Non-inertial quantum clock frames lead to non-Hermitian dynamics

Ismael L. Paiva,[1,*] Amit Te'eni,[1] Bar Peled,[1] Eliahu Cohen,[1] and Yakir Aharonov[2,3,4]

[1] *Faculty of Engineering and the Institute of Nanotechnology and Advanced Materials, Bar-Ilan University, Ramat Gan 5290002, Israel*

[2] *Schmid College of Science and Technology, Chapman University, Orange, California 92866, USA*

[3] *Institute for Quantum Studies, Chapman University, Orange, California 92866, USA*

[4] *School of Physics and Astronomy, Tel Aviv University, Tel Aviv 6997801, Israel*

Recently, there have been many attempts to extend the notion of proper time to quantum mechanics with the use of quantum clocks. Using a similar idea combined with the relativistic mass-energy equivalence, we consider an accelerating massive quantum particle with an internal clock system. We show that the ensuing evolution from the perspective of the particle's internal clock is non-Hermitian. This result does not rely on specific implementations of the clock. As a particular consequence, we prove that the effective Hamiltonian of two gravitationally interacting particles is non-Hermitian from the perspective of the clock of either particle.

Time is an intriguing physical concept that can be connected to most — if not all — fundamental issues in physics. A good example of that concerns how a refined understanding of time is associated with the revolution brought by the introduction of relativistic theories. In fact, Lorentz transformations, which were independently introduced by Voigt [1] and Lorentz [2] and named as such by Poincaré [3], were already known for some time before Einstein's introduction of special relativity [4]. However, they had never been taken to their full mechanical consequences prior to Einstein's work. He did so by considering that clocks — and rods for that matter — are physical objects and, hence, subject to physical laws.

In quantum mechanics, the issue of time was discussed since the early days of the theory, and understandably so: while measurements are essential element of it, the theory does not readily allow the description of measurements of time since it contains time as a parameter. One could, then, wonder about the possibility of constructing a time operator. However, arguably, much progress in this regard stagnated due to Pauli's well-known objection to such an operator [5], which is based on the argument that the Hamiltonian, canonically conjugate to a time operator, would have to be unbounded from below. Although discussions about time still continued to exist in the literature, to the best of our knowledge, a time operator returned to the scene with a discussion by Aharonov and Bohm that involved the idea of *Heisenberg's cut* in a special measurement scheme [6]. Later, Garrison and Wong introduced a time operator in such a way that overcomes Pauli's objection [7].

Remarkably, an isolated quantum system is constrained by the Wheeler-DeWitt equation, i.e., it does not evolve in time [8]. For such systems, Page and Wootters introduced a formalism in which a non-interacting subsystem works as a reference for time (i.e., a clock) for the remaining parts [9]. With this scheme, which can be studied in the general context of *quantum reference frames* [10–14], they showed that the usual unitary evolution given by the Schrödinger equation can be recovered. Their formalism has attracted much attention, especially during the last few years [15–33]. In particular, if an arbitrary interaction term between the clock and the remaining parts is considered, a generalized Schrödinger equation is obtained [19].

Nevertheless, the resulting evolution was found to be unitary in a vast quantity of scenarios investigated in the literature, even when gravitationally interacting clocks were considered [18, 19, 24]. Generally, such clocks present a particular type of time dilation and can also undergo decoherence, losing their ability to behave as good clocks [18]. However, it is still argued that in circumstances for which they still work as clocks, the evolution of the rest of the systems from their perspective is unitary [24].

That said, non-unitary evolution does manifest itself in energy measurements of clock systems [31]. More precisely, the dynamics of the rest of the system from the perspective of a clock undergoing a von Neumann measurement of energy is non-unitary even when the final "collapse" (or update) of the state of the system is not taken into account.

In this Letter, we state the fundamental (and perhaps inevitable) aspect of non-unitarity arising from the Page and Wooters framework. Specifically, we show that the resulting evolution from the perspective of the proper time (i.e., internal clock) of an accelerating massive particle is non-unitary. We also analyze gravitationally interacting clocks from this new perspective, explaining how to reconcile our results with previous ones, even though they may seem to be at odds. Important in our approach is a post-Newtonian correction to the mass: the mass-energy equivalence. Such a correction has previously led to other worth-noting results [18, 19, 25, 34–36].

Time evolution given by quantum clocks—In the study of quantum systems with quantum clocks, a physical system, which we will denote by system A, represents a quantum clock. If H_A is the system's Hamiltonian, time states are built as

$$|t_A + t'_A\rangle \equiv e^{-iH_A t'_A/\hbar}|t_A\rangle, \tag{1}$$

which implies that

$$\frac{\partial}{\partial t_A}|t_A\rangle = -\frac{i}{\hbar}H_A|t_A\rangle. \tag{2}$$

From these states, a time operator T_A is constructed. If different time states are orthogonal to each other, i.e., $\langle t_A|t'_A\rangle = \delta(t_A - t'_A)$ for every t_A and t'_A, T_A is Hermitian and, moreover, it is canonically conjugate to H_A, i.e., $[T_A, H_A] = i\hbar I$. However, the resulting time states are not always orthogonal to each other [7, 37–40]. These represent more realistic clocks, with the lack of orthogonality reflecting the absence of infinite resolution of the clock. In this work, clocks are not assumed to be ideal.

Besides clock A, let the other relevant systems be represented by the index M. Also, assume that the joint system is isolated, i.e., it is in a state $|\Psi\rangle\rangle \in \mathcal{H}_A \otimes \mathcal{H}_M$ that obeys the Wheeler-DeWitt equation

$$H|\Psi\rangle\rangle = 0, \tag{3}$$

where H is the Hamiltonian of the joint system. Here, the double-ket notation is used to identify the entire system, which does not evolve with respect to an external time. This means that not every vector in $\mathcal{H}_A \otimes \mathcal{H}_M$ represents a valid physical state. Instead, physical states are represented by vectors that satisfy Eq. (3).

If H_M denotes the Hamiltonian of the system of interest and H_{int}, an arbitrary interaction between A and M, we have

$$H = H_A + H_M + H_{int}. \tag{4}$$

Moreover, define $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$. As a result, $|\Psi\rangle\rangle$ can be written as

$$|\Psi\rangle\rangle = \int dt_A\, |t_A\rangle \otimes |\psi(t_A)\rangle. \tag{5}$$

Also, Eq. (3) implies that $\langle t_A|H|\Psi\rangle\rangle$ vanishes, which, in turn, can be written as [19]

$$i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle = H_M|\psi(t_A)\rangle + \int dt'_A\; K(t_A, t'_A)|\psi(t'_A)\rangle, \tag{6}$$

giving rise to a generalized Schrödinger equation. Here, $K(t_A, t'_A) \equiv \langle t_A|H_{int}|t'_A\rangle$.

In many instances, it is desirable to consider scenarios where $|\Psi\rangle\rangle$ is composed of various subsystems and, in particular, multiple clocks. In cases with multiple clocks, the dynamics can be studied from the perspective of any of them [24, 40, 41].

Accelerating clock frames—Now, we connect the acceleration of clocks to the observance of non-unitary evolution. First, a discussion of how acceleration affects the Hamiltonian of a free particle is needed. In fact, classically, this influence should be taken into account with the addition of a potential V. Restricting our study to potentials given by a function of the position x and recalling from Newtonian physics that $ma = -dV/dx$, where m is the mass of the system and a is its acceleration, we conclude that

$$V(x) = -m\int_{x_0}^{x} a(x')dx'. \tag{7}$$

In the above, it was assumed for simplicity that a can be parametrized by x and $V(x_0) = 0$. Now, a post-Newtonian correction can be added to V by using the mass-energy equivalence [42], i.e., $m \mapsto m + H/c^2$, where H is the Hamiltonian of the particle. For simplicity, we neglect the term with the static mass m and consider only H/c^2. Then, redefining f to absorb the constant $1/c^2$, we can write $V(x) = Hf(x)$.

Observe that the mass-energy equivalence has been also used in the study of quantum systems [18, 19, 25, 34–36]. It implies that the potential (associated with $f(X)$) also couples to internal degrees of freedom of the system [34]. Then, if a quantum clock is an internal degree of freedom of an accelerating quantum particle, the above discussion leads to the change

$$H_T \mapsto H_T + \frac{1}{2}\left[f(X_M)H_T + H_T f(X_M)\right], \tag{8}$$

where H_T is the full quantum Hamiltonian of the system composed by a particle M and its internal clock A before it accelerates and X_M is the system's position operator. It can be checked that this leads to a non-unitary evolution from the perspective of clock A. In fact, assuming that the only interaction between A and M is due to the coupling arising from the particle's acceleration, the total Hamiltonian of the system is

$$\begin{aligned} H =& H_A + H_M \\ &+ \frac{1}{2}\left[f(X_M)(H_A + H_M) + (H_A + H_M)f(X_M)\right]. \end{aligned} \tag{9}$$

Then, defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$ and using Eqs. (2) and (3), we obtain the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle = H^A_{eff}|\psi(t_A)\rangle, \tag{10}$$

where

$$H^A_{eff} \equiv H_M - \frac{1}{2}[I + f(X_M)]^{-1}[f(X_M), H_M] \tag{11}$$

is the effective Hamiltonian of system M with respect to clock A. Generally speaking, H^A_{eff} is non-Hermitian since $[I + f(X_M)]^{-1}$ does not always commute with $[f(X_M), H_M]$. A detailed derivation of the above expression can be seen in the Supplemental Material.

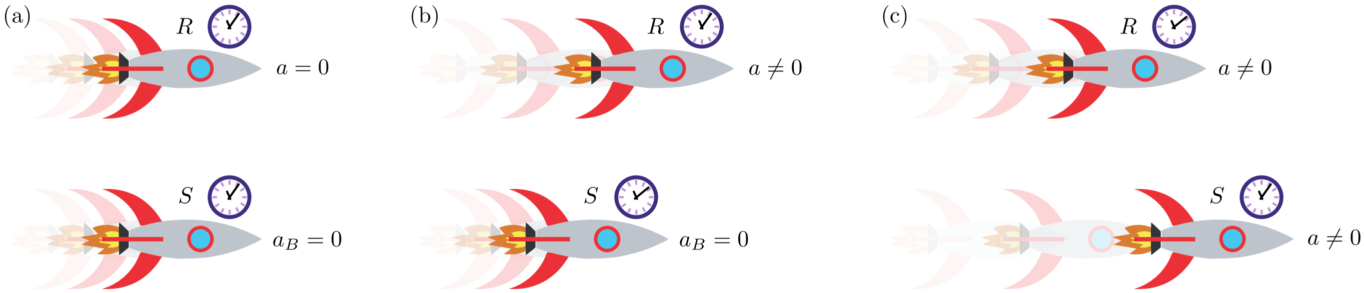

FIG. 1. Representation of non-interacting travelling rocket ships, each with its own internal quantum time. (a) If both rockets are at rest or moving with constant speed, the time evolution of systems described by either of them is unitary. (b) However, if rocket R starts accelerating, then the time evolution of systems from its clock's perspective is, generally speaking, non-unitary, while the evolution from the perspective of rocket S's clock remains unitary. (c) Finally, the evolution given by none of the clocks is unitary if both rockets are accelerating.

It is worth noting that, if an external non-interacting clock to M is included in the analysis, the only change to the total Hamiltonian of the joint system is the addition of the Hamiltonian of this clock since it does not get coupled to M. As a consequence, it follows by direct computation that the effective Hamiltonian from the perspective of the external clock is Hermitian. More precisely, it just corresponds to the Hamiltonian in Eq. (9).

To summarize and evidence the significance of the result just presented, we discuss the case of two rocket ships R and S, each with their own internal clock, as shown in Fig. 1. Initially, we assume that there is no interaction between the rockets nor between each rocket's external degree of freedom with its clock. Thus, if the two rockets are inertial, as represented in Fig. 1(a), the evolution from the perspective of either clock is unitary. Now, suppose rocket R starts accelerating while rocket S remains inertial, as illustrated in Fig. 1(b). Then, according to the result just presented, the evolution given by rocket R's clock should generally be non-unitary. Yet, the evolution from the perspective of rocket S's clock is unitary. However, if both rockets are accelerating, as in Fig. 1(c), the evolution given by either clock is, generally, non-unitary.

Important in the above reasoning is the assumption of non-interaction between degrees of freedoms associated with different rockets. The analysis would be modified if this was not the case, as will be shown next.

Gravitationally interacting clocks—We now focus our attention on accelerations due to gravitational interaction between massive systems, each with their own clock. This will allow us to reconsider previous results on gravitationally interacting clocks. In these studies, it was shown that gravitational interactions lead to a subtle form of time dilation, although the unitarity of the evolution persists [18, 19, 24, 35, 36].

In those analyses, it was used that the Newtonian gravitational potential is written as $V(r) = -Gm_A m_B/r$ together with the post-Newtonian mass-energy equivalence correction, as we have done to obtain our main result. Then, the gravitational interaction between two clocks A and B was added to the Hamiltonian as a term proportional to the product of their free Hamiltonians, i.e., $\lambda H_A H_B$, where $\lambda = -G/c^4 r$.

It can be noticed that in these previous works the distance between the clocks was assumed to remain constant. This justifies the fact that, despite the gravitational effects, both clocks were found to yield unitary evolution. In fact, both are inertial frames. Here, however, we allow the relative position of the clocks to change as a result of the gravitational interaction.

More precisely, we assume that clocks A and B are internal degrees of freedom of massive particles M and N, respectively. Then, the gravitational potential can be written as $V(x_N - x_M) = -Gm_M m_N/|x_N - x_M|$. For simplicity, if S is much more massive than R, we can assume that $x_N - x_M \approx x_N - x_0$, where x_0 is the initial position of system M. Letting the latter vanish, we have $V(x_N) = -Gm_M m_N/|x_N|$.

Now, using the mass-energy equivalence and defining $S \equiv A + M$ and $R \equiv B + N$, we write $V(X_N) = -G(X_N^{-1}H_R + H_R X_R^{-1})H_S/2c^4$, where X_N is assumed to only take positive values. For simplicity and to make sure any change to the ticking of either clock is due to the gravitational interaction between R and S, we assume $H_S = H_A + H_M$ and $H_R = H_B + H_N$. This means that the total Hamiltonian of the system composed by S and R is

$$H = [I + f(X_N, H_R)]H_S + H_R, \tag{12}$$

where $f(X_N, H_R) = -G(X_N^{-1}H_R + H_R X_R^{-1})/2c^4$. As a result, the dynamics from the perspective of clock A is given by an expression similar to Eq. (10) with

$$H^A_{eff} = H_M + [I + f(X_N, H_B + H_N)]^{-1}(H_B + H_N), \tag{13}$$

which is non-Hermitian since $f(X_N, H_B + H_N)$ does not commute with H_N. This is so in spite of system S being assumed to be approximately inertial. This might seem surprising in view of the example with the rocket

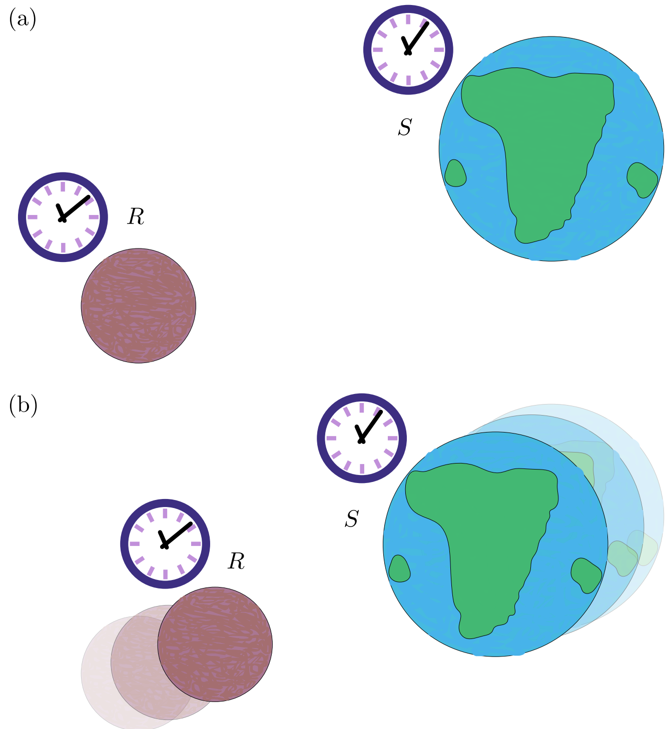

FIG. 2. Representation of gravitational interaction between two massive systems, each having its own internal quantum time. (a) If the relative spatial distance between the systems is kept constant, the time evolution from the perspective of either of their clocks is unitary, even though the gravitational interaction causes a type of time dilation in the clocks. (b) However, if the relative spatial distance between the systems changes due to the gravitational attraction, the description of their time evolution given by either of their clocks is generally non-unitary. This is the case even if one of the systems is assumed to be much more massive than the other and, therefore, can be approximated as inertial.

ships. However, a crucial aspect in this example is that the rockets had no interaction between them whatsoever. Here, system S interacts with the non-inertial system R. Moreover, the effective Hamiltonian from the perspective of clock B is also non-Hermitian, as expected. More precisely, it is

$$H_{eff}^{B} = H_N + \left[I - gX_N^{-1}H_S\right]^{-1}\left(I + \frac{g}{2}[X_N^{-1}, H_N]\right)H_S, \tag{14}$$

where $g \equiv G/c^4$. Details of the derivation of Eqs. (13) and (14) can be found in the Supplemental Material.

Discussion—We have studied how accelerations of massive particles lead to the emergence of non-unitary evolution from the perspective of quantum clocks internal to them. The non-unitarity comes as a consequence of the coupling between external and internal degrees of freedom of a system, which include a clock system (associated with the system's proper time). This is a general feature arising from the Page and Wootters formalism, not relying on specific implementations of the clocks or even on them being ideal.

By the equivalence principle, accelerating massive particles are equivalent to systems under gravitational forces. To evidence it, we have conducted an explicit analysis of gravitationally interacting systems. This allowed us to explain why non-unitary evolutions were not observed in previous theoretical treatments of the problem [18, 19, 24, 35, 36]. Moreover, the relation between our results and gravitational effects is particularly emblematic: since, ultimately, any system can be addressed as being inertial only up to a certain order, the results presented here suggest that unitarity can only be recovered as an approximation in an eventual quantum theory incorporating gravitation. In other words, the time evolution of a system should be, in general, non-unitary in those theories.

This result can be assimilated in two different ways. In one way, it is possible to conclude that a non-unitary evolution will indeed be a fundamental characteristic of relativistic quantum theories with an operational approach for time — and, in particular, to a yet to be constructed consistent theory of quantum gravity. In this case, it is necessary to develop an understanding of the physical meaning of such evolution. For instance, by the construction of the state $|\psi(t_A)\rangle$ according to Page and Wootters' recipe in their framework, the state $|\psi(t_A)\rangle$ in Eq. (10) is a vector (i.e., a pure state) for every t_A. The difference between unitary and non-unitary dynamics in this context is, then, that the norm of the vector changes in time within the latter. Knowing that, how can the usual operational meaning of quantum mechanics be recovered? More, since the Schrödinger and the Heisenberg pictures are unitarily equivalent, how can the Heisenberg picture be recovered in this case?

A possible answer to these questions may lie within a method to treat non-Hermitian Hamiltonians introduced by Dirac [43] and further studied by Pauli [44] and others [45–47]. The method consists of introducing a new metric to the Hilbert space of the system, which modifies its inner product. This new metric should be such that the new norm of the vector is kept constant throughout its evolution. This, however, comes at a price: the choice of a new metric is not unique [48] and, most disturbingly, it is not guaranteed *a priori* that there always exist a positive-definite metric. This means that the theory may have states with negative norms, known as "ghost states" since they do not have an operational meaning.

Then, one may question what are the physical consequences of the change of inner product. While there is much to be investigated in this regard, some hints may be found in the literature of (non-Hermitian) $\mathcal{PT}$-symmetric quantum mechanics [49–54]. For instance, it is known that $\mathcal{PT}$-symmetric systems can evolve faster than Hermitian ones [55, 56]. Therefore, quantum bounds that rely on inner products may, in general, be modified.

The other way to look at the results presented in this article consists of seeing them as a limitation of the Page and Wootters framework. Then, the framework should be modified/extended in such a way that unitarity is restored. This will likely require a reevaluation of the foundations of the framework.

In either case, this Letter reveals challenges for devising operational approaches of time in relativistic quantum theories. These challenges, in turn, constitute new research directions that may lead to a better understanding of time and relativistic structures in quantum mechanics.

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* ismaellpaiva@gmail.com

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SUPPLEMENTAL MATERIAL TO THE MANUSCRIPT "NON-INERTIAL QUANTUM CLOCK FRAMES LEAD TO NON-HERMITIAN DYNAMICS"

In this supplemental material, we provide details for the derivation of the Hamiltonians in Eqs. (11), (13), and (14).

First, we derive H_{eff}^{A} in Eq. (11). To start, observe that

$$\frac{1}{2}[f(X_M)H_M + H_M f(X_M)] = f(X_M)H_M - \frac{1}{2}[f(X_M), H_M]. \tag{15}$$

Then, defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$ and using Eqs. (2), (3), and (9), we obtain

$$\begin{aligned}
0 &= \langle t_A|H|\Psi\rangle\rangle \\
&= \langle t_A| \left\{ [I + f(X_M)]\, H_A + [I + f(X_M)]\, H_M - \frac{1}{2}[f(X_M), H_M] \right\} |\Psi\rangle\rangle \\
&= [I + f(X_M)]\, (\langle t_A|H_A)\, |\Psi\rangle\rangle + [I + f(X_M)]\, H_M \langle t_A|\Psi\rangle\rangle - \frac{1}{2}[f(X_M), H_M]\langle t_A|\Psi\rangle\rangle \\
&= [I + f(X_M)] \left(-i\hbar \frac{\partial}{\partial t_A} \langle t_A| \right) |\Psi\rangle\rangle + \left\{ [I + f(X_M)]\, H_M - \frac{1}{2}[f(X_M), H_M] \right\} |\psi(t_A)\rangle \\
&= -i\hbar\, [I + f(X_M)]\, \frac{\partial}{\partial t_A} |\psi(t_A)\rangle + \left\{ [I + f(X_M)]\, H_M - \frac{1}{2}[f(X_M), H_M] \right\} |\psi(t_A)\rangle.
\end{aligned} \tag{16}$$

Assuming that $I + f(X_M)$ is invertible and reorganizing the terms, we obtain Eq. (10) with H_{eff}^{A} given by Eq. (11).

Now, the same derivation process leads to the Hamiltonian in Eq. (13). Defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$ and using Eqs. (2), (3), and (12), we obtain

$$\begin{aligned}
0 &= \langle t_A|\, \{[I + f(X_N, H_R)](H_A + H_M) + H_R\}\, |\Psi\rangle\rangle \\
&= [I + f(X_N, H_R)]\, (\langle t_A|H_A)\, |\Psi\rangle\rangle + [I + f(X_N, H_R)]\, H_M \langle t_A|\Psi\rangle\rangle + H_R \langle t_A|\Psi\rangle\rangle \\
&= [I + f(X_N, H_R)] \left(-i\hbar \frac{\partial}{\partial t_A} \langle t_A| \right) |\Psi\rangle\rangle + \{[I + f(X_N, H_R)]\, H_M + H_R\}\, |\psi(t_A)\rangle \\
&= -i\hbar\, [I + f(X_N, H_R)]\, \frac{\partial}{\partial t_A} |\psi(t_A)\rangle + \{[I + f(X_N, H_R)]\, H_M + H_R\}\, |\psi(t_A)\rangle.
\end{aligned} \tag{17}$$

Assuming that $I + f(X_N, H_R)$ is invertible and reorganizing the terms, we obtain Eq. (10) with H_{eff}^{A} given by Eq. (13).

Finally, a similar reasoning leads to Eq. (14). To start, observe that the Hamiltonian in Eq. (12) can be rewritten as

$$H = \left(I - gX_N^{-1}H_S\right)(H_B + H_N) + \left(I + \frac{g}{2}[X_N^{-1}, H_N]\right) H_S, \tag{18}$$

where $g \equiv G/c^4$. Then, defining $|\phi(t_B)\rangle \equiv \langle t_B|\Psi\rangle\rangle$ and using Eq. (3) and the analog of Eq. (2) for clock B, we have

$$\begin{aligned}
0 &= \langle t_B| \left\{ \left(I - gX_N^{-1}H_S\right)(H_B + H_N) + \left(I + \frac{g}{2}[X_N^{-1}, H_N]\right) H_S \right\} |\Psi\rangle\rangle \\
&= \left(I - gX_N^{-1}H_S\right)(\langle t_B|H_B)\, |\Psi\rangle\rangle + \left(I - gX_N^{-1}H_S\right) H_N \langle t_B|\Psi\rangle\rangle + \left(I + \frac{g}{2}[X_N^{-1}, H_N]\right) H_S \langle t_B|\Psi\rangle\rangle \\
&= \left(I - gX_N^{-1}H_S\right) \left(-i\hbar \frac{\partial}{\partial t_B} \langle t_B| \right) |\Psi\rangle\rangle + \left\{ \left(I - gX_N^{-1}H_S\right) H_N + \left(I + \frac{g}{2}[X_N^{-1}, H_N]\right) H_S \right\} |\phi(t_B)\rangle \\
&= -i\hbar \left(I - gX_N^{-1}H_S\right) \frac{\partial}{\partial t_B} |\phi(t_B)\rangle + \left\{ \left(I - gX_N^{-1}H_S\right) H_N + \left(I + \frac{g}{2}[X_N^{-1}, H_N]\right) H_S \right\} |\phi(t_B)\rangle.
\end{aligned} \tag{19}$$

Assuming that $I - gX_N^{-1}H_S$ is invertible and reorganising the terms, we obtain a Schrödinger equation with the Hamiltonian given by Eq. (14).