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The strange case of a Black Hole,
which is classical, and yet has quantum traits

Lo strano caso del Buco Nero,
che è classico con aspetti quantistici

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Contents

Introduction

Our experience of the world is well-described by classical physics; however, the universe seems to be fundamentally quantum. This contradiction is the cornerstone of two of the most debated issues of modern theoretical physics: the description of the quantum to classical crossover and the search for a theory of quantum gravity. Although from different viewpoints, both issues try to answer the same questions: *is it possible to match classical and quantum physics? And if yes, how?* In fact, the approach to the issues above is quite different. In the first case the goal is that of explaining how the classical reality emerges from a quantum underlying structure; in the second, one goes in the opposite direction, trying to find a quantum description for general relativity. In this framework, some objects are attracting special attention: amongst them, Black Holes (BH).

Aim of this thesis is to explore the boundary between classical and quantum mechanics, a goal we plan to accomplish by finding a way to identify systems belonging to this region, here dubbed "quasi-classical", and building an appropriate toolkit to describe them. Once our formal apparatus is complete, we test it in a setting that has always been challenging due to the presence of both quantum and classical features: BHs evaporation.

Coming to the details, we firstly review the group-theoretic construction of Generalized Coherent States (GCS). These are a set of quantum states that can be described referring to a manifold with the same properties of a classical phase-space, and are known to provide a useful tool for describing the quantum to classical crossover. In particular, using these states makes possible to formally show that a globally symmetric quantum system, made of a very large number of subsystems, can be described as a classical system whose phase-space is just the GCS's manifold. The procedure leading to this result, known as the large-N limit, is explored in order to understand how to adapt it to our scopes.

Our quest to explore the bounday between quantum and classical mechanics starts by wondering which microscopic description a macroscopic quantum system should have so as to retain some quantum features even when it behaves as if it were classical. By looking at the microscopic structure of a system we find that it is possible to choose a description of it as a collection of independent quantum subsystems whose collective behaviour is classical. Aside from being easy to deal with, the most valuable feature of such description is the mutual independence of each constituent, as this allows one to perform quantum measurements upon single parts of the global system without destroying its emerging classical character. Moreover, after we find the request for the actual $N \to \infty$ limit not necessary for the emergence of an effective classical description, we show how a system goes from displaying a classical behaviour to feature some genuinely quantum traits while crossing a blurry boundary between two regimes, when repeated quantum measurements are performed.

In the second part of this work we make use of the previously developed tools to describe BHs evaporation. Because of their dual identity as classical objects described by general relativity and quantum systems that can emit thermal radiation, we rekon BHs as being in the quasi-classical regime we defined. As our construction is developed to deal with classical-like systems that still display some quantum features, we use our formal apparatus to define and describe what we call *quantum black hole* (QBH). Thanks to the freedom in choosing its microscopic description, we define a QBH so that it can lose some of its quantum components, namely emitting radiation according to a quantum process, while retaining its classical significance. Moreover, in order to keep the most relevant property of its classical description, we require the QBH to be only partially accessible for measurements by outside observers. In this way we manage to describe Hawking radiation and dynamical BH evaporation. Ultimately, from the observation that measuring BH evaporation imposes a univocal direction to the arrow of time we propose a further development of this work aimed at understanding the role that BH might have in setting the irreversibility of our universe.

Brief outline of the contents

This thesis is made of two parts. In the first part we summarize the theory of GCS and, by exploiting their classical-like structure, we develop a framework that permits to explore the border between quantum and classical mechanics. In the second part, we recap some notions about BH physics and propose our model for BH evaporation.

Part I
In Cap. 1, we define the GCS and we introduce the large-N limit to describe the quantum to classical crossover. In Cap. 2, we familiarize with the pseudospin coherent states, which are the main tool of this thesis. In Cap. 3, we explore the classical to quantum crossover, finding how a classical system, whose phase-space is known, can emerge from a large quantum theory made of non-interacting quantum systems. In fact, we find a lower bound on the size that a quantum system must have to be described as if it were classical.

Part II
In Cap. 4, we briefly introduce results from BH physics and BH thermodynamics, focusing on the features needed in the following chapter: the Hawking temperature and the thermal form of the Hawking radiation. In Cap. 5, we propose our model of QBH: a collection of non-interacting systems whose leading feature is to be only in part accessible to quantum observations. This model allows us to find a description of the Hawking radiation whose traits matches those usually found via quantum field theory in curved spacetime. By combining our results from Cap.3 with our description of the evaporation process, we make our proposal about the last stages of the life of a BH. Lastly, we discuss how the observation of Hawking radiation can provide a direction to the arrow of time.

Conventions and notation

Throughout this thesis, we use the natural units $\hbar = c = k_B = 1$ almost everywhere, and sometimes $G = 1$. The spacetime signature is taken to be $(-,+,+,+)$ and the Dirac's bracket notation is always used when dealing with Hilbert spaces. We denote with the mathematical italic letters $\mathcal{Q}$ and $\mathcal{C}$ quantum and classical theories respectively. Differential manifolds are usually denoted by the letter $\mathcal{M}$ and $\mathcal{N}$ everywhere denotes a classical phase-space. The quanticity parameter, denoted by κ, as well as the parameters δ and ϵ (denoting some observer's accuracies) have usually to be considered small. Quite the opposite, the number of subsystems denoted by N is most likely to be considered big. The greek letter β usually, but not always, denotes the inverse temperature of a system. Lastly, the letters $\mathcal{O}$ and $\mathcal{I}$ denote respectively the exterior and the interior of a black hole.

Acronyms

BH	Black Hole
CP	Carter-Penrose
CSS	Coherent States System
GCS	Generalized Coherent States
GR	General Relativity
PaW	Page and Wootters
QBH	Quantum Black Hole
QM	Quantum Mechanics
SBH	Schwarzschild Black Hole

Part I

From Quantum to Classical (and back)

Chapter 1

The crossover: from Quantum to Classical

The question of how the classical world emerges from a quantum reality has been at the core of a continuous debate since the birth of Quantum Mechanics (QM). For a long time, in the absence of a compelling criterion to tell apart systems described by quantum and classical rules, macroscopicity has been used as a sufficient condition for classicality. However, the advancements in experimental settings have put strain on this idea, and a lot of theoretical work has been consequently developed in order to understand the so-called *quantum to classical crossover*.
In this chapter, we review the apparatus developed in the framework of quantum field theories, and usually referred to as the "Large-N" limit. In particular, we will use the formulation of the procedure proposed by L.G. Yaffe in 1982, as described in Ref. [1] and recently reviewed in Ref. [2]. Aim of this procedure is to elucidate how, and under which conditions, a quantum theory that describes a very large (infinite) number of subsystems can emerge as a classical theory. This is done by exploiting the features of a particular set of quantum states, called the Generalized Coherent States, which are presented in the second section of this chapter. These states are in one-to-one correspondence with points of a differentiable manifold whose structure is identical to that of a classical phase-space. In this sense, GCS are the "most classical" quantum states and make the perfect tool for studying the quantum to classical crossover. In the third section, we review the Large-N limit to show how a classical structure can emerge from the quantum description of a macroscopic system endowed with a global symmetry.
An important role in this discussion is played by Lie algebras, Lie groups and their representations. All the necessary definitions and conventions used in this thesis are reported in Appendix A.

1.1 Quantum Theories and Classical Theories

The first thing we have to do in order to describe the differences between classical and quantum physics is to clearly define what we call a quantum and a classical theory.
In this thesis, we define a *quantum theory* $\mathcal{Q}$ via three basic ingredients: a Lie algebra $\mathfrak{g}$, an irreducible representation $r = (\pi, \mathcal{H})$ of said algebra on a separable Hilbert space, and a Hamiltonian operator $\hat{H} \in \pi(\mathfrak{g})$. The propagators of the theory are elements of a unitary irreducible representation $R = (\Pi, \mathcal{H})$ of the Lie group G, sometimes called *dynamical group*, as obtained from the Lie algebra $\mathfrak{g}$ by the exponential map. We will refer to a quantum theory $\mathcal{Q}$ as the triplet

$$\mathcal{Q} \equiv \{\mathfrak{g}, r = (\pi, \mathcal{H}), \hat{H}\} \ . \tag{1.1}$$

The set of all operators from $\mathcal{H}$ to itself will be indicated by $op(\mathcal{H})$, and the only physically relevant outputs of the theory will be the expectation values

$$O(x) = \langle x| \hat{O} |x\rangle \ , \ \ |x\rangle \in \mathcal{H} \tag{1.2}$$

of the so-called *observable operators*, i.e. the elements $\hat{O} \in op(\mathcal{H})$ that make Eq.(1.2) real, and therefore are hermitian.
We define a *Hamiltonian classical theory*, or *Hamiltonian system*, the triplet

$$\mathcal{C} = \{\mathcal{N}, \omega, h\} \tag{1.3}$$

where $\mathcal{N}$ is a $2n$ dimensional symplectic manifold[1] with ω as symplectic form and h a smooth real function over $\mathcal{N}$, called *Hamiltonian function.* The symplectic form define the *Poisson bracket* of two smooth real functions over $\mathcal{N}$ as

$$\{f, g\}_{\mathrm{PB}} \equiv \omega(X_f, X_g),\ \forall f, g \in C^{\infty}(\mathcal{N}) \ , \tag{1.4}$$

where X_f and X_g are the Hamiltonian vector fields generated by f and g respectively. Once we choose a special chart, called *Darboux chart*, that associate to every $p \in \mathcal{N}$ a point $(q_1, ..., q_n, p_1, ..., p_n) \in \mathbb{R}^{2n}$ for which the symplectic 2-form takes the form

$$\omega = \sum_i dq_i \wedge dp_i \ , \tag{1.5}$$

we get the Poisson brackets in their standard form

$$\{f, g\}_{\mathrm{PB}} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right),\ \forall f, g \in C^{\infty}(\mathcal{N}) \ . \tag{1.6}$$

Real functions on $\mathcal{N}$ are the physically relevant outputs of the theory. Ultimately, the main difference in the mathematical structures (Lie algebras vs. Differential manifolds) we use in the these settings can be brought back to the non-commutativity character of QM.

1.2 Generalized Coherent States

In this section, we introduce the theoretical-group construction of *Generalized Coherent States* (GCS). These states can be understood as a generalization of the well-known bosonic coherent states with which they share some of their most relevant properties.

Definition of the GCS

Let us consider a quantum theory $\mathcal{Q}$ with Lie algebra $\mathfrak{g}$, and let $R = (\Pi, \mathcal{H})$ be a unitary irreducible representation of the corresponding dynamical group G. After having chosen a *reference vector state* $|\psi_0\rangle \in \mathcal{H}$, we can characterize a subgroup $F \subset G$, usually called the *stabilizer subgroup*, by requiring that its elements leave the reference vector state invariant up to a phase factor,

$$\Pi(f) |\psi_0\rangle = e^{i\alpha(f)} |\psi_0\rangle ,\ \forall f \in F \ . \tag{1.7}$$

If we define the set $\{|\psi_g\rangle\}$ as the collection of elements

$$|\psi_g\rangle = \Pi(g) |\psi_0\rangle \ \ g \in G \ , \tag{1.8}$$

it is straightforward to note that, by construction, two vectors $|\psi_{g_1}\rangle$ and $|\psi_{g_2}\rangle$ are equal up to a phase factor $|\psi_{g_1}\rangle = e^{i\alpha} |\psi_{g_2}\rangle$ only if $g_1 = g_2 f$, for $f \in F$. In fact, as

$$|\psi_{g_1}\rangle = |\psi_{g_2}\rangle \ \ \Rightarrow \ \Pi(g_1) |\psi_0\rangle = \Pi(g_2) |\psi_0\rangle \tag{1.9}$$

we can use the properites of group homomorphisms to get

$$\Pi(g_2^{-1})\Pi(g_1) |\psi_0\rangle = e^{i\alpha} |\psi_0\rangle \ \ \Rightarrow \ \Pi(g_2^{-1} g_1) |\psi_0\rangle = e^{i\alpha} |\psi_0\rangle \ . \tag{1.10}$$

[1] A symplectic manifold is a pair $(\mathcal{N}, \omega)$, where $\mathcal{N}$ is an even-dimensional manifold and ω is a closed nondegenerate differential 2-form called symplectic form.

This means that $g_2^{-1} g_1 \in F$ and therefore the statement. Given the coset G/F, a generalized coherent state $|x\rangle$ is then defined as

$$|x\rangle \equiv \Pi(x)\,|\psi_0\rangle \;\;,\;\; x \in G/F \;; \tag{1.11}$$

different elements of the same equivalence class $[x]$ with respect to the isotopy subgroup will be equal to the coherent state $|x\rangle$ times a physically irrelevant phase factor. We call Coherent State System (CSS) the set

$$\mathbb{S} = \{|x\rangle\}_{x\in G/F} \;. \tag{1.12}$$

G/F as a differential manifold

It can be shown, by the *quotient manifold theorem*[2], that elements in G/F are univocally associated to points of a manifold $\mathcal{M}$, via a 1-1 correspondence. Since every element $\Omega \in G/F$ is related to an element of the CSS by definition, we can build a correspondence between $\mathbb{S}$ and $\mathcal{M}$ as in the following diagram

$$\begin{tikzcd} \Omega \in G/F \arrow[r] \arrow[d] & |\Omega\rangle \in \mathbb{S} \arrow[dl, dashed] \\ p_\Omega \in \mathcal{M} & \end{tikzcd} \tag{1.13}$$

Thanks to this correspondence, in what follows we will often refer to the CSS and the manifold $\mathcal{M}$ without distinction.

Complex structure of G/F

Whenever we choose a quantum theory $\mathcal{Q} \equiv \{\mathfrak{g}, (\pi, \mathcal{H}), \hat{H}\}$ with semisimple $\mathfrak{g}$ it is possible to write $\mathfrak{g}$ via the Cartan decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$, where $\mathfrak{f}$ is the Lie algebra associated to the isotopy subgroup F and $\mathfrak{p}$ its orthogonal complement. Given the Cartan-Weyl basis as in (A.22), it is possible to choose the operators $\pi(h_i) = \hat{H}_i$ diagonal and Hermitian, and the so-called *shift operators* $\pi(g_\alpha) = \hat{E}_\alpha$ such that

$$\hat{E}_\alpha^\dagger = \hat{E}_{-\alpha} \;. \tag{1.14}$$

Moreover, we can write any element in $\mathfrak{p}$ as

$$\hat{\Omega} = \sum_\alpha \left(\Omega_\alpha \hat{E}_\alpha - \Omega_\alpha^* \hat{E}_{-\alpha}\right) ,\; \Omega_\alpha \in \mathbb{C} \;, \tag{1.15}$$

and, using the exponential map, the action of G/F on the reference state gives

$$|\Omega\rangle = e^{\sum_\alpha \left(\Omega_\alpha \hat{E}_\alpha - \Omega_\alpha^* \hat{E}_\alpha^\dagger\right)}\,|\psi_0\rangle \;. \tag{1.16}$$

This makes explicit the correspondence mentioned in (1.13): every element of $\mathbb{S}$ can be written in terms of a collection of $m = \dim(G/F)/2$ complex numbers, that are recognised as coordinates of the points $p_\Omega \in \mathcal{M}$.
Once the complex vector $\Omega = (\Omega_1, ..., \Omega_m)$ is assigned, the new coordinates

$$\zeta = \Omega \frac{\sin\sqrt{\Omega^\dagger \Omega}}{\sqrt{\Omega^\dagger \Omega}} \quad \text{for compact } \mathcal{M} \;, \tag{1.17}$$

$$\zeta = \Omega \frac{\sinh\sqrt{\Omega^\dagger \Omega}}{\sqrt{\Omega^\dagger \Omega}} \quad \text{for non}-\text{compact } \mathcal{M} \;; \tag{1.18}$$

[2] The quotient manifold theorem states that a group G that acts smoothly, freely and properly on a manifold $\mathcal{M}$ has a smooth manifold as orbit space. By choosing the stabilizer F as the group acting on G by the right or left multiplication map, we obtain that the orbit space G/F is a smooth manifold. The full discussion can be found in Ref. [3].

can be defined. A further complex projection transform them into the complex projective coordinates

$$\tau = \zeta(1-\zeta^\dagger\zeta)^{-1/2} \quad \text{for compact } \mathcal{M} \ , \tag{1.19}$$
$$\tau = \zeta(1+\zeta^\dagger\zeta)^{-1/2} \quad \text{for non} - \text{compact } \mathcal{M} \ . \tag{1.20}$$

Riemannian structure of G/F

In Ref. [4] it is shown that $\mathcal{M}$ inherits a metric from G/F; when expressed in terms of the complex projective coordinates, this reads

$$ds^2 = g_{\alpha\beta} d\tau^\alpha d\tau^{\beta *} \tag{1.21}$$

with metric tensor

$$g_{\alpha\beta} = \frac{\partial^2 F(\tau,\tau^*)}{\partial\tau^\alpha \partial\tau^{\beta *}} \ ; \tag{1.22}$$

the function $F(\tau,\tau^*)$ is

$$F(\tau,\tau^*) = \log N(\tau,\tau^*), \ N(\tau,\tau^*) = \langle\tilde{\tau}|\,\tilde{\tau}\rangle \tag{1.23}$$

and the vector $|\tilde{\tau}\rangle$ is a non-normalized coherent state obtained from the normalized one via

$$|\Omega(\tau)\rangle\rangle = e^{\sum_\beta(\Omega_\beta(\tau)\hat{E}_\beta + h.c.)}\,|\psi_0\rangle = N^{-\frac{1}{2}} e^{\sum_\beta \Omega_\beta(\tau)\hat{E}_\beta}\,|\psi_0\rangle = N^{-\frac{1}{2}}\,|\tilde{\tau}\rangle \ . \tag{1.24}$$

From the metric, one can obtain the left-invariant measure of the coset space, that reads

$$d\mu(\tau) \propto \det(g)\prod_\alpha d\tau^\alpha d\tau^{*\alpha} \ . \tag{1.25}$$

Symplectic structure of G/F

The coset space is equipped with a symplectic structure that can be transferred to $\mathcal{M}$. The symplectic form on $\mathcal{M}$ has the coordinate representation

$$\omega = -i\sum_{\alpha,\beta} g_{\alpha\beta} d\tau^\alpha \wedge d\tau^{*\beta} \tag{1.26}$$

and can be used to define the Poisson brackets as

$$\{f,g\} = \omega(X_f,X_g) = i\sum_{\alpha,\beta} g^{\alpha\beta}\left(\frac{\partial f}{\partial\tau^\alpha}\frac{\partial g}{\partial\tau^{*\beta}} - \frac{\partial g}{\partial\tau^\alpha}\frac{\partial f}{\partial\tau^{*\beta}}\right) \ . \tag{1.27}$$

To recover their standard form one can transform back τ to the ζ-coordinates and, using

$$\zeta_\beta = \frac{1}{\sqrt{2}}\left(q_\beta + ip_\beta\right), \ q_\alpha, p_\alpha \in \mathbb{R} \ , \tag{1.28}$$

get

$$\{f,g\} = \sum_\alpha\left(\frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p^\alpha} - \frac{\partial g}{\partial q^\alpha}\frac{\partial f}{\partial p^\alpha}\right) \ . \tag{1.29}$$

Overcompleteness of the coherent states

Since $\hat{\Omega}^\dagger = \hat{\Omega}^{-1}$, the elements of $\mathbb{S}$ are normalized

$$\langle\Omega|\,\Omega\rangle = \langle\psi_0|\,\hat{\Omega}^\dagger\hat{\Omega}\,|\psi_0\rangle = \langle\psi_0|\,\psi_0\rangle = 1\ , \tag{1.30}$$

but non-orthogonal

$$\langle\Omega|\,\Omega'\rangle = \langle\psi_0|\,\hat{\Omega}^{-1}\hat{\Omega}'\,|\psi_0\rangle = \langle\psi_0|\,\hat{\Omega}''\,|\psi_0\rangle \neq 0\ . \tag{1.31}$$

By using the invariance of the group measure $d\mu(\Omega)$ and the Shur's lemma it is possible to show that $\mathbb{S}$ resolves the identity on $\mathcal{H}$,

$$\mathbb{I} \propto \int_{G/F} d\mu(\Omega)\,|\Omega\rangle\,\langle\Omega|\ . \tag{1.32}$$

For these reasons, GCS are said to be *overcomplete*.

1.3 The large-N limit

Despite related to points on a classical phase-space, states on the CSS manifold should not be considered classical states even if only because they are not orthogonal.
In this section, we review a procedure to exploit the CSS manifold to obtain a formal description of the quantum to classical crossover, called large-N limit. Given a family of quantum theories, each equipped with a CSS, the idea is that of finding what mathematical objects stay well-defined, with the physical properties they represent remaining meaningful, when a positive real parameter that labels each theory, and is called *quanticity parameter*, vanishes. In Ref. [1], Yaffe lists some assumptions that must be made upon this family of quantum theories if a classical structure is to emerge when the quanticity parameter goes to zero. Following the procedure outlined in Ref. [2], we will show how the classical description of a macroscopic quantum system (e.g. with a considerable number N of *d.o.f.*) is constructed, by establishing a connection between a single large-N quantum theory (featuring a global symmetry) and the elements of the aforementioned family. We underline that, despite performing the limit be necessary for this construction, most of the steps taken retain their validity when the quanticity parameter stays finite. This consideration is what will lead us, in the 3rd chapter, to develop a finite-N description of the procedure which is presented here for $N \to \infty$.
Before getting into details, let us briefly discuss what N refers to in this thesis. In literature N often describes the number of dynamical *d.o.f.* of the system, thus we retain this definition within this chapter. A similar but more precise definition it is to call N the number of components a quantum system has. Even if these definitions may coincide (e.g. if at each component is associated only one *d.o.f.*) we will use the latter in most of this thesis, as better fits our scopes.

1.3.1 The classical limit of a family of quantum theories

Let $\kappa \in \mathbb{R}^+$ be a positive number called quanticity parameter. We define a family of quantum theories $\{\mathcal{Q}_\kappa\}$ in which any element:

- shares the same Lie algebra $\mathfrak{g}$ with every other element of the family.
- may be represented by a different homomorphism on a different Hilbert space for each value of κ,
$$t_\kappa : \mathfrak{g} \to \mathcal{H}_k\ . \tag{1.33}$$
- has Hamiltonian $\hat{H}_\kappa$, depending on the quanticity parameter.

With these choices and referring to the definition (1.1), a specific value of κ defines the quantum theory

$$\mathcal{Q}_\kappa \equiv \{\mathfrak{g}, r_\kappa, \hat{H}_\kappa\}, \text{ with } r_\kappa = (t_\kappa, \mathcal{H}_\kappa) \ . \tag{1.34}$$

In each $\mathcal{Q}_\kappa$ we select a reference $|0_\kappa\rangle \in \mathcal{H}_\kappa$ and build the CSS $\mathbb{S}_\kappa = \{|\Omega_\kappa\rangle\}$ whose elements are in 1-1 correspondence with points on the manifold $\mathcal{M}_\kappa$, as described in the previous section. Given any operator $\hat{A}_\kappa : \mathcal{H}_\kappa \to \mathcal{H}_\kappa$, we define its *symbol* as

$$A_\kappa(\Omega_\kappa) = \langle\Omega_\kappa| \hat{A}_\kappa |\Omega_\kappa\rangle \tag{1.35}$$

and a set $\mathcal{K}$ of operators, dubbed *classical operators*, as the collection of operators such that

$$\lim_{\kappa\to 0} \frac{\langle\Omega_\kappa| \hat{A}_\kappa |\Omega'_\kappa\rangle}{\langle\Omega_\kappa| \Omega'_\kappa\rangle} < \infty, \quad \forall\, |\Omega_\kappa\rangle\,, |\Omega'_\kappa\rangle \in \mathbb{S}_\kappa \ . \tag{1.36}$$

By means of the equivalence relation

$$|\Omega_\kappa\rangle \sim |\Omega'_\kappa\rangle \text{ if } \lim_{\kappa\to 0} A_\kappa(\Omega_\kappa) = \lim_{\kappa\to 0} A_\kappa(\Omega'_\kappa) < \infty, \quad \forall \hat{A}_\kappa \in \mathcal{K} \tag{1.37}$$

we build the quotient space $\cup_\kappa \mathbb{S}_\kappa / \sim$ which connects CSS of $\mathcal{Q}_\kappa$ for different values of κ. We indicate the equivalence classes obtained thorough $\sim$ by

$$[|\Omega_\kappa\rangle]_\sim \equiv |\Omega\rangle_\kappa \ . \tag{1.38}$$

This notation makes it explicit the fact that when ordered by means of Eq.(1.37) the states belonging to different theories $\mathcal{Q}_\kappa$ can be labelled by the same complex number parametrizing the points of the CSS manifold.

Assumptions upon the $\kappa \to 0$ limit

In this paragraph, we list the assumptions that must be made upon the collection $\{\mathcal{Q}_\kappa\}$ in order to obtain a well defined classical theory when κ goes to zero.

A1

Each representation $R_\kappa = (\Pi_\kappa, \mathcal{H}_\kappa)$ of the group G acts irreducibly on its Hilbert space.

Using Shur's lemma it is possible to show that this hypothesis forces the identity to be a sum of projectors weighted by the left-invariant group measure

$$\hat{\mathbb{I}}_\kappa = c_\kappa \int_{(G/F_\kappa)/\sim} d\mu(\Omega)\, |\Omega\rangle \langle\Omega|_\kappa \,, \tag{1.39}$$

where c_κ is a constant that depends on the normalization of the group measure.

A2

Zero is the only operator whose symbol identically vanishes.

This assumption implies that two different operators cannot have the same symbol and allows one to establish a correspondence between operators and symbols, that become functions on the classical manifold when $\kappa \to 0$.

A3

The function

$$\phi(\Omega,\Omega') = \lim_{\kappa\to 0} \phi(\Omega,\Omega')_\kappa = -\lim_{\kappa\to 0} \kappa \log \langle \Omega_\kappa | \, \Omega'_\kappa \rangle \tag{1.40}$$

exists for every $|\Omega_\kappa\rangle, |\Omega'_\kappa\rangle$ and satisfies

- $\mathfrak{Re}[\phi(\Omega,\Omega')] > 0$ when $|\Omega_\kappa\rangle \nsim |\Omega'_\kappa\rangle$.
- $\mathfrak{Re}[\phi(\Omega,\Omega')] = 0$ when $|\Omega_\kappa\rangle \sim |\Omega'_\kappa\rangle$.

This means that when $\kappa \to 0$ the overlap of inequivalent coherent states goes to zero as

$$\langle \Omega_\kappa | \, \Omega'_\kappa \rangle \simeq e^{-\frac{\phi(\Omega,\Omega')_\kappa}{\kappa}} \; . \tag{1.41}$$

A3 also implies that

$$\lim_{\kappa\to 0} [(AB)_\kappa(\Omega_\kappa) - A_\kappa(\Omega_\kappa) B_\kappa(\Omega_\kappa)] = 0 \quad \forall \hat{A}, \hat{B} \in \mathcal{K} \text{ and } \forall \, |\Omega_\kappa\rangle \;\; , \tag{1.42}$$

which means that the symbols of any two operators commute in the $\kappa \to 0$ limit.

A4

$\kappa \hat{H}_\kappa$ is a classical operator: coupling constants in the Hamiltonian are scaled to be finite as κ vanishes.

If the assumptions **A1** through **A4** are made, the family of quantum theories $\{\mathcal{Q}_\kappa\}$ flow into the classical theory

$$\mathcal{C} = \{\mathcal{M}, \omega, h\} \tag{1.43}$$

as $\kappa \to 0$, where $\mathcal{M}$ is the CSS manifold defined in (1.13), ω is the symplectic form defined in (1.26) and h is a Hamiltonian function given by

$$h(p_\beta, q_\beta) = \lim_{\kappa\to 0} \kappa \hat{H}_\kappa(\Omega) \; , \tag{1.44}$$

with (p_β, q_β) the coordinates defined in (1.28).

1.3.2 Orthonormal states and the classical limit

Following Ref. [5], we here describe what happens to the overlaps between elements of the CSS, $\{|\Omega\rangle\}$, and elements of orthonormal bases, $\{|\xi\rangle\}$, in the classical limit. The square modulus of $\langle \Omega | \, \xi \rangle$ can be interpreted as the probability of observing the element $|\xi\rangle$ of the basis when measuring $|\Omega\rangle$ by a projective measurement identified by $\{|\xi\rangle\}$. Since the GCS are overcomplete, for any $\kappa \neq 0$, this probability is always finite. By defining S_ξ as the set of points on $\mathcal{M}$ where $|\langle \Omega \, | \xi \rangle| > 0$, this property can be formally stated as

$$S_\xi \cap S_{\xi'} \neq 0 \; , \quad \forall \kappa \neq 0 \tag{1.45}$$

However, it is possible to show that when $\kappa = 0$ these overlaps vanish. In fact, being $|\xi\rangle$ and $|\xi'\rangle$ orthonormal, using (1.39) we can write

$$\delta_{\xi\xi'} = \langle \xi | \, \xi' \rangle = c_\kappa \int_{\mathcal{M}} d\mu(\Omega) \langle \xi \, | \Omega \rangle \, \langle \Omega | \, \xi' \rangle \tag{1.46}$$

for any value of κ, including $\kappa = 0$. On the other hand, in Ref. [5] it is shown that **A3** can be written as

$$\lim_{\kappa\to 0} \frac{1}{\kappa} | \, \langle \Omega | \, \Omega' \rangle|^2 = \delta(\Omega - \Omega') \; ; \tag{1.47}$$

therefore, one can fix c_κ as the inverse of κ by using

$$1 = \langle\Omega|\,\Omega\rangle = \lim_{\kappa\to 0} c_\kappa \kappa \int_{\mathcal{M}} d\mu(\Omega)\delta(\Omega-\Omega') \;\Rightarrow\; c_\kappa = \frac{1}{\kappa}\;, \tag{1.48}$$

and take the limit of Eq.(1.46) to find

$$\delta_{\xi\xi'} = \lim_{\kappa\to 0}\int_{\mathcal{M}} \frac{1}{\kappa} d\mu(\Omega)\langle\xi\,|\Omega\rangle\,\langle\Omega|\,\xi'\rangle\;, \tag{1.49}$$

which is only possible if the two overlaps are never simultaneously finite on a set of finite measure. A correspondence between elements of orthonormal bases and sets on $\mathcal{M}$ is therefore established: orthogonality of vectors $|\xi\rangle$ is translated into the disjointness of the sets S_ξ.

1.3.3 Symmetries and reduction of *d.o.f.*

We say that

$$\mathcal{Q} = \{\mathfrak{g}, (\phi,\mathcal{H}), \hat{H}\} \tag{1.50}$$

has a symmetry when it is possible to select a group $\mathbb{X}$ (that defines the symmetry) whose elements $\mathcal{U}$, when represented on $\mathcal{H}$, commute with all the relevant operators of the theory

$$[\hat{\mathcal{U}}, \hat{A}] = 0,\ \forall\hat{\mathcal{U}}\ . \tag{1.51}$$

When this is the case, one can define the equivalence relation of the symmetric vectors under the action of $\mathbb{X}$

$$|x\rangle \overset{\mathbb{X}}{\sim} |x'\rangle \quad\text{if}\quad |x'\rangle = \hat{\mathcal{U}}\,|x\rangle\,,\ \hat{\mathcal{U}}\in\mathbb{X}\ , \tag{1.52}$$

and the related equivalence classes

$$[|x\rangle]_{\mathbb{X}} = \{|x'\rangle\in\mathcal{H} \text{ such that } |x'\rangle \overset{\mathbb{X}}{\sim} |x\rangle\}\ . \tag{1.53}$$

The dimension of the quotient space will be equal to the dimension of the full Hilbert space minus the dimension of the orbit spaces of the group action: we can reduce the *d.o.f.* of the theory by collapsing every equivalent state into its representative

$$|x\rangle\in\mathcal{H} \longrightarrow [|x\rangle]_{\mathbb{X}} \in \mathcal{H}/\overset{\mathbb{X}}{\sim}\ . \tag{1.54}$$

A symmetry is said to be *global* if it acts on the whole target Hilbert space and its action is the same on every vector.

1.3.4 The large-N limit and the global symmetry

We are now ready to build the strategy that aims at finding the classical limit of a macroscopic quantum theory $\mathcal{Q}_N$. In principle, we may proceed by starting from the dynamical group G_N and then construct the related GCS $|\Omega_N\rangle$ and the manifold $\mathcal{M}_N$. However, as far as we don't have any way to reduce the *d.o.f.* of the theory, this scheme might not work since it can be hard to deal with a possibly very large number of variables.
As mentioned in the previous paragraph, the way to reduce the number of *d.o.f* of the theory is provided by the presence of a global symmetry $\mathbb{X}$. In fact, any large-N quantum theory with a symmetry defines a family $\{\mathcal{Q}_\kappa\}$ of theories with the same algebra but represented onto smaller Hilbert spaces. By asking the relation between N and κ to be

$$\kappa = \frac{1}{N^\alpha}, \tag{1.55}$$

with α a positive number that depends on the properties of $\mathcal{Q}_N$, we obtain that κ goes to zero as N grows to infinity. Defining

$$|x\rangle_N \in \mathcal{H}_N/\overset{\mathbb{X}}{\sim}\ , \tag{1.56}$$

the correspondence is established in a way such that

$$\lim_{N\to\infty} \langle x_N | \hat{A}_N | x_N \rangle = \lim_{\kappa\to 0} \langle \Omega_\kappa | \hat{A}_\kappa | \Omega_\kappa \rangle = A(\Omega) \tag{1.57}$$

for any relevant operator defined on $\mathcal{H}_N$. Note that it is possible to define equivalence classes *w.r.t.* the symmetry for any value of N, and then group the classes for different values of N under the same name, thus factoring out the label N from definition (1.56), as

$$[|x_N\rangle]_{\underset{\sim}{\mathbb{X}}} \equiv |x\rangle_N \ . \tag{1.58}$$

This is equivalent to what we have done in (1.38) for the GCS equivalence classes as from different $\mathcal{Q}_\kappa$ theories. In this manner, the equivalence classes provided by the symmetry take the place of those defined in (1.37), and it is possible to show that they satisfy the four assumptions **A1**-**A4**. In conclusion, when the $N \to \infty$ limit is performed, a classical theory emerges from $\mathcal{Q}_{N\to\infty}$. This theory is identical to that emerging from $\mathcal{Q}_{\kappa\to 0}$, which means that it is defined on a symplectic manifold $\mathcal{N}$ that has the same geometrical and topological properties of $\mathcal{M}_{\kappa\to 0}$, as well as the same symplectic structure, with Poisson brackets defined as in (1.29) and a Hamiltonian function

$$h(q,p) = \lim_{N\to\infty} \kappa(N) H_N(x) \ , \tag{1.59}$$

as defined in **A4**, when $\kappa(N)$ is fixed by (1.55).

Chapter 2

Spin and pseudo-spin coherent states

In this chapter, we show how to build the CSS manifold for two selected Lie algebras: $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. They both are well-known examples of GCS and are widely discussed in the literature. While we report upon $\mathfrak{su}(2)$ just as to provide a clear example, the algebra $\mathfrak{su}(1,1)$ is fundamental in this thesis as a tool for investigating quantum features of BHs.

Before proceeding, let us highlight some aspects that we did not mention in the previous chapter. The two main approaches to the CSS construction are those proposed by Perelomov (see i.e. Ref. [6]) and Gilmore (see i.e. Ref. [4]). In Ref. [4] is also presented an analysis of the differences between the two approaches, which is summarized in different choices of three basic elements:

- the group structure G, that in this thesis we require to be a Lie group (or equivalently, the algebra structure $\mathfrak{g}$, that we require to be a Lie algebra).
- the irreducible representation $r = (\phi, \mathcal{H})$, that we here demand to be unitary.
- the reference state vector, that it is here chosen as the lowest-weight vector of $\mathcal{H}$.

For any given quantum theory $\mathcal{Q}$, the first two choices are often defined *a priori* by physical motivations: G is determined by the algebraic properties of the Hamiltonian and the space of the physical states determines the carrier space of the unitary irreducible representation r. Therefore, the choice of the reference state vector is what ultimately characterizes the construction, since it determines the structure of the CSS and the geometry of the manifold (thanks to the 1-1 correspondence given in (1.13)). As we will see at the end of this chapter, this choice also permits to easily treat quantum theories with more general Hamiltonians (that is, not only those that can be written as the representation of an element of the Lie algebra).

In the first section of this chapter, we present the coherent states of the Lie algebra $\mathfrak{su}(2)$, called *spin coherent states*. In the second section, we describe those relative to the $\mathfrak{su}(1,1)$ Lie algebra, called *pseudo-spin coherent states*. In both cases, we follow the steps presented in Cap. 1. We then present one of the main tools for this thesis: the *two-modes representation* of the pseudo-spin coherent states. In the third section, we briefly analyze how the choice of the reference state vector as the lowest-weight state permits, under certain conditions, to consider quantum theories with Hamiltonians that are finite polynomials of the Lie algebra's generators. This is to make it clear that the large-N limit can actually generate a range of classical Hamiltonians which is wider than that obtained only considering Hamiltonian operators obtained from the Lie algebra.

2.1 The spin coherent states

The Lie algebra $\mathfrak{su}(2)$ is the space of vectors $\mathbf{J} = (J_0, J_1, J_2)$ with the Lie brackets

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ . \tag{2.1}$$

We can choose the linear combinations $J_\pm = J_1 \pm iJ_2$ to build a Cartan-Weyl basis of $\mathfrak{su}(2)$, that satisfies

$$\begin{cases} [J_+, J_-] &= 2J_0 \\ [J_0, J_\pm] &= \pm J_\pm \ . \end{cases} \tag{2.2}$$

The irreducible representations $(\phi_j, \mathcal{H}_j)$ of this algebra are labelled by a half-integer number j, associated with the Casimir operator according to

$$\hat{\mathbf{J}}^2 = j(j+1)\mathbb{I}_{\mathcal{H}_j} \ ; \tag{2.3}$$

once the j-representation is fixed, we call m the eigenvalue of the diagonal element of the Cartan-Weyl basis

$$\hat{J}_0 \left|j, m\right\rangle = m \left|j, m\right\rangle \ . \tag{2.4}$$

The action of the shift operators follows from that of $\hat{\mathbf{J}}^2$ and $\hat{J}_0$; the Hilbert space can be built from any element $|j, m\rangle$ by applying these operators as

$$\mathcal{H}_j = \{\left|j, k\right\rangle, \ m = -j, ..., j\} \ . \tag{2.5}$$

The dimension of such space is fixed by the Casimir's eigenvalue as $\dim(\mathcal{H}_j) = 2j + 1$. As described in Sec. 1.2, the choice of $|j, m = -j\rangle$ as reference state vector automatically selects the isotopy subgroup by

$$e^{i\delta\hat{J}_0} \left|j, -j\right\rangle = e^{-i\delta j} \left|j, -j\right\rangle, \ \delta \in \mathbb{R} \tag{2.6}$$

as $F = U(1)$, and consequently the coset space as $SU(2)/U(1)$. The *spin coherent states* are defined as

$$\left|\Omega\right\rangle = \hat{\Omega} \left|j, -j\right\rangle \ , \tag{2.7}$$

where $\hat{\Omega}$ is

$$\hat{\Omega} = e^{\Omega\hat{J}_+ - \Omega^*\hat{J}_-} \ . \tag{2.8}$$

In order to show that the manifold $\mathcal{M}$ associated with the coset space $SU(2)/U(1)$ is isomorphic to a sphere, we work with $j = 1/2$, for the sake of simplicity. In this representation the elements of $\mathfrak{su}(2)$ are given by

$$\hat{J}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{J}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{J}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.9}$$

and $\hat{\Omega}$ is

$$\hat{\Omega} = \exp\begin{pmatrix} 0 & \Omega \\ -\Omega^* & 0 \end{pmatrix} = \begin{pmatrix} \cos|\Omega| & \frac{\Omega}{|\Omega|}\sin|\Omega| \\ \frac{-\Omega^*}{|\Omega|}\sin|\Omega| & \cos|\Omega| \end{pmatrix} \ . \tag{2.10}$$

By introducing the polar coordinates

$$\Omega = \frac{\theta}{2}e^{-i\phi} \quad \text{with} \quad (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \ , \tag{2.11}$$

we identify the points of $SU(2)/U(1)$ with points of on a sphere S_2, called *Bloch sphere*. Using the changes of variables (1.17) and (1.19) for compact coset spaces

$$\Omega(\theta, \phi) \ \longrightarrow \ \zeta(\theta, \phi) = \sin\frac{\theta}{2}e^{-i\phi} \ \longrightarrow \ \tau(\theta, \phi) = \tan\frac{\theta}{2}e^{-i\phi} \ . \tag{2.12}$$

the coherent states can be expressed via the stereographic-projection coordinates $\tau \in \mathbb{C}$ as

$$\left|\tau\right\rangle = \hat{\tau}\left|j, -j\right\rangle = \frac{1}{(1 + |\tau|^2)^j}e^{\tau\hat{J}_+}\left|j, -j\right\rangle \ , \tag{2.13}$$

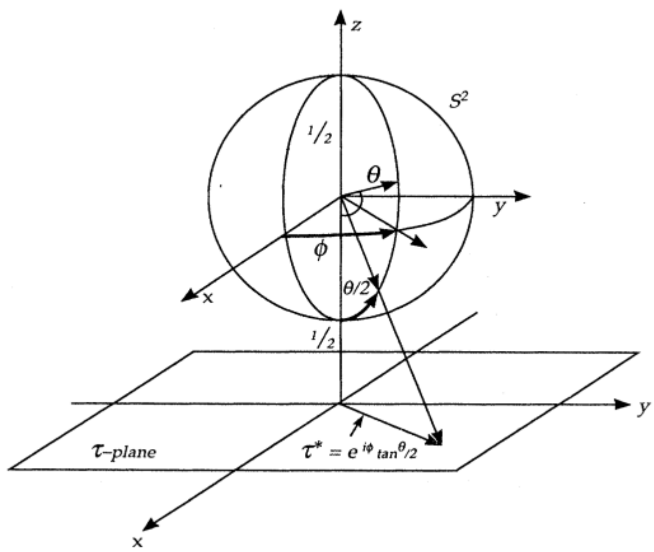


Figure 2.1: Bloch sphere and its stereographic projection (from Ref. [4]).

where we have restored the j-dependence and used the BCH formula for $\mathfrak{su}(2)$ (see Ref. [4]). The form (2.13) has the advantage to make explicit the term $N(\tau,\tau^*)$, that is necessary to get the coset measure. The metric on the manifold is obtained using (1.21-1.23) from

$$F(\tau,\tau^*) = \log \langle\tilde{\tau}|\,\tilde{\tau}\rangle = \log(1+|\tau|^2)^{2j} \tag{2.14}$$

as

$$g(\tau,\tau^*) = \frac{2j}{(1+|\tau|^2)^2} \tag{2.15}$$

and the coset measure is

$$ds^2 = \frac{2j}{(1+|\tau|^2)^2} d\tau d\tau^* \;\Rightarrow\; d\mu = \frac{2j+1}{4\pi}\frac{d\tau d\tau^*}{(1+|\tau|^2)^2} \tag{2.16}$$

or equivalently,

$$ds^2 = \frac{j}{2}(d^2\theta + \sin^2\theta d^2\phi) \;\Rightarrow\; d\mu = \frac{2j+1}{4\pi} d\theta \sin\theta d\phi \;. \tag{2.17}$$

The decomposition of a coherent state onto the $\{|j,m\rangle\}$ basis is

$$|\tau\rangle = \sum_{n=-j}^{j} c_n(\tau)\,|j,n\rangle \tag{2.18}$$

with

$$c_n(\tau) = \langle j,n|\,\tau\rangle = \begin{pmatrix} 2j \\ j+n \end{pmatrix}^{1/2} (1+|\tau|^2)^{-j}\tau^{j+n} = \tag{2.19}$$

$$= \begin{pmatrix} 2j \\ j+n \end{pmatrix}^{1/2} \left(\cos\frac{\theta}{2}\right)^{j-n} \left(\sin\frac{\theta}{2}\right)^{j+n} e^{-i\phi(j+n)} \;. \tag{2.20}$$

The Poisson brackets in the form (1.29) are obtained by defining the (q,p) coordinates as

$$\zeta = \frac{1}{\sqrt{4j}}(q-ip),\; \zeta^* = \frac{1}{\sqrt{4j}}(q+ip) \;. \tag{2.21}$$

The overlap of two different coherent states is

$$\langle\Omega|\,\Omega'\rangle = \sum_{n=-j}^{j} c_n(\Omega)^* c_n(\Omega') = \left[\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{-i(\phi-\phi')}\right]^{2j} \tag{2.22}$$

in (θ, ϕ)-coordinates, or

$$\langle \tau | \tau' \rangle = \frac{(1+\tau^* \tau')^j}{(1+|\tau|^2)^j (1+|\tau'|^2)^j} \tag{2.23}$$

in τ-coordinates (these formulas are prooved in Appendix B, where we present a way to find the overlap of two coherent states that holds for any Lie algebra of rank 1).

2.2 The pseudo-spin coherent states

In this section, we repeat the same path we have followed in Sec. 2.1 but for a different algebra: $\mathfrak{su}(1,1)$. This algebra is defined by the space of vectors $\mathbf{K} = (K_0, K_1, K_2)$ with the Lie brackets

$$[K_\alpha, K_\beta] = i\epsilon_{\alpha\beta\gamma} K^\gamma \ , \tag{2.24}$$

where the indices α, β, γ are raised and lowered by the 3-dimensional Minkowski metric and the sum is expressed using Einstein's notation. By the linear combinations $K_\pm = K_1 \pm iK_2$ we can obtain the Cartan-Weyl basis $\{K_0, K_+, K_-\}$ with commutators given by

$$\begin{cases} [K_+, K_-] = -2K_0 \\ [K_0, K_\pm] = \pm K_\pm \ . \end{cases} \tag{2.25}$$

The Casimir element

$$\mathbf{K}^2 = -K_\alpha K^\alpha = K_0^2 - K_1^2 - K_2^2 = \tag{2.26}$$

$$= K_0^2 - \frac{1}{2}(K_+ K_- + K_- K_+) \tag{2.27}$$

can be used to classify the irreducible representations[1] $(\phi_k, \mathcal{H}_k)$ by means of its eigenvalues: being

$$\hat{\mathbf{K}}^2 = k(k-1)\mathbb{I}_{\mathcal{H}_k} \tag{2.28}$$

we use $k \in \mathbb{R}^+$, called *Bargmann index*, to label the representations and their carrying Hilbert spaces

$$\mathcal{H}_k = \{|k; m\rangle \ , \ m \in \mathbb{N}\} \ . \tag{2.29}$$

In Eq.(2.29) m denotes the eigenvalue of $\hat{K}_0$,

$$\hat{K}_0 |k; m\rangle = (k+m) |k; m\rangle \ . \tag{2.30}$$

The action of $\hat{K}_\pm$ follows from the commutators and from Eqs. (2.29-2.30). After choosing a k-representation and a reference state vector $|k, 0\rangle$ in $\mathcal{H}_k$, we build the CCS as in Sec. 1.2. The isotopy subgroup is recognized, by

$$e^{i\delta \hat{K}_0} |k; 0\rangle = e^{i\delta k} |k; 0\rangle \ , \ \delta \in \mathbb{R} \ , \tag{2.31}$$

as $F = U(1)$, and therefore the coset space is $SU(1,1)/U(1)$. The *pseudo-spin coherent states* are obtained from $|0\rangle$ by the action of

$$\hat{\Omega} = e^{\Omega \hat{K}_+ - \Omega^* \hat{K}_-} \tag{2.32}$$

and are denoted by

$$|\Omega\rangle = \hat{\Omega} |k; 0\rangle \ . \tag{2.33}$$

[1] Since the $\mathfrak{su}(1,1)$ algebra is non-compact, its unitary irreducible representations are infinite-dimensional (see Ref. [7]). As a consequence, the number m in (2.29) is not bounded, at variance that appearing in representations of $\mathfrak{su}(2)$.

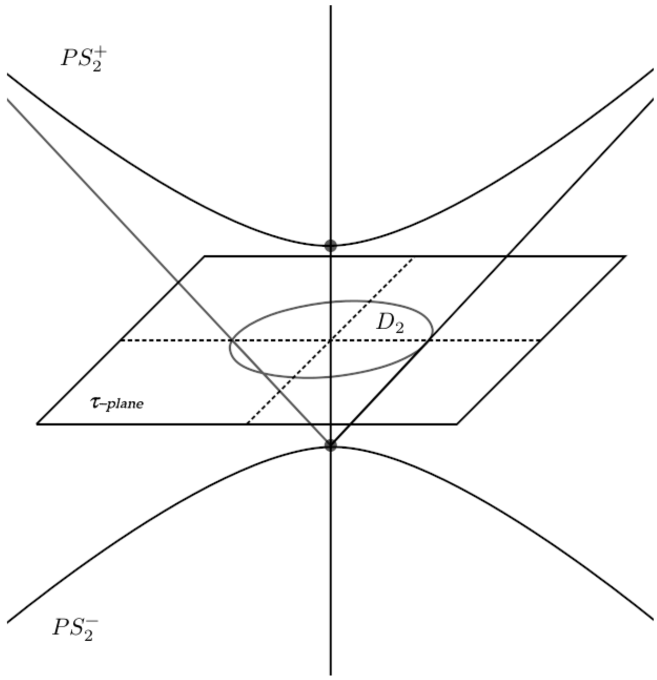


Figure 2.2: The hyperboloids $PS^{\pm}$ and the stereographic projection of the upper fold on the Poincaré disk D_2.

The manifold $\mathcal{M}$ associated to $SU(1,1)/U(1)$ is isomorphic to the 2-dimensional manifold PS^2, called *Bloch pseudo-sphere.* Being formed by a two-fold hyperboloid, this manifold is non-compact and not simply connected: these features are inherited from the group structure (that is likewise non-compact and not simply connected). We can parametrize the upper hyperboloid by the polar coordinates $(\rho, \phi) \in \mathbb{R}^+ \times [0, 2\pi]$, as

$$i\Omega = \frac{\rho}{2} e^{-i\phi} \ . \tag{2.34}$$

Points of $SU(2)/U(1)$ are identified with points of the hyperboloid PS_2. Using the changes of variables (1.18) and (1.20) for non-compact coset spaces

$$\Omega(\rho, \phi) \ \longrightarrow \ \zeta(\rho, \phi) = \sinh \frac{\rho}{2} e^{-i\phi} \ \longrightarrow \ \tau(\rho, \phi) = \tanh \frac{\rho}{2} e^{-i\phi} \tag{2.35}$$

we obtain the stereographic projection of the upper hyperboloid PS_2^+ onto the disk

$$D^2 = \{\tau \in \mathbb{C} \ : \ |\tau| < 1\} \ , \tag{2.36}$$

which is called *Poincaré disk* (see Fig. 2.2). Using the BCH formulas for $\mathfrak{su}(1,1)$, we get

$$|\tau\rangle = \hat{\tau} \, |k; 0\rangle = (1 - |\tau|^2)^k e^{\tau \hat{k}_+} \, |k; 0\rangle \ \ , \tag{2.37}$$

that has the advantage to make explicit the normalization

$$N(\tau, \tau*) = (1 - |\tau|^2)^{-2k} \ , \tag{2.38}$$

which is necessary in order to get the coset measure. Using Eqs. (1.21-1.23), the metric on the manifold is obtained from

$$F(\tau, \tau^*) = \log \langle \tilde{\tau} | \, \tilde{\tau} \rangle = -2k \log(1 - |\tau|^2) \tag{2.39}$$

and reads

$$g(\tau, \tau^*) = \frac{2k}{(1 - |\tau|^2)^2} \ ; \tag{2.40}$$

furthermore, the coset measure is

$$ds^2 = \frac{2k}{(1-|\tau|^2)^2} d\tau d\tau^* \;\Rightarrow\; d\mu = \frac{2k-1}{\pi} \frac{d\tau d\tau^*}{(1-|\tau|^2)^2} \tag{2.41}$$

in τ-coordinates, or

$$ds^2 = \frac{k}{2}(d^2\rho + \sin^2\rho d^2\phi) \;\Rightarrow\; d\mu = \frac{2k-1}{\pi} d\rho \sin\rho d\phi \tag{2.42}$$

in (θ,ϕ)-coordinates. These results hold for any $k \geq 1/2$ [6]. A procedure for finding the standard form of the Poisson brackets is reported in Ref. [8] and we just sketch the general idea behind such procedure. Starting from $\tau \in D_2$ one obtains (w,v) by the Möbius transformation

$$z = \frac{i+\tau}{i-\tau} \tag{2.43}$$

from D_2 to $\mathbb{H}$ (called Poincaré half-plane), followed by the change of coordinates

$$z = \frac{k^2}{w} - iv\ . \tag{2.44}$$

The pair (w,v) then realizes the standard Poisson brackets

$$\{f,g\} = \frac{\partial f}{\partial v}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial v}\frac{\partial f}{\partial w},\ \forall f,g \in C^\infty(\mathcal{M})\ . \tag{2.45}$$

2.2.1 Representations

In this section, we present two useful representations of the $\mathfrak{su}(1,1)$ algebra, the second of which is of main importance in our discussion about black-holes evaporation in the last chapter.

One-mode representation

If we choose the bosonic Fock space $\mathcal{F}$ as the Hilbert space carrying the representation, then the rules

$$\begin{cases} \hat{\phi}_1(K_0) = \frac{1}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\ \hat{\phi}_1(K_+) = \frac{1}{2}(\hat{a}^\dagger)^2 \\ \hat{\phi}_1(K_-) = \frac{1}{2}(\hat{a})^2 \end{cases} \tag{2.46}$$

describe a possible choice for translating the $\mathfrak{su}(1,1)$ elements into bosonic operators, as it is easy to show by computing the commutation rules. The Casimir operator

$$\hat{\mathbf{K}}^2 = -\frac{3}{16}\mathbb{I} \tag{2.47}$$

tells us to which value of κ this representation corresponds. As Eq. (2.47) has two solutions, $k = 1/4$ and $k = 3/4$, $(\phi_1, \mathcal{F})$ corresponds to two different representations. A possible choice of basis for $\mathcal{F}$ is

$$\left\{ |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \right\}\ , \tag{2.48}$$

with even and odd admitted values of n in the representations $k = 1/4$ and $k = 3/4$, respectively; these representations are called *one-mode representations*. The operator

$$\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*(\hat{a})^2 - \xi(\hat{a}^\dagger)^2} \tag{2.49}$$

that generates coherent states in this representation is called *squeeze operator* and it is often used in quantum optics.

Two-modes representation

Similarly, the *two-modes representation* is defined by choosing

$$\mathcal{F}\otimes\mathcal{F}=\{|m\rangle\otimes|n\rangle \quad : \quad m,n\in\mathbb{N}\} \tag{2.50}$$

as the carrier space of the representation. The $\mathfrak{su}(1,1)$ elements are translated in $op(\mathcal{F}\otimes\mathcal{F})$ by the algebra homomorphism ϕ_2 as

$$\begin{cases} \hat{\phi}_2(K_0)=\dfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{b}\hat{b}^\dagger) \\ \hat{\phi}_2(K_+)=\hat{a}^\dagger\hat{b}^\dagger \\ \hat{\phi}_2(K_-)=\hat{a}\hat{b}\ . \end{cases} \tag{2.51}$$

By defining the *number operators* $\hat{N}_a=\hat{a}^\dagger\hat{a}$ and $\hat{N}_b=\hat{b}^\dagger\hat{b}$, $\hat{\phi}_2(K_0)$ takes the form

$$\hat{\phi}_2(K_0)=\frac{1}{2}(\hat{N}_a+\hat{N}_b+1)\ . \tag{2.52}$$

Being the action of the Casimir operator,

$$\hat{\mathbf{K}}^2\,|m\rangle\otimes|n\rangle=\frac{1}{4}\left((\hat{N}_a-\hat{N}_b)^2-1\right)|m\rangle\otimes|n\rangle\ , \tag{2.53}$$

tantamount to a multiplication by $k(k-1)$, the relation

$$k=\frac{1}{2}(n_0+1)\ , \tag{2.54}$$

with n_0 eigenvalue of $\hat{N}_a-\hat{N}_b$, selects the representation. The carrier space is built from the lowest-weight state $|n_0,0\rangle$ by the action of the raising operator $\hat{\phi}_2(K_+)$, resulting in

$$\mathcal{H}_k=\{|k;n+n_0,n\rangle \text{ with } n\in\mathbb{N} \text{ and } k=(n_0+1)/2,\ n_0\in\mathbb{N}\}\ . \tag{2.55}$$

The action of the creation and annihilation operators are defined by

$$\hat{a}\,|n,m\rangle=\sqrt{n}\,|n-1,m\rangle,\ \ \hat{a}^\dagger\,|n,m\rangle=\sqrt{n+1}\,|n+1,m\rangle\ , \tag{2.56}$$

$$\hat{b}\,|n,m\rangle=\sqrt{m}\,|n,m-1\rangle,\ \ \hat{b}^\dagger\,|n,m\rangle=\sqrt{m+1}\,|n,m+1\rangle\ . \tag{2.57}$$

Coherent states in the one- and two-modes representations

A generic coherent state in τ-coordinates is given by Eq. (2.37), which reads, using the one-mode representation

$$|\tau\rangle=\left(1-|\tau|^2\right)^{1/4}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tau}{2}\right)^n\sqrt{(2n)!}\,|2n\rangle\ , \tag{2.58}$$

for $k=1/4$, and

$$|\tau\rangle=\left(1-|\tau|^2\right)^{3/4}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tau}{2}\right)^n\sqrt{(2n+1)!}\,|2n+1\rangle\ . \tag{2.59}$$

for $k=3/4$. For any permitted value of $k=(n_0+1)/2$, Eq. (2.37) in the two-modes representations becomes

$$|\tau\rangle=(1-|\tau|^2)^k\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\left(\frac{(n_0+n)!}{n_0!}\right)^{\frac{1}{2}}(n!)^{\frac{1}{2}}\,|n_0+n,n\rangle\ . \tag{2.60}$$

The overlap of coherent states is

$$\langle\tau|\,\tau'\rangle=\frac{(1-|\tau|^2)^k(1-|\tau'|^2)^k}{(1-\tau^*\tau')^{2k}}\ , \tag{2.61}$$

as shown in Appendix B.

2.3 More general Hamiltonians

In the introduction of this chapter we mentioned that the choice of the lowest-weight vector as reference state allows one to consider quantum theories whose Hamiltonians are finite polynomials (i.e. of order n) in the elements of the algebra $\mathfrak{g}$. This means that the Hamiltonian of a quantum theory $\mathcal{Q}$ with Lie algebra $\mathfrak{g}$ can be an element of the universal enveloping algebra $H \in \cup_{r=0}^{n} U(\mathfrak{g})^r$ with coefficients

$$H = c + \sum_i c_i g_i + \sum_{ij} c_{ij} g_i g_j + ... + \sum_{i_1} .. \sum_{i_n} c_{i_1,...i_n} g_{i_1} ... g_{i_n} \tag{2.62}$$

and it is therefore represented on $\mathcal{H}$ by

$$\hat{H} = c + \sum_i c_i \hat{\pi}(g_i) + \sum_{ij} c_{ij} \hat{\pi}(g_i)\hat{\pi}(g_j) + ... + \sum_{i_1} .. \sum_{i_n} c_{i_1,...i_n} \hat{\pi}(g_{i_1}) ... \hat{\pi}(g_{i_n}) \ , \tag{2.63}$$

where the map π is that defining the Lie-algebra representation. Since the Hamiltonian never enters in the CSS construction, the only thing one should check to extend the previous construction to such theories is that the symbol of (2.63) be well defined (i.e. finite everywhere on $\mathcal{M}$). We define the *normal-ordered Hamiltonian* as the operator that, using commutation relations, is obtained from $\hat{H}$ by shuffling the algebra elements so that every component of the sum (2.63) is in the form

$$\hat{\pi}(g_{-\alpha_1})^{l_1} \hat{\pi}(h_1)^{n_1} \hat{\pi}(g_{\alpha_1})^{m_1} \dots \hat{\pi}(g_{-\alpha_R})^{l_R} \hat{\pi}(h_R)^{n_R} \hat{\pi}(g_{\alpha_R})^{m_R} \ . \tag{2.64}$$

Since any Lie algebra has commutators $[g_i, g_j] = \sum c_{ijk} g_k$ then moving the operators inside a product can only generate terms that are of the same order of the product itself, or less. We can consider without loss of generality a rank 1 algebra so that the only possible terms appearing in the normal ordered Hamiltonian are

$$\hat{K}_-^a \hat{K}_0^b \hat{K}_+^c \tag{2.65}$$

with $\hat{K}_- = \hat{\pi}(g_{-\alpha})$, $\hat{K}_0 = \hat{\pi}(h_1)$ and $\hat{K}_+ = \hat{\pi}(g_\alpha)$. Being $|\psi_n\rangle$ the normalized vectors obtained from the lowest-weight state via the action of the raising operator $\hat{K}_+$ to the nth power, the symbol of (2.65) is

$$\langle\tau| \hat{K}_-^a \hat{K}_0^b \hat{K}_+^c |\tau\rangle = \sum_{n,m} \langle\tau| \hat{K}_-^a |\psi_n\rangle \langle\psi_n| \hat{K}_0^b |\psi_m\rangle \langle\psi_m| \hat{K}_+^c |\tau\rangle \ . \tag{2.66}$$

This is obviously finite for $\mathfrak{su}(2)$ for any finite choice of (a, b, c) since the space is bounded from both above and below. However, when the space is bounded only from below we have to check, case by case, if the series in the RHS of (2.66) converges.
Choosing $\hat{K}_- |\psi_n\rangle = m(n) |\psi_{n-1}\rangle$, with $m(0) = 0$, and $\hat{K}_0 |\psi_n\rangle = z(n) |\psi_n\rangle$ it is

$$c_n(\tau) = \langle n| \tau\rangle = \frac{\tau^n}{n!} \prod_{k=1}^{n} m^*(k) \ , \tag{2.67}$$

and therefore Eq. (2.66) can be made explicit as

$$\langle\tau| \hat{K}_-^a \hat{K}_0^b \hat{K}_+^c |\tau\rangle = \tau^{a-c} \sum_{n=0}^{\infty} z^b(n+a) \frac{\|\tau\|^{2(n-1)}}{(n-1)!(n+a-c-1)!} \prod_{i=1}^{n+a} \|m(i)\|^2 \tag{2.68}$$

if $a \geq c$, or

$$\langle\tau| \hat{K}_-^a \hat{K}_0^b \hat{K}_+^c |\tau\rangle = \tau^{c-a} \sum_{n=0}^{\infty} z^b(n+c) \frac{\|\tau\|^{2(n-1)}}{(n-1)!(n+c-a-1)!} \prod_{i=1}^{n+c} \|m(i)\|^2 \tag{2.69}$$

if $c \geq a$. In the case of $\mathfrak{su}(1,1)$ in the κ-representation, assuming without loss of generality $a \geq c$, the symbol

$$\langle\tau| \hat{K}_-^a \hat{K}_0^b \hat{K}_+^c |\tau\rangle = \tau^{a-c} \sum_{n=0}^{\infty} (n+a+k)^b \frac{\|\tau\|^{2(n-1)}}{(n-1)!(n+a-c-1)!} ((n+a)!)^2 \tag{2.70}$$

converges for any finite (a, b, c), as easily shown by ratio test ,

$$\lim_{n\to\infty} \|\tau\|^2 \left(1 + \frac{1}{n+a+k}\right)^b \left(1 + \frac{1+a}{n}\right) \left(1 + \frac{1+c}{n+a-c}\right) = \|\tau\|^2 < 1 \; . \tag{2.71}$$

Therefore, any quantum theory

$$\mathcal{Q} = \{\mathfrak{su}(1,1), (\phi_k, \mathcal{H}_k), \hat{H}\} \tag{2.72}$$

with Hamiltonian that is a finite polynomial in the elements $\hat{\phi}_k(\mathfrak{g})$ is well defined and can be described by a CSS system.

Chapter 3

Back to the crossover: from Classical to Quantum

Our perception of reality is classical. We describe what we see by naked eye with the tools provided by classical mechanics: real parameters, symplectic manifolds, Hamiltonian functions etc. The universe, though, seems to be fundamentally quantum. There are two essentially different ways to tackle this disturbing contradiction: either one "quantizes" a classical picture or she looks for some classical limit of a quantum description. While the former approach, usually called *quantization*, is as old as QM, finding the quantum theory underlying some given classical one is very challenging (and not necessarily possible). We here propose a third possibility, that will bring us into a "boundary region", where classical and quantum features coexist.

Our starting point is the observation that many different quantum theories have the same classical limit, which allows us to choose a particularly convenient quantum theory when studying a target classical one. Our pick is a theory that globally emerges as a classical one, despite its N components stay locally quantum, even in the large-N limit. The relevance of this choice is twofold: in the first place, the global theory will be a rescaled replica of each microscopic quantum theory, so that the structure of the classical theory will be directly derived from that of the microscopic ones. Secondly, the fact that the microscopic theories are non-interacting will allow us to probe the quantum structure via local measurements without damaging the global symmetry, and hence keeping the classical limit safe. Once such theory can manifest the classical and quantum behaviours at once, allowing us to explore a regime that we here call *quasi-classical*. In order to identify whether a system is in this quasi-classical regime or not, we propose an accuracy-dependent lower bound N_d over the number of microscopic subsystems: if N is greater than N_d, the theory can surely be threatened as classical.

In the first section of this chapter, we discuss how to use the large-N limit to obtain a collection of quantum theories corresponding to the same classical one. In the second section, starting from a large-N quantum theory we show how is it always possible to find a microscopic description that is conveniently simple (viz. a collection of non-interacting theories). The third section shows how the classical behaviour can manifest itself even for finite values of N. The reverse is also true: a large-N system described by its classical limit that for any reason reduces its size while keeping its global symmetry, can display quantum features when N gets sufficiently small. In the last section, we apply the tools previously developed to the $\mathfrak{su}(1,1)$ algebra, both as an exercise and because it turns out to be useful in the second part of this work.

Ahead of what we will see in the next chapters,BHs can be regarded as systems in the quasi-classical regime, i.e. defined as classical objects and found to evolve via the emission of quantum particles. By looking at BHs as globally symmetric systems composed of a large number of subsystems, the construction here presented allows one to study whether or not black-hole evaporation process (by means of local measurements made upon parts of the black hole representing Hawking radiation) can arise from the quasi-classical setting. For these reasons, this chapter contains the essential tools that we have developed to deal with black-holes evaporation and details relevant for the later discussion will be pointed out in the text.

3.1 How to get a quantum theory from a classical one

The large-N limit brings one from the quantum to the classical world. One might then ask if it is possible to reverse the process, so as to obtain, given a classical theory $\mathcal{C}$, the quantum theory whose large-N limit corresponds to $\mathcal{C}$ itself. This turns out to be possible, although, not surprisingly, with a certain degree of arbitrariness, so that the underlying quantum theory is not at all unique.

Let us consider a classical theory $\mathcal{C} = \{\mathcal{N}, \omega, h\}$, as defined in Sec. 1.1, and suppose that its phase-space $(\mathcal{N}, \omega)$ is associated, through a symplectomorphism[1]

$$f : (\mathcal{M}, \omega) \to (\mathcal{M}, \omega') \ , \tag{3.1}$$

to the symplectic manifold $(\mathcal{M}, \omega')$ emerging from the GCS construction of a semisimple Lie algebra $\mathfrak{g}$, as reported in Sec. 1.2. If this is possible, $\mathcal{C}$ is related to a family of quantum theories

$$\mathcal{Q}_\kappa = \{\mathfrak{g}, r_\kappa, \hat{H}_\kappa\} \tag{3.2}$$

that satisfy assumptions **A1-A4** of Sec. 1.3.1 such that

- the CSS manifold of $\mathcal{Q}_\kappa$ is $\mathcal{M}$, $\forall\kappa$.
- $\lim_{\kappa\to 0} \kappa H_\kappa(\Omega) = h$

We then choose another family of quantum theories, $\mathcal{Q}_N$, that are invariant under some global symmetry, whose equivalence classes of states can be related to the GCS defined on $\mathcal{M}_\kappa$ by Eq. (1.54). The large-N limit of any such $\mathcal{Q}_N$ is, by definition, the classical theory $\mathcal{C}$.

In Fig. 3.1, the procedures leading from $\mathcal{Q}_N$ to $\mathcal{C}$ (left) and from $\mathcal{C}$ to $\mathcal{Q}_N$ (right) via $\mathcal{Q}_\kappa$ are summarized. As briefly mentioned above, however, while the large-N limit of $\mathcal{Q}_N$ can be a well defined classical theory $\mathcal{C}$, the same $\mathcal{C}$ can be the large-N limit of many different $\mathcal{Q}_N$. In what follows we review what are the possible sources of arbitrariness.

The Hamiltonian

Once the Hamiltonians $\hat{H}_\kappa$ are chosen, any term

$$\kappa^\delta \hat{H}' \ , \tag{3.3}$$

with $\delta > -1$, can be added to $\hat{H}_\kappa$ without modifying its classical limit. It is then clear that the family of quantum theories $\{\mathcal{Q}_\kappa\}$ we can choose to reproduce $\mathcal{C}$ is not unique. Even more freedom emerges from the choice in the order of operators appearing in the Hamiltonian, if that is an element of $U(\mathfrak{g})$. Since the symbol of a product of operators in the form (2.65) becomes the product

$$K_-^a(\Omega) K_0^b(\Omega) K_+^c(\Omega) \ , \tag{3.4}$$

of symbols, which clearly are commuting functions, any Hamiltonian with differently ordered products gives the same Hamiltonian function h in the large-N limit. This also implies that, for what concerns the classical limit, Hamiltonians can be safely normal-ordered. This is very similar to what happens in the standard quantization scheme for the $\hat{r}$ and $\hat{p}$ operators.

[1] An isomorphism f between two symplectic manifolds $(\mathcal{N}, \omega)$ and $(\mathcal{M}, \omega')$ is said to be a symplectomorphism if the action of the pullback f^* on ω' is $f^*\omega' = \omega$.

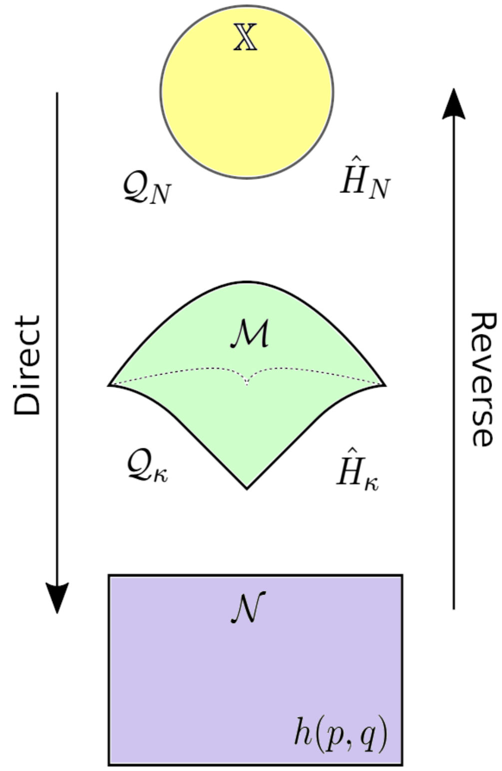


Figure 3.1: Direct and reverse large-N limit.

The Lie algebra

Even if discussing this point goes beyond the scopes of this thesis, we mention that some flexibility in inferring $\mathcal{Q}_N$ from $\mathcal{C}$ also arises from the choice of the Lie algebra. In fact, the algebra whose CSS manifold satisfies the condition (3.1) might be non-unique.

The microscopic structure

Referring to Fig. 3.1, once we have chosen the $\mathcal{Q}_\kappa$ theories that make the correspondence (3.1), and their Hamiltonians $\mathcal{H}_\kappa$, we still have some arbitrariness, which is due to the possibly different choice of the symmetry $\mathbb{X}$ and that of the subsystems composing the one described via $\mathcal{Q}_N$. These two choices are not independent, as we will see in the next section.

Is this a quantization procedure?

In conclusion, even if a process to obtain $\mathcal{Q}_N$ from $\mathcal{C}$ is possible, the degree of arbitrariness implied by such procedure reduces its relevance. In fact:

- the quantum Hamiltonian corresponding to the classical one is determined modulus an infinite collection of terms that are suppressed in the large-N limit, and Hamiltonians with differently ordered products all give the same classical limit meaning that the actual quantum dynamic stays essentially undefined.

- the quantum Hamiltonian is built starting from the set of operators that are representation of elements of a Lie algebra, whose structure is encoded in the phase-space: no information other than that is provided, and a lot of other details about the quantum structure (i.e. hidden parts of the Lie algebra) may not appear in the classical setting and yet be relevant in the microscopic description.

- knowing the global Hamiltonian and the set of operators from which is built, does not provide the details either of the microscopic interactions nor of the global symmetry which is necessary for the classical limit to emerge.

3.2 The microscopic world underlying large-N quantum theories

In the previous section we have seen that the arbitrariness in reversing the reverse large-N limit can make the description of a "classical to quantum transition" very tough. However, if traces of some quantum features somehow survive in the classical regime we might get valuable clues about the quantum world that lays beneath the classical surface. To show how these clues can be recognised, let us explore the connection between the global symmetry $\mathbb{X}$ and the subsystems of a large-N quantum theory.

Let us consider a large-N quantum theory $\mathcal{Q}_N = \{\mathfrak{g}_N, (\phi_N, \mathcal{H}_N), \hat{H}_N\}$ described by a rank-1 Lie algebra $\mathfrak{g}_N$. We suppose the Hamiltonian is

$$\hat{H}_N = A_N \hat{K}_0 + B_N \hat{K}_+ + B_N^* \hat{K}_- \ , \quad A_N \in \mathbb{R}, \ B_N \in \mathbb{C} \ , \tag{3.5}$$

where we have used the decomposition on the Weyl-Cartan basis $\{g_0, g_+, g_-\}$ of $\mathfrak{g}_N$ and the representation

$$\hat{\phi}_N(g_0) = \hat{K}_0, \ \hat{\phi}_N(g_+) = \hat{K}_+, \ \hat{\phi}_N(g_-) = \hat{K}_- \ . \tag{3.6}$$

The coefficients in Eq.(3.5) have been chosen so that $\hat{H}$ is Hermitian. We take the Lie brackets of the algebra as

$$[h, g_\pm] = \pm\alpha_N g_\pm, \ [g_+, g_-] = \beta_N h \ , \tag{3.7}$$

for some values α_N and β_N, depending on the specific $\mathfrak{g}_N$ considered.

Requiring a system with many subsystems to be described by a Lie algebra of not only finite but actually small dimension is already equivalent to ask for some global symmetry $\mathbb{X}$ to exist so as to allow the small-κ versus large-N correspondence to be established and the classical limit defined. That being said, the Hamiltonian (3.5) does neither specify which subsystems form the global system, nor their possible interactions. In fact this Hamiltonian does neither suffices to define the global symmetry.

In fact, as far as the classical theory is concerned, any microscopic theory with the right global symmetry can work: as long as we obtain the same Hamiltonian function $h \in C^\infty(\mathcal{N})$, we cannot tell apart which microscopic theory is actually ruling the system. However, when the quantum character somehow emerges together with the classical one, we might be able at least to rule out some of the microscopic descriptions. When the behaviour of a quantum system is such that a classical structure for a theory that describes it is already well established and yet the system can be regarded as quantum for some aspects, we will hereafter say that the system is *quasi-classical.* Note that neither the large-N classical limit nor the fully quantum description can efficiently describe this system in that the former will completely miss fundamental consequences of its quanto-mechanical nature, and the latter will be lost in the details, without highlighting its effectively classical behaviour. In order to deal with this situation, we need some specific tools, that will be defined in the remaining part of this chapter.

Different choices of representations of the global algebra in terms of *local* operators, i.e. operators acting onto the Hilberts spaces of the many subsystems composing $\mathcal{Q}_N$, give different microscopic descriptions of the global theory. For example, the Hamiltonian (3.5) can be obtained by a microscopic theory for

- non-interacting subsystems. In fact, it is always possible to define a collection of $3N$ operators $\{\hat{K}^i_+, \hat{K}^i_0, \hat{K}^i_-\}_{i=1,\ldots N}$ such that

$$\hat{H}_N = \sum_i \hat{H}_i, \text{with } \hat{H}_i = A_N \hat{K}^i_0 + B_N \hat{K}^i_+ + B^*_N \hat{K}^i_- \ , \tag{3.8}$$

  and which has permutation invariance as global symmetry $\mathbb{X}$. We will refer to one such case as the *free-theory* description.

- interacting bosons. In fact, the generators of any Lie algebra that has a matrix representation can be realized as quadratic forms in bosonic creation and annihilation operators[2]; by choosing such form one can describe the quantum theory via a bosonic description. For example if $\mathfrak{g} = \mathfrak{su}(1,1)$, and one chooses the two modes representation,

$$\hat{H}_N = \frac{A}{2}\left(\hat{a}^\dagger \hat{a} + \hat{b}\hat{b}^\dagger\right) + B\hat{a}^\dagger \hat{b}^\dagger + B^* \hat{a}\hat{b} \ . \tag{3.9}$$

  This Hamiltonian does not yet display any microscopic structure, however, replacing the bosonic operators according to

$$\hat{a} = \sum_i \hat{a}_i, \ \hat{a}^\dagger = \sum_i \hat{a}^\dagger_i, \ \hat{b} = \sum_i \hat{b}_i, \ \hat{b}^\dagger = \sum_i \hat{b}^\dagger_i \ , \tag{3.10}$$

  a microscopic theory of $2N$ interacting bosons is obtained.

- interacting fermions. In fact, what written above for bosons also holds for fermionic creation and annihilation operators, thus making it possible a description in terms of interacting fermions. We will not work out the details of this possibility as we aim at obtaining a model for Hawking radiation, which is known to be largely composed of massless bosons[3] (photons).

These are just some examples from what is in fact an infinite zoo of possibilities, which makes a question naturally arises: *if all these theories are classically equivalent, which one should we use?*
In this thesis, we choose the free-theory description. This is not due to some fundamental features we expect the global theory to display but just because of some advantages this choice implies; In fact:

- the microscopic Hamiltonians can be easily written.
- the global Lie algebra is a replica of the local ones, after a proper rescaling. This means that the classical structure is already present in the local theories, encoded into their Lie algebras and Hamiltonians.
- since any set of a permutation invariant set is still permutation invariant, one can remove some subsystems from the large-N one without substantially altering the global symmetry. This means that one can "cut out" pieces of the macroscopic system to perform quantum local measurement upon its components without altering the global theory that describes the overall system, except from a rescaling.

As a last note, let us observe that choosing the free theory description does not rule out the presence of entanglement between the subsystems although this aspect is not relevant in what follows.

[2] The two modes representations of Sec. 2.2.1 for $\mathfrak{su}(1,1)$ and the so-called *Holstein–Primakoff transformation* for $\mathfrak{su}(2)$ are some examples. For a general result see i.e. Ref. [9].

[3] As a historical note, in Ref. [10] D.Page calculated the percentage composition of Hawking radiation and found a large amount of it to be neutrinos (roughly 80%). That is because at the time neutrinos were thought to be massless. More recent calculations show that Hawking radiation is mainly composed of photons and, in a smaller amount, by gravitons.

3.2.1 Free theories

The take-home message of the previous section is that it is always possible, and is often convenient, to choose a non-interacting microscopic quantum theory endowed with permutation invariance, i.e. the free theory introduced above, to describe a system upon which the only constraint we have is that relative to the classical theory that makes its large-N limit. In particular, the three operators

$$\begin{cases} \hat{K}_0 &= \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \hat{K}_0^i \ , \\ \hat{K}_+ &= \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \hat{K}_+^i \ , \\ \hat{K}_- &= \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \hat{K}_-^i \ , \end{cases} \tag{3.11}$$

close the commutation relations (3.7) if the *local* operators, acting on each subsystem and denoted by i, close the same commutation rules with properly rescaled coefficients α_N and β_N. In fact,

$$\begin{aligned} [\hat{K}_0, \hat{K}\pm] = \frac{1}{N^2}\sum_{ij}[\hat{K}_0^i, \hat{K}_\pm^j] = \frac{1}{N^2}\sum_{i,j}\delta_{ij}(\pm\alpha)\hat{K}_\pm^i = \pm\alpha\frac{1}{N^2}\sum_i \hat{K}_\pm^i = \frac{1}{N}(\pm\alpha)\hat{K}_\pm = \\ \stackrel{\alpha/N=\alpha_N}{\longrightarrow} \pm\alpha_N\hat{K}_\pm \ , \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} [\hat{K}_+, \hat{K}_-] = \frac{1}{N^2}\sum_{ij}[\hat{K}_+^i, \hat{K}_-^j] = \frac{1}{N^2}\sum_{i,j}\delta_{ij}\beta\hat{K}_0^i = \beta\frac{1}{N^2}\sum_i \hat{K}_0^i = \frac{1}{N}\beta\hat{K}_0 = \\ \stackrel{\beta/N=\beta_N}{\longrightarrow} \beta_N\hat{K}_0 \ . \end{aligned} \tag{3.13}$$

Using Eqs. (3.11) the global Hamiltonian (3.5) becomes

$$\hat{H}_N = \sum_i \hat{H}_i \ , \tag{3.14}$$

where

$$\hat{H}_i = a_N\hat{K}_0^i + b_N\hat{K}_+^i + b_N^*\hat{K}_-^i \ , \tag{3.15}$$

with $a_N = A_N/N$ and $b_N = B_N/N$. This Hamiltonian is invariant under the action of the permutation group; therefore, we can choose the permutation symmetry as the global symmetry $\mathbb{X}$ required for the large-N limit to establish a well defined classical theory.

The coefficients a_N and b_N are fixed by requiring that the global theory satisfies **A4** in Sec. 1.3.1. In fact, since the symbols of $\hat{K}_0$ and $\hat{K}_\pm$ must be finite for $N \to \infty$, to guarantee that

$$\lim_{N\to\infty}\frac{1}{N}H_N(\Omega) = \lim_{N\to\infty}\left(\frac{A_N}{N}K_0(\Omega) + \frac{B_N}{N}K_+(\Omega) + \frac{B_N^*}{N}K_+(\Omega)\right) \tag{3.16}$$

stays finite, it must be

$$\begin{aligned} \lim_{N\to\infty}\frac{A_N}{N} &= \lim_{N\to\infty} a_N < \infty \ , \\ \lim_{N\to\infty}\frac{B_N}{N} &= \lim_{N\to\infty} b_N < \infty \ . \end{aligned}$$

Therefore, we can choose

$$\hat{H}_i = a\hat{K}_0^i + b\hat{K}_+^i + b^*\hat{K}_-^i \tag{3.17}$$

with $a = a_N$ and $b = b_N$ independent of N, and obtain a well behaved classical limit. In conclusion, each one of the local subsystems is described by a quantum theory with $\mathfrak{g}$ as Lie algebra and with (3.17) as Hamiltonian. The global system made of their collection is described by

$$\mathcal{Q}_N = \{\mathfrak{g}_N, (\Phi, \mathcal{H}_N), \hat{H}_N\} \tag{3.18}$$

where:

- $\mathfrak{g}_N$ has structure constants
$$(c_{ijk})_N = \frac{1}{N} c_{ijk} \ , \tag{3.19}$$
with c_{ijk} structure constants of $\mathfrak{g}$.
- $(\Phi_N, \mathcal{H}_N)$ is such that
$$\Phi_N(g) = \frac{1}{N}\phi(g) \otimes \mathbb{I}^{\otimes N-1} + \mathbb{I} \otimes \frac{1}{N}\phi(g) \otimes \mathbb{I}^{\otimes N-2} + \ldots + \mathbb{I}^{\otimes N-1} \otimes \frac{1}{N}\phi(g) \tag{3.20}$$
where $(\phi, \mathcal{H})$ is a representation of the local Lie algebra $\mathfrak{g}$ and $\mathcal{H}_N$ is a permutation invariant subspace of $\mathcal{H}^{\otimes N}$.
- The Hamiltonian is
$$\hat{H}_N = \sum_i \left(a\hat{K}^i_0 + b\hat{K}^i_+ + b^*\hat{K}^i_- \right) \tag{3.21}$$

As previously mentioned, the free-theory description presented above is of particular interest if one wants to model a local measurement upon just one part of the global system, without changing the overall quantum state: if the system is made of a large number of subsystems that do not interact, one can observe each one of these parts while leaving the others undisturbed. The observed part will "pop out" from the system due to the quantum measurement interaction, and will possibly start belonging to another system (the one made of the measuring apparatus and/or the observer itself). This picture will be useful for discussing our prototype of quantum black hole, where the local measurements will be interpreted as made upon the Hawking radiation.

Rescaled Lie algebras and their coherent states

We denote as $\mathfrak{g}_\Gamma$ the Lie algebra with elements $\{K_0, K_+, K_-\}$ that satisfies the same Lie brackets as $\mathfrak{g} = \{k_0, k_+, k_-\}$ except for the rescaling

$$\begin{cases} [K_+, K_-] = \beta \frac{1}{\Gamma} K_0 \ , \\ [K_0, K_\pm] = \alpha_\pm \frac{1}{\Gamma} K_\pm \ . \end{cases} \tag{3.22}$$

We call $\mathfrak{g}_\Gamma$ the *rescaled Lie algebra* of $\mathfrak{g}$. From any representation $r = (\phi, \mathcal{H})$ of $\mathfrak{g}$ we can obtain a representation for $\mathfrak{g}_\Gamma$. This is done by defining a map $f : \mathfrak{g}_\Gamma \to \mathfrak{g}$ from the rescaled algebra to the original one, acting on the elements as

$$K_0 \mapsto k_0 \text{ and } K_\pm \mapsto k_\pm \ .$$

The representation is then constructed as $r_\Gamma = (\phi_\Gamma, \mathcal{H})$, where

$$\phi_\Gamma(G) = \frac{1}{\Gamma}\phi(f(G)) \ , \tag{3.23}$$

and the Hilbert space is the same as that of r. The GCS relative to $\mathfrak{g}_\Gamma$, denoted by a proper lower index outside the *ket* in what follows, can be related with those relative to $\mathfrak{g}$, via

$$|\Omega\rangle_\Gamma = |\Omega/\Gamma\rangle \ . \tag{3.24}$$

Note that the global algebra $\mathfrak{g}_N$ previously defined for the free-theory description is the rescaled Lie algebra of the (all identical) local Lie algebras $\mathfrak{g}$.

An example: $\mathfrak{su}(1,1)$ rescaled coherent states

Let us show in some detail how to determine the relation (3.24) between GCS for $\mathfrak{su}(1,1)$ (with elements denoted by lowercase letters) and GCS for its rescaled Lie algebra (with elements denoted by uppercase letters). This is an example that will be useful for our discussion about black-holes evaporation. Without loss of generality[4] we choose the 2-dimensional representation of $\mathfrak{su}(1,1)_\Gamma$ in which the 3 generators are mapped into $op(\mathbb{C}^2)$ as

$$\begin{cases} \hat{K}_\pm = i\dfrac{1}{2\Gamma}\sigma_\pm \ , \\ \hat{K}_0 = \dfrac{1}{2\Gamma}\sigma_3 \ , \end{cases} \tag{3.25}$$

where $\hat{\sigma}_i, \ i = 1,2,3$ are the Pauli operators and $\hat{\sigma}_\pm = \hat{\sigma}_1 \pm i\hat{\sigma}_2$. It is easy to show that the right commutation relations (3.22) follow from those of $\{\hat{\sigma}_i\}$. The operator

$$\hat{\Omega}_\Gamma = e^{\Omega_\Gamma \hat{K}_+ - \Omega_\Gamma^* \hat{K}_-} \tag{3.26}$$

generates the GCS $|\Omega\rangle_\Gamma$ acting upon the reference state. Its matrix representation on the basis of the eigenvectors of $\hat{\sigma}_3$ reads

$$\begin{aligned} \hat{\Omega}_\Gamma &= \exp\left\{\Omega_\Gamma \frac{i}{2\Gamma}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - \Omega_\Gamma^* \frac{i}{2\Gamma}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\right\} = \\ &= \begin{pmatrix} \cosh|\Omega_\Gamma/\Gamma| & i\frac{\Omega_\Gamma/\Gamma}{|\Omega_\Gamma/\Gamma|}\sinh|\Omega_\Gamma/\Gamma| \\ -i\frac{\Omega_\Gamma^*/\Gamma}{|\Omega_\Gamma/\Gamma|}\sinh|\Omega_\Gamma/\Gamma| & \cosh|\Omega_\Gamma/\Gamma| \end{pmatrix} \ . \end{aligned} \tag{3.27}$$

In order to better understand the relation between the standard pseudo-spin coherent states and the rescaled ones, we notice that the expression (3.27) is identical to the representation on $\mathbb{C}^2$ of

$$\hat{\Omega} = e^{\Omega \hat{k}_+ - \Omega^* \hat{k}_-} \ , \tag{3.28}$$

via the representation

$$\begin{cases} \hat{k}_\pm = i\dfrac{1}{2}\sigma_\pm \ , \\ \hat{k}_0 = \dfrac{1}{2}\sigma_3 \ , \end{cases} \tag{3.29}$$

except for the rescaling

$$\Omega = \Omega_\Gamma/\Gamma \ . \tag{3.30}$$

Therefore, the rescaled pseudo-spin coherent states and the standard ones are related by

$$\hat{\Omega}_\Gamma = e^{\Omega_\Gamma \hat{K}_+ - \Omega_\Gamma^* \hat{K}_-} = \hat{\Omega} = e^{\Omega \hat{k}_+ - \Omega^* \hat{k}_-} \ , \tag{3.31}$$

and hence

$$|\Omega\rangle_\Gamma = |\Omega\rangle = |\Omega_\Gamma/\Gamma\rangle \ . \tag{3.32}$$

3.3 Classical behaviour at finite N

We have seen that a quantum theory describing a large number N of subsystems can flow into a well defined classical theory when $N \to \infty$. On the contrary, systems can display a classical behaviour even for finite, though large, values of N. A question then arises: *is it possible to obtain the very same classical limit for finite, and more physical, values of N?*

[4] As mentioned in [6], since the parameters are independent of the selected representation, it is sufficient to show that our results are valid for just one convenient choice.

We propose that a classical behaviour emerges any time the number of subsystems of the originally quantum object, is neither infinite nor "big" without any reference, but rather when it is big enough to allow is to distinguish different GCS with some given accuracy.
Starting from the usual large-N limit, we observe that, as the number of subsystems N increases, the overlap between any two GCS tends to a δ-like form

$$N|\langle\Omega|\,\Omega'\rangle|^2 \xrightarrow[N\to\infty]{} \delta(\Omega-\Omega')\ , \tag{3.33}$$

meaning that they become more and more distinguishable. This is indeed the characterizing feature of the classical limit: vectors represented by points on the CSS manifold become orthogonal and, in the limit, can be treated as classical states. The idea we pursue in this section is to find a lower bound for N above which we cannot anymore distinguish a large-N quantum system from its classical limit.
When combined with the previous discussion about free theories, this result has two consequences. First, if enough local subsystems are somehow removed, a system made of non-interacting subsystems can go from behaving as it were classical back to its originally quantum nature. Second, we can use the free-theory description together with this finite-N constraint to describe a system in the above mentioned quasi-classical regime.

3.3.1 The setup

When we observe a system, be it classical or quantum, we are limited by the instruments we use for our investigation. Within our setting, these limitations result in:

- a finite distance between points on the CSS manifold which is too small to be measured.
- a finite overlap between non-orthogonal states, that we cannot detect.

Therefore, not only there is no need to reach the $N\to\infty$ limit (thus having exact orthogonality between different GCS) to possibly observe a classical behaviour, but, even if so, we would not be able to tell apart such case form the finite-N one, whenever N is sufficiently large.

Finite distance

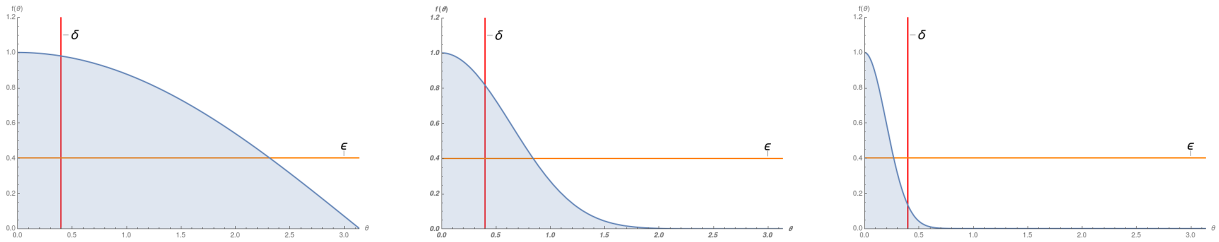
Given $\Omega\in\mathcal{M}$ we define its δ-neighbourhood $I_D^N(\Omega,\delta)$ as the set of points

$$I_D^N(\Omega,\delta)=\{\Omega'\in\mathcal{M}\ :\ D(\Omega,\Omega')\leq\delta\}\ , \tag{3.34}$$

where $D:\mathcal{M}\times\mathcal{M}\to\mathbb{R}^+$ is some distance that needs being specified (and might depend on N). If $\Omega'\in I_D^N(\Omega,\delta)$, then we say that Ω and Ω' are unδ-distinguishable.
As an extra requirement, we ask the D distance to be equal to the "natural" distance on the CSS manifold (defined in Box 3.1) when $N\to\infty$. This is to recover the distance between points on the classical phase-space $\mathcal{N}$ in the $N\to\infty$ limit. The question about which is the right metric to choose for finite N has several possible answers, each bearing advantages and weaknesses. In this thesis, we use the so-called *Monge distance* D_M, introduced in the quantum setting in Ref. [11] and discussed in some detail in Ref. [12]. When considering two GCS, this choice has the advantage to flow in the natural distance[5] as $N\to\infty$ thus automatically satisfying our requirement. The drawback of D_M is that it is quite hard to evaluate in general.

The original definition of Monge distance dates back to 1781 when it was defined to solve the following problem: *which is the best way to move a pile of soil from a mount in a given shape Q_1 to another configuration Q_2?* Some details about this problem and its solution can be found

[5] This is true also without performing the classical limit when one considers the Heisenberg algebra's coherent states.



(a) $N = 1$ (b) $N = 15$ (c) $N = 50$

Figure 3.2: General idea of the procedure: the GCS whose overlap is higher than the horizontal orange line (which represents the accuracy ϵ) will be increasingly closer as N grows bigger. When the more distant GCS with overlap greater then ϵ is closer than the red vertical line (which represents the distance δ), the classical behaviour emerges. The function $f(\theta)$ represents the overlap between $|0\rangle$ and $|\Omega(\theta, \phi)\rangle$ for the $\mathfrak{su}(2)$ GCS.

in Box 3.2. The relevant feature for us is that, by analysing how we can perform this task, a notion of distance between probability densities arises: D_M is defined as the total displacement we need in order to transform Q_1 into Q_2 by using the most convenient protocol to do it.

The Monge distance can be generalized to a quantum one thanks to the so-called called *Husimi functions*. For any GCS[6] $|\Omega\rangle$ we define its Husimi function

$$H_{|\Omega\rangle\langle\Omega|}(\omega) = |\langle\omega|\,\Omega\rangle|^2 \ , \tag{3.35}$$

where ω is another GCS. Clearly enough, $H_{|\Omega\rangle\langle\Omega|}$ is a positive function defined on the CSS and becomes a well-behaved probability distribution on $\mathcal{M}$ when read by means of diagram (1.13). Since Husimi functions are in one-to-one correspondence with GCS, one can define the Monge distance between $|\Omega\rangle$ and $|\Omega'\rangle$ as

$$D_M(\Omega, \Omega') \equiv D_M(H_{|\Omega\rangle\langle\Omega|}, H_{|\Omega'\rangle\langle\Omega'|}) \ . \tag{3.36}$$

In fact, since we are just interested in finding an upper bound for $D(\Omega, \Omega')$, we will not calculate the Monge distance exactly, which is, in general, a difficult goal to accomplish. What we actually need is to show that

$$D_M(\Omega, \Omega') < d(\Omega, \Omega') \ , \tag{3.37}$$

which means that the Monge metric between GCS is always smaller than the natural one. This holds for any compact CSS manifold. Furthermore, it holds that

$$D_M(\Omega, \Omega') \xrightarrow[N\to\infty]{} d(\Omega, \Omega') \tag{3.38}$$

(see Ref. [12]). We can intuitively understand this property (that follows from translation invariance of both distances) by considering the fact that surely exist the way to move a GCS to another by simply translating the distribution along the geodesics of $\mathcal{M}$, but there might still better ways to do so, hence the inequality.

[6]The definition of Husimi function is also valid for mixed states. However, since we do not need this generality we limit ourselves to the definition for GCS.

Box 3.1: the “natural” distance

For any two points Ω and Ω' on the CSS manifold $\mathcal{M}$ a set Γ of smooth curves $\gamma : \mathbb{R} \to \mathcal{M}$ with Ω as starting point and Ω' as ending one, can be defined

$$\Gamma = \{\gamma(t),\ t \in [a, b]\ :\ \gamma(a) = \Omega,\ \ \gamma(b) = \Omega'\}\ . \tag{3.39}$$

The “natural” distance between Ω and Ω' is defined as

$$d(\Omega, \Omega') = \inf_{\gamma\in\Gamma} \left\{ \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt \right\}\ , \tag{3.40}$$

which is the length of the shortest path connecting Ω to Ω', where the metric is independent of N.

Finite overlap

Suppose we are unable to discriminate by a quantum measurement, whether two states $|x\rangle$ and $|y\rangle$ are orthogonal or just such that

$$|\langle x|\, y\rangle| \leq \epsilon\ , \tag{3.41}$$

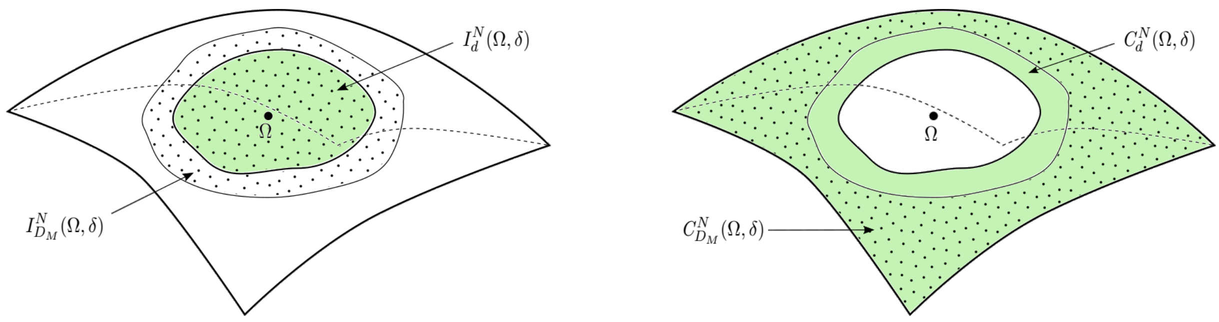
for a given $\epsilon > 0$. This means that, as fas as our experimental analysis is concerned, two states are distinguishable even if their overlap is finite. Two such states can be called ϵ-orthogonal. In particular, we will say that two GCS are ϵ-orthogonal if

$$|\langle \Omega|\, \Omega'\rangle| \leq \epsilon\ . \tag{3.42}$$

Clearly enough there is a relation between δ as introduced in (3.34) and ϵ as in (3.42), so that $|\Omega\rangle$ and $|\Omega'\rangle$ are unδ-distinguishable iff $|\langle\Omega|\,|\Omega'\rangle| > \epsilon$.

Our plan

Our plan is to find a lower bound for N such that the system can be treated as if it were classical whenever N is larger than such bound. This construction will depend on the two given parameters δ and ϵ, representing the smallest distance on the GCS manifold we are able to measure and the accuracy with which we can differentiate orthogonal and non-orthogonal states, respectively. Fig. 3.2 represents a sketch of unδ-distinguishability and ϵ-orthogonality.
For a given GCS $|\Omega\rangle$ we evaluate the overlap $|\langle\Omega|\,\Omega'\rangle|$ with all other coherent states, $|\Omega'\rangle$. If this is less than ϵ for all GCS which are more distant than δ (as obtained by the Monge distance D_M), then we say that the $|\Omega\rangle$ is “well separated”, and we expect a classical behaviour to emerge. As previously mentioned, calculating D_M is a daunting task; on the other hand, the distance d naturally defined on the manifold can be easily calculated. Therefore, we propose to use the latter instead of the former. This comes at a price: d is greater than D_M for any finite N and using d results in a larger value of N at which the theory becomes effectively classical, compared to that one we would obtain using D_M (see Fig. 3.4). We will call N_d and N_M the lower bounds found and say that when N is smaller than N_M the theory will retain its quantum character, while if N is larger than N_d the theory is effectively classical. The range $N_M < N < N_d$ defines a region in which the quantum or classical character of the theory depends on the choice of the specific distance.



(a) δ-neighbourhoods

(b) Complements

Figure 3.3: δ-neighbourhoods and their complements. In panel (a), the δ-neighbourhoods using Monge distance and the natural one are reported. The smaller one is the latter. In panel (b), the complements for the δ-neighbourhoods are reported, showing that $C_{D_M}^N(\Omega,\delta) \subseteq C_d^N(\Omega,\delta)$.

Box 3.2: the Monge distance

Let us consider two probability densities Q_1 and Q_2 defined in an open set O of a metric space $\mathcal{M}$ with measure μ and note that it is $Q_i \geq 0$ and $\int_O Q_i(x)dx = 1$, for $i = 1, 2$. We define V_1 and V_2 as

$$V_i = \{(x, y) \in O \times \mathbb{R}^+ \ : \ 0 \leq y \leq Q_i(x)\}, \ i = 1, 2 \ . \tag{3.43}$$

The sets V_i describe the *mounts of soil* with shape given by the probability densities Q_i. The many ways to move the first pile into the second one are described by C^1 one-to-one maps $T : O \to O$ that are volume-preserving. This condition is made explicit by

$$\int_O Q_1(x)d\mu(x) = \int_{T^{-1}(O)} Q_2(x)d\mu(x) \ . \tag{3.44}$$

The Monge distance is defined as

$$D_M(Q_1, Q_2) = \inf_T \left\{ \int_O d(x, T(x))Q_1(x)d\mu(x) \right\} \ . \tag{3.45}$$

The transformation that realizes the minimum in (3.45) is called *Monge plan*. As already mentioned, finding the Monge distance can be hard. An exact solution has been found [13] when $\mathcal{M}$ is $\mathbb{R}$ with the Euclidean distance. By defining $F_i(x) = \int_{-\infty}^x Q_i(t)dt$ it is possible to show that

$$D_M(Q_1, Q_2) = \int_{-\infty}^{\infty} |F_1(x) - F_2(x)|dx \ . \tag{3.46}$$

Some problems defined on $\mathbb{R}^2$ (and on some 2-dimensional manifolds) can also be solved, by using Eq. (3.46) plus some symmetry considerations [12].

3.3.2 Finding the lower bound

In this section, we look for the value of N that separates the classical regime from the quantum one. To this aim, we define the sets $I_d^N(\Omega,\delta)$ and $I_{D_M}^N(\Omega,\delta)$ as the collections of points that are at a distance smaller than δ from Ω (for fixed N), using the natural distance and the Monge distance respectively, i.e.

$$I_d^N(\Omega,\delta) = \{\Omega' \in \mathcal{M} \ : \ d(\Omega,\Omega') \leq \delta\} \ , \tag{3.47}$$

$$I_{D_M}^N(\Omega,\delta) = \{\Omega' \in \mathcal{M} \ : \ D_M(\Omega,\Omega') \leq \delta\} \ . \tag{3.48}$$

From the properties (3.37-3.38) it follows that[7]

$$I_d^N(\Omega,\delta) \subseteq I_{D_M}^N(\Omega,\delta) \ \forall N \ , \tag{3.50}$$

and

$$I_d^\infty(\Omega,\delta) = I_{D_M}^\infty(\Omega,\delta) \ . \tag{3.51}$$

By defining the complements

$$C_d^N(\Omega,\delta) = \overline{I_d^N(\Omega,\delta)} \ \text{ and } \ \overline{I_{D_M}^N(\Omega,\delta)} = C_{D_M}^N(\Omega,\delta) \ ,$$

from (3.50) it follows that

$$C_{D_M}^N(\Omega,\delta) \subseteq C_d^N(\Omega,\delta) \ . \tag{3.52}$$

We also define the set of points on $\mathcal{M}$ that correspond to GCS which are ϵ-orthogonal, i.e.

$$O_\epsilon^N(\Omega) = \{\Omega' \in \mathcal{M} \ : \ |\langle\Omega|\,\Omega'\rangle| \leq \epsilon\} \ . \tag{3.53}$$

Consistently with these definitions, we call N_d the smallest value of N such that

$$\Omega' \in C_d^N(\Omega,\delta) \Rightarrow \Omega' \in O_\epsilon^N(\Omega), \ \forall\Omega, \ \forall N \geq N_d \ . \tag{3.54}$$

and N_M the smallest value of N such that

$$\Omega' \in C_{D_M}^N(\Omega,\delta) \Rightarrow \Omega' \in O_\epsilon^N(\Omega), \ \forall\Omega, \ \forall N \geq N_M \ . \tag{3.55}$$

From (3.52) it follows that $N_d \geq N_M$. In fact, let us consider $\tilde{N} \geq N_d$: the following implication holds

$$\Omega' \in C_d^{\tilde{N}}(\Omega,\delta) \Rightarrow \Omega' \in O_\epsilon^{\tilde{N}}(\Omega), \ \forall\Omega \ . \tag{3.56}$$

On the other hand, as $\Omega' \in C_d^{\tilde{N}}(\Omega,\delta)$ implies $\Omega' \in C_{D_M}^{\tilde{N}}(\Omega,\delta)$, it also holds true that

$$\Omega' \in C_{D_M}^{\tilde{N}}(\Omega,\delta) \Rightarrow \Omega' \in O_\epsilon^{\tilde{N}}(\Omega), \ \forall\Omega \ , \tag{3.57}$$

which implies $\tilde{N} \geq N_M$. The proposition $\tilde{N} \geq N_d \Rightarrow \tilde{N} \geq N_M$ means $N_d \geq N_M$, as expected. In conclusion, for fixed values of δ and ϵ we obtain different values of N to discriminate the classical from the quantum behaviour: if $N \geq N_d$, the theory we are considering is still quantum but can be treated as classical (within our accuracy). Having two definitions of distance on the manifold means that we find two different bounds on N:

- if $N < N_M$ the theory is surely (which means using both distance definitions) quantum.
- if $N > N_d$ the theory is surely (which means using both distance definitions) classical.
- if $N_M < N < N_d$ the theory is classical if considering the actual distance between states, as defined by the Monge metric, while it is quantum if using the natural distance, which is not exact but can be evaluated analytically.

[7] This can be shown as follows. If $\Omega' \in I_d(\Omega,\delta)$ then $d(\Omega,\Omega') \leq \delta$. Since

$$D_M(\Omega,\Omega') \leq d(\Omega,\Omega') \leq \delta \tag{3.49}$$

then $\Omega' \in I_d(\Omega,\delta)$ implies $\Omega' \in I_{D_M}(\Omega,\delta)$: all the points of $I_d(\Omega,\delta)$ are also points of $I_{D_M}(\Omega,\delta)$. The converse is not necessarily true.

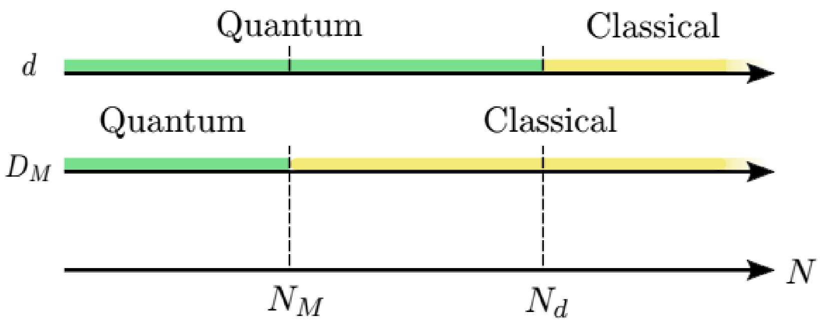


Figure 3.4: N_d and N_M lower bounds arising from the natural distance (N_d) and the Monge one (N_M). Since $D_M < d$, the bound obtained using with the Monge distance is smaller than that corresponding to the natural one.

3.3.3 An example: N_d for the $\mathfrak{su}(2)$ coherent states

In order to better understand the meaning of the above discussion, let us consider an explicit example. We consider a large-N theory whose Lie algebra is $\mathfrak{su}(2)$; we know from Sec. 2.1 that the proper CSS manifold $\mathcal{M}$ is the Bloch sphere. Without getting into the details, let us assume that it is possible to obtain a large-N quantum theory from the microscopic one just rescaling $j \to jN$[8].
Referring to Ω and Ω' as introduced in Sec. 3.3.1, we choose $|\Omega\rangle = |0\rangle$, with $|0\rangle$ the reference state used to construct the GCS, i.e. the eigenstate of $\hat{\sigma}_3$ with eigenvalue -1, for the evaluation of the δ-neighbourhood and the CSS overlaps. The natural distance induced by Eq. (2.17) between the point on $\mathcal{M}$ associated with $|0\rangle$ and that one of $|p\rangle = |\theta, \phi\rangle$ is

$$d(0,p) = \theta \ , \tag{3.58}$$

where we have normalized the Bloch-sphere radius to 1. By means of the correspondence (1.13), the δ-neighbourhood of $|0\rangle$ is then

$$I_d(0,\delta) = \{(\theta,\phi) \in \mathcal{M} \ : \ \theta < \delta\} \ , \tag{3.59}$$

meaning that, when read as points of $\mathcal{M}$, any GCS at an angle larger than δ should be δ-distinguishable from $|0\rangle$, as $N > N_d$. Note that the ϕ coordinate does not enter in this definition, as the distance of any point from the south pole of the sphere does not depend on the azimuth angle. Since the overlap between any two spin-coherent states is given by Eq. (2.22), when we choose one of them to be $|0\rangle$ we get

$$|\langle 0|\, \Omega(\theta,\phi)\rangle| = \left(\cos\frac{\theta}{2}\right)^{jN} \ . \tag{3.60}$$

Finally, by requiring that the overlap with $|0\rangle$ is smaller than ϵ for all $|\theta,\phi\rangle$ such that $(\theta,\phi) \in C_d^{N_d}(0,\delta)$, we get

$$\cos\frac{\theta}{2} < \cos\frac{\delta}{2} \Rightarrow \left(\cos\frac{\theta}{2}\right)^{jN_d} < \epsilon \tag{3.61}$$

and the natural distance is thus seen to induce

$$N_d = \frac{1}{j}\frac{\log\epsilon}{\log\cos(\delta/2)} \ . \tag{3.62}$$

[8]This is explicitly shown for the $\mathfrak{su}(1,1)$ Lie algebra at the end of this chapter and in a way that can be easily adapted to the present case, without any relevant difference.

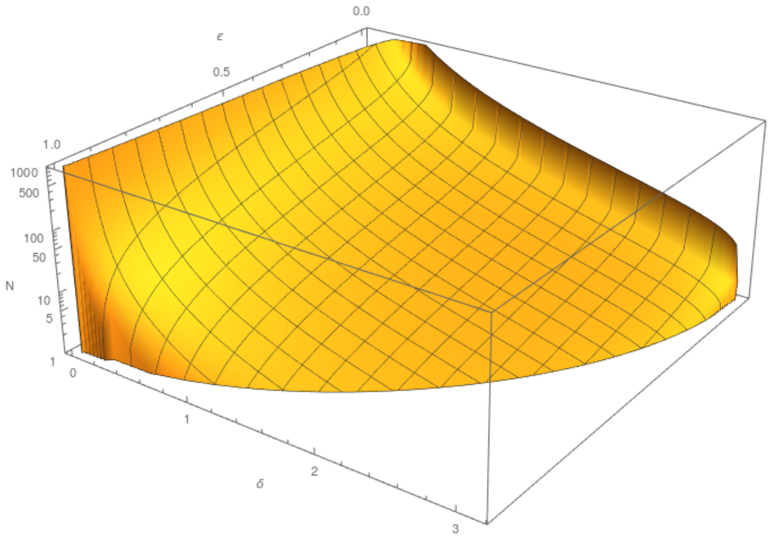


Figure 3.5: $\delta - \epsilon$ plot of N_d for the case of a theory with $\mathfrak{g} = \mathfrak{su}(2)$ (see text for details). The vertical axis is in log scale.

More generally, to find N_d we should look for the largest value that RHS in equation (3.62) assumes, on different points of $\mathcal{M}$, for fixed δ and ϵ. This can be done by rotating the manifold so as to bring any given point p on the north pole[9]. Then we are allowed to apply the above argument.

The expression (3.62) means that different choices of ϵ and δ give different values for N_d. As ϵ and δ become smaller, this bound increases: an observer with good accuracies will perceive even a large system as being quantum. This is shown in Fig. 3.5, where N_d is shown as a function of δ and ϵ.

3.3.4 From classical to quantum

The lower bounds N_d and N_M we found in the previous section show their power when combined with the free-theory description proposed in this chapter. In fact, since one can cut out from the overall system some of its constituent subsystems, without altering the main features of the collective behaviour, an effective classical theory that describes a system made by a large and yet finite number N of subsystems (for some given δ and ϵ parameters) can be made to unveil its underlying quantum structure by consecutive cuttings.

Before entering the details of one such procedure, let us qualitatively describe how it might work. The phases of the process should be as follows:

- when $N \simeq \infty$, the system is classical.
- when $N \gtrsim N_d$, the system is still classical but can manifest quantum features when investigated by local quantum measurements.
- when $N_M < N < N_d$ the system is quantum for any observer using the d distance to tell apart points on the CSS manifold.
- when $N < N_M$ the system is quantum.

This piecewise process is exactly what we are pursuing, namely a sort of classical-to-quantum crossover, with a quasi-classical regime emerging in between the small- and the large-N behaviours. Given the relevance of this mechanism for our scopes, in this section we discuss it in some more detail.

[9]As this is a translation, it does not change the quantities of our interest (see Ref. [12]).

Let us consider a quantum theory $\mathcal{Q}_N = \{\mathfrak{g}_N, r_N, \hat{H}_N\}$ with finite N such that an observer provided with some δ-distinguishability and ϵ-orthogonality parameters experiences it as a classical theory $\mathcal{C}$. Let us suppose the microscopic theory underlying $\mathcal{Q}_N$ is a free theory. By randomly selecting one of the N subsystems, labelled by j_1, and changing its couplings, we break the N-permutation symmetry. However, since we have selected the free-theory description, if the system is large enough we can leave out the j_1-th system and use the $N-1$ subsystems to get a large-N limit which is identical to the one we would have obtained before the cut. By repeating this procedure p times the global Hamiltonian $\hat{H}_N$ goes to

$$\hat{H}_{N-p} = \sum_{i \neq j_1,\ldots,j_p}^{N} \left(a\hat{K}_0^i + b\hat{K}_+^i + b^*\hat{K}_i^- \right) + \sum_{k=1}^{p} \left(a_{j_k}\hat{K}_0^{j_k} + b_{j_k}\hat{K}_+^{j_k} + b_{j_k}^*\hat{K}_-^{j_k} \right) . \tag{3.63}$$

The new set of global operators

$$\begin{cases} \hat{K^p}_0 = \dfrac{1}{N-p} \displaystyle\sum_{i \neq j_1,\ldots,j_p} \hat{K}_0^i \\ \hat{K^p}_\pm = \dfrac{1}{N-p} \displaystyle\sum_{i \neq j_1,\ldots,j_p} \hat{K}_\pm^i \end{cases} \tag{3.64}$$

close the Lie brackets

$$[K_0^p, K_\pm^p] = \frac{1}{N-p}(\pm\hat{K}_\pm^p) \tag{3.65}$$

$$[K_+^{(L)}, K_-^{(L)}] = \frac{1}{N-p}(-2\hat{K}_0^p) , \tag{3.66}$$

and thus they form a $\mathfrak{g}_{N-p}$ algebra. After removing p subsystems, the effective value of N to be used when performing the large-N limit, decreases as

$$N \to N_p = N - p . \tag{3.67}$$

If N_d is the lower bound above which a theory $\mathcal{Q}_{N>N_d}$ can be effectively considered as a classical one, $\mathcal{C}$, after a finite number $p > N - N_d$ of removals it is

$$\mathcal{Q}_{N-p<N_d} \tag{3.68}$$

and the original quantum nature of $\mathcal{Q}_{N-p<N_d}$ must be retained: in other terms, the effective description of $\mathcal{Q}_N$ through $\mathcal{C}$ ceases to be valid and the system recovers its quantum character. It is important to remark that this classical-to-quantum crossover is observer-dependent, as the parameters that define N_p are fixed by the accuracy of the tools used to probe the system.

3.4 An example: $\mathfrak{su}(1,1)$ free theories and classical to quantum transition

In this last section of the chapter, we summarize our analysis by means of one example: finding the free-theory description of a large-N quantum theory whose Lie algebra is $\mathfrak{su}(1,1)_N$ and discussing its quasi-classical regime. This, besides being interesting by its own, establish a link with the next part of the thesis, where this analysis will play a relevant role in the discussion on QBH and their evaporation.
Let us consider the large-N quantum theory

$$\mathcal{Q}_N = \{\mathfrak{su}(1,1)_N, r_N, \hat{H}_N\} . \tag{3.69}$$

Following Sec. 3.2.1, we define $r_N = (\Phi_N, \mathcal{H}^{\otimes N})$ using a representation $r = (\phi, \mathcal{H})$ of $\mathfrak{su}(1,1)$ as

$$\begin{cases} \hat{\Phi}_N(K_0) = \hat{K}_0 = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \hat{\phi}_i(k_0) = \dfrac{1}{N}\sum_{i=1}^{N} \hat{k}_0^i \ , \\ \hat{\Phi}_N(K_\pm) = \hat{K}_\pm = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \hat{\phi}_i(k_\pm) = \dfrac{1}{N}\sum_{i=1}^{N} \hat{k}_\pm^i \ , \end{cases} \tag{3.70}$$

where the uppercase K_α indicates the elements (and their representations when marked by the hat) of the rescaled algebra, while the lowercase k_α the elements of each original one and where i denotes the local subsystems. The global Hamiltonian is defined as in equation (3.5) and the microscopic Hamiltonians as in equation (3.17). Thus this is a free-theory description of a pseudo-spin system made by N pseudo-spin subsystems.

$N \to \infty$ limit

In this paragraph, we build the GCS for $\mathcal{Q}_N$ to check if this description has a well behaved $N \to \infty$ limit which is a classical system, i.e. if assumptions **A1-A4** are verified. $\mathfrak{g}_N$ can be constructed from a reference state $|Z\rangle_N$ which is the tensor product of the local reference states, $|0_i\rangle$, i.e.

$$|Z\rangle_N = \otimes_{i=1}^{N} |0_i\rangle \ , \tag{3.71}$$

as

$$|\Omega\rangle_N = e^{\Omega \hat{K}_+ - \Omega^* \hat{K}_-} |Z\rangle_N = \bigotimes_{i=1}^{N} e^{\frac{\Omega}{N}\hat{K}_+^i + \frac{\Omega^*}{N}\hat{K}_-^i} |0_i\rangle \tag{3.72}$$

or, using the Poicaré disk coordinates, as

$$|T\rangle_N = \bigotimes_{i=1}^{N} |\tau_i\rangle \ , \tag{3.73}$$

where $|\tau_i| = |T|/N$, and i denotes the local subsystems. Note that the modulus of τ_i is indipendent of i. In this description, the only possible GCS are tensor products of N identical coherent states, each relative to one subsystem: a large part of the total Hilbert space $\mathcal{H}^{\otimes N}$ is not accessible because we are already required the presence of the global (permutation) symmetry $\mathbb{X}$ on $\mathcal{Q}_N$. The components of the algebra elements (3.11) evaluated on the GCS have the form

$$\begin{aligned} \frac{\langle T|\hat{K}_0|T'\rangle_N}{\langle T|T'\rangle_N} &= \frac{1}{N}\sum_i \frac{\langle \tau_i|\hat{K}_0^i|\tau_i'\rangle}{\langle \tau_i|\tau_i'\rangle} = \frac{\langle \tau|\hat{K}_0|\tau'\rangle}{\langle \tau|\tau'\rangle} = k\frac{1+\tau^*\tau'}{1-\tau^*\tau'} \ , \\ \frac{\langle T|\hat{K}_1|T'\rangle_N}{\langle T|T'\rangle_N} &= \frac{1}{N}\sum_i \frac{\langle \tau_i|\hat{K}_1^i|\tau_i'\rangle}{\langle \tau_i|\tau_i'\rangle} = \frac{\langle \tau|\hat{K}_1|\tau'\rangle}{\langle \tau|\tau'\rangle} = 2k\frac{Re(\tau)}{1-\tau^*\tau'} \ , \\ \frac{\langle T|\hat{K}_2|T\rangle_N}{\langle T|T'\rangle_N} &= \frac{1}{N}\sum_i \frac{\langle \tau_i|\hat{K}_2^i|\tau_i'\rangle}{\langle \tau_i|\tau_i'\rangle} = \frac{\langle \tau|\hat{K}_2|\tau'\rangle}{\langle \tau|\tau'\rangle} = -2k\frac{Im(\tau)}{1-\tau^*\tau'} \ , \end{aligned} \tag{3.74}$$

and they are finite in the large-N limit. Therefore all $\hat{K}_\alpha$ are classical operators and the symbols

$$\begin{cases} K_0(T) &= k\dfrac{1+|\tau|^2}{1-|\tau|^2} \ , \\ K_1(T) &= 2k\dfrac{Re(\tau)}{1-|\tau|^2} \ , \\ K_2(T) &= -2k\dfrac{Im(\tau)}{1-|\tau|^2} \ , \end{cases} \tag{3.75}$$

can be shown [8] to be different only for states $|T\rangle$ in different equivalence classes (as required by **A2**). While **A1** and **A4** are trivial, from the overlap of two GCS

$$\langle T|\,T'\rangle = \langle \tau|\,\tau'\rangle^N = \frac{(1-|\tau|^2)^{Nk}(1-|\tau'|^2)^{Nk}}{(1-\tau^*\tau')^{2Nk}} \ , \tag{3.76}$$

one can define

$$\phi(T,T') = -\frac{1}{N}\log\langle T|\,T'\rangle = k\log\left(\frac{(1-\tau^*\tau')^2}{(1-|\tau|^2)(1-|\tau'^2|)}\right) \tag{3.77}$$

and, since $\mathfrak{Re}(\phi(T,T)) = 0$ and

$$\mathfrak{Re}(\phi(T,T')) = k\left(2\log|1-\tau^*\tau'| - \log(1-|\tau|^2) - \log(1-|\tau'|^2)\right) > 0 \ , \quad \forall k > 0 \ , \tag{3.78}$$

assumption **A3** is verified. Lastly, the limit

$$\lim_{N\to\infty}\frac{1}{N}\hat{H}_N \tag{3.79}$$

is finite, as discussed in Sec. 3.2. Therefore, the $N\to\infty$ limit of the quantum theory $\mathcal{Q}_N$, Eq. (3.69), is a well defined classical theory.

Finite N classical theory

Following the path outlined in the previous section, we call N_d the minimum number of non-interacting $\mathfrak{su}(1,1)$ theories required to effectively regard $\mathcal{Q}_N$ as a classical theory. At finite N, from Eq. (3.76) it is

$$|\langle T|\,T'\rangle| = |\langle\tau|\,\tau'\rangle|^N = f(\tau,\tau')^N \ , \tag{3.80}$$

with

$$f(\tau,\tau') = (1-|\tau|^2)^k(1-|\tau'|^2)^k|1-\tau^*\tau'|^{-2k} \ ; \tag{3.81}$$

this function is smaller than 1 for any two different local GCS, i.e. $\tau\neq\tau'$, and it is one if $\tau=\tau'$. The δ-neighbourhood of the point associated with the GCS $|\tau\rangle$ is

$$I_d(\tau,\delta) = \{\tau'\in D_2 \ : \ d(\tau,\tau')\leq\delta\} \ , \tag{3.82}$$

where, following Ref. [14], the distance d over the Poincarè disk D_2 is defined as

$$d(\tau,\tau') = \log\left(\frac{|1-\bar{\tau}\tau'|+|\tau-\tau'|}{|1-\bar{\tau}\tau'|-|\tau-\tau'|}\right) \ . \tag{3.83}$$

By means of the Möbius transformation $T_p : D_2 \to D_2$, defined by

$$T_p(z) = \frac{z-p}{1-z\bar{p}} \ , \tag{3.84}$$

it is possible to bring any point $p\in D_2$ into the origin and, since the distance (3.83) is invariant under T_p, it is

$$d(\tau,\tau') = d(0,T_\tau(\tau')) = \log\left(\frac{1+|T_\tau(\tau')|}{1-|T_\tau(\tau')|}\right) \ . \tag{3.85}$$

To obtain N_d, for any given point on $\mathcal{M}$ we firstly perform the transformation (3.84), and then evaluate the δ-neighbourhood $I_d(0,\delta)$ which results in

$$I_d(0,\delta) = \left\{\tau'\in D_2 \ : \ \frac{1+|\tau'|}{1-|\tau'|}\leq e^\delta\right\} \ . \tag{3.86}$$

We finally require N_d to be the smallest integer such that

$$\tau\in\overline{I_d(0,\delta)} \ \Rightarrow \ |\langle 0|\,\tau\rangle| < \epsilon \tag{3.87}$$

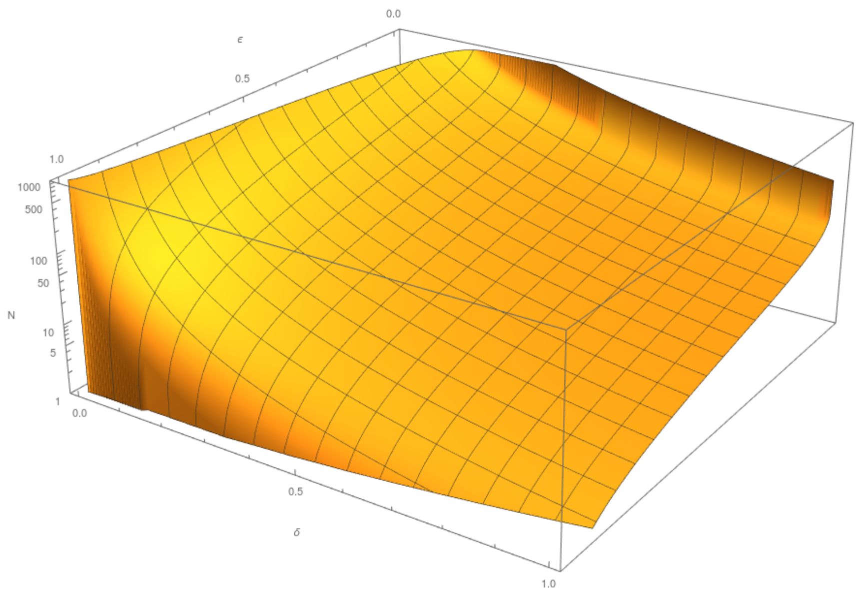


Figure 3.6: $\delta - \epsilon$ plot of the N_d bounds (with $k = 1/2$). Note that the vertical axis is in log scale.

or, equivalently, such that

$$\frac{1+|\tau|}{1-|\tau|} > e^{\delta} \;\Rightarrow\; (1-|\tau|^2)^{Nk} < \epsilon \; . \tag{3.88}$$

This provides us with

$$N_d = \frac{\log \epsilon}{2k \log(\frac{2e^{\delta/2}}{1+e^{\delta}})} \tag{3.89}$$

as a function of δ and ϵ. In Fig. 3.6, N_d is shown vs. δ and ϵ. We remind that a more appropriate[10] bound would be N_M, which is lower than N_d in general. However, as we are only interested in the quasi-classical regime, knowing for how many N the theory can be safely replaced by an effective classical one is already quite satisfactory.

[10]Even if the discussion in Ref. [12] is developed only for compact CSS manifolds, the reasoning there presented should apply even better to non-compact manifolds. However, a formal proof is missing.

Part II

Black Holes Evaporation

Chapter 4

Black Holes

Black holes (BH) are puzzling objects. Born from the collapse of sufficiently massive stars, BHs are characterized by the fact that everything can go in, but nothing (including light) can come out. Despite having been considered exotic and unphysical for a long time, in the last decades BHs were detected by observing the trajectories of stars moving in their surroundings, measuring the emission of gravitational waves formed when they merge (see Ref. [15]), and detecting the way they bend light rays (see Ref. [16] and successive papers). Thus, BHs should not be considered theoretical oddities anymore but rather actual physical entities.

BHs, at least when quantum effects are negligible, are properly described by General Relativity (GR). However, it is commonly accepted that GR is not the end of the story of gravity and a more general theory has been long sought after. This theory is expected to solve GR's issues by referring to QM, for which it is dubbed *quantum gravity.* In fact, some of the problematic features of GR are direct consequences of the presence of BH: GR predicts that BHs have an unphysical singularity at their centre and that they can evaporate due to quantum effects. This second feature sparks our curiosity: *being BHs objects that manifest both a classical and a quantum behaviour, shouldn't they fall in the quasi-classical regime we have investigated?*

In this chapter, referring to App. C and Refs. [17] for further details, we review some notions about BHs that will be useful to set up the stage of our description of QBH. In the first section we describe some features of the so-called Schwarzschild solution to Einstein equations and its limiting case, called Schwarzschild Black Hole (SBH for short). In the second section, we discuss the connection between BHs and thermodynamics. Even if these two realms of physics are very different some of their laws are strikingly similar. Moreover, when one tries to define a temperature for BHs finds the use of Planck's constant as necessary, suggesting that something quantum is missing from the classical description of BHs. This hint is taken, in the third section, to discuss the so-called *black hole's evaporation*, a process described by S.Hawking in Ref. [18] that emerges due to an effective combination of QM and GR. In the last section, we shortly expose the work done in Ref. [8] to describe SBHs as large-N quantum systems, which is a prelude to the next chapter.

4.1 Schwarzschild Black Holes

Schwarzschild's solution of Einstein's Equations describes the spacetime outside a spherically symmetric distribution of mass with no angular momentum and no electric charge. These requests upon the source of spacetime curvature translate into the request of having a spherically symmetric spacetime outside of it or, in more mathematical terms, into the request, on the isometry group acting on the manifold $\mathcal{M}$ to contain an $SO(3)$ subgroup whose orbits are 2-

spheres. This means that elements of the Lie algebra $\mathfrak{so}(3) = \{L_1, L_2, L_3\}$, with Lie brackets

$$[L_i, L_j] = \epsilon_{ijk} L_k \ , \tag{4.1}$$

are present in the collection of elements K, dubbed *Killing vectors*, such that

$$\mathcal{L}_K(g) = 0 \ , \tag{4.2}$$

where $\mathcal{L}_K(g)$ denotes the Lie derivative of the metric tensor with respect to K (see App. C). These vector fields do not exhaust the isometries of Schwarzschild's spacetime: as an additional feature is time-independent and thus a timelike Killing vector can be found. In the Schwarzschild coordinates (t, r, θ, ϕ) the metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{4.3}$$

where M is the mass of the source of gravitational field and G is gravitational constant. From Eq. (4.3) it follows that the timelike Killing vector has the simple form

$$\xi = \frac{\partial}{\partial t} \ . \tag{4.4}$$

This solution applies to the spacetime outside any astrophysical object, even if its radius is time-dependent, as long as its mass does not change and the spherical symmetry is preserved as in the case for collapsing stars. Moreover, a theorem due to Birkhoff states that the Schwarzschild's solution is the only spherically symmetric solution to Einstein's equations possible in vacuum (meaning that no time-dependent vacuum and spherically symmetric solutions actually exist).

Limits

As the r coordinate approaches infinity, the metric becomes the Minkowski's one

$$ds^2 \xrightarrow[r\to\infty]{} -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{4.5}$$

A spacetime with this property is said to be "asymptotically flat": observers that are at a sufficiently large distance from the source of curvature see the physics as described by special relativity. The same result can be obtained in the $M \to 0$ limit, meaning that if the mass goes to zero, then the spacetime becomes flat.

Singularities

The metric (4.3) is divergent at two different values of the r coordinate: $r = 0$ and the Schwarzschild radius, $r_s = 2GM$. The singularity in r_s is removable by a change of coordinates and it is possible to show that an observer passing through the $r = r_s$ surface, called *event horizon*[1], does not experience anything strange. However, the event horizon still presents some unique features: light cannot reach infinity from points on and beyond it, and hence observers outside this region cannot get knowledge of events there happening. The singularity in $r = 0$ is intrinsic and thus it cannot be removed. The fact that GR predicts intrinsic singularities in its own solutions is regarded as a signature of it containing the seeds of its own failure.

We call Schwarzschild Black Hole (SBH) a solution of Einstein's equations in which the Schwarzschild solution holds for $r > 0$. Since the spacetime is not singular on the event horizon, it is possible to find a set of coordinates to describe the region $r \leq r_s$, called *Kruskal coordinates*. These

[1]Note that not all spherically symmetric solutions present an event horizon. In fact, the Schwarzschild metric is valid only outside the body generating the curvature and if its external surface lays at a radius greater than $2GM$ then there is no horizon: the solution is not a BH one.

coordinates describe the full geometry of the SBH, including two asymptotically flat regions and two different horizons, called *future event-horizon* $\mathcal{H}^+$ and *past event-horizon* $\mathcal{H}^-$, respectively. The causal structure of the SBH is shown in Fig. 4.1, where the Carter-Penrose diagram for the SBH is shown.

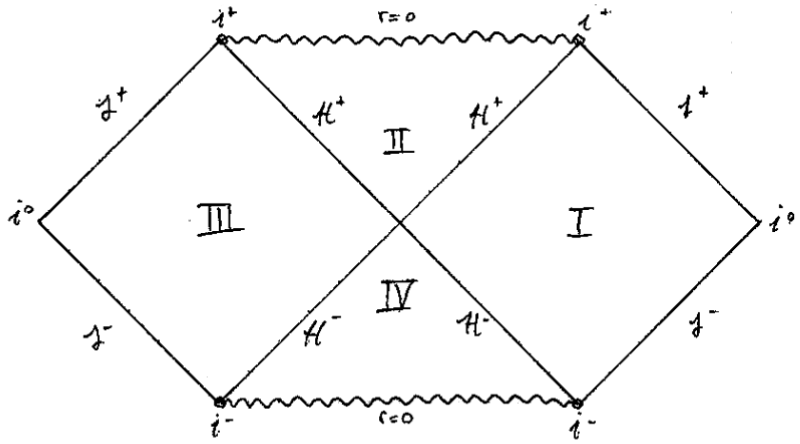


Figure 4.1: Carter-Penrose (CP) conformal diagram for the SBH: in this diagram each point represents a 2-sphere, and light cones are crossing lines inclined by 45 degrees (see App. C). The two waving lines represent the $r = 0$ singularities and the two crossing lines separating the regions I,II,III, and IV are the event horizons at $r = r_s$. As mentioned in the text, when an observer falls inside region II there is no way to leave.

Contraction of time

An observer near the event horizon of a BH experiences a different flow of time when compared with the one experienced by another observer that is elsewhere (i.e. at $r = \infty$).
To describe this phenomenon, let us consider two observers[2]: Alice A and Bob B, that lay at different fixed points on the spacelike slices: (r_A, θ, ϕ) and (r_B, θ, ϕ), respectively, with $r_A < r_B$. The only way that the two observes have to compare their times is to send two distinct signals (i.e. photons) to each other. Let Alice send two signals to Bob separated by a previously agreed time interval $\Delta\tau_A$. After receiving the signals, B can check the difference between $\Delta\tau_A$ and that he observes, $\Delta\tau_B$.
Since the spacelike slices do not change over time, the two time-separated photons emitted by A follow the same path. The photons emitted by Alice are separated by a time interval $\Delta\tau_A$ that, in Schwarzschild time, becomes

$$\Delta\tau_A = \int_0^{\Delta t} dt\sqrt{-g_{00}(r_A)} = \Delta t\sqrt{1 - 2GM/r_A} \;\Rightarrow\; \Delta t = \frac{\Delta\tau_A}{\sqrt{1 - 2GM/r_A}} \;. \tag{4.6}$$

Bob receives the two signals at a different time distance, given by

$$\Delta\tau_B = \int_0^{\Delta t} dt\sqrt{-g_{00}(r_B)} = \Delta t\sqrt{1 - 2GM/r_B} = \Delta\tau_A\sqrt{\frac{1 - 2GM/r_B}{1 - 2GM/r_A}} \;. \tag{4.7}$$

The ratio between the two time intervals is

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - 2GM/r_B}{1 - 2GM/r_A}} > 1 \;, \tag{4.8}$$

[2]Alice and Bob are the most widely used names for observers in quantum information theory. For being consistent with the later discussion we start using these names here.

which makes clear that $\Delta\tau_B > \Delta\tau_A$, as claimed. Note that as r_A goes to $r_s = 2GM$, $\Delta\tau_A$ as seen by Bob diverges.

4.2 Surface Gravity and the Hawking temperature

We call *Killing horizon* for some Killing vector field ξ a hypersurface where ξ becomes null. Even if Killing horizons and event horizons are two different concepts, it can be shown that if the spacetime is static and asymptotically flat, then every event horizon in that spacetime is also a Killing horizon for a Killing vector field that represents time translations at infinity. If we call $\mathcal{N}$ the Killing horizon, it is possible to show that

$$\nabla_\xi \xi|_{\mathcal{N}} = k\xi|_{\mathcal{N}} \ , \tag{4.9}$$

where k is called *surface gravity* of the horizon[3]. Note that if ξ is a Killing vector, then also any $\xi' = c\xi$ is a possible Killing vector that defines a rescaled k, according to

$$\xi^\mu \nabla_\mu \xi^\nu = k\xi^\nu \longrightarrow \xi'^\mu \nabla_\mu \xi'^\nu = ck\xi'^\nu \ . \tag{4.10}$$

For a well-defined description of the surface gravity it is necessary to fix a normalization and a sign for ξ. Consequently, we require ξ to be future-directed and to satisfy

$$\lim_{r\to\infty} \xi^2 = -1 \ , \tag{4.11}$$

which means that $\xi \to \partial_t$ at $r = \infty$.
Since we do not have any physical understanding of equation (4.9), one might ask why is that k is called "surface gravity". To answer this question let us consider a static observer and ask what is its acceleration as seen from an observer at infinity. A static observer is an observer that moves along a world-line with proper velocity u^μ proportional to the Killing vector. We choose

$$\xi^\mu = V(x)u^\mu \ , \tag{4.12}$$

where the function $V(x)$ is fixed by the fact that $u^\mu u_\mu = -1$, as

$$V(x) = \sqrt{-\xi^\mu \xi_\mu} \ . \tag{4.13}$$

For example, in the Schwarzschild black hole, this function is

$$V(r) = \sqrt{-\xi^\mu \xi_\mu} = \sqrt{-g_{00}} = \sqrt{1 - \frac{2GM}{r}} \ , \tag{4.14}$$

that is the same factor determining time contraction at the denominator of (4.6), called *redshift factor*. The four-acceleration of an observer at distance r is

$$a^\mu = u^\nu \nabla_\nu u^\mu \ . \tag{4.15}$$

Using the fact that $\nabla_\xi \xi^2 = 0$ holds for Killing vectors, we get the expression

$$a^\mu = -\frac{\xi^\nu \nabla_\nu \xi^\mu}{\xi^2} \Rightarrow |a| = \frac{\sqrt{(\nabla_\xi \xi)^2}}{V} \ , \tag{4.16}$$

for the proper acceleration of an observer at fixed distance r, which is divergent when $r \to 2GM$; however this acceleration remains finite when seen from another point at $\tilde{r} > 2GM$

$$A^\mu_{\tilde{r}} = \frac{d\tau}{dt_{\tilde{r}}} a^\mu \Rightarrow |A_{\tilde{r}}| = \frac{V(r)}{V(\tilde{r})}|a(r)| \ . \tag{4.17}$$

[3] Thanks to the Frobenius theorem it is possible to show that k is constant over $\mathcal{N}$.

By choosing $r \to 2GM$ (where the definition (4.9) holds) and $\tilde{r} = \infty$ we get

$$|A_\infty| = k \ . \tag{4.18}$$

This result answers our initial question: the acceleration of a static observer at r_s as seen from infinity is equal to the surface gravity. We can find the explicit value of the four acceleration[4] of an observer at distance r as

$$a_\mu = u^\nu \nabla_\nu u_\mu = \nabla_\mu \log V \ . \tag{4.19}$$

with modulus

$$a = \sqrt{a_\mu a^\mu} = \frac{1}{V}\sqrt{\nabla_\mu V \nabla^\mu V} \ . \tag{4.20}$$

This means that

$$k = V|a|_{r_s} = \sqrt{g^{rr}\partial_r V \partial_r V}|_{r_s} = \frac{GM}{r^2}|_{r=2M} = \frac{1}{4GM} \tag{4.21}$$

for SBHs.

Hawking Temperature

Classical thermodynamics describes the behaviour of closed systems made of a very large number of components (generally, this number is of the order of magnitude of Avogadro's constant, N_A). The laws ruling this area of physics have been known for a long time and therefore they have various forms. In box 4.1 we report those laws in a way that makes the analogy with the laws that govern BHs quasi-equilibrium dynamics (as described in box 4.2) explicit. In fact, it is shown in Ref. [19] that if one identifies the surface gravity κ with some temperature and the area of the event horizon[5], $\mathcal{A}$, with some entropy, then box 4.1 and 4.2 become very similar.

Box 4.1: Laws of thermodynamics

0th law: The temperature through a system in thermal equilibrium is constant.
1st law: If a closed system in thermal equilibrium at temperature T with internal energy U undergoes an infinitesimally reversible transformation such that it reaches another equilibrium state characterized by internal energy $U + \delta U$ then

$$dU = TdS \tag{4.22}$$

holds.

2nd law: the total entropy of an isolated system can never decrease with time, that is

$$\frac{dS}{dt} \geq 0 \tag{4.23}$$

[4]This expression of the four-acceleration can be found from the definition of the four-velocity $u^\mu u_\mu = -1$ and the Killing equation. Since $\nabla_\nu(u^\mu u_\nu) = 0$ we have that $u_m u \nabla_\nu u^\mu = 0$; by multiplying the Killing equation

$$\begin{aligned}\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\\ \nabla_\mu(u_\nu V) + \nabla_\nu(u_\mu V) = 0\\ V\nabla_\mu u_\nu + u_\nu \nabla_\mu V + V\nabla_\nu u_\mu + u_\mu \nabla_\nu V = 0\end{aligned}$$

by $u^\mu u^\nu$, we get that $u^\mu \nabla_\mu V = 0$, and hence that

$$\begin{aligned}Vu^\nu \nabla_\mu u_\nu - \nabla_\mu V = 0\\ -\nabla_\mu V + Vu^\mu \nabla_\nu u_\mu = 0 \ \Rightarrow \ a_\mu = u^\nu \nabla_\nu u_\mu = \frac{\nabla_\mu V}{V} = \nabla_\mu \log V\end{aligned}$$

[5]We define the area of the event horizon $\mathcal{A}$ as the area of the spherical surface $r = r_s$ and $t = const.$ (see App. C). The event-horizon area for a SBH is $\mathcal{A} = 16\pi M^2$.

Box 3.2: Laws of black holes mechanics

0th law: If $T^{\mu\nu}$ satisfies the *Dominant Energy Condition* (that is, roughly speaking, if energies are non-negative and move at most at the speed of light) then the surface gravity is constant over the (future) event horizon.
1st law: If a stationary non-rotating and electrically neutral black hole[a] with mass M is perturbed in such a way that it reaches another state, still non-rotating and electrically neutral, but with mass $M + \delta M$, then

$$dM = \frac{k}{8\pi G} d\mathcal{A} \tag{4.24}$$

holds.
2nd law: If $T^{\mu\nu}$ satisfies the *Weak Energy Condition* (that is, roughly speaking, if every observer can see only positive energies) and assuming the *Cosmic Censorship Conjecture* (see App.C), the area A of the black hole is a non-decreasing function of time, that is

$$\frac{d\mathcal{A}}{dt} \geq 0 \tag{4.25}$$

[a] The 1st law also covers the case of rotating and electrically charged BHs, but since we don't need those features, here we have set $J = Q = 0$ for the sake of simplicity.

Despite this resemblance being quite surprising, these analogies are not sufficient to claim that a black hole is a thermodynamical system: some differences are worth pointing out. Firstly, thermodynamical laws hold for systems with a large number of degrees of freedom, while it has been shown that BHs are described by just their mass M, their charge Q and their angular momentum J. Furthermore, while the laws in the first box hold for closed systems, it has been shown that the laws in box 4.2 retain their validity also for open systems (i.e. BHs with matter fields in their surroundings) with little modifications. Lastly, if we take into account only classical physics, BHs cannot have a temperature because they do not radiate; this issue, already known in Ref. [19], was later addressed by S.Hawking. Even though making explicit the connections

$$T \propto k \text{ and } S \propto \mathcal{A}/G ,$$

the previous analogies do not provide the proportionality constants between the thermodynamical quantities and the BHs related ones: this is made evident by the fact that $\mathcal{A}/G$ and k have not the dimensions of an entropy and a temperature respectively. In fact

$$\frac{[\mathcal{A}]}{[G]} = [L]^2 \frac{[\hbar]}{[L]^2} = [\hbar] \Rightarrow \left[\frac{\mathcal{A}}{G}\right] = [E][L] \tag{4.26}$$

and $[k] = [L]^{-1}$, being an acceleration. Nevertheless the product

$$\left[\frac{\mathcal{A}}{8\pi G k}\right] = \frac{[E]}{[L]}[L] = [E] \tag{4.27}$$

does have the right dimensions, as expected. To factor two parts with dimensions $[S]$ and $[T]$ from (4.24) in Box 3.2, it is sufficient to multiply and divide by $\hbar/k_B$; then, by grouping the constants in a convenient way, we get the famous definitions

$$S_{BH} \doteq \frac{A k_B}{4\hbar G} , \tag{4.28}$$

called *Bekenstein-Hawking entropy*, and

$$T_H \doteq \frac{k\hbar}{2\pi k_B} , \tag{4.29}$$

called *Hawking temperature*[6]. It is important to note that Planck's constant appears in Eq. (4.29) form a fully classical description: this remarkable fact is regarded as a clue for the intrinsic quantum character of BHs temperature. Finally, if we accept T_H to be the temperature of a BH as seen from infinity, then the temperature seen by an observer at finite distance is given by Tolman's law (see Ref. [20]) as

$$T_H(r) = \frac{T_H}{\sqrt{-\xi^\mu \xi_\mu}} = \frac{T_H}{\sqrt{1-\frac{2M}{r}}} \ . \tag{4.30}$$

4.3 Unruh effect and Hawking Radiation

An even stronger argument for identifying the surface gravity of a BH with its temperature stems from the evidence that BHs emit thermal radiation. This phenomenon is an inevitable result of the combination of GR and QM, as shown by S.Hawking in Ref. [18]. The fact that this radiation, called *Hawking radiation*, is emitted exactly at temperature T_H is a good reason to conclude that BHs mechanics do have some similarity with classical thermodynamics. However, the presence of thermal radiation due to quantum effects makes us wonder a question about the nature of BHs: *are they classical, quantum, or something in between?*

Particle creation

Particle creation in flat and curved spacetime is the effect according to which observers that cannot agree on a unique definition of time also have different notions of what a vacuum state is. In special relativity, this means that accelerated observers measure thermal radiation where inertial ones detect the vacuum, and in GR that observers in a non-stationary spacetime see some particles in what they would have called the vacuum in the past.

The main tool we use to investigate particle creation is the so-called *Bogoliubov transformation.* This transformation is a way to express a set of creation and annihilation operator $\{\hat{a}_i, \hat{a}_i^\dagger\}$ in terms of another set $\{\hat{b}_i, \hat{b}_i^\dagger\}$ of given operators, where the index i runs over the possible modes (here taken as discrete to highlight the general idea without unnecessary complication).

We denote the vacuum of the modes a_i as $|0\rangle_{a_i}$, defined as usual to be such that

$$\hat{a}_i \left|0\right\rangle_{a_i} = 0 \ \forall i \ . \tag{4.31}$$

The number operator $\hat{N}_i^a = \hat{a}_i^\dagger \hat{a}_i$ satisfies

$$\left\langle n_i\right|_{a_i} \hat{N}_i^a \left|n_i\right\rangle_{a_i} = n_i \ , \tag{4.32}$$

and the total number operator

$$\hat{N}^a = \sum_{i=1}^{d} \hat{N}_i^a \tag{4.33}$$

gives the number of total excitations (or particles) regardless of their frequency. We define the vacuum and the number operator for the second set of operators $\hat{b}_i$ likewise. Writing the latter operators as linear combinations of the former

$$\begin{cases} \hat{b} = A\hat{a} + B\hat{a}^\dagger \\ \hat{b}^\dagger = B^*\hat{a} + A^*\hat{a}^\dagger \ , \end{cases} \tag{4.34}$$

for some A and B square matrices, and imposing the commutation rules

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \ [\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij} \ , \tag{4.35}$$

[6]To arrange the numerical factors a more precise discussion is needed. Since an explanation of BHs thermodynamics is not in the scopes of this thesis, we will take them for granted.

one finds that the condition

$$AA^\dagger - BB^\dagger = \mathbb{I}_d \tag{4.36}$$

must be satisfied. The a-vacuum can be evaluated by the b-number operator as

$$\langle 0|_{a_i} \hat{N}_i^{(b)} |0\rangle_{a_i} = \langle 0|_{a_i} \hat{b}_i^\dagger \hat{b}_i |0\rangle_{a_i} = \tag{4.37}$$
$$= \langle 0|_{a_i} (-B_{ik}\hat{a}_k + A_{ik}\hat{a}_k^\dagger)(A_{il}^*\hat{a}_l - B_{il}^*\hat{a}_l^\dagger) |0\rangle_{a_i} = \tag{4.38}$$
$$= \langle 0|_{a_i} B_{ik}B_{il}^*\delta_{kl} |0\rangle_{a_i} = \tag{4.39}$$
$$= B_{ik}B_{ki}^* = (BB^\dagger)_{ii} \tag{4.40}$$

which is, in general, non-zero. The expectation value for the total number operator (4.33) defined for b-modes on the a-vacuum is

$$N^b = \sum_{i=1}^{d} (BB^\dagger)_{ii} = Tr(BB^\dagger) \ . \tag{4.41}$$

The Bogoliubov transformations are essential when considering quantum fields defined by observers that have different definitions of the timelike Killing vector. A massless scalar field is defined as an operator $\hat{\phi}(x)$ that depends on the point of the (globally hyperbolic) spacetime and that satisfies the equation[7]

$$\Box\phi(x) = 0 \ , \tag{4.43}$$

called the *Klein-Gordon equation in curved spacetime*. Any field can be expanded on different bases

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x)\right) = \sum_j \left(\hat{b}_j g_j(x) + \hat{b}_j^\dagger g_j^*(x)\right) \ , \tag{4.44}$$

where $\{f_i, f_i^*\}$ and $\{g_j, g_j^*\}$ must form complete sets. Positive modes are defined by

$$\mathcal{L}_K f_i(x) = K^\mu \partial_\mu f_i(x) = -i\omega f_i(x), \ \omega > 0 \ , \tag{4.45}$$

where K is the Killing vector field associated to time translations. If two regions of spacetime (or two different definitions of the same spacetime given by different observers) have different timelike Killing vector, then the two bases of functions $\{f_i, f_i^*\}$ and $\{g_j, g_j^*\}$ will in general be different. With an appropriate definition of scalar product between fields (see Ref. [21]) we get the rule

$$g_j = \sum_i (A_{ji} f_i + B_{ji} f_i^*) \ , \tag{4.46}$$

connecting the two bases that, case by case, define the Bogoliubov coefficients. We remark that, besides all complications that might arise, particle production in special and general relativity happens whenever different observers cannot agree on a univocal definition of time, and hence on a unique definition of vacuum: observers that have different definitions of the former will have different notions of the latter.

4.3.1 Unruh effect and Hawking radiation

Rindler spacetime

Let us consider the Minkowski spacetime and suppress the angular part for the sake of simplicity, thus ending up with a spacetime whose metric is $ds^2 = -dt^2 + dx^2$. An observer A that moves

[7] The operator $\Box$ is defined as

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \ . \tag{4.42}$$

in this spacetime along the trajectory

$$\begin{cases} t(\tau) = \dfrac{1}{\alpha}\sinh(\alpha\tau) \ , \\ x(\tau) = \dfrac{1}{\alpha}\cosh(\alpha\tau) \ , \end{cases} \tag{4.47}$$

has acceleration $a = \pm\alpha$. The trajectory of A, as seen from a static observer, is a hyperbole expressed by

$$-t^2 + x^2 = \frac{1}{\alpha^2} \ , \tag{4.48}$$

and reported in Fig. 4.2.

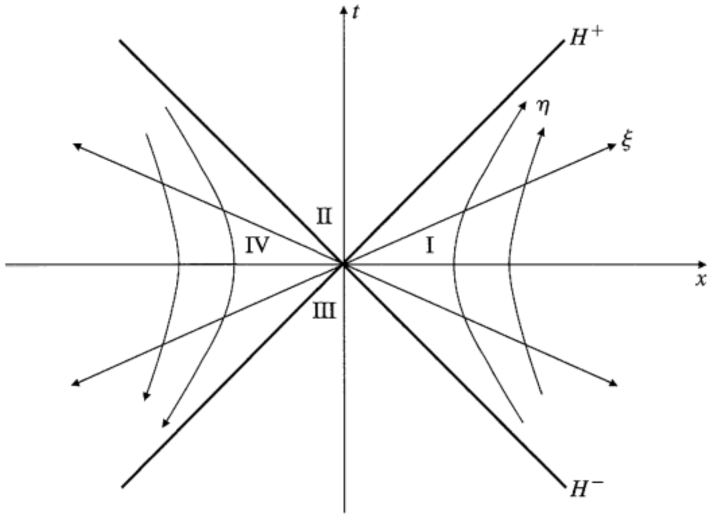


Figure 4.2: Trajectories of the accelerated observer: the lines $t = \pm x$ are *acceleration horizons* for A, meaning that she has no way to access the information contained in the regions II and III. (from Ref. [17])

The observer A can define two different sets of coordinates, depending on whether she is in the right ($x > |t|$) or in the left ($x < -|t|$) wedge of the Minkowski spacetime. Without loss of generality we suppose she is in the right wedge, where we define the coordinates (η, ξ) so that the relations

$$\begin{cases} t = \dfrac{1}{a}e^{a\xi}\sinh(a\eta) \\ x = \dfrac{1}{\alpha}e^{a\xi}\cosh(a\eta) \end{cases} \tag{4.49}$$

hold. Using these coordinates, the trajectory of A is the worldline

$$\begin{cases} \eta = \tau \ , \\ \xi = 0 \ , \end{cases} \tag{4.50}$$

meaning that she stays in a fixed point without moving: (η, ξ) are “her” coordinates. With this choice, the metric becomes

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \ , \tag{4.51}$$

called *Rindler* metric, and has ∂_η as Killing vector. Likewise, the coordinates in the left wedge are

$$\begin{cases} t = -\dfrac{1}{a}e^{a\xi}\sinh(a\eta) \\ x = -\dfrac{1}{\alpha}e^{a\xi}\cosh(a\eta) \end{cases} \tag{4.52}$$

and thus define the same metric as in (4.51), but with Killing vector opposite in direction.

Unruh effect

The presence of an accelerating observer allows one to define three different timelike Killing vectors, implying three different definitions of time. Referring to Fig. 4.3, the Killing vector $K = \partial_t$ is defined everywhere on the spacetime, while the accelerated observer can define the two Killing vectors

$$\begin{cases} K = -\partial_\eta \text{ on } \mathcal{L} \\ K = \partial_\eta \text{ on } \mathcal{R} \end{cases} \tag{4.53}$$

depending on which region she can see.

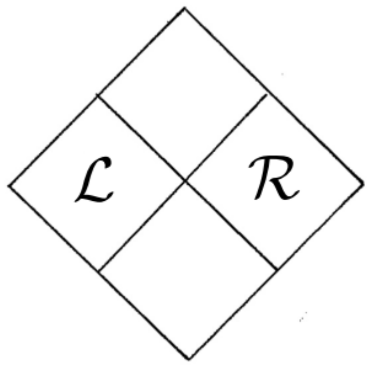


Figure 4.3: Carter-Penrose diagram of the Minkowski spacetime with left ($\mathcal{L}$) and right ($\mathcal{R}$) Rindler wedges made evident.

When field quantization is performed (see Ref. [17] and Ref. [21]) the change of basis between field modes defined on the Minkowski's spacetime and those defined in the right wedge[8] gives the Bogoliublov coefficients

$$A_{\omega\omega'} = -e^{\pi\omega/a} B_{\omega\omega'} \ . \tag{4.54}$$

According to A, the expectation value of the number of particles with frequency ω in the vacuum defined by a static (Minkowski) observer is

$$N_\omega = |B|^2 = \frac{1}{e^{2\pi\omega/a} - 1} \ , \tag{4.55}$$

as obtained from equation (4.36). This is identical to the mean number of particles composing a Bose-Einstein gas with zero chemical potential: such statistical ensemble is said to have a black-body spectrum[9]. The accelerated observer is thus "immersed" in a thermal bath of particles with temperature

$$T_R = \frac{a\hbar}{2\pi k_B} \ , \tag{4.56}$$

or

$$T_R = \frac{a}{2\pi} \ , \tag{4.57}$$

in natural units. The phenomenon for which an accelerated observer sees a thermal radiation where an inertial observer sees nothing is called the *Unruh effect.*

[8] The basis for expanding the field must include also modes from $\mathcal{L}$ to satisfy the need of a complete set defined over a Cauchy surface of the whole Minkowski spacetime.

[9] A black-body is an object that can absorb electromagnetic waves of any frequency. The correct description of the emission of such object was given at the birth of QM and it depends only on its temperature.

Hawking radiation

In this section, we finally describe the quantum effect that makes BHs evaporate. To this aim, let us consider the spacetime outside a spherically symmetric body collapsing into a black hole. This spacetime, shown in Fig. 4.4 through its CP diagram, is not stationary as a whole, but it is stationary in the far future $\mathscr{J}^+$ and in the far past $\mathscr{J}^-$, where it is possible to define the Killing vectors associated to time translations. An observer at $\mathscr{I}^+$ that looks the vacuum defined in $\mathscr{I}^-$, see a collection of particles that he interprets as emitted by the BH. This phenomenon, called *Hawking radiation*, emerges in this setting due to the non-stationary character of the collapse.

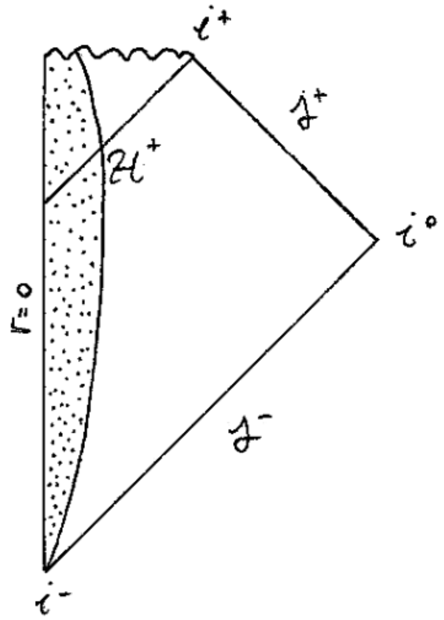


Figure 4.4: Carter-Penrose diagram of a collapsing star.

Even though the physics behind particle production in the Unruh effect and in this scenario is quite different, the result (if one overlooks the backreaction of the Schwarzschild potential) is essentially the same: an observer at $\mathscr{I}^+$ sees a thermal bath of particles with temperature

$$T_H = \frac{k\hbar}{2\pi k_B} \ , \tag{4.58}$$

or

$$T_H = \frac{k}{2\pi} = \frac{1}{8\pi M} \ , \tag{4.59}$$

in natural units. These particles are "emitted" by the black hole as a constant flux of thermal radiation. Thus, the picture of a stationary eternal BH is inconsistent: due to quantum effects the BH gradually loses energy and decreases its size, i.e. it *evaporates*[10].

Summarizing, even if BHs are classical objects they give rise to Hawking radiation, a phenomenon that is due to quantum effects. This double identity could match with what we have dubbed quasi-classical regime in the previous part of this thesis. Therefore we ask ourselves: *could we explore BHs by using the tools of chapters 1-3 and referring to the classical to quantum crossover?*

[10]BH evaporation raises a serious issue called *black holes information paradox*. In short, the problem is the following: assuming that when something falls into the event horizon the information carried by that object is lost, what does it happen when the BH eventually disappear? Since analysing this problem is not among the scopes of this thesis, we mention Ref. [22] and Ref. [23] to get an overview of the subject.

4.4 A prelude to Quantum Black Holes

A first step for connecting BHs and the quantum to classical crossover has been done in Ref. [8], where a way of describing SBH through a large-N theory is presented. Following the same steps we have exposed in Sec. 3.1, and requiring the classical theory $\mathcal{C}$ to be the Hamiltonian description of a test particle's orbits around the SBH permits to find a quantum theory whose limit is $\mathcal{C}$. In the remaining part of this chapter we describe the main points of this procedure, referring to the original work for further details.

Let us consider a globally $O(N)$-invariant quantum theory that describes a collection of N distinguishable free particles without spin. By performing its $N \to \infty$ limit we can reproduce, when a specific Hamiltonian is selected, the geodesic motion around a SBH, reported later in this paragraph. This is done via two different choices of Hamiltonians, called $\hat{H}_N^I$ and $\hat{H}_N^{II}$, that can reproduce the dynamics near and everywhere outside the event horizon, respectively. Let us briefly show how.

Starting from the metric (4.3), it is possible to derive the effective Hamiltonian for a test particle of mass m moving in the spacetime, which reads

$$h^{II}(p,r) = \frac{p^2}{2m} + \frac{1}{2m}\left(\frac{L^2}{r^2} + \mu^2\right)\left(1 - \frac{r_s}{r}\right) , \tag{4.60}$$

where we have included the index II for later convenience. By requiring the test particle to move at a radius close to r_s, we can reduce the previous Hamiltonian to its first-order form

$$h^{I}(p,r) = \frac{p^2}{2m} + \frac{1}{2m}\frac{\delta}{r_s}\left(\frac{L^2}{r_s^2} + \mu^2\right) + o\left((\delta/r_s)^2\right) \tag{4.61}$$

in the variable $\delta = r - r_s$.

Let us consider the collection of free spinless particles with positions $\hat{q}_i$ and momenta $\hat{p}_i$. By imposing on these operators the commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\frac{1}{N}\delta_{ij}, \ \forall i,j = 1,\dots,N , \tag{4.62}$$

we can derive the $O(N)$-invariant operators

$$\begin{cases} \hat{A} = \frac{1}{2}\sum_i \hat{q}_i^2 \\ \hat{B} = \frac{1}{2}\sum_i (\hat{q}_i\hat{p}_i + \hat{p}_i\hat{q}_i) \\ \hat{C} = \frac{1}{2}\sum_i \hat{p}_i^2 \end{cases} \tag{4.63}$$

that close the commutation relations

$$[\hat{A},\hat{B}] = \frac{2i}{N}\hat{A}, \ [\hat{A},\hat{C}] = \frac{i}{N}\hat{B}, \ [\hat{B},\hat{C}] = \frac{2i}{N}\hat{C} . \tag{4.64}$$

This structure naturally leads to the pseudo-spin Lie algebra. In fact, the linear combinations

$$\hat{K}_0 = \frac{1}{2}(\hat{A}+\hat{C}), \ \hat{K}_1 = \frac{\hat{B}}{2}, \ \hat{K}_2 = \frac{1}{2}(\hat{A}-\hat{C}), \tag{4.65}$$

satisfy the rescaled Lie algebra $\mathfrak{su}(1,1)_N$ commutation rules: a theory of $O(N)$-globally invariant free particles can be seen as having the rescaled $SU(1,1)$ group as dynamical group instead of the one generated by the operators (4.63). The procedure to obtain the GCS described in Cap. 1 can then be used.

Furthermore, one can express the Casimir operator $\hat{K} = -\hat{K}_\alpha \hat{K}^\alpha$, which is constant over the selected representation, as

$$\hat{K}^2 = \frac{1}{4}\left(\hat{L}^2 + \frac{1}{4} - \frac{1}{N}\right) , \tag{4.66}$$

where

$$\hat{L}^2 = \frac{1}{2}\sum_{ij} \hat{L}_{ij}^2 = \frac{1}{2}\sum_{ij} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i) \, , \; \forall i, j = 1, \dots, N \tag{4.67}$$

is the N particles angular momentum. In Ref. [2] it is shown that the symbol of the operator (4.66) (that is the fixed parameter k^2) becomes, in the $N \to \infty$ limit, proportional to a (conserved) angular momentum $\tilde{l}$.
The $N \to \infty$ limit of the Hamiltonian operator

$$\hat{H}_N^I = N\left(\hat{C} + \frac{1}{2r_S^2 m^2}\left(\frac{3\tilde{l}^2}{r_S^2} + \mu^2\right)\hat{A} - \frac{1}{4m}\left(\frac{5\tilde{l}^2}{r_S^2} + \mu^2\right)\mathbb{I}\right) \tag{4.68}$$

is the Hamiltonian function (4.61): this means that a quantum theory $\mathcal{Q}_N$ with (4.68) as Hamiltonian $\hat{H}_N^I$, flows into a classical theory $\mathcal{C}$ whose dynamics is identical to that of a classical particle orbiting near r_s. Its conjugated variables (q, p) are obtained by the standard Poisson brackets coordinates (w, v) by the canonical transformation

$$\begin{cases} w = \dfrac{r^2 m}{2} \, , \\ v = \dfrac{p}{rm} \, . \end{cases} \tag{4.69}$$

In addition, by writing the operators $\hat{A}$, $\hat{B}$, $\hat{C}$ in terms of the two-modes operators, it is possible to write Eq. (4.68) as

$$\hat{H}_N^I = N\left(\frac{a+1}{2}(\hat{N}_a + \hat{N}_b) + d\hat{\mathbb{I}} + \frac{a-1}{2i}\left(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}\right)\right) , \tag{4.70}$$

where

$$a = \frac{1}{2r_s^2 m^2}\left(\frac{3\tilde{l}^2}{r_S^2} + \mu^2\right) , \; d = -\frac{1}{4m}\left(\frac{5\tilde{l}^2}{r_S^2} + \mu^2\right) . \tag{4.71}$$

Being the operators $\hat{a} = \sum_i \hat{a}_i$ and $\hat{b} = \sum_i \hat{b}_i$ made of local operators satisfying

$$[\hat{a}_i, \hat{a}_j] = [\hat{b}_i, \hat{b}_j] = \delta_{ij}\hat{\mathbb{I}}/N^2 , \tag{4.72}$$

the Hamiltonian (4.70) describes a collection of interacting bosons. The difference between the number of the two-modes excitations is fixed by

$$\hat{K}^2 = \frac{1}{4}\left(\hat{N}_a - \hat{N}_b - \frac{1}{N^2}\right) = k(k-1)\mathbb{I} \tag{4.73}$$

over the k-representation. The Hamiltonian $\hat{H}_N^I$ make it possible to describe the effects of a SBH by referring to a collection of bosons.
In the very same way, when the N set of free particles has Hamiltonian

$$\hat{H}_N^{II} = \frac{N}{2m}\left(\frac{1}{4}(\hat{A}\hat{B})^2 + (4\hat{K}^2\hat{A}^2 + \mu^2)(\hat{\mathbb{I}} - r_S\hat{A})\right) \tag{4.74}$$

its $N \to \infty$ limit flows into a classical theory with (4.60) as Hamiltonian.
In conclusion, Ref. [8] shows how the $\mathfrak{su}(1,1)_N$ Lie algebra can be a suitable choice to obtain the geodesic motion around a SBH thanks to the $N \to \infty$ limit. This is the take-home message of this section for the next chapter. What we plan to do is to develop the details about Hawking radiation from a purely quantum mechanical setting which involves the rescaled pseudospin coherent states, meanwhile preserving some classical behaviour.

Chapter 5

Quantum Black Holes

Explaining the physics of BHs has emerged as a test for any theory aiming at reconciling GR and QM. The approach we developed in the first part of this thesis proposes that quasi-classical systems have to emerge from a quantum reality as finite-N globally symmetric systems, and we believe that BHs might fall into this scenario. In this chapter we give an *ad hoc* definition of what a Quantum Black Hole (QBH) might be and describe its evaporation as a quantum process, quite different from the standard one. Finally, we attempt to provide a dynamical description of QBHs' evaporation and end up speculating about the nature of time.

In the first section of this chapter, we define a model for QBHs that makes it possible both the $N \to \infty$ limit and the evaporation process. In the second section, we show how a notion of temperature can emerge from the pseudospin coherent states. This leads us to the third section, where we propose a model for BHs evaporation as a sequential partitioning process. In the fourth and last section, we discuss some connections we found between the evaporation of BHs and the arrow of time.

5.1 QBH: definition as a theory of free couples

When proposing an alternative model for a physical object with a well-established behaviour, one should put particular care in reproducing the already known phenomena characterizing such behaviour. Regarding BHs, our attention will be focused on getting their classical description meanwhile properly characterizing the Hawking radiation.

We propose that a *quantum black hole* is described by a quantum theory

$$\mathcal{Q}_N = \{\mathfrak{su}(1,1)_N, r, \hat{H}_N\} \ , \tag{5.1}$$

with Hamiltonian

$$\hat{H}_N = N(\alpha \hat{K}_0 + \beta \hat{K}_+ + \beta^* \hat{K}_-) \ , \tag{5.2}$$

where the global operators $\{\hat{K}_0, \hat{K}_+, \hat{K}_-\}$ are representation of the elements of $\mathfrak{su}_N(1,1)$. As discussed in Cap. 3, we can choose this theory to emerge from N local (microscopic) identical theories

$$\mathcal{Q}_i = \{\mathfrak{su}(1,1), (\phi_2^i, \mathcal{F} \otimes \mathcal{F}), \hat{H}_i\} \ , \tag{5.3}$$

each one with Hamiltonian

$$\hat{H}_i = \alpha\phi_2^i(K_0) + \beta\phi_2^i(K_+) + \beta^*\phi_2^i(K_-) = \frac{\alpha}{2}(\hat{a}_i^\dagger \hat{a}_i + \hat{b}_i^\dagger \hat{b}_i + \mathbb{I}_i) + \beta \hat{a}_i^\dagger \hat{b}_i^\dagger + \beta^* \hat{a}_i \hat{b}_i \ . \tag{5.4}$$

when considering the system they individually describe as non-interacting. Note that the local theories labelled with the i index are all identical. In particular, and referring to Sec. 3.4, the global operators will be given in terms of the local ones, by

$$\begin{cases} \hat{K}_0 = \dfrac{1}{N}\displaystyle\sum_i^N \dfrac{1}{2}(\hat{a}_i^\dagger \hat{a}_i + \hat{b}_i^\dagger \hat{b}_i + \mathbb{I}_i) \ , \\ \hat{K}_+ = \dfrac{1}{N}\displaystyle\sum_i^N \hat{a}_i^\dagger \hat{b}_i^\dagger \ , \\ \hat{K}_- = \dfrac{1}{N}\displaystyle\sum_i^N \hat{a}_i \hat{b}_i \ . \end{cases} \tag{5.5}$$

While the local theories have non-vanishing commutators for every value of N (since they are defined regardless of how many other copies of each theory are considered), the global-theory's Lie brackets

$$\begin{cases} [\hat{K}_0, \hat{K}_\pm] = \pm\dfrac{1}{N}\hat{K}_\pm \\ [\hat{K}_+, \hat{K}_-] = -\dfrac{2}{N}\hat{K}_0 \end{cases} \tag{5.6}$$

vanish in the $N \to \infty$ limit, thus making $\mathfrak{su}(1,1)_\infty$ abelian. We remind that this is typical of the quantum-to-classical crossover: the parts making the composite system retain their quantum character, but their collection behaves according to the laws of classical physics as N increases. In the free-theory case the global symmetry that permits the quantum-to-classical crossover, is the invariance *w.r.t.* any permutation of the local systems: the Hilbert space explored by a system described by $\mathcal{Q}_N$, including what we propose a QBH might be, is only the permutation-invariant part of $\mathcal{H}^{\otimes N}$: its states will be denoted as

$$|Q_N\rangle \in \mathcal{S}_N = Symm(\mathcal{H}_N) \tag{5.7}$$

in the following. Note that the full Hilbert space is much bigger than the subspace we consider here. This reduction is justified by the evidence that BHs feature classical aspects, implying that they should possess a global symmetry that makes it possible the large-N versus small-κ correspondence, as presented in Cap. 1.

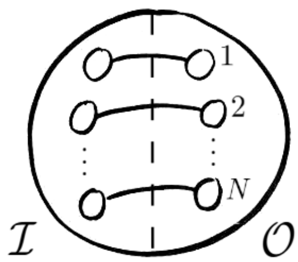


Figure 5.1: Free-theory description of the QBH using two modes local theories. The left part represents the inaccessible partition of the QBH, while the right is the accessible one.

With these definitions, we can alternatively think of the QBH as equivalently made of

- N couples that do not interact with each other;
- $2N$ single subsystems, grouped into non-interacting couples, with partners of each couple interacting and entangled;
- two interacting and entangled composite systems, denoted by $\mathcal{I}$ and $\mathcal{O}$, each made of N single partners of the above couples.

The last option turns out to be the most relevant to model the QBH evaporation: if we interpret $\mathcal{O}$ as the accessible part of the system and $\mathcal{I}$ as the inaccessible one, identifying $\mathcal{O}$ with the exterior and of $\mathcal{I}$ with the interior of the BH's event horizon, is straightforward. The idea is summarized in Fig. 5.1. To make this feature explicit, we add to our definition of QBH that it does not exists a way to observe the subsystems in $\mathcal{I}$. Note that our proposal does not explain *why* the black hole should be made like this, as it just aims at explaining the BH's properties after its formation.

It is legitimate to ask if and why one such system should be called a BH. In fact, despite this system being fundamentally different from what is usually called BH, it still has the potential to behave in the same way usually expected for a BH, at least as far as two aspects are concerned. Firstly, if the parameters α and β in the local Hamiltonians (5.4) are properly chosen, the near-horizon potential is reproduced in the $N \to \infty$ limit, as described in Sec. 4.4. Secondly, briefly discussed in Appendix D, by choosing $\beta = 0$ and α as a specific function of a positive real parameter, it seems possible to reproduce both the contraction of time described in Sec. 4.1 and the Tolman's law, Eq. (4.30). This ideas, however, are still subjects of "work in progress".

5.2 From $\mathfrak{su}(1,1)$ coherent states to thermal states

Since the path leading from $\mathfrak{su}(1,1)_N$ algebras to the proper classical theory has already been displayed in Sec. 3.4, our main concern here is to show how thermal radiation, i.e. what Hawking radiation actually is, can be emitted by a QBH. In order to do this, we first have to find a connection between $\mathfrak{su}(1,1)$ coherent states and thermal states. Let us then consider a coherent state of $\mathfrak{su}(1,1)$ in two-mode representation with Bargmann index k, and refer to it with the complex number τ defined on the Poincarè disk, as usual. Its explicit expression in terms of two-mode operators is

$$|\tau\rangle = (1-|\tau|^2)^k e^{\tau \hat{a}^\dagger \hat{b}^\dagger} |k; n_0, 0\rangle \ , \tag{5.8}$$

i.e. expanding the exponential and using the definitions of $\hat{a}^\dagger$ and $\hat{b}^\dagger$ in terms of their action upon $|n_0, 0\rangle$,

$$|\tau\rangle = (1-|\tau|^2)^k \sum_{l=0}^{\infty} \frac{\tau^l}{l!} (\hat{a}^\dagger)^l (\hat{b}^\dagger)^l |n_0, 0\rangle = \tag{5.9}$$

$$= (1-|\tau|^2)^k \sum_{l=0}^{\infty} \frac{\tau^l}{l!} \left(\frac{(n_0+l)!}{n_0!}\right)^{\frac{1}{2}} (l!)^{\frac{1}{2}} |n_0+l, l\rangle \ , \tag{5.10}$$

As it is well known (see i.e. Ref. [24]), the state of an isolated quantum system can be equivalently expressed by a normalized vector $|\psi\rangle \in \mathcal{H}$ or by a density operator $\rho = |\psi\rangle \langle\psi|$. The main advantage of choosing the second possibility is that the density-operator formalism allows one to deal with composite and non-isolated quantum systems, in a framework usually called of *open quantum systems*. The density operator for the state (5.10) is

$$\rho = |\tau\rangle\langle\tau| = (1-|\tau|^2)^{2k} \sum_{l,l'=0}^{\infty} \frac{\tau^l \tau^{*l'}}{\sqrt{l! l'!}} \left(\frac{(n_0+l)!}{n_0!}\right)^{\frac{1}{2}} \left(\frac{(n_0+l')!}{n_0!}\right)^{\frac{1}{2}} |n_0+l, l\rangle \langle n_0+l', l'| \ . \tag{5.11}$$

By interpreting the bosonic operators $\hat{a}$ and $\hat{b}$ as related to two different subsystems $\mathcal{I}$ and $\mathcal{O}$ respectively, and hence the global state (5.11) as that of the composite system $\mathcal{I}+\mathcal{O}$, we can write the state of one subsystem only by the partial trace onto the Hilbert space of the other

$$Tr_i : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_j \text{ with } i,j = \mathcal{O}, \mathcal{I} \text{ and } i \neq j \ . \tag{5.12}$$

Note that, even these subsystems will be later somehow related to the couples composing the QBH, here they are defined regardless of the previous discussion. When doing the partial trace on the Fock space of $\mathcal{O}$, whose bosonic operators are $\hat{b}$ and $\hat{b}^\dagger$, the density operator for $\mathcal{I}$ reads

$$\rho_{\mathcal{I}} = \mathrm{Tr}_{\mathcal{O}}(\rho) = \tag{5.13}$$

$$= (1-|\tau|^2)^{2k} \sum_{n=0}^{\infty} \sum_{l,l'=0}^{\infty} \frac{\tau^l \tau^{*l'}}{\sqrt{l!l'!}} \frac{\sqrt{(n_0+l)!(n_0+l')!}}{n_0!} \left|n_0+l\right\rangle \left\langle n_0+l'\right| \delta_{n,l}\delta_{n,l'} = \tag{5.14}$$

$$= (1-|\tau|^2)^{2k} \sum_{n=0}^{\infty} |\tau|^{2n} \frac{(n_0+n)!}{n!n_0!} \left|n_0+n\right\rangle \left\langle n_0+n\right| \ . \tag{5.15}$$

Vice versa, by tracing out the Hilbert space of $\mathcal{I}$ one gets

$$\rho_{\mathcal{O}} = (1-|\tau|^2)^{2k} \sum_{n=0}^{\infty} |\tau|^{2n} \frac{(n+n_0)!}{n!n_0!} \left|n\right\rangle \left\langle n\right| \ , \tag{5.16}$$

which is the state of the subsystem $\mathcal{O}$. The density operators $\rho_{\mathcal{I},\mathcal{O}}$ can be expressed in a way similar to that one of a system which is at thermal equilibrium with a heat-bath, with temperature related to $|\tau|^2$. The procedure for getting this result is different depending on $n_0 = 0$ or not, that is, it depends on the choice of representation.

Case $n_0 = 0$

By choosing[1] $n_0 = 0$ (or equivalently, $k = 1/2$), we can write (5.15) and (5.16) as

$$\rho_{\mathcal{I}} = \rho_{\mathcal{O}} = (1-|\tau|^2) \sum_{n=0}^{\infty} |\tau|^{2n} \left|n\right\rangle \left\langle n\right| = \frac{1}{Z_0} \sum_{n=0}^{\infty} e^{-n \log \frac{1}{|\tau|^2}} \left|n\right\rangle \left\langle n\right| \ , \tag{5.17}$$

where

$$Z_0 = \sum_n |\tau|^{2n} = \frac{1}{1-|\tau|^2} \ , \tag{5.18}$$

converges for any $|\tau| \in D_2$. We call Z_0 the *partition function* of the system (we have introduced the index "0" to denote what is valid for $n_0 = 0$, for later convenience). Note that, with this choice, the density operators $\rho_{\mathcal{I}}$ and $\rho_{\mathcal{O}}$ have the same expression.
By defining the two parameters

$$\begin{cases} E_n \doteq \omega n \\ \beta \doteq \dfrac{1}{\omega} \log \dfrac{1}{|\tau|^2} \end{cases} , \tag{5.19}$$

Eq. (5.17) can be written as

$$\rho_{\mathcal{I}} = \rho_{\mathcal{O}} = \frac{1}{Z_0} \sum_{n=0}^{\infty} e^{-\beta E_n} \left|n\right\rangle \left\langle n\right| \ , \tag{5.20}$$

while the square modulus of τ reads

$$|\tau|^2 = e^{-\omega\beta} \ . \tag{5.21}$$

The density operators (5.20) have the same form of that of a system (in particular, a harmonic oscillator) in thermal equilibrium with a heat-bath at a temperature $T = 1/\beta$, often called *thermal state*. Note that we can add a constant to E_n without modifying the result. The expectation value of the energy of one such system is

$$\langle E \rangle_0 = Tr(\rho \hat{H}) = -\frac{\partial}{\partial \beta} \log Z_0 = \frac{\omega}{e^{\omega\beta} - 1} \tag{5.22}$$

[1]This and any other choice of n_0 can be related to the choice of an angular momentum for the global system, if the $O(N)$-symmetric model presented in Sec. 4.4 is considered.

and the expectation value of the number operator,

$$\langle N_i \rangle_0 = Tr(\rho \hat{N}_i) = \frac{1}{e^{\omega\beta} - 1} \text{ for } i = a, b \ , \tag{5.23}$$

have the same form of the expectation value of the number of particles emitted from a BH as Hawking radiation at frequency ω, when no back-reaction from the Schwarzschild potential is considered.
This is what leads us to the description of the evaporation process as the emission of single subsystems of $\mathcal{O}$, say o_i: if the QBH emits its microscopic quantum subsystems one at a time, it can retain some classical structure, if the overall N is still sufficiently large, and yet manifest its "quantumness" in the details of such emission, i.e. of the Hawking radiation.

Case $n_0 \neq 0$

Choosing a different representation for the Lie algebra does not substantially modify the results of the case $n_0 = 0$. For any $n_0 \neq 0$ representation, we can write the factor in front of the sums (5.15-5.16) as a partition function, but the density operator is less obviously interpreted. In fact, defining

$$Z_{n_0} = \sum_{n=0}^{\infty} x^n \frac{(n_0 + n)!}{n! n_0!} \tag{5.24}$$

with $x = |\tau|^2 < 1$, and observing that it is

$$\frac{(n_0 + n)!}{n! n_0!} = \frac{1 \dots n(n+1) \dots (n+n_0)}{1 \dots n 1 \dots n_0} = \frac{(n+1)}{1} \frac{(n+2)}{2} \dots \frac{(n+n_0)}{n_0} = \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) \tag{5.25}$$

we can prove[2] that Eq. (5.24) can be rewritten as

$$Z_{n_0} = \frac{1}{(1 - |\tau|^2)^{n_0+1}} = \frac{1}{(1 - |\tau|^2)^{2k}} \ . \tag{5.26}$$

This allows us to write Eqs. (5.15) and (5.16) as

$$\begin{cases} \rho_{\mathcal{I}} = \dfrac{1}{Z_{n_0}} \displaystyle\sum_{n=0}^{\infty} |\tau|^{2n} \frac{(n_0 + n)!}{n! n_0!} \left| n_0 + n \right\rangle \left\langle n_0 + n \right| \ , \\ \rho_{\mathcal{O}} = \dfrac{1}{Z_{n_0}} \displaystyle\sum_{n=0}^{\infty} |\tau|^{2n} \frac{(n + n_0)!}{n! n_0!} \left| n \right\rangle \left\langle n \right| \ , \end{cases} \tag{5.27}$$

[2]To prove this relation we proceed by induction. For $n_0 = 1$ the relation holds true. In fact

$$Z_1 = \sum_{n=0}^{\infty} x^n \left(1 + \frac{n}{1}\right) = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^n n = \frac{1}{1-x} + \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2} \ .$$

Assuming that the same holds for n_0, then

$$\begin{aligned} Z_{n_0+1} &= \sum_{n=0}^{\infty} x^n \prod_{r=1}^{n_0+1} \left(1 + \frac{n}{r}\right) = \sum_{n=0}^{\infty} x^n \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) \left(1 + \frac{n}{n_0 + 1}\right) = \\ &= \sum_{n=0}^{\infty} x^n \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) + \sum_{n=0}^{\infty} x^n \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) \frac{n}{n_0 + 1} = \\ &= \frac{1}{(1-x)^{n_0+1}} + \frac{1}{n_0 + 1} x \frac{\partial}{\partial x} \sum_{n=0}^{\infty} x^n \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) = \\ &= \frac{1}{(1-x)^{n_0+1}} + \frac{1}{n_0 + 1} x \frac{\partial}{\partial x} \frac{1}{(1-x)^{n_0+1}} = \frac{1}{(1-x)^{n_0+1}} + \frac{x}{(1-x)^{n_0+2}} = \\ &= \frac{1}{(1-x)^{n_0+2}} \ . \end{aligned}$$

Therefore, since Eq. (5.25) is true for $n_0 = 1$ and for $n_0 + 1$, $\forall n_0$, than it holds true for all values of n_0.

and, using Eqs. (5.25) and (5.19) we obtain

$$\rho_{\mathcal{I}} = \frac{1}{Z_{n_0}} \sum_{n=0}^{\infty} e^{-\beta E_n} f_{n_0}(n) \left| n_0 + n \right\rangle \left\langle n_0 + n \right| \tag{5.28}$$

and

$$\rho_{\mathcal{O}} = \frac{1}{Z_{n_0}} \sum_{n=0}^{\infty} e^{-\beta E_n} f_{n_0}(n) \left| n \right\rangle \left\langle n \right| \ , \tag{5.29}$$

with

$$f_{n_0}(n) = \prod_{r=1}^{n_0} \left(1 + \frac{n}{r}\right) = \frac{(n_0+n)!}{n_0! n!} \ . \tag{5.30}$$

This function is equal to

$$F_{l=n_0+1}(n) = \frac{\Gamma(n+l)}{\Gamma(n+1)(l-1)!} \ , \tag{5.31}$$

that express the number of different ways in which one can obtain n by adding $l = n_0+1$ smaller integers. By interpreting $f_{n_0}(n)$ as a degeneracy on the energy levels[3], (5.29) can be recognised as the state of a collection of $n_0 + 1$ copies of the system described in the $n_0 = 0$ case. Note that the same result is obtained by recognising that the partition function (5.26) is the product of $n_0 + 1$ partition functions in the form (5.18). In fact, the density operator of a collection of $n_0 + 1$ harmonic oscillators at thermal equilibrium with a heat-bath at $T = 1/\beta$ has the same form of (5.29).
The explicit expressions (5.28) and (5.29) make it clear that $\mathcal{I}$ and $\mathcal{O}$ have different mean energies, in this picture. In fact,

$$\langle E_{\mathcal{O}} \rangle = -\frac{\partial}{\partial \beta} \log Z_{n_0} = (n_0+1)\langle E \rangle_0 \ , \tag{5.32}$$

where $\langle E \rangle_0$ is the mean energy in the $n_0 = 0$ case, while

$$\langle E_{\mathcal{I}} \rangle = \frac{1}{Z_{n_0}} \sum_{n=0}^{\infty} (E_n + E_{n_0}) f_{n_0}(n) e^{-\beta E_n} = \langle E_{\mathcal{O}} \rangle + E_{n_0} \ . \tag{5.33}$$

This means that each $n_0 + 1$ subsystem described by $\rho_{\mathcal{O}}$ is identical to the single subsystem described in the $n_0 = 0$ case as it has the same mean energy $\langle E \rangle_0$ and the same expectation value of the number operator, $\langle N \rangle_0$, as seen by

$$\langle \hat{N}_{\mathcal{O}} \rangle = (n_0+1)\langle \hat{N} \rangle_0 \ . \tag{5.34}$$

On the other hand, the mean energy of $\mathcal{I}$ equals that one of $\mathcal{O}$ plus a factor

$$\Delta \langle E \rangle = \omega n_0 \ . \tag{5.35}$$

The expectation value of the number operator of $\mathcal{I}$ is

$$\langle \hat{N}_{\mathcal{I}} \rangle = (n_0+1)\langle \hat{N} \rangle_0 + n_0 \ , \tag{5.36}$$

which helps clarifying the picture: the $n_0 > 0$ representation is equivalent to considering $n_0 + 1$ copies of the $n_0 = 0$ representation one with only one difference: the ground state of the system $\mathcal{I}$ is not the vacuum *w.r.t.* the particles created by the operators $\hat{a}_i^\dagger$, as it already contains n_0 of such particles. Since the physics of this case is not much different from that the $n_0 = 0$ case, in the following we will consider exclusively the latter.

[3] Note that since $f_0(n) = 1$ for all n, the previous expressions are valid also for $n_0 = 0$.

Entropy

Even it is not relevant to our proposal, we here briefly comment upon the entropy of the Hawking radiation as it emerges from this model. A quantum system in a state ρ has a von Neumann entropy

$$S(\rho) = -Tr(\rho \log \rho) \ . \tag{5.37}$$

Since for any bipartite system the entropy of one of the two parts is equal to the entropy of the other, we can evaluate S for the interior or the exterior of the QBH, only. In the case $n_0 = 0$,

$$S_0(\rho_{\mathcal{I}}) = -\mathrm{Tr}(\rho_A \log \rho_A) = \beta \langle E \rangle_0 + \log Z_0 = S_0(\rho_{\mathcal{O}}) \tag{5.38}$$

takes the form of the entropy of a system in thermal equilibrium with a heat-bath at $T = \beta^{-1}$. By expressing (5.38) in terms of β and the energy-levels separation, one obtains

$$S_0 \equiv S_0(\rho_{\mathcal{I}}) = S_0(\rho_{\mathcal{O}}) = \frac{\beta\omega}{e^{\beta\omega} - 1} - \log(1 - e^{-\beta\omega}) \ . \tag{5.39}$$

For $n_0 \neq 0$, choosing the simpler density operator to evaluate S_{n_0}, i.e. $\rho_{\mathcal{I}}$, and get

$$S_{n_0}(\rho_{\mathcal{I}}) = \beta \langle E \rangle + \log Z_{N_0} = \beta(n_0 + 1)\langle E \rangle_0 + \log Z_0^{n_0+1} = (n_0 + 1) S_0 \ . \tag{5.40}$$

Therefore, the entropy for the $n_0 \neq 0$ case is

$$S_{n_0}(\rho_{\mathcal{I}}) = S_{n_0}(\rho_{\mathcal{O}}) = (n_0 + 1) S_0 \tag{5.41}$$

i.e. $(n_0 + 1)$ times at $n_0 = 0$. After the discussion above, this should not surprise: the $n_0 \neq 0$ representation is effectively described as if we had multiple copies of the $n_0 = 0$ one, and its entropy increases accordingly.

A remark

It is important to note that the $\mathfrak{su}(1,1)$ two-mode description herein presented can reproduce the density operator of a thermal emission only for one selected frequency ω. We believe that this incompleteness in our discussion comes from considering the free-theory description. Even if useful for our scopes, and reasonably simple to deal with, we were expecting this choice to have a price. However, we propose to turn this weakness into the strength of our QBH model. As we will see by the end of next section, our scheme for QBH evaporation involves observers as fundamental players in the scene, and we propose that the restriction upon ω can arise as a consequence of the specific measurement the observer performs. Postponing some comments upon this point, from here on we choose $\omega = 1$, for the sake of simplicity.

5.3 Black holes evaporation by sequential partitioning

In this section we finally present our model for QBHs evaporation. This is based on viewing the evaporation as a sequential partitioning of a QBH, realized by repeated cycles of 4 steps. The steps are as follows:

1. One of the N couples, say c_1, is randomly selected.

2. The external partner o_1 of the couple c_1 interacts with the external world, which breaks the global symmetry of the overall system, although as discussed in Sec. 3.3.4, in a mild way. In fact, after the removal of o_1, the QBH is still invariant under the permutations of the remaining $N - 1$ parts.

3. An external observer looks at o_1, and this is the step where the actual "evaporation" occurs: o_1 becomes entangled with the measurement apparatus, according to the quantum mechanical description of the measurement process. Note that an observer needs to implement some sort of apparatus to detect the QBH subsystems emitted as Hawking radiation (in a way similar to what we have described in Sec. 3.3.4) that surely has some practical limitation to the information that can be gathered. From this viewpoint we can interpret the single ω appearing in the last section as due to the choice of the instrument the observer uses to measure the system.

4. we restart from step 1.

Since $\mathcal{I}$ is defined by the fact that it is impossible to interact with it, when the measurement of the (randomly selected) o_1 takes place, its i_1 partner becomes un-detectable: everything we can learn about it is encoded in the outcoming partner and, once the measurement happens, i_1 effectively disappears[4].
Finally, as each cycle the size of the system, i.e. the number of systems that make it, is reduced by one, we propose this procedure to be efficient in describing BHs evaporation as long as the QBH system as a whole has a macroscopically classical behaviour, that is, as long as N is sufficiently large to make it meaningful the $N \to \infty$ limit for an outside observer. In what follows we describe the above steps in more details.

Step 1 and 2

Let us suppose that the large-N system we recognise as the QBH is in a global coherent state of the algebra $\mathfrak{su}_N(1,1)$. This ensures it shows a classical behaviour for $N > N_d$. Recalling Sec. 3.4, we can then write the pure state of such large-N system as

$$|Q_0\rangle = |\Omega\rangle_N = e^{\Omega\hat{K}_+ + \Omega^*\hat{K}_-}\,|Z\rangle_N = e^{\sum_i \frac{\Omega}{N}\hat{K}^i_+ + \frac{\Omega^*}{N}\hat{K}^i_-}\,|0\rangle_1 \otimes \ldots \otimes |0\rangle_N = \bigotimes_{i=1}^{N} |\Omega/N\rangle \ , \tag{5.42}$$

for some $\Omega \in \mathbb{C}$ that characterizes the QBH state. In the first step of the evaporation process, we remove one randomly chosen pair and, by simply cutting it out, we end up with the state described by

$$|Q_1\rangle = \bigotimes_{i=1}^{N-1} |\Omega/N\rangle \ , \tag{5.43}$$

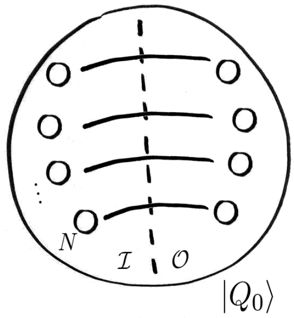
which is again a coherent state, but for a different global algebra. To show this, let us define the global operators $\{\hat{L}^1_\pm, \hat{L}^1_0\}$ on the Hilbert space $\mathcal{H}_{N-1}$, as

$$\begin{cases} \hat{L}^1_\pm = \dfrac{1}{N-1}\displaystyle\sum_{i=1}^{N-1} \hat{K}^i_\pm \ , \\ \hat{L}^1_0 = \dfrac{1}{N-1}\displaystyle\sum_{i=1}^{N-1} \hat{K}^i_0 \ ; \end{cases} \tag{5.44}$$

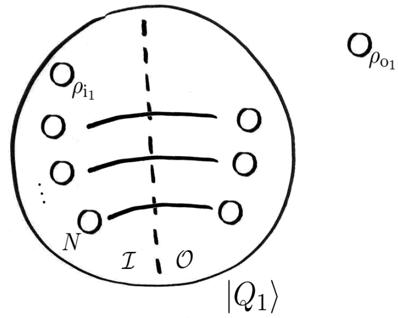
it is easy to show that they satisfy

$$[\hat{L}^1_+, \hat{L}^1_-] = \frac{1}{N-1}(-2\hat{L}^1_0), \quad [\hat{L}^1_0, \hat{L}^1_\pm] = \frac{1}{N-1}(\pm\hat{L}^1_\pm) \ , \tag{5.45}$$

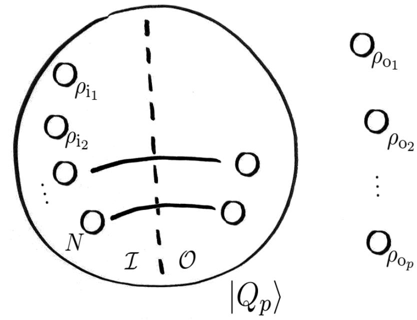
[4] In our opinion, saying that something is not detectable is equivalent to say that it does not exist. Alternatively, one could think that after the measurement of o_1, the i_1 subsystem lays inside the QBH and remains there, forever undetected. We value these to approaches equivalent for all practical purposes.



(a) Initial state



(b) After 1 emission



(c) After p emissions

Figure 5.2: Sequential partitioning of the QBH: After p steps we identify the Black Hole as only the remaining $N-p$ subsystems. The p subsystems that once belonged to $\mathcal{O}$ are free to be detectable and the ones that belonged to $\mathcal{I}$ are "non-existent" for all practical purposes.

and thus they define a $\mathfrak{su}(1,1)_{N-1}$ rescaled Lie algebra. By means of these operators, we find that the QBH is still in a coherent state, namely

$$\begin{aligned} |Q_1\rangle &= \bigotimes_{i=1}^{N-1} e^{\frac{\Omega_0}{N}\hat{K}_+ - \frac{\Omega_0^*}{N}\hat{K}_-} |0_i\rangle = \\ &= e^{\frac{\Omega_0}{N}(N-1)\hat{L}_+ - \frac{\Omega_0^*}{N}(N-1)\hat{L}_-} |Z\rangle_{N-1} = |\Omega(N-1)/N)\rangle_{N-1} \ . \end{aligned} \tag{5.46}$$

The randomly chosen pair i_i-o_1 is, by definition, in a $\mathfrak{su}(1,1)$ coherent state, and thus its external part o_1 is in the thermal state

$$\rho_{\mathcal{O}_1} = \frac{1}{Z_0} \sum_{n=0}^{\infty} e^{-\beta E_n} |n\rangle \langle n| \ , \tag{5.47}$$

where, as previously mentioned, we have chosen the $k=1/2$ representation. Since subsystems in such state can be regarded as thermal emissions outcoming from the QBH, they make the Hawking radiation. By repeating the cycle p times the QBH ends up having a Hilbert space $\mathcal{H}_{N-p}$, on which the operators

$$\begin{cases} \hat{L}_\pm^p = \dfrac{1}{N-p} \displaystyle\sum_{i=1}^{N-p} \hat{K}_\pm^i \\ \hat{L}_0^p = \dfrac{1}{N-p} \displaystyle\sum_{i=1}^{N-p} \hat{K}_0^i \end{cases} \tag{5.48}$$

are defined. These operators again define a rescaled Lie algebra, with factor $N-p$; After p cycles the QBH will be in the global coherent state

$$|Q_p\rangle = |\Omega(N-p)/N\rangle_{N-p} \ . \tag{5.49}$$

relative to the above rescaled algebra. Since

$$|\Omega\rangle_N = |\Omega/N\rangle \Rightarrow |\Omega(N-1)/N\rangle_{N-1} = |\Omega/N\rangle \Rightarrow \ \dots \ |\Omega(N-p)/N\rangle_{N-p} = |\Omega/N\rangle \ , \tag{5.50}$$

the complex number that defines the global coherent state, when regarded as a $\mathfrak{su}(1,1)$ GCS, never changes during the sequential partitioning. This means that the quasi-classical behaviour still emerges, and it stays the same during the whole evaporation. On the other hand, the QBH's Hilbert space and Lie algebra change at each cycle. As for the outside world, after p cycles there will be p microscopic systems o_i with state described by Eq. (5.47) available for observation.

Step 3

Let us now suppose an observer exists, say Alice, that have access to the external part of the QBH, $\mathcal{O}$, which is by definition the only components that Alice can test of the QBH. Every once in a while, she detects a particle the partner o_i of the randomly chosen couple c_i emitted by the black hole. Furthermore, Alice knows nothing about the BH, except for the fact that its emissions are identical density operators. In fact, these density operators have form (5.47), which is not known to her. This setup makes it possible for Alice to investigate the emitted systems via quantum-state tomography.

Note that, while we can think that Alice "awaits" the emissions of the QBH in a time-dependent picture, a dynamics is actually not necessarily involved in this process. Our approach implies looking at the evaporation as if it were a collection of "pictures" each defined by a different partitioning of the QBH and hence such that Alice has access to different subsets of the QBH in each picture. An actual dynamics might be obtained by putting these frames together, while ordering them according to some logical argument. How this ordering of successive detections can be defined is a delicate question, and we will come back to it at the end of this chapter.

For now, let us stick to the time-dependent picture and let us suppose that Alice observes one subsystem after another. In general, she should measure its state via a selected informationally complete[5] set of operators. In order to keep the focus on the physics, we here consider this set to be the energy eigenbasis. Note that this set is not informationally complete, but we rely on "our" knowledge, that is not hers, on the form of the "to-be-observed" density operators to reassure Alice that this choice is sufficient to complete her task.

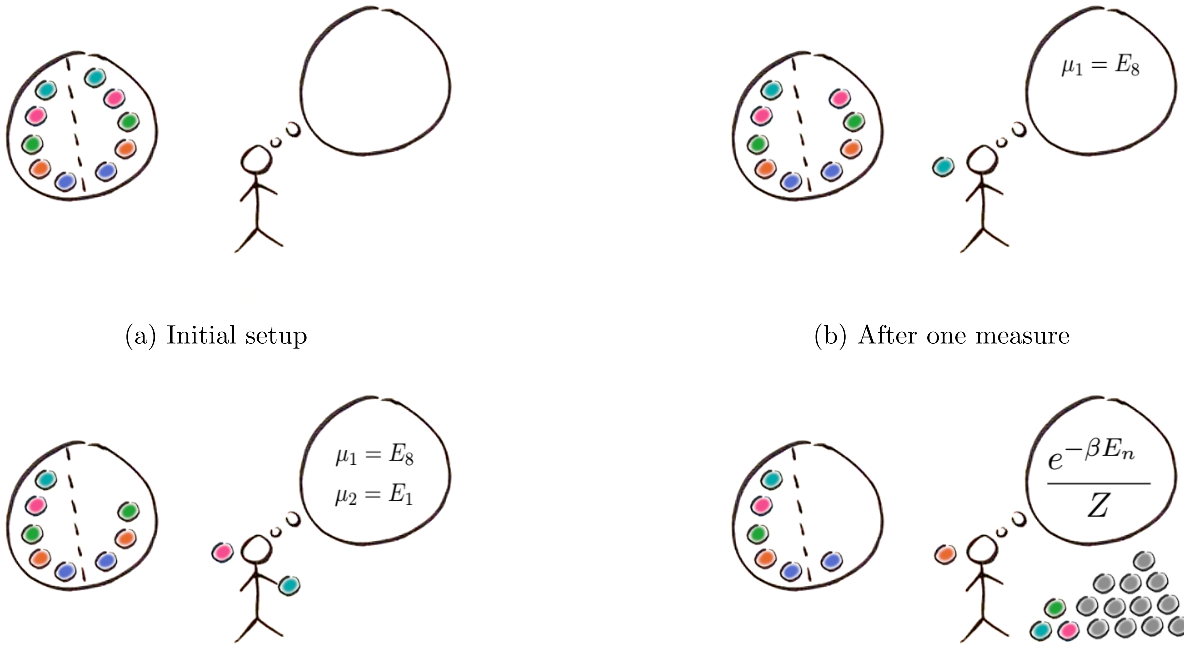
As the sequential emission process proceeds, and Alice piece-wisely collects her results μ_i, she may start to realize that the energies she is measuring are distributed accordingly to a thermal distribution[6]. As soon as Alice has collected enough results, she can claim that the observed subsystems are coming from a source at some temperature $T = 1/\beta$: the QBH, as observed by an external observer, manifests an inverse temperature β. A question arises: *when can Alice become aware of this?* If at each step she tries her best to infer the form of the density operator, at a given step p, describing the system she is observing her guess will be

$$\rho^{(p)} = \sum_{n=0}^{\infty} \frac{F_n(p)}{p} |n\rangle \langle n| \ , \tag{5.51}$$

where $F_p(n)$ is the fraction of total p measurements with outcome E_n. For example, if the first energy measurement gives E_8 as a result (see Fig. 5.3b), the best thing Alice can say is that the

[5]A POVM $\{\Pi_i\}$ is said to be *informationally complete* if is composed of at least d^2 operators.

[6]Since Alice cannot have a piece of prior knowledge about the Hawking radiation, the way she manages the data must only rest on the permutation invariance we have requested. We believe that a full description of how she can build her probability distribution should be based on the *quantum De Finetti theorem*, whose hypotheses matches our QBH model and whose aim is exactly to build a description for a collection of unknown density operators that have permutation invariance and that are expected to be part of a larger collection of other identical density operators (see Ref. [25]).



(a) Initial setup

(b) After one measure

(c) After two measures

(d) After infinite measures

Figure 5.3: Sequential measurement of a QBH: the pairing in colours denotes the two parts of a couple. It is only after an infinite amount of measurements that the observed probability distribution converges to the thermal one.

system she's observing is emitting subsystems in the state

$$\rho^{(1)} = |8\rangle \langle 8| \ , \tag{5.52}$$

with probability 1. It is only when she performs the second measurement (see Fig. 5.3c) and obtains, say E_1, that she can guess a different state for the system i.e.

$$\rho^{(2)} = \frac{1}{2} \left(|8\rangle \langle 8| + |1\rangle \langle 1| \right) \ , \tag{5.53}$$

and she can start seeing a probability distribution to emerge. According to the measurement postulate of QM, the probability of obtaining an outcome n when measuring the density operator ρ is given by

$$p(n) = Tr(\Pi_n \rho) \ , \quad \text{with } \Pi_n = |n\rangle \langle n| \ . \tag{5.54}$$

Only by repeating the same measurement many times, upon a large collection of identical systems, the probability (5.54) emerges. In fact, it is only for large number of emissions p that the factor in (5.51) become similar to the thermal distribution

$$\frac{F_n(p)}{p} \simeq \frac{e^{-\beta E_n}}{Z} \ , \tag{5.55}$$

with the equality holding for $p \to \infty$. This means that Alice can claim with confidence that the source emitting the subsystems she is observing behaves like a blackbody at inverse temperature

$$\beta = \log \frac{1}{|\tau|^2} \ , \tag{5.56}$$

only a large number of measurements (see Fig. 5.3d). In fact, for whatever finite p she cannot be sure to be witnessing a thermal emission. However, if p is sufficiently large, the observed

distribution will be similar to the thermal one, and Alice will possibly claim with some confidence that the QBH is at temperature β^{-1}.
Note that if another observer starts looking at the QBH after Alice has observed it, he will detect the same temperature: this is because the observed temperature is related to the coherent state of the QBH as expressed by the $\mathfrak{su}(1,1)$ algebra, and thus, as shown by (5.50), it doesn't change during the evaporation. By matching Eq. (5.56) with Hawking's temperature,

$$\beta = \log \frac{1}{|\tau|^2} = \frac{1}{T_H} = 8\pi M \ , \tag{5.57}$$

we obtain that the state describing the QBH can be related to the mass of the black hole by

$$|\tau|^2 = e^{-8\pi M} \ . \tag{5.58}$$

This result, despite its simplicity, hides some subtleties that we have only partially analysed (see App. D), and that we expect may be relevant for future developments.
In conclusion, the temperature associated with the QBH is, in our model, the temperature resulting from an infinite number of measurements made by an observer upon what is effectively accessible for observation in the QBH. This is not some temperature of the QBH itself, that we would not be able to define unless admitting the internal parts of it not being isolated.

Evaporation

Once the measurements occur, the observed subsystems o_1 become disentangled from their respective partners i_i and hence from the QBH, meaning that they cannot be considered as parts of the original large-N system. As a result, the QBH slowly loses subsystems and energy, on average at a rate

$$N \longmapsto N-1 \quad \text{and} \quad E \longmapsto E - \langle E \rangle = E - \frac{1}{e^{\beta}-1} \tag{5.59}$$

for every emission. Note that even if the QBH loses energy (and thus its mass decreases), its temperature is fixed during the whole process. Even if in contrast with Hawking's temperature definition, we think this is consistent with the current state of BH thermodynamics, since (4.29) is demonstrated to be valid only for equilibrium BHs, while the process here presented is a non-equilibrium phenomenon. We call this process the *QBH evaporation*.

5.3.1 Death of the Quantum Black Hole: remnants

One of the possible ways black holes are believed to end their lives is by becoming *remnants*. A remnant is a small quantum system left after a BH has evolved evaporating to a state in which for somewhat reason, it cannot any more loose energy. Various remnants with different features have been defined in the literature, and they all have in common being stable or long-lived quantum systems.
Based on what we found in Sec. 3.4, we here propose our picture of the last stage of a QBH's life, after evaporation has occurred in terms of many of the above-described cycles.
The sequential partitioning described in the previous section gradually reduces the number of subsystems that collectively compose the QBH. After each measurement performed by Alice, the total number of subsystems N is reduced by 1, and we expect the inequality

$$N - p < N_d \ , \tag{5.60}$$

for a given δ-distinguishability and ϵ-orthogonality, to come true at some point. When this happens, the QBH exits the quasi-classical regime and needs to be described as a thoroughly quantum system. This suggests that the QBH ends its life as a purely quantum system, similar to a remnant. Note that the evaporation process does not have to stop at this stage, as it may continue until $p = N$.

The most widely employed argument against BH remnants (for instance see Ref. [26]) is the fact that these objects need being very small but largely entangled with the emitted Hawking radiation, which seems to break Eq. (4.28). However, as shown in Ref. [27], properly taking into account the observer, without which the Hawking radiation itself would not exist, the total entropy is continuously balanced by the information entropy that the observer gets by measuring the outcoming radiation. Moreover, once the measurement takes place the entanglement between the $\mathcal{O}$ and $\mathcal{I}$ is zero, due to the post-measurement collapse of the quantum state.

Our proposal is then that a QBH can evaporate as long as $N > 0$. At some point, during this process, a point that depends on ϵ and δ, as discussed in Cap. 3, the QBH stops behaving as if it were classical, i.e. it exits the quasi-classical regime (when the lower bound N_d is surpassed), and it cannot be considered as a black hole anymore.

5.4 Quantum Black Holes and Quantum Time

In this last section, we explore an aspect that has been previously postponed: the picture we have proposed about the QBH evaporation involves, in a vague way, a notion of dynamical evolution. Therefore, in this last section, we ask ourself: *How a sequence of frames (or a "ticking" of time) can emerge from the sequential partitioning process?*
Please note that the following reasoning is in a preliminary stage and we plan to further investigate this topic.

5.4.1 QBHs evaporation and the arrow of time

Since the observation of emitted subsystems makes the observer evolve irreversibly, we ask whether this evolution might be regarded as the imposition of an arrow of time by the QBH, or not. The reasoning here presented is similar to that developed by L.Maccone in Ref. [28].

Let us take our model-universe as made solely of Alice and the QBH. Let us then suppose that Alice can perform the cycles described above: she randomly chooses the i-th couple o_i-i_i, and measures o_i via a projective measurement on the energy eigenbasis,

$$\Pi_n = |E_n\rangle \langle E_n| \ . \tag{5.61}$$

After each measurement, Alice will observe one of the possible outcomes, say E_m, and infer that the observed system is now in the pure state

$$|E_m\rangle \langle E_m| \ . \tag{5.62}$$

It is only after many measurements that Alice will be able to test the thermal distribution of what she is observing. Let us now enlarge our universe to include another observer, say Bob. We require Bob to know every detail of the QBH and of the evaporation process: this also includes knowing everything about Alice and her interaction with the Hawking radiation. From Bob's viewpoint, Alice does not see *one* of the outcomes but, by measuring the system, she entangles herself with the black hole. This means that, starting from the total[7] separable state

$$|\psi_0\rangle = |Q_0\rangle \otimes |\mathcal{A}\rangle \ , \tag{5.63}$$

the first measurement produces an overall state

$$|\psi_1\rangle = |Q_1\rangle \otimes \rho_{\mathrm{i}_1} \otimes \left(\sum_m p(m) \, |m\rangle \otimes |\mathcal{A} \text{ sees } m\rangle \right) \ , \tag{5.64}$$

[7]Since Bob is only a non-interacting witness for the process, he is not included in this total state.

where $p(m) = e^{-\beta E_m}/Z_0$. This writing has two possible way of being read. The first, according to the *many-worlds interpretation* of Everett, is that after the measurement Alice exists in multiple universes, each related to a different outcome of her measurement. The second is that Alice is just "the" measuring apparatus. Even if we do not consider these two approaches substantially different, we deem the second to be more physical. Since choosing one of the twos does not change anything relevant, in what follows we will keep the second interpretation, meaning that, after p measurements the overall state will be

$$|\psi_p\rangle = |Q_p\rangle \bigotimes_{i=1}^{p} \rho_{\mathrm{i}_i} \otimes \left(\sum_{m_1,\ldots m_p} p(m_1)\ldots p(m_p)\, |m_1\rangle \ldots |m_p\rangle \otimes |\mathcal{A} \text{ sees } m_1, m_2 \ldots m_p\rangle \right) . \tag{5.65}$$

If we now ask Alice what happened during the sequential measurements, her answer will be that her measurements broke unitary evolution. In contrast, if we ask Bob the same question, he will argue that the system (Alice included) has evolved unitarily. The expression (5.65) is just the statement that while Alice believes to have seen a certain outcome, from the viewpoint of an external observer she just entangled herself with the observed system, ending up in a superposition of states different from her initial state $|\mathcal{A}\rangle$. Moreover, if Bob knows every detail of the interaction between Alice and the QBH, the process that Alice experiences as irreversible might be perfectly reversible for Bob. Indeed, he might decide to reverse the dynamics[8] by applying a specific unitary evolution to the system and reset the measurement performed by Alice. As showed in Ref. [28], this process necessarily deletes the memory of Alice, rescuing the 2nd law of thermodynamics.

However, since our proposal of the QBH was constructed to be such that no observer can have access to the $\mathcal{I}$ partition, no observer such as Bob can actually exist. Therefore, observing the QBH imposes an arrow of time as Alice can, by measuring the Hawking radiation, make the universe evolve by a non-reversible evolution.

This seems to resolve the issue about how to order the "frames" emerging from the sequential partitioning process: since once a measurement is performed there is no way to get back, the time-order should be dictated by the number of subsystems extracted from the QBH and detected. This justifies the name "sequential partitioning" that we gave to this process. In this sense, the QBH can induce a discrete "ticking of time".

[8]Bob can also perform a different operation that is not a time-reversal of the dynamics. This has the very same effect on Alice's memory: it gets decorrelated from the observed systems.

Conclusions

In this thesis we have analyzed physical systems in a regime here called quasi-classical. These systems are characterized by displaying both classical and quantum properties. As neither the classical nor the quantum formalism allows treating this type of systems, starting from the procedure known as the large-N limit, we have developed the following tools to deal with the quasi-classical regime. In particular,

- A procedure to perform a classical to quantum crossover has been discussed.
- A particular microscopic description has been chosen, called *free-theory description*, that formally permits classical and quantum features to co-exist, This description embodies the possibility of performing quantum measurements on parts of a large-N system even if this effectively behaves classically.
- A lower bound over the number of subsystems a large-N theory should describe to give rise to a behaviour that can be recognised as classical. This permits to roughly identify the number of subsystems a system should be made of to fall into the quasi-classical regime.

Starting from the observation that BH belongs to the class of systems here called quasi-classical, we have built a quantum description of BH through the free-theory description previously developed. This has allowed us to study the evaporation process and, in particular

- To design a (dynamical) process for BH evaporation.
- To find a thermal form for the states of the systems emitted by the BH during the evaporation. The temperature arising in this picture is not proper to the QBH itself, which is not a thermodynamical system, but rather to the probability distribution recorded by observers outside its event horizon.
- To discuss how including an observer is fundamental to accurately describe phenomena involving quantum systems, such as the QBH evaporation process.

Finally, the last observation made us wonder what it means for the QBH evaporation to be dynamical and explore, even if only at a preliminary level, the relation between QM and time. The idea that BH, via their discrete emissions, can induce a direction upon the arrow of time of outside observers, has thus emerged.

Let us conclude this thesis by briefly discussing some possible future developments. The choice of the free-theory description, even if convenient, is not the only possible one. In particular, we believe that extending our description of QBH to interacting microscopic theory can permit to deal with different observers that employ different measurement schemes to probe the system's features. From this scheme, we expect the BH to emerge as an *objective reality* in the sense of *Quantum Darwinism*: a group of observes might be able to gather information about the system's state in different ways, still ending up concording about the overall state of the QBH. Since it is how Hawking's radiation is measured that determines the flow of time for these multiple observers, we ask ourself: *could it be that different observers measuring the same QBH end up having different notions of time?*

This question opens a vast landscape of exciting opportunities of which we, in this thesis, have only seen a tiny portion.

Appendices

Appendix A

Lie groups, Lie algebras and their representations

In this appendix, we review some elements of the theory of Lie groups and Lie algebras. The two main references that we use for this scope are the book written by A.O. Barut and R.Raczka [7] and the one by B. Hall [29].

Lie groups and Lie algebras

We call an abstract group G a *Lie group* of dimension d if it is a differentiable manifold of dimension d and if the group multiplication and inverse, defined respectively by

$$(\cdot,\cdot) : G \times G \to G\ ,\ (g_1, g_2) \mapsto g_1 g_2,\ \forall g_1, g_2 \in G \tag{A.1}$$

and

$$\cdot^{-1} : G \to G, g \mapsto g^{-1},\ \forall g \in G\ , \tag{A.2}$$

are differentiable. Being H a Lie subgroup of G, which means it is a closed subset of G under the action of the two operations (A.1-A.2), we define an equivalence relation $\sim$ on the group by saying that two of its elements are equivalent when there exist $h \in H$ such that $g' = gh$. With this definition[1], we can build the equivalence classes $[g]$ as

$$[g] = \{g' \in G \mid g' = gh,\ h \in H\}\ . \tag{A.3}$$

and the set of all equivalence classes

$$G/H = \{[g] \mid g \in G\}, \tag{A.4}$$

called *Coset Space*.

Let $\mathfrak{g}$ be a finite dimensional vector space over a field K. We call $\mathfrak{g}$ a *Lie algebra* over K if there exist an operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\ , \tag{A.5}$$

called Lie Bracket, that has the following three properties

- Linearity: $\forall a, b \in K$ and $\forall X, Y, Z \in \mathfrak{g}$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]\ . \tag{A.6}$$

[1] It is easy to show that $\sim$ is a well defined equivalence relation thanks to the group structure of H. In fact, $a \sim a$ since $1_G \in H$, $a \sim b \Rightarrow b \sim a$ since $h \in H \Rightarrow h^{-1} \in H$ and if $a \sim b \wedge b \sim c$ implies $a \sim c$ since $h, h' \in H \Rightarrow hh' \in H$.

- Antisymmetry: $\forall X, Y \in \mathfrak{g}$
$$[X, Y] = -[Y, X] \ . \tag{A.7}$$

- Jacobi associativity: $\forall X, Y, Z \in \mathfrak{g}$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ . \tag{A.8}$$

If K is the field of real (complex) numbers, then $\mathfrak{g}$ is said to be a real (complex) Lie algebra. Given a basis $\{X_i\}$ of $\mathfrak{g}$ we define the *structure constants* of the algebra as the coefficients c_{ij}^k such that
$$[X_i, X_j] = \sum_k c_{ijk} X_k \ . \tag{A.9}$$
It is worth mention that Lie brackets' antisymmetry implies that also the structure constants are antisymmetric under the exchange of any two indices.

From Lie algebras to Lie groups

Lie groups and Lie algebras are tightly connected: the latter can be seen as tangent spaces of former. Conversely, there exist a natural way to obtain the Lie group corresponding to a given Lie algebra: for any $X \in \mathfrak{g}$ there exist a map $\theta_X : \mathbb{R} \to G$ such that
$$\dot{\theta}_X(t)|_{t=0} = X \ . \tag{A.10}$$
This authorizes us to define the map
$$\exp : \mathfrak{g} \to G \ , \ \ X \mapsto \exp X \tag{A.11}$$
in such a way that
$$\exp X = \theta_X(1) \ ; \tag{A.12}$$
we call it the *exponential map*. It is important to note that if G is a Lie group associated with the Lie algebra $\mathfrak{g}$ through exp there only exists a neighbourhood U of $0 \in \mathfrak{g}$ and a neighbourhood V of $1_G \in G$ such that the exponential map takes U homomorphically onto V. Despite being valid only locally, this result is of main importance since it allows us to write the elements of G in terms of the ones of $\mathfrak{g}$.

Killing form

A *Killing form* is an application
$$(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to K \tag{A.13}$$
that acts on the elements of the Lie algebra as
$$(X, Y) = \mathrm{Tr}(ad_X ad_Y) \ , \tag{A.14}$$
where ad_X is the adjoint action defined by
$$ad_X(Y) = [X, Y] \ . \tag{A.15}$$
By choosing a basis $\{X_i\}$ of $\mathfrak{g}$ and thus the coefficients appearing in (A.9) it is possible to show that the matrix elements of the Killing form are
$$g_{ij} = (X_i, X_j) = \sum_{m,n} c_{imn} c_{jnm} \tag{A.16}$$
The Killing form can be used to define a metric on $\mathfrak{g}$, called *Cartan-Killing metric.*

Semisimple Lie Algebras and the Cartan-Weyl basis

Given a Lie algebra $\mathfrak{g}$ it is possible to define a Lie subalgebra $\mathfrak{h}$ as a subspace of $\mathfrak{g}$ that is closed under the action of the Lie brackets. A subalgebra $\mathfrak{i}$ is said to be an *ideal* of $\mathfrak{g}$ if

$$[g,h] \in \mathfrak{i} \ , \forall g \in \mathfrak{g}, \ \forall h \in \mathfrak{i} \ . \tag{A.17}$$

An algebra is called *semisimple* if it can be expressed as a direct sum of *simple* subalgebras, i.e. non abelian subalgebras whose only ideals are $\{0\}$ and $\mathfrak{g}$ itself. Complex semisimple Lie algebras are useful for our discussion because they allow the so-called *Cartan-Weyl basis.*
Given a complex semisimple Lie algebra $\mathfrak{g}$ we select the maximal subalgebra $\mathfrak{h}$ such that its adjoint action is diagonalizable, called *Cartan subalgebra.* We call *rank* of $\mathfrak{g}$ the dimension of $\mathfrak{h}$ and we call a *root* of $\mathfrak{g}$ an element α of $\mathfrak{h}^*$ such that the set

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid ad_h(g) = \alpha(h)g, \ \forall h \in \mathfrak{h}\} \ . \tag{A.18}$$

is not empty. We denote with Φ the collection of all roots. It is worth noting that for every root

$$\dim(\mathfrak{g}_\alpha) = 1 \tag{A.19}$$

and that roots come in pairs, which means that if $a \in \Phi$ then $-\alpha \in \Phi$ and that no other nonzero multiple of a root is a root. This means that Φ can be splitted as

$$\Phi = \Phi^+ \cup \Phi^- \ . \tag{A.20}$$

Lastly, for every root $\alpha \in \Phi$ there exists unique element t_α in $\mathfrak{h}$ such that

$$(h, t_\alpha) = \alpha(h), \ \forall h \in \mathfrak{h} \ . \tag{A.21}$$

With these definitions, it is possible to find a basis $\{h_i, g_\alpha\}$, called the *Cartan-Weyl basis* of $\mathfrak{g}$, with the following commutation rules:

$$\begin{cases} [h_i, h_j] = 0 \\ [g_\alpha, g_{-\alpha}] = (g_\alpha, g_{-\alpha})t_\alpha \\ [h_i, g_\alpha] = \alpha(h_i)g_\alpha, \\ [g_\alpha, g_\beta] = c_{\alpha\beta}g_{\alpha+\beta} \end{cases} \tag{A.22}$$

where $c_{\alpha\beta} \neq 0$ only if $\alpha + \beta \in \Phi$. This procedure provides a decomposition of the given algebra as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \right) \ . \tag{A.23}$$

Representations

Representation theory is the tool that, when dealing with Lie groups and Lie algebras, allows us to switch from the difficulties of abstract algebra to the easiness of linear algebra. Given a vector space V and a homomorphism $\pi : \mathfrak{g} \to GL(V)$, we define a representation of the algebra $\mathfrak{g}$ as the pair $r = (\pi, V)$. We refer to the dimension of V as the dimension of the representation. These definitions hold the same for Lie groups and we will refer to the representation generated by the homomorphisms from groups to $GL(V)$ with capital letters.
A representation is said to be *irreducible* if V does not include non-trivial invariant subspaces under the action of $\pi(g)$ $\forall g \in \mathfrak{g}$. Moreover, a representation on a complex vector space V is said to be *unitary* if

$$\pi(g)^\dagger \pi(g) = \pi(g)\pi(g)^\dagger = \mathbb{I}_V, \ \forall g \in \mathfrak{g} \ , \tag{A.24}$$

that is, if every element of $\mathfrak{g}$ is represented by a unitary operator.

The Universal Enveloping Algebra

Given any Lie algebra $\mathfrak{g}$, we call any associative algebra with identity $\mathcal{A}$ an *enveloping algebra* if there exists a linear map $j : \mathfrak{g} \to \mathcal{A}$ such that

$$j([X,Y]) = j(X)j(Y) - j(Y)j(X) \ , \tag{A.25}$$

$\forall X, Y \in \mathfrak{g}$. For any there exist a special associative algebra with identity $U(\mathfrak{g})$ togheter with a linear application $i : \mathfrak{g} \to U(\mathfrak{g})$ such that

- $\forall X, Y \in \mathfrak{g}$
$$i([X,Y]) = i(X)i(Y) - i(Y)i(X) \ . \tag{A.26}$$
- $U(\mathfrak{g})$ is generated by elements of the form
$$i(X), \ X \in \mathfrak{g} \ , \tag{A.27}$$
meaning that the smallest subalgebra with identity containing all $i(X)$ is $U(\mathfrak{g})$ itself.
- for any given enveloping algebra $\mathcal{A}$ there exists an algebra homomorphism $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $\phi(1_{U(\mathfrak{g})}) = 1_{\mathcal{A}}$ and that $\phi(i(X)) = j(X), \ \forall X \in \mathfrak{g}$. This is equivalent to say that for any $j : \mathfrak{g} \to \mathcal{A}$ that satisfies (A.25) the diagram
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "j"] \arrow[d, "i"] & \mathcal{A} \\ U(\mathfrak{g}) \arrow[ur, "\phi"] & \end{tikzcd}$$
commutes.

We call this algebra the *universal enveloping algebra* of $\mathfrak{g}$; the additional adjective *universal* for $U(\mathfrak{g})$ comes from the fact that if $\mathcal{A}$ is a generic enveloping algebra and ϕ is the homomorphism between $U(\mathfrak{g})$ and $\mathcal{A}$, then the latter is isomorphic to the quotient $U(\mathfrak{g})/\ker(\phi)$ which means, roughly speaking, that $U(\mathfrak{g})$ contains every other enveloping algebra of $\mathfrak{g}$.
One intuitive way to build $U(\mathfrak{g})$ is to start from the *Free Tensor Algebra* over $\mathfrak{g}$

$$\tau = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} = K \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots \ , \tag{A.28}$$

that is an associative algebra with abstract multiplication law given by the tensor product and identity given as the identity element over K. By defining J as the smallest two-sided ideal[2] generated by the elements of the form

$$X \otimes Y - Y \otimes X - [X,Y], \ X, Y \in \mathfrak{g} \tag{A.29}$$

it is possible to show (as, for example, is done in Ref. [29]) that $U(\mathfrak{g})$ is isomorphic to τ/J. This gives a more easy way to look at the universal enveloping algebra: since the quotient space τ/J is the vector space obtaines from τ by "collapsing" J to zero, this building embeds the Lie algebra into an associative algebra in such a way that the commutator $X \otimes Y - Y \otimes X$ in said algebra becomes equal to the Lie bracket $[X,Y]$ of $\mathfrak{g}$.

Futhermore, it is possible to show that any given representation (π, V) of $\mathfrak{g}$ induces a unique algebra homomorphism $\tilde{\pi} : U(\mathfrak{g}) \to GL(V)$ such that $\tilde{\pi}(1_{U(\mathfrak{g})}) = \mathbb{I}_V$, $\tilde{\pi}(X) = \pi(X)$ if $X \in \mathfrak{g}$ and

$$\tilde{\pi}(X_1 \otimes \dots \otimes X_r) = \pi(X_1) \dots \pi(X_r) \tag{A.30}$$

[2] A two-sided ideal of τ is any subspace $J \subset \tau$ such that $\forall \alpha \in \tau$ and $\forall \beta \in j$ the elements $\alpha\beta$ and $\beta\alpha$ belong to J

where $X_1 \otimes \ldots \otimes X_r$ is an element of the subspace $U(\mathfrak{g})^r$ spanned by all the elements $X_{i_1} \otimes \ldots \otimes X_{i_r}$, with $X_{i_k} \in \mathfrak{g}$. In other words, since $GL(V)$ is an enveloping algebra of $\mathfrak{g}$ with $\pi : \mathfrak{g} \to GL(V)$ it follows that any representation (π, V) is included in the universal enveloping algebra and $\tilde{\pi}$ acts as

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "i"] & GL(V) \\ U(\mathfrak{g}) \arrow[ur, "\tilde{\pi}"] & \end{tikzcd}$$

Within this seeting, a theorem due to Poincaré, Birkhoff and Witt called *PBW theorem* is of main importance: it states that for any finite dimensional[3] Lie algebra $\mathfrak{g}$ spanned by $\{X_1, \ldots, X_D\}$, the elements

$$i(X_1)^{n_1} i(X_2)^{n_2} \ldots i(X_D)^{n_D} \tag{A.31}$$

are linearly independent for every choice of non negative integers $n_1, \ldots, n_D$. Therefore a collection of these elements composed of every possible combination of said integers such that

$$n_1 + \ldots + n_D = r \tag{A.32}$$

can be used as a basis for $U(\mathfrak{g})^r$. As an additional result, by selecting $n_i = 1, n_{j \neq i} = 0$ for all i we get that the elements

$$i(X_1), \ldots, i(X_D) \tag{A.33}$$

are linearly independent, which means that the map i is injective.

The Casimir element

For any given Lie algebra $\mathfrak{g}$ we can select an element C of its universal enveloping algebra, called *Casimir* element, that is of great importance for classifing irreducible representations. Given an orthonormal basis $\{X_i\}$ of $\mathfrak{g}$ we define the Casimir element as

$$C = \sum_i X_i^2 \in U(\mathfrak{g}) \ . \tag{A.34}$$

From the antisymmetry of the structure constants, we get that

$$[X_i, C] = \sum_j [X_i, X_j^2] = \sum_{j,k} \left(c_{ijk} X_k X_j + c_{ijk} X_j X_k \right) = \tag{A.35}$$

$$= \sum_j \left(c_{ijk} + c_{ikj} \right) X_k X_j = \sum_j \left(c_{ijk} - c_{ijk} \right) X_k X_j = 0 \ . \tag{A.36}$$

By extension of the fact that C commutes with every element of $\mathfrak{g}$, we can say that it also commutes with every element of $U(\mathfrak{g})$. The explicit form of C in any given finite-dimensional irreducible representation $r = (\pi, V)$ of $\mathfrak{g}$ (obtained through the extension of π to $\tilde{\pi}$ as described previously) is fixed by Schur's lemma as

$$\tilde{\pi}(C) = c_\pi \mathbb{I}_V \tag{A.37}$$

where c_π is a constant that depends only on our choice of r, and therefore can be used to classify different representations.

[3]The PBW theorem work also for infinite dimensional Lie algebras but since we will not encounter them in this work we have prefered to leave them out for the sake of simplicity.

Appendix B

Coherent states overlaps

In this appendix we present a new tecnique to evaluate the overlap of two coherent stats that works for a wide class of Lie algebras, that includes amog the others both $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. Let $\mathfrak{g}$ be a rank 1 Lie algebra with Weyl-Cartan basis $\{g_\alpha, g_0, g_{-\alpha}\}$ and with commutation rules

$$[g_0, g_\pm] = \alpha_\pm g_\pm \ , \ [g_+, g_-] = \beta g_0 \ . \tag{B.1}$$

Once a representation and a reference state vector are choosen, we can build the coherent states as

$$|\tau\rangle = N^{-1/2}(\tau)\,|\tilde{\tau}\rangle = N^{-1/2}(\tau)\exp(\tau\hat{K}_+)\,|\psi_0\rangle \ , \tag{B.2}$$

with the prescriptions given in the first and second chapters. The overlap between any two coherent states is

$$\langle\tau|\,\tau'\rangle = \big(N(\tau)N(\tau')\big)^{-1/2}\,\langle\psi_0|\exp(\tau^*\hat{K}_-)\exp(\tau'\hat{K}_+)\,|\psi_0\rangle \equiv \frac{M(\tau,\tau')}{(N(\tau)N(\tau'))^{1/2}} \ . \tag{B.3}$$

The inner product of two non-normalized GCS

$$M(\tau,\tau') = \langle\psi_0|\sum_{n=0}^{\infty}\frac{(\tau^*)^n}{n!}\hat{K}_-^n\sum_{m=0}^{\infty}\frac{(\tau')^m}{m!}\hat{K}_+^m\,|\psi_0\rangle = \tag{B.4}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\tau^*)^n}{n!}\frac{(\tau')^m}{m!}\,\langle\psi_0|\,\hat{K}_-^n\hat{K}_+^m\,|\psi_0\rangle \tag{B.5}$$

can be recasted in the form

$$M(\tau,\tau') = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\tau^*)^n}{n!}\frac{(\tau')^m}{m!}\,\langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+^m]\,|\psi_0\rangle \tag{B.6}$$

since the expectation value of $[\hat{K}_-^n,\hat{K}_+^m]$ satifies

$$\langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+^m]\,|\psi_0\rangle = \langle\psi_0|\,\hat{K}_-^n\hat{K}_+^m - \hat{K}_+^m\hat{K}_-^n\,|\psi_0\rangle = \langle\psi_0|\,\hat{K}_-^n\hat{K}_+^m\,|\psi_0\rangle \ , \tag{B.7}$$

as it is easy to show by recalling that the reference state vector is choosen to be the lowest weight. By using the property of the commutation rules, we have that

$$\langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+^m]\,|\psi_0\rangle = \langle\psi_0|\,\hat{K}_+[\hat{K}_-^n,\hat{K}_+^{m-1}] + [\hat{K}_-^n,\hat{K}_+]\hat{K}_+^{m-1}\,|\psi_0\rangle = \tag{B.8}$$

$$= \langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+]\hat{K}_+^{m-1}\,|\psi_0\rangle \ . \tag{B.9}$$

Since

$$[\hat{K}_+,\hat{K}_-^n] = \left(n\beta\hat{K}_0 - \frac{n(n-1)}{2}\beta\alpha_-\right)\hat{K}_-^{n-1} \ , \tag{B.10}$$

as it is possible to show by induction[1], equation (B.8) is equal to

$$\langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+^m]\,|\psi_0\rangle = \Sigma(n)\,\langle\psi_0|\,\hat{K}_-^{n-1}\hat{K}_+^{m-1}\,|\psi_0\rangle \tag{B.11}$$

where we have defined the function

$$\Sigma(n) = \frac{n(n-1)}{2}\beta\alpha_- - n\beta\nu_0 \ , \tag{B.12}$$

assuming $\hat{K}_0\,|\psi_0\rangle = \nu_0\,|\psi_0\rangle$. By iterating (B.7), we have that (B.8) is nonzero only for $m = n$, where it assumes the value

$$\begin{aligned}\langle\psi_0|\,[\hat{K}_-^n,\hat{K}_+^n]\,|\psi_0\rangle &= \Sigma(n)\,\langle\psi_0|\,\hat{K}_-^{n-1}\hat{K}_+^{n-1}\,|\psi_0\rangle = \\ &= \Sigma(n)\Sigma(n-1)\,\langle\psi_0|\,\hat{K}_-^{n-2}\hat{K}_+^{n-2}\,|\psi_0\rangle = \\ &= ... = \Sigma(n)...\Sigma(1) = \prod_{k=1}^{n}\Sigma(k) \ .\end{aligned} \tag{B.13}$$

This quantity, herein called $P_{\mathfrak{g}}^{\nu_0}(n)$, depends only on the Lie algebra's structure constants, on the value ν_0 specific of the representation, and on the vaule of n; its explicit expression is

$$P_{\mathfrak{g}}^{\nu_0}(n) = \left(\frac{\beta\alpha_-}{2}\right)^n \prod_{k=1}^{n} k\left(k - \left(\frac{2\nu_0}{\alpha_-}+1\right)\right) = n!\left(\frac{\beta\alpha_-}{2}\right)^n(-2\nu_0/\alpha_-)_n \tag{B.14}$$

where $(x)_n$ is the *Pochhammer symbol*[2]. Finally, the general result for coherent states overlap in τ coordinates is

$$\langle\tau|\,\tau'\rangle = \big(N(\tau)N(\tau')\big)^{-1/2}\sum_{n=0}^{\infty}\frac{(\tau^*\tau')^n}{n!}\left(\frac{\beta\alpha_-}{2}\right)^n(-2\nu_0/\alpha_-)_n \ . \tag{B.17}$$

Application to $\mathfrak{su}(2)$

For $\mathfrak{g} = \mathfrak{su}(2)$ in j-representation we have $\alpha_\pm = \pm 1$, $\beta = 2$, $\nu_0 = -j$. By applying

$$P_{\mathfrak{su}(2)}^{-j}(n) = (-1)^n n!(-2j)_n = n!(2j-n+1)_n \tag{B.18}$$

to (B.5) we get

$$M(\tau,\tau') = \sum_{n=0}^{2j}\frac{(\tau^*\tau')^n}{n!}(2j-n+1)_n = \tag{B.19}$$

$$= \sum_{n=0}^{2j}\frac{(\tau^*\tau')^n}{n!}\frac{(2j)!}{(2j-n)!} = \tag{B.20}$$

$$= \sum_{n=-j}^{j}\binom{2j}{m+j}(\tau^*\tau')^{m+j} \ . \tag{B.21}$$

[1]The proof is given in the last paragraph of this appendix.

[2]The Pochhammer symbol is defined by

$$(x)_n = x(x+1)...(x+n-1) = \frac{(x+n-1)!}{(x-1)!} \ . \tag{B.15}$$

If x is a negatve integer it hold true that

$$(x)_n = 0 \ \forall \ n > |x|+1 \tag{B.16}$$

and $(-|x|)_n = (-1)^n(|x|-n+1)_n$.

By inserting (B.21) in (B.3) we obtain

$$\langle\tau|\,\tau'\rangle = \left(N(\tau)N(\tau')\right)^{-1/2}\sum_{m=-j}^{j}\binom{2j}{m+j}(\tau^*\tau')^{m+j}\ , \tag{B.22}$$

that is the well known result for $\mathfrak{su}(2)$ coherent states. In τ coordinates the result is

$$\langle\tau|\,\tau'\rangle = \frac{(1+\tau^*\tau')^j}{(1+|\tau|^2)^j(1+|\tau'|^2)^j}\ , \tag{B.23}$$

while in (θ,ϕ) coordinates it is

$$\langle\tau|\,\tau'\rangle = \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{i(\phi'-\phi)}\right)^{2j}\ . \tag{B.24}$$

Application to $\mathfrak{su}(1,1)$

For $\mathfrak{g}=\mathfrak{su}(1,1)$ in one of the two modes representations we have that $\alpha_\pm=\pm1$, $\beta=-2$, $\nu_0=k=(n_0+1)/2$, that gives

$$P^k_{\mathfrak{su}(1,1)}(n) = n!(n_0+1)_n = n!\frac{(n_0+n)!}{n_0!}\ . \tag{B.25}$$

The function $M(\tau,\tau')$ is

$$M(\tau,\tau') = \sum_{n=0}^{\infty}\frac{(n_0+n)!}{n_0!n!}(\tau^*\tau')^n \tag{B.26}$$

and the overlap results in

$$\langle\tau|\,\tau'\rangle = (1-|\tau|^2)^k(1-|\tau'|^2)^k\sum_{n=0}^{\infty}\frac{(n_0+n)!}{n_0!n!}(\tau^*\tau')^n\ . \tag{B.27}$$

Since the sum

$$\sum_{n=0}^{\infty}\frac{(n_0+n)!}{n_0!n!}x^n = \frac{1}{(1-x)^{(1+n_0)}} \tag{B.28}$$

converges if $|x|<1$. Since $\tau\in D^2$, we obtain

$$\langle\tau|\,\tau'\rangle = \frac{(1-|\tau|^2)^k(1-|\tau'|^2)^k}{(1-\tau^*\tau')^{2k}} \tag{B.29}$$

in τ coordinates.

Proof of equation (B.11)

The only thing left to do is to prove that

$$[\hat{K}_+,\hat{K}_-^n] = \left(n\beta\hat{K}_0-\frac{n(n-1)}{2}\beta\alpha_-\right)\hat{K}_-^{n-1}\ . \tag{B.30}$$

As already mentioned, this can be done by induction. For $n=1$ the formula gives the Lie bracket $[\hat{K}_+,\hat{K}_-]=\beta\hat{K}_0$ translated in the representation. Supposing it true for $n-1$

$$[\hat{K}_+,\hat{K}_-^{n-1}] = \left((n-1)\beta\hat{K}_0-\frac{(n-1)(n-2)}{2}\beta\alpha_-\right)\hat{K}_-^{n-2}\ , \tag{B.31}$$

we get that it is true for n. In fact

$$
\begin{aligned}
[\hat{K}_+, \hat{K}_-^n] &= [\hat{K}_+, \hat{K}_-\hat{K}_-^{n-1}] = \qquad \text{(B.32)}\\
&= \hat{K}_-[\hat{K}_+, \hat{K}_-^{n-1}] + [\hat{K}_+, \hat{K}_-]\hat{K}_-^{n-1} = \\
&= \hat{K}_-\left((n-1)\beta\hat{K}_0 - \frac{(n-1)(n-2)}{2}\beta\alpha_-\right)\hat{K}_-^{n-2} + \beta\hat{K}_0\hat{K}_-^{n-1} = \\
&= \left((n-1)\beta + \beta\right)\hat{K}_0\hat{K}_-^{n-1} - \beta\alpha_-\left((n-1) + \frac{(n-1)(n-2)}{2}\right)\hat{K}_-^{n-1} = \\
&= n\beta\hat{K}_0\hat{K}_-^{n-1} - \beta\alpha_-\frac{n(n-1)}{2}\hat{K}_-^{n-1} \;, \qquad \text{(B.33)}
\end{aligned}
$$

as expected. For completeness we note that it is possible to show in the very same way that

$$
[\hat{K}_+^n, \hat{K}_-] = \left(n\beta\hat{K}_0 - \frac{n(n-1)}{2}\beta\alpha_+\right)\hat{K}_+^{n-1} \;. \tag{B.34}
$$

Appendix C

Elements of General Relativity and Black Holes physics

In order to make the contents of Cap. 4 clearer, in this appendix we briefly present some elements of General Relativity (GR) and BH physics. Our main references are *Spacetime and Geometry: An Introduction to General Relativity* by S.Carroll [17], *The Large Scale Structure of Space-Time* by S.W.Hawking and G.F.R.Ellis [30], and *General Relativity* by R.Wald [31].
GR is a geometric theory of gravity. By describing reality as a set of events distributed on a four-dimensional manifold, called *spacetime*, it successfully explains most of the astrophysical phenomena we observe. The spacetime is defined as the collection of all (past, present and future) events. Formally, we express this idea using the pair $(\mathcal{M}, g)$, made of a connected smooth manifold $\mathcal{M}$ of dimension 4 and a metric tensor g with Lorentzian signature. The metric tensor g is usually made explicit by its components as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{C.1}$$

when a chart (U, ϕ) of $\mathcal{M}$ with coordinates $\{x^\mu\}$ is chosen. Thanks to the metric on $\mathcal{M}$, we classify the elements of the tangent bundle $\mathcal{TM}$ of $\mathcal{M}$, called tangent vectors, by the sign of their norm:

- if $v \in \mathcal{T}_p\mathcal{M}$ is such that $g_p(v, v) > 0$, then v is said to be *space-like* in $p \in \mathcal{M}$;
- if $v \in \mathcal{T}_p\mathcal{M}$ is such that $g_p(v, v) = 0$, then v is said to be *null* or *light-like* in $p \in \mathcal{M}$;
- if $v \in \mathcal{T}_p\mathcal{M}$ is such that $g_p(v, v) < 0$, then v is said to be *time-like* in $p \in \mathcal{M}$;

where $g_p(\cdot, \cdot)$ is the inner product induced by the metric onto the tangent space $\mathcal{T}_p\mathcal{M}$. A curve on the manifold is said to be spacelike, null or timelike according to the nature of its tangent vectors; conversely, a submanifold is said to be spacelike if its normal vectors are timelike and vice-versa. Lastly, a submanifold is said to be null if its normal vectors are null as well. A defining feature of GR is that nothing can move faster than light. This, in formal terms, means that an observer can only move along timelike curves. Furthermore, null curves are possible trajectories for a massless object, since only such objects can move as fast as light.
If there exists a non-vanishing timelike vector field v on $\mathcal{M}$ then the spacetime is said to be time-orientable, as for any given timelike vector we can say whether it has the same direction as v or not, by the sign of its scalar product with v. This is what is usually done to define an *arrow of time* on a spacetime.

Symmetries and conserved quantities in GR

A spacetime is said to be symmetric under some transformation if moving all the points of the manifold along the integral curves of the vector field K specific to the symmetry leaves some property of the spacetime unchanged. This is formally expressed by saying that the Lie derivative of the metric tensor

$$[\mathcal{L}_K(g)]_{\mu\nu} = K^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu K^\lambda + g_{\lambda\mu}\partial_\nu K^\lambda \tag{C.2}$$

vanishes for all $\mu, \nu = 1, ..., 4$. If this happens, K is called a *Killing vector field* and it satisfies the condition

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ , \tag{C.3}$$

called *Killing equation*. The "moving" that sends $\mathcal{M}$ into itself keeping the metric unchanged is called an *isometry*. Furthermore, if a symmetry is present then it is possible to show that the quantity

$$Q = K^\mu \dot{x}_\mu \tag{C.4}$$

is conserved along the trajectories $x(\tau)$ on $\mathcal{M}$. Finally, if the metric tensor does not depend on some of the coordinates, then it is obviously invariant under translations of these coordinates. This, for example, is what happens for the Schwarzschild spacetime and the time coordinate: the manifold is invariant under time translations since Eq. (4.3) does not depend explicitly on t.

Causal structure and the Carter-Penrose Diagrams

A non-trivial feature of a spacetime is how events can influence each other, defining what is called the spacetime's *causal structure*. As any signal can travel at most at the speed of light, this aspect is encoded into null geodesics.
A way to visually understand the causal structure of a given spacetime $(\mathcal{M}, g)$ is given by its *Carter-Penrose* (CP) diagram, a conformal compactification of the whole manifold into a finite size picture (that can usually fit into a piece of paper).

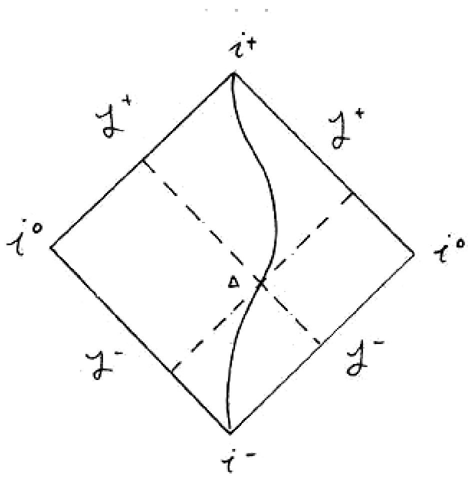
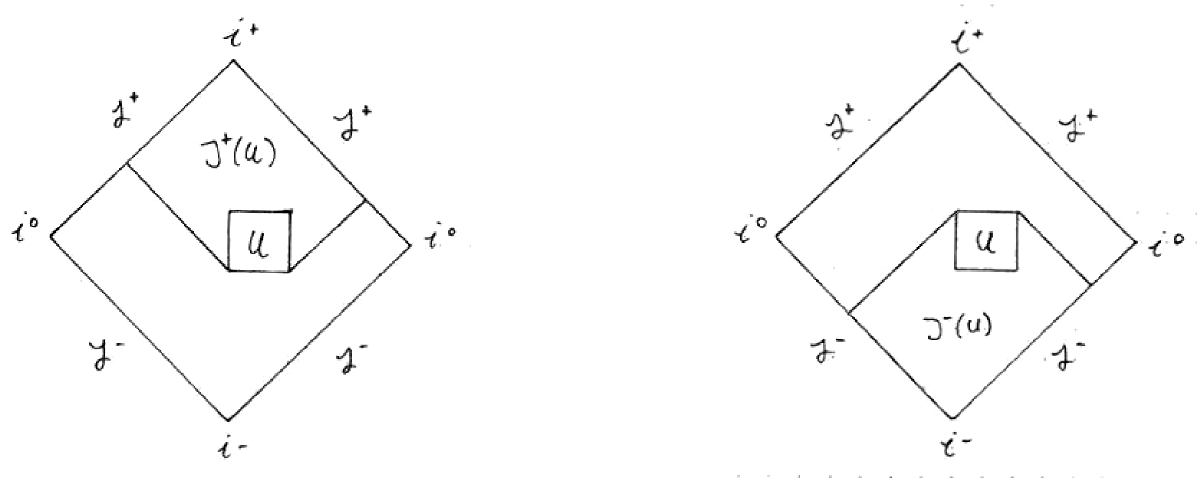


Figure C.1: CP diagram for the Minkowski's spacetime: the curve represents a physical trajectory for an observer. The dashed lines crossing in A represent the light-cone of the observer when it passes in A point.

This is done by defining a new metric tensor g' such that the points that are at infinite distance using the metric g become at finite distance using the new one. Moreover, since we don't want to mess the causal structure, we ask that the new metric preserves the normalization of the vectors: this means, for instance, that curves with spacelike tangents vectors are still impossible trajectories for observers after the metric is transformed. These two requests are satisfied by defining the *Weyl transformation*

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Lambda^2(r,t) g_{\mu\nu} \ , \tag{C.5}$$



(a) $J^+(\mathcal{U})$ (b) $J^-(\mathcal{U})$

Figure C.2: CP diagram with $J^+(\mathcal{U})$ (panel a) and $J^-(\mathcal{U})$ (panel b) for a domain $\mathcal{U}$ in the Minkowski's spacetime.

where $\Lambda^2(r,t)$ is a real function[1] defined on $\mathcal{M}$, such that

- $\lim_{r\to\infty} \Lambda^2(t,r) = 0$,
- $\lim_{t\to\pm\infty} \Lambda^2(t,r) = 0$,

and $\Lambda(t,r) \neq 0$ everywhere on $\mathcal{M}$ (note that $r = \infty$ and $t = \pm\infty$ are not on $\mathcal{M}$). It is possible to show that null geodesics are preserved by the Weyl transformation, meaning that the space-/null-/time-like character of the tangent vectors does not change, as required. Without entering into the details, let us present the Carter-Penrose diagram for the Minkowski's spacetime, shown in Fig. C.1.

Referring to the figure, the points $i^\pm$ correspond to $r = const.$ and $t = \pm\infty$; they are called *future* and *past timelike infinity*, respectively; the point i^0 corresponds to $t = const.$ and $r = \infty$ and is called *spacelike infinity.* The lines $\mathscr{I}^\pm$ correspond to $r = \infty$ and $t = \pm\infty$ and are called *past* and *future null infinity*, respectively. Since in a CP diagram every null geodesic is a 45° inclined line, reading the causal structure becomes easy. In the next paragraph, we show a meaningful application of these diagrams to identify event horizons and BHs.

Black holes

Without dwelling into technicalities, we can observe an important difference between Fig. C.1 and Fig. 4.1: while in the former it is possible to reach $\mathscr{I}^+$ from any point of the diagram by means of a timelike or null curve, this is not possible in the latter. This feature is what defines the black hole's event horizon as a cut between causal regions. In fact, once in region II (of Fig. 4.1) there is no way to reach $\mathscr{I}^+$, and any possible curve followed by an observer (or a signal) leads to $r = 0$: nothing that happens inside region II can influence the region I.

More formally, for any given subset $\mathcal{U} \subseteq \mathcal{M}$ we can define the set of points that can be reached from $\mathcal{U}$ propagating null or timelike curves "backwards" in time via $J^-(\mathcal{U})$, and $J^+(\mathcal{U})$ as the set of points that can be reached from $\mathcal{U}$ propagating null or timelike curves "forward" in time. These two sets are called *causal past* and *causal future* of $\mathcal{U}$, respectively. One can define the boundary of the closure of $J^\pm(\mathcal{U})$ as

$$j^\pm(\mathcal{U}) = \partial\overline{J^\pm(\mathcal{U})} = \overline{J^\pm(\mathcal{U})} - J^\pm(\mathcal{U}) \ . \tag{C.6}$$

[1] Note that here we have introduced a set of spherical coordinates (t, r, θ, ϕ) on the manifold, for the sake of simplicity.

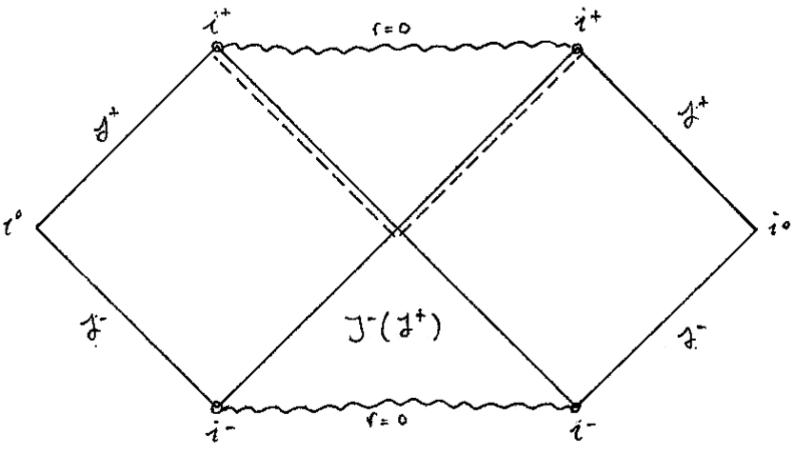


Figure C.3: How to show that the SBH presents an event horizon, and thus a BH. Referring to Eq. (C.9), the causal past of $\mathscr{I}^+$ is not the whole spacetime $\mathcal{M}$.

An example explaining how to build these sets is reported in Fig C.2. Using these tools, one can give a formal rephrasing to the intuitive concept of an event horizon. A spacetime is said to have an event horizon if

$$\mathscr{H}^+ \equiv j(\mathscr{I}^+) \neq \{0\} \ , \tag{C.7}$$

and a *past event horizon* if

$$\mathscr{H}^- \equiv j(\mathscr{I}^-) \neq \{0\} \ . \tag{C.8}$$

Equivalently, a spacetime $\mathcal{M}$ presents an event horizon if

$$\overline{J^-(\mathscr{I}^+)} \neq \mathcal{M} \ . \tag{C.9}$$

Being a submanifold that separates causal regions, an event horizon is always a null surface. Finally, let us define what we mean by "area" of an event horizon $\mathcal{A}$. Thinking of how an observer would define the surface of a BH, we say that $\mathcal{A}$ is the area of a spherical surface of radius r_s, at $t = const.$, which reads

$$\mathcal{A} = \int_0^{2\pi} \int_0^{\pi} d\theta d\phi \sqrt{det\gamma} = 16\pi M^2 \ , \tag{C.10}$$

where γ is the angular part of Eq. (4.3), for a BH of mass M.

The Cosmic Censorship Conjecture

Since we refer to it in the text, let us briefly mention what is the so-called Cosmic Censorship Conjecture (CCC) formulated by Penrose. Quoting Ref. [31], the CCC states

> *The complete gravitational collapse of a body always results in a black hole rather than a naked singularity; i.e. all singularities of gravitational collapse are "hidden" within BHs, where they cannot be "seen" by distant observers.*

This means that we cannot observe a singularity without passing through an event horizon. Note that this conjecture is not true in general, and it must be enriched by some conditions on the matter fields within $\mathcal{M}$.

Appendix D

QBHs, Hawking temperature and Quantum time

In this appendix, we discuss some possible strategies to fix the parameters (5.19). The values we assign, and the interpretation we give, to these parameters is of main relevance to complete the correspondence between the temperature an observer measures as outcoming from the QBH, Eq. (5.56), and the Hawking temperature (4.29).
The most naive thing we might try to do is to match β and $1/T_H$ directly[1], according to

$$\beta = \frac{1}{\omega} \log \frac{1}{|\tau|^2} = \frac{1}{T_H} = 8\pi M \ , \tag{D.1}$$

which results in the relation

$$|\tau|^2 = e^{-8\pi M\omega} \tag{D.2}$$

between the QBH's state, parametrized by τ, and its mass M. The value of ω in (D.2) is still a free parameter and various strategies to fix it are presented in what follows.

The observer chooses ω

Since in the free-theory description we have complete freedom over the choice of the N microscopic Hamiltonians (5.4), we require $\hat{H} = \hat{K}_0$. This is tantamount to say that the instrument we use to observe the subsystems can only detect subsystems with $\omega = 1$. After this choice, the formula (D.2) becomes

$$|\tau|^2 = e^{-8\pi M} \ . \tag{D.3}$$

This choice is consistent with the rest of the thesis, as discussed in Sec. 5.2.

The Hamiltonian is fixed so as to restore the geodesic motion around the SBH in the $N \to \infty$ limit

One might require the Hamiltonian of the system to be that defined in (4.68) or in (4.74). As already mentioned, these choices make the QBH's theory flow, when $N \to \infty$ into the classical theory with effective Hamiltonians (4.61) and (4.60), respectively. Despite having this desirable property, this choice presents a problem when trying to derive Hawking temperature. In fact, as choosing Eq. (4.68) as the QBH's Hamiltonian, the energy levels become

$$E_n = \langle n| \hat{H}_I |n\rangle = \frac{N}{2}(a+1)n + d \ , \tag{D.4}$$

[1] We retain this try not in contrast with our claim that the QBH is out of equilibrium. In fact, we might think to have an equilibrium stage where the evaporation has not started yet that characterize both the BH's mass and temperature and its quantum state τ. The temperature is then fixed and never changes, as shown in Sec. 5.3.

and the level-spacing reads

$$\omega = \frac{N}{2}(a+1) = \frac{N}{4r_s^2 m^2}\left(\frac{3\tilde{l}^2}{r_S^2} + \mu^2 + 2r_s^2 m^2\right) , \tag{D.5}$$

which makes it hard to interpret (D.2) as defining the QBH state. That is because ω contains the mass of a test particle which is not a parameter of the QBH itself and should not have a role in defining its state. As one can imagine, the identification becomes even harder using Eq. (4.74).
However, we believe it might be possible to solve this issue (and some work in this direction is ongoing).

QBH as clocks

Lastly, we briefly present an approach based on the so-called PaW mechanism. Our idea is to obtain some classical features of SBH (but not the right geodesic motion) together with the correct expression of Hawking's temperature expressed by Tolman's law by requiring the QBH to behave as a clock. This section, being based on PaW mechanism (see Ref. [32]) and its extension by the parametric representation via environmental coherent states (or PRECS), does not claims to be exhaustive. Indeed, we refer to Ref. [33] for a complete discussion about the PRECS and its application in the PaW mechanism. The idea we present here is still in its early stages and it might be of some interest a future development.

Every time we describe the dynamics of an object (be it classical or quantum), the strategy is to relate the values of its observables to those of another system, usually called a "clock". To review the PaW mechanism, which formalizes this idea in the context of QM, let us consider an isolate bipartite system $\Psi = \Gamma + C$, where Γ and C are the "principal system" and the "clock", respectively. If C and Γ are correlated, i.e. entangled, one can describe the evolution of the observables of the latter by means of those of the former. Moreover, to avoid that C influences the dynamic of Γ, we require Γ and C to evolve independently. Thus the overall system Ψ is in the state

$$|\psi\rangle = \sum_\gamma c_\gamma |\gamma\rangle \otimes |\xi_\gamma\rangle \ , \tag{D.6}$$

with Schmidt rank larger than 1, and evolves with Hamiltonian

$$\hat{H} = \hat{H}_\Gamma \otimes \hat{\mathbb{I}}_C + \hat{\mathbb{I}}_\Gamma \otimes \hat{H}_C \ . \tag{D.7}$$

We then require the system to be in the zero eigenvalue of the Hamiltonian (D.7). This condition, made explicit as

$$\hat{H}|\psi\rangle = 0 \ , \tag{D.8}$$

is called Wheeler-de Witt equation[2].
Let the clock's Lie algebra to have rank 1 and Hamiltonian to be the diagonal element of that algebra. Once we build the CSS for the clock system we choose $\tau = |\tau|e^{i\phi}$ to be a parametrization of the points on the manifold. One can then manipulate Eq. (D.8) by means of the PRECS to find the von Neumann-like equation

$$i\alpha_C \frac{d}{d\phi}\tilde{\rho}_\Gamma(\tau) = [\hat{H}_\Gamma, \tilde{\rho}_\Gamma(\tau)] \tag{D.9}$$

for the non-normalized density operator

$$\tilde{\rho}_\Gamma(\tau) = \sum_{\gamma,\gamma'} c_\gamma c^*_{\gamma'} |\gamma\rangle\langle\gamma| \langle\tau|\,\xi_\gamma\rangle\langle\xi_{\gamma'}|\,\tau\rangle \ , \tag{D.10}$$

[2] This condition can be relaxed. In fact, if the system is found in an eigenstate different from 0 we can get back to the Wheeler-de Witt equation just by rescaling (D.7).

with α_C defined as in Eqs. (3.7). As described in Ref. [33], one can normalize Eq. (D.10) and, by taking the $N \to \infty$ limit for the clock obtain

$$\sum_{\gamma} c_{\gamma}^{2} \left\{ [\hat{H}_{\Gamma}, \tilde{\rho}_{\Gamma}^{\gamma}(\phi)] - i\alpha_{c} \frac{d}{d\phi} \tilde{\rho}_{\Gamma}^{\gamma}(\phi) \right\} = 0 \tag{D.11}$$

where $\tilde{\rho}_{\Gamma}^{\gamma}(\phi)$ is the density operator (D.10) normalized and integrated over the clock's CSS manifold. By noting the similarity with the Liouville-Von Neumann equation, one can then interpret the phase of the CSS label parametrizing the clock's state as a "time parameter" for Γ.

The idea we propose is to consider a PaW-like system and replicate it an infinite amount of times, labelling each replica with a positive-valued real parameter r. Moreover, we allow the clock's Hamiltonians (or equivalently, the principal system's Hamiltonians) to vary with respect to r. This is done in order to describe a collection of identical principal systems placed at different distances r, that use the same QBH as clock.
Assuming C to be a QBH, its Lie algebra is $\mathfrak{su}(1,1)_N$. Therefore, we can choose $\hat{H}_C^{(r)}$ to be proportional to the diagonal element as

$$\hat{H}_C^{(r)} = \Delta(r) \hat{K}_0 \ , \tag{D.12}$$

where $\Delta(r)$ is a non-vanishing function of the label r. Dividing Eq. (D.8) by $\Delta(r)$ we get

$$\sum_{\gamma} c_{\gamma}(r) \frac{1}{\Delta(r)} \hat{H}_{\Gamma}^{(r)} \left|\gamma\right\rangle \otimes \left|\xi_{\gamma}\right\rangle + \sum_{\gamma} c_{\gamma}(r) \left|\gamma\right\rangle \otimes \hat{K}_0 \left|\xi_{\gamma}\right\rangle = 0 \ . \tag{D.13}$$

Because we want the principal systems and their correlations with the QBH to be all identical, we choose

$$\hat{H}_{\Gamma}^{(r)} \to \hat{H}_{\Gamma} \quad , \quad c_{\gamma}(r) \to c_{\gamma} \ . \tag{D.14}$$

Therefore Eq.(D.13) reads

$$\sum_{\gamma} c_{\gamma} \frac{1}{\Delta(r)} \hat{H}_{\Gamma} \left|\gamma\right\rangle \otimes \left|\xi_{\gamma}\right\rangle + \sum_{\gamma} c_{\gamma} \left|\gamma\right\rangle \otimes \hat{K}_0 \left|\xi_{\gamma}\right\rangle = 0 \ . \tag{D.15}$$

We then propose to interpret the r label as the distance between the BH and the specific principal system considered. The request for the Γs to be identical is then that to have the same system at different distances.
By repeating the previous steps, we get

$$\sum_{\gamma} c_{\gamma}^{2} \left\{ [\hat{H}_{\Gamma}/\Delta(r), \tilde{\rho}_{\Gamma}^{\gamma}(\phi)] - i\alpha_{c} \frac{d}{d\phi} \tilde{\rho}_{\Gamma}^{(r)}(\phi) \right\} = 0 \ , \tag{D.16}$$

which describes a von Neumann-like equation for a collection of principal systems that we require being at different distances r from the QBH. Note that the Hamiltonian (and thus the time parameter) is rescaled inversely with $\Delta(r)$, which is the coupling of the clock's Hamiltonian in the r system.
From the above discussion, the following statements follow:

- principal systems at different distances r have the same dynamics as long as they do not compare their evolutions with other identical systems at different distances. If they do so, they discover that their times are rescaled as

$$\frac{\tau_A}{\Delta(r_A)} = \frac{\tau_B}{\Delta(r_B)} \ , \tag{D.17}$$

  where A and B are two of those principal systems.

- When an observer at distance r measures the temperature of an evaporating QBH finds

$$\beta = \frac{1}{\Delta(r)} \log \frac{1}{|\tau|^2} \tag{D.18}$$

By fixing

$$\Delta(r) = \frac{1}{\left(1 + \frac{2M}{r}\right)^{\frac{1}{2}}} \tag{D.19}$$

we get that:

- Eq. (D.17) is identical to Eq. (4.6), which express the contraction of time around a SBH.
- Eq. (D.18) is identical to Eq. 4.30, which express Hawking's temperature seen by an observer at finite distance from a SBH once that

  $$|\tau|^2 = e^{-8\pi M} \tag{D.20}$$

  is fixed.

Therefore, regarding QBH as clocks for external observers gives both the correct contraction of time formula predicted by GR and the Hawking temperature. Despite these advantages, this construction presents a problem: the parameter representing the time label ϕ cannot be observed by outside observers, as they can only interact with the o_i partitions and gather information about the modulus of τ.
Whether the observability of the time label is or not a requirement for a good clock, we still do not know and some further developments in this direction can be intriguing.

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