# Chapter 11
# Conservation Laws

In the previous chapters we have studied numerical methods for ODEs. In the next couple of chapters we will develop numerical methods for partial differential equations (PDEs), which arise in many different physical systems.

## 41 Self-Assessment

**Before** reading this chapter, you may wish to review...

- Differential Calculus [Reference to wherever this is taught]
- Divergence Theorem [Reference to wherever this is taught]

**After** reading this chapter you should be able to...

- write conservation laws in integral and differential form
- understand the behaviour of a convection equation
- understand the behaviour of a diffusion equation
- understand the behaviour of a convection-diffusion problem and how it varies with the Peclet number

**Relevant self-assessment exercises:** 1

## 42 Conservation Laws in Integral and Differential Form

In most engineering applications, the physical system is governed by a set of conservation laws. For example, the governing equations in gas dynamics correspond to the conservation of mass, momentum, and energy. These conservation laws are often written in integral form for a fixed physical domain. Suppose we have a physical domain, $\Omega$, with the boundary of the domain, $\partial\Omega$. Then, the canonical conservation equation assuming that the physical domain is fixed is of the form,

$$\frac{d}{dt}\int_{\Omega} U\,d\mathbf{x} + \int_{\partial\Omega} \mathbf{F}(U)\cdot\mathbf{n}\,ds = \int_{\Omega} S(U,t)\,d\mathbf{x}, \tag{75}$$

where $U$ is the conserved state, $\mathbf{F}$ is the flux of the conserved state, $\mathbf{n}$ is the outward pointing unit normal on the boundary of the domain, and $S$ is a source term.

This conservation law can be written as a partial differential equation by applying the divergence theorem which states that,

$$\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\Omega} \nabla\cdot\mathbf{F}\,d\mathbf{x}. \tag{76}$$

Thus, Equation 75 becomes,

$$\frac{d}{dt}\int_\Omega U\,d\mathbf{x} + \int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}\,ds = \int_\Omega S\,d\mathbf{x},$$

$$\int_\Omega \frac{dU}{dt}\,d\mathbf{x} + \int_\Omega \nabla\cdot\mathbf{F}\cdot\mathbf{n}\,ds = \int_\Omega S\,d\mathbf{x},$$

$$\int_\Omega \left(\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{F} - S\right) d\mathbf{x} = 0.$$

Since this last equation must be valid for any arbitrary domain, $\Omega$, this means that the integrand must be zero everywhere, or, equivalently,

$$\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{F} = S \tag{77}$$

Equation 77 is the conservation law written as a partial differential equation.

*Example 1.* Conservation of Mass for a Compressible Fluid

One of the simplest examples of a conservation law is the conservation of mass for a compressible fluid. Let the fluid density and velocity be $\rho(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$, respectively. The conservation of mass for the fluid may be written in integral form as:

$$\frac{d}{dt}\int_\Omega \rho\,d\mathbf{x} + \int_{\partial\Omega}(\rho\mathbf{v})\cdot\mathbf{n}\,ds = 0. \tag{78}$$

where (78) has the same form as Equation 75, with the conserved state, $U = \rho$, flux, $\mathbf{F} = \rho\mathbf{v}$ and source, $S = 0$. The differential corresponding differential form for the conservation of mass is:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{79}$$

*Example 2.* Euler Equations for a Compressible Fluid

Often we wish to consider systems of conservation laws. For example the Euler equations governing an inviscid compressible flow correspond to the conservation of mass, momentum, and energy of the fluid. The state $U$, flux $F$, and source $S$ for the two-dimensional Euler equations are,

$$U = \begin{pmatrix}\rho \\ \rho u \\ \rho v \\ \rho E\end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix}\rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH\end{pmatrix}\mathbf{i} + \begin{pmatrix}\rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH\end{pmatrix}\mathbf{j} \qquad S = \mathbf{0}. \tag{80}$$

where

$$\rho = \text{density}$$
$$u = \text{x-component of the velocity}$$
$$v = \text{y-component of the velocity}$$
$$E = \text{total energy per unit mass,}$$
$$p = \text{static pressure,}$$
$$H = E + \frac{p}{\rho} = \text{total enthalpy per unit mass.}$$

The conserved states are the density, $\rho$; x- and y- momenta, $\rho u$ and $\rho v$; and total energy, $\rho E$. Thus, the first row of 80 corresponds to the conservation of mass, the second and third rows correspond to the conservation of $x$ and $y$ momentum respectively, while the fourth row corresponds to conservation of energy. This system of equations is not quite complete, however, since the number of conservation laws does not equal the number of dependent variables in the equations. In particular, note that we have given four conservation equations and the definition of total enthalpy, while the number of dependent variables is six ($\rho$, $u$, $v$, $p$, $E$, and $H$). To complete the set of equations we define an equation of state. Often, we assume an ideal gas and use the ideal gas law. In terms of the dependent variables we have

introduced, the ideal gas law can be written as,

$$p = (\gamma - 1) \left[ \rho E - \frac{1}{2} \rho (u^2 + v^2) \right], \tag{81}$$

where $\gamma$ is the ratio of specific heats (for air, $\gamma \approx 1.4$). You may be more familiar with the ideal gas law in the form, $p = \rho RT$ where $R$ is the gas constant and $T$ is the temperature. Equation 81 is equivalent to $p = \rho RT$ but Equation 81 is used since $p = \rho RT$ introduces a new dependent variable (i.e. the temperature) and would therefore require yet another state equation to complete the system.

**Exercise 1.** The one-dimensional Burger's equation is given in differential form as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

What is the conserved state and corresponding flux?

(a) $U = u$, $\mathbf{F} = u$
(b) $U = u$, $\mathbf{F} = u^2$
(c) $U = \frac{1}{2}u$, $\mathbf{F} = u^2$
(d) $U = u$, $\mathbf{F} = \frac{1}{2}u^2$

# 43 Convection

In many applications, especially those in fluid dynamics, convection is the dominant physical transport mechanism over much of the domain of interest. While diffusion is always present, often its effects are small except in limited regions (often near solid boundaries where boundary layers form due to the combined effects of convection and diffusion).

In this section, we will derive the convection equation using the conservation law as given in Equation 75. Specifically, let $U$ be the 'conserved' scalar quantity and let the fluxes be given by,

$$F = \mathbf{v}U, \qquad S = 0, \tag{82}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the known velocity vector. Note, a non-zero source term could be included, but for simplicity is assumed to be zero. This scalar conservation law may be written as the first order partial differential equation

$$\frac{\partial U}{\partial t} + \nabla \cdot (\mathbf{v}U) = 0. \tag{83}$$

Expanding the spatial derivatives gives,

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U + (\nabla \cdot \mathbf{v})U = 0 \tag{84}$$

Often a reasonable assumption is that the velocity field is divergence free ($\nabla \cdot \mathbf{v} = 0$). In this case, we arrive at what is commonly referred to as the convection equation,

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U = 0. \tag{85}$$

Physically, this equation states that following along the streamwise direction (i.e. convecting with the velocity), the quantity $U$ does not change.

In developing numerical methods for convection-dominated problems, we will often rely on insight that can be gained from the convection equation for the specific case when the velocity field is a constant value. Consider the flow with constant velocity field, i.e. $\mathbf{v}(\mathbf{x},t) = \mathbf{v}$. In this situation, the solution to Equation 85 has the following form,

$$U(\mathbf{x},t) = U_0(\xi) \quad \text{where} \quad \xi = \mathbf{x} - \mathbf{v}t, \tag{86}$$

and $U_0(\mathbf{x})$ is the distribution of $U$ at time $t = 0$. By substitution, we can confirm that this indeed is a solution of Equation 85. Using the chain rule,

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U = \frac{\partial}{\partial t} U_0(\xi) + \mathbf{v} \cdot \nabla_{\mathbf{x}} U_0(\xi)$$
$$= \nabla_\xi U_0 \cdot \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \left( \nabla_\xi U_0 \cdot \nabla_{\mathbf{x}} \xi \right)$$

where $\nabla_{\mathbf{x}}$ and $\nabla_\xi$ denote, respectively, the gradients with respect to $\mathbf{x}$ and $\xi$. The partial derivatives of $\xi$ with respect to $\mathbf{x}$ and $t$ are,

$$\nabla_{\mathbf{x}} \xi = I, \quad \frac{\partial \xi}{\partial t} = -\mathbf{v}, \tag{87}$$

where $I$ is the identity tensor. Upon substitution of these partial derivatives,

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U = \nabla_\xi U_0 \cdot (-\mathbf{v}) + \mathbf{v} \cdot \left( \nabla_\xi U_0 \cdot I \right) \tag{88}$$
$$= \nabla_\xi U_0 \cdot (-\mathbf{v}) + \mathbf{v} \cdot \left( \nabla_\xi U_0 \right) \tag{89}$$
$$= 0. \tag{90}$$

Thus, $U(\mathbf{x},t) = U_0(\mathbf{x} - \mathbf{v}t)$ is a solution to the convection equation.

*Example 3.* One-dimensional Convection

To illustrate the behavior of the convection equation, we consider a simple one-dimensional convection problem

$$\frac{\partial U}{\partial t} + u \frac{\partial U}{\partial x} = 0 \tag{91}$$

on the domain $\Omega = [0, 2]$, with constant flow velocity $u = 1$. The initial distribution $U_0$ is

$$U_0(x) = 0.75 e^{-(\frac{x-0.5}{0.1})^2}$$

Figure 3 plots the initial solution $U_0$ (i.e. at $t = 0$) as well as $U$ at $t = 1$.
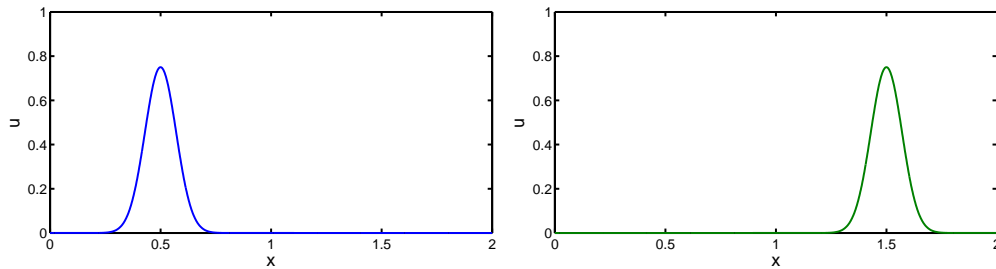


**Fig. 15** Distribution of $U$ at $t = 0$ and at $t = 1$ for a one-dimensional convection problem with velocity $u = 1$.

Looking at the solution for the one-dimensional convection problem we observe that $U(t)$ has the same shape as $U_o$, except the solution has been shifted by a distance $u \times t$. Using the definition of the total derivative, $U$ evolves in time as,

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\frac{dx}{dt}.$$

Comparing this to Equation 91, we see that if $\frac{dx}{dt} = u$, then $\frac{dU}{dt} = 0$ (i.e., $U$ = constant). We call the lines $x(t)$, such that $\frac{dx}{dt} = u$, **characteristic lines** or simply **characteristics**. Figure 43 shows the characteristic lines for the one-dimensional convection problem (in dashed lines) with the solutions at $t = 0$ and $t = 1$ superimposed. As we can see from Figure 43, the solution at $(x, t)$ is simply obtained by following the characteristic line back to $t_0$ and evaluating the initial condition.
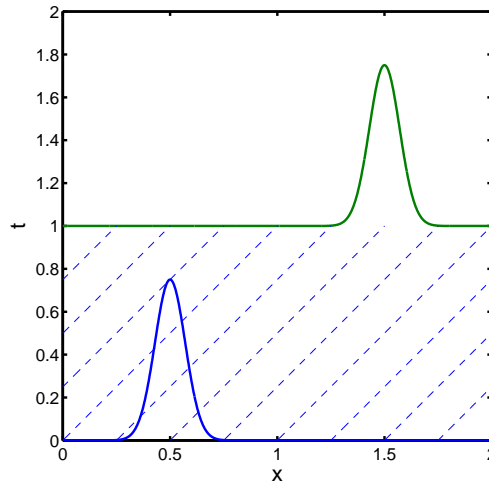


**Fig. 16** Characteristic lines for a one-dimensional convection problem with velocity $u = 1$.

We may extend the idea of characteristics to any conservation law which is a first order partial differential equation. Namely, we consider conservation laws as in Equation 77 where $\mathbf{F}$ and $S$ may be functions of $U$ but not its derivatives. These conservation laws exhibit convection like behavior. The total derivative of $U$ is:

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \nabla U \cdot \frac{d\mathbf{x}}{dt}$$

Along the line $\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{F}}{dU}$ the total derivative of $U$ is:

$$\frac{dU}{dt} = S(U) - \nabla \cdot \mathbf{F} + \nabla U \cdot \frac{d\mathbf{F}}{dU}$$

$$\frac{dU}{dt} = S(U) \tag{92}$$

which is an ordinary differential equation for $U$. As in the simple one-dimensional convection problem, the lines $\mathbf{x}(t)$ which satisfy $\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{F}}{dU}$ are known as the characteristic lines.

Notice, that when there is a source term, $U$ is not constant along a characteristic line. However, we may may evaluate $U$ by simply integrating the ODE in Equation 92 along the characteristic line starting from $U_o$. The solution at a particular point in space and time $U(\mathbf{x}, t)$ depends only upon the points on characteristic line which goes through

the point $(\mathbf{x}, t)$. For any partial differential equation, we call the region which affects the solution at $(\mathbf{x}, t)$ the **domain of dependence**. For convection, the domain of dependence for $(\mathbf{x}, t)$ is simply the characteristic line, $\mathbf{x}(t)$, $s < t$.

**Exercise 2.** Consider the one-dimensional Burger's equation (given in exercise 1). What is the equation for the characteristics?

(a) $\dfrac{dx}{dt} = u$

(b) $\dfrac{dx}{dt} = 2u$

(c) $\dfrac{dx}{dt} = u^2$

(d) $\dfrac{dx}{dt} = \frac{1}{2}u^2$

**Exercise 3.** Consider the one-dimensional Burger's equation on the domain $[0, 1]$ with initial condition $u_0 = \frac{x-1}{2}$. What happens to the characteristic lines at $t$ increases.

(a) The characteristics are parallel
(b) The characteristics converge
(c) The characteristics diverge
(d) All of the above

# 44 Diffusion

In many engineering applications the dominant physical transport phenomenon is modeled as diffusion. This section presents the conservation law for diffusion in differential form and discuss the behavior of problems modeled with diffusion.

Diffusion is characterized by a flux, $\mathbf{F}$, of the form:

$$\mathbf{F} = -\mu \nabla U$$

where $\mu$ is the diffusion coefficient and $U$ is the state. The conservation law for the state, $U$, in a domain $\Omega$ may be written as:

$$\frac{d}{dt} \int_\Omega U \, d\mathbf{x} + \int_{\partial \Omega} \mathbf{F} \cdot \mathbf{n} \, ds = \int_\Omega S(U, t) \, d\mathbf{x},$$

The corresponding differential form is:

$$\frac{\partial U}{\partial t} - \nabla \cdot (\mu \nabla U) = S \tag{93}$$

Equation 93 is a second-order partial differential equation often called the diffusion equation or heat equation. Partial differential equations of the form 93 arise in many applications including molecular diffusion and heat conduction.

*Example 4.* One-dimensional Diffusion

We now illustrate the behavior of the diffusion equation considering a simple one-dimensional model problem. Consider the one-dimensional diffusion equation with constant $\mu$. The diffusion equation simplifies to

$$\frac{\partial U}{\partial t} + \mu \frac{\partial^2 U}{\partial x^2} = 0$$

Figure 4 show the initial solution

$$U_0(x) = 0.75 e^{-\left(\frac{x-0.5}{0.1}\right)^2},$$

as well as the solution after a short time $t = 0.05$. The behavior of the diffusion equation is markedly different from what we have seen for the convection equation. The effect of the diffusion equation is to "smooth" out the conserved state in regions of steep gradients.
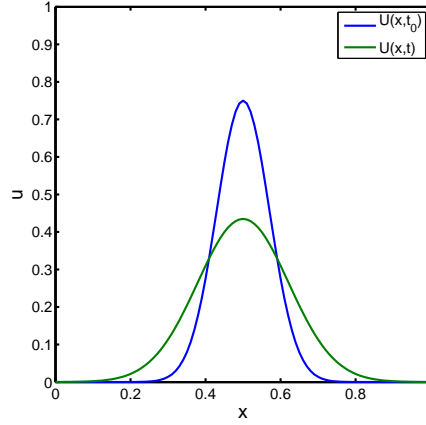


**Fig. 17** One-dimensional diffusion problem, $\mu = 1$

Unlike in the case of convection, (where the domain of dependence is along characteristics), the solution at any point $(\mathbf{x}, t)$ for the diffusion equation depends on the solution everywhere else in the domain. Thus the domain of dependence at any point $(\mathbf{x}, t)$ is the entire domain at all previous time.

## 45 Convection-Diffusion

In practice, both convection and diffusion are important phenomenon governing fluid dynamics. While convection may be the dominant transport mechanism in most of a flow, accounting for diffusion effect near solid boundaries may be essential for accurately computing engineering quantities such as drag.

The convection-diffusion equation is derived from a general conservation law including both convective and diffusive fluxes. In differential form, the convection-diffusion equation is

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U - \nabla \cdot (\mu \nabla U) = S. \tag{94}$$

The convection-diffusion equation arises in many engineering applications ranging from heat transfer to statistical mechanics to financial engineering.

Solutions of the convection-diffusion equation exhibit behavior which is the combined effect of both convection and diffusion. A key question which naturally arises is how significant are the relative effects of convection and diffusion. This is governed by the non-dimensional Peclet number

$$Pe = \frac{\mu}{|\mathbf{v}| L}$$

where $L$ is a characteristic length scale for the problem of interest. When $Pe \ll 1$ diffusion effects are dominant, and convection plays a relatively small role. On the other hand, when $Pe \gg 1$ convection is the dominant transport mechanism.