B-SPLINE CURVES

Robby T. Tan

A Lecture Note of 3D Modeling Course, 2011-2012

1 Introduction

1. ”B” in B-spline stands for basis. The mathematical term ”spline” means a piecewise polynomial function.
2. Compared to a Bezier curve, a b-spline curve has a few distinct features:
   (a) The degree of the curve is independent from the total number of the control points
   (b) It is made out of several curve segments that are joined smoothly.
   (c) It locally propagates (while a Bezier curve globally propagates)

2 Definition

A b-spline curve is mathematically defined as:

$$\mathbf{Q}(u) = \sum_{j=0}^{n} \mathbf{P}_j B_{j,d}(u) \tag{1}$$

$$t_{d-1} \leq u \leq t_{n+1} \tag{2}$$

where P_j is a control point. j is the index of the control points. $n+1$ is the number of the control points. d is the number of the control points that control a segment. Implying, $d-1$ is the degree of the polynomial, for example: $d=2$ is linear, $d=3$ is quadratic, $d=4$ is cubic, etc. The value of n must be larger or equal to d. t_j is a knot value. A few knot values form a knot vector $\mathbf{t}$.

2.1 B-spline's Basis Functions

There are two mathematical definitions of b-spline's basis functions: when $d=1$, and when $d>1$. We will immediatealy see that these two definitions depend on the knot vector $\mathbf{t}$, where $\mathbf{t} = (t_0, t_1, t_2, ...)$, which will be explained in the subsequent section.

1. **when** $d = 1$

$$B_{j,1}(u) = \begin{cases} 1 & \textbf{if } t_j \leq u < t_{j+1} \\ 0 & \textbf{otherwise} \end{cases} \tag{3}$$

2. **when** $d > 1$

$$B_{j,d}(u) = \left(\frac{u - t_j}{t_{j+d-1} - t_j}\right) B_{j,d-1}(u) + \left(\frac{t_{j+d} - u}{t_{j+d} - t_{j+1}}\right) B_{j+1,d-1}(u) \tag{4}$$

2.2 Knot Vector

The knot vector, as can be observed in the last two equations, affects the values of the basis functions. Moreover, different ways to build the vector create two different types of b-spline: **uniform** and **non-uniform**. The number of the knot values (or the size of the knot vector) is determine by:

$$m = n + d + 1 \tag{5}$$

Hence, if $n = 2$ and $d = 2$ then $m = 5$, $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4)$.

1. **The knot vector for a uniform b-spline curve** requires the following condition:

$$t_{j+1} = t_j + k \tag{6}$$

   where k is any constant scalar value.

   Example: $n = 2, d = 2, k = 1 \rightarrow \mathbf{t} = (t_0, t_1, t_2, t_3, t_4) = (0, 1, 2, 3, 4)$.

2. **The knot vector for a non-uniform b-spline curve** requires the following general condition:

$$t_{j+1} \geq t_j \tag{7}$$

   One of the forms of a non-uniform knot vector is as follows:

$$t_j = 0 \quad \text{if} \quad j < d \tag{8}$$
$$t_j = j - d + 1 \quad \text{if} \quad d \leq j \leq n \tag{9}$$
$$t_j = n - d + 2 \quad \text{if} \quad j > n \tag{10}$$

   Example: $n = 2, d = 2$ produces $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4) = (0, 0, 1, 2, 2)$.

3 Linear Uniform B-spline Curves

In b-spline, linear means that $d = 2$. Suppose we have 3 control points, implying $n = 2$. Hence, we can write the linear uniform b-spline curves as:

$$\mathbf{Q}(u) = \sum_{j=0}^{2} \mathbf{P}_j B_{j,2}(u) \tag{11}$$
$$= \mathbf{P}_0 B_{0,2}(u) + \mathbf{P}_1 B_{1,2}(u) + \mathbf{P}_2 B_{2,2}(u) \tag{12}$$

To be able to do the computation of the curves, we have to do the following steps:

1. Compute the knot vector, $\mathbf{t}$.
2. Compute $B_{j,d}(u)$, namely: $B_{0,2}$, $B_{1,2}$, $B_{2,2}$.
3. Compute $\mathbf{Q}(u)$ from $\mathbf{P}_j$'s and the all computed $B_{j,d}(u)$.

3.1 Computing the knot vector

By considering that $n = 2$ and $d = 2$, we can have $m = n + d + 1 = 5$, which means $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4)$. Since it is a uniform b-spline curve and we can assume $k = 1$, then:

$$\mathbf{t} = (t_0, t_1, t_2, t_3, t_4) = (0, 1, 2, 3, 4) \tag{13}$$

3.2 Computing $B_{j,d}(u)$

Considering the basis functions' definition in Eq.(4) and the knot vector in Eq.(13) we can have the definition of the uniform basis functions in the case of $d = 2$ and $n = 2$ as:

1. $B_{0,2}$:

$$
\begin{aligned}
B_{0,2}(u) &= \frac{u - t_0}{t_1 - t_0} B_{0,1}(u) + \frac{t_2 - u}{t_2 - t_1} B_{1,1}(u) && (14) \\
&= u B_{0,1}(u) + (2 - u) B_{1,1}(u) && (15)
\end{aligned}
$$

2. $B_{1,2}$:

$$
\begin{aligned}
B_{1,2}(u) &= \frac{u - t_1}{t_2 - t_1} B_{1,1}(u) + \frac{t_3 - u}{t_3 - t_2} B_{2,1}(u) && (16) \\
&= (u - 1) B_{1,1}(u) + (3 - u) B_{2,1}(u) && (17)
\end{aligned}
$$

3. $B_{2,2}$:

$$
\begin{aligned}
B_{2,2}(u) &= \frac{u - t_2}{t_3 - t_2} B_{2,1}(u) + \frac{t_4 - u}{t_4 - t_3} B_{3,1}(u) && (18) \\
&= (u - 2) B_{2,1}(u) + (4 - u) B_{3,1}(u) && (19)
\end{aligned}
$$

As can be seen in Eq.(15),(17), and (19), to have a complete definition of the basis functions, we need the definition of $B_{0,1}(u), B_{1,1}(u), B_{2,1}(u)$, and $B_{3,1}(u)$. Hence, we have to evaluate them, which are based on Eq.(3):

1. $B_{0,1}$:

$$
B_{0,1}(u) = \begin{cases} 1 & \text{if } t_0 \leq u < t_1 \text{, implying } 0 \leq u < 1 \\ 0 & \text{otherwise} \end{cases} \tag{20}
$$

2. $B_{1,1}$:

$$
B_{1,1}(u) = \begin{cases} 1 & \text{if } t_1 \leq u < t_2 \text{, implying } 1 \leq u < 2 \\ 0 & \text{otherwise} \end{cases} \tag{21}
$$

3. $B_{2,1}$:

$$
B_{2,1}(u) = \begin{cases} 1 & \text{if } t_2 \leq u < t_3 \text{, implying } 2 \leq u < 3 \\ 0 & \text{otherwise} \end{cases} \tag{22}
$$

4. $B_{3,1}$:

$$
B_{3,1}(u) = \begin{cases} 1 & \text{if } t_3 \leq u < t_4 \text{, implying } 3 \leq u < 4 \\ 0 & \text{otherwise} \end{cases} \tag{23}
$$

Having obtained the definition of $B_{0,1}(u), B_{1,1}(u), B_{2,1}(u)$, and $B_{3,1}(u)$, we can write the complete definition of $B_{0,2}$, $B_{1,2}$, $B_{2,2}$ as:

1. $B_{0,2}$:

$$
B_{0,2}(u) = u 1_{(0 \leq u < 1)} + (2 - u) 1_{(1 \leq u < 2)} \tag{24}
$$

2. $B_{1,2}$:

$$B_{1,2}(u) = (u-1)1_{(1 \leq u < 2)} + (3-u)1_{(2 \leq u < 3)} \tag{25}$$

3. $B_{2,2}$:

$$B_{2,2}(u) = (u-2)1_{(2 \leq u < 3)} + (4-u)1_{(3 \leq u < 4)} \tag{26}$$

If we plot the function of $B_{0,2}$ in a 2D space with $x-axis$ representing u and $y-axis$ representing $B_{0,2}(u)$, then we can obtain:

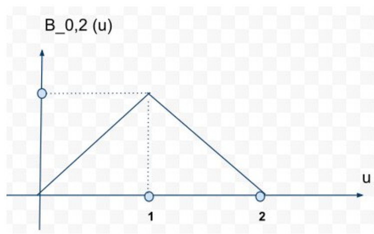

Figure 1: *The plot of the basis function* $B_{0,2}(u)$

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If we plot all basis functions: $B_{0,2}(u)$, $B_{1,2}(u)$,and $B_{2,2}(u)$, we will obtain:

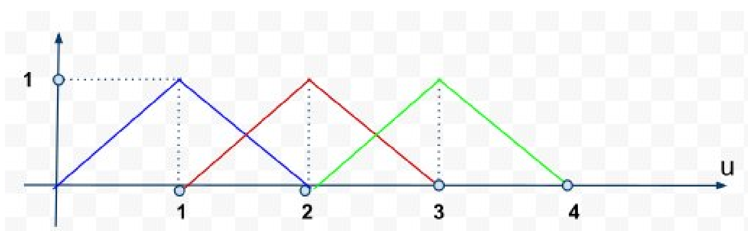


Figure 2: *The plot of all basis functions* $B_{0,2}(u)$*,* $B_{1,2}(u)$ *and* $B_{2,2}(u)$*, which are represented by blue, red, and green lines, respectively. This is the reason why this b-spline is called uniform.*

3.3 Computing $\mathbf{Q}(u)$

Given $\mathbf{P}_j$'s and the all computed $B_{j,d}(u)$, the last step of computing the curve is to obtain the value of $\mathbf{Q}(u)$:

$$\begin{aligned}
\mathbf{Q}(u) &= \sum_{j=0}^{2} \mathbf{P}_j B_{j,2}(u) && (27)\\
&= \mathbf{P}_0 B_{0,2}(u) + \mathbf{P}_1 B_{1,2}(u) + \mathbf{P}_2 B_{2,2}(u) && (28)\\
&= \mathbf{P}_0\big(u1_{(0 \leq u < 1)} + (2-u)1_{(1 \leq u < 2)}\big) + \mathbf{P}_1\big((u-1)1_{(1 \leq u < 2)} + (3-u)1_{(2 \leq u < 3)}\big) + && (29)\\
&\quad \mathbf{P}_2\big((u-2)1_{(2 \leq u < 3)} + (4-u)1_{(3 \leq u < 4)}\big)
\end{aligned}$$

According to Eq.(2) the range of u is:

$$\begin{aligned}
t_{d-1} &\leq u \leq t_{n+1} && (30)\\
t_1 &\leq u \leq t_3 && (31)
\end{aligned}$$

Alternatively we can write Eq.(29) in a better way:

- $0 \leq u < 1$: $\quad \mathbf{Q}(u) = \mathbf{P}_0 u$
- $1 \leq u < 2$: $\quad \mathbf{Q}(u) = \mathbf{P}_0(2-u) + \mathbf{P}_1(u-1)$
- $2 \leq u < 3$: $\quad \mathbf{Q}(u) = \mathbf{P}_1(3-u) + \mathbf{P}_2(u-2)$
- $3 \leq u < 4$: $\quad \mathbf{Q}(u) = \mathbf{P}_2(4-u)$

Note:

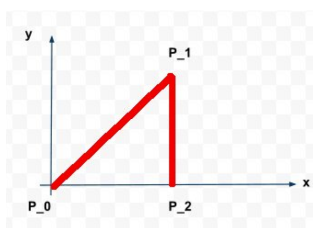
1. Aside from the identical shape for all basis functions as shown in Fig.(2), "uniform" also means that each control point affects the same number of segments. For instance, $\mathbf{P}_0$ affects two segments: segment $0 \leq u < 1$ and segment $1 \leq u < 2$. Like $\mathbf{P}_0$, both $\mathbf{P}_1$ and $\mathbf{P}_2$ also affect two segments.

3.4 Linear Uniform B-spline: Example

Given $\mathbf{P}_0 = (0,0)$, $\mathbf{P}_1 = (1,1)$ and $\mathbf{P}_2 = (1,0)$, the curve can be generated by computing:

- $0 \leq u < 1$: $\quad \mathbf{Q}(u) = \mathbf{P}_0 u = \begin{pmatrix} 0 \\ 0 \end{pmatrix} u$
- $1 \leq u < 2$: $\quad \mathbf{Q}(u) = \mathbf{P}_0(2-u) + \mathbf{P}_1(u-1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}(2-u) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(u-1)$
- $2 \leq u < 3$: $\quad \mathbf{Q}(u) = \mathbf{P}_1(3-u) + \mathbf{P}_2(u-2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(3-u) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(u-2)$
- $3 \leq u < 4$: $\quad \mathbf{Q}(u) = \mathbf{P}_2(4-u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(4-u)$

Recall that the range of u as defined in Eq.(31) is $t_1 \leq u \leq t_3$, implying $1 \leq u \leq 3$. Hence, the curve, $\mathbf{Q}(u)$, will look like:



4 Linear Non-uniform B-spline Curves

Similar to the linear uniform b-spline curves explained in Sect. 3, and assuming $n = 2$ (hence, 3 control points), we can write the non-uniform b-spline equation for $d = 2$:

$$\begin{aligned} \mathbf{Q}(u) &= \sum_{j=0}^{2} \mathbf{P}_j B_{j,2}(u) && (32) \\ &= \mathbf{P}_0 B_{0,2}(u) + \mathbf{P}_1 B_{1,2}(u) + \mathbf{P}_2 B_{2,2}(u) && (33) \end{aligned}$$

To have the curve, we have to do the following steps:

1. Compute the knot vector, $\mathbf{t}$.
2. Compute $B_{j,d}(u)$, namely: $B_{0,2}$, $B_{1,2}$, $B_{2,2}$.
3. Compute $\mathbf{Q}(u)$ given $\mathbf{P}_j$'s and based on the all computed $B_{j,d}(u)$.

4.1 Computing the knot vector

By considering that $n = 2$ and $d = 2$, we can have $m = n + d + 1 = 5$, which means $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4)$. For a non-uniform b-spline curve as explained in Sect. 2.2, we can follow the following rules:

$$t_j = 0 \quad \text{if} \quad j < d \tag{34}$$
$$t_j = j - d + 1 \quad \text{if} \quad d \leq j \leq n \tag{35}$$
$$t_j = n - d + 2 \quad \text{if} \quad j > n \tag{36}$$

Based on the rules, for $n = 2, d = 2$, we can obtain $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4) = (0, 0, 1, 2, 2)$.

4.2 Computing $B_{j,d}(u)$

Considering the basis functions' definition in Eq.(4) and the knot vector $\mathbf{t} = (t_0, t_1, t_2, t_3, t_4) = (0, 0, 1, 2, 2)$, we can have the definition of the uniform basis functions in the case of $d = 2$ and $n = 2$ as written below. Note that in computing the basis functions we have to define: $\left(\frac{u}{0} = 0\right)$, or anything divided by zero equals to zero.

1. $B_{0,2}$:

$$\begin{aligned} B_{0,2}(u) &= \frac{u - t_0}{t_1 - t_0} B_{0,1}(u) + \frac{t_2 - u}{t_2 - t_1} B_{1,1}(u) &(37)\\ &= \frac{u}{0} B_{0,1}(u) + \frac{(1-u)}{1} B_{1,1}(u) &(38)\\ &= (1-u) B_{1,1}(u) &(39)\end{aligned}$$

2. $B_{1,2}$:

$$\begin{aligned} B_{1,2}(u) &= \frac{u - t_1}{t_2 - t_1} B_{1,1}(u) + \frac{t_3 - u}{t_3 - t_2} B_{2,1}(u) &(40)\\ &= (u-1) B_{1,1}(u) + (2-u) B_{2,1}(u) &(41)\end{aligned}$$

3. $B_{2,2}$:

$$\begin{aligned} B_{2,2}(u) &= \frac{u - t_2}{t_3 - t_2} B_{2,1}(u) + \frac{t_4 - u}{t_4 - t_3} B_{3,1}(u) &(42)\\ &= (u-1) B_{2,1}(u) + \frac{(2-u)}{0} B_{3,1}(u) &(43)\\ &= (u-1) B_{2,1}(u) &(44)\end{aligned}$$

Unlike the uniform counterpart, the above equations require only the definition of $B_{1,1}, B_{2,1}$:

1. $B_{1,1}$:

$$B_{1,1}(u) = \begin{cases} 1 & \text{if } t_1 \leq u < t_2, \text{ implying } 0 \leq u < 1 \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

2. $B_{2,1}$:

$$B_{2,1}(u) = \begin{cases} 1 & \text{if } t_2 \leq u < t_3, \text{ implying } 1 \leq u < 2 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

Having obtained the definition of $B_{1,1}(u)$ and $B_{2,1}(u)$, we can write the complete definition of $B_{0,2}$, $B_{1,2}$, $B_{2,2}$ as:

1. $B_{0,2}$:

$$B_{0,2}(u) = (1-u)1_{(0 \leq u < 1)} \tag{47}$$

2. $B_{1,2}$:

$$B_{1,2}(u) = u1_{(0 \leq u < 1)} + (2-u)1_{(1 \leq u < 2)} \tag{48}$$

3. $B_{2,2}$:

$$B_{2,2}(u) = (u-1)1_{(1 \leq u < 2)} \tag{49}$$

If we plot all basis functions: $B_{0,2}(u)$, $B_{1,2}(u)$,and $B_{2,2}(u)$, we will obtain:

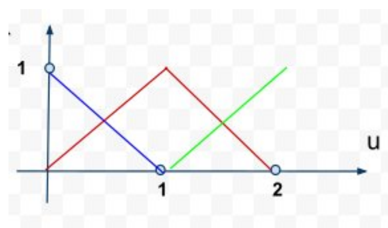


Figure 3: *The plot of all basis function $B_{0,2}(u)$, $B_{1,2}(u)$ and $B_{2,2}(u)$. This is the reason why this b-spline is called non-uniform.*

4.3 Computing $\mathbf{Q}(u)$

Given $\mathbf{P}_j$'s and the all computed $B_{j,d}(u)$, the last step of computing the curve is to obtain the value of $\mathbf{Q}(u)$:

$$\begin{aligned} \mathbf{Q}(u) &= \sum_{j=0}^{2} \mathbf{P}_j B_{j,2}(u) && (50) \\ &= \mathbf{P}_0 B_{0,2}(u) + \mathbf{P}_1 B_{1,2}(u) + \mathbf{P}_2 B_{2,2}(u) && (51) \\ &= \mathbf{P}_0\big((1-u)1_{(0 \leq u < 1)}\big) + \mathbf{P}_1\big(u1_{(0 \leq u < 1)} + (2-u)1_{(1 \leq u < 2)}\big) + && (52) \\ &\quad \mathbf{P}_2\big((u-1)1_{(1 \leq u < 2)}\big) \end{aligned}$$

Alternatively we can write the last equation in a clearer way:

- $0 \leq u < 1$: $\qquad \mathbf{Q}(u) = \mathbf{P}_0(1-u) + \mathbf{P}_1 u$

- $1 \leq u < 2$: $\mathbf{Q}(u) = \mathbf{P}_1(2-u) + \mathbf{P}_2(u-1)$

Notes:

1. Aside from the different shapes of the basis functions as shown in Fig.(3), ”non-uniform” also means that each control point affects a different number of segments. For instance, $\mathbf{P}_0$ affects only one segments: segment $0 \leq u < 1$, and $\mathbf{P}_1$ affects two segments: segment $0 \leq u < 1$ and segment $1 \leq u < 2$.
2. The latest point ($\mathbf{P}_2$) does not give any influence to the first segment (segment $0 \leq u < 1$).