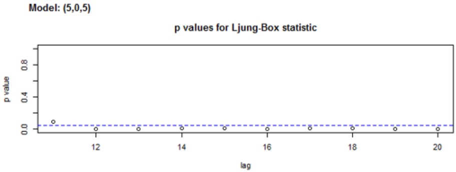
- H_0 : x_t values are i.i.d. **p-value NOT small**
- H_a : x_t values are not i.i.d. **p-value small**

Box-Ljung test

- H_0 : data are independent **p-value NOT small**
- H_a : data are not independent **p-value small**



GOOD



BAD

SARIMA

- Multiplicative seasonal autoregressive integrated moving average model $ARIMA(p,d,q)\ x\ (P,D,Q)_s$

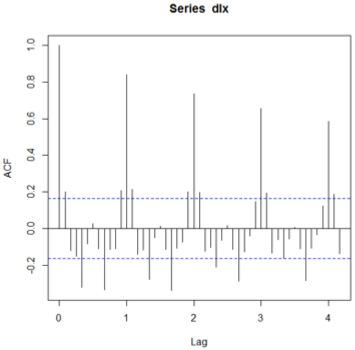
$$\Phi_p(B^s)\phi(B)\nabla_s^D\nabla^d x_t = \delta + \Theta_Q(B^s)\theta(B)w_t$$

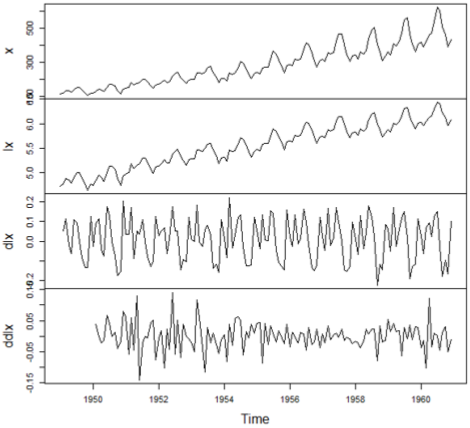
$$\nabla_s^D = (1 - B^s)^D$$

- How to identify SARIMA?
  1. Perform differencing first (trend)
  2. Investigate ACF $\rightarrow$ slowly decays at peaks?
     1. Yes $\rightarrow$ Additional differencing by ∇_s^D
  3. Model non-seasonal part
  4. Model seasonal part (check peaks), check ACF and PACF of residuals

SARIMA

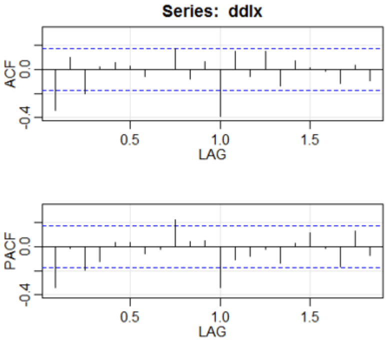
- Example: Air passangers





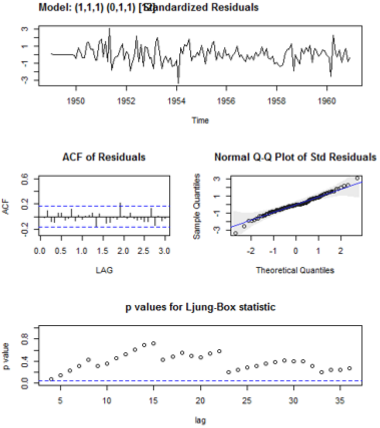
SARIMA

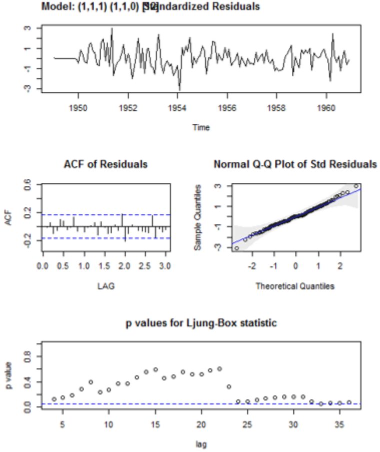
- Example: Air passangers



$$(0,1,1)_{12} \text{ or } (1,1,0)_{12}$$

SARIMA


Model: (1,1,1) (0,1,1) [12] Standardized Residuals
Time
ACF of Residuals
ACF
LAG
Normal Q-Q Plot of Std Residuals
Sample Quantiles
Theoretical Quantiles
p values for Ljung-Box statistic
p value
lag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Time
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ACF
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Normal Q-Q Plot of Std Residuals
Sample Quantiles
Theoretical Quantiles
p values for Ljung-Box statistic
p value
lag

SARIMA

- Remove AR term!

Is one model much better the other one?

$(1,1,1)\ x\ (1,1,0)_{12}$

```
> m1$fit

Call:
stats::arima(x = xdata, order = c(p, d, q), seasonal = list(order = c(P, D,
    Q), period = S), include.mean = !no.constant, optim.control = list(trace = trc,
    REPORT = 1, reltol = tol))

Coefficients:
         ar1      ma1     sar1
      0.0547  -0.4886  -0.4731
s.e.  0.2161   0.1933   0.0800

sigma^2 estimated as 0.001425:  log likelihood = 241.73,  aic = -475.47
```

$(1,1,1)\ x\ (0,1,1)_{12}$

```
> m2$fit

Call:
stats::arima(x = xdata, order = c(p, d, q), seasonal = list(order = c(P, D,
    Q), period = S), include.mean = !no.constant, optim.control = list(trace = trc,
    REPORT = 1, reltol = tol))

Coefficients:
         ar1      ma1     sma1
      0.1960  -0.5784  -0.5643
s.e.  0.2475   0.2132   0.0747

sigma^2 estimated as 0.001341:  log likelihood = 244.95,  aic = -481.9
```

State space modelling

- State space models
  - Linear and Gaussian state space models
  - Kalman filtering, Kalman smoothing and Forecasting
  - Maximum likelihood estimate of the state space models
  - Stochastic volatility

Vectorized AR(2) model **Whiteboard**

Consider an AR(2) model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$$

Let $\mathbf{z}_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}$ and $e_t = \begin{bmatrix} w_t \\ 0 \end{bmatrix}$.

Show that we rewrite the AR(2) model in the state space form:

$$\begin{aligned} \mathbf{z}_t &= \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \mathbf{z}_{t-1} + e_t \\ x_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{z}_t, \end{aligned}$$

ARIMA models in State Space form **Whiteboard**

$$\phi^p(B)x_t = \theta^q(B)w_t$$

Can we rewrite any model of this form as a state space model?

$$\mathbf{z}_t = A\mathbf{z}_{t-1} + e_t,$$
$$\mathbf{x}_t = C\mathbf{z}_t + \nu_t,$$

ARIMA models in State Space form **Whiteboard**

$$\phi^p(B)x_t = \theta^q(B)w_t$$

Outline of the solution:

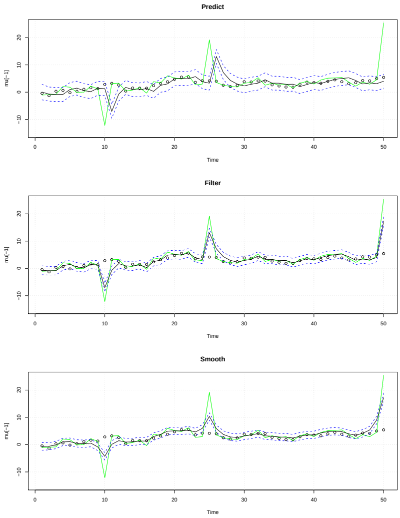
Let $r = \max(p, q+1)$,
$\phi^r(B) = 1 - \phi_1 B - \cdots - \phi_r B^r$,
$\theta^r(B) = 1 + \theta_1 B - \cdots - \theta_{r-1} B^{r-1}$,
$\phi^r(B)(\theta^r(B))^{-1} x_t = w_t$. Hence, for $z_t = (\theta^r(B))^{-1} x_t$ we can have
$\phi^r(B) z_t = w_t$

$$\mathbf{z}_t = \begin{bmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-r+1} \end{bmatrix} \text{ and } \mathbf{z}_t = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_r \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \mathbf{z}_{t-1} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$x_t = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_r \end{bmatrix} \mathbf{z}_t$$

Robustness to outliers:filter versus smoother

Live example in Rstudio



Stochastic Volatility : Gaussian sum filter

The problem is finding the filtering distribution of $\mathbf{z}_t | \mathbf{x}_{1:t}$ when

$$\begin{aligned}\mathbf{z}_t &= A\mathbf{z}_{t-1} + w_t \\ \mathbf{x}_t &= C\mathbf{z}_t + \eta_t\end{aligned}$$

and

$$\begin{aligned}w_t &\sim iidN(0, Q) \\ \eta_t &\sim \pi_0 N(\mu_0, R_1) + \pi_1 N(\mu_1, R_2)\end{aligned}$$

where $\pi_0 + \pi_1 = 1$

Examination

- Most of the examination will be your Computer labs and assignments from the teaching sessions with a twist.
- You need to have a deep knowledge of the subjects covered in the lectures to get a B+ score.
- Study them over and over and make sure you have the correct solutions with you on the examination day.