22/10/2020

CATENA DI ISING QUANTISTICA

[fisso $J=1$ e $h>0$]

mi interessa il limite termodinamico $N\to+\infty$

$$\hat{H} = -J\sum_j \hat{\sigma}_j^x\hat{\sigma}_{j+1}^x - h\sum_j \hat{\sigma}_j^z \qquad \text{spin-}1/2$$

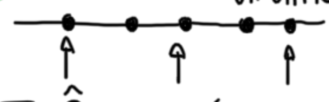
$\downarrow$ J-W

Fermioni

$$(*)\quad \hat{H}_f = -t\sum_j\left(\hat{c}_j^\dagger\hat{c}_{j+1} + \underbrace{\hat{c}_j^\dagger\hat{c}_{j+1}^\dagger + \text{h.c.}}\right) - 2h\sum_j\left(\hat{c}_j^\dagger\hat{c}_j - \frac{1}{2}\right)$$

$$\hat{c}_{j+1}^\dagger\hat{c}_j + \hat{c}_{j+1}\hat{c}_j$$

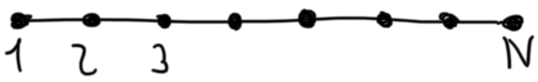
(*1) $J>0$ e $J<0$ equivalenti a meno di un'unitaria



$\hat{U} = \prod_j \hat{R}_{j\,\text{dispari}}(\hat{z},\pi)$

$\hat{\sigma}^x_{j\,\text{dispari}} \to -\hat{\sigma}^x_{j\,\text{dispari}}$

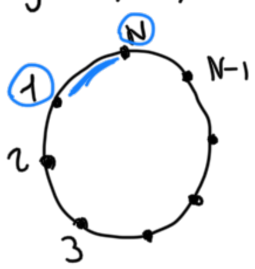
- Condizioni al contorno $\to$ aperte $j=1,\dots,N-1$ nella prima somma



(OBC)

- periodiche $j=1,\dots,N$ nella prima somma $\hat{\sigma}^\alpha_{N+1}\equiv\hat{\sigma}^\alpha_1$ $(\alpha=x,y,z)$

(PBC)



$$\hat{\sigma}_N^x\hat{\sigma}_1^x \xrightarrow{\text{J.-W.}} \Bigg[\underbrace{\Big(\prod_{j<N}\big(\overbrace{-1+2\hat{n}_j}^{(\hat{\sigma}_j^z)}\big)\Big)}_{\text{la stringa rimane } (+1\,|-1)}\left(\hat{c}_N^\dagger+\hat{c}_N\right)\Bigg]\left(\hat{c}_1^\dagger+\hat{c}_1\right)$$

$$(\hat{\sigma}^x = \hat{\sigma}^+ + \hat{\sigma}^-)$$

$\downarrow$

(*2) parità fermionica $(-1)^{N_F}$

(*) $\hat{H}_f$ è quadratica negli operatori $(\hat{c}_j^\dagger, \hat{c}_j)$.

$\to$ diagonalizzazione analitica (rotazione di Bogoliubov)

- se $\hat{H}_f$ è invariante per traslazioni [(t,h) costanti e PBC per $\hat{H}_f$]

$\to$ è utile fare una trasf. di Fourier

$$\hat{d}_k = \frac{1}{\sqrt{N}}\sum_j \hat{c}_j e^{-\frac{2\pi}{N}ikj} \quad ; \quad \hat{d}_k^\dagger = \frac{1}{\sqrt{N}}\sum_j \hat{c}_j^\dagger e^{+\frac{2\pi}{N}ikj}$$

$$\left(k = -\frac{N}{2}+1,\dots,\frac{N}{2}\right)$$

$\hat{d}_k, \hat{d}_k$ sono Fermioni (anticommutano)

$$\Rightarrow \hat{H}_f = -\sum_k\left\{h + f_k\hat{d}_k\hat{d}_k^\dagger - g_k\left(\hat{d}_{-k}\hat{d}_k - \hat{d}_k^\dagger\hat{d}_{-k}^\dagger\right)\right\}$$

$$f_k = 2\left[\cos\left(\frac{2\pi k}{N}\right) - h\right] \quad ; \quad g_k = -i\sin\left(\frac{2\pi k}{N}\right)$$

$$\hat{H}_f = -\sum_k\left\{h + \left(\hat{d}_k^\dagger \;\; \hat{d}_{-k}\right)\underbrace{\begin{pmatrix}0 & g_k\\ -g_k & f_k\end{pmatrix}}_{\text{matrice }2\times2}\begin{pmatrix}\hat{d}_k\\ \hat{d}_{-k}^\dagger\end{pmatrix}\right\}$$

ϑ_k è tale che:

$$\sin\vartheta_k = -i\,\frac{g_k}{\varepsilon_k}$$

$$\cos\vartheta_k = \frac{f_k/2}{\varepsilon_k}$$

$$\Pi_k = \tfrac{1}{2}(\mathbb{1}-\sigma^z) f_k + i g_k \sigma^y$$

$$\hat{R}_x(\vartheta_k) = e^{-i\frac{\vartheta_k}{2}\hat{\sigma}^x} = \cos\left(\frac{\vartheta_k}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\vartheta_k}{2}\right)\hat{\sigma}^x$$

dove $\varepsilon_k = \sqrt{\left(\frac{f_k}{2}\right)^2 - g_k^2} = \sqrt{1+h^2-2h\cos\left(\frac{2\pi k}{N}\right)}$ legge di dispersione dei modi normali

definisco i modi normali: (quasi-particelle)

$$\begin{pmatrix}\hat{b}_k\\ \hat{b}^\dagger_{-k}\end{pmatrix} = R_x(\vartheta_k)\begin{pmatrix}\hat{d}_k\\ \hat{d}^\dagger_{-k}\end{pmatrix} = \begin{cases} u_k \hat{d}_k - i v_k \hat{d}^\dagger_{-k} \\ u_k \hat{d}^\dagger_k - i v_k \hat{d}_k \end{cases}$$

$$\Rightarrow \hat{H}_f = -\sum_k \left\{ h + \begin{pmatrix}\hat{b}^\dagger_k & \hat{b}_{-k}\end{pmatrix}\begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}\begin{pmatrix}\hat{b}_k\\ \hat{b}^\dagger_{-k}\end{pmatrix}\right\} \qquad \begin{pmatrix} u_k = \cos\frac{\vartheta_k}{2} \\ v_k = \sin\frac{\vartheta_k}{2}\end{pmatrix}$$

$$h_{1,2} = \pm i g_k \sin\vartheta_k + \frac{f_k}{2}(1\mp\cos\vartheta_k)$$

alla fine: $\hat{H}_f = 2\sum_k \varepsilon_k\left(\hat{b}^\dagger_k\hat{b}_k - \tfrac{1}{2}\right)$

usare $h_1 - h_2 = 2\varepsilon_k$

$\Downarrow$

$$\hat{H}_f = -\sum_k\left(h + h_1\hat{b}^\dagger_k\hat{b}_k + h_2\hat{b}_{-k}\hat{b}^\dagger_{-k}\right)$$

- stato fondamentale è il vuoto rispetto ai "modi normali" $|\Omega\rangle$ tale che $\boxed{\hat{b}_k|\Omega\rangle = 0 \quad (\forall k)}$

$$E_{g.s.} = -\sum_k \varepsilon_k$$

$$\langle\Omega|\hat{H}|\Omega\rangle = 2\sum_k\varepsilon_k\left(0-\tfrac{1}{2}\right)$$

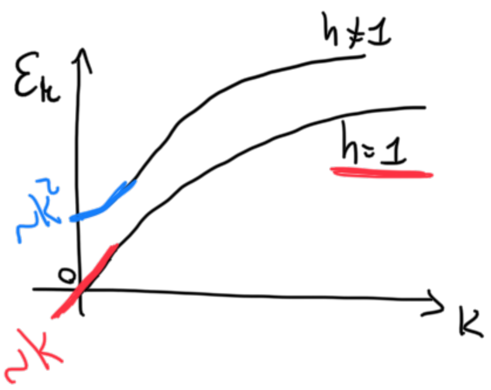
$$\hat{n}_k = \hat{b}^\dagger_k\hat{b}_k \qquad \langle\Omega|\hat{n}_k|\Omega\rangle = 0$$

- 1 stato eccitato: energia minima è per $k=0$ $\quad \varepsilon_{k=0} = |1-h|$

$\quad \| \quad \hat{b}^\dagger_{k=0}|\Omega\rangle$

$$E_1 = E_{g.s.} + 2\varepsilon_{k=0} = \left(-\sum_k\varepsilon_k\right) + 2\varepsilon_0$$

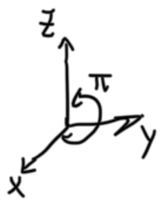
gap 1° e g.s. $\quad \Delta(0) = 2|1-h|$



$$\varepsilon_k = \sqrt{1+h^2-2h\cos\left(\frac{2\pi k}{N}\right)}$$

$$h=1 \Rightarrow \varepsilon_{k=0} = 0$$

(*1) $\hat{U} = \hat{R}_1(z,\pi)\,\hat{R}_3(z,\pi)\,\hat{R}_5(z,\pi),\ldots$



rotazione di π su z

$x \to -x$

$y \to -y$

$$\hat{U}\hat{\sigma}^x_j\hat{U}^\dagger = (-1)^j\hat{\sigma}^x_j$$

$$\hat{U}\hat{\sigma}_j^y\hat{U}^\dagger = (-1)^j\hat{\sigma}_j^y$$
$$\hat{U}\hat{\sigma}_j^z\hat{U}^\dagger = \hat{\sigma}_j^z$$

$$H = -J\sum_j \sigma_j^x\sigma_{j+1}^x - h\sigma_j^z$$
$$\hat{U}\hat{H}\hat{U}^\dagger = +J\sum_j \hat{\sigma}_j^x\hat{\sigma}_{j+1}^x - h\hat{\sigma}_j^z$$
$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{H}_t \qquad \hat{H} \text{ ferro } (J>0)$$
$$\Downarrow$$
$$\hat{H}_t \text{ antiferro } (J<0)$$

Osservabili: $\langle\hat{\sigma}_j^z\rangle$

$|\Omega\rangle$ vuoto rispetto a $\{\hat{b}_k^{(\dagger)}\}$

→ a Temperatura 0 $\quad \langle\Omega|\hat{\sigma}_j^z|\Omega\rangle \qquad \hat{b}_k|\Omega\rangle = 0 \;\; \forall k$

$$\hat{\sigma}_j^z \xrightarrow{J.W.} \{\hat{c}_j^\dagger, \hat{c}_j\} \xrightarrow{F} \{\hat{d}_k^\dagger, \hat{d}_k\} \xrightarrow{Bog.} \{\hat{b}_k^\dagger, \hat{b}_k\}$$

$$\begin{cases} \langle\Omega|\hat{b}_k\hat{b}_{k'}^\dagger|\Omega\rangle = \delta_{kk'} \\ \langle\Omega|\hat{b}_k\hat{b}_{k'}|\Omega\rangle = 0 \\ \langle\Omega|\hat{b}_k^\dagger\hat{b}_{k'}|\Omega\rangle = 0 \\ \langle\Omega|\hat{b}_k^\dagger\hat{b}_{k'}^\dagger|\Omega\rangle = 0 \end{cases}$$

→ sullo stato termico a T finita

$$\delta_{kk'}\langle\hat{n}_k\rangle = \langle\hat{b}_k^\dagger\hat{b}_{k'}\rangle = \delta_{kk'}\cdot\frac{1}{1+e^{\beta\varepsilon_k}} = \delta_{kk'}\left(1-\langle\hat{b}_k\hat{b}_{k'}^\dagger\rangle\right)$$

$$\left(\begin{array}{c} \hat{b}_k|\Omega\rangle = 0 \\ \Updownarrow \\ \langle\Omega|\hat{b}_k^\dagger = 0 \end{array}\right)$$

$$\langle\hat{b}_k\hat{b}_{k'}\rangle = \langle\hat{b}_k^\dagger\hat{b}_{k'}^\dagger\rangle = 0$$

$$\to \langle\Omega|\hat{\sigma}_j^x|\Omega\rangle \sim \langle\Omega|(\hat{c}^\dagger + c)|\Omega\rangle = 0 \qquad \langle\Omega|\hat{c}|\Omega\rangle = 0$$

ma se facciamo $\langle\Omega|\hat{\sigma}_j^x\hat{\sigma}_{j+r}^x|\Omega\rangle \equiv C^{xx}(r)$?

j Non lo metto (assumo invarianza per traslazioni)

$$\hat{\sigma}^x\hat{\sigma}^x \sim (\hat{c}^\dagger + \hat{c})(\hat{c}^\dagger + \hat{c})$$

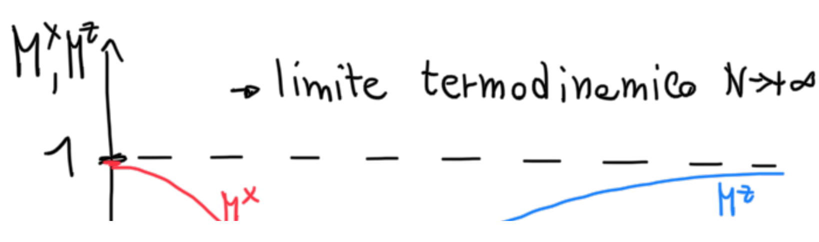
$$M^x \equiv \langle\Omega|\hat{\sigma}_j^x|\Omega\rangle = \left(\lim_{r\to+\infty} C^{xx}(r)\right)^{1/2} \qquad \langle\hat{\sigma}_j^x\hat{\sigma}_{j+r}^x\rangle \overset{r \text{ grande}}{\sim} \langle\hat{\sigma}_j^x\rangle\langle\hat{\sigma}_{j+r}^x\rangle$$

$$C^{xx}(r) = \langle\hat{\sigma}_j^x\hat{\sigma}_{j+r}^x\rangle \xrightarrow{J.W.} \Big\langle(\hat{c}_j^\dagger + \hat{c}_j)\Big[\prod_{j\le \ell<j+r}\underbrace{\hat{n}_\ell^z}_{\hat{c}_\ell^\dagger\hat{c}_\ell}\Big](\hat{c}_{j+r}^\dagger + \hat{c}_{j+r})\Big\rangle$$

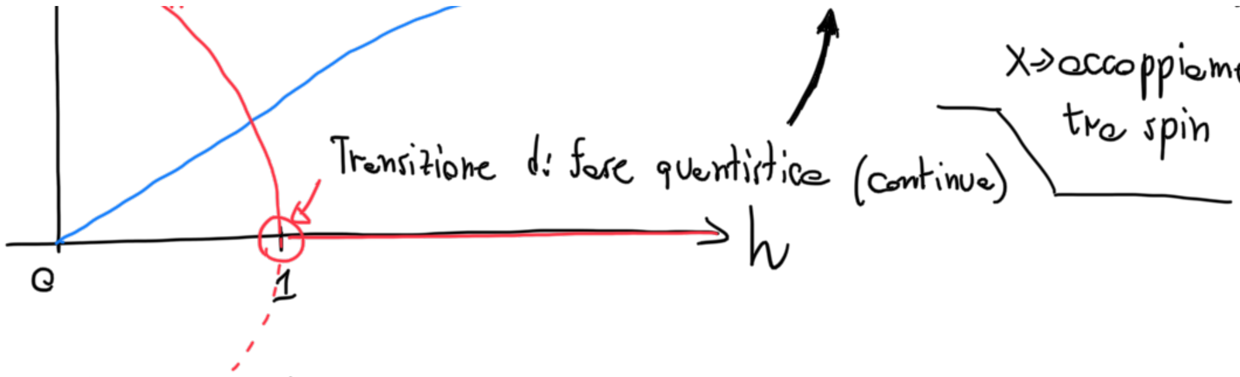
Teorema di Wick

→ determinanti di matrici $O(2N\times 2N)$

→ Pfaffiani (matrici antisimmetriche)



$\boxed{T=0}$ $\qquad z \to$ Campo magnetico



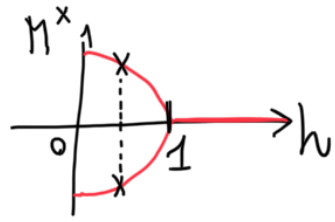
- esponenti critici quantistici in 1D stessi classici in 2D

  Ising quantistico 1D $\leftrightarrow$ Ising classico 2D
  $(h=1)$ $\qquad\qquad$ (T_c)

$$M^x = \begin{cases} (1-h^2)^{1/8} & \text{se } h<1 \\ 0 & \text{se } h>1 \end{cases} \sim |(h+h_c)(h-h_c)|^{1/8} \quad (\beta = 1/8)$$

$h<1$ ferromagnetico (x)

$h>1$ paramagnetico (disordinato x)



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- si possono diagonalizzare Hamiltoniane quadratiche nei fermioni:

$$\hat H = \sum_{ij} \hat c_i^\dagger A_{ij} \hat c_j + \frac{1}{2}\sum_{ij}\left\{ \hat c_i^\dagger B_{ij} \hat c_j^\dagger + h.c.\right\}$$

(Rotazione di Bogoliubov generalizzata)

A hermitiana $(A = A^\dagger)$

B antisimmetrica

$\Rightarrow \hat H^\dagger = \hat H$

$$(*) \quad \hat\sigma_N^x \hat\sigma_1^x = \underbrace{\left(\prod [2\hat n_i - 1]\right)(\hat c^\dagger_N + \hat c_N)}_{\hat\sigma_N^x}\underbrace{(\hat c_1^\dagger + \hat c_1)}_{\hat\sigma_1^x}$$

$$\sigma_N \, \sigma_1 = \underbrace{\left(\prod_{j<N} [2n_j - 1]\right)}\left(c_N^\dagger + c_N\right)\left(c_1^\dagger + c_1\right)$$

$$(2n_1 - 1)(2n_2 - 1) \cdot \ldots \cdot (2n_{N-1} - 1) = \pm \underline{1}$$

correlato con $(-1)^{N_F}$

$$\hat{c}_j^\dagger \hat{c}_j = \hat{n}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{n}_j = \frac{1}{2}\left(\hat{\mathbb{1}} + \hat{\sigma}_j^z\right)$$

$$\hat{\sigma}_j^z = 2\hat{n}_j - \hat{\underline{\mathbb{1}}}$$