Superselection Rules

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Abstract

This note provides a summary of the meaning of the term 'Superselection Rule' in Quantum Mechanics and Quantum-Field Theory. It is a slightly extended version of a contribution to the *Compendium of Quantum Physics: Concepts, Experiments, History and Philosophy*, edited by Friedel Weinert, Klaus Hentschel, Daniel Greenberger, and Brigitte Falkenburg.

General Notion

The notion of **superselection rule** (henceforth abbreviated SSR) was introduced in 1952 by Wick (1909-1992), Wightman, and Wigner (1902-1995) [13] in connection with the problem of consistently assigning intrinsic parity to elementary particles. They understood an SSR as generally expressing "restrictions on the nature and scope of possible measurements".

The concept of SSR should be contrasted with that of an ordinary **selection rule** (SR). The latter refers to a dynamical inhibition of some transition, usually due to the existence of a conserved quantity. Well known SRs in Quantum Mechanics concern radiative transitions of atoms. For example, in case of electric dipole radiation they take the form $\Delta J = 0, \pm 1$ (except $J = 0 \to J = 0$) and $\Delta M_J = 0, \pm 1$. It says that the quantum numbers J, M_J associated with the atom's total angular momentum may at most change by one unit. But this is only true for electric dipole transitions, which, if allowed, represent the leading-order contribution in an approximation for wavelengths much larger than the size of the atom. The next-to-leading-order contributions are given by magnetic dipole and electric quadrupole transitions, and for the latter $\Delta J = \pm 2$ *is* possible. This is a typical situation as regards SRs: They are valid for the leading-order modes of transition, but not necessarily for higher order ones. In contrast, a SSR is usually thought of as making a more rigorous statement. It not only forbids certain transitions through particular modes, but altogether as a matter of some deeper lying principle; hence

the "Super". In other words, transitions are not only inhibited for the particular dynamical evolution at hand, generated by the given Hamiltonian operator, but for all conceivable dynamical evolutions.

More precisely, two states ψ_1 and ψ_2 are separated by a SR if $\langle\psi_1 \mid H \mid \psi\rangle = 0$ for the given Hamiltonian H. In case of the SR mentioned above, H only contains the leading-order interaction between the radiation field and the atom, which is the electric dipole interaction. In contrast, the states are said to be separated by a SSR if $\langle\psi_1 \mid A \mid \psi_2\rangle = 0$ for *all* (physically realisable) observables A. This means that the relative phase between ψ_1 and ψ_2 is not measurable and that coherent superpositions of ψ_1 and ψ_2 cannot be verified or prepared. It should be noted that such a statement implies that the set of (physically realisable) observables is strictly smaller than the set of all self-adjoint operators on Hilbert space. For example, $A = \mid\psi_1\rangle\langle\psi_2 \mid + \mid\psi_2\rangle\langle\psi_1 \mid$ is clearly self-adjoint and satisfies $\langle\psi_1 \mid A \mid \psi_2\rangle \neq 0$. Hence the statement of a SSR always implies a restriction of the set of observables as compared to the set of all (bounded) self-adjoint operators on Hilbert space. In some sense, the existence of SSRs can be formulated in terms of observables alone (see below).

Since all theories work with idealisations, the issue may be raised as to whether the distinction between SR and SSR is really well founded, or whether it could, after all, be understood as a matter of degree only. For example, dynamical decoherence is known to provide a very efficient mechanism for generating apparent SSRs, without assuming their existence on a fundamental level [15][10].

Elementary Theory

In the most simple case of only two **superselection sectors**, a SSR can be characterised by saying that the Hilbert space $\mathcal{H}$ decomposes as a direct sum of two orthogonal subspaces, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, such that under the action of each observable vectors in $\mathcal{H}_{1,2}$ are transformed into vectors in $\mathcal{H}_{1,2}$ respectively. In other words, the action of observables in Hilbert space is reducible, which implies that $\langle\psi_1 \mid A \mid \psi_2\rangle = 0$ for each $\psi_{1,2} \in \mathcal{H}_{1,2}$ and all observables A. This constitutes an inhibition to the superposition principle in the following sense: Let $\psi_{1,2}$ be normed vectors and $\psi_+ = (\psi_1 + \psi_2)/\sqrt{2}$, then

$$\langle\psi_+ \mid A \mid \psi_+\rangle = \tfrac{1}{2}\big(\langle\psi_1 \mid A \mid \psi_1\rangle + \langle\psi_2 \mid A \mid \psi_2\rangle\big) = \mathrm{Tr}(\rho A)\,, \qquad (1)$$

where

$$\rho = \tfrac{1}{2}\big(\mid\psi_1\rangle\langle\psi_1 \mid + \mid\psi_2\rangle\langle\psi_2 \mid\big)\,. \qquad (2)$$

Hence, considered as state (expectation-value functional) on the given set of observables, the density matrix ρ corresponding to ψ_+ can be written as non-trivial convex combination of the (pure) density matrices for ψ_1 and ψ_2 and therefore defines a mixed state rather than a pure state. Relative to the given observables, coherent superpositions of states in $\mathcal{H}_1$ with states in $\mathcal{H}_2$ do not exist.

In direct generalisation, a characterisation of **discrete SSRs** can be given as follows: There exists a finite or countably infinite family $\{P_i \mid i \in I\}$ of mutually orthogonal ($P_iP_j = 0$ for $i \neq j$) and exhaustive ($\sum_{i\in I} P_i = 1$) projection operators ($P_i^\dagger = P_i$, $P_i^2 = P_i$) on Hilbert space $\mathcal{H}$, such that each observable commutes with all P_i. Equivalently, one may also say that states on the given set of observables (here represented by density matrices) commute with all P_i, which is equivalent to the identity

$$\rho = \sum_i P_i \rho P_i \,. \tag{3}$$

We define $\lambda_i := \mathrm{Tr}(\rho P_i)$ and let $I' \subset I$ be the subset of indices i for which $\lambda_i \neq 0$. If we further set $\rho_i := P_i\rho P_i/\lambda_i$ for $i \in I'$, then (3) is equivalent to

$$\rho = \sum_{i\in I'} \lambda_i \rho_i \,, \tag{4}$$

showing that ρ is a non-trivial convex combination if I' contains more than one element. The only pure states are the projectors onto rays within a single $\mathcal{H}_i$. In other words, only vectors (or rays) in the union (not the linear span) $\bigcup_{i\in I} \mathcal{H}_i$ can correspond to pure states. If, conversely, *any* non-zero vector in this union defines a pure state, with different rays corresponding to different states, one speaks of an **abelian superselection rule**. The $\mathcal{H}_i$ are then called **superselection sectors** or **coherent subspaces** on which the observables act irreducibly. The subset Z of observables commuting with all observables is then given by $Z := \left\{\sum_i a_i P_i \mid a_i \in \mathbb{R}\right\}$. They are called **superselection-** or **classical observables**.

In the general case of **continuous SSRs** $\mathcal{H}$ splits as direct integral of an uncountable set of Hilbert spaces $\mathcal{H}(\lambda)$, where λ is an element of some measure space Λ, so that

$$\mathcal{H} = \int_\Lambda \mathrm{d}\mu(\lambda)\mathcal{H}(\lambda) \,, \tag{5}$$

with some measure $\mathrm{d}\mu$ on Λ. Observables are functions $\lambda \mapsto O(\lambda)$, with $O(\lambda)$ acting on $\mathcal{H}(\lambda)$. Closed subspaces of $\mathcal{H}$ left invariant by the observables are precisely given by

$$\mathcal{H}(\Delta) = \int_\Delta \mathrm{d}\mu(\lambda)\mathcal{H}(\lambda) \,, \tag{6}$$

where $\Delta \subset \Lambda$ is any measurable subset of non-zero measure. In general, a single $\mathcal{H}(\lambda)$ will not be a subspace (unless the measure has discrete support at λ).

In the literature, SSRs are discussed in connection with a variety of superselection-observables, most notably univalence, overall mass (in non-relativistic QM), electric charge, baryonic and leptonic charge, and also time.

Algebraic Theory

In Algebraic Quantum Mechanics, a system is characterised by a C^*–algebra $\mathfrak{C}$. Depending on contextual physical conditions, one chooses a faithful representation

$\pi : \mathfrak{C} \to \mathcal{B}(\mathcal{H})$ in the (von Neumann) algebra of bounded operators on Hilbert space $\mathcal{H}$. After completing the image of π in the weak operator-topology on $\mathcal{B}(\mathcal{H})$ (a procedure sometimes called **dressing** of $\mathfrak{C}$ [5]) one obtains a von Neumann sub-algebra $\mathfrak{N} \subset \mathcal{B}(\mathcal{H})$, called the **algebra of (bounded) observables**. The physical observables proper correspond to the self-adjoint elements of $\mathfrak{N}$.

The **commutant** $\mathcal{S}'$ of any subset $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is defined by

$$\mathcal{S}' := \{A \in \mathcal{B}(\mathcal{H}) \mid AB = BA\,, \forall B \in \mathcal{S}\}, \tag{7}$$

which is automatically a von Neumann algebra. One calls $\mathcal{S}'' := (\mathcal{S}')'$ the von Neumann algebra generated by $\mathcal{S}$. It is the smallest von Neumann sub-algebra of $\mathcal{B}(\mathcal{H})$ containing $\mathcal{S}$, so that if $\mathcal{S}$ was already a von Neumann algebra one has $\mathcal{S}'' = \mathcal{S}$; in particular, $\big(\pi(\mathfrak{C})\big)'' = \mathfrak{N}$

SSRs are now said to exists iff[1] the commutant $\mathfrak{N}'$ is not trivial, that is, iff $\mathfrak{N}'$ is different from multiples of the unit operator. Projectors in $\mathfrak{N}'$ then define the sectors. **Abelian SSRs** are characterised by $\mathfrak{N}'$ being abelian. The significance of this will be explained below. $\mathfrak{N}'$ is often referred to as **gauge algebra**. Sometimes the algebra of physical observables is *defined* as the commutant of a given gauge algebra. That the gauge algebra is abelian is equivalent to $\mathfrak{N}' \subseteq \mathfrak{N}'' = \mathfrak{N}$ so that $\mathfrak{N}' = \mathfrak{N} \cap \mathfrak{N}' =: \mathfrak{N}^c$, the centre of $\mathfrak{N}$. An abelian $\mathfrak{N}'$ is equivalent to **Dirac's requirement**, that there should exist a complete set of commuting observables [9] (cf. Chap. 6 of [10]).

In finite-dimensional Hilbert spaces Dirac's requirement is equivalent to the hypothesis that there be sufficiently many pairwise commuting self-adjoint elements of $\mathfrak{N}$ so that the simultaneous eigenspaces are one-dimensional. In other words, each array of eigenvalues ('quantum numbers'), one for each self-adjoint element, uniquely determines a pure quantum state (a ray in $\mathcal{H}$). This implies the existence of a self-adjoint $A \in \mathfrak{N}$ with a simple spectrum (pairwise distinct eigenvalues). It then follows that any other self-adjoint $B \in \mathfrak{N}$ commuting with A must then be a polynomial function (of degree $n-1$ if $n = \dim(\mathcal{H})$) of A and that there exists a vector $\psi \in \mathcal{H}$ so that any other $\phi \in \mathcal{H}$ is obtained by applying a polynomial (of degree $n-1$) in A to ψ. The vector ψ is called a cyclic vector for $\mathfrak{N}$ and may be chosen to be any vector with non-vanishing components in each simultaneous eigenspace for the complete set of commuting observables; see Chap. 6 of [10] for details.

The algebraic theory allows to translate these statements to the general situation. Here, the existence of a 'complete' set of commuting observables is interpreted as existence of a 'maximal' abelian subalgebra $\mathfrak{A} \subset \mathfrak{N}$. Here it is crucial that 'maximal' is properly understood, namely as 'maximal in $\mathcal{B}(\mathcal{H})$' and not just maximal in $\mathfrak{N}$, which would be a rather trivial requirement (given Zorn's lemma, a maximal abelian subalgebra in $\mathfrak{N}$ always exists). Now, it is easy to see that $\mathfrak{A}$ is maximal abelian (in $\mathcal{B}(\mathcal{H})$) iff it is equal to its commutant (in $\mathcal{B}(\mathcal{H})$):

$$\mathfrak{A} \text{ max. abelian } \Leftrightarrow\ \mathfrak{A} = \mathfrak{A}'\,. \tag{8}$$

[1] Throughout we use 'iff' as abbreviation for 'if and only if'.

This is true since $\mathfrak{A} \subseteq \mathfrak{A}'$ certainly holds due to $\mathfrak{A}$ being abelian. On the other hand, $\mathfrak{A} \supseteq \mathfrak{A}'$ also holds since it just expresses the maximality requirement that $\mathfrak{A}$ already contains all elements of $\mathcal{B}(\mathcal{H})$ commuting with each element of $\mathfrak{A}$.

Moreover, it can be shown that the existence of a maximal abelian subalgebra $\mathfrak{A}$ in $\mathfrak{N}$ is equivalent to $\mathfrak{N}'$ being abelian:

$$\mathfrak{A} \text{ max. abelian } \Leftrightarrow\ \mathfrak{N}' \subseteq \mathfrak{N}'' = \mathfrak{N}\,. \tag{9}$$

The proof of this important statement is easy enough to be reproduced here: Suppose first that $\mathfrak{A} = \mathfrak{A}'$, then $\mathfrak{N} \supseteq \mathfrak{A} = \mathfrak{A}' \supseteq \mathfrak{N}'$ and hence $\mathfrak{N}' \subseteq \mathfrak{N} = \mathfrak{N}''$, implying that $\mathfrak{N}'$ is abelian. Conversely, suppose $\mathfrak{N}'$ is abelian:

$$\mathfrak{N}' \subseteq \mathfrak{N} \qquad (\mathfrak{N}' \text{ is abelian})\,. \tag{10}$$

Choose an abelian subalgebra $\mathfrak{A} \subseteq \mathfrak{N}$ which is maximal in $\mathfrak{N}$:

$$\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{N} \qquad (\mathfrak{A} \text{ max. abelian in } \mathfrak{N})\,. \tag{11}$$

As already noted above, Zorn's lemma guarantees the existence of $\mathfrak{A}$ satisfying (11). We show that $\mathfrak{A}$, albeit only required to be maximal in $\mathfrak{N}$, is in fact maximal in $\mathcal{B}(\mathcal{H})$ due to $\mathfrak{N}'$ being abelian. Indeed, since $\mathfrak{A} \subseteq \mathfrak{N}$ trivially implies $\mathfrak{N}' \subseteq \mathfrak{A}'$, we have

$$\mathfrak{N}' \overset{(10)}{=} \mathfrak{N} \cap \mathfrak{N}' \subseteq \mathfrak{N} \cap \mathfrak{A}' \overset{(11)}{=} \mathfrak{A}\,. \tag{12}$$

Since $\mathfrak{N}' \subseteq \mathfrak{A}$ trivially implies $\mathfrak{A}' \subseteq \mathfrak{N}$, equation (11) immediately leads to $\mathfrak{A} = \mathfrak{A}'$. This shows that Dirac's requirement is equivalent to the hypothesis of abelian SSRs.

Another requirement equivalent to Dirac's is that there should exist a **cyclic vector** $\psi \in \mathcal{H}$ for $\mathfrak{N}$. This means that the smallest closed subspace of $\mathcal{H}$ containing $\mathfrak{N}\psi := \{A\psi \mid A \in \mathfrak{N}\}$ is $\mathcal{H}$ itself.

In Quantum Logic a quantum system is characterised by the lattice of propositions (corresponding to the closed subspaces, or the associated projectors, in Hilbert-space language). The subset of all propositions which are compatible with all other propositions is called the **centre of the lattice**. It forms a Boolean sublattice. A lattice is called **irreducible** iff its centre is trivial (i.e. just consists of 0, the smallest lattice element). The presence of SSRs is now characterised by a non-trivial centre. Propositions in the centre are sometimes called **classical**.

SSRs and Conserved Additive Quantities

Let Q be the operator of some charge-like quantity that behaves additively under composition of systems and also shares the property that the charge of one subsystem is independent of the state of the complementary subsystem (here we restrict attention to two subsystems). This implies that if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}$ is the Hilbert space of the total system and $\mathcal{H}_{1,2}$ those of the subsystems, Q must be of the form

$Q = Q_1 \otimes 1 + 1 \otimes Q_2$, where $Q_{1,2}$ are the charge operators of the subsystems. We also assume Q to be conserved, i.e. to commute with the total Hamiltonian that generates time evolution on $\mathcal{H}$. It is then easy to show that a SSR for Q persists under the operations of composition, decomposition, and time evolution: If the density matrices $\rho_{1,2}$ commute with $Q_{1,2}$ respectively, then, trivially, $\rho = \rho_1 \otimes \rho_2$ commutes with Q. Likewise, if ρ (not necessarily of the form $\rho_1 \otimes \rho_2$) commutes with Q, then the reduced density matrices $\rho_{1,2} := \mathrm{Tr}_{2,1}(\rho)$ (where Tr_i stands for tracing over $\mathcal{H}_i$) commute with $Q_{1,2}$ respectively. This shows that if states violating the SSR cannot be prepared initially (for whatever reason, not yet explained), they cannot be created though subsystem interactions [14]. This has a direct relevance for measurement theory, since it is well known that an exact von Neumann measurement of an observable P_1 in system 1 by system 2 is possible only if P_1 commutes with Q_1, and that an approximate measurement is possible only insofar as system 2 can be prepared in a superposition of Q_2 eigenstates [2].

Let us see how to prove the second to last statement for the case of discrete spectra. Let S be the system to be measured, A the measuring apparatus and $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$ the Hilbert space of the system plus apparatus. The charge-like quantity is represented by the operator $Q = Q_S \otimes 1 + 1 \otimes Q_A$, the observable of S by $P \in \mathcal{B}(\mathcal{H}_S)$. Let $\{|s_n\rangle\} \subset \mathcal{H}_S$ be a set of normalised eigenstates for P so that $P|s_n\rangle = p_n|s_n\rangle$. Let $U \in \mathcal{B}(\mathcal{H})$ be the unitary evolution operator for the von Neumann measurement and $\{|a_n\rangle\} \subset \mathcal{H}_A$ a set of normalised 'pointer states' with neutral pointer-position a_0, so that

$$U\big(|s_n\rangle|a_0\rangle\big) = |s_n\rangle|a_n\rangle \tag{13}$$

We assume the total Q to be conserved during the measurement, i.e. $[U, Q] = 0$. Clearly $\langle a_n \mid a_m\rangle \neq 1$ if $n \neq m$, for, otherwise, this process is not a measurement at all, since $\langle a_n \mid a_m\rangle = 1$ iff $|a_n\rangle = |a_n\rangle$. Let now $n \neq m$, then the following lines prove the claim:

$$\begin{aligned}
(p_n - p_m)\langle s_n|Q_S|s_m\rangle &= (p_n - p_m)\langle s_n|\langle a_0|\, Q\, |s_m\rangle|a_0\rangle \\
&= (p_n - p_m)\langle s_n|\langle a_0|\, U^\dagger Q U\, |s_m\rangle|a_0\rangle \\
&= (p_n - p_m)\langle s_n|\langle a_n|\, Q_S \otimes 1 + 1 \otimes Q_A\, |s_m\rangle|a_m\rangle \\
&= \langle a_n|a_m\rangle\, (p_n - p_m)\langle s_n|Q_S|s_m\rangle\,.
\end{aligned} \tag{14}$$

The first and fourth equality follow from $\langle s_n|s_m\rangle = 0$, the second from $[U, Q] = 0$ and unitarity of U, and the third from (13). Equality of the left-hand side with the last expression on the right-hand side, taking into account $\langle a_n|a_m\rangle \neq 1$, is possible iff $p_n \neq p_m$ implies $\langle s_n|Q_S|s_m\rangle = 0$, which means that Q_s is reduced by (i.e. acts within) each eigenspace of P, which in turn implies that Q_s commutes with P, as was to be shown.

As already indicated, the reasoning above does not explain the actual existence of SSRs in the presence of conserved additive quantities, for it does not imply anything about the *initial* nonexistence of SSR violating states. In fact, there are many

additive conserved quantities, like momentum and angular momentum, for which certainly no SSRs is at work. The crucial observation here is that the latter quantities are physically always understood as *relative* to a system of reference that, by its very definition, must have certain localisation properties which exclude the total system to be in eigenstate of *relative* (linear and angular) momenta. Similarly it was argued that one may have superpositions of relatively charged states [1]. A more complete account of this conceptually important point, including a comprehensive list of references, is given in Chap. 6 of [10].

SSRs and Symmetries

Symmetries in Quantum Mechanics are often implemented via **unitary ray-representations** rather than proper unitary representations (here we discard anti-unitary ray-representations for simplicity). A unitary ray-representation is a map U from the symmetry group G into the group of unitary operators on Hilbert space $\mathcal{H}$ such that the usual condition of homomorphy, $U(g_1)U(g_2) = U(g_1g_2)$, is generalised to

$$U(g_1)U(g_2) = \omega(g_1, g_2)\, U(g_1g_2)\,, \tag{15}$$

where $\omega : G \times G \to U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$ is the so-called **multiplier** that satisfies

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_1, g_2g_3)\omega(g_2, g_3)\,, \tag{16}$$

for all g_1, g_2, g_3 in G, so as to ensures associativity: $U(g_1)\big(U(g_2)U(g_3)\big) = \big(U(g_1)U(g_2)\big)U(g_3)$. Any function $\alpha : G \to U(1)$ allows to redefine $U \mapsto U'$ via $U'(g) := \alpha(g)U(g)$, which amounts to a redefinition $\omega \mapsto \omega'$ of multipliers given by

$$\omega'(g_1, g_2) = \tfrac{\alpha(g_1)\alpha(g_2)}{\alpha(g_1\, g_2)}\, \omega(g_1, g_2)\,. \tag{17}$$

Two multipliers ω and ω' are called **similar** iff (17) holds for some function α. A multiplier is called **trivial** iff it is similar to $\omega \equiv 1$, in which case the ray-representation is, in fact, a proper representation in disguise.

The following result is now easy to show: Given unitary ray-representations $U_{1,2}$ of G on $\mathcal{H}_{1,2}$, respectively, with non-similar multipliers $\omega_{1,2}$, then no ray-representation of G on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ exists which restricts to $U_{1,2}$ on $\mathcal{H}_{1,2}$ respectively. From this a SSR follows from the requirement that the Hilbert space of pure states should carry a ray-representation of G, since such a space cannot contain invariant linear subspaces that carry ray-representations with non-similar multipliers.

An example is given by the SSR of univalence, that is, between states of integer and half-integer spin. Here G is the group $SO(3)$ of proper spatial rotations. For integer spin it is represented by proper unitary representations, for half integer spin with non-trivial multipliers. Another often quoted example is the Galilei group, which is implemented in non-relativistic quantum mechanics by non-trivial unitary ray-representations whose multipliers depend on the total mass of the system and are not similar for different masses.

Such derivations have sometimes been criticised (e.g. in [12]) for depending crucially on ones prejudice of what the symmetry group G should be. The relevant observation here is the following: Any ray-representation of G can be made into a proper representation of a larger group $\bar{G}$, which is a *central extension* of G. But no superselection rules follow if $\bar{G}$ rather than G were required to be the acting symmetry group on the set of pure states. For example, in case of the rotation group, $G = SO(3)$, it is sufficient to take $\bar{G} = SU(2)$, its double (and universal) cover. For G the 10-parameter inhomogeneous Galilei group it is sufficient to take for $\bar{G}$ an extension by the additive group $\mathbb{R}$, which may even be motivated on classical grounds [7].

SSRs in Local Quantum Field Theories

In Quantum Field Theory SSRs can arise from the restriction to (quasi) local observables. Charges which can be measured by fluxes through closed surfaces at arbitrarily large spatial distances must then commute with all observables. A typical example is given by the total electric charge, which is given by the integral over space of the local charge density ρ. According to Maxwell's equations, the latter equals the divergence of the electric field $\vec{E}$, so that Gauß' theorem allows to write

$$Q = \lim_{R \mapsto \infty} \int_{\|\vec{x}\|=R} (\vec{n} \cdot \vec{E}) d\sigma \,, \tag{18}$$

where $\vec{n}$ is the normal to the sphere $\|\vec{x}\| = R$ and $d\sigma$ its surface measure. If A is a local observable its support is in the *causal complement* of the spheres $\|\vec{x}\| = R$ for sufficiently large R. Hence, in the quantum theory, A commutes with Q. It is possible, though technically far from trivial, that this formal reasoning can indeed be justified in Local Quantum Field Theory [11]. For example, one difficulty is that Gauß' law does not hold as an operator identity.

In modern Local Quantum-Field Theory [8], representations of the quasi-local algebra of observables are constructed through the choice of a preferred state on that algebra (GNS-construction), like the Poincaré invariant vacuum state, giving rise to the **vacuum sector**. The superselection structure is restricted by putting certain selection conditions on such states, like e.g. the Doplicher-Haag-Roberts (DHR) selection criterion for theories with mass gap (there are various generalisations [8]), according to which any representation should be unitarily equivalent to the vacuum representation when restricted to observables whose support lies in the causal complement of a sufficiently large (causally complete) bounded region in spacetime. Interestingly this can be closely related to the existence of gauge groups whose equivalence classes of irreducible unitary representations faithfully label the superselection sectors. Recently, a systematic study of SSRs in 'Locally Covariant Quantum Field Theory' was started in [6]. Finally we mention that SSRs may also arise as a consequence of non-trivial spacetime topology [3].

Environmentally Induced SSRs

The ubiquitous mechanism of decoherence effectively restricts the *local* verification of coherences [10]. For example, scattering of light on a particle undergoing a two-slit experiment *delocalises* the relative-phase information for the two beams along with the escaping light. Hence effective SSRs emerge locally in a practically irreversible manner, albeit the correlations are actually never destroyed but merely delocalised. The emergence of effective SSRs through the dynamical process of decoherence has also been called **einselection** [15]. For example, this idea has been applied to the problem of why certain molecules naturally occur in eigenstates of chirality rather than energy and parity, i.e. why sectors of different chirality seem to be superselected so that chirality becomes a classical observable. This is just a special case of the general question of how classical behaviour can emerge in Quantum Theory. It may be asked whether *all* SSRs are eventually of this dynamically emergent nature, or whether strictly fundamental SSRs persist on a kinematical level [10]. The complementary situation in theoretic modelling may be characterised as follows: Derivations of SSRs from axiomatic formalisms lead to exact results on models of only approximate validity, whereas the dynamical approach leads to approximate results on more realistic models.

SSRs in Quantum Information

In the theory of Quantum Information a somewhat softer variant of SSRs is defined to be a restriction on the allowed local operations (completely positive and trace-preserving maps on density matrices) on a system [4]. In general, it therefore leads to constraints on (bipartite) entanglement. Here the restrictions considered are usually not thought of as being of any fundamental nature, but rather for mere practical reasons. For example, without an external reference system for the definition of an overall spatial orientation, only 'rotationally covariant' operations $\mathcal{O} : \rho \mapsto \mathcal{O}(\rho)$ are allowed, which means that $\mathcal{O}$ must satisfy

$$\mathcal{O}\big[U(g)\rho U^{\dagger}(g)\big] = U(g)\mathcal{O}(\rho)U^{\dagger}(g) \quad \forall g \in SO(3)\,, \tag{19}$$

where U is the unitary representation of the group $SO(3)$ of spatial rotations in Hilbert space. Insofar as the local situation is concerned, this may be rephrased in terms of the original setting of SSRs, e.g. by regarding $SO(3)$ as gauge group, restricting local observables and states to those commuting with $SO(3)$. On the other hand, one also wishes to consider situations in which, for example, a local bipartite system (Alice and Bob) is given a state that has been prepared by a third party that is *not* subject to the SSR.

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