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Entanglement of quantum clocks via gravity

Page and Wootters formalism

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Contents

Introduction to Quantum Gravity

When we talk about **Quantum Gravity**, rather than a theory, we refer to a **family of problems** about the relation between our fundamental theories:

Quantum Mechanics and General Relativity

These problems lead physicists to pursue different approaches toward the "complete" picture, but all these paths originated from the same question:

If spacetime is a dynamical field and dynamical fields are quantized, then should we expect a kind of quantum spacetime?[1]
there's something wrong in our foundations?

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In the present picture, our fundamental theories are able to describe with astonishing precision a huge amount of natural phenomena, but, in the words of Einstein[2]:

> *"One is struck [by the fact] that the theory [of special relativity]... introduces two kinds of physical things, i.e., measuring rods and clocks, all other things, e.g., the electromagnetic field, the material point, etc. This, in a certain sense, is inconsistent..."*

and so, as pointed out by C. Rovelli[3]:

> *"The search for a quantum theory of gravity raises once more old questions such as: What is space? What is time? What is the meaning of "moving"? Is motion to be defined with respect to objects or with respect to space? And also: What is causality? What is the role of the observer in physics?"*

---

[2]Cfr. ref. [2] for the sources.
[3]Cfr. section 1.1.3 of ref. [1].

Our work is based on a different way of conceiving time, not more as a classical parameter, but defining it from an operative point of view as the observable measured by the clock of an observer.

As pointed out in ref. [2]:

> *"For the sake of consistency, it is natural to assume that the clocks, being physical, behave according to the principles of our most fundamental physical theories."*

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How can we take care of the physical, observer dependent, nature of time?

Timeless Quantum Mechanics

Here we follow the approach introduced by Page and Wootters[4], and then modified by Giovannetti, Lloyd and Maccone[5] to define the concept of "Quantum Time" as the one measured via the "Time Operator".

We can describe a clock as a quantum system C, such that

$$|t\rangle_C \in \mathcal{H}_C \qquad \hat{T}\,|t\rangle_C = t\,|t\rangle_C \qquad \hat{H}_C = \hbar\hat{\Omega} \tag{1}$$

where $\hat{\Omega}$ is the conjugate variable to $\hat{T}$, i.e. $[\hat{T},\hat{\Omega}] = i\mathbb{1}_C$. We also have

$${}_C\langle t'|t\rangle_C = \delta(t'-t) \quad and \quad {}_C\langle t'|\,\hat{\Omega}\,|t\rangle_C = \delta(t'-t)\partial_t \tag{2}$$

---

[4] Cfr. ref. [3] for the original approach to this framework.
[5] Cfr. ref. [4] for the renewed approach without ambiguities.

PaW framework

Quantum Time

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In this framework, we assume that the time measured by **the clock** C is used by an observer to make measures about the system S.
It follows that C **has to be regarded as a part of the quantum system** $S+C$, subject to a global constraint $\hat{H}$ in the form of a **Wheeler-DeWitt Equation**, such that

$$|\Psi\rangle \in \mathcal{H} := \mathcal{H}_C \otimes \mathcal{H}_S \qquad \hat{H}\,|\Psi\rangle = 0 \;\; \textit{with} \;\; \hat{H} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{3}$$

This allows an observer-dependent and timeless description of QM!

Note that we are assuming C to be a perfect clock, i.e. isomorphic to a particle on a line, and $\hat{H}_S$ to be time independent. For generalization to the time dependent case, cfr. ref. [4].

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Indeed, we recover the **Schrödinger Equation** from equation 3 conditioning on time via projection on the generalized eigenvectors of the $\hat{T}$ operator. Using equations 2, we find:

$$ {}_C\langle t|\,\hat{H}\,|\Psi\rangle = 0 \iff \partial_t\,|\psi(t)\rangle_S = \hat{H}_S\,|\psi(t)\rangle_S \tag{4} $$

where $|\psi(t)\rangle_S := {}_C\langle t|\Psi\rangle$ is the **Schrödinger State** we know from QM.[6]

A general solution of equation 3 is called **history state** and can be written as

$$ |\Psi\rangle = \int \mathrm{d}t\,|t\rangle_C \otimes |\psi(t)\rangle_S \tag{5} $$

It incorporates all of the information of the quantum system.

[6] Some definitions may vary because of the introduction of a normalization factor into the time generalized eigenvectors. Note that here they're improper states. Cfr. ref. [4].

Cfr. equation 15 of ref. [5] for an alternative definition of the history state.

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At this point:

- How to describe measures and probabilities in this framework?
- Which criticisms have been raised about the clock model?
- Which is the interpretation of the emerging picture?

In ref. [4], the **correct expression** for the n-times **propagator** has been recovered. For the derivation, they start adopting the **von Neumann formulation of the measurement apparatus** and then they cast the outcome in the time dependent PaW framework.

It is a "purification" of the procedure, **introducing a memory system** M for each measure. This allows us to **write the probability** of n-outcomes in n-times **accordingly to the Bayes rule** for conditional probabilities.

As an example, the conditional probability of getting b at time t given a at time $t_0 < t$ reads

$$P(b, t|a, t_0) = \frac{P(b, a, t)}{P(a, t_0)} = \frac{\left\| \left({}_C\langle t| \otimes {}_{M_b}\langle b| \otimes {}_{M_a}\langle a| \right) |\Psi\rangle \right\|^2}{\left\| \left({}_C\langle t_0| \otimes {}_{M_a}\langle a| \right) |\Psi\rangle \right\|^2} \qquad (6)$$

Timeless QM
Conditional probabilities

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Here we describe the **von Neumann formulation** of the measurement apparatus.[7]

We consider the system S as the sum of the **subsystem of interest** Q and a **memory system** M. Then, we describe the **process of measuring** as an **instantaneous transformation** which induces the unitary mapping

$$|\psi(t_i)\rangle_Q \otimes |r\rangle_M \longmapsto \sum_a \hat{K}_a |\psi(t_i)\rangle_Q \otimes |a\rangle_M \tag{7}$$

in which $\{\hat{K}_a\}$ are called **Kraus operators** and fulfill the normalization condition $\sum_a \hat{K}_a^\dagger \hat{K}_a = \mathbb{1}$. The transformation can be expressed as an **interaction term** between Q and M in the Hamiltonian of S as

$$\hat{H}_S(t) = \hat{H}_Q + \hat{f}(t) \qquad \hat{f}(t) = \delta(t - t_i)\hat{h}_{QM} \tag{8}$$

[7] More details in section "Measurements" of ref. [4].
In eq. 7, the memory state "r" stands for "ready" and "a" stands for "after".
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Then, we define the unitary operator $\hat{V}_{QM} := e^{-\frac{i}{\hbar}\hat{h}_{QM}}$.[8] It is responsible of the mapping in equation 7 via the unitary evolution operator

$$\hat{U}_S(t, t_0) = \begin{cases} \hat{U}_Q(t, t_0) & \forall t < t_1 \\ \hat{U}_Q(t, t_1)\hat{V}_{QM}\hat{U}_Q(t_1, t_0) & \forall t > t_1 \end{cases} \tag{9}$$

Putting together equations 5, 7 and 9, we have

$$|\Psi\rangle = \int_{-\infty}^{t_1} \mathrm{d}t \, |t\rangle_C \otimes |\psi(t)\rangle_Q \otimes |r\rangle_M + \int_{t_1}^{+\infty} \mathrm{d}t \, |t\rangle_C \otimes \hat{U}_Q(t, t_1) \sum_a \hat{K}_a \, |\psi(t)\rangle_Q \otimes |a\rangle_M \tag{10}$$

[8]The fact that it's unitary depends on the experimental apparatus via $\hat{h}_{QM}$.

After the measure, the probability of getting the outcome a at time t is then

$$P(a,t) := \left\| \left({}_C \langle t| \otimes {}_M \langle a| \right) |\Psi\rangle \right\|^2 = \left\| \hat{K}_a |\psi(t)\rangle_Q \right\|^2$$

and the corresponding state of the subsystem Q after the collapse is then

$$|\phi_a\rangle_Q := \frac{1}{\sqrt{P(a,t)}} \hat{K}_a |\psi(t)\rangle_Q$$

Here we describe the measuring process in such a way that the evolution is still unitary after the measure using the many-world approach. According to equation 7 we have

$$\hat{V}_{QM} |\psi(t_i)\rangle_Q \otimes |r\rangle_M = \sum_a \sqrt{P(a,t_i)}\, |\phi_a\rangle_Q \otimes |a\rangle_M$$

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As we've just seen with the arbitrariness in the measuring process, **the present picture is not free of ambiguities**. Another criticism that may be crucial in our work, is about the clock's Hamiltonian:

The **clock model** we are using in this work **is unrealistic** and has to be intended as an approximation of a n-level system. More realistic choices of the clock, with lower bounded Hamiltonian, led to closed time-like curves. Other choices are a straightforward modification of the above theory and some of them can be found in references [24,29] of ref. [4].

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Finally, as a criticism, some physicists argued that this framework describes a **stationary universe**. The most interesting observation pointed out by the authors of ref. [4] is that we can define two points of view:

- the **external observer**, who sees the whole universe as a static system whose state is an eigenstate of its global Hamiltonian, i.e. equation 3.
- the **internal observer**, who can anyway observe a dynamics, i.e. time dependence, and Born-rule induced wave-function collapses.

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- the **internal observer**, who can anyway observe a dynamics, i.e. time dependence, and Born-rule induced wave-function collapses.

This is a kind of generalization of the **relativity principle**, which lead us to the introduction of the concept of **Time Reference Frame**.

What is a "Time Reference Frame"?

According to ref. [5],

- We define a "Time Reference Frame" as a quantum temporal reference frame associated to a quantum clock.
- Temporal localization of events, i.e. quantum measurements on the system, is defined by the time reference frame of an observer.
- In the presence of more than one clock, each of them describes the evolution of the rest system according to its proper time[9].

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Suppose to have a system made up of two clocks $I = A, B$ and a subsystem of interest S with a trivial Hamiltonian $\hat{H}_S = 0$. No interaction is considered between A, B, S.

Accordingly to equations 3 and 8, we introduce in S an experimental apparatus with two memory systems, each associated to one clock.

$$\hat{H} = \hat{H}_A \otimes \mathbb{1}_{\bar{A}} + \hat{H}_B \otimes \mathbb{1}_{\bar{B}} + \hat{f}_A(\hat{T}_A) + \hat{f}_B(\hat{T}_B) \tag{11}$$

Expressing equation 5 in terms of A and B reference frames, we have

$$|\Psi\rangle = \int \mathrm{d}t_I \, |t_I\rangle_I \otimes |\psi_I(t_I)\rangle_{\bar{I}} = \int \mathrm{d}t_I \, |t_I\rangle_I \otimes \hat{U}(t_I) \, |\psi_I(0)\rangle_{\bar{I}}$$

Using the relation

$$|\psi_A(t_A)\rangle_{\bar{A}} = \hat{U}_{\bar{A}}(t_A)\hat{S}_{AB}\hat{U}^{\dagger}_{\bar{B}}(t_B) \, |\psi_B(t_B)\rangle_{\bar{B}}$$

in which

$$\hat{S}_{AB} = \int \mathrm{d}t_B \, |t_B\rangle_B \otimes \langle t_A = 0|_A \, \hat{U}_{\bar{B}}(t_B) e^{i\hat{T}_A \hat{H}_S}$$

We can express them in terms of each others as

$$|\Psi\rangle = \int \mathrm{d}t_A \, |t_A\rangle_A \otimes e^{-it_A \hat{H}_B} \mathcal{T}\{e^{-i\int_0^{t_A} \mathrm{d}s (\hat{f}_A(s) + \hat{f}_B(s + \hat{T}_B))}\} \, |\psi_A(0)\rangle_{\bar{A}}$$

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Interacting quantum clocks

Interacting quantum clocks

Ref. PNAS [2].

Perspectives

Perspective

Future research directions are:

- classical limit

Acknowledgments and References

References I

[1] Daniele Oriti et al.
Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter.
Cambridge University Press, 2009.

[2] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner.
Entanglement of quantum clocks through gravity.
Proceedings of the National Academy of Sciences, 114(12):E2303–E2309, 2017.

[3] Don N. Page and William K. Wootters.
Evolution without evolution: Dynamics described by stationary observables.
Phys. Rev. D, 27:2885–2892, Jun 1983.

[4] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone.
Quantum time.
Phys. Rev. D, 92:045033, Aug 2015.

References II

[5] Esteban Castro-Ruiz, Flaminia Giacomini, A. Belenchia, and Časlav Brukner.
Quantum clocks and the temporal localisability of events in the presence of gravitating quantum systems.
Nature Commun., 11(1):2672, 2020.

Thank you for your attention!