Algebra - Seminar 5

1. Det. the generated subspaces.

(i) $\langle 1, X, X^2\rangle$ in the real vector space $\mathbb{R}[X]$

$\langle 1, X, X^2\rangle = \{ k_1 1 + k_2 \cdot X + k_3 X^2 \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{ k_1 + k_2 X + k_3 X^2 \mid k_1, k_2, k_3 \in \mathbb{R}\} = \mathbb{R}_2[X]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$

$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle =$

$= \left\{ k_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$

$= \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} \mid k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$

$= \left\{ \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \mid k_1, k_2, k_3, k_4 \in \mathbb{R}\right\} = M_2(\mathbb{R})$

2. Write the following subspaces of the real vector space $\mathbb{R}^3$ with a minimal nr. of generators.

(i) $A = \{ (x, y, z) \in \mathbb{R}^3 \mid x = 0\} = \{ (0, y, z) \mid y, z \in \mathbb{R}\} = \{ (0, y, 0) + (0, 0, z) \mid y, z \in \mathbb{R}\} = \{ y(0,1,0) + z(0,0,1) \mid y, z \in \mathbb{R}\} = \langle (0,1,0), (0,0,1)\rangle$

(ii) $B = \{ (x, y, z) \in \mathbb{R} \mid x + y + z = 0\}$

$x + y + z = 0 \rightarrow x = -y - z$

$B = \{ (-y-z, y, z) \mid y, z \in \mathbb{R}\} = \{ (-y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{ y(-1,1,0) + z(-1,0,1) \mid y, z \in \mathbb{R}\} = \langle (-1,1,0), (-1,0,1)\rangle$

(iii) $C = \{ (x, y, z) \in \mathbb{R} \mid x = y = z\} = \{ (x, x, x) \mid x \in \mathbb{R}\} = \{ x(1,1,1) \mid x \in \mathbb{R}\}$

$= \langle (1,1,1)\rangle$

3. $a=(-2,1,3)$; $b=(3,-2,-1)$, $c=(1,-1,2)$, $d=(-5,3,4)$, $e=(-9,5,10)$

$\langle a,b\rangle = \langle c,d,e\rangle$?

$\langle a,b\rangle = \{k_1\cdot a + k_2\cdot b \mid k_1,k_2\in\mathbb{R}\}$

$= \{k_1(-2,1,3)+k_2(3,-2,-1) \mid k_1,k_2\in\mathbb{R}\}$

$= \{(-2k_1, k_1, 3k_1)+(3k_2,-2k_2,-k_2) \mid k_1,k_2\in\mathbb{R}\}$

$= \{(-2k_1+3k_2,\ k_1-2k_2,\ 3k_1-k_2) \mid k_1,k_2\in\mathbb{R}\}$

$\langle c,d,e\rangle = \{(k_3,-k_3,2k_3)+(-5k_4,3k_4,4k_4)+(-9k_5,5k_5+10k_5) \mid k_{3,4,5}\in\mathbb{R}\}$

$= \{(k_3-5k_4-9k_5$

$a+b=(-2+3,\ 1-2,\ 3-1) = (1,-1,2)=c$

$a-b=(-2-3,\ 1+2,\ 3+1)=(-5,3,4)=d$

$2d+c=(-10,6,8)-(1,-1,2)=(-9,5,10)=e = 2(a-b)+(a+b) = 3a-b$

$\langle c,d,e\rangle = \{k_1\cdot c+k_2\cdot d+k_3\cdot e \mid k_1,k_2,k_3\in\mathbb{R}\}$

$= \{k_1(a+b)+k_2(a-b)+k_3(3a-b) \mid k_1,k_2,k_3\in\mathbb{R}\}$

$= \{a(k_1+k_2+3k_3)+b(k_1-k_2-k_3) \mid k_1,k_2,k_3\in\mathbb{R}\} =$

Let $k_4 = k_1+k_2+3k_3\in\mathbb{R}$

$k_5 = k_1-k_2-k_3\in\mathbb{R}$

$= \{k_5 a + k_6 b \mid k_5,k_6\in\mathbb{R}\} = \langle a,b\rangle$

4. $S=\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

$T=\{(x,y,z)\in\mathbb{R}^3 \mid x=y=z\}$

(i) $S,T \leq \mathbb{R}^3$

$(0,0,0)\in S \to S\neq\emptyset$

$\forall k_1,k_2\in\mathbb{R}, v_1=(x_1,y_1,z_1), v_2=(x_2,y_2,z_2)\in S$, $k_1v_1+k_2v_2\in S$

$k_1(x_1,y_1,z_1)+k_2(x_2,y_2,z_2) = (k_1x_1+k_2x_2,\ k_1y_1+k_2y_2,\ k_1z_1+k_2z_2)\in S$

$k_1x_1+k_2x_2+k_1y_1+k_2y_2+k_1z_1+k_2z_2 = k_1(x_1+y_1+z_1)+k_2(x_2+y_2+z_2)=$

$=k_1\cdot 0+k_2\cdot 0=0$

T...

(ii) $\mathbb{R}^3 = S \oplus T$?

$\mathbb{R}^3 = S \oplus T \Leftrightarrow \forall (x,y,z) \in \mathbb{R}^3,\ \exists! (x_s, y_s, z_s) \in S,\ (x_t, y_t, z_t) \in T$ s.t.

$(x,y,z) = (x_s, y_s, z_s) + (x_t, y_t, z_t)$

$S \cap T = \{(x,y,z) \mid x+y+z=0 \text{ and } x=y=z\}$

$= \{(0,0,0)\} = 0$

Let $v = (x,y,z) \in \mathbb{R}^3$, $s = (x_s, y_s, z_s) \in S$, $t = (x_t, y_t, z_t) \in T$

$s \in S \rightarrow x_s + y_s + z_s = 0 \rightarrow x_s = -y_s - z_s$

$t \in T \rightarrow x_t = y_t = z_t \rightarrow t = (x_t, x_t, x_t)$

$v = s + t \Leftrightarrow (x,y,z) = (x_s + x_t,\ y_s + y_t,\ z_s + z_t) \Leftrightarrow$

$\Leftrightarrow (x,y,z) = (-y_s - z_s + x_t,\ y_s + x_t,\ z_s + z_t)$

$$\Leftrightarrow \begin{cases} x = -y_s - z_s + x_t \\ y = y_s + x_t \\ z = z_s + x_t \end{cases} \rightarrow$$

$x + y + z = 3x_t \Leftrightarrow x_t = \frac{x+y+z}{3} = y_t = z_t$

$x = x_s + x_t \rightarrow x_s = x - x_t = x - \frac{x+y+z}{3}$

$x_s = \frac{2x-y-z}{3}$

$y = y_s + y_t \rightarrow y_s = y - y_t = \frac{-x+2y-z}{3}$

$z_s = y - y_t = \frac{-x-y+2z}{3}$

$$\rightarrow \left.\begin{array}{l} s = \left(\frac{2x-y-z}{3}, \frac{-x+2y-z}{3}, \frac{-x-y+2z}{3}\right) \\ t = \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right) \end{array}\right\} \rightarrow$$

$\rightarrow \forall v \in \mathbb{R}^3, \exists! s \in S, t \in T$ s.t. $v = s + t \rightarrow \mathbb{R}^3 = S \oplus T$

6. $f, g: \mathbb{R}^2 \to \mathbb{R}^2$, $h: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y) = (x+y, x-y)$

$g(x,y) = (2x-y, 4x-2y)$

$h(x,y,z) = (x-y, y-z, z-x)$

$f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$, $h \in End_{\mathbb{R}}(\mathbb{R}^3)$?

* f - $\mathbb{R}$ linear map $\Leftrightarrow$ $f(k_1v_1 + k_2v_2) = k_1 f(v_1) + k_2 f(v_2)$, $\forall\, k_1, k_2 \in \mathbb{R}$, $\forall\, v_1, v_2 \in$

$\cancel{f(k_1v_1 + k_2v_2) = f(k_1(x_1, y_1, z_1) + k_2(x_2, y_2, z_2)) = f((k_1x_1, k_1y_1, k_1z_1) + (k_2x_2, k_2y_2, k_2z_2)) =}$

$=$

$f(k_1v_1 + k_2v_2) = f(k_1(x_1,y_1) + k_2(x_2,y_2)) = f((k_1x_1, k_1y_1) + (k_2x_2, k_2y_2)) =$

$= f((k_1x_1 + k_2x_2, k_1y_1 + k_2y_2)) = (k_1x_1 + k_2x_2 + k_1y_1 + k_2y_2, \; k_1x_1 + k_2x_2 - k_1y_1 - k_2y_2)$

$= (k_1(x_1+y_1) + k_2(x_2+y_2), \; k_1(x_1-y_1) + k_2(x_2-y_2))$

$= k_1 f(x_1,y_1) + k_2 f(x_2,y_2)$
f - linear $\Big\} \to f \in End_{\mathbb{R}}(\mathbb{R}^2)$

* $g(k_1(x_1,y_1) + k_2(x_2,y_2)) = g((k_1x_1 + k_2x_2, k_1y_1 + k_2y_2)) =$

$= (2(k_1x_1 + k_2x_2) - (k_1y_1 + k_2y_2), \; 4(k_1x_1 + k_2x_2) - 2(k_1y_1 + k_2y_2))$

$= (k_1(2x_1 - y) + k_2(2x_2 - y_2), \; k_1(4x_1 - 2y_1) + k_2(4x_2 - 2y_2))$

$= k_1 g(x_1,y_1) + k_2 g(x_2,y_2)$
g - linear $\Big\} \to g \in End_{\mathbb{R}}(\mathbb{R}^2)$

* $h(k_1(x_1,y_1,z_1) + k_2(x_2,y_2,z_1)) = h((k_1x_1 + k_2x_2, k_1y_1 + k_2y_2, k_1z_1 + k_2z_2)) =$

$= (k_1x_1 + k_2x_2 - k_1y_1 - k_2y_2, \; k_1y_1 + k_2y_2 - k_1z_1 - k_2z_2, \; k_1z_1 + k_2z_2 - k_1x_1 - k_2x_2)$

$= (k_1(x_1 - y_1) + k_2(x_2 - y_2), \; k_1(y_1 - z_1) + k_2(y_2 - z_2), \; k_1(z_1 - x_1) + k_2(z_2 - x_2))$

$= k_1 h(x_1,y_1,z_1) + k_2 h(x_2,y_2,h_2)$
h - linear $\Big\} \to h \in End_{\mathbb{R}}(\mathbb{R}^3)$

2. 11. 2023

Algebra - Seminar 5

- Let V be a vector space and $X \subseteq V$.

  $\langle X \rangle = \cap \{ S \leq V \mid X \subseteq S \}$

  $\hookrightarrow$ subspace generated by X

- V = v.s., $v_1, v_2, \dots, v_n \in V$, $n \in \mathbb{N}^*$

  $k_1, \dots, k_n \in K$ (K - field)

  A linear combination is $k_1v_1 + \dots + k_nv_n$

- V = v.s. over K, $\emptyset \neq X \subseteq V$. Then

  $\langle X \rangle = \{ k_1v_1 + \dots + k_nv_n \mid k_i \in K, v_i \in V, i = \overline{1,n} \}$

  $\hookrightarrow$ the set of all linear combinations

  $\langle x_1, \dots, x_n \rangle = \{ k_1x_1 + \dots + k_nx_n \mid k_i \in K, x_i \in V, i = \overline{1,n} \}$

1. Det. the following generated subsaces:

(i) $\langle 1, X, X^2 \rangle$ in ${}_{\mathbb{R}}\mathbb{R}[X]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in ${}_{\mathbb{R}}M_2(\mathbb{R})$

(i) $\langle 1, X, X^2 \rangle = \{ k_1 \cdot 1 + k_2 \cdot X + k_3 X^2 \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ k_1 + k_2 X + k_3 X^2 \mid k_1, k_2, k_3 \in \mathbb{R} \} =$

$= \{ X^2 k_3 + X k_2 + k_1 \mid k_1, k_2, k_3 \in \mathbb{R} \} = \mathbb{R}_2[X]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle =$

$= \left\{ k_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R} \right\} = M_2(\mathbb{R})$

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3. $a = (-2,1,3)$, $b = (3,-2,-1)$; $c = (1,-1,2)$, $d = (-5,3,4)$, $e = (-9,5,10)$

$\langle a, b\rangle = \langle c, d, e\rangle$?

we show that c, d, e can be written as a linear combination of a & b

$\cancel{c = a + b = a +}$

$c = k_1 a + k_2 b$

$(1,-1,2) = (-2k_1 + 3k_2,\ k_1 - 2k_2,\ 3k_1 - k_2)$

$$\begin{cases} -2k_1 + 3k_2 = 1 \\ k_1 - 2k_2 = -1 \rightarrow k_1 = 2k_2 - 1 \\ 3k_1 - k_2 = 2 \end{cases}$$

$\rightarrow -2(2k_2 - 1) + k_2 = 1$

$-4k_2 + 2 + 3k_2 = 1$

$-k_2 = -1 \rightarrow k_2 = 1 \rightarrow$

$\rightarrow k_1 = 2 - 1 = 1$

$c = 1 \cdot a + 1 \cdot b$

$d = k_3 a + k_4 b$

$(-5,3,4) = (-2k_3 + 3k_4,\ k_3 - 2k_4,\ 3k_3 - k_4)$

$$\begin{cases} -2k_3 + 3k_4 = -5 \\ k_3 - 2k_4 = 3 \rightarrow k_3 = 2k_4 + 3 \\ 3k_3 - k_4 = 4 \end{cases}$$

$\rightarrow -4k_4 - 6 + 3k_4 = -5$

$-k_4 = 1 \rightarrow k_4 = -1$

$k_3 = -2 + 3 = 1$

$d = 1 \cdot a + (-1) \cdot b = a - b$

$e = k_5 a + k_6 b = 3a - b$

$(-9,5,10) = (-2k_5 + 3k_6,\ k_5 - 2k_6,\ 3k_5 - k_6)$

$$\begin{cases} -2k_5 + 3k_6 = -9 \\ k_5 - 2k_6 = 5 \rightarrow k_5 = 5 + 2k_6 \\ 3k_5 - k_6 = 10 \end{cases}$$

$\rightarrow -10 - 4k_6 + 3k_6 = -9$

$-k_6 = 1 \rightarrow k_6 = -1$

$k_5 = 5 + 2(-1)$

$= 5 - 2 = 3$

$\langle c, d, e\rangle = \{k_1 c + k_2 d + k_3 e \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{k_1 a + k_1 b + k_2 a - k_2 b + 3k_3 \; e - k_3 b \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{a(k_1 + k_2 + 3k_3) + b(k_1 - k_2 - k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \langle a, b \rangle$

4. $S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

$S, T \leq \mathbb{R}^3$ (Hw)

$\mathbb{R}^3 = S \oplus T$

$S + T = \{s + t \mid s \in S, t \in T\}$

If $S \cap T = \{0\}$, then we denote it by $S \oplus T$ (direct sum

$S \cap T = \{(x, y, z) \mid x + y + z = 0 \; \& \; x = y = z\} = \{(0,0,0)\}$

$\forall v \in \mathbb{R}^3, \exists! s \in S, t \in T : v = s + t$

$v = (x, y, z)$

$s = (x_1, y_1, z_1) \in S \rightarrow x_1 + y_1 + z_1 = 0$

$t = (x_2, x_2, x_2) \in T \quad (x = y = z)$

$v = s + t \Leftrightarrow$

$\Leftrightarrow (x, y, z) = (x_1 + x_2, y_1 + x_2, z_1 + x_2)$

$\Leftrightarrow \begin{cases} x_1 + x_2 = x \\ y_1 + x_2 = y \\ z_1 + x_2 = z \\ x_1 + y_1 + z_1 = 0 \rightarrow x_1 = -y_1 - z_1 \end{cases} \Leftrightarrow \begin{cases} -y_1 - z_1 + x_2 = x \\ y_1 + x_2 = y \\ z_1 + x_2 = z \end{cases}$ +

$3x_2 = x + y + z \rightarrow$

$\rightarrow x_2 = \dfrac{x+y+z}{3} = y_2 = z_2$

$x_1 = x - x_2 = x - \dfrac{x+y+z}{3} = \dfrac{2x - y - z}{3}$

$y_1 = \dfrac{2y - x - z}{3}$

$z_1 = \dfrac{2z - x - y}{3}$

$\rightarrow$ the sol is unique

The sistem

$$\begin{cases} x_1 + x_2 = x \\ x_2 + y_2 = y \\ x_2 + z_2 = z \\ x_1 + y_1 + z_1 = 0 \end{cases}$$

has unique sol. because $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix} \neq 0$

5. S = the set of all even functions in $\mathbb{R}^{\mathbb{R}}$ ($f(-x) = f(x)$)

T = the set of all odd functions in $\mathbb{R}^{\mathbb{R}}$ ($f(-x) = -f(x)$)

Hw $S, T \leq \mathbb{R}^{\mathbb{R}}$

Show that $\mathbb{R}^{\mathbb{R}} = S \oplus T$

* $S \cap T = \{ f \in \mathbb{R}^{\mathbb{R}}$ s.t. $f(-x) = f(x)$ and $f(-x) = -f(x) \} = \{0\}$

$f(x) = -f(x) \to 2f(x) = 0 \to f(x) = 0$

* $\mathbb{R}^{\mathbb{R}} = S \oplus T \Leftrightarrow \forall f \in \mathbb{R}^{\mathbb{R}}, \exists! g \in S$ (g-even) $\exists! h \in T$ (h-odd)

s.t. $f = g + h$

$f(x) = g(x) + h(x)$

$f(-x) = g(-x) + h(-x)$

$f(-x) = g(x) - h(x)$

$$\begin{cases} f(x) = g(x) + h(x) \\ f(-x) = g(x) - h(x) \end{cases} \quad +$$

$f(x) + f(-x) = 2g(x) \to g(x) = \dfrac{f(x) + f(-x)}{2}$

$f(x) = \dfrac{f(x) + f(-x)}{2} + h(x) \to \cancel{2f(x) = f(x) + f(-x) + 2h(x)} \to$

$\Rightarrow h(x) = \dfrac{2f(x) - f(x) - f(-x)}{2} = \dfrac{f(x) - f(-x)}{2}$

$\to$ has a unique sol.

$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(x) + f(-x)}{2} = g(x) \to g - \text{even} \to g \in S$

$h(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x) \to$

$\to h(x) - \text{odd} \to h(x) \in T$

$\Rightarrow \mathbb{R}^{\mathbb{R}} = S \oplus T$

6. Hw: g, h

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+y, x-y)$

Show that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$

$f: V \to V'$ is a linear map if $\begin{cases} f(v_1 + v_2) = f(v_1) + f(v_2) \\ f(kv) = k f(v) \end{cases}$ $\forall v_1, v_2 \in$, $\forall k \in K$

$\Leftrightarrow f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$

• $f: V \to V'$ is an edomorphism if $V = V'$

Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

$k_1, k_2 \in \mathbb{R}$

$f(k_1 v_1 + k_2 v_2) = f(k_1(x_1, y_1) + k_2(x_2, y_2)) = f((k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2))$

$= (k_1 x_1 + k_2 x_2 + k_1 y_1 + k_2 y_2,\ k_1 x_1 + k_2 x_2 - k_1 y_1 - k_2 y_2)$

$= (k_1(x_1 + y_1) + k_2(x_2 + y_2),\ k_1(x_1 - y_1) + k_2(x_2 - y_2))$

$= (k_1(x_1 + y_1), k_1(x_1 - y_1)) + (k_2(x_2 + y_2), k_2(x_2 - y_2))$

$= k_1(x_1 + y_1, x_1 - y_1) + k_2(x_2 + y_2, x_2 - y_2)$

$\left.\begin{array}{l} = k_1 f(v_1) + k_2 f(v_2) \\ f: \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{R}^2 \end{array}\right\} \to f \in End_{\mathbb{R}}(\mathbb{R}^2)$

7., 8. H

8. $f(x,y) = (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$

↳ counterclockwise of angle α

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix}$$

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9 Det. the kernel and the image of

a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+y, x-y)$

b) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (2x-y, 4x-2y)$

c) $h: \mathbb{R}^3 \to \mathbb{R}^3$, $h(x,y,z) = (x-y, y-z, z-x)$

a) $Ker(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = (0,0)\}$

$(x+y, x-y) = (0,0) \to \begin{cases} x+y=0 \\ x-y=0 \end{cases} \to x = y = 0 \to$

$\to Ker(f) = \{(0,0)\} \dashrightarrow f$ - injective

$Im(f) = \{f(x,y) \mid x,y \in \mathbb{R}\} = \{(x+y, x-y) \mid x,y \in \mathbb{R}\}$

$= \{(x,x) + (y,-y) \mid x,y \in \mathbb{R}\}$

$= \{x(1,1) + y(1,-1) \mid x,y \in \mathbb{R}\} = \langle (1,1), (1,-1) \rangle$

b) $Ker(g) = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = (0,0)\}$

$g(x,y) = 0 \leftrightarrow (2x-y, 4x-2y) = 0$

$\begin{cases} 2x-y=0 \to y = 2x \\ 4x-2y=0 \to 4x - 2\cdot 2x = 0 \to 4x - 4x = 0, \forall x \in \mathbb{R} \end{cases}$

$Ker(g) = \{(x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

$Im(g) = \{g(x,y) \mid x,y \in \mathbb{R}\} = \{(2x-y, 4x-2y) \mid x,y \in \mathbb{R}\}$

$= \{(2x, 4x) + (-y, -2y) \mid x,y \in \mathbb{R}\}$

$= \{x(2,4) + y(-1,-2) \mid x,y \in \mathbb{R}\} =$

$= \{2x(1,2) + (-y)(1,2) \mid x,y \in \mathbb{R}\} = \langle (1,2) \rangle$

c) $Ker(h) = \{(x,x,x) \mid x \in \mathbb{R}\}$

$Imf(h) = \langle (-1,1,0), (-1,0,1) \rangle$