Department of Physics "E. Fermi" University of Pisa

Entanglement of quantum clocks via gravity

Page and Wootters formalism

Dario Cafasso Link to GitHub d.cafasso@studenti.unipi.it

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Contents



Introduction to Quantum Gravity

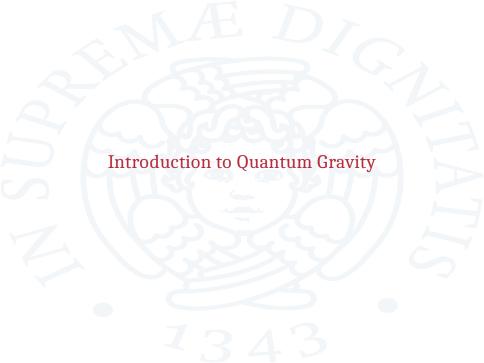
Timeless Quantum Mechanics

Page and Wootters framework Consequences of the framework More than one quantum clock

Interacting quantum clocks

Perspectives

Acknowledgments and References



Introduction to Quantum Gravity Different approaches



When we talk about **Quantum Gravity**, rather than a theory, we refer to a **family of problems** about the relation between our fundamental theories:

Quantum Mechanics and General Relativity

These problems lead physicists to pursue different approaches toward the "complete" picture, but all these paths originated from the same question:

If spacetime is a dynamical field and dynamical fields are quantized, then should we expect a kind of quantum spacetime?¹ there's something wrong in our foundations?

¹Cfr. section 1.1 of ref. [1]

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Introduction to Quantum Gravity

Underlying structures



In the present picture, our fundamental theories are able to describe with astonishing precision a huge amount of natural phenomena, but, in the words of Einstein²:

"One is struck [by the fact] that the theory [of special relativity]... introduces two kinds of physical things, i.e., measuring rods and clocks, all other things, e.g., the electromagnetic field, the material point, etc. This, in a certain sense, is inconsistent..."

and so, as pointed out by C. Rovelli³:

"The search for a quantum theory of gravity raises once more old questions such as: What is space? What is time? What is the meaning of "moving"? Is motion to be defined with respect to objects or with respect to space? And also: What is causality? What is the role of the observer in physics?"

²Cfr. ref. [2] for the sources.

³Cfr. section 1.1.3 of ref. [1].

Introduction to Quantum Gravity The role of time



Our work is based on a different way of conceiving time, not more as a classical parameter, but defining it from an operative point of view as the observable measured by the clock of an observer.

As pointed out in ref. [2]:

"For the sake of consistency, it is natural to assume that the clocks, being physical, behave according to the principles of our most fundamental physical theories."

How can we take care of the physical, observer dependent, nature of time?

Introduction to Quantum Gravity The role of time

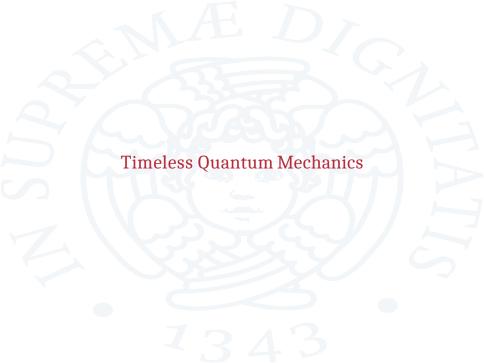


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Quantum Time



Here we follow the approach introduced by Page and Wootters⁴, and then modified by Giovannetti, Lloyd and Maccone⁵ to define the concept of "Quantum Time" as the one measured via the "Time Operator".

We can describe a clock as a quantum system C, such that

$$|t\rangle_C\in\mathcal{H}_C$$
 $\hat{T}|t\rangle_C=t\,|t\rangle_C$ $\hat{H}_C=\hbar\hat{\Omega}$ (1)

where $\hat{\Omega}$ is the conjugate variable to \hat{T} , i.e. $[\hat{T},\hat{\Omega}]=i\mathbb{1}_{\mathcal{C}}.$ We also have

$$_{C}\langle t'|t\rangle_{C}=\delta(t'-t)$$
 and $_{C}\langle t'|\hat{\Omega}|t\rangle_{C}=\delta(t'-t)\partial_{t}$ (2)

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In this framework, we assume that the time measured by **the clock** *C* is used by an observer to make measures about the system *S*.

It follows that C has to be regarded as a part of the quantum system S+C, subject to a global constraint \hat{H} in the form of a Wheeler-DeWitt Equation, such that

$$|\Psi\rangle\in\mathcal{H}:=\mathcal{H}_{C}\otimes\mathcal{H}_{S} \quad \hat{H}\,|\Psi\rangle=o \text{ with } \hat{H}=\hat{H}_{C}\otimes\mathbb{1}_{S}+\mathbb{1}_{C}\otimes\hat{H}_{S} \quad \ \ (3)$$

This allows an observer-dependent and timeless description of QM!

Note that we are assuming C to be a perfect clock, i.e. isomorphic to a particle on a line \hat{H}_S to be time independent. For generalization to the time dependent case, cfr. ref. [4].



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Schrödinger Equation



Indeed, we recover the **Schrödinger Equation** from equation 3 conditioning on time via projection on the generalized eigenvectors of the \hat{T} operator. Using equations 2, we find:

$$_{\text{C}}\left\langle t\right|\hat{H}\left|\Psi\right\rangle =\text{O}\iff\partial_{t}\left|\psi(t)\right\rangle _{\text{S}}=\hat{H}_{\text{S}}\left|\psi(t)\right\rangle _{\text{S}}$$

where $|\psi(t)
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angle$ is the **Schrödinger State** we know from QM. 6

A general solution of equation 3 is called history state and can be written as

$$|\Psi\rangle = \int \mathrm{d}t \, |t\rangle_{\mathcal{C}} \otimes |\psi(t)\rangle_{\mathcal{S}}$$
 (5)

It incorporates all of the information of the quantum system.

Cfr. equation 15 of ref. [5] for an alternative definition of the history state.

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$$_{C}\langle t|\hat{H}|\Psi\rangle = 0 \iff \partial_{t}|\psi(t)\rangle_{S} = \hat{H}_{S}|\psi(t)\rangle_{S}$$
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At this point:

- ▶ How to describe measures and probabilities in this framework?
- ▶ Which criticisms have been raised about the clock model?
- Which is the interpretation of the emerging picture?



In ref. [4], the **correct expression** for the n-times **propagator** has been recovered. For the derivation, they start adopting the **von Neumann formulation of the measurement apparatus** and then they cast the outcome in the time dependent PaW framework.

It is a "purification" of the procedure, introducing a memory system M for each measure. This allows us to write the probability of n-outcomes in n-times accordingly to the Bayes rule for conditional probabilities.

As an example, the conditional probability of getting b at time t given a at time $t_{\rm o} < t$ reads

$$P(b,t|a,t_o) = \frac{P(b,a,t)}{P(a,t_o)} = \frac{\left\| \left(c \left\langle t \right| \otimes_{M_b} \left\langle b \right| \otimes_{M_a} \left\langle a \right| \right) \left| \Psi \right\rangle \right\|^2}{\left\| \left(c \left\langle t_o \right| \otimes_{M_a} \left\langle a \right| \right) \left| \Psi \right\rangle \right\|^2} \tag{6}$$



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Here we describe the **von Neumann formulation** of the measurement apparatus.⁷

We consider the system *S* as the sum of the **subsystem of interest** *Q* and a **memory system** *M*. Then, we describe the **process of measuring** as an **instantaneous transformation** which induces the unitary mapping

$$|\psi(t_i)\rangle_Q \otimes |r\rangle_M \longmapsto \sum_a \hat{K}_a |\psi(t_i)\rangle_Q \otimes |a\rangle_M$$
 (7)

in which $\{\hat{K}_a\}$ are called **Kraus operators** and fulfill the normalizatior condition $\sum_a \hat{K}_a^\dagger \hat{K}_a = \mathbb{1}$. The transformation can be expressed as an **interaction term** between Q and M in the Hamiltonian of S as

$$\hat{H}_{S}(t) = \hat{H}_{Q} + \hat{f}(t) \qquad \hat{f}(t) = \delta(t - t_{i})\hat{h}_{QM}$$
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In eq. 7, the memory state "r" stands for "ready" and "a" stands for "after".

Eq. 7 defines the statistical properties of a Positive Operator Valued Measure (POVM)

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Then, we define the unitary operator $\hat{V}_{QM}:=e^{-\frac{i}{\hbar}\hat{h}_{QM}.8}$ It is responsible of the mapping in equation 7 via the unitary evolution operator

$$\hat{U}_{S}(t, t_{o}) = \begin{cases} \hat{U}_{Q}(t, t_{o}) & \forall t < t_{1} \\ \hat{U}_{Q}(t, t_{1}) \hat{V}_{QM} \hat{U}_{Q}(t_{1}, t_{o}) & \forall t > t_{1} \end{cases}$$
(9)

Putting together equations 5, 7 and 9, we have

$$\begin{split} |\Psi\rangle &= \int_{-\infty}^{t_{1}} \mathrm{d}t \, |t\rangle_{C} \otimes |\psi(t)\rangle_{Q} \otimes |r\rangle_{M} \, + \\ &\int_{t_{1}}^{+\infty} \mathrm{d}t \, |t\rangle_{C} \otimes \hat{U}_{Q}(t,t_{1}) \sum_{a} \hat{K}_{a} \, |\psi(t)\rangle_{Q} \otimes |a\rangle_{M} \quad \text{(10)} \end{split}$$

⁸The fact that it's unitary depends on the experimental apparatus via \hat{h}_{OM} .



After the measure, the probability of getting the outcome a at time t is then

$$P(a,t) := \left\| \left({_C} \left\langle t \right| \otimes_M \left\langle a \right| \right) \left| \Psi \right\rangle \right\|^2 = \left\| \hat{K}_a \left| \psi(t) \right\rangle_Q \right\|^2$$

and the corresponding state of the subsystem Q after the collapse is then

$$|\phi_a\rangle_Q := \frac{1}{\sqrt{P(a,t)}}\hat{K}_a |\psi(t)\rangle_Q$$

Here we describe the measuring process in such a way that the evolution is still unitary after the measure using the many-world approach. According to equation 7 we have

$$\hat{\mathsf{V}}_{\mathsf{QM}} \left| \psi(\mathsf{t}_{\mathsf{i}})
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As we've just seen with the arbitrariness in the measuring process, the present picture is not free of ambiguities. Another criticism that may be crucial in our work, is about the clock's Hamiltonian:

The clock model we are using in this work is unrealistic and has to be intended as an approximation of a n-level system. More realistic choices of the clock, with lower bounded Hamiltonian, led to closed time-like curves. Other choices are a straightforward modification of the above theory and some of them can be found in references [24,29] of ref. [4].

Timeless QM Overcoming criticism



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- ▶ the internal observer, who can anyway observe a dynamics, i.e. time dependence, and Born-rule induced wave-function collapses.



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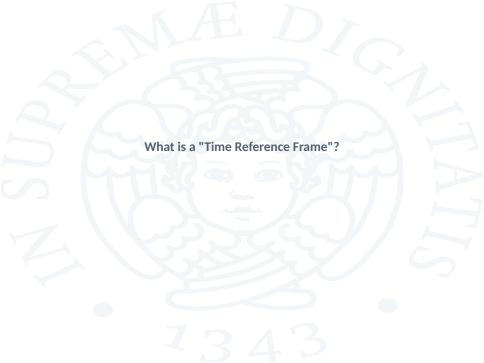


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This is a kind of generalization of the **relativity principle**, which lead us to the introduction of the concept of **Time Reference Frame**.





- We define a "Time Reference Frame" as a quantum temporal reference frame associated to a quantum clock.
- ► Temporal localization of events, i.e. quantum measurements on the system, is defined by the time reference frame of an observer.
- In the presence of more than one clock, each of them describes the evolution of the rest system according to its proper time?

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Suppose to have a system made up of two clocks I = A, B and a subsystem of interest S with a trivial Hamiltonian $\hat{H}_S = o$. No interaction is considered between A, B, S.

Accordingly to equations 3 and 8, we introduce in *S* an experimental apparatus with two memory systems, each associated to one clock.

$$\hat{H} = \hat{H}_A \otimes \mathbb{1}_{\bar{A}} + \hat{H}_B \otimes \mathbb{1}_{\bar{B}} + \hat{f}_A(\hat{T}_A) + \hat{f}_B(\hat{T}_B) \tag{11}$$



Expressing equation 5 in terms of A and B reference frames, we have

$$|\Psi\rangle = \int dt_I \, |t_I\rangle_I \otimes |\psi_I(t_I)\rangle_{\overline{I}} = \int dt_I \, |t_I\rangle_I \otimes \hat{U}(t_I) \, |\psi_I(o)\rangle_{\overline{I}}$$

Using the relation

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in which

$$\hat{S}_{AB} = \int \mathrm{d}t_B \left| t_B \right\rangle_B \otimes \left\langle t_A = 0 \right|_A \hat{U}_{\bar{B}}(t_B) e^{i\hat{T}_A \hat{H}_S}$$

We can express them in terms of each others as

$$|\Psi\rangle = \int \mathrm{d}t_\mathrm{A} \left|t_\mathrm{A}\right\rangle_\mathrm{A} \otimes e^{-it_\mathrm{A}\hat{H}_\mathrm{B}} \Im\{e^{-i\int_\mathrm{o}^{t_\mathrm{A}}\mathrm{d}s(\hat{f}_\mathrm{A}(s)+\hat{f}_\mathrm{B}(s+\hat{T}_\mathrm{B}))}\} \left|\psi_\mathrm{A}(\mathrm{O})\right\rangle_{\bar{A}}$$

Time Reference Frames

Change of Time Reference Frame



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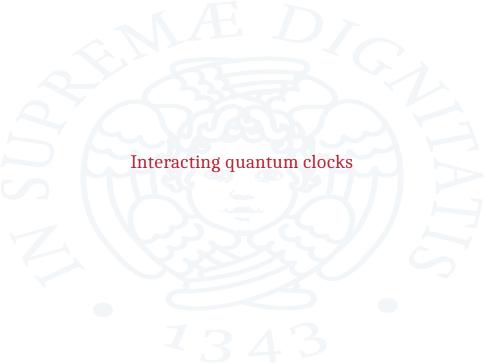
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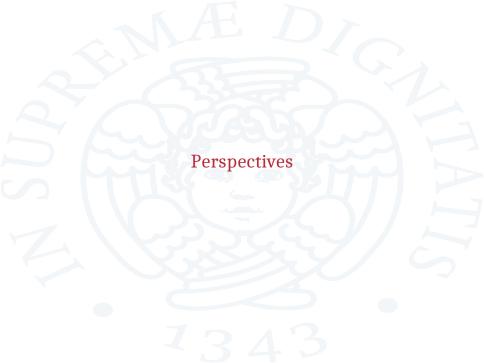
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Interacting quantum clocks



Ref. PNAS [2].

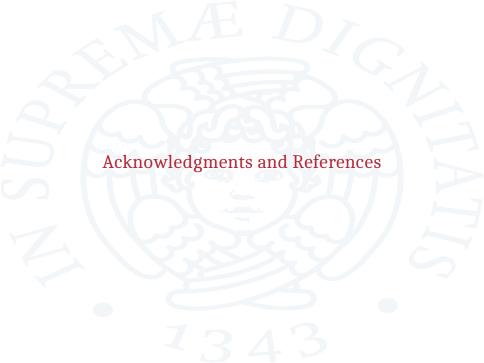


Perspective



Future research directions are:

classical limit



References L



[1] Daniele Oriti et al.

Approaches to Quantum Gravity: Toward a New Understanding of Space. Time and Matter. Cambridge University Press. 2009.

- [2] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner. Entanglement of quantum clocks through gravity. Proceedings of the National Academy of Sciences, 114(12):E2303-E2309, 2017.
- [3] Don N. Page and William K. Wootters. Evolution without evolution: Dynamics described by stationary observables
 - Phys. Rev. D. 27:2885-2892, Jun 1983.
- [4] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Quantum time. Phys. Rev. D. 92:045033, Aug 2015.

References II



[5] Esteban Castro-Ruiz, Flaminia Giacomini, A. Belenchia, and Časlav Brukner.

Quantum clocks and the temporal localisability of events in the presence of gravitating quantum systems.

Nature Commun., 11(1):2672, 2020.

Thank you for your attention!