Calculus - Homework 11 - Impropriate integrals

1. a) $f:(-1,1)\to\mathbb{R},\ f(x)=\frac{1}{\sqrt{1-x^2}}$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$C=0 \to F(x)=\arcsin x$$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1_+}^{c} f(x)\,dx + \int_{c}^{1_-} f(x)\,dx = \lim_{\substack{t\to-1\\t>-1}} (F(c)-F(t)) + \lim_{\substack{t\to1\\t<1}} (F(t)-F(c))$$

$$= \cancel{\arcsin c} - \lim_{\substack{t\to-1\\t>-1}} \arcsin t + \lim_{\substack{t\to1\\t<1}} \arcsin t - \cancel{\arcsin c}$$

$$= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi\in\mathbb{R}\to C$$

b) $f:[1,\infty)\to\mathbb{R},\ f(x)=\frac{1}{x(x+1)}$

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}-\frac{1}{x+1}\,dx = \int\frac{1}{x}\,dx - \int\frac{1}{x+1}\,dx + C = \ln|x| - \ln|x+1| + C =$$
$$= \ln\left|\frac{x}{x+1}\right| + C$$

$$\frac{1}{x(x+1)} = \frac{A}{x}+\frac{B}{x+1}$$

$$1 = x(A+B)+A \to \left.\begin{array}{l}A=1\\A+B=0\end{array}\right\}\to B=-1$$

$$C=0\to F(x)=\ln\left|\frac{x}{x+1}\right|$$

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty}(F(t)-F(1)) = \lim_{t\to\infty}\left(\ln\frac{t}{t+1}-\ln\frac{1}{2}\right) =$$

$$= \ln 1 - \ln\frac{1}{2} = 0-\ln 2^{-1} = \ln 2\in\mathbb{R}\to C$$

c) $f:(0,1]\to\mathbb{R},\ f(x)=\ln x$

$$\int \ln x\,dx = \int \ln x\,(x)'\,dx = x\ln x - \int\frac{1}{x}\cdot x\,dx = x\ln x - x + C = x(\ln x-1)+C$$

$$C=0\to F(x)=x(\ln x-1)$$

$$\int_{0_+} f(x)\,dx = \lim_{\substack{t\to0\\t>0}}(F(1)-F(t)) = \lim_{\substack{t\to0\\t>0}}(-1-t(\ln t-1)) = -1-\lim_{\substack{t\to0\\t>0}} t(\ln t-1)$$

$$= -1-\lim_{\substack{t\to0\\t>0}} t\ln t + \underbrace{\lim_{\substack{t\to0\\t>0}} t}_{0} = -1-\lim_{\substack{t\to0\\t>0}} \ln t^t = -1-0 = -1\in\mathbb{R}\to C$$

d) $f:[0,1)\to\mathbb{R},\ f(x)=\dfrac{\arcsin x}{\sqrt{1-x^2}}$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}dx = \int \arcsin x\ (\arcsin x)'\,dx = \frac{1}{2}\arcsin^2 x + C$$

$$C=0 \to F(x)=\frac{1}{2}\arcsin^2 x$$

$$\int_0^{1^-} f(x)\,dx = \lim_{\substack{t\to 1\\ t<1}} (F(t)-F(0)) = \lim_{\substack{t\to 1\\ t<1}} \Big(\frac{1}{2}\arcsin^2 t - \underbrace{\frac{1}{2}\arcsin^2 0}_{0}\Big) = \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}\in\mathbb{R}$$

$\to C$

e) $f:(0,1]\to\mathbb{R},\ f(x)=\dfrac{\ln x}{\sqrt{x}}$

$$\int \frac{\ln x}{\sqrt{x}}dx = \int \ln x\ (2\sqrt{x})'\,dx = 2\sqrt{x}\ln x - \int \frac{2\sqrt{x}}{x}dx = 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}dx =$$

$$= 2\sqrt{x}\ln x - 4\sqrt{x} = 2\sqrt{x}(\ln x - 2) + C$$

$$C=0 \to F(x) = 2\sqrt{x}(\ln x - 2)$$

$$\int_{0_+}^{1} f(x)\,dx = \lim_{\substack{t\to 0\\ t>0}} (F(1)-F(t)) = -4 - \lim_{\substack{t\to 0\\ t>0}} (2\sqrt{t}\ln t - \underbrace{4\sqrt{t}}_{0}) =$$

$$= -4 - \lim_{\substack{t\to 0\\ t>0}} \ln t^{2\sqrt{t}} = -4\in\mathbb{R} \to C$$

f) $f:[e,\infty)\to\mathbb{R},\ f(x)=\dfrac{1}{x(\ln x)^3}$

$$\int \frac{1}{x(\ln x)^3}dx = \int \frac{1}{t^3}dt = \frac{-1}{2t^2} = \frac{-1}{2(\ln x)^2} + C$$

$$\ln x = t \to \frac{1}{x}dx = dt \qquad C=0 \to F(x) = -\frac{1}{2}\cdot\frac{1}{(\ln x)^2}$$

$$\int_e^\infty f(x)\,dx = \lim_{t\to\infty}(F(t)-F(e)) = \lim_{t\to\infty} -\frac{1}{2}\cdot\underbrace{\frac{1}{(\ln t)^2}}_{\downarrow\, 0} + \frac{1}{2} = \frac{1}{2}\in\mathbb{R}\to C$$

g) $f:\left(\dfrac{1+\sqrt{3}}{2},2\right]\to\mathbb{R},\ f(x)=\dfrac{1}{x\sqrt{2x^2-2x-1}}$

$$\int \frac{1}{x\sqrt{2x^2-2x-1}}dx = \int \frac{1}{x\sqrt{(\sqrt{2}x)^2-2\cdot\sqrt{2}\cdot\frac{1}{\sqrt{2}}\cdot x+\left(\frac{1}{\sqrt{2}}\right)^2-\frac{3}{2}}}dx = \int \frac{1}{x\sqrt{\left(\sqrt{2}x-\frac{1}{\sqrt{2}}\right)^2-\frac{3}{2}}}dx =$$

$$= \int \frac{1}{x\sqrt{\frac{1}{2}(2x-1)^2-\frac{3}{2}}}dx = \int \frac{1}{\frac{1}{\sqrt{2}}x\sqrt{(2x-1)^2-3}}dx = \sqrt{2}\int \frac{1}{x\sqrt{(2x-1)^2-3}}dx =$$

$$2x-1=t \to 2dx = dt$$

$$x=\frac{t+1}{2}$$

$$= \frac{\sqrt{2}}{2}\int \frac{1}{\frac{t+1}{2}}\cdot\frac{1}{\sqrt{t^2-3}}dt = \sqrt{2}\int \frac{1}{(t+1)\sqrt{t^2-(\sqrt{3})^2}}dt$$

h) $f:[0,\infty)\to\mathbb{R},\ f(x)=\frac{\pi}{2}-\operatorname{arctg}x$

$$\int \frac{\pi}{2}-\operatorname{arctg}x\,dx = \frac{\pi}{2}x - \int \operatorname{arctg}x\cdot(x)'dx = \frac{\pi}{2}x - x\operatorname{arctg}x +$$

$$+\int \frac{x}{x^2+1}dx = \frac{\pi}{2}x - x\operatorname{arctg}x + \frac{1}{2}\ln(x^2+1)+C$$

$$F(x) = \frac{\pi}{2}x - x\operatorname{arctg}x + \frac{1}{2}\ln(x^2+1)$$

$$\int_0^\infty f(x)dx = \lim_{t\to\infty}(F(t)-F(0)) = \lim_{t\to\infty}\left(\frac{\pi}{2}t - t\operatorname{arctg}t + \frac{1}{2}\ln(t^2+1) - 0\right)$$

$$= \lim_{t\to\infty}\left(\frac{\pi}{2}t - \frac{\pi}{2}t + \frac{1}{2}\ln(t^2+1)\right) = \frac{1}{2}\lim_{t\to\infty}\ln(t^2+1) = \infty \notin \mathbb{R} \to D$$

i) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\frac{1}{1+x^2}$

$$\int\frac{1}{1+x^2}dx = \operatorname{arctg}x + C$$

$$F(x) = \operatorname{arctg}x$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx = \cancel{F(c)} - \lim_{t\to-\infty}F(t) + \lim_{t\to\infty}F(t) - \cancel{F(c)}$$

$$= -\lim_{t\to-\infty}\operatorname{arctg}t + \lim_{x\to\infty}\operatorname{arctg}t = -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi \in\mathbb{R}\to C$$

j) $f:\left(\frac{1}{3};3\right]\to\mathbb{R},\ f(x)=\frac{1}{\sqrt[3]{3x-1}}$

$$\int\frac{1}{\sqrt[3]{3x-1}}dx = \frac{1}{3}\int(3x-1)'\cdot(3x-1)^{-\frac{1}{3}}dx = \frac{1}{\cancel{3}}\cdot\frac{(3x-1)^{\frac{2}{3}}}{\frac{2}{\cancel{3}}}+C = \frac{1}{2}\sqrt[3]{(3x-1)^2}+C$$

$$F(x) = \frac{1}{6}\sqrt[3]{(3x-1)^2}$$

$$\int_{\frac{1}{3}+}^{3} f(x)dx = F(3) - \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}}F(t) = \frac{1}{2}\cdot 8^{\frac{2}{3}} - \lim_{\substack{t\to\frac{1}{3}\\ t>\frac{1}{3}}}\frac{1}{2}\sqrt[3]{(3x-1)^2} = \frac{1}{2}\cdot 2^2 - 0 = 2\in\mathbb{R}$$
$\to C$

k) $f:[1,\infty)\to\mathbb{R},\ f(x)=\frac{x}{(1+x^2)^2}$

$$\int\frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int t^{-2}dt = -\frac{1}{2}\cdot\frac{1}{t}+C = -\frac{1}{2}\cdot\frac{1}{1+x^2}+C$$

$1+x^2=t \to 2x\,dx = dt$

$$F(x) = -\frac{1}{2}\cdot\frac{1}{1+x^2}$$

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} F(t) - F(1) = \lim_{t\to\infty} -\frac{1}{2}\cdot\frac{1}{t^2+1} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} - 0 = \frac{1}{4} \in \mathbb{R} \to C$$

2. a) $f:[1,\infty)\to\mathbb{R},\ f(x) = \dfrac{1}{S\sqrt{1+x^2}}$

$$L = \lim_{x\to\infty} x^P \cdot f(x) = \lim_{x\to\infty} \frac{x^P}{S\sqrt{1+x^2}} = \begin{cases} 0, & P<2 \\ \frac{1}{S}, & P=2 \\ \infty, & P>2 \end{cases}$$

for $p = 2 > 1$

- $S \neq 0 \to L \in (0,\infty) \to C$
- $S = 0 \to L = \infty$

$p > 1$

$f(x) = \infty,\ \forall\ x \in [1,\infty)$

b) $f:[0,\frac{\pi}{2})\to\mathbb{R},\ f(x) = \dfrac{1}{\cos x}$

$$L = \lim_{\substack{x\to\frac{\pi}{2} \\ x<\frac{\pi}{2}}} \frac{(\frac{\pi}{2}-x)^P}{\cos x} \overset{\frac{0}{0}}{\underset{\text{l'H}}{=}} \lim_{x\to\frac{\pi}{2}} \frac{-P(\frac{\pi}{2}-x)^{P-1}}{-\sin x} = \lim_{x\nearrow\frac{\pi}{2}} P\left(\frac{\pi}{2}-x\right)^{P-1} \overset{P=1}{=} 1 \in (0,\infty) \to D$$

c) $f:(0,\infty)\to\mathbb{R},\ f(x) = \left(\dfrac{\operatorname{arctg} x}{x}\right)^2$

Let $g:(0,\infty)\to\mathbb{R},\ g(x) = \dfrac{1}{x^2}$

$$\frac{f(x)}{g(x)} = \operatorname{arctg}^2 x \in \left(0, \frac{\pi^2}{4}\right)$$

$$0 < \frac{f(x)}{g(x)} < \frac{\pi^2}{4} \in (0,\infty) \to \int_{0_+}^\infty f(x)\,dx \sim \int_{0_+}^\infty g(x)\,dx$$

$$G(x) = \int \frac{1}{x^2} = -\frac{1}{x}$$

$$\int_{0_+}^\infty g(x)\,dx = \int_{0_+}^c g(x)\,dx + \int_c^\infty g(x)\,dx = \cancel{G(c)} - \lim_{x\searrow 0} G(x) + \lim_{x\nearrow\infty} G(x) - \cancel{G(c)} =$$

$$= -\lim_{x\searrow 0}\left(-\frac{1}{x}\right) + \lim_{x\to\infty}\frac{1}{x} = -(-\infty) + 0 = \infty \to D \to \text{ii } f(x) - D$$

d) $f:(1,\infty)\to\mathbb{R},\ f(x) = \left(\dfrac{\ln x}{x\sqrt{x^2-1}}\right)^2$

$$\frac{\ln^2 x}{x^2(x^2-1)} > \frac{1}{x^2(x^2-1)} = -\frac{1}{x^2} + \frac{1}{x^2-1} = g(x)$$

$$\frac{1}{x^2(x^2-1)} = \frac{A}{x^2} + \frac{B}{x^2-1}$$

$1 = x^2(A+B) - A \to A = -1$

$A + B = 0 \to B = 1$

$$G(x) = \int \frac{1}{x^2-1} - \frac{1}{x^2}\,dx = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{x}$$

$$\int_{1^+}^{\infty} g(x)\,dx = \int_{1^+}^{c} g(x)\,dx + \int_{c}^{\infty} g(x)\,dx = -\lim_{t\searrow 1} G(t) + \lim_{t\to\infty} G(t) =$$

$$= -\lim_{t\searrow 1}\left(\frac{1}{2}\ln\frac{t-1}{t+1} + \frac{1}{t}\right) + \lim_{t\to\infty}\left(\frac{1}{2}\ln\left(\underbrace{\left|\frac{t-1}{t+1}\right|}_{\downarrow\, 1}\right) + \underbrace{\frac{1}{t}}_{\searrow 0}\right)$$

$$= -(-\infty) + 0 = \infty \to D \xrightarrow{C/C} \therefore f - D$$

e) $f: (0,1) \to \mathbb{R},\ f(x) = \left(\frac{\ln x}{x\sqrt{1-x^2}}\right)^2$

$$\frac{\ln^2 x}{x^2(1-x^2)} > \frac{1}{x^2(1-x^2)} = \frac{1}{x^2} + \frac{1}{1-x^2} = g(x)$$

$$\frac{1}{x^2(1-x^2)} = \frac{A}{x^2} + \frac{B}{1-x^2}$$

$1 = x^2(B-A) + A \to A = 1$

$A - B = 0 \to B = 1$

$$G(x) = \int g(x)\,dx = -\frac{1}{x} - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$$

$$\int_{0^-}^{1^+} g(x)\,dx = \int_{0^-}^{c} g(x)\,dx + \int_{c}^{1^+} g(x)\,dx = -\lim_{t\searrow 0} G(t) + \lim_{t\nearrow 1} G(t) =$$

$$= -\lim_{t\searrow 0}\left(-\frac{1}{t} - \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) + \lim_{t\nearrow 1}\left(-\frac{1}{t} - \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) =$$

$$= -(-\infty) + (-1-(-\infty)) = \infty + \infty = \infty \to D \xrightarrow{C/C} \therefore f - D$$