CSC 212: Data Structures and Abstractions

Merge Sort

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Divide and Conquer

- **Divide** the problem into <u>smaller</u> subproblems

- **Conquer** recursively
  - … each subproblem

- **Combine** Solutions

Example

10	2	3	7	4	13	11	9

- sorting with insertion sort is $\Theta(n^2)$
- we can divide the array into two halves and sort them separately

2	3	7	10

4	9	11	13

- each subproblem could be sorted in $\approx n^2/4$
- sorting both halves will require $\approx 2(n^2/4)$ 🤔
- we need an additional operation to combine both solutions

Time "reduced" from $\approx n^2$ to $\approx n^2/2 + n$

Merge Sort

- **Divide** the array into **two** halves
  - just need to calculate the mid point
- Conquer **Recursively** each half
  - call Merge Sort on each half (i.e. solve 2 smaller problems)
- **Merge** Solutions
  - after both calls are finished, proceed to **merge** the solutions

Divide and Conquer

0	1	2	3	4	5	6	7
4	2	1	5	6	3	7	8

Merge Sort: pseudocode

```
if (hi <= lo) return;

int mid = lo + (hi - lo) / 2;

mergesort(A, lo, mid);
mergesort(A, mid+1, hi);

merge(A, lo, mid, hi);
```

Merge Sort

```
void r_mergesort(int *A, int *aux, int lo, int hi) {
    // base case (single element or empty list)
    if (hi <= lo) return;
    // divide
    int mid = lo + (hi - lo) / 2;
    // recursively sort halves
    r_mergesort(A, aux, lo, mid);
    r_mergesort(A, aux, mid+1, hi);
    // merge results
    merge(A, aux, lo, mid, hi);
}
```

```
void mergesort(int *A, int n) {
    int *aux = new int[n];
    r_mergesort(A, aux, 0, n-1);
    delete [] aux;
}
```

Merging two sorted arrays

1	2	5	10	15

i

3	6	9	16	20

j

1	2	3	5	6	9	10	15	16	20

A secondary array is necessary to guarantee a **linear time** operation

Merge

```
void merge(int *A, int *aux, int lo, int mid, int hi) {
    // copy array
    std::memcpy(aux+lo, A+lo, (hi-lo+1) * sizeof(int));
    // merge
    int i = lo, j = mid + 1;
    for (int k = lo ; k <= hi ; k ++) {
        if (i > mid) A[k] = aux[j++];
        else if (j > hi) A[k] = aux[i++];
        else if (aux[j] < aux[i]) A[k] = aux[j++];
        else A[k] = aux[i++];
    }
}
```

Analysis (recurrence)

```
if (hi <= lo) return;

int mid = lo + (hi - lo) / 2;

mergesort(A, lo, mid);
mergesort(A, mid+1, hi);

merge(A, lo, mid, hi);
```

Worst case?

Average case?

Best case?

Recursion Tree (trace)

```
void mergesort(int *A, int n) {
    int *aux = new int[n];
    r_mergesort(A, aux, 0, n-1);
    delete [] aux;
}
```

```
void r_mergesort(int *A, int *aux, int lo, int hi) {
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    r_mergesort(A, aux, lo, mid);
    r_mergesort(A, aux, mid+1, hi);
    merge(A, aux, lo, mid, hi);
}
```

Animation

https://www.toptal.com/developers/sorting-algorithms/merge-sort

Comments on Merge Sort

- Major disadvantage
  - it is not **in-place**
  - in-place algorithm exists but it is complex and inefficient
- Improvements
  - use insertion sort for small arrays
    - avoid overhead on small instances (~10 elements)
  - stop if already sorted
    - avoids unnecessary merge
    - works well with partially sorted arrays

In-place Sorting

Example

- Think about reversing an array or string

  - **solution 1**: use an additional array of equal size
    - what is the required extra memory?

  - **solution 2**: exchange first and last and work recursively on the inner part
    - can do it iteratively as well
    - what is the required extra memory?

In-place sorting

- A sorting algorithm is **in-place** if it uses **O(log n)** extra memory

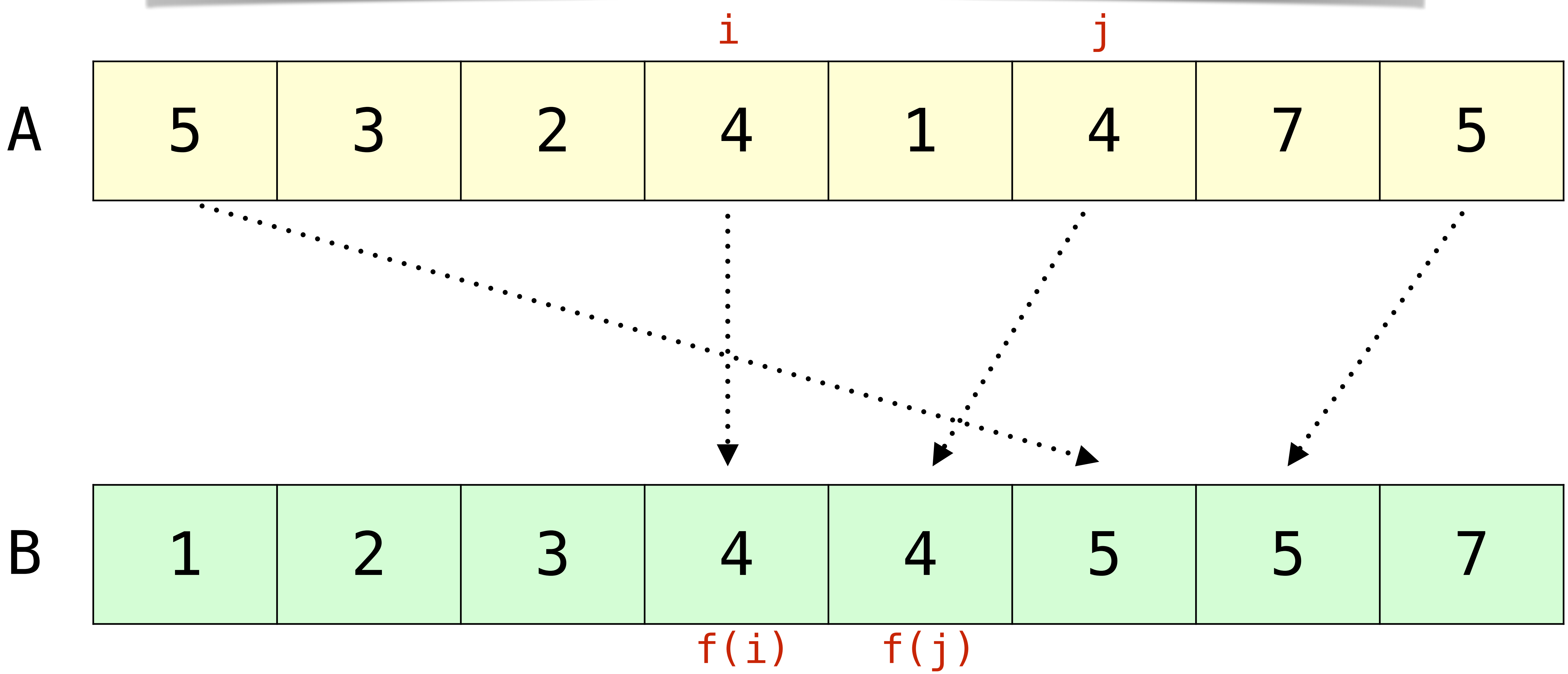
- Are selection and insertion sorts **in-place?**

Stable Sorting

Stability

- A sorting algorithm is **stable** if it preserves the order of **equal** elements

Consider sorting (in ascending order) a list **A** into a sorted list **B**. Let **f(i)** be the index of element **A[i]** in **B**. The sorting algorithm is stable if for any pair **(i,j)** such that **A[i] = A[j]** and **i < j**, then **f(i) < f(j)**.



DL 2273	Detroit	5:30 am	Departed
WN 6240	Chicago - MDW	5:55 am	Departed
AA 489	Philadelphia	6:00 am	Departed
DL 1263	Atlanta	6:00 am	Departed
UA 6208	Washington - IAD	6:00 am	Departed
WN 1138	Baltimore	6:05 am	Departed
AA 5202	Washington - DCA	6:14 am	Departed
B6 475	Orlando	6:15 am	Departed
UA 4894	New York/Newark	6:15 am	Departed
AA 1703	Charlotte	6:17 am	Departed
WN 28	Orlando	6:55 am	Departed
AA 3410	Chicago - ORD	7:02 am	Departed
WN 6235	Tampa	7:05 am	Departed
UA 3615	Chicago - ORD	7:30 am	Departed
AA 1735	Philadelphia	8:02 am	Departed
AA 632	Charlotte	8:07 am	At 9:45 am
WN 6247	Fort Lauderdale	8:30 am	Departed
WN 2640	Washington - DCA	8:45 am	Departed
WN 3420	Chicago - MDW	8:45 am	Departed
AA 4280	Washington - DCA	8:49 am	At 10:20 am
WN 846	Baltimore	9:20 am	Departed
DL 305	Detroit	10:40 am	On time
AA 774	Philadelphia	10:51 am	On time
AA 1981	Charlotte	11:01 am	On time
WN 3020	Baltimore	11:20 am	On time
AA 5524	Washington - DCA	11:46 am	At 2:35 pm
AC 7379	Toronto	11:50 am	On time
AA 5550	Charlotte	11:54 am	On time
DL 5090	Detroit	12:32 pm	On time
WN 6296	Baltimore	12:35 pm	On time
DL 2225	Atlanta	12:48 pm	On time
AA 4424	Washington - DCA	1:38 pm	On time

sort then sort again

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DL 2225	Atlanta	12:48 pm	On time
WN 1138	Baltimore	6:05 am	Departed
WN 846	Baltimore	9:20 am	Departed
WN 3020	Baltimore	11:20 am	On time
WN 6296	Baltimore	12:35 pm	On time
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stable

Stability

- Is **selection sort** stable? ❌
  - long distance swaps
  - try: 5 1 2 4 4 3 2 1

- Is **insertion sort** stable? ✅
  - equal items never pass each other (depends on correct implementation)

Sorting Algorithms

	Best-Case	Average-Case	Worst-Case	Stable?	In-place?
Selection Sort	$\theta(n^2)$	$\theta(n^2)$	$\theta(n^2)$	No	Yes
Insertion Sort	$\theta(n)$	$\theta(n^2)$	$\theta(n^2)$	Yes	Yes
Merge Sort	$\theta(n\log n)$	$\theta(n\log n)$	$\theta(n\log n)$	Yes	No