Quantum Time

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We give a consistent quantum description of time, based on Page and Wootters' conditional probabilities mechanism, that overcomes the criticisms that were raised against similar previous proposals. In particular we show how the model allows to reproduce the correct statistics of sequential measurements performed on a system at different times.

Time in quantum mechanics appears as a classical parameter in the Schrödinger equation. Physically it represents the time shown by a "classical" clock in the laboratory. Even though this is acceptable for all practical purposes, it is important to be able to give a fully quantum description of time. Many such proposals have appeared in the literature (e.g. [1–11]), but none seem entirely satisfactory [7, 12–16]. One of these is the Page and Wootters (PaW) mechanism [6] (see also [2, 17–20]) which considers "time" as a quantum degree of freedom by assigning to it a Hilbert space $\mathcal{H}_T$. The "flow" of time then consists simply in the correlation (entanglement) between this quantum degree of freedom and the rest of the system, a correlation present in a global, time-independent state $|\Psi\rangle\rangle$. An internal observer will see such state as describing normal time evolution: the familiar system state $|\psi(t)\rangle$ at time t arises by conditioning (via projection) the state $|\Psi\rangle\rangle$ to a time t (Fig. 1), it is a conditioned state. The PaW mechanism was criticized in [7, 12] and a proposal that overcomes these criticisms [21, 22] used Rovelli's evolving constants of motion [4, 23] parametrized by an arbitrary parameter that is then averaged over to yield the correct propagators. Although the end result matches the quantum predictions [24], the averaging used there amounts to a statistical averaging which is typically reserved to unknown physical degrees of freedom rather than to parameters with no physical significance. (A different way of averaging over time was also presented in [25] to account for some fundamental decoherence mechanism.)

Here we use a different strategy: we show that the same criticisms can be overcome by carefully formalizing measurements through the von Neumann prescription [26] (which we extend to generalized observables, POVMs). We show how this implies that all quantum predictions can be obtained by conditioning the global, timeless state $|\Psi\rangle\rangle$: this procedure gives the correct quantum propagators and the correct quantum statistic for measurements performed at different times, features that were absent in the original PaW mechanism [7, 16]. We also show how the PaW mechanism can be extended to give the time-independent Schrödinger equation and give a physical interpretation of the mechanism.

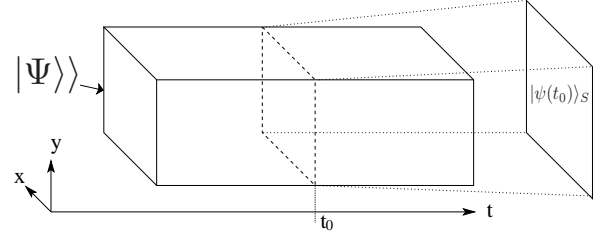What is the physical significance of the quantized time

FIG. 1. Pictorial representation of the global state $|\Psi\rangle\rangle$. The Hilbert space of the system $\mathcal{H}_S$ is represented by the x, y axes, the time Hilbert space $\mathcal{H}_T$ by the horizontal axis. The state of the system $|\psi(t_0)\rangle$ at time t_0 of the conventional formulation of quantum mechanics (dashed lines) is obtained by conditioning $|\Psi\rangle\rangle$ to having time t_0.

in the PaW representation? One is free to consider the time quantum degree of freedom either as an abstract purification space without any physical significance or as a dynamical degree of freedom connected to some system, or collection of systems, that represents a clock that is used to define time. The latter point of view may describe an operational definition of time [27, 28] that is appropriate for proper time: it entails defining proper time as "what is read on a clock", where a clock is a specific physical system (described by the Hilbert space $\mathcal{H}_T$). In what follows we do not make commitment on any of these interpretations: our aim is only to elucidate some technical aspects of the representation and to clarify how it can be used to reproduce the predictions of standard quantum mechanics.

A. Review and revision of PaW

Our proposal is an extension of PaW's mechanism [6, 16, 29]. It consists in enlarging the Hilbert space $\mathcal{H}_S$ of the system under consideration to $\mathfrak{H} := \mathcal{H}_T \otimes \mathcal{H}_S$ with $\mathcal{H}_T$ the space of an ancillary system T (we shall call it the "clock" system) that we assume to be isomorphic to the Hilbert space of a particle on a line (other choices are possible [24, 29]). The latter is equipped with canonical coordinates $\hat{T}$ and $\hat{\Omega}$ with $[\hat{T}, \hat{\Omega}] = i$, that represent position and momentum and (under the following restrictions) can be interpreted as the time and energy indicator

of the evolving system. Next we introduce what we may call the *constraint* operator of the model, i.e.

$$\hat{\mathbb{J}} := \hbar \hat{\Omega} \otimes \mathbb{1}_S + \mathbb{1}_T \otimes \hat{H}_S \,, \tag{1}$$

with $\hat{H}_S$ the system Hamiltonian, and $\mathbb{1}_S$ and $\mathbb{1}_T$ the identity operators on $\mathcal{H}_S$ and $\mathcal{H}_T$. By construction $\hat{\mathbb{J}}$ is selfadjoint and has a continuous spectrum that includes all possible real values as generalized eigenvalues. Next we identify a special set of vectors $|\Psi\rangle\rangle$ which we call the *physical vectors* of the model and which, as will be clear in the following, provides a compact, yet static, representation of the full history of the system S. They are identified by the eigenvector equation associated with the null eigenvalue of $\hat{\mathbb{J}}$, i.e.

$$\hat{\mathbb{J}}|\Psi\rangle\rangle = 0 \,, \tag{2}$$

where the double-ket notation reminds us that $|\Psi\rangle\rangle$ is defined on $\mathcal{H}_T \otimes \mathcal{H}_S$. More precisely Eq. (2) defines generalized eigenvectors which (as in the case of the position operator of a particle) are not proper elements of $\mathfrak{H}$ but still possesses a scalar product with all the elements of such space, inducing a representation of it.

One may interpret Eq. (2) as a constraint that forces the physical vectors to be eigenstates of the total 'Hamiltonian' $\hat{\mathbb{J}}$ with null eigenvalue, consistently with a Wheeler-DeWitt equation [13, 30]. Accordingly, in this model the $|\Psi\rangle\rangle$'s are "static" objects which do not evolve. The conventional state $|\psi(t)\rangle_S$ of the system S at time t can then be obtained by conditioning a solution $|\Psi\rangle\rangle$ of Eq. (2) on having the time t via projection with the generalized eigenvectors of the time operator $\hat{T}$ (Fig. 1), i.e.

$$|\psi(t)\rangle_S = {}_T\langle t|\Psi\rangle\rangle \,, \tag{3}$$

with

$$\hat{T}|t\rangle_T = t|t\rangle_T \,, \qquad {}_T\langle t'|t\rangle_T = \delta(t - t') \,. \tag{4}$$

By writing (2) in the 'position' representation in $\mathcal{H}_T$, one can easily verify that such vector indeed obeys Schrödinger equation, i.e. [6, 29]

$$_T\langle t|\hat{\mathbb{J}}|\Psi\rangle\rangle = 0 \quad \Leftrightarrow \quad i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S = \hat{H}_S|\psi(t)\rangle_S \,, \tag{5}$$

where we used the fact that $\hat{\Omega}$ is described by the differential operator. In a similar way we can identify the eigenvectors of $\hat{H}_S$ by projecting $|\Psi\rangle\rangle$ on the (generalized) eigenstates of $\hat{\Omega}$ (i.e. the vectors $|\omega\rangle_T = \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}|t\rangle_T$ with $\omega \in \text{Re}$). Specifically, given

$$|\psi(\omega)\rangle_S = {}_T\langle\omega|\Psi\rangle\rangle \,, \tag{6}$$

with

$$\hat{\Omega}|\omega\rangle_T = \omega|\omega\rangle_T \,, \qquad {}_T\langle\omega'|\omega\rangle_T = \delta(\omega - \omega') \,, \tag{7}$$

we have that

$$_T\langle\omega|\hat{\mathbb{J}}|\Psi\rangle\rangle = 0 \iff \hat{H}_S|\psi(\omega)\rangle_S = -\hbar\omega|\psi(\omega)\rangle_S \,, \tag{8}$$

which shows that the momentum representation (6) of a physical vector $|\Psi\rangle\rangle$ that solves Eq. (2) obeys the Schrödinger eigenvector equation – more precisely for ω such that $-\hbar\omega$ equals an element of the spectrum of H_S, then $|\psi(\omega)\rangle_S$ is an eigenvector of H_S at that eigenvalue, otherwise $|\psi(\omega)\rangle_S = 0$.

Exploiting the fact that both $\{|t\rangle_T\}_t$ and $\{|\omega\rangle_T\}_\omega$ provide a decomposition for the identity operator on $\mathcal{H}_T$, any solution of Eq. (2) can be expressed as

$$|\Psi\rangle\rangle = \int dt \, |t\rangle_T \otimes |\psi(t)\rangle_S \tag{9}$$

$$= \int d\mu(\omega) \, |\omega\rangle_T \otimes |\psi(\omega)\rangle_S \,, \tag{10}$$

with $d\mu(\omega)$ a measure on the real axis which selects those ω's that admit a non trivial solution for Eq. (8). The identity (9) shows that the vectors $|\Psi\rangle\rangle$ provide a complete description of the temporal evolution of the system S by representing it in terms of correlations between the latter and the degree of freedom of the ancillary system T. In particular, introducing the unitary operator $\hat{U}_S(t) = e^{-\frac{i}{\hbar}\hat{H}_S t}$ which solves Eq. (5), we get

$$|\Psi\rangle\rangle = \int dt \, |t\rangle_T \otimes \hat{U}_S(t)|\psi(0)\rangle_S \tag{11}$$

$$= \hat{\mathbb{U}} \, |T_{ime}L_{ine}\rangle_T \otimes |\psi(0)\rangle_S \,, \tag{12}$$

where $|\psi(0)\rangle_S$ is the state of S at time $t = 0$, where $|T_{ime}L_{ine}\rangle_T$ is the improper state of $\mathcal{H}_T$ obtained by superposing all vectors $|t\rangle_T$, i.e.

$$|T_{ime}L_{ine}\rangle_T := \int dt \, |t\rangle_T = \sqrt{2\pi} \, |\omega = 0\rangle_T \,, \tag{13}$$

and where $\hat{\mathbb{U}}$ is the unitary operator

$$\hat{\mathbb{U}} := \int dt|t\rangle_T\langle t| \otimes \hat{U}_S(t) \tag{14}$$

$$= \hat{U}_S(\hat{T}) = e^{-i\hat{T}\otimes\hat{H}_S/\hbar}. \tag{15}$$

Before proceeding further we comment on some important technical aspects of the PaW representation.

The Zero-eigenvalue:— In the construction of the PaW model, the zero eigenvalue of $\hat{\mathbb{J}}$ seems to play a special role in identifying the physical vectors $|\Psi\rangle\rangle$ but this is not the case. Indeed up to an irrelevant global phase, the physical vectors $|\Psi\rangle\rangle$ can be identified also by imposing the constraint

$$\hat{\mathbb{J}}|\Psi\rangle\rangle = \epsilon|\Psi\rangle\rangle \,, \tag{16}$$

with real ϵ. Indeed Eq. (16) can be cast in the form (2) by rigidly shifting the spectrum of $\hat{H}_S$ by ϵ.

Time-dependent Hamiltonians:— The relevance of Hamiltonians $\hat{H}_S(t)$ which exhibit an explicit time dependence may be questioned at fundamental level. Still it is well known that the possibility of dealing with these models is extremely useful in simplifying the analysis of systems where effective time dependent Hamiltonians

arise from the interplay between local degree of freedom and an external, complex environment characterized by a semi-classical behavior (e.g. a measurement apparatus) – see e.g. [31] and references therein. Notably the PaW representation can be extended to incorporate also these examples by simply replacing the constraint operator (1) with

$$\hat{\mathbb{J}} := \hbar \hat{\Omega} \otimes \mathbb{1}_S + \hat{H}_S(\hat{T}) \,, \qquad (17)$$

with $\hat{H}_S(\hat{T})$ now an operator that acts not trivially on both $\mathcal{H}_S$ and $\mathcal{H}_T$, obtained by formally promoting the variable t which appears in $\hat{H}_S(t)$ into the canonical coordinate operator $\hat{T}$. Selecting the physical states $|\Psi\rangle\rangle$ as in Eq. (2) it then follows that the decompositions (9), (11), and (14) still hold with the operator $\hat{U}_S(t,0)$ defined as

$$\hat{U}_S(t,t_0) = \begin{cases} \overleftarrow{\exp}[-(i/\hbar)\int_{t_0}^t dt' \hat{H}_S(t')] & \forall t \geq t_0 \,, \\[2mm] \overrightarrow{\exp}[(i/\hbar)\int_t^{t_0} dt' \hat{H}_S(t')] & \forall t < t_0 \,, \end{cases} \qquad (18)$$

where $\overleftarrow{\exp}[\int_{t_0}^t dt'...]$ (resp. $\overrightarrow{\exp}[\int_{t_0}^t dt'...]$) indicates the time-ordering (resp. anti-ordering) of the associated integral.

The initial time:— The choice of $t = 0$ as the reference time in Eqs. (11) and (12) is just a matter of convention. Indeed an equivalent way to express $|\Psi\rangle\rangle$ is the following

$$|\Psi\rangle\rangle = \int dt \, |t\rangle_T \otimes \hat{U}_S(t,t_0)|\psi(t_0)\rangle_S \qquad (19)$$

$$= \hat{\mathbb{U}}_{t_0} \, |T_{ime}L_{ine}\rangle_T \otimes |\psi(t_0)\rangle_S \,, \qquad (20)$$

where now $|\psi(t_0)\rangle$ is the state of the system at time t_0 and where

$$\hat{\mathbb{U}}_{t_0} := \int dt |t\rangle_T \langle t| \otimes \hat{U}_S(t,t_0) = \hat{\mathbb{U}} \left[\mathbb{1}_T \otimes \hat{U}_S(t_0,0)\right] \,, \qquad (21)$$

(the identity is valid also in the case of time-dependent Hamiltonian $\hat{H}_S(t)$).

Propagators:— As anticipated, the physical vectors $|\Psi\rangle\rangle$ give a compact description of the system dynamical evolution in terms of a superposition of components each associated with a different time measured by an external clock described by $\mathcal{H}_T$. In particular, suppose we want to calculate the propagator between a state $|I\rangle_S$ at time t_I and a state $|F\rangle_S$ at time t_F, i.e. the quantity $\mathcal{G}(F,t_F;I,t_I) := {}_S\langle F|\hat{U}_S(t_F,t_I)|I\rangle_S$. In the PaW formalism this can be obtained by simply identifying t_0 with the time t_I and $|\psi(t_0)\rangle_S$ with $|I\rangle_S$ in Eq. (19) (this fixes the initial condition of the system trajectory) and then projecting the associated $|\Psi\rangle\rangle$ on $|t_F\rangle_T \otimes |F\rangle_S$, i.e.

$$\mathcal{G}(F,t_F;I,t_I) = ({}_T\langle t_F| \otimes {}_S\langle F|)|\Psi\rangle\rangle \,. \qquad (22)$$

One of the criticisms to the PaW mechanism is the fact that it did not seem to be able to reproduce the correct propagators [7]. He we have shown how the correct propagators can emerge.

About conditioning:— In the PaW representation the physical vectors identified by Eq. (2) ideally should describe a joint state of S and of the clock system T. Accordingly, given $\{|a\rangle_S\}$ a complete orthonormal basis for S, the quantities $({}_T\langle t| \otimes {}_S\langle a|)|\Psi\rangle\rangle$ should correspond to proper amplitude joint-probability distributions associated with the probability of finding $|a\rangle_S$ on S *and* $|t\rangle_T$ on T. In this framework it makes sense to interpret the vector (3) as the conditioned state of S obtained by forcing T to be on $|t\rangle_T$. Similarly one would like to interpret ${}_S\langle a|\Psi\rangle\rangle$ as the state of the clock conditioned by forcing S to be on $|a\rangle_S$. This last assumption however is problematic because, being $|\Psi\rangle\rangle$ an improper element of $\mathfrak{H}$, the vector ${}_S\langle a|\Psi\rangle\rangle$ as well as the function $|({}_T\langle t|\otimes {}_S\langle a|)|\Psi\rangle\rangle|^2$, do not admit a proper normalization, forcing us to assign a uniform distribution to the time variable t – no other choice being allowed by the representation. One can fix this by replacing Eq. (11) with a normalized element of $\mathfrak{H}$, i.e. with vectors of the form

$$|\Phi\rangle\rangle = \int dt \, \phi(t) \, |t\rangle_T \otimes |\psi(t)\rangle_S \,, \qquad (23)$$

with $|\psi(t)\rangle_S$ a normalized vector of $\mathcal{H}_S$ (i.e. $\||\psi(t)\rangle_S\| = 1$) and with $\phi(t)$ a square integrable function that guarantees the normalization condition for $|\Phi\rangle\rangle$, i.e. $\||\Phi\rangle\rangle\|^2 = \int dt|\phi(t)|^2 = 1$. Note that $|\Phi\rangle\rangle$ of Eq. (23) is the most general state of $\mathfrak{H} = \mathcal{H}_S \otimes \mathcal{H}_T$. Imposing next $|\psi(t)\rangle_S$ to describe the evolution of the initial state $|\psi(0)\rangle_S$ under the action of the system Hamiltonian $\hat{H}_S$ we can then write

$$|\Phi\rangle\rangle = \hat{\mathbb{U}} \, |\phi\rangle_T \otimes |\psi(0)\rangle_S \,, \qquad (24)$$

which replaces Eq. (12) by substituting the improper vector $|T_{ime}L_{ine}\rangle_T$ with the properly normalized state $|\phi\rangle_T := \int dt \, \phi(t)|t\rangle_T$ – the operator $\hat{\mathbb{U}}$ is still defined as in Eq. (14). For any assigned $\phi(t)$, the identity (3) is now replaced by

$$|\psi(t)\rangle_S = {}_T\langle t|\Phi\rangle\rangle/\phi(t) \qquad (25)$$

which, by noticing that $\phi(t)$ is the amplitude probability of finding the clock on $|t\rangle_S$ when measuring $|\Phi\rangle\rangle$, makes explicit the conditioning nature of $|\psi(t)\rangle_S$: such vector is obtained as an application of a Bayes rule for probability amplitudes, where the numerator gives the joint statistics of measurement on S and on t and the denominator describes the statistics of measurement on t only. At variance with the state of Eq. (11), the representation (24) finally allows for a proper definition of the state of the clock: the reduced density matrix $\mathrm{Tr}_S[|\Phi\rangle\rangle\langle\langle\Phi|]$ is now well behaved. These considerations imply that $\phi(t)$ is the weight that represents the probability amplitude that the system is found at time t, namely, in a sense, it is the probability amplitude that the system "exists" at such time. [Clearly, a suitable regularization is implicit in the expression (25), to avoid Borel-Kolmogorov type paradoxes that arise when one conditions on something that has null probability.]

In view of the above results, we can interpret the representation $|\Phi\rangle\rangle$ of (24) as a regularized version of the original PaW representation $|\Psi\rangle\rangle$ of (9), since it satisfies the normalization $\langle\langle\Phi|\Phi\rangle\rangle = 1$ on the joint system, which is the Stückelberg normalization [1]. In fact, following the conventional technique used for regularizing the eigenstates of operators with continuous spectrum, the PaW state (9) can be replaced, for example, by a normalized state (23) with Gaussian weight $\phi(t) \equiv \phi_n(t) = (2/n\pi)^{1/4}\exp(-t^2/n)$. Then, using the Weyl criterion [32], one can conclude that $\lambda = 0$ is an essential eigenvalue of the self-adjoint operator $\hat{\mathbb{J}}$, since

$$\|(\hat{\mathbb{J}} - \lambda)|\Phi_n\rangle\rangle\| \to 0 \text{ for } n \to \infty \qquad (26)$$

where $|\Phi_n\rangle\rangle = \int dt\, \phi_n(t)|t\rangle_T|\psi(t)\rangle_S$ is [33] a Weyl sequence, i.e. a normalized sequence of Hilbert space vectors that converges weakly to 0, namely $\langle\langle\theta|\Phi_n\rangle\rangle \to 0$ for $n \to \infty$ for all $|\theta\rangle\rangle \in \mathcal{H}_T \otimes \mathcal{H}_S$. Moreover, the unnormalized PaW state $|\Psi\rangle\rangle$ is obtained for $n \to \infty$ as $(n\pi/2)^{1/4}|\Phi_n\rangle\rangle \to |\Psi\rangle\rangle$.

The representation (24) allows for a constraint description analogous to (2) obtained by adding $\hat{H}_S$ to the non-Hermitian correction term $i\dot{\phi}(\hat{T})/\phi(\hat{T})$ (the dot representing time derivation) yielding

$$\left((\hbar\hat{\Omega} + i\hbar\frac{\dot{\phi}(\hat{T})}{\phi(\hat{T})}) \otimes \mathbb{1}_S + \mathbb{1}_T \otimes \hat{H}_S\right)|\Phi\rangle\rangle = 0 \text{ , i.e.} \quad (27)$$

$$\left((\phi(\hat{T})\hbar\hat{\Omega} + [\phi(\hat{T}), \hbar\Omega]) \otimes \mathbb{1}_S + \phi(\hat{T}) \otimes \hat{H}_S\right)|\Phi\rangle\rangle = 0.$$

Alternatively, one can still retain the constraint equation (2) if one supposes that the Schrödinger equation applies also to non-normalized states $|\psi'(t)\rangle = \phi(t)|\psi(t)\rangle$ as $\hat{H}_S|\psi'(t)\rangle = i(\partial/\partial t)|\psi'(t)\rangle$, in analogy to the action of the momentum operator on non-normalized wavefunctions (such as the components of spinors). Both of these approaches are extensions of conventional quantum mechanics, which deals only with states that are normalized at all times.

We stress that, while working on this theoretical framework may be have some appeal, the approach is not fully satisfactory as for instance the choice of $\phi(t)$ is completely arbitrary and there is no indication in the conventional theory on how to fix it. The fact that $\phi(t)$ is non-unique is a consequence of the freedom that one has in quantum mechanics to choose any vector of the Hilbert space as representing a valid state of the system, as long as it does not violate physical or dynamical constraints.

Physical interpretation:— We briefly comment here on the physical interpretation of the additional Hilbert space $\mathcal{H}_T$. One can interpret it as an abstract 'purification' space without physical relevance. However, an operational definition of proper time [27] as "such that is measured by a clock" requires some physical system that acts as a clock. In contrast to the conventional formulation of quantum theory, the above formalism naturally accommodates it: $\mathcal{H}_T$ is the Hilbert space of such system. Clearly the particular form of $\mathcal{H}_T$ employed above is an idealization where the clock is isomorphic to a particle on a line [11]. Other choices [24, 29] are a straightforward modification of the above theory. This approach is consistent with a relational point of view, where the only physically relevant quantities are events defined as coincidences in spacetime [34] such as the correlations between observables and what is shown on a local clock (e.g. [23], Sec. 2.3).

Is the above physical definition of proper time sufficient to identify time, i.e. coordinate time? It is for Newtonian mechanics (coordinate time = proper time) and for special relativity (coordinate time = proper time of a static inertial observer). In general relativity any observer can identify the coordinate time from its own proper time if the metric is known and considered as a classical degree of freedom [28, 35], even though the coordinate time has no physical meaning [34] and it is impossible to synchronize local clocks meaningfully (i.e. so that two clocks synchronized to a master clock are synchronized among themselves) [27]. The case in which the metric is considered as a quantum degree of freedom is currently an open problem and clearly beyond the scope of the present work.

When one considers time as a dynamical variable, an apparent contradiction arises ([36], sec. 8-6): if one interprets momentum as the generator of space translations and the Hamiltonian as the generator of time translations, then one would expect that the Hamiltonian always commutes with the momentum, since these two translations are independent. Why is this untrue in general? In the conventional formalism, time is not a dynamical variable, so the unitary transformations generated by the Hamiltonian are not symmetries of the system. In contrast, in the PaW formalism, time *is* a dynamical variable, but the generator of its translations is $\hat{\Omega}$, not the system Hamiltonian $\hat{H}_S$, and $\hat{\Omega}$ indeed commutes with the system momentum (it acts on a different Hilbert space). The above apparent contradiction is thus resolved in a different manner.

B. Measurements

At variance with what is typically believed (e.g. [7, 16]), the PaW formalism, appears to be particularly well suited to describe in a compact form the statistics of measurements which are performed sequentially on a system of interest.

To show this explicitly let us first analyze the case where a measurement is performed at time t_1 on the system Q. We begin adopting the von Neumann formulation of a measurement apparatus [26], describing the process in terms of a memory system M that is in a fiducial state "ready" $|r\rangle_M$ before the measurement and which will be in a state $|a\rangle_M$ that contains the measurement outcome after. In other words we describe the measurement as an instantaneous transformations which at time t_1 induces

the following unitary mapping:

$$|\psi(t_1)\rangle_Q \otimes |r\rangle_M \longrightarrow \sum_a \hat{K}_a|\psi(t_1)\rangle_Q \otimes |a\rangle_M , \quad (28)$$

where $\{\hat{K}_a\}$ are Kraus operators fulfilling the normalization condition $\sum_a \hat{K}_a^\dagger \hat{K}_a = \hat{\mathbb{1}}$. Projective nondegenerate von Neumann measurements are the special case in which $\hat{K}_a = |a\rangle\langle a|$ are projectors on the eigenspaces relative to the eigenstates $|a\rangle$ of the observable. In this specific case, Eq. (28) becomes [26]

$$|\psi(t_1)\rangle_Q \otimes |r\rangle_M \longrightarrow \sum_a \psi_a(t_1)|a\rangle_Q \otimes |a\rangle_M , \quad (29)$$

with $\psi_a(t_1) := \langle a|\psi(t_1)\rangle$. Accordingly, the probability of getting the outcome a is given by

$$P(a|t_1) := \|\hat{K}_a|\psi(t_1)\rangle_Q\|^2 , \quad (30)$$

with $\||v\rangle\| = \sqrt{\langle v|v\rangle}$ the norm of the vector $|v\rangle$, and

$$|\phi_a\rangle_Q := \hat{K}_a|\psi(t_1)\rangle_Q/\sqrt{P(a|t_1)} , \quad (31)$$

is the vector which describes the state of the system Q immediately after such event has been recorded by the memory M. In the general setting, Eq. (28) defines the statistical properties of a Positive Operator Valued Measure (POVM), see e.g. [37].

The process described above can now be cast in the PaW formalism by redefining S to include both the system to be measured Q and the ancillary memory system M. In this context we shall assume no interactions between Q and M apart from a strong (impulsive) coupling between Q and M at time t_1 that is responsible for the mapping (28). Adopting the time-dependent description (18) we write

$$\hat{H}_S(t) = \hat{H}_Q(t) + \delta(t - t_1)\hat{h}_{QM} , \quad (32)$$

where $\hat{H}_Q(t)$ is the (possibly time-depedent) free Hamiltonian of Q, where $\delta(x)$ is the Dirac delta function, while $\hat{h}_{QM}$ is related to the unitary $\hat{V}_{QM}$ responsible of the mapping Eq. (28) via the identity $\hat{V}_{QM} := e^{-\frac{i}{\hbar}\hat{h}_{QM}}$ (since M is a memory we assume no free dynamics for it). With this choice

$$\hat{U}_S(t, t_0) = \begin{cases} \hat{U}_Q(t, t_0) & \forall t < t_1 , \\ \hat{U}_Q(t, t_1)\hat{V}_{QM}\hat{U}_Q(t_1, t_0) & \forall t > t_1 , \end{cases} \quad (33)$$

where $\hat{U}_Q(t, t')$ is the operator which gives the free evolution of Q defined as in Eq. (18) through the Hamiltonian $\hat{H}_Q(t)$ [38]. Accordingly, Eq. (19) becomes

$$|\Psi\rangle\rangle = \int_{-\infty}^{t_1} dt\, |t\rangle_T \otimes |\psi(t)\rangle_Q \otimes |r\rangle_M$$
$$+ \int_{t_1}^{\infty} dt\, |t\rangle_T \otimes \sum_a \hat{U}_Q(t, t_1)\hat{K}_a|\psi(t_1)\rangle_Q \otimes |a\rangle_M , \quad (34)$$

where for $t < t_1$, $|\psi(t)\rangle_Q = \hat{U}_Q(t, t_0)|\psi(t_0)\rangle_Q$ is the state of Q at time t prior of the measurement stage. In this framework the probability that, at a given time t measured by the ancillary system T, a certain outcome a will be registered by the memory M can be formally expressed as [38]

$$P(a|t) = \|(_T\langle t| \otimes {_M}\langle a|)|\Psi\rangle\rangle\|^2. \quad (35)$$

As a consequence of the impulsive coupling we have assumed in describing the measurement process, Eq. (35) is a step function which exhibits a sharp transition at the measurement time $t = t_1$: for smaller values of t, the probability of getting a certain outcome a on M does not depend upon Q yielding $P(a|t) = |_M\langle a|r\rangle_M|^2$, the resulting statistics being only associated with the the ready state of the memory; for $t > t_1$ instead, $P(a|t)$ coincides with the value (30): it only depends upon the statistical uncertainty of the state of the system Q at time t_1 and it remains constant in time due to the fact that we have explicitly suppressed any dynamical evolution on M.

The above framework immediately extends to the case where different measurements are performed at different times, giving the correct transition probabilities. This was lacking [7] in the PaW proposal. In fact, the global state of a system where a measurement of A at time t_1 and of B at a later time $t_2 > t_1$ is performed can be expressed within the formalism by adding an extra memory element M' which stores the information associated with the second measurement. Accordingly, we replace Eq. (32) with

$$\hat{H}_S(t) = \hat{H}_Q(t) + \delta(t - t_1)\hat{h}_{QM} + \delta(t - t_2)\hat{h}_{QM'} , \quad (36)$$

with $\hat{h}_{QM'}$ responsible for the unitary coupling $\hat{V}_{QM'}$ associated with the measurement of B. With this choice for all $t > t_2$, Eq. (33) gets replaced by $\hat{U}_Q(t, t_2)\hat{V}_{QM'}\hat{U}_Q(t_2, t_1)\hat{V}_{QM}\hat{U}_Q(t_1, t_0)$ while the state $|\Psi\rangle\rangle$ becomes

$$|\Psi\rangle\rangle = \int_{-\infty}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{\infty} dt\, |t\rangle_T \otimes \quad (37)$$
$$\sum_{ab} \hat{U}_Q(t, t_2)\hat{K}_b\hat{U}_Q(t_2, t_1)\hat{K}_a|\psi(t_1)\rangle_Q \otimes |a\rangle_M \otimes |b\rangle_{M'} ,$$

where the first two integrals have the same integrands as the left-hand-side of (34), M' is the memory where the B outcome is stored, b and $\hat{K}_b$ the corresponding outcomes and Kraus operators. It is worth observing that the probability $P(a|t)$ of getting an outcome a at time t is not affected by the presence of the second measurement: this quantity can still be computed by projecting $|\Psi\rangle\rangle$ on $|t\rangle_T \otimes |a\rangle_M$ and assumes the same value given in Eq. (35). Similarly the joint probability that at time t the two memories will record a certain outcome a and b, respectively, can be computed as

$$P(b, a|t) = \|(_T\langle t| \otimes {_M}\langle a| \otimes {_{M'}}\langle b|)|\Psi\rangle\rangle\|^2 . \quad (38)$$

As in the case of Eq. (35) also this is a step function. In particular for $t \geq t_2$ it assumes the value

$$P(b,a|t) = \|\hat{K}_b \hat{U}_Q(t_2,t_1)\hat{K}_a|\psi(t_1)\rangle_Q\|^2$$
$$= \|\hat{K}_b|\phi_a(t_2,t_1)\rangle_Q\|^2 \, \|\hat{K}_a|\psi(t_1)\rangle_Q\|^2 \, , \quad (39)$$

where in the second line we used Eqs. (30) and (31) and where $|\phi_a(t_2,t_1)\rangle_Q = \hat{U}_Q(t_2,t_1)|\phi_a\rangle_Q$ is the evolved via $H_Q(t)$ of the state $|\phi_a\rangle_Q \propto \hat{K}_a|\psi(t_1)\rangle_Q$ of the system Q when the first measurement yields the outcome a. The quantity $\|\hat{K}_b|\phi_a(t_2,t_1)\rangle_Q\|^2$ is nothing but that the conditional probability $P\big[(b|t)\big|(a|t_1)\big]$ of getting the outcome b when measuring B on Q given that the outcome a was registered by the first measurement performed at time t_1. Invoking Eq. (35) we notice that it obeys the identity

$$P\big[(b|t)\big|(a|t_1)\big] = \frac{P(b,a|t)}{P(a|t_1)} \, . \quad (40)$$

This allows us to identify $P(b,a|t)$ with the joint probability $P\big[(b|t),(a|t_1)\big]$ of getting b on M' at time t and a on M at time t_1. In fact we have

$$P\big[(b|t),(a|t_1)\big] = P\big[(b|t)\big|(a|t_1)\big]P(a|t_1) = P(b,a|t) \, , \quad (41)$$

where in writing the first identity we used the Bayes rule. It is worth stressing that Eq. (38) can also be applied for times t prior than t_2. In this case we get $P(b,a|t) = P(a|t)\,|_{M'}\langle b|r\rangle_{M'}|^2$ with $P(a|t)$ as in (35) and with $|_{M'}\langle b|r\rangle_{M'}|^2$ accounting for the statistical distribution of the ready state of M'. Similarly we can extend Eq. (41) for $t \in]t_2,t_1]$ – indeed one can easily verify that in this case $P[(b|t),(a|t_1)] = P(b,a|t) = P(b,a|t_1)$. From the above expressions we can finally compute the probability $P(b|t)$ of getting an outcome b at time $t \geq t_2$, irrespective of the outcome of the A measurement. This is given by the marginal distribution obtained by tracing $P(b,a|t)$ with respect to the a variable, i.e.

$$P(b|t) = \sum_a P(b,a|t) = \|(_T\langle t| \otimes {}_{M'}\langle b|)|\Psi\rangle\rangle\|^2 \, , \quad (42)$$

where the second identity is a consequence of the fact that $\{|a\rangle_M\}$ is a complete set for M. For $t < t_2$ (i.e. prior then the measurement event B) this is just $P(b|t) = |_{M'}\langle b|r\rangle_{M'}|^2$, while for $t > t_2$ (see Eq. (39)) we get

$$P(b|t) = \sum_a \|\hat{K}_b \hat{U}_Q(t_2,t_1)\hat{K}_a|\psi(t_1)\rangle_Q\|^2 \, . \quad (43)$$

Equations (35), (38) and (42) are the main results of this section and are summarized in Table I. More generally, consider the case where Q undergoes to a sequence of measurements $A_1, A_2, \cdots, A_N$ performed at times $t_1, t_2, \cdots, t_N$ which, for convenience, we can assume to be ordered so that $t_{n+1} > t_n$ for all $n = 1, 2, \cdots, N$. We describe this by adding N memory systems $M_1, M_2, \cdots, M_N$, each initialized into a ready

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Joint probability of getting b and a at time t: $P(b,a|t) = \|(_T\langle t| \otimes {}_M\langle a| \otimes {}_{M'}\langle b|)|\Psi\rangle\rangle\|^2$
Probability of getting a at time t: $P(a|t) = \sum_b P(b,a|t) = \|(_T\langle t| \otimes {}_M\langle a|)|\Psi\rangle\rangle\|^2$ Probability of getting b at time t: $P(b|t) = \sum_a P(b,a|t) = \|(_T\langle t| \otimes {}_{M'}\langle b|)|\Psi\rangle\rangle\|^2$
Joint prob. of getting b at time t'' and a at time $t'(<t'')$: $P[(b|t''),(a|t')] = P(b,a|t'') = \|(_T\langle t''| \otimes {}_M\langle a| \otimes {}_{M'}\langle b|)|\Psi\rangle\rangle\|^2$ Cond. prob. of getting b at time t'' given a at time $t'(<t'')$: $P[(b|t'')|(a|t')] = \frac{P(b,a|t'')}{P(a|t')} = \frac{\|(_T\langle t''|\otimes_M\langle a|\otimes_{M'}\langle b|)|\Psi\rangle\rangle\|^2}{\|(_T\langle t'|\otimes_M\langle a|)|\Psi\rangle\rangle\|^2}$

TABLE I. Probability distribution associated with a two measurement events. The first thee identities hold irrespectively from the ordering of the events (i.e. first A and then B, or first B and then A). The last two instead assume a specific ordering, i.e. first A and then B.

state $|r\rangle_{M_n}$ and which couple with Q though the time-dependent Hamiltonian

$$\hat{H}_S(t) = \hat{H}_Q(t) + \sum_{n=1}^N \delta(t - t_n)\hat{h}_{QM_n} \, . \quad (44)$$

Via Eqs. (18) and (19) this defines the physical vector $|\Psi\rangle\rangle$ of the problem. In this context the joint probability that at time t the memory will register a certain string $\vec{a} := (a_1, a_2, \cdots, a_N)$ of outcomes can then be computed as

$$P(\vec{a}|t) = \|(_T\langle t| \otimes \langle \vec{a}|)\,|\Psi\rangle\rangle\|^2 \, , \quad (45)$$

with $|\vec{a}\rangle := |a_1\rangle_{M_1} \otimes |a_2\rangle_{M_2} \otimes \cdots \otimes |a_N\rangle_{M_N}$. Exploiting the Bayes rule argument one can also observe that, given a collection of times $t'_1 < t'_2 < \cdots < t'_N$, Eq. (45) provides the joint probability associated with the events of obtaining the outcome a_n at time t'_n, i.e.

$$P\big[(a_1|t'_1),(a_2|t'_2),\cdots,(a_N|t'_N)\big] = P(\vec{a}|t'_N). \quad (46)$$

Similarly given a subset $M_{j_1}, M_{j_2}, \cdots, M_{j_K}$ formed by $K \leq N$ different memories, the joint probability $P(\vec{a}^{(K)}|t)$ that at time t they will record certain events $\vec{a}^{(K)} := (a_{j_1}, a_{j_2}, \cdots, a_{j_K})$ is obtained by considering the associated marginal of (45), i.e.

$$P(\vec{a}^{(K)}|t) = \sum P(\vec{a}|t) = \|(_T\langle t| \otimes \langle \vec{a}^{(K)}|)\,|\Psi\rangle\rangle\|^2 \, , \quad (47)$$

where in the first identity the sum is performed over all components of $\vec{a}$ which are not involved in the definition

of $\vec{a}^{(K)}$ and where $|\vec{a}^{(K)}\rangle := |a_{j_1}\rangle_{M_{j_1}} \otimes |a_{j_2}\rangle_{M_{j_2}} \otimes \cdots \otimes |a_{j_K}\rangle_{M_{j_K}}$.

C. Overcoming criticisms

Here we give an overview of the main criticisms to the PaW mechanism and to the conditional probability interpretation and show how our proposal overcomes them.

There are two main criticisms to the PaW mechanism [7, 12, 21]. The first refers to superselection [39–41]: the observables of a theory must commute with the theory's constraints. Whenever one of the constraints is the total energy, such as in canonical general relativity, then all observables must be stationary as they commute with the Hamiltonian. In the Schrödinger picture this translates into static physical states, which contrasts with obvious experimental evidence and is the root of the problem of time [7, 13, 14]. The second refers to the fact that the PaW mechanism is not able to provide the correct propagators, or the correct two-time correlations [7]: after the first time measurement, the clock remains "stuck". We have already shown how these criticisms can be overcome: the first is solved by using a global state $|\Psi\rangle\rangle$ that is independent of time and observing that internal observers will use conditioned states, the second by using conditioning through a von Neumann description of the measurement interaction. In a sense, our prescription fulfills Page's desiderata [16] in showing that the second objection can be overcome by interpreting a measurement at two different times (or, equivalently a preparation followed by a measurement) as a single measurement that acts both on the system and on the degrees of freedom that store the earlier measurement outcome. It is a sort of purification of the time measurements and implements Wheeler's operationalist stance that "the past has no existence except as it is recorded in the present". [16, 42].

Further criticisms were proposed in [12], where it was noted that *(i)* interpretive problems cannot be alleviated incorporating observers into the theory; *(ii)* in a constrained theory where one of the constraints is the energy (such as the Hamiltonian formulation of General Relativity), all observables commute with the Hamiltonian and no time dependence is possible. This is true also for two-time correlation functions and propagators [7]; *(iii)* no dynamical variable can correlate monotonically with 'Heraclitian' time if the Hamiltonian is lower bounded; *(iv)* only time is appropriate for conditioning the state: for example, space may be inappropriate for setting the conditions since a system may occupy the same position multiple times or never.

Our mechanism replies to *(ii)* by indeed carefully incorporating the observers into the theory, thereby overturning *(i)*. In fact, there are two points of view: the external observer (clearly, a hypothetical entity whenever the whole universe is considered) and the internal observer. The Hamiltonian constraint refers to the external observer's point of view, who sees the whole laboratory (or universe) as a static system whose state is an eigenstate of its global Hamiltonian. That, however, does not prevent the internal observer from observing evolving systems, time-dependent measurement outcomes and Born-rule induced wave-function collapses, as shown above. In a sense, the "relativity" philosophy is extended also to quantum mechanics: states and measurements are relative to the observer [43, 44], just as time and space are relative. Indeed, we showed above how internal observers recover the correct two-time correlations and propagators. As regards to objection *(iii)*, indeed if we want to describe a non-periodic time variable that takes all values (the 'Heraclitian' time), we must use an unbounded 'Hamiltonian': if one considers Eq. (2) as a sort of Wheeler-De Witt equation, that Hamiltonian is unbounded (it contains a 'momentum' operator $\hat{\Omega}$). We remark that other choices may lead to 'periodic time' coordinates, but that is acceptable in specific cosmologies: it is certainly not surprising that a system with finite global energy will have periodic evolution. In these cases, except as an approximation internal observers will not be able to use a Schrödinger equation, as predicted in [12]. They must employ a more general dynamical equation. As regards to point *(iv)*, time's role in the conditioning to achieve conventional quantum mechanics is made transparent by our formulation, which can be used to show its identical role to space regarding conditioning. In fact, just as for space, it is possible that a system never occupies a given time, or that it occupies the same time at two different locations if it follows a closed timelike curve, whose existence is predicted by general relativity [45] and studied also in the context of quantum mechanics [46]. So, while indeed only time is the appropriate quantity for conditioning to obtain the conventional theory, quantizing time with our mechanism is a viable pathway to the unconditioned theory.

D. Conclusions

Here we modified the PaW mechanism to give a quantization of time, and showed how the conventional quantum mechanics and the correct quantum predictions (e.g. regarding propagators and measurement statistics) arise from a quantum Bayes rule by conditioning the global state $|\Psi\rangle\rangle$ to a specific time. We emphasize that our approach can quantize time for completely arbitrary quantum systems $|\psi(t)\rangle_S$. As such, we can also provide a description of quantum field theory with a quantum clock.

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