15.12.2023
Analysis - Lecture 10

Leibniz - Newton Theorem

Consider $a < b \in \mathbb{R}$
$f : [a,b] \to \mathbb{R}$ s.t. • f is Riemann integrable on $[a,b]$
• f has anti-derivatives on $[a,b]$

Then $\forall F : [a,b] \to \mathbb{R}$ an anti-derivative of f, the following holds:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: We prove that

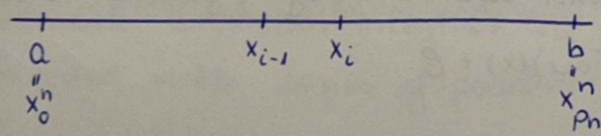
$\forall (\Delta^n) \subseteq \text{Part}[a,b]$ with $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$

$\forall (\xi^n)$ s.t. $\xi^n \in \text{Part}(\Delta^n)$

$$\lim_{n\to\infty} \sigma(f, \Delta^n, \xi^n) = F(b) - F(a)$$

$$\left.\begin{array}{l}\text{Consider } (\Delta^n) \subseteq \text{Part}[a,b] \text{ with } \lim\limits_{n\to\infty} \|\Delta^n\| = 0 \\ (\xi^n) \text{ s.t. } \xi^n \in \text{Part}(\Delta^n),\ \forall n \in \mathbb{N}\end{array}\right\} \text{randomly chosen}$$

$$\sigma(f, \Delta^n, \xi^n) = \sum_{i=1}^{p_n} f(\xi_i^n)(x_i^n - x_{i-1}^n), \quad x_i^n \in \mathbb{R},\ i = \overline{1, p_n}$$



$$\left.\begin{array}{l}F \text{ is continuous on } [x_{i-1}, x_i] \\ \text{differentiable on } (x_{i-1}, x_i)\end{array}\right\} \xrightarrow{\text{Lagrange}} \exists\, c_i^n \in (x_{i-1}^n, x_i^n) \text{ s.t.}$$

$$\text{s.t. } F(x_i^n) - F(x_{i-1}^n) = \overbrace{F'(c_i^n)}^{f(c_i^n)} \cdot (x_i^n - x_{i-1}^n) \quad (*)$$

We may replace the procedure

$i = \overline{1, p_n} \to \exists\, c^n = (c_1^n, c_2^n, \dots, c_{p_n}^n) \in Tp(\Delta^n)$

$$\sigma(f, \Delta^n, c^n) = \sum_{i=1}^{p_n} f(c_i^n)(x_i^n - x_{i-1}^n)$$

$$\overset{(*)}{=} \sum_{i=1}^{p_n} \left(F(x_i^n) - F(x_{i-1}^n)\right) = F(\cancel{x_1^n}) - F(x_0^n) + F(\cancel{x_2^n}) - F(\cancel{x_1^n}) + \dots +$$

$$+ F(x_{p_n}^n) - F(\cancel{x_{p_n-1}^n}) = F(x_{p_n}^n) - F(x_0^n) = F(b) - F(a)$$

$\to \forall n \in \mathbb{N},\ \sigma(f, \Delta^n, c^n) = F(b) - F(a)$

$$\xrightarrow[n\to\infty]{\lim} \sigma(f, \Delta^n, c^n) = F(b) - F(a) \quad (**)$$

f is Riemann integrable on $[a,b] \Leftrightarrow \exists$ a unique $I \in \mathbb{R}$ s.t.

$I = \lim_{n\to\infty} \tau(f, \Delta^n, \xi^n)$ (***) $\forall (\Delta^n) \in Part[a,b]$ with $\lim_{n\to\infty} \|\Delta^n\| = 0$

$\forall (\xi^n)$ with $\xi^n \in Ip(\Delta^n)$

(**) & (x **) $\rightarrow$ $I = F(b) - F(a)$

Remark: we use $F(x)\Big|_a^b = F(b) - F(a)$

Theorem 1 (concerning parts integration)

$f: I \to \mathbb{R}$

I - interval

$f, g: I \to \mathbb{R}$ s.t. • f and g are diff. on I
• f' and g' are cont. on I $\rightarrow$ the functions fg' & $f'g$ have anti-derivatives on I

and $\boxed{\int_a^b f(x)\, g'(x)\, dx = f(x) \cdot g(x)\Big|_a^b - \int_a^b f'(x) \cdot g(x) dx}$

Theorem 2 (the first th. on change of variable)

$I, J \in \mathbb{R}$ - two intervals
$f: J \to \mathbb{R}$
$u: I \to J$

s.t. u - diff on I
f - has anti-derivatives on J

a) Then $(f \circ u) \cdot u'$ has anti-derivatives on I

b) If $F: J \to \mathbb{R}$ is an anti-derivative of $(f \circ u) \cdot u'$ then

$\int (f \circ u)(x) \cdot u'(x)\, dx = (F \circ u)(x) + C$

Example:

$\int \sin(x^2) \cdot 2x\, dx = \int \sin(x^2)\, (x^2)'\, dx = -\cos(x^2) + C$

$f(t) = \sin t$ $\quad g(x) = x^2$ $\quad F(t) = -\cos t$

Theorem 3 (the second theorem on change of variable)

a) $I, J \subseteq \mathbb{R}$ intervals

$f: I \to \mathbb{R}$
$u: J \to I$

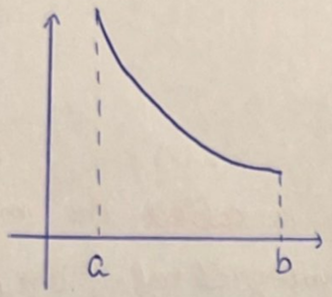
s.t. • u is bijective
• u is diff on J and $u'(x) \neq 0$, $\forall x \in J$
• $(f \circ u) \cdot u'$ has anti-derivatives on J $\rightarrow$

$\rightarrow$ f has anti-derivatives on J

b) If $H: J \to \mathbb{R}$ is an anti-derivative of $(f \circ u) \cdot u'$ then $H \circ u^{-1}$ is an anti-derivative of f on I and $\int f(x) dx = (H \circ u^{-1})(x) + C$

! Th. 2 & 3 must be applied in exercices (don't memorise them)!

Improper Integrals



$f : [a,b] \to \mathbb{R}$

f is locally Riemann integrable on $[a,b]$

? $\lim\limits_{\substack{t \to a \\ t > a}} \int_t^b f(x)\,dx$

We study improper integrals on set such as:

I. $(a,b]$ where $-\infty \le a < b < \infty$ $\quad \int_{a_+}^{b} f(x)\,dx$

II. $[a,b)$ where $-\infty \le a < b \le \infty$ $\quad \int_{a}^{b_-} f(x)\,dx$

III. (a,b) where $-\infty \le a < b \le \infty$

Remark: • For the case III we usually express the integral with the help of an intermediate point $c \in (a,b)$ $\quad ((a,b) = (a,c] \cup [c,b))$

• By using the additivity property of the integral,

$$\int_{a_+}^{b-} f(x)\,dx = \int_{a_+}^{c} f(x)\,dx + \int_{c}^{b-} f(x)\,dx$$

• Moreover, $\int_a^b f(x)\,dx = \int_b^a -f(x)\,dx$

Remark: Improper integrals are sometimes very similar in behaviour with series of positive terms

Theorem:

$f : [a,b] \to [0,\infty]$ $\left.\begin{array}{l} \circ \text{ continuous} \\ \circ \text{ decreasing} \end{array}\right\} \to \int_a^\infty f(x)\,dx$ has the same nature with $\sum\limits_{n=k}^{\infty} f(n)$, $k \in \mathbb{N} \cap [a,\infty)$

$[k,\infty)$ $\quad k \in \mathbb{N}$

$\Delta = (k, k+1, \dots, n, n+1, \dots) \in \mathrm{Part}([k,\infty))$

$\xi = (k, k+1, \dots, n, n+1, \dots) \in \mathrm{Tp}(\Delta)$

$$\sigma(f, \Delta, \xi) = \sum_{n=k}^{\infty} \underset{\xi_i}{f(n)}\underbrace{(n+1-n)}_{\substack{1 \\ x_i - x_{i-1}}} = \sum_{n=k}^{\infty} f(n)$$

Example: Study the nature of $\int_1^{\infty} \frac{1}{\sqrt{x^4+x^2+1}}\,dx$

we have $f:[1,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{\sqrt{x^4+x^2+1}}$ continuous & decreasing $\to$

$\to \int_1^{\infty} f(x)\,dx \sim \sum_{n=1}^{\infty} f(n) \sim \sum_{n=1}^{\infty} \frac{1}{n^2}$ - C

Def: f is called Riemann integrable

- If $\exists \lim_{\substack{t\to a\\ t>a}} \int_t^b f(x)\,dx \in \overline{\mathbb{R}} \overset{\text{not}}{=} \int_{a_+}^b f(x)\,dx$ is called the improper integral of f on $(a,b]$

- If $\exists \lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx \in \overline{\mathbb{R}} \overset{\text{not}}{=} \int_a^{b_-} f(x)\,dx$ is called the improper integral of f on $[a,b)$

- If for a random $c\in(a,b)$
  $\exists \int_{a_+}^{c} f(x)\,dx + \int_c^{b_-} f(x)\,dx \in \overline{\mathbb{R}} = \int_{a_+}^{b_-} f(x)\,dx$ is called the improper integral of f on (a,b)

Remark: The improper integrals are C if $\lim \in \mathbb{R}$, D otherwise $\Big\}$ just like in the case for limits of functions

Example: · Study the improper integrability of

$f:[a,b]\to\mathbb{R}$, $f(x)=\frac{1}{(b-x)^p}$, $\forall x\in[a,b]$, $p\in\mathbb{R}$ - constant

we need $\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx =$

I. First we compute the indefinite integral

- $\int \frac{1}{(b-x)^p}\,dx = -\int \frac{(b-x)'}{(b-x)^p}\,dx = -\int (b-x)'(b-x)^{-p}\,dx = -(-p-1)^{-1}(b-x)^{-p+1} + \mathcal{C}$

$= \frac{1}{(p-1)(b-x)^{p-1}} + \mathcal{C}$

- $p=1 \to \int \frac{1}{b-x}\,dx = -\ln(b-x)+\mathcal{C}$

II. $\lim_{\substack{t\to b\\ t<b}} F(t) = \lim_{\substack{t\to b\\ t<b}} \frac{1}{(p-1)(b-t)^{p-1}} = \frac{1}{p-1} \lim_{\substack{t\to b\\ t<b}} \underbrace{(b-t)}_{\sim 0_+}{}^{1-p} = \frac{1}{p-1} \begin{cases} 0_+, & 1-p>0 \\ \infty, & 1-p<0 \end{cases}$

$= \begin{cases} 0_+, & 1>p \\ \infty, & 1<p \end{cases}$

$$\rightarrow \int_a^b f(x)dx = \lim_{\substack{t\to b\\ t<b}} (F(t)-F(a)) = \begin{cases} 0 - F(a), & 1>p \\ \infty - F(a), & 1<p \end{cases} = \begin{cases} -F(a), & 1>p & -C \\ \infty, & 1<p & -D \end{cases}$$

$$p=1 \rightarrow F(x) = -\ln(b-x) + C \rightarrow \lim_{\substack{t\to b\\ t<b}} F(t) = -\ln(0_+) = -(-\infty) = \infty$$

$$\rightarrow \int_a^b f(x)\,dx = \begin{cases} C, & 1>p \\ D, & 1\le p \end{cases}$$

- $f:(a,b] \rightarrow \mathbb{R}$, $f(x) = \frac{1}{(x-a)^p}$, $\forall x \in (a,b]$, $b \in \mathbb{R}$ - Hw

(should get the same results as the one above)

- $f:[a,\infty) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x^p}$, $\forall x \in [a,\infty)$

$a>0$

$$\mathrm{I}\ \int f(x)dx = \int \frac{1}{x^p}dx = \begin{cases} \ln x, & p=1 \\ \frac{x^{-p+1}}{-p+1}, & p\ne 1 \end{cases}$$

$$\mathrm{II}\ \lim_{t\to\infty} F(t) = ?$$

- $p=1 \rightarrow \lim_{t\to\infty} F(t) = \lim_{t\to\infty} \ln(t) = \infty$
- $p\ne 1 \rightarrow \lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{1}{1-p}\cdot\frac{1}{x^{p-1}} = \begin{cases} \frac{1}{1-p}\cdot 0, & 1-p<0 \\ \frac{1}{1-p}\cdot\infty, & 1-p>0 \end{cases} = \begin{cases} \infty, & 1>p \\ 0, & 1<p \end{cases}$

$$\rightarrow \int_a^\infty f(x)dx = \begin{cases} \infty, & 1\ge p \\ -F(a), & 1<p \end{cases} = \begin{cases} \infty, & p\le 1 & -D \\ -\ln a, & p>1, & C \end{cases}$$

General conclusion

	$p<1$	$p\ge 1$
$[a,b)$	C	D
$(a,b]$	C	D
$(a,\infty]$	D (for $p=1$ too)	C (not for $p=1$)

Comparison criteria for improper integrals

Theorem:

$f:[a,b)\to\mathbb{R}$ - locally Riemann integrable on $[a,b]$. Then:

1. if $\exists\, c\in(a,b)$ s.t. $\forall x\in[c,b]$ $f(x)\le g(x)$ then

$$\boxed{\int_a^{b-} g \text{ - C} \to \int_a^{-b} f \text{ - C}}$$
$$\boxed{\int_a^{b-} f \text{ - D} \to \int_a^{-b} g \text{ - D}}$$

$\sim C_1C$

2. if $\exists\, c\in(a,b)$ ~~s.t. $g(x)>0, \forall x\in[c,b)$~~

~~$\bullet\ \exists L=\lim\limits_{\substack{t\to b\\ t<b}}\frac{f(t)}{g(t)}\in(0,\infty)$~~

$\exists\ 0<\alpha<\beta<\infty$ and $\forall x\in[c,b)$, $\alpha\le\frac{f(x)}{g(x)}\le\beta$. Then

$$\boxed{\int_a^{b-} f\sim\int_a^{b-} g}$$

$\sim C_2C$

3. if $\exists\, c\in(a,b)$ s.t. $\bullet\ g(x)>0,\ \forall x\in[c,b)$

$\bullet\ \exists\, L=\lim\limits_{\substack{t\to b\\ t<b}}\frac{f(t)}{g(t)}\in(0,\infty)$

Then $\int_a^{b-} f\sim\int_a^{b-} g$

Remark: Similar theorems may be stated for $\int_{a+}^{b}$ and $\int_{a+}^{b-}$

Remark: When solving exercices with the help of the comparison criteria we use the example detailed above

The algorithm is the following

Step 1: Compute $L=\lim\limits_{\substack{x\to\text{problem}\\ x\neq\text{problem}}}\frac{f(x)}{g(x)}=\begin{cases}\lim\limits_{\substack{x\to b\\ x<b}}(b-x)^p\cdot f(x) & : [a,b)\\[2ex] \lim\limits_{\substack{x\to a\\ x>a}}(x-a)^p f(x) & : (a,b]\\[2ex] \lim\limits_{x\to\infty} x^p f(x) & : [a,\infty)\end{cases}$

We may use the following guiding table:

interval	L	P	Nature
$[a,b)$	$<\infty$	<1	C
$(a,b]$	>0	≥ 1	D
$[a,\infty)$	$<\infty$	>1	C
	>0	≤ 1	D

Remark: Our goal should be, just like in the case applying C2C for SPT, to get the limit $L \in (0,\infty)$, because then the table delivers conclusions $\forall p \in \mathbb{R}$.

However, this is not always achivable

The cases uncovered by the tables are $\boxed{L=\infty,\ P<1}$, $\boxed{L=0,\ P\geq 1}$ for finite problem, $\boxed{L=\infty,\ P>1}$, $\boxed{L=0,\ P\leq 1}$ for ∞

Exercices: Study the improper integrability of f

- $f:[0,1)\to\mathbb{R}$, $f(x)=\dfrac{1}{\sqrt[4]{1-x^4}}$

Solution: $f(x)>0$

We conclude $L=\lim\limits_{\substack{x\to 1\\ x<1}}(1-x)^P\cdot f(x)=\lim\limits_{\substack{x\to 1\\ x<1}}\dfrac{(1-x)^P}{\sqrt[4]{1-x^4}}=$

$$=\lim_{\substack{x\to 1\\ x<1}}\frac{(1-x)^P}{\sqrt[4]{(1-x)(1+x)(1+x^2)}}\ \overset{P=\frac{1}{4}}{=\!=}\ \lim_{\substack{x\to 1\\ x<1}}\frac{1}{\sqrt[4]{(1+x)(1+x^2)}}=\frac{1}{\sqrt[4]{4}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$L\in(0,\infty)\to$ The table may be applied (the first/finite cases)

$P=\frac{1}{4}<1\to$ the improper integral is C $\to \int_0^{1^-} f(x)\,dx\in\mathbb{R}$

Remarks on the terminology:

- Study the ii (improper integrability) of a function = study the nature of an impropriate integral $\to$ CONVERGENT / $\to$ DIVERGENT

- For functions whose image $\subseteq[0,\infty)$, just like in the case of SPT $\int_c^\infty f(x)\,dx$ <u>exists</u>, so we should just see if it's finite or not.

Examples: Study the ii of the function

$f: [\frac{\pi}{2}, \pi) \to \mathbb{R}, \quad f(x) = \frac{1}{\sin x}$

Method I: directly

I. We start with an anti-derivative from the indefinite integral

$$\int \frac{1}{\sin x} dx = \int \frac{\sin x}{1-\cos^2 x} dx = -\int \frac{(\cos x)'}{1-\cos^2 x} dx = -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1} =$$

$$= \frac{1}{2} \ln \left| \frac{1-\cos x}{1+\cos x} \right| + C = \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x} + C$$

We choose $F(x) = \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x}$

II. We compute

$$\lim_{\substack{x \to \pi \\ x < \pi}} \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x} = \lim_{\substack{x \to \pi \\ x < \pi}} \frac{1}{2} \ln(1-\cos x) - \frac{1}{2} \ln(1+\cos x) =$$

$$= \frac{1}{2} \ln 2 \underbrace{- \frac{1}{2} \ln(0_+)}_{-(-\infty)} = \infty$$

$$\to \exists \lim_{\substack{x \to \pi \\ x < \pi}} F(x) = \infty \to \exists \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sin x} dx = \infty - F\left(\frac{\pi}{2}\right) = \infty$$

↳ but it is divergent

Method II: by using comparison criteria

$$L = \lim_{\substack{x \to pb \\ x + pb}} O_+^P f(x) = \lim_{\substack{x \to \pi \\ x < \pi}} (\pi - x)^P \frac{1}{\sin x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{\substack{x \to \pi \\ x < \pi}} \frac{P(\pi-x)^P}{\cos x} = \lim_{\substack{x \to \pi \\ x < \pi}} p(\pi - x)^P =$$

$\underset{=}{P=1}$ $1 \in (0, \infty) \to$ the table may be applied $\to$ improper integral is Divergent

- $f: [1, \infty) \to \mathbb{R}, \quad f(x) = \frac{1}{\sqrt{1+x^2}}$

$f(x) > 0, \ \forall x \in [1, \infty)$

$$L = \lim_{x \to \infty} x^P f(x) = \lim_{x \to \infty} \frac{x^P}{\sqrt{1+x^2}} \overset{P=1}{=} \lim_{x \to \infty} \frac{x}{\sqrt{1+x^2}} = 1 \in (0, \infty)$$

$\to$ the table may be applied $\to$ ii is Divergent