High-Dimensional Expanders Research Group

July 17, 2020

Roughly speaking high-dimensional expanders (HDXs) are hypergraph expanders having similar properties to complete simplicial complexes. This is analogous to the 1-dimensional case in which expander graphs have similar properties to complete graphs. Over the past 40 years expander graphs flourished as an important field at the intersection of theoretical computer science (TCS) and mathematics [28]. Building on top of this rich expander graph theory, the young field of HDXs has already been instrumental in recent breakthroughs (e.g. counting bases of matroids [5], derandomizing direct product tests [20], Gromov's topological overlap property [33], etc) and there is a strong hope that HDXs may turn out to be fundamental objects as their one dimensional analogues. Currently, there is no shortage of open problems about HDXs ranging from their very construction (we only know a few of them) to applications in coding theory, optimization and hardness of approximation.

An ambitious line of work in the TCS community is to use HDXs to achieve three goals: linear size Probabilistically Checkable Proofs (PCPs), completely explicit PCP constructions and random PCP constructions.

In this line of work some natural questions arise (according to the author's biases). Towards the first goal it might be instructive to address the following problems.

1.1 When are CSPs on HDXs hard for our best algorithms (e.g. Sum-of-Squares)? The answer might suggest good parameters for PCPs on HDXs.

1.2 What codes can be obtained from HDXs? In the classical PCP theory, codes and PCPs are intimately connected.

1.3 How do these codes can be used for hardness?

The second goal might involve the following problems.

2.1 What combinatorial constructions of HDXs can we obtain?

2.2 Is there a generalized zig-zag [53] product for HDXs?

2.3 How does representation theory help?

The third goal could be a major breakthrough in average case hardness. It might benefit from answers to the following questions.

3.1 What are good random models for HDXs? This problem seems surprisingly more involved than the graph case.

3.2 Are random CSPs coming from these models hard for our best algorithms (e.g. Sum-of-Squares)?

Another question is: what HDX partitioning can be obtained assuming HDX special property? The one dimensional theory is rich with such results [43].

1 HDX and Related Papers

In the following we briefly describe some papers about or related to HDXs. Lubotzky has a HDX survey in ICM 2018 [47]. Please note that this list of papers is intended to serve only as starting point for our reading group, but we do not need and should not be restricted to it.

1.1 HDXs Applications to TCS

The HDX field started in pure mathematics around 2005 and only very recently has seen applications in TCS. This freshness provides a variety of research opportunities.

1.1.1 Agreement Testing

- In [37], Kaufman and Mass develop a technique dubbed the variance method to derive agreement expasion results.

- In [13], Dikstein and Dinur generalized the agreement test on HDXs [20] to more general families of "layered" set systems.

- In [20], Dinur and Kaufman showed that HDXs are "agreement expanders". As a consequence HDXs can be used to derandomize product test which is an important test in the PCP literature. (Presented by Fernando.)

- In [17], Dinur et al. study higher order agreement testing generalizing direct product testing. Their generalized agreement testing is applied to p-biased Boolean function applications.

1.1.2 Coding Theory

- In [15], Dikstein et al. propose an approach towards good LTC using HDXs. They also use HDX as a unifying language for LTCs.

- In [30], Jeronimo et al. provide unique decoding algorithm for ϵ-balanced codes near the Gilbert–Varshamov bound. (Presented by Fernando.)

- In [23], Evra et al. use HDXs to obtain LDPC quantos codes with distance larger than $\sqrt{n}$.

- In [1], Alev et al. show how to list decode direct sum codes on HDXs and expander walks. (Presented by Fernando.)

- In [18], Dinur et al. show how to list decode product codes using double samplers. Currently, the only known constructions of double samplers are based on HDXs. (Presented by Dylan.)

- In [19], Dinur et al. use agreement testing to make local testing "robust" of certain codes.

1.1.3 Constraint Satisfaction Problems

- In [2], Alev et al. show how to approximate k-CSPs on HDXs using the Sum-of-Squares hierarchy. This result can be viewed as a generalization to k-CSPs (for $k \geq 2$) of the work of Barak, Raghavendra and Steurer [8] for approximating 2-CSPs on low threshold rank graphs.

1.1.4 Counting and Sampling

- In [4], Anari et al. use distributions yielding HDXs to sample independent sets from the hardcore model.

- In [3], Alev and Lau give sharp spectral bounds for the up and down walks improving the analysis of [39]. As an application, they show how to sample independent sets.

- In [5], Anari et al. presented an efficient scheme to count bases of a matroid solving a longstanding open problem of Mihail and Vazirani. (Presented by Akash.)

1.1.5 Sparsification

- In [6], Bansal et al. explore new notion of graph and hypergraph sparsifier.

1.2 HDXs Applications to Mathematics

- In [36], Kaufman and Mass obtained good distance lattices using high-dimensional expanders.

1.3 HDXs Properties

- In [40], Kaufman and Oppenheim study a higher order notion of edge expansion.

- In [14], Dinur et al. introduce a machinery of expanding posets clarifying the spectral theory of an important class of HDXs. Their machinery shows that natural random walks on these HDXs behave similarly to their counterpart in the complete complex. (Presented by Shashank.)

- In [39], Kaufman and Oppenheim obtain similar spectral results those obtained in [14]. However, the language adopted in these two papers is different.

- In [52], Parzanchevski et al. develop isoperimetric inequalities for HDXs (e.g. they have a notion of high-dimensional Cheeger's inequality).

- In [35], Kaufman and Mass (KM) present some notion of high-dimensional expansion. Spectrally, their result is weaker than [14] and [39]. However, KM result has a stronger combinatorial flavor.

- In [32, 33], Kaufman et al. address a conjecture of Gromov about a property called topological overlap which generalizes geometric overlap.

- In [22], Evra and Kaufman provide constructions of topological expanders of every dimension generalizing [32].

- In [34], Kaufman and Lubotzky show that certain notions of high-dimensional expansion are equivalent to testability of some properties.

1.4 HDXs Constructions

- Harsha and Saptharishi [27] provide a self-contained exposition of Kaufman–Oppenheim HDX construction which avoids representation theory.

- Karni and Kaufman [31] use a connection with zig-zag product to improve the expansion of some 3-uniform hypegraph constructions (e.g., Conlons and CLP).

- Liu et al. [45] gave a combinatorial construction of HDXs using expander graphs as a starting point.

- Conlon et al. [11] gave a combinatorial construction of HDXs with an **almost** linear number of hyperedges. Currently, it might be the simplest non-trivial construction of HDXs.

- Chapman et al. study and construct expander graphs whose neighborhood of every vertex is also expanding [9].

- In [38], Kaufman and Oppenheim give a linear size construction of one-sided link HDXs. The construction is algebraic and uses representation theory. (Presented by Tushant.)

- In [50, 49], Lubotzky et al. construct Ramanujan complexes generalizing Ramanujan graphs. Their construction is algebraic.

1.5 HDXs Random Models

- In [48], Lubotzky et al. present the first bounded degree random model of HDXs.

- In [44], Linial and Meshulam present a dense random model of HDXs similar to the $\mathcal{G}_{n,p}$ random graph model.

1.6 Complete Complex

Roughly, HDXs provide sparse approximations to complete complexes. In some cases, understanding properties of complete complexes may be important towards understanding HDXs. Contrary to complete graphs, properties of complete complexes can be far from trivial.

- In [24], Filmus develop a theory of Boolean functions over a slice of the Boolean cube. Part of his result can be seen as a complete complex analogue of [14].

- In [21], Dinur and Steurer show direct product tests for complete complexes. Their result is used as black box in [20]. (Presented by Mrinal.)

- In [12], Dinur et al. show a direct sum test for complete complexes. Similarly, their result is used as black box in [20].

1.7 Classical Results

- In [10], Cohen and Yankovitz use expanders to perform query-efficient distance amplification of LDCs.

- Ta-Shma gives an explicit almost optimal (i.e., close to the GV bound) construction of epsilon-balanced codes in [55]. (Presented by Shashank.)

- In [51], Sidhanth, O'Donnell and Paredes construct explicit near Ramanujan graphs of every degree.

- By "derandomizing" the long code, Barak et al. proved that a graph family is a small set expander but has reasonably large threshold rank [7]. These properties were then used to show hardness results.

- In [25], Garg et al. show a matrix Chernoff bound for expander graphs.

- Finding forbidden minors: one-sided [41] and two-sided testers [42]. (Presented by Akash.)

- Expander graph survery [28] of Hoory, Linial and Wigderson.

- In [29], Impagliazzo et al. give a PCP construction using direct product. Further derandomizing their construction using HDXs may yield linear size PCPs.

- In [53], Reingold et al. give a combinatorial construction of HDXs using a graph operation called zig-zag product which has connections to semi-direct product on groups [46]. It is an open problem to find combinatorial constructions of HDXs.

- In [16], Dinur gives an influential combinatorial PCP using expander graphs. Her ideas might be useful in obtaining a linear size PCP using HDXs.

- In [43], Kowk et al. provide graph partitioning based on higher eigenvalues generalizing Cheeger's inequality.

- Some natural walks in the complete complex are related to the Johnson scheme [26]

- Spherical buildings [54] arise as substructures of Ramanujan complexes.

2   Format of the Reading Group

For each paper presentation, we also require a LaTeX document clearly stating the main result, explaining the architecture of its proofs and ideally providing a few proofs that illuminate the main result (possibly with simpler parameters).

To encourage collaboration papers can be read and presented by two people. Nonetheless, single person presentation is also welcome.

To improve the quality of the final document we will designate a reviewer in case of single person presentation. The reviewer is expected to assist in the clarity of the document and point out typos. For this reason, you might be expected to review a few documents (possibly one or at most two). We hope that the review process might also encourage collaboration.

The presentation follows a similar structure to KTH TCS seminar. The presentation has two parts. The first hour is intended to motivation, overview of the main result and simpler proofs. It is followed by a 10 minutes break. Finally, we allocate one hour and 30 minutes for more technical and detailed proofs.

The goal is to have weekly presentations starting at 3:30pm. The precise day of the week is to be defined according to majority vote (most likely Friday this time).

References

[1] Vedat Levi Alev, Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, and Madhur Tulsiani. List decoding of direct sum codes. In *Proceedings of the 31st ACM-SIAM Symposium on Discrete Algorithms*, 2020.

[2] Vedat Levi Alev, Fernando Granha Jeronimo, and Madhur Tulsiani. Approximating constraint satisfaction problems on high-dimensional expanders. 2019.

[3] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *Proceedings of the 52nd ACM Symposium on Theory of Computing*, 2020.

[4] Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model, 2020.

[5] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. 2019.

[6] N. Bansal, O. Svensson, and L. Trevisan. New notions and constructions of sparsification for graphs and hypergraphs. In *Proceedings of the 60th IEEE Symposium on Foundations of Computer Science*, pages 910–928, 2019.

[7] Boaz Barak, Parikshit Gopalan, Johan Håstad, Raghu Meka, Prasad Raghavendra, and David Steurer. Making the long code shorter. In *Proceedings of the 53rd IEEE Symposium on Foundations of Computer Science*, pages 370–379, 2012.

[8] Boaz Barak, Prasad Raghavendra, and David Steurer. Rounding semidefinite programming hierarchies via global correlation. In *Proceedings of the 52nd IEEE Symposium on Foundations of Computer Science*, pages 472–481, 2011.

[9] Michael Chapman, Nati Linial, and Yuval Peled. Expander graphs – both local and global. 2019.

[10] Gil Cohen and Tal Yankovitz. Rate amplification and query-efficient distance amplification for locally decodable codes, 2020.

[11] David Conlon, Jonathan Tidor, and Yufei Zhao. Hypergraph expanders of all uniformities from Cayley graphs. *arXiv e-prints*, page arXiv:1809.06342, September 2018.

[12] Roee David, Irit Dinur, Elazar Goldenberg, Guy Kindler, and Igor Shinkar. Direct sum testing. pages 327–336, New York, NY, USA, 2015. ACM.

[13] Yotam Dikstein and Irit Dinur. Agreement testing theorems on layered set systems. In *Proceedings of the 60th IEEE Symposium on Foundations of Computer Science*, 2019.

[14] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. Boolean function analysis on high-dimensional expanders. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA*, pages 38:1–38:20, 2018.

[15] Yotam Dikstein, Irit Dinur, Prahladh Harsha, and Noga Ron-Zewi. Locally testable codes via high-dimensional expanders, 2020.

[16] Irit Dinur. The pcp theorem by gap amplification. In *Proceedings of the Thirty-eighth Annual ACM Symposium on Theory of Computing*, Proceedings of the 38th ACM Symposium on Theory of Computing, pages 241–250, New York, NY, USA, 2006. ACM.

[17] Irit Dinur, Yuval Filmus, and Prahladh Harsha. Analyzing boolean functions on the biased hypercube via higher-dimensional agreement tests: [extended abstract]. In *Proceedings of the 30th ACM-SIAM Symposium on Discrete Algorithms*, pages 2124–2133, 2019.

[18] Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. List decoding with double samplers. 2019.

[19] Irit Dinur, Prahladh Harsha, Tali Kaufman, and Noga Ron-Zewi. From local to robust testing via agreement testing. pages 29:1–29:18, 2019.

[20] Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders. In *Proceedings of the 58th IEEE Symposium on Foundations of Computer Science*, pages 974–985, 2017.

[21] Irit Dinur and David Steurer. Direct product testing. In *Proceedings of the 2014 IEEE 29th Conference on Computational Complexity*, Proceedings of the 29th IEEE Conference on Computational Complexity, pages 188–196, 2014.

[22] Shai Evra and Tali Kaufman. Bounded degree cosystolic expanders of every dimension. Proceedings of the 48th ACM Symposium on Theory of Computing, pages 36–48, 2016.

[23] Shai Evra, Tali Kaufman, and Gilles Zmor. Decodable quantum ldpc codes beyond the $\sqrt{n}$ distance barrier using high dimensional expanders, 2020.

[24] Yuval Filmus. An orthogonal basis for functions over a slice of the boolean hypercube. *Electr. J. Comb.*, 23(1):P1.23, 2016.

[25] Ankit Garg, Yin Tat Lee, Zhao Song, and Nikhil Srivastava. A matrix expander chernoff bound. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, Proceedings of the 50th ACM Symposium on Theory of Computing, pages 1102–1114, 2018.

[26] Christopher Godsil and Karen Meagher. *Erdős-Ko-Rado Theorems: Algebraic Approaches*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2015.

[27] Prahladh Harsha and Ramprasad Saptharishi. A note on the elementary hdx construction of kaufman-oppenheim, 2019.

[28] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc.*, 43(04):439562, August 2006.

[29] Russell Impagliazzo, Valentine Kabanets, and Avi Wigderson. New direct-product testers and 2-query pcps. In *Proceedings of the Forty-first Annual ACM Symposium on Theory of Computing*, Proceedings of the 41st ACM Symposium on Theory of Computing, pages 131–140, 2009.

[30] Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, and Madhur Tulsiani. Unique decoding of explicit ϵ-balanced codes near the gilbert–varshamov bound. 2020.

[31] Eyal Karni and Tali Kaufman. High dimensional expansion using zig-zag product, 2020.

[32] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Ramanujan complexes and bounded degree topological expanders. In *Proceedings of the 55th IEEE Symposium on Foundations of Computer Science*, pages 484–493, 2014.

[33] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Isoperimetric inequalities for ramanujan complexes and topological expanders. *Geometric and Functional Analysis*, 26(1):250–287, Feb 2016.

[34] Tali Kaufman and Alexander Lubotzky. High dimensional expanders and property testing. pages 501–506, 2014.

[35] Tali Kaufman and David Mass. High dimensional random walks and colorful expansion. pages 4:1–4:27, 2017.

[36] Tali Kaufman and David Mass. Good distance lattices from high dimensional expanders. *CoRR*, abs/1803.02849, 2018.

[37] Tali Kaufman and David Mass. Local-To-Global Agreement Expansion via the Variance Method. 2020.

[38] Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, Proceedings of the 50th ACM Symposium on Theory of Computing, pages 773–786. ACM, 2018.

[39] Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA*, pages 47:1–47:17, 2018.

[40] Tali Kaufman and Izhar Oppenheim. Bounding the High Order Edge Expansion via High Order Radius. *arXiv e-prints*, Jul 2019.

[41] A. Kumar, C. Seshadhri, and A. Stolman. Finding forbidden minors in sublinear time: A n^1/2+o(1)-query one-sided tester for minor closed properties on bounded degree graphs. In *Proceedings of the 59th IEEE Symposium on Foundations of Computer Science*, pages 509–520, 2018.

[42] A. Kumar, C. Seshadhri, and A. Stolman. Random walks and forbidden minors ii: A poly($d\epsilon^{-1}$)-query tester for minor-closed properties of bounded degree graphs. 2019.

[43] Tsz Chiu Kwok, Lap Chi Lau, Yin Tat Lee, Shayan Oveis Gharan, and Luca Trevisan. Improved cheegers inequality: analysis of spectral partitioning algorithms through higher order spectral gap. In *Proceedings of the 45th ACM Symposium on Theory of Computing*, pages 11–20. ACM, 2013.

[44] Nathan Linial and Roy Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, Aug 2006.

[45] Siqi Liu, Sidhanth Mohanty, and Elizabeth Yang. High-dimensional expanders from expanders. *CoRR*, abs/1907.10771, 2019.

[46] A. Lubotzky. Semi-direct product in groups and zig-zag product in graphs: Connections and applications. In *Proceedings of the 42Nd IEEE Symposium on Foundations of Computer Science*, Proceedings of the 42nd IEEE Symposium on Foundations of Computer Science, 2001.

[47] Alexander Lubotzky. High dimensional expanders. In *ICM*, 2018.

[48] Alexander Lubotzky, Zur Luria, and Ron Rosenthal. Random steiner systems and bounded degree coboundary expanders of every dimension. *Discrete & Computational Geometry*, Apr 2018.

[49] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of ramanujan complexes of type ad. *Eur. J. Comb.*, 26(6):965–993, August 2005.

[50] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Ramanujan complexes of typeãd. *Israel Journal of Mathematics*, 149(1):267–299, Dec 2005.

[51] Sidhanth Mohanty, Ryan O'Donnell, and Pedro Paredes. Explicit near-ramanujan graphs of every degree. 2019.

[52] Ori Parzanchevski, Ron Rosenthal, and Ran J. Tessler. Isoperimetric inequalities in simplicial complexes. *Combinatorica*, 36(2):195–227, Apr 2016.

[53] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders and extractors. In *Proceedings 41st Annual Symposium on Foundations of Computer Science*, pages 3–13, Nov 2000.

[54] M Roman. *Lectures on Buildings*. University of Chicago Press, 2009.

[55] Amnon Ta-Shma. Explicit, almost optimal, epsilon-balanced codes. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, Proceedings of the 49th ACM Symposium on Theory of Computing, pages 238–251. ACM, 2017.