13. 10 2023
Analysis - Lecture 2

Neighborhoods, Open sets, Closed sets.

1. Balls and Neighborhoods

Def: Let $x \in \mathbb{R}$, $r>0 \in \mathbb{R}$, the OPEN BALL centered at x of radius r is the set $B(x,r) = \{ y \in \mathbb{R} \mid |y-x| < r \} = (x-r, x+r)$

• the CLOSED BALL centered at x of radius r is $\bar{B} = \{ y \in \mathbb{R} \mid |y-x| \leq r \} = [x-r; x+r]$

Def: Let $x \in \mathbb{R}$. A set $V \subseteq \mathbb{R}$ is said to be a neighborhood of x if $\exists\, r>0$ s.t. $B(x,r) \subseteq V$

Properties of neighborhoods:

1. $\mathcal{V}(x) = \{ V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x \}$
2. $\phantom{\mathcal{V}(x)} = \{ V \subseteq \mathbb{R} \mid \exists r_V > 0 \text{ s.t. } B(x, r_V) \subseteq V \}$

$\forall\, V \in \mathcal{V}(x) \rightarrow x \in V$

3. If $V \in \mathcal{V}(x)$, $W \subseteq \mathbb{R}$ s.t. $V \subseteq W \rightarrow W \in \mathcal{V}(x)$

4. If $x \neq y \in \mathbb{R} \rightarrow \exists\, V \in \mathcal{V}(x)$, $\exists\, U \in \mathcal{V}(y)$ s.t. $V \cap U = \emptyset$

5. If $V \in \mathcal{V}(x) \rightarrow \exists\, T \in \mathcal{V}(x)$ s.t. $V = \mathcal{V}(t), \forall\, t \in T$

Proof:

(1) $\to$ def

(2) Consider $V \in \mathcal{V}(x)$ a random neighborhood.

$$\xrightarrow{\text{def}} \left.\begin{array}{l} \exists\, r_V > 0 \text{ s.t. } B(x, r_V) \subseteq V \\ x \in B(x, r_V) \end{array}\right\} \to x \in V$$

V was chosen randomly $\to \forall\, V \in \mathcal{V}(x) \ldots$

(3) Let $\left.\begin{array}{l} V \in \mathcal{V}(x) \\ W \subseteq \mathbb{R},\ \text{s.t. } V \subseteq W \end{array}\right\}$ both random

def

$$\underbrace{\underline{\exists\, r_V > 0} \quad \left.\begin{array}{l} \text{s.t. } B(x, r_V) \subseteq V \\ V \subseteq W \end{array}\right\} \to \underline{B(x, r_V) \subseteq W}}_{}$$

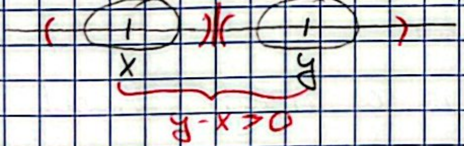
$\downarrow$ def

$W \in \mathcal{V}(x)$

(4) Consider $x \neq y \in \mathbb{R}$ random

$x \neq y \xrightarrow[\mathbb{R}\text{- totaly ordered (lecture 1)}]{} x < y$ or $y < x$

Without loosing the generality, assume that $x < y$



$B(x, \frac{y-x}{2}) \in \mathcal{V}(x)$

$B(y, \frac{y-x}{2}) \in \mathcal{V}(y)$ and $B(x, \frac{y-x}{2}) \cap B(y, \frac{y-x}{2}) = \emptyset$

(5) We proof that $\forall\, V \in \mathcal{V}(x)$, $\underline{\exists\, T \in \mathcal{V}(x)}$ s.t. $V \in \mathcal{V}(t), \forall\, t \in T$

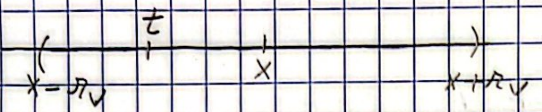
Consider $V \in \mathcal{V}(x)$ a random neighborhood

$\downarrow$ def

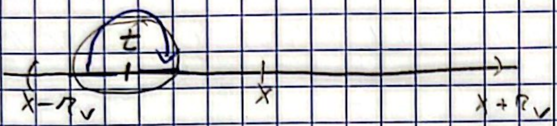
$\exists\, r_V > 0$ s.t. $B(x, r_V) \subseteq V$

The $B(x, r_V)$ is an example of such a neighborhood T.

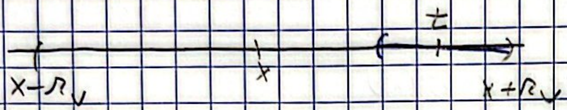
We prove that $\forall\, t \in B(x, r_V)$, $V \in \mathcal{V}(x)$



$B(t, r_t) \subseteq B(x, r_V)$



$r_t = t - (x - r_V)$ if $t \leq x$



if $t > x$, $x + r_V - t$

$\exists \quad r_t = \begin{cases} t-(x-r_V), & t \leq x \\ x+r_V-t, & t > x \end{cases}$

$B(t, r_t) \subset B(x, r_V) \subseteq V \rightarrow V \in \mathcal{V}(t)$

t - random $\rightarrow \forall$

V - random $\rightarrow \forall$

Obs: $\forall\, r > 0$, $B(x, r) \in \mathcal{V}(x)$

Examples

$x = 10$

$\mathbb{N} \overset{?}{\in} \mathcal{V}(10)$

$(5, 15) \overset{?}{\in} \mathcal{V}(10)$ ✓

$(9, 11) \overset{?}{\in} \mathcal{V}(10)$ ✓

$[1, 20) \overset{?}{\in} \mathcal{V}(10)$ ✓

$(6; 10) \overset{?}{\in} \mathcal{V}(10)$

$\mathbb{R} \setminus (10; 10)$

$\mathbb{R} \setminus \mathbb{Z}$ x , $10 \notin$

$\mathbb{R} \setminus \mathbb{Q}$ x , $10 \notin$

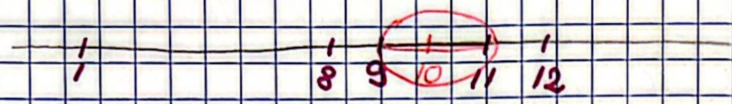
$\mathbb{R} \setminus \{10\}$ x , $10 \notin$

$(-\infty; 11) \cup \mathbb{N}$

$(-\infty; 11) \cap \mathbb{Z}$

$(-\infty; 11) \cap \mathbb{Q}$

• $\mathbb{N}$



Remark: In Calculus $\mathbb{N}=\{1,2,3,\dots\}$, $!\ 0 \notin \mathbb{N}$

$\forall\, r>0$, $B(10,r)=(10-r;10+r)\not\subseteq\mathbb{N} \to \mathbb{N}\notin \mathcal{V}(10)$

$\mathbb{Z}\notin\mathcal{V}(10)$

• $(6,10)\overset{?}{\in}\mathcal{V}(10)$

$10\notin(6;10) \to (6;10)\notin\mathcal{V}(10)$

• $\mathbb{R}\setminus(10;11) = (-\infty;10]\cup[11;+\infty)$

$\forall\, r>0$, $B(x,r)\subseteq \mathbb{R}\setminus(10;11) \to \notin \mathcal{V}(10)$

• $(-\infty;11)\cup\mathbb{N}$

$\exists\, r=1>0$ s.t. $B(10,r)\subseteq(-\infty,11)\cup\mathbb{N}$, $\to (-\infty,11)\cup\mathbb{N}\in\mathcal{V}(x)$

• $(-\infty;11)\cap\mathbb{Z}\notin\mathcal{V}(10)$ (subset of $\mathbb{Z}$ and $\mathbb{Z}$ - is not a neigh.)

• $(-\infty;11)\cap\mathbb{Q}$

$\mathbb{Q}=\{\frac{p}{q} \mid p,q\in\mathbb{Z},\ q\neq 0\}$ → the set of RATIONAL NUMBERS

$=\{\frac{p}{q} \mid p\in\mathbb{Z},\ q\in\mathbb{N}\}$

$\mathbb{R}\setminus\mathbb{Q}$ → the set of IRRATIONAL NUMBERS

→ $\mathbb{Q}\cap\mathbb{R}\setminus\mathbb{Q}=\emptyset$

$\mathbb{Q}\cup\mathbb{R}\setminus\mathbb{Q}=\mathbb{R}$

The DENSITY AXIOM OF $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$

$\forall\, x<y \in \mathbb{R}$, $\exists\, t \in \mathbb{Q}$ s.t. $x<t<y$
$\exists\, s \in \mathbb{R}\setminus\mathbb{Q}$ $x<s<y$

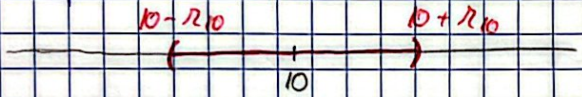
→ Between two distinct real nr.:

- there are infinitely many rational ones
- " " " irrational ones

Remark $\mathbb{Q}$ is not a neighborhood for any of its points.

Proof

We prove that $\mathbb{Q} \notin \mathcal{V}(10)$



$\in$ belongs
$\subseteq$ is included

1. $10 \in \mathbb{Q}$
2. Assume by contradiction that $\mathbb{Q} \in \mathcal{V}(x) \longrightarrow$

def → $\exists\, r_{10} > 0$ s.t. $B(10, r_{10}) \subseteq \mathbb{Q}$ ①

$10 - r_{10} < 10 + r_{10} \xrightarrow{\text{density}} \exists\, s \in \mathbb{R}\setminus\mathbb{Q}$ ②

$10 - r_{10} < s < 10 + r_{10} \rightarrow s \in B(x, r_{10}) \rightarrow$

$\rightarrow s \in \mathbb{Q}$ ③

$\frac{②}{③} \rightarrow \mathbb{Q} \cap \mathbb{R}\setminus\mathbb{Q} = \emptyset \rightarrow$ contradiction $\rightarrow \mathbb{Q} \notin \mathcal{V}(10)$

Remark $\mathbb{R}\setminus\mathbb{Q}$ is not a neighborhood for any of its points.

Remark However $(-\infty, 11) \cup \mathbb{Q} \in \mathcal{V}(10)$

2. Open and closed sets

Def A set $A \subseteq \mathbb{R}$ is said to be open if $\forall a \in A,\ A \in \mathcal{V}(a) \Leftrightarrow \forall a \in A, \exists r_a > 0$ st $B(a, r_a) \subseteq A$

A set is open if it is a neighborhood for all of its points.

Examples

O_1: $\forall a < b \in \mathbb{R}$, (a, b) is an open set

O_2: $\mathbb{R}$

O_3: $\forall a \in \mathbb{R}$; $(-\infty; a)$ and $(a; +\infty)$ are open set

O_4: All reunions of open sets are open

O_5: All intersections of a finite nr. of open sets are open (See Hw1, ex 2)

If we intersect infinite many open sets the result varies: it may be open, closed or none.

Def $B \subseteq \mathbb{R}$ is said to be a closed set if $\mathbb{R} \setminus B$ is open

Examples

C_1: $\forall a < b \in \mathbb{R}$, $[a, b]$ is a closed set (because $\mathbb{R} \setminus [a, b] = (-\infty; a) \cup (b; \infty)$)

C_2: $\{a\}$

C_3: $\mathbb{N}$ (because $\mathbb{R} \setminus \mathbb{N} = (-\infty; 1) \cup (1; 2) \cup \dots =$

$\mathbb{Z}$ $= (-\infty; 1) \cup \bigcup_{\substack{n \in \mathbb{N} \\ n \geq 2}} (n, n+1)$)

C_4: $\{a_i \mid a_i \in \mathbb{R}, i \in I, I \subseteq \mathbb{N}, \text{card } I < \infty\}$

C_5: $(-\infty; a]$ (because $\mathbb{R} \setminus (-\infty; a] = (a; \infty)$)

$[a; +\infty)$

O mulțime poate să nu fie nici open nici closed

Ex 1. $\mathbb{Q}$ - is not open nor closed ~~sau reuniune de closed or open~~

2. $(1, 2]$

Remark Not all subsets of $\mathbb{R}$ are either closed or open

Ex: $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$

as well as $(a;b]$ or $[a,b)$

or $(-\infty; a) \cup \mathbb{N}$

? $(-\infty; a] \cap \mathbb{N} \to$ always closed

Remark The only subsets of $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$

Proff
- $\mathbb{R}$ is open because $\forall\, x \in \mathbb{R}$, $\exists\, r = 1 > 0$ s.t. $B(x,1) \subseteq \mathbb{R}$
- $\emptyset = \underset{\text{open}}{\mathbb{R}} \setminus \mathbb{R} \xrightarrow{\text{def}} \emptyset$ is closed
- $\emptyset$ is open

  Assume by contradiction that $\emptyset$ is not open

  $\to \exists\, a \in \emptyset$ s.t. $\emptyset \notin \mathcal{V}(a)$ (but there are no elements in $\emptyset$) $\to$ contradiction $\to \emptyset$ - open $\to \mathbb{R} \setminus \emptyset$ - closed

C6: Each intersection of closed sets is closed

C7: Each reunion of a finite nr of number sets is closed

Remark If we reunite an infinite nr of closed sets the result is random. (Hw 1, ex 2)

3. Interior, exterior, boundary, closure, accumulation points

Def Consider $\emptyset \neq A \subseteq \mathbb{R}$. Then

- $\text{int } A = \{a \in A \mid A \in \mathcal{V}(a)\}$

  $= \{a \in A \mid \exists\, r_a > 0 \text{ s.t. } B(a, r_a) \subseteq A\}$

  $\to$ The INTRIOR of A

$\boxed{\text{int } A \subseteq A}$

- $\text{ext } A = \text{int}(\mathbb{R} \setminus A)$

$\boxed{\text{ext } A \subseteq \mathbb{R} \setminus A}$

- $\text{cl } A = \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

$= \{x \in \mathbb{R} \mid \forall r > 0,\ B(x, r) \cap A \neq \emptyset\}$

$\to$ the CLOSURE of A

$\boxed{\text{int } A \subseteq A \subseteq \text{cl } A}$

- $\text{bd } A = \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \text{ and } V \cap (\mathbb{R} \setminus A) \neq \emptyset\}$

$= \{x \in \mathbb{R} \mid \forall r > 0,\ B(x, r) \cap A \neq \emptyset \text{ and } B(x, r) \cap (\mathbb{R} \setminus A) \neq \emptyset\}$

$\to$ the BOUNDARY of A

- $A' = \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x),\ V \cap A \setminus \{x\} \neq \emptyset\}$

$\boxed{A' \subset \text{cl } A}$ $\to$ the set of ACCUMULATION POINTS

- $\text{izo } A = \{a \in A \mid \exists V \in \mathcal{V}(a) \text{ s.t. } V \cap A = \{a\}\}$

$= \{a \in A \mid \exists r > 0 \text{ s.t. } B(a, r) \cap A = \{a\}\}$

$\to$ the set of the isolated points of A

$\boxed{\text{iso } A \subseteq A}$

Properties

1. $A' = \text{cl } A \setminus \text{izo } A$
2. $\text{int } A \cap \text{bd } A = \emptyset$

   $\text{ext } A \cap \text{bd } A = \emptyset$ but $\text{int } A \cup \text{bd } A \cup \text{ext } A = \mathbb{R}$

   $\text{int } A \cap \text{ext } A = \emptyset$
3. $\text{cl } A = \text{int } A \cup \text{bd } A \to \text{bd } A = \text{cl } A \setminus \text{int } A$

Examples: $A = [1; 2) \cup \{3\}$

$\text{int } A = (1, 2)$ ✓ $A' = [1, 2]$

$\text{bd } A = \{1, 2, 3\}$ ✓

$\text{cl } A = [1; 2] \cup \{3\}$ $2 \in \text{cl } A,\ \forall V \in \mathcal{V}(2),\ V \cap A \neq \emptyset$

$\text{ext } A = (-\infty; 1) \cup (2; \cancel{+\infty}\ 3) \cup (3; +\infty) = \mathbb{R} \setminus \text{cl } A$

$\text{izo } A = \{3\}$ ✓