Chapter 14
Stability of Finite Difference Methods

In this lecture, we analyze the stability of finite difference discretizations. First, we will discuss the Courant-Friedrichs-Levy (CFL) condition for stability of finite difference methods for hyperbolic equations. Then we will analyze stability more generally using a matrix approach.

51 Self-Assessment

Before reading this chapter, you may wish to review...

- Convection Equation 11
- Finite Difference Approximations 13
- Eigenvalue Stability 7

After reading this chapter you should be able to...

- describe the Courant-Friedrichs-Levy (CFL) condition
-

Relevant self-assessment exercises: [LIST SELF-ASSESSMENT EXERCISES HERE]

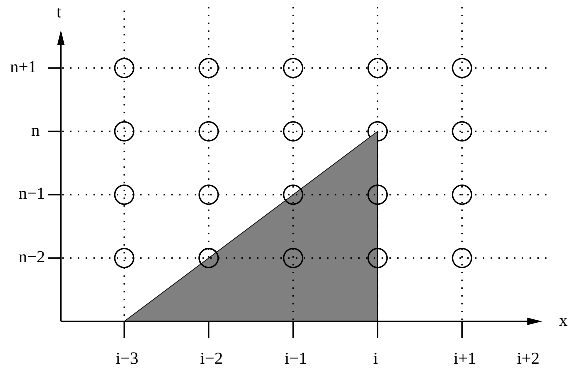
52 The CFL condition

From Exercise 4 we notice that as we increase the time step while keeping the mesh size fixed (or decrease the mesh size while keeping the time step fixed) the FTBS method eventually becomes unstable. Clearly, the choice of time step cannot be independent of the mesh size. Thus, we want to know how we must change the time step with changes in mesh size in order to maintain stability.

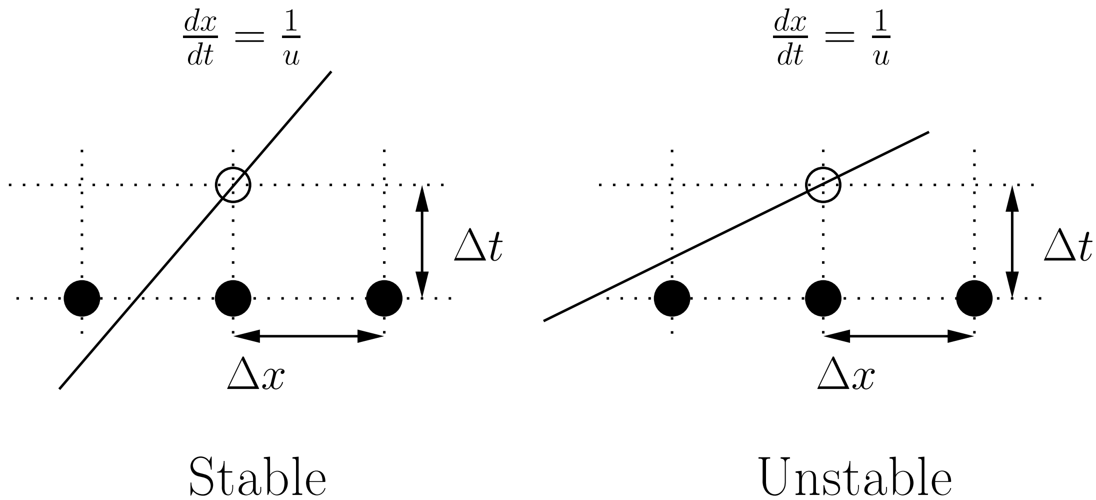
Recall from Chapter 11 that the domain of dependence for the convection equation at (x,t) is the characteristic $x(s < t)$. We can also consider the numerical domain of dependence of the solution at (x_i, t^n). Figure 22(a) shows the numerical domain of dependence of the FTBS method. A necessary condition for the convergence of a finite difference method for a hyperbolic PDE is that the numerical domain of dependence contains the mathematical domain of dependence. This requirement is known as the **Courant-Friedrichs-Levy** or **CFL condition**, named after the authors who first described this requirement.

For the one-dimensional convection equation discretized using the first-order upwind scheme, the CFL condition requires that for stability

$$CFL \equiv \frac{|u|\Delta t}{\Delta x} \leq 1. \tag{109}$$



(a) Numerical domain of dependence



(b) CFL condition

Fig. 22 Numerical domain of dependence and CFL condition for first order upwind scheme.

The non-dimensional number $\frac{|u|\Delta t}{\Delta x}$ is called the **CFL Number** or just the **CFL**. In general, the stability of explicit finite difference methods will require that the CFL be bounded by a constant which will depend upon the particular numerical scheme.

Exercise 1. Suppose we wish to solve the 1-D convection equation with velocity $u = 2$ on a mesh with $\Delta x = \frac{1}{10}$ using the FTBS method. What is the largest possible time step for which this scheme is stable?

(a) $\Delta t = \frac{1}{5}$
(b) $\Delta t = \frac{1}{10}$
(c) $\Delta t = \frac{1}{20}$
(d) $\Delta t = \frac{1}{50}$

Exercise 2. Suppose we solve the 1-D convection equation from the previous exercise with $\Delta x = \frac{1}{10}$ and $\Delta t = \frac{1}{50}$. Now suppose that we increase our spatial resolution by refining $\Delta x = \frac{1}{20}$. What must the time step be to maintain the same CFL?

(a) $\Delta t = \frac{1}{20}$
(b) $\Delta t = \frac{1}{25}$
(c) $\Delta t = \frac{1}{50}$
(d) $\Delta t = \frac{1}{100}$

53 Matrix Stability for Finite Difference Methods

As we saw in Section 47, finite difference approximations may be written in a semi-discrete form as,

$$\frac{dU}{dt} = AU + b. \tag{110}$$

While there are some PDE discretization methods that cannot be written in that form, the majority can be. So, we will take the semi-discrete Equation (110) as our starting point.

Let $U(t)$ be the exact solution to the semi-discrete equation. Then, consider perturbation $e(t)$ to the exact solution such that the perturbed solution, $V(t)$, is:

$$V(t) = U(t) + e(t).$$

The questions that we wish to resolve are: (1) can the perturbation $e(t)$ grow in time for the semi-discrete problem, and (2) what the stability limits are on the timestep for a chosen time integration method.

First, we substitute $V(t)$ into Equation (110),

$$\begin{aligned}\frac{dV}{dt} &= AV + b\\ \frac{d(U+e)}{dt} &= A(U+e) + b\\ \frac{de}{dt} &= Ae.\end{aligned}$$

Thus, the perturbation must satisfy the homogeneous equation, $e_t = Ae$. Having studied the behavior of linear system of equations in Section ?? we know that $e(t)$ will grow unbounded as $t \to \infty$ if any of the real parts of the eigenvalues of A are positive.

The problem is that determining the eigenvalues of A can be non-trivial. In fact, for a general problem finding the eigenvalues of A can be about as hard as solving the specific problem. So, while the matrix stability method is quite general, it can also require a lot of time to perform. Still, the matrix stability method is an indispensible part of the numerical analysis toolkit.

As we saw in the eigenvalue analysis of ODE integration methods, the integration method must be stable for all eigenvalues of the given problem. One manner that we can determine whether the integrator is stable is by plotting the eigenvalues scaled by the timestep in the complex $\lambda\Delta t$ plane and overlaying the stability region for the desired ODE integrator. Then, Δt can be adjusted to attempt to bring all eigenvalues into the stability region for the desired ODE integrator.

Example 1. Matrix Stability of FTCS for 1-D convection

In Example 1, we used a forward time, central space (FTCS) discretization for 1-d convection,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + u_i^n \delta_{2x} U_i^n = 0. \tag{111}$$

Since this method is explicit, the matrix A does not need to be constructed directly, rather Equation (111) can be used to find the new values of U at each point i. The Matlab script given in Example 1 does exactly that. However, if we are interested in calculating the eigenvalues to analyze the eigenvalue stability, then the A matrix is required. The following script does exactly that (i.e. calculates A, determines the eigenvalues of A, and then plots the eigenvalues scaled by Δt overlayed with the forward Euler stability region). The script can set either the periodic boundary conditions described in Example 1, or can set the inflow/outflow boundary conditions described in Exercise 2.

We will look at the eigenvalues of both cases.

```
% This Matlab script solves the one-dimensional convection
% equation using a finite difference algorithm.  The
% discretization uses central differences in space and forward
% Euler in time.
%
% periodic bcs are set if periodic flag == 1
%

clear all;
close all;

periodic_flag = 1;

% Number of points
Nx = 20;
x  = linspace(0,1,Nx+1);
dx = 1/Nx;

% velocity
u = 1;

% Set timestep
CFL = 1;
dt = CFL*dx/abs(u);

% Allocate matrix to hold stiffness matrix (A).
A = zeros(Nx,Nx);

% Construct A except for first and last row
for i = 2:Nx-1,
  A(i,i-1) =  u/(2*dx);
  A(i,i+1) = -u/(2*dx);
end

if (periodic_flag == 1), % Periodic bcs

  A(1 ,2   ) = -u/(2*dx);
  A(1 ,Nx  ) =  u/(2*dx);
  A(Nx,1   ) = -u/(2*dx);
  A(Nx,Nx-1) =  u/(2*dx);

else % non-periodic bc's

  % At the first interior node, the i-1 value is known (UL).
  % So, only the i+1 location needs to be set in A.
  A(1,2) = -u/(2*dx);

  % Outflow boundary uses backward difference
  A(Nx,Nx-1) =  u/dx;
  A(Nx,Nx  ) = -u/dx;
end

% Calculate eigenvalues of A
lambda = eig(A);

% Plot lambda*dt
plot(lambda*dt,'*');
```

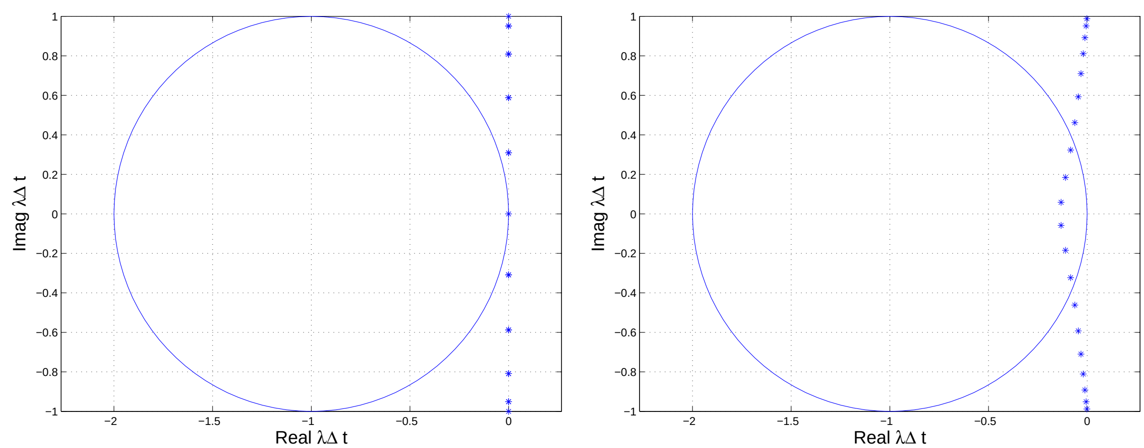
```
xlabel('Real \lambda\Delta t');
ylabel('Imag \lambda\Delta t');

% Overlay Forward Euler stability region
th = linspace(0,2*pi,101);
hold on;
plot(-1 + sin(th),cos(th));
hold off;
axis('equal');
grid on;
```

Figure 23 shows a plot of $\lambda\Delta t$ for a CFL set to one. Recall that for this one-dimensional problem, the CFL number was defined as,

$$CFL = \frac{|u|\Delta t}{\Delta x}.$$

The periodic boundary conditions give purely imaginary eigenvalues which approach $\pm i$ as the move away from the origin. Note that the periodic boundary conditions actually give a zero eigenvalue so that the matrix A is actually singular. In the inflow/outflow boundary condition case the eigenvalues lie slightly inside the negative real half-plane. However, as they move away from the origin, they also approach the imaginary axis at $\pm i$.

Regardless, what we see is that for a $CFL = 1$, some $\lambda\Delta t$ exist which are outside of the forward Euler stability region. We could try to lower the timestep to bring all of the $\lambda\Delta t$ into the stability region, however that will prove to be practically impossible since the extreme eigenvalues approach $\pm i$ (i.e. they are purely imaginary). Thus, no finite value of Δt exists for which these eigenvalues can be brought inside the circular stability region of the forward Euler method (i.e. the FTCS is unstable for convection).

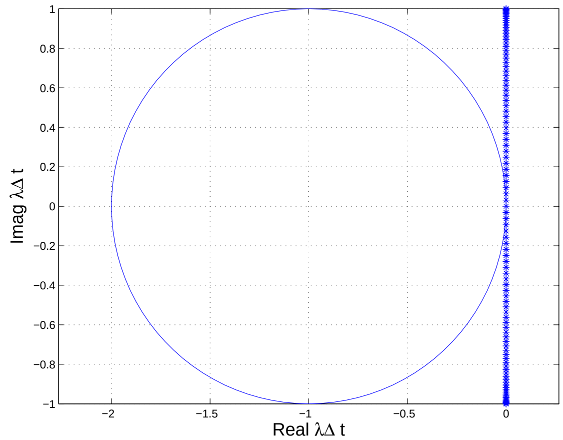


(a) Periodic BCs

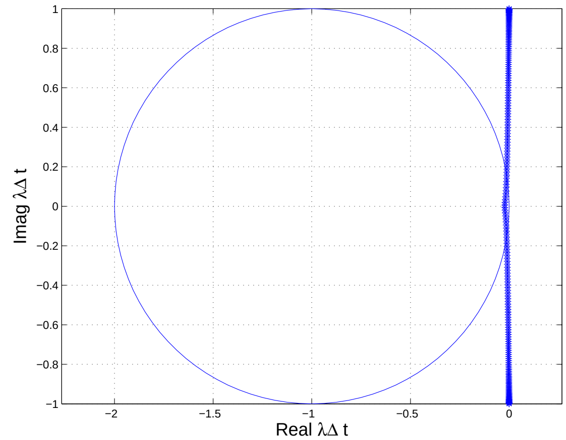
(b) Dirichlet inflow and upwinded outflow

Fig. 23 $\lambda\Delta t$ distribution for one-dimensional convection example using two different boundary conditions. Note: Δt set such that $CFL = 1$.

We may also be interested in what happens to the eigenvalue spectrum of A when we change Δx. From Section 52 we know that for the convection equation as we refine Δx we must refine Δt, keeping the CFL constant in order to maintain stability. Thus, for a fixed CFL we would expect that the eigenvalue spectrum does not change significantly with the number of points. Figure 24 plots the eigenvalue spectrum for both periodic and Dirichlet boundary conditions refining Δx by a factor of 10. For the periodic BC case, we again see purely imaginary eigenvalues approaching $\pm i$ (though, of course, we have more eigenvalues as A is now a larger system). For the Dirichlet case, the eigenvalues again lie slightly inside the negative real half-plane, though in this case closer to the imaginary axis than for the coarser system in Figure 23(b). In fact, the eigenvalues of the Dirichlet problem approach those of the periodic problem in the limit as $\Delta x \to 0$.



(a) Periodic BCs



(b) Dirichlet inflow and upwinded outflow

Fig. 24 $\lambda \Delta t$ distribution for one-dimensional convection example using two different boundary conditions. Note: $\Delta x = \frac{1}{20}$ and Δt set such that $CFL = 1$.

Exercise 3. Download the matlab code for Example 1. Modify this code to compute the eigenvalues of the FTBS method with periodic boundary conditions. For CFL=1 where are there eigenvalues of A?

(a) On the imaginary axis in the range $[-i, i]$
(b) On the real axis in the range $[-1, 1]$
(c) On the real axis in the range $[-2, 0]$
(d) On the unit circle about $(-1, 0)$

In Example 1 we saw that the eigenvalues for the problem with Dirichlet boundary conditions approaches that of the periodic BC case as we refine the mesh. We can take advantage of this property and consider the stability of our finite difference methods without regard for the exact boundary conditions implementation.

Though the eigenvalues of A typically require numerical techniques for the general problem, a special case of practical interest occurs when the matrix is ‘periodic’. That is, the column entries shift a column every row. Thus, the matrix has the form,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_N \\ a_N & a_1 & a_2 & \dots & a_{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{bmatrix}$$

This type of matrix is known as a circulant matrix. Circulant matrices have eigenvalues given by,

$$\lambda_n = \sum_{j=1}^{N} a_j e^{i2\pi(j-1)\frac{n}{N}} \qquad \text{for} \quad n = 0, 1, \dots, N-1 \tag{112}$$

Example 2. As we saw in Example 1, when periodic boundary conditions are assumed, the central space discretization of one-dimensional convection gives purely imaginary eigenvalues, and when scaled by a timestep for which the CFL number is one, the eigenvalues stretch along the axis until $\pm i$. Since for a convection problem with constant velocity and periodic boundary conditions gives a circulant matrix, we can use Equation (112) to determine the eigenvalues analytically. We begin by finding the coefficients, a_j. For a central space discretization, we find,

$$a_2 = -\frac{u}{2\Delta x}, \quad a_N = \frac{u}{2\Delta x}, \text{ and for all other } j, \quad a_j = 0.$$

Then, substituting these a_j into Equation (112) gives,

$$\begin{aligned}
\lambda_n &= -\frac{u}{2\Delta x}e^{i2\pi\frac{n}{N}} + \frac{u}{2\Delta x}e^{i2\pi(N-1)\frac{n}{N}}, \\
&= -\frac{u}{2\Delta x}e^{i2\pi\frac{n}{N}} + \frac{u}{2\Delta x}e^{i2\pi n}e^{-i2\pi\frac{n}{N}}.
\end{aligned}$$

Since $e^{i2\pi n} = 1$ (because n is an integer), then,

$$\begin{aligned}
\lambda_n &= -\frac{u}{2\Delta x}e^{i2\pi\frac{n}{N}} + \frac{u}{2\Delta x}e^{-i2\pi\frac{n}{N}}, \\
&= -\frac{u}{2\Delta x}\left(e^{i2\pi\frac{n}{N}} - e^{-i2\pi\frac{n}{N}}\right), \\
&= -i\frac{u}{\Delta x}\sin\left(2\pi\frac{n}{N}\right).
\end{aligned}$$

Multiplying by the timestep,

$$\lambda_n\Delta t = -i\frac{u\Delta t}{\Delta x}\sin\left(2\pi\frac{n}{N}\right).$$

As observed in Example 1, the eigenvalues are purely imaginary and will extend to $\pm i$ when $CFL = |u|\Delta t/\Delta x = 1$.

Exercise 4. Compute analytically the eigenvalues for the FTBS method with periodic boundary conditions.

Challenge Problem

Consider the 1-D diffusion equation

$$\frac{\partial U}{\partial t} - \mu\frac{\partial^2 U}{\partial^2 x} = 0$$

a) Modify the matlab code from Example 1 to solve the 1-D diffusion problem starting from the same initial condition. Using $\mu = 1$, $\Delta x = \frac{1}{20}$ and $\Delta t = \frac{1}{1000}$ solve the finite difference problem and plot the solution at $t = 0.00, 0.05$ and 0.10. Does the FTCS method appear stable for this problem?
b) Reduce the mesh size and time step by a factor of 2 such that $\Delta x = \frac{1}{40}$ and $\Delta t = \frac{1}{2000}$. Does the FTCS method appear stable for this problem?
c) Determine the analytically the eigenvalues of the FTCS method for 1-D diffusion. Based on your analysis, how should Δt scale with μ and Δx in order to maintain stability? d) Compute numerically the eigenvalues of the FTCS scheme from parts a) and b). Plot $\lambda\Delta t$ and the stability region for forward Euler in the complex plane. Explain your observations from a) and b) in reference to these plots
e) What is the maximum time step that you can use with $\Delta x = \frac{1}{40}$. Plot $\lambda\Delta t$ for this case. Where are the extrema of the eigenvalue spectrum located?
f) Now increase the diffusivity μ by just 1%. Run the finite difference simulation to $t = 0.5$. Explain what happened?