Appendix A: Analysis of the coordinate t

In this section we give an operational definition of the time coordinate t introduced in the main text. It is defined as the proper time measured by an observer who is sufficiently far away from the clocks, such that the different states of his/her local clock, corresponding to different energy configurations of the clocks under study, are almost overlapping. In other words, the coordinate distance from the observer to the clocks is such that the different metrics originating from different energy eigenstates of the clocks are operationally indistinguishable at his/her location. The analysis given here considers only two clocks, but its generalisation to an arbitrary number of clocks is straightforward.

To give an estimate of such distance, consider that the observer measures time with a spin-j coherent state clock, labeled by C, situated at a (finite) coordinate distance r from the clocks under study, labeled by A and B. We assume that $r \gg x$, where x is the coordinate distance between A and B, and compare the state of the observer's clock in the two extreme situations, i.e. when both A and B are in the ground state and when both are in the excited state. In the first case, the state of C at time t is $|\psi_{00}\rangle = \left|\vartheta = \frac{\pi}{2}, \varphi = -\frac{t\varepsilon}{\hbar}\right\rangle$, where t is the time measured by a clock that is far away enough so that the gravitational field is effectively zero at its location, and ε is the energy gap of the clock C. (For simplicity we put $E_0 = 0$, and therefore $E_1 = E_1 - E_0 = \Delta E$, as in the main text.) In the second case, where both A and B are in an excited state, the state of C is $|\psi_{11}\rangle = \left|\vartheta = \frac{\pi}{2}, \varphi = -\frac{t\varepsilon}{\hbar}(1 - \frac{2G\Delta E}{c^4 r})\right\rangle$. The overlap between these two states is

$$|\langle\psi_{00}|\psi_{11}\rangle|^2 = \frac{1}{4^j}\left(1 + \cos\frac{2G\Delta E\varepsilon t}{\hbar c^4 r}\right)^{2j} \approx 1 - 2j\left(\frac{2G\Delta E\varepsilon t}{\hbar c^4 r}\right)^2. \tag{A1}$$

Consider now a finite measurement accuracy (i.e. a finite capability of distinguishing two quantum states) given by δ, defined in such a way that that if $|\langle\psi_{00}|\psi_{11}\rangle|^2 > 1-\delta$ then both states are effectively undistinguishable. Given this accuracy, indistinguishability is achieved for a distance r satisfying

$$r > \frac{2\sqrt{2j}G\Delta E\varepsilon t}{\hbar c^4\sqrt{\delta}}. \tag{A2}$$

For these distances we can use the label t for the time coordinate in both cases. This procedure gives an operational meaning to coordinate time.

Appendix B: Heuristic derivation of the two-clock Hamiltonian from superposition principle and mass-energy equivalence

In the following we discuss how the Hamiltonian for the evolution of an internal degree of freedom in a fixed static background [15] can be generalised to the case where the background is not fixed but, rather, is set by a quantum superposition of energies. Our basic assumptions are that the quantum mechanical principle of superposition and the laws of general relativity are valid. If this were not the case, we would be forced to conclude that at least one of the theories breaks down in this regime and new physics should emerge. We proceed iteratively by considering first the two-clock case and then extending to the three-clock case. The generalisation to higher number of clocks is then straightforward.

Consider two quantum clocks, labeled by A and B, to be in the general state

$$|\psi_{in}\rangle = (\alpha\,|0\rangle_A + \beta\,|1\rangle_A)\,|\psi\rangle_B\,. \tag{B1}$$

From Ref. [15] we know that for the amplitude α, $|\psi\rangle_B$ evolves in the background produced by the state $|0\rangle_A$. Similarly, for the amplitude β, $|\psi\rangle_B$ evolves in the background produced by the state $|1\rangle_A$. Focusing on $|\psi\rangle_B$, without loss of generality, the evolution is given by

$$|\psi(t)\rangle_B = e^{-\frac{\mathrm{i}t}{\hbar}\dot{\tau}\hat{H}_B}\,|\psi\rangle_B, \tag{B2}$$

where $\hat{H}_B$ is the Hamiltonian of the internal degree of freedom of particle B and $\dot{\tau}$ is the derivative of the proper time τ with respect to the coordinate time t. The operational meaning of t is discussed in Appendix (A) above. In the lowest-order approximation to the solution for the metric, we can write [12,15]

$$\dot{\tau} \approx 1 + \frac{\Phi(x)}{c^2}, \tag{B3}$$

where $\Phi(x) = 0$ for the state $|0\rangle_A$ and $\Phi(x) = -G\Delta E/(c^2 x)$ for the state $|1\rangle$.

In this way, $|\psi\rangle_B$ evolves as

$$|\psi\rangle_B \longrightarrow e^{-\frac{it}{\hbar}\hat{H}_B} |\psi\rangle_B \quad \text{for} \quad |0\rangle_A, \tag{B4}$$

$$|\psi\rangle_B \longrightarrow e^{-\frac{it}{\hbar}\hat{H}_B(1-\frac{G\Delta E}{c^4 x})} |\psi\rangle_B \quad \text{for} \quad |1\rangle_A. \tag{B5}$$

The phases in the previous equations already include the gravitational interaction between particles A and B, as well as the free evolution of B. Therefore, the evolution of $|0\rangle_A$ ($|1\rangle_A$) is merely given by the phase corresponding to E_0 (E_1), that is, $|0\rangle_A \longrightarrow |0\rangle_A$ and $|1\rangle_A \longrightarrow e^{-\frac{it}{\hbar}\Delta E} |1\rangle_A$.

Now, applying the superposition principle, we write the solution for the evolved state as a linear combination of the solutions for each energy:

$$|\psi_{in}\rangle \longrightarrow \alpha |0\rangle_A e^{-\frac{it}{\hbar}\hat{H}_B} |\psi\rangle_B + \beta e^{-\frac{it}{\hbar}\Delta E} |1\rangle_A e^{-\frac{it}{\hbar}\hat{H}_B(1-\frac{G\Delta E}{c^4 x})} |\psi\rangle_B = |\psi_{fin}\rangle. \tag{B6}$$

This evolution can be expressed in terms of the Hamiltonian (4), that is,

$$|\psi_{fin}\rangle = e^{-\frac{it}{\hbar}(\hat{H}_A+\hat{H}_B-\frac{G}{c^4 x}\hat{H}_A\hat{H}_B)} |\psi_{in}\rangle. \tag{B7}$$

We now extend the result of the two-clock case to the case where we have three clocks, labeled by A, B and C. We assume that the coordinate distance between each pair of particles is x, according to the observer far away. The idea is to single out one of the particles, say A, as the one whose energy eigenstates set the metric background, and use the result of the two particle case for the remaining particles. We write the initial state as

$$|\psi_{in}\rangle = (\alpha |0\rangle_A + \beta |1\rangle_A) |\psi\rangle_{BC}. \tag{B8}$$

Following the steps for the two particle case, we write

$$|\psi_{in}\rangle \longrightarrow \alpha |0\rangle_A e^{-\frac{it}{\hbar}H_{BC}} |\psi\rangle_{BC} + \beta e^{-\frac{it}{\hbar}\Delta E} |1\rangle_A e^{-\frac{it}{\hbar}H_{BC}(1-\frac{G\Delta E}{c^4 x})} |\psi\rangle_{BC} = |\psi_{fin}\rangle, \tag{B9}$$

where $H_{BC} = \hat{H}_B + \hat{H}_C - \frac{G}{c^4 x}\hat{H}_B\hat{H}_C$ is the joint Hamiltonian for the particles B and C, derived from the analysis of two particles. Explicitly, the phase corresponding to the state $|1\rangle_A$ is

$$\frac{t}{\hbar}H_{BC}(1-\frac{G\Delta E}{c^4 x}) = \frac{t}{\hbar}(\hat{H}_B + \hat{H}_C - \frac{G}{c^4 x}(\hat{H}_B\hat{H}_C + \Delta E\hat{H}_B + \Delta E\hat{H}_C) + \frac{G^2}{c^8 x^2}\Delta E\hat{H}_B\hat{H}_C). \tag{B10}$$

Note that the last term in the last equation is higher order in c^{-2} and therefore is neglected at the first level of approximation. Therefore we have

$$|\psi_{fin}\rangle = \alpha |0\rangle_A e^{-\frac{it}{\hbar}\left(\hat{H}_B+\hat{H}_C-\frac{G}{c^4 x}\hat{H}_B\hat{H}_C\right)} |\psi\rangle_{BC} + \beta e^{-\frac{it}{\hbar}\Delta E} |1\rangle_A e^{-\frac{it}{\hbar}\left(\hat{H}_B+\hat{H}_C-\frac{G\Delta E}{c^4 x}(\Delta E\hat{H}_B+\Delta E\hat{H}_C+\hat{H}_B\hat{H}_C)\right)} |\psi\rangle_{BC} \tag{B11}$$

$$= e^{-\frac{it}{\hbar}H_{ABC}} |\psi_{in}\rangle, \tag{B12}$$

where

$$H_{ABC} = \hat{H}_A + \hat{H}_B + \hat{H}_C - \frac{G}{c^4 x}(\hat{H}_A\hat{H}_B + \hat{H}_A\hat{H}_C + \hat{H}_B\hat{H}_C). \tag{B13}$$

It is clear now that the generalisation for N particles is given by the Hamiltonian

$$H = \sum_a \hat{H}_A - \frac{G}{c^4}\sum_{a<b}\frac{\hat{H}_A\hat{H}_B}{|x_a - x_b|}, \tag{B14}$$

which reduces to Eq. (6) under the approximation $|x_a - x_b| \approx x$ for all a, b.

Appendix C: Two clock Hamiltonian from Quantum Field Theory approach

The Hamiltonian (4) can also be obtained from a quantum field theory in the weak-field limit by restricting to the two particle subspace, as done in [16], and using the mass-energy equivalence. In the following we sketch a derivation of this fact using natural units ($c = 1$ and $\hbar = 1$). For a detailed presentation, the reader may consult [16]. We implement the mass energy equivalence, in the sense of the main text, from the beginning of our calculation by considering our composite particle to emerge from the interaction of two scalar fields φ_1 and φ_2. The Lagrangian density of the field coupled to gravity reads

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\left(\sum_A g^{\mu\nu}\left(\partial_\mu\varphi_A\right)\left(\partial_\nu\varphi_A\right) + \sum_{AB} M_{AB}^2\varphi_A\varphi_B\right), \tag{C1}$$

where g denotes the determinant of the metric $g_{\mu\nu}$ and M_{AB} is a symmetric matrix that couples the fields φ_1 and φ_2 (c.f. [S1]). This interaction gives rise to the mass of the composite particle. We denote the eigenvalues of M_{AB} by m and $m + \Delta E$. It is important to stress that, as noted in the main text, there is fundamentally no difference between mass and interaction energy in a relativistic theory. The distinction between static mass and dynamical mass, i.e. the mass that arises from the interaction of internal degrees of freedom, is only an effective one, depending on the energy scale with which the system is probed. In this sense, the matrix M_{AB} can be interpreted as a sum of static mass (m) contribution and internal energy or dynamical mass (with eigenvalue ΔE) contribution. It provides an effective description of a composite particle with different energy levels. A full treatment of the dynamics of composite particles in quantum field theory is a research area on its own (see for example [S2]) and is beyond the scope of our paper.

We write the Lagrangian density in the form (C1) to make explicit the fact that we will take superpositions of different energy eigenstates of the internal Hamiltonian, in the sense of the main text. Since M_{AB} is symmetric, there exists an orthogonal matrix C_{AB} that diagonalises it. We assume a metric field in the weak field limit $\mathrm{d}s^2 = -(1+2\Phi(\mathbf{x}))\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$, and calculate, via a Legendre transformation, the Hamiltonian in this approximation

$$H = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\,(1+\Phi(\mathbf{x}))\left(\sum_A\left(\pi_A^2 + (\nabla\varphi_A)^2\right) + \sum_{AB} M_{AB}^2\varphi_A\varphi_B\right). \tag{C2}$$

Here $\pi_A = \dot{\varphi_A}$ denotes the canonical conjugate momentum to φ_A. The gravitational potential Φ satisfies the equation $\nabla^2\Phi = -4\pi\rho$, where $\rho = \sum_A\left(\pi_A^2 + (\nabla\varphi_A)^2\right) + \sum_{AB} M_{AB}^2\varphi_A\varphi_B$ is the energy density of the matter field.

In order to quantise the field, we first write the Hamiltonian (C2) in the basis in which M_{AB} is diagonal, $H = \frac{1}{2}\sum_A\int \mathrm{d}^3\mathbf{x}\,(1+\Phi(\mathbf{x}))\left(p_A^2 + (\nabla\Psi_A)^2 + \mu_A^2\Psi_A^2\right)$, where $\Psi_A = \sum_B C_{AB}\varphi_B$, C_{AB} is the matrix that diagonalises M_{AB}, and p_A is the momentum conjugate to Ψ_A. The matrix M_{AB} has eigenvalues μ_A, for $A = 1,\,2$.

We then Fourier-expand Ψ and p_A

$$\Psi_A(\mathbf{x}) = \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\sqrt{2\omega_{A,\mathbf{k}}}}\left(\mathrm{e}^{-\mathrm{i}k_Ax}b_{A,\mathbf{k}} + \mathrm{e}^{\mathrm{i}k_Ax}b_{A,\mathbf{k}}^\dagger\right) \tag{C3a}$$

$$p_A(\mathbf{x}) = \mathrm{i}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sqrt{\frac{\omega_{A,\mathbf{k}}}{2}}\left(\mathrm{e}^{-\mathrm{i}k_Ax}b_{A,\mathbf{k}} + \mathrm{e}^{\mathrm{i}k_Ax}b_{A,\mathbf{k}}^\dagger\right), \tag{C3b}$$

where $k_Ax = -\omega_{A,\mathbf{k}}t + \mathbf{k}\cdot\mathbf{x}$, and $\omega_{A,\mathbf{k}}^2 = \mathbf{k}^2 + \mu_A^2$, and impose the commutation relations $\left[b_{A,\mathbf{k}}, b_{\mathbf{A'k'}}^\dagger\right] = \delta_{A,A'}\delta^3(\mathbf{k}-\mathbf{k'})$ Next we insert the expressions (C3) in the Hamiltonian and then take the slow velocity approximation of the fields

$$\Psi_A(\mathbf{x}) \approx \frac{1}{\sqrt{2\mu_a}}\left(\chi_A(\mathbf{x}) + \chi_A^\dagger(\mathbf{x})\right) \tag{C4a}$$

$$p_A(\mathbf{x}) \approx \mathrm{i}\sqrt{\frac{\mu_a}{2}}\left(\chi_A(\mathbf{x}) - \chi_A^\dagger(\mathbf{x})\right), \tag{C4b}$$

where $\chi_A(\mathbf{x}) = (2\pi)^{-3}\int\mathrm{d}^3\mathbf{k}\mathrm{e}^{\mathrm{i}(\mu_At - \mathbf{k}\cdot\mathbf{x})}b_{A,\mathbf{k}}$. Then we solve the Poisson equation for the potential

$$\Phi(\mathbf{x}) = -G\int\mathrm{d}^3\mathbf{x}'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{C5}$$

insert the expression (C5) into (C2), and keep terms within the slow velocity approximation. The (normal-ordered) Hamiltonian then reads

$$H = \sum_{AB}\int\mathrm{d}^3\mathbf{x}\left(M_{AB}\phi_A^\dagger\phi_B - \frac{1}{2}M_{AB}^{-1}\phi_A^\dagger\nabla^2\phi_B\right) - G\sum_{ABCD}\int\mathrm{d}^3\mathbf{x}d^3\mathbf{x}'\frac{(M_{AB}\phi_A^\dagger\phi_B)(M_{CD}\phi_C^\dagger\phi_D)}{|\mathbf{x}-\mathbf{x}'|}, \tag{C6}$$

where $\phi_A = \sum_B C_{AB}^{-1}\chi_B$. The expression $\rho(\mathbf{x}) = \sum_{AB} M_{AB}\phi_A^\dagger(\mathbf{x})\phi_B(\mathbf{x})$ has the interpretation of a mass-energy density of the field. However, since it involves the product of two field operators at the same point, it is not a well-defined operator and leads to divergencies. This problem is handled by a suitable regularisation procedure and leads to a renormalisation of the mass [16]. The regularised mass density is given by $\rho_{reg}(\mathbf{x}) = \sum_{AB} \int \mathrm{d}^3\mathbf{x}' f_\delta(\mathbf{x} - \mathbf{x}')M_{AB}\phi_A^\dagger(\mathbf{x}')\phi_B(\mathbf{x}')$, where f_δ is a normalised, positive function dependent on a regularisation parameter δ in such a way that $f_\delta(\mathbf{x}) \longrightarrow \delta^3(\mathbf{x})$ as $\delta \longrightarrow 0$.

In order to obtain the restriction of (C6) to the two-particle sector of the Fock space, we compute the matrix element $\langle \xi^{(1)}, \eta^{(1)} \,|\, H \,|\xi^{(2)}, \eta^{(2)}\rangle$, where $|\xi^{(i)}, \eta^{(i)}\rangle = 2^{-1/2} \sum_{AB} \int \mathrm{d}^3\mathbf{x} d^3\mathbf{x}' \xi_A(\mathbf{x})\eta_B(\mathbf{x}')\phi_A^\dagger(\mathbf{x})\phi_B^\dagger(\mathbf{x}') \,|0\rangle$ is a two-particle state for $i = 1, 2$ (here $|0\rangle$ denotes the vacuum state of the field). From this matrix element we can then read off the form of the two particle Hamiltonian, which we can write as

$$\hat{H} = \hat{M}_{ren} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{M}_{ren} + \frac{1}{2}\hat{M}^{-1}\hat{\mathbf{p}}^2 \otimes \mathbb{1} + \frac{1}{2}\hat{M}^{-1}\mathbb{1} \otimes \hat{\mathbf{p}}^2 - G\frac{\hat{M} \otimes \hat{M}}{|\hat{\mathbf{x}} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{\mathbf{x}}|}, \tag{C7}$$

where $\langle \xi \,|\, \hat{M} \,|\eta\rangle = \sum_{AB} \int \mathrm{d}^3\mathbf{x}\bar{\xi}_A(\mathbf{x})M_{AB}\eta_B(\mathbf{x})$, and $\hat{M}_{ren} = \hat{M} - (\pi\delta^2)^{-1/2}G\hat{M}^2$ is the renormalised mass matrix. The Hamiltonian (C7) is equivalent to the Hamiltonian (4) together with the kinetic part, which we have ignored in the main text. The Hamiltonian for an arbitrary number of particles can be obtained from this approach by projecting in the corresponding subspace. In the same way as for the matrix M_{AB}, the operator $\hat{M}$ can be interpreted as a sum of static mass and internal energy.

Appendix D: Derivation of the Master Equation

In this section we derive the master equation for the evolution of a single coherent state clock, labeled by B, interacting gravitationally with another coherent state clock, labeled by A. We follow closely the derivation presented in Ref. [15]. The Hamiltonian of the system is given by Eq. (11), which we write here as

$$\hat{H} = \hat{H}_A + \hat{H}_B + \lambda\hat{H}_A\hat{H}_B \tag{D1}$$

$$= \hat{H}_0 + \hat{H}_{int}, \tag{D2}$$

where $\hat{H}_0 = \hat{H}_A + \hat{H}_B$ and $\hat{H}_{int} = \lambda\hat{H}_A\hat{H}_B$. In our case $\lambda = -G/(c^4x)$, but the derivation is completely general. We assume that the state at $t = 0$ is uncorrelated: $\rho(0) = \rho_A(0) \otimes \rho_B(0)$. The evolution for the full state is given by

$$\mathrm{i}\hbar\dot{\rho}(t) = [\hat{H}, \rho(t)]. \tag{D3}$$

For a general operator $\hat{A}$, we define

$$\tilde{A} = e^{\frac{\mathrm{i}t}{\hbar}(\hat{H}_0 + \hat{h})}\, \hat{A}\, e^{-\frac{\mathrm{i}t}{\hbar}(\hat{H}_0 + \hat{h})}, \tag{D4}$$

where $\hat{h} = \lambda\bar{E}_A\hat{H}_B$, and $\bar{E}_A = \mathrm{Tr}(\rho_A\hat{H}_A)$. Now we apply this transformation to the total density operator of the system. In terms of $\tilde{\rho}$, the equation of motion is

$$\mathrm{i}\hbar\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = [\tilde{H}_{int} - \tilde{h}, \tilde{\rho}]. \tag{D5}$$

The implicit solution to this equation is

$$\tilde{\rho}(t) = \tilde{\rho}(0) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}s[\tilde{H}_{int} - \tilde{h}, \tilde{\rho}(s)]. \tag{D6}$$

Substituting (D6) into (D5) yields

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\tilde{H}_{int} - \tilde{h}, \tilde{\rho}(0)] - \frac{1}{\hbar^2}\int_0^t \mathrm{d}s[\tilde{H}_{int} - \tilde{h}, [\tilde{H}_{int} - \tilde{h}, \tilde{\rho}(s)]]. \tag{D7}$$

We now approximate the last equation to the second order in $\tilde{H}_{int}$ and replace $\tilde{\rho}(s)$ by $\rho_A(0) \otimes \tilde{\rho}_B(s)$. With this approximation we trace over the A clock and obtain an equation for $\tilde{\rho}_B$:

$$\frac{\mathrm{d}\tilde{\rho}_B}{\mathrm{d}t} \approx -\frac{\mathrm{i}}{\hbar}\mathrm{Tr}_A\left([\tilde{H}_{int} - \tilde{h}, \rho_A(0) \otimes \tilde{\rho}_B(0)]\right) - \frac{1}{\hbar^2}\int_0^t \mathrm{d}s\mathrm{Tr}_A\left([\tilde{H}_{int} - \tilde{h}, [\tilde{H}_{int} - \tilde{h}, \rho_A(0) \otimes \tilde{\rho}_B(s)]]\right) \tag{D8}$$

$$= -\left(\frac{\lambda\Delta E_A}{\hbar}\right)^2 \int_0^t \mathrm{d}s[\hat{H}_B, [\hat{H}_B, \tilde{\rho}(s)]], \tag{D9}$$

where we have defined $\Delta E_A = \sqrt{\mathrm{Tr}\left(\rho_A(\hat{H}_A - \bar{E}_A \mathbb{1}_A)^2\right)}$ and have taken into account the fact that $\hat{H}_B = \tilde{H}_B$. Changing back to the original density operator ρ_B and writing the value of λ explicitly, we find

$$\frac{\mathrm{d}\rho_B}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\left[\hat{H}_B\left(1 + \frac{G j_A \Delta E}{c^4 x}\right), \rho_B\right] - \left(\sqrt{\frac{j_A}{2}}\frac{G\Delta E}{2c^4 x}\right)^2 \int_0^t \mathrm{d}s \left[\hat{H}_B, \left[\hat{H}_B, \rho_B\right]_s\right], \tag{D10}$$

where

$$\left[\hat{H}_B, \rho_B\right]_s = e^{\frac{-\mathrm{i}s}{\hbar}\hat{H}_B}[\hat{H}_B, \rho_B] e^{\frac{\mathrm{i}s}{\hbar}\hat{H}_B}, \tag{D11}$$

which coincides with Eq. (14).

[S1] Peskin, M.E. Schroeder, D.V. An Introduction to Quantum Field Theory (Westview Press, 1955), p 351.
[S2] Stumpf, H. Borne, T. Composite Particle Dynamics in Quantum Field Theory, (Springer, 1994).