Department of Physics "E. Fermi"
University of Pisa

Entanglement of quantum clocks via gravity

Page and Wootters formalism

Dario Cafasso
Link to GitHub
d.cafasso@studenti.unipi.it

September 19, 2022

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Introduction to Quantum Gravity

When we talk about **Quantum Gravity**, rather than a theory, we refer to a **family of problems** about the relation between our fundamental theories:

Quantum Mechanics and General Relativity

These problems lead physicists to pursue different approaches toward the "complete" picture, but all these paths originated from the same question:

If spacetime is a dynamical field and dynamical fields are quantized, then should we expect a kind of quantum spacetime?[1]
or there's something wrong with our foundations?

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[1] Cfr. section 1.1 of ref. [1]

In the present picture, our fundamental theories are able to describe with astonishing precision a huge amount of natural phenomena, but, in the words of Einstein[2]:

> *"One is struck [by the fact] that the theory [of special relativity]... introduces two kinds of physical things, i.e., measuring rods and clocks, all other things, e.g., the electromagnetic field, the material point, etc. This, in a certain sense, is inconsistent..."*

and so, as pointed out by C. Rovelli[3]:

> *"The search for a quantum theory of gravity raises once more old questions such as: What is space? What is time? What is the meaning of "moving"? Is motion to be defined with respect to objects or with respect to space? And also: What is causality? What is the role of the observer in physics?"*

[2] Cfr. ref. [2]
[3] Cfr. section 1.1.3 of ref. [1]

Our work starts from the considerations about the fundamental nature of time as measured by the clock of an observer.

As pointed out in reference [2]:

> *"For the sake of consistency, it is natural to assume that the clocks, being physical, behave according to the principles of our most fundamental physical theories."*

How can we take care of the physical, observer dependent, nature of time?

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How can we take care of the physical, observer dependent, nature of time?

Timeless Quantum Mechanics

Here we follow the approach introduced by Page and Wootters[4], and then modified by Giovannetti, Lloyd and Maccone[5] to define the concept of "Quantum Time" as the one measured via the "Time Operator".

We can describe a clock as a quantum system C, such that

$$|t\rangle_C \in \mathcal{H}_C \qquad \hat{T}\,|t\rangle_C = t\,|t\rangle_C \qquad \hat{H}_C = \hbar\hat{\Omega} \tag{1}$$

where $\hat{\Omega}$ is the conjugate variable to $\hat{T}$, i.e. $[\hat{T}, \hat{\Omega}] = i\mathbb{1}_C$.

We also observe that

$$_C\langle t'|t\rangle_C = \delta(t'-t) \quad and \quad _C\langle t'|\,\hat{\Omega}\,|t\rangle_C = \delta(t'-t)\partial_t \tag{2}$$

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PaW framework
Wheeler-DeWitt Equation

If the time measured by the clock C is used by an observer to make measures about the system S, then C has to be regarded as a part of the quantum system $S+C$, subject to a global constraint $\hat{H}$ in the form of a Wheeler-DeWitt Equation, such that

$$|\Psi\rangle \in \mathcal{H} := \mathcal{H}_C \otimes \mathcal{H}_S \qquad \hat{H}\,|\Psi\rangle = 0 \;\; \textit{with} \;\; \hat{H} = \hat{H}_C \otimes \mathbb{1}_S + \mathbb{1}_C \otimes \hat{H}_S \tag{3}$$

It allows an observer-dependent and timeless description of QM!

Note that we are assuming C to be a perfect clock, i.e. isomorphic to a particle on a line, and $\hat{H}_S$ to be time independent. For generalization to the time dependent case, cfr. ref. [4].

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Indeed, we recover the Schrödinger Equation from equation 3 via projection with the generalized eigenvectors of the time operator. Inserting relations 2, we find:

$$ {}_C\langle t|\, \hat{H}\, |\Psi\rangle = 0 \iff \partial_t \,|\psi(t)\rangle_S = \hat{H}_S\, |\psi(t)\rangle_S \tag{4} $$

where $|\psi(t)\rangle_S := {}_C\langle t|\Psi\rangle$ is the Schrödinger State we are familiar with.[6]

A general solution of equation 3 is called **history state** and can be written as

$$ |\Psi\rangle = \int \mathrm{d}t\, |t\rangle_C \otimes |\psi(t)\rangle_S \tag{5} $$

[6] Some definitions may vary because of the normalization factor of time generalized eigenvectors. Note that here they're improper states. Cfr. ref. [4].

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In reference [4], the correct expression for the n-times propagator has been recovered. For the derivation, we start adopting the von Neumann formulation of the measurements apparatus, introducing concept of POVM and then we cast them in the time dependent PaW framework.

The "purification" of the procedure, introducing a memory system M for each measure, allows us to write the probability of n-outcomes in n-times accordingly to the Bayes rule for conditional probabilities.

As an example, the conditional probability of getting b at time t given a at time $t_0 < t$ reads

$$P(b, t|a, t_0) = \frac{P(b, a, t)}{P(a, t_0)} = \frac{\left\| \left({}_C \langle t| \otimes {}_{M_b} \langle b| \otimes {}_{M_a} \langle a| \right) |\Psi\rangle \right\|^2}{\left\| \left({}_C \langle t_0| \otimes {}_{M_a} \langle a| \right) |\Psi\rangle \right\|^2} \tag{6}$$

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Slide of implementation of the procedure above from section "Measurements" in reference [4].

The present picture is not completely clear or free of inconsistencies. The objection that may be crucial in our work, is about the clock's hamiltonian.

The clock model we are using in this work is unrealistic and has to be intended as an approximation of a n-level system. More realistic choices of the clock, with lower bounded hamiltonians, led to closed-time curve. Other choices are a straightforward modification of the above theory and some of them can be found in references [24,29] in reference [4].

Timeless QM
Overcoming criticism

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Some physicists argued that this picture describes a static universe, and the most interesting observation pointed out by the authors of reference [4] is that we can define two points of view:

- the external observer, who sees the whole universe as a static system whose state is an eigenstate of its global Hamiltonian, i.e. equation 3.
- the internal observer, who can anyway observe evolving systems, time-dependent measurement outcomes and Born-rule induced wave-function collapses.

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This is a kind of generalization of the relativity principle, which lead us to the introduction of the concept of **Time Reference Frame**.

So, according to reference [5]

- We define a "Time Reference Frame" as a quantum temporal reference frame associated to a quantum clock.
- Temporal localization of events, i.e. quantum measurements on the system, is defined by the time reference frame of an observer.
- In the presence of more than one clock, each of them describes the evolution of the complementary system according to its "proper time"[7].

[7] The relation with the definition of proper time in GR has to be clarified.

Time Reference Frames
Definitions

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Suppose to have a system made up of two clocks $I = A, B$ and a subsystem of interest S with a trivial hamiltonian $\hat{H}_S = 0$. No interaction is considered between A, B, S.

The experimental apparatus introduces the instantaneous interaction because of the process of measure, described above. Analogously to equation 3, we have

$$\hat{H} = \hat{H}_A \otimes \mathbb{1}_{\bar{A}} + \hat{H}_B \otimes \mathbb{1}_{\bar{B}} + \hat{f}_A(\hat{T}_A) + \hat{f}_B(\hat{T}_B) \tag{7}$$

in which $\hat{f}_I(\hat{T}_I)$ is the same described above in "measurements".

Expressing equation 5 in terms of A and B reference frames, we have

$$|\Psi\rangle = \int dt_I \, |t_I\rangle_I \otimes |\psi_I(t_I)\rangle_{\bar{I}} = \int dt_I \, |t_I\rangle_I \otimes \hat{U}(t_I) \, |\psi_I(0)\rangle_{\bar{I}}$$

If we want to express them in terms of each others, we have

$$|\Psi\rangle = \int dt_A \, |t_A\rangle_A \otimes e^{-it_A \hat{H}_B} \mathcal{T}\{e^{-i \int_0^{t_A} ds (\hat{f}_A(s) + \hat{f}_B(s + \hat{T}_B))}\} \, |\psi_A(0)\rangle_{\bar{A}}$$

using the relations (credo sia C barra alla U dagger)

$$|\psi_A(t_A)\rangle_{\bar{A}} = \hat{U}_{\bar{A}}(t_A) \hat{S}_{AC} \hat{U}^\dagger_{\bar{A}}(t_C) \, |\psi_C(t_C)\rangle_{\bar{C}}$$

and

$$\hat{S}_{AC} = \int dt_C \, |t_C\rangle_C \otimes \langle t_A = 0|_A \, \hat{U}_{\bar{C}}(t_C) e^{i\hat{T}_A \hat{H}_S}$$

Interacting quantum clocks

Interacting quantum clocks

Reference PNAS [2].

Perspectives

Perspective

Future research directions are:

- classical limit

Acknowledgments and References

References I

[1] Daniele Oriti et al.
Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter.
Cambridge University Press, 2009.

[2] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner.
Entanglement of quantum clocks through gravity.
Proceedings of the National Academy of Sciences, 114(12):E2303–E2309, 2017.

[3] Don N. Page and William K. Wootters.
Evolution without evolution: Dynamics described by stationary observables.
Phys. Rev. D, 27:2885–2892, Jun 1983.

[4] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone.
Quantum time.
Phys. Rev. D, 92:045033, Aug 2015.

References II

[5] Esteban Castro-Ruiz, Flaminia Giacomini, A. Belenchia, and Časlav Brukner.
Quantum clocks and the temporal localisability of events in the presence of gravitating quantum systems.
Nature Commun., 11(1):2672, 2020.

Thank you for your attention!