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Algebra - Seminar 13

1. Consider a (63, 56) - code

(i) the no of digits in the message before

(ii) the no of check digits

(iii) the ...

(i) $k = 56$

(ii) $n-k = 63-56 = 7$

(iii) $\frac{k}{n} = \frac{56}{63}$

(iv) $2^{n-k} = 2^7$

2. $H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

Syndrome	000	001	010	011	101	110	111
coset leader	000000	001000	010000	000010	000101	000100	000001

Let $\gamma : \mathbb{Z}_2^k \to \mathbb{Z}_2^k$, $\eta : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ the linear maps corresponding to G, H resp. $V = \operatorname{Im} \gamma = \ker \eta$

$v \in V$ is a code vector, $e \in \mathbb{Z}_2^n$ is an error, then the recived word $u = v + e$

[syndrome of u] $= H \cdot [u]$

coset leader = the most likely error pattern

$e = u - v$

decode: 101110, 011000, 001011, 111111, 110011

To decode a message:

$u_1 = 101110$ $\quad [u_1] = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

1. Calculate the syndrome of u_1

$H \cdot [u_1] = 000$ (DU)

2. We find the coset leader that coresponds to 000 (in the table)

$000 \longmapsto 000000 \; (= e_1)$

3. The word will be: $v_1 = u_1 + e_1 = \boxed{101|110}$ check digits

4. Drop the check digits to obtain the messag: 110

- $u_2 = 011000$

  $S_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

  $e_2 = 000010$

  $v_1 = 011010 \to$ the m: 010

- $u_3 = 001011$

  $s_3 = 101$

  $v_3 = u_3 + e_3 = 001101 \to m = 101$

- $u_4 = 011111$

  $s_4 = 101$

  $v_4 = 111001 \to m = 001$

- $u_5 = 110011$

  $s_5 = 010$

  $v_5 = 1000011 \to m = 011$

3. A (7,4)-code is defined by $\begin{cases} u_1 = u_4 + u_5 + u_7 \\ u_2 = u_4 + u_6 + u_7 \\ u_3 = u_4 + u_5 + u_6 \end{cases}$, where

u_4, u_5, u_6, u_7 = message digits, u_1, u_2, u_3 = check digits

Write G, H. Decode: 0000111, 0001111

$$H = \left(\begin{array}{ccc|cccc} \multicolumn{3}{c|}{\overbrace{\qquad\quad}^{I_3}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad}^{P}} \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \qquad G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$H \cdot [u_1] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to u_1$ - code word

$H[u_2] = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\to v_2 = u_2 + e_2 = 0001111 + 0001000 = 00\underbrace{10111}$ $\to$

$\to m_2 = 111$

Syndrome	111
Coset leader	0001000

pt. a det coset leader ne uităm prima dată dacă nu avem pe vreo coloană din H syndrome-ul. Dacă avem, în coset leader vom avea 1 pe poziția coloanei respective, și 0 în rest. Dacă nu, încercăm să facem sume de coloane...

4. (3, 2) - parity check code

(3, 1) - reapeating code

for (3, 2) parity code: 2^{3-2} syndroms = 2

for m = 01 ⇝ 101

• m = 10 ⇝ 110

$G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \to H = (111)$ P

$000 \to H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$

$001 \to H \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$

$010 \to H \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$

011 → 1

100 → 1

101 → 2

110 → 2

111 → 3

for (3, 1) reapiting

$2^{3-1} = 4$ sydroms

m = 1 ⇝ $\boxed{111}$

$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ →P $H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

000 → 00

001 → 11

010 → 01

011 → 10

100 → 10

101 → 01

110 → 11

111 → 00

5. Construct a table of cosit leaders and syndroms for the (7, 4) - code with $H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$

$2^{7-4} = 2^3 = 8$

Syndromes	000	001	010	011	100	101	110	111
Coset leader	0000000	0010000	0100000	0000001	1000000	0000010	0000100	0001000

6. Det. H, all syndromes and coset leaders of the $(5,3)$ code with $G = \begin{pmatrix} P \\ I_3 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$2^{5-3} = 4$ syndromes

$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

Syndromes	00	01	10	11
Coset leader	00000	01000	10000	00010

Construct a table of coset leaders and syndromes for the

7. $(3,1)$ code generated by $= 1 + X + X^2 \in \mathbb{Z}_2[X]$

The encoder is an injective linear map

$\gamma : \mathbb{Z}_2^k \to \mathbb{Z}_2^n$

$G = [\gamma]_{EE'}$ $\quad E =$ canonical basis of $\mathbb{Z}_2^k$

$E' =$ —— // —— $\mathbb{Z}_2^n$

$k = 1, n = 3$

$\gamma : \mathbb{Z}_2 \to \mathbb{Z}_2^3$

- $m = 1$

$p_1 = (1)$

$m \cdot X^2 = X^2$ $\quad u_1 = 1 + X + X^2 = \boxed{111}$

$$\begin{array}{l|l} X^2 & 1+X+X^2 \\ \underline{X^2+X+1} & 1 \\ X+1 & \end{array}$$

$G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Syndromes	00	01	10	11
Coset leader	000	010	100	001

$H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

8. $(7,3)$ - code generated by $p = 1 + x^2 + x^3 + x^4 = \mathbb{Z}_2[x]$

$f: \mathbb{Z}_2^3 \to \mathbb{Z}_2^7$

- $100 \to m = 1$

$m \cdot x^4 = x^4$

$$
\begin{array}{l|l}
x^4 & x^4 + x^3 + x^2 + 1 \\
\hline
x^4 + x^3 + x^2 + 1 & 1 \\
\hline
x^3 + x^2 + 1 &
\end{array}
$$

$x^3 + x^2 + 1 \to v = 1 + x^2 + x^3 + x^4 \to \boxed{1011|100}$

- $010 \to m = x$

$m \cdot x^4 = x^5$

$$
\begin{array}{l|l}
x^5 & x^4 + x^3 + x^2 + 1 \\
\hline
x^5 + x^4 + x^3 + x & x + 1 \\
\hline
x^4 + x^3 + x & \\
x^4 + x^3 + x^2 + 1 & \\
\hline
x^2 + x + 1 &
\end{array}
$$

$x^2 + x + 1 \to v = 1 + x + x^2 + x^5 \to \boxed{1110|100}$

- $001 \to m = x^2$

$m \cdot x^4 = x^6$

$$
\begin{array}{l|l}
x^6 & x^4 + x^3 + x^2 + 1 \\
\hline
x^6 + x^5 + x^4 + x^2 & x^2 + x \\
\hline
x^5 + x^4 + x^2 & \\
x^5 + x^4 + x^3 + x & \\
\hline
x^3 + x^2 + x &
\end{array}
$$

$x^3 + x^2 + x \to v = x + x^2 + x^3 + x^6 \to \boxed{0111|001}$

$$
G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to P \qquad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}
$$

Syndroms	0000	0001	0010	0011	0100	0101	0110	0111	1000
Coset Coder	0000000	0001000	0010000	0011000	0100000	0000110	0110000	0000001	1000000

1001	1010	1011	1100	1101	1110	1111
			00001000	110 1000	0000010	0001010