

Sum-of-Squares Reading

July 17, 2020

1 Introduction

Roughly speaking, the Sum-of-Squares (SOS) hierarchy is a semi-definite programming hierarchy to reason about polynomial optimization subject to a system of polynomial equalities and inequalities constraints [BPT12, Las15]. Given a concrete instance of a polynomial optimization problem and a degree parameter ¹ d , the SOS hierarchy ² at level d specifies a semi-definite relaxation program, whose size grows with d . The larger the degree d the tighter this relaxation becomes as additional consistency constraints are imposed by the hierarchy and a harsher positive semi-definite condition must be satisfied. The versatility of polynomials as a language to model combinatorial optimization, statistics and machine learning problems makes the SOS hierarchy quite handy. More importantly, SOS gives the best known polynomial time guarantees for several of these problems [FKP19]. This striking success as a powerful algorithmic tool makes SOS lower bounds quite appealing as evidence of hardness. This is specially true for some average case problems for which NP-hardness may not be an option.

2 Algorithms

The following is a list of some SOS (or SOS-related) algorithms.

- In [CMY20], Cherapanamjeri–Mohanty–Yau show how to do list decode mean estimation in nearly linear time. This improves the running time of the SOS based result for mean estimation of [RY20].
- In [BK20], Bakshi and Kothari give a list decoding algorithm for subspace recovery.
- In [BBK⁺20], Bafna et al. show how to approximate unique games on constraint graphs whose small-set expansion can be certified.
- In [RY20], Raghavendra and Yau give list decoding algorithms for learning (regression and mean estimation).
- In [KKK19], Karmalkar–Klivans–Kothari also give list decoding algorithms for learning (regression and mean estimation).

¹Sufficiently large to for the SDP to be well-defined.

²This is dual view of SOS hierarchy.

- In [AJQ⁺20], Alev et al. give list decoding algorithm for distance amplified codes using HDX and expander graphs.
- In [AJT19], Alev et al. give approximation algorithms for k -CSPs on HDXs extending the result of Barak–Raghavendra–Steurer [BRS11].
- In [RSS18], Raghavendra–Schramm–Steurer surveys the growing literature of SOS for high-dimensional estimation (deducing parameters of distributions given their samples).
- In [BKS17], Barak–Kothari–Steurer give a $\exp(\tilde{O}(\sqrt{n}))$ -time algorithm to find a rank one operator acting on an input subspace of ambient dimension n . This gives an $\exp(\tilde{O}(\sqrt{n}))$ -time algorithm for QMA(2) with each witness dimension of n .
- In [BGG⁺17], Bhattiprolu et al. give approximation algorithms to compute the maximum absolute value of a multivariate homogeneous polynomial over the sphere. Their result is based on a weak decoupling result (de Finetti like result).
- In [HKP⁺17], Hopkins et al. show that that spectral algorithms using matrices whose entries are low degree polynomials on the input variables provide similar guarantees of SOS. They also show nearly tight SOS lower bounds for tensor and sparse PCA.
- In [RRS17], Raghavendra–Rao–Schramm give SOS refutation algorithms for random CSP (see [KMOW17] for nearly matching lower bounds).
- In [HSSS16], Hopkins et al. show faster spectral algorithms based on SOS proofs. Although (standard) SOS is not the fastest algorithmic tool (in terms of finer polynomial running time), its insights can lead to more efficient implementation.
- In [BKS14], Barak–Kelner–Steurer design some reasonably general rounding scheme for SOS. In particular, they apply their techniques to polynomial optimization, with non-negative coefficients, over the sphere.
- In [BRS11], Barak–Raghavendra–Steurer use the SOS hierarchy to give an approximation algorithm for 2-CSPs with guarantees based on the threshold rank of a graph. They also show that degree- $n^{\text{poly}(\epsilon)}$ SOS can decide unique games. Qualitatively speaking their results have some similarities to [GS11].
- In [GS11], Guruswami and Sinop use the SOS hierarchy to give approximation algorithms with guarantees based on the threshold rank of a graph. They also show subexponential time algorithm to decide unique games. Qualitatively speaking their results have some similarities to [BRS11].

Although the SOS hierarchy specifies how to write an SDP program relaxation given a polynomial optimization problem and a degree d , it does not refrain the user from rounding if an integral solution is need. Rounding can be quite challenging and it is not always clear what is the best rounding scheme. Therefore, an ambitious open-ended question is the following.

Open Problem 2.1 (Open-ended Problem). *Is there a general theory of SOS rounding?*

The SOS hierarchy had some early successes in approximating CSPs, where under the Unique Games Conjecture it provides the best approximation guarantees. It can decide unique games in subexponential time and give efficient good approximations for CSPs on reasonably structured graphs (low threshold rank). More recently, it has been successfully applied to machine learning, some (robust) statistics and coding theory problems.

Open Problem 2.2 (Open-ended Problem). *Which fields and algorithmic questions can (algorithmically) benefit from the SOS hierarchy?*

3 Lower Bounds

The following is a list of some SOS lower bounds.

- In [MRX20], Mohanty–Raghavendra–Xu quite surprisingly show how to lift degree-2 lower bounds to degree-4 lower bounds under suitable conditions.
- In [Pot18], Potechin shows nearly tight upper and lower bounds on the ordering principle.
- In [KOS19], Kothari, O’Donnell and Schramm show how to incorporate global cardinality constraints in SOS lower bounds. They obtain strong degree lower bounds for Min-Bisection and Max-Bisection.
- In [Pot19], Potechin explores symmetry to give a framework for SOS lower bounds for symmetric problems.
- In [KM18], Kothari and Mehta show SOS hardness of finding any equilibrium rather than for those optimizing some objective function.
- In [KMOW17], Kothari et al. obtain nearly tight SOS lower bounds for refuting CSPs whose predicates support a t -wise uniform distribution. Their PSD analysis is based on a Gram–Schmidt orthogonalization process.
- In a breakthrough result [BHK⁺16], Barak et al. show nearly tight SOS lower bounds for the planted clique problem. Their result introduces the general framework of pseudocalibration to find good candidate integrality gap solutions.
- In [LRS15], Lee–Raghavendra–Steurer introduce reasonably general SDP lower bounds techniques. In particular, they obtain that for MAX CSPs any polynomial-size SDP is equivalent in power to degree- $O(1)$ SOS.
- In [Tul09], Tulsiani obtains SOS integrality gaps for MAX k -CSPs. Then using reductions Tulsiani obtains SOS gaps for Max Independent Set, Approximate Coloring, Chromatic Number and Min Vertex Cover.

SOS lower bounds are oftentimes obtained in a case-by-case manner, though some general elegant tools and techniques have emerged such as graph matrices and pseudo-calibration. Proving that a given candidate solution is PSD can be quite challenging. One might ambitiously (and possibly very naively) ask the following.

Open Problem 3.1 (Open-ended Problem). *Is there a general theory of SOS lower bounds?*

The preceding open problem may be too ambitious to ever see a positive resolution. However, more modestly and of great interest is the following.

Open Problem 3.2 (Open-ended Problem). *More tools and techniques for SOS lower bound.*

4 SOS Machinery

A noteworthy tool that has emerged in the study of SOS lower bounds are the so-called graph matrices. They provided an elegant mix of Fourier analysis, combinatorics and random matrix theory with (possibly) non independent entries.

- In [AMP16], Kwangjun et al. give a detailed account of the graph matrices machinery, which is a powerful tool used in some SoS lower bound results. Graph matrices can be seen as a matrix analogue of Fourier characters. Spectral norm bounds follow from simple combinatorial properties of underlying graphs.
- In [CP20], Cai and Potechin pin-point the singular values spectra of some graph matrices called Z -shaped.

5 Misc

- In [RSST18], Raymond et al. show an equivalence of (the semi-definite method of) flag algebras and (symmetric) Sum-of-Squares. Their techniques are based on the representation theory of the symmetric group.
- In [GP04], Gatermann and Parrilo show how to explore symmetries of a polynomial optimization problem to greatly reduce the size of the corresponding SOS SDP program. They use representation theory to find this succinct description.
- In [BBH⁺12], Barak et al. give an SOS proof of the hypercontractivity inequality obtaining several applications. In particular, they rule out the short-code graph as a hard (super polynomial) instance for SOS.

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