Department of Physics "E. Fermi"
University of Pisa

Entanglement of quantum clocks via gravity

Page and Wootters formalism

Dario Cafasso
Link to GitHub
cafasso.dario@gmail.com

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Contents

Introduction to Quantum Gravity

When we talk about **Quantum Gravity**, rather than a theory, we refer to a **family of problems** about the relation between our fundamental theories:

Quantum Mechanics and General Relativity

These problems lead physicists to pursue different approaches toward the "complete" picture, but all these paths originated from the same question:

If spacetime is a dynamical field and dynamical fields are quantized, then should we expect a kind of quantum spacetime?[1]
or rather there's something wrong in our foundations?

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> *"One is struck [by the fact] that the theory [of special relativity]... introduces two kinds of physical things, i.e., measuring rods and clocks, all other things, e.g., the electromagnetic field, the material point, etc. This, in a certain sense, is inconsistent..."*

and so, as pointed out by C. Rovelli[3]:

> *"The search for a quantum theory of gravity raises once more old questions such as: What is space? What is time? What is the meaning of “moving”? Is motion to be defined with respect to objects or with respect to space? And also: What is causality? What is the role of the observer in physics?"*

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Our work is based on a different way of conceiving time, not more as a classical parameter, but defining it from an operative point of view as the observable measured by the clock of an observer.

As pointed out in ref. [Castro-Ruiz et al., 2017]:

> *"For the sake of consistency, it is natural to assume that the clocks, being physical, behave according to the principles of our most fundamental physical theories."*

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Timeless Quantum Mechanics

PaW framework
Quantum Time

Here we follow the approach introduced by Page and Wootters[4], and then discussed by Giovannetti, Lloyd and Maccone[5] to define the concept of "Time Observable" via correlations.

We can describe a clock as a quantum system C, such that

$$|t\rangle_C \in \mathcal{H}_C \qquad \hat{T}\,|t\rangle_C = t\,|t\rangle_C \qquad \hat{H}_C = \hbar\hat{\Omega} \tag{1}$$

where $\hat{\Omega}$ is the conjugate operator to $\hat{T}$, i.e. $[\hat{T}, \hat{\Omega}] = i\mathbb{1}_C$. We also have

$${}_C\langle t'|t\rangle_C = \delta(t'-t) \quad and \quad {}_C\langle t'|\,\hat{\Omega}\,|t\rangle_C = \delta(t'-t)\partial_t \tag{2}$$

[4] Cfr. ref. [Page and Wootters, 1983, Wootters, 1984, Marletto and Vedral, 2016].
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In this framework, we assume that the time measured by **the clock** C is used by an observer to make measures on the system S.

It follows that C **has to be regarded as a part of the quantum system** $S + C$, subject to a global constraint $\hat{H}$ in the form of a **Wheeler-DeWitt Equation**, such that

$$|\Psi\rangle \in \mathcal{H} := \mathcal{H}_C \otimes \mathcal{H}_S \qquad \hat{H}\,|\Psi\rangle = 0 \ \textit{with} \ \hat{H} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{3}$$

This allows an observer-dependent and timeless description of QM!

Note that we are making several assumptions such as that C is a perfect clock, i.e. isomorphic to a particle on a line, and $\hat{H}_S$ is time-independent, i.e. not interacting.

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Indeed, we recover the **Schrödinger Equation** from equation 3 conditioning on time via projection on the generalized eigenvectors of the $\hat{T}$ operator. Using equations 2, we find:

$$_C \langle t | \hat{H} | \Psi \rangle = 0 \iff \partial_t | \psi(t) \rangle_S = \hat{H}_S | \psi(t) \rangle_S \tag{4}$$

where $|\psi(t)\rangle_S := {}_C\langle t | \Psi \rangle$ is the **Schrödinger State** we know from QM.[6]

A general solution of eq. 3 is called **history state**[7] and can be written as

$$|\Psi\rangle = \int dt \, |t\rangle_C \otimes |\psi(t)\rangle_S \tag{5}$$

It incorporates all of the information of the quantum system.

[6] Cfr. ref. [Giovannetti et al., 2015]. Some definitions may vary because of the introduction of a normalization factor for the time generalized eigenvectors (they're improper states).

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Consequences of the framework

At this point:

- How to describe measurements and probabilities?
- Which criticisms have been argued?
- How to interpret the emerging picture?

In ref. [Giovannetti et al., 2015], the **correct expression** for the n-times **propagator** has been recovered. For the derivation, they start adopting the **von Neumann formulation of the measurement apparatus** and then they cast the outcome in the time dependent PaW framework.

It is a "purification" of the procedure, **introducing a memory system** M for each measure. This allows us to **write the probability** of n-outcomes in n-times **accordingly to the Bayes rule** for conditional probabilities.

As an example, the conditional probability of getting b at time t given a at time $t > t_0$ reads

$$P(b,t|a,t_0) = \frac{P(b,a,t)}{P(a,t_0)} = \frac{\left\|\left({}_C\langle t| \otimes {}_{M_b}\langle b| \otimes {}_{M_a}\langle a|\right)|\Psi\rangle\right\|^2}{\left\|\left({}_C\langle t_0| \otimes {}_{M_a}\langle a|\right)|\Psi\rangle\right\|^2} \quad (6)$$

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Here we describe the **von Neumann formulation** of the measurement apparatus.[8]

We consider the system S as the sum of the **subsystem of interest** Q and a **memory system** M. Then, we describe the **process of measuring** as an **instantaneous transformation** which induces the unitary mapping

$$|\psi(t_I)\rangle_Q \otimes |ready\rangle_M \longmapsto \sum_a \hat{K}_a |\psi(t_I)\rangle_Q \otimes |a\rangle_M \tag{7}$$

in which $\{\hat{K}_a\}$ are called **Kraus operators** and fulfill the normalization condition $\sum_a \hat{K}_a^\dagger \hat{K}_a = \mathbb{1}$. The transformation can be expressed as an **interaction term** between Q and M in the Hamiltonian of S as

$$\hat{H}_S(t) = \hat{H}_Q + \hat{f}(t) \qquad \hat{f}(t) = \delta(t - t_I)\hat{h}_{QM} \tag{8}$$

[8] Cfr. ref. [Giovannetti et al., 2015], section "Measurements", for more details.
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Let's define the unitary operator $\hat{V}_{QM} := e^{-\frac{i}{\hbar}\hat{h}_{QM}}$. [9] It is responsible of the mapping in equation 7 and its action can be expressed as

$$\hat{V}_{QM} |\psi(t)\rangle_Q \otimes |r\rangle_M = \sum_a \langle a|_M \hat{V}_{QM} |r\rangle_M |\psi(t)\rangle_Q \otimes |a\rangle_M = \sum_a \hat{K}_a |\psi(t)\rangle_Q \otimes |a\rangle_M = \sum_a \sqrt{P(a,t)}\, |\phi_a\rangle_Q \otimes |a\rangle_M \tag{9}$$

in which, according to equation 6, $|\phi_a\rangle_Q$ is the state of Q after the collapse

$$|\phi_a\rangle_Q := \frac{1}{\sqrt{P(a,t)}} \hat{K}_a |\psi(t)\rangle_Q \qquad P(a,t) := \left\| \hat{K}_a |\psi(t)\rangle_Q \right\|^2$$

[9] Its relation with the experimental apparatus may be far away simple.

The evolution of the system is then described by the unitary operator

$$\hat{U}_S(t, t_0) = \begin{cases} \hat{U}_Q(t, t_0) & \forall t < t_I \\ \hat{U}_Q(t, t_I)\hat{V}_{QM}\hat{U}_Q(t_I, t_0) & \forall t > t_I \end{cases} \tag{10}$$

Putting together equations 7, 9 and 10, the history state in 5 becomes

$$|\Psi\rangle = \int_{-\infty}^{t_I} \mathrm{d}t \, |t\rangle_C \otimes |\psi(t)\rangle_Q \otimes |r\rangle_M + \int_{t_I}^{+\infty} \mathrm{d}t \, |t\rangle_C \otimes \hat{U}_Q(t, t_I) \sum_a \hat{K}_a \, |\psi(t_I)\rangle_Q \otimes |a\rangle_M \tag{11}$$

Problems:

- Instantaneous transformations are unrealistic and a generalization to **short-time interaction** is necessary for a realistic description.

  Introducing an interaction like in ref. [Ozawa, 1997] could be a solution.

- The **clock model** we are using **is unrealistic** and has to be intended as an approximation of a n-level system.

  Indeed, a lower bounded Hamiltonian leads to closed time-like curves and for a realistic description one may look at different clock models, like in ref. [Wootters, 1984, Moreva et al., 2013].

- The introduction of an **interaction term** leads to the **"clock ambiguity"**.

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Timeless QM
Overcoming criticism

Insights:

In ref. [Marletto and Vedral, 2016], section "**The appearance of the flow of time**", new insights about the interpretation of **history** and **memory** arise.
As pointed out in ref. [Wootters, 1984, Giovannetti et al., 2015] we can define two points of view:

- the **external observer**, who sees the whole universe as a static system whose state is an eigenstate of its global Hamiltonian, i.e. equation 3;
- the **internal observer**, who can anyway observe a dynamics, i.e. time dependence, and Born-rule induced wave function collapses.

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- the **internal observer**, who can anyway observe a dynamics, i.e. time dependence, and Born-rule induced wave-function collapses.

This is a kind of generalization of the **relativity principle**, which lead us to the introduction of the concept of **Time Reference Frame**.

What is a "Time Reference Frame"?

Time Reference Frames
Definition and basic concepts

According to ref. [Castro-Ruiz et al., 2020],

- We define a "Time Reference Frame" as a quantum temporal reference frame associated to a quantum clock.
- Temporal localization of events, i.e. quantum measurements on the system, is defined by the time reference frame of an observer.
- In the presence of more than one clock, each of them describes the evolution of the rest system according to its "proper time".

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Time Reference Frames
Introducing the second clock

Suppose to have a system made up of two clocks $I = A, B$ and a subsystem of interest S with a trivial Hamiltonian $\hat{H}_S = 0$. No interaction is considered between A, B, S.

Accordingly to equations 3 and 8, we introduce in S an experimental apparatus with two memory systems, each associated to one clock

$$\hat{H} = \hat{H}_A \otimes \mathbb{1}_{\bar{A}} + \hat{H}_B \otimes \mathbb{1}_{\bar{B}} + \hat{f}_A(\hat{T}_A) + \hat{f}_B(\hat{T}_B) \tag{12}$$

in which, defining R_I as the rest of $I + M_I + Q$, we have

$$\hat{f}_I(\hat{T}_I) = |t_I\rangle\langle t_I| \otimes \hat{h}_{QM_I} \otimes \mathbb{1}_{\bar{R}_I}$$

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Expressing equation 5 in terms of $I = A, B$ reference frames, we have

$$|\Psi\rangle = \int \mathrm{d}t_I \, |t_I\rangle_I \otimes |\psi_I(t_I)\rangle_{\bar{I}} = \int \mathrm{d}t_I \, |t_I\rangle_I \otimes \hat{U}(t_I) \, |\psi_I(0)\rangle_{\bar{I}}$$

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in which

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We can express them in terms of each others as

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Time Reference Frames
Change of Time Reference Frame

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$$|\psi_A(t_A)\rangle_{\bar{A}} = \hat{U}_{\bar{A}}(t_A)\hat{S}_{AB}\hat{U}^{\dagger}_{\bar{B}}(t_B) \, |\psi_B(t_B)\rangle_{\bar{B}}$$

in which

$$\hat{S}_{AB} = \int \mathrm{d}t_B \, |t_B\rangle_B \otimes \langle t_A = 0|_A \, \hat{U}_{\bar{B}}(t_B) e^{i\hat{T}_A \hat{H}_S}$$

We can express them in terms of each others as

$$|\Psi\rangle = \int \mathrm{d}t_A \, |t_A\rangle_A \otimes e^{-it_A \hat{H}_B} \mathcal{T}\{e^{-i\int_0^{t_A} \mathrm{d}s(\hat{f}_A(s) + \hat{f}_B(s + \hat{T}_B))}\} \, |\psi_A(0)\rangle_{\bar{A}}$$

Interacting quantum clocks

Interacting quantum clocks

Ref. PNAS [Castro-Ruiz et al., 2017].

Conclusion

Conclusion

What we achieved:

- something

Conclusion

What we achieved:

- something

What we learned:

- something else

Conclusion

What we achieved:

- something

What we learned:

- something else

- **In the end!**

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Thank you for your attention!