10.11.2023

Analysis - Lecture 6

Limits of functions

- 2nd part -

Def: Consider $\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A'$ (in $\overline{\mathbb{R}}$). f is said to have a limit at A if $\forall (a_n) \in A \setminus \{a\}$ s.t. $\lim_{n\to\infty} a_n = a$, the sequence $\boxed{(f(a_n)) \text{ has a limit}}$. (1)

so the lim can change if we take another seq

Property: Consider $\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A'$. Then f has a limit at $a \Leftrightarrow$ (if and only if) $\exists\, \ell \in \overline{\mathbb{R}}$ s.t. $\forall (a_n) \subseteq A \setminus \{a\}$, with the $\lim_{n\to\infty} a_n = a$ it holds $\boxed{\lim_{n\to\infty} f(a_n) = \ell}$ (2)

remains the same for all sequences

Remark: The previous property is useful when trying to prove that the limit of a function at a point does not exist.

- We emphasize two sequences $(a_n) \subseteq A \setminus \{a\}$, $(b_n) \subseteq A \setminus \{a\}$ for which $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$ but $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$

- or: We could also emphasize $(c_n) \in A \setminus \{a\}$ with $\lim_{n\to\infty} c_n = a$ for which $\lim_{n\to\infty} f(c_n) \nexists$

Examples:

a) Prove that $\lim_{x\to 4} x^2 = 16$

We have $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

domain $\boxed{A = \mathbb{R}}$

$\boxed{A' = \overline{\mathbb{R}}}$, $4 \in \overline{\mathbb{R}}$

We prove that $\forall\ (a_n) \subseteq \mathbb{R}\setminus\{4\}$, with $\lim_{n\to\infty} a_n = 4$, $\lim_{n\to\infty} f(a_n) = 16$

Step 1: Consider $(a_n) \subset \mathbb{R}\setminus\{4\}$ with $\lim_{n\to\infty} a_n = 4$, random

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n^2 = \left(\lim_{n\to\infty} a_n\right)^2 = 4^2 = 16$$

$(a_n) \to$ random $\to \forall$

$\to \lim_{x\to 4} x^2 = 4^2 = 16$

Remark: 4 may be replaced by a random $a \in \overline{\mathbb{R}}$

b) Prove that $\nexists \lim_{x\to\infty} \sin x$

We use the remark and emphasize 2 sequences

$$\left.\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ f(x) = \sin x \\ A = \mathbb{R} \to A' = \overline{\mathbb{R}} \\ \qquad \infty \in \overline{\mathbb{R}} \end{array}\right\} \text{framework}$$

$a_n = 2\pi n \in \mathbb{R}\setminus\{\infty\}$, $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin(2\pi n) = 0$

$=$ $\qquad\qquad\qquad \neq$

$b_n = 2\pi n + \frac{\pi}{2} \in \mathbb{R}\setminus\{\infty\}$, $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \sin(2\pi n + \frac{\pi}{2}) = 1$

$\to \nexists \lim_{x\to\infty} \sin x$

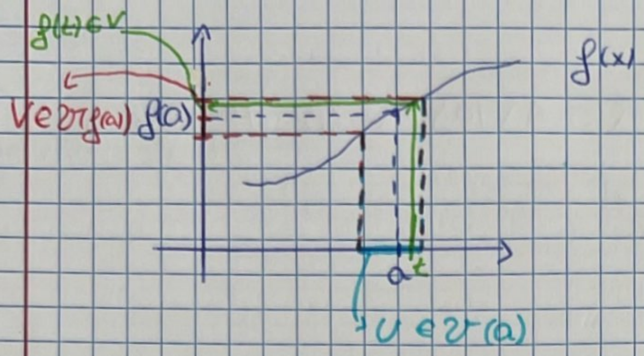
c) $\lim_{n\to -4} \ln x$

$f: (0,\infty) \to \mathbb{R}$

$f(x) = \ln x$

$A = (0,\infty) \to A' = [0,\infty]$

$-4 \notin A' \to$ the $\lim_{x\to -4} f(x)$ can not be defined!



Theorem (Characterisation of limits with neighbourhoods)

Framework: $\emptyset \neq A \subseteq \mathbb{R}$

$f: A \to \mathbb{R}$

$a \in A'$

$\ell \in \overline{\mathbb{R}}$

$$\ell = \lim_{x \to a} f(x) \iff \forall V \in \mathcal{V}(\ell),\ \exists U \in \mathcal{V}(a) \text{ s.t. }$$
$$\forall x \in U \cap A \setminus \{a\},\ f(x) \in V$$

Remark: As previosly mentioned each theorem which is expressed in terms of neighbourhoods can be equivalently rewritten in terms of balls.

For limits, as a convention V is replaced ($\sim$) by $B(\ell, \varepsilon)$ $\to (\ell-\varepsilon, \ell+\varepsilon)$, if $\ell \in \mathbb{R}$

$\to (\varepsilon, \infty]$, $\ell = \infty$

$\to [-\infty, -\varepsilon)$, if $\ell = -\infty$

$U \sim B(a, \delta)$ $\to (a-\delta, a+\delta)$, if $a \in \mathbb{R}$

$\to (\delta, \infty]$, if $a = \infty$

$\to [-\infty, -\delta)$, if $a = -\infty$

Theorem (Characterisation with ε & δ for the limit of a function)

Framework: $\left.\begin{array}{l} \emptyset \neq A \subseteq \mathbb{R} \\ f: A \to \mathbb{R} \\ a \in A' \\ \ell \in \overline{\mathbb{R}} \end{array}\right\}$ Then $\ell = \lim_{x \to a} f(x) \iff$

- $a, \ell \in \mathbb{R}$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x \in A \setminus \{a\}$ with $|x - a| < \delta$ ($\hookrightarrow x \in U \cap A \setminus \{a\}$)

it holds $|f(x) - \ell| < \varepsilon$ ($\hookrightarrow f(x) \in V$)

- $a = -\infty$, $\ell \in \mathbb{R}$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $x < -\delta$ it holds $|f(x) - \ell| < \varepsilon$

- $a = \infty$, $\ell = -\infty$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $x > \delta$ it holds $f(x) < -\varepsilon$

Remark: If in the definition of a limit at a point we impose the extra condition:

- $a_n < a \to$ the left limit to a

$\overset{\text{not}}{=} \lim\limits_{\substack{x \to a \\ x < a}} f(x) \overset{\text{or}}{=} \lim\limits_{x \nearrow a} f(x)$

- $a_n > a \to$ the right limit to a

$\overset{\text{not}}{=} \lim\limits_{\substack{x \to a \\ x > a}} f(x) \overset{\text{or}}{=} \lim\limits_{x \searrow a} f(x)$

Property: a) If $\exists \lim\limits_{x \searrow a} f(x)$, $\exists \lim\limits_{x \nearrow a} f(x)$ and they are $=$ $\Big\} \to \exists \lim\limits_{x \to a} f(x) =$ both of them

b) If both side limits are well defined, $\exists \lim\limits_{x \to a} f(x)$ $\Big\} \to$

$\to \exists \lim\limits_{x \searrow a} f(x) = \lim\limits_{x \nearrow a} f(x) = \lim\limits_{x \to a} f(x)$

Remark: $\cdot\ \infty \in A'$ then $\lim\limits_{x \to \infty} f(x) = \lim\limits_{\substack{x \to \infty \\ x < \infty}} f(x)$

$\cdot\ 0 \in A'$ then $\lim\limits_{x \to 0} \ln x = \lim\limits_{\substack{x \to 0 \\ x > 0}} \ln x$ (because $\lim\limits_{\substack{x \to 0 \\ x < 0}} \ln x$ is not well defined)

$\cdot$ If both sides limits are well defined and $\lim\limits_{\substack{x \to a \\ x < a}} f(x) \neq \lim\limits_{\substack{x \to a \\ x > a}} f(x) \to \nexists \lim\limits_{x \to a} f(x)$

Example:

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

$$\left.\begin{aligned} &\lim_{\substack{x \to 0 \\ x < 0}} \frac{1}{x} = \frac{1}{-0} = -\infty \\ &\lim_{\substack{x \to 0 \\ x > 0}} \frac{1}{x} = \frac{1}{+0} = +\infty \end{aligned}\right\} \to \nexists \lim_{x \to 0} f(x)$$

$\cdot\ \lim\limits_{\substack{x \to 0 \\ x < 0}} f(x)$

$(a_n) \subseteq \mathbb{R}$ with $\lim\limits_{n \to \infty} a_n = 0$

$!\ a_n < 0$

$a_n = -\frac{1}{n} \to \lim\limits_{n \to \infty} a_n = 0,\ a_n < 0 \to \lim\limits_{n \to \infty} f(a_n) = \lim\limits_{n \to \infty} -n = -\infty$

$\cdot\cdot\ \lim\limits_{\substack{x \to 0 \\ x > 0}} f(x)$

$(b_n) \subseteq \mathbb{R}$ with $\lim\limits_{n \to \infty} b_n = 0$

$!!\ b_n > 0$

$b_n = \frac{1}{n} \to \lim\limits_{n \to \infty} f(b_n) = \lim\limits_{n \to \infty} n = \infty$

$(a_n), (b_n) \xrightarrow[\text{def}]{\text{contradiction}} \nexists \lim\limits_{x \to 0} f(x)$

Continuous functions

Def: $\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A$ $\Big\}$ f is said to be continuous at a if $\forall (a_n) \subseteq A$ with $\lim\limits_{n\to\infty} (a_n) = a$ it holds $\lim\limits_{n\to\infty} f(a_n) = f(a)$

Theorem (with neighbourhoods)

Framework: $\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A$ $\Big\}$ f is continuous at $a \Leftrightarrow \forall V \in \mathcal{V}(f(a)), \exists U \in \mathcal{V}(a)$ s.t. $\forall x \in U \cap A$, $f(x) \in V$

Theorem (with ε & δ)

Framework: —〃— $\quad$ f is continuous at $a \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$, s.t. $\forall x \in A$ with $|x - a| < \delta$, to hold $|f(x) - f(a)| < \varepsilon$

Remark: The difference between the frameworks for limits of functions & continuity lies in the set where the point a is considered:

LF: $a \in A'$

C: $a \in A$

Example:

a) $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$

$a = 0 \notin \mathbb{R}\setminus\{0\}$ & $\nexists \lim\limits_{x\to 0} f(x)$

but we cannot say that f is not CONTINEOUS at 0 because the continuity of f is not well defined at 0

R: In order for f not to be continuous at a point, that point should be in the domain (it should satisfy the framework for which the notion of continuity is considered). • However, if we consider a new function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ t, & x = 0 \end{cases}$$

$A = \mathbb{R}$, and $0 \in$ domain $\to$ the notion of continuity is well defined

... $\to$ g is not continuous at 0

Remark:

If we want to prove that f is not continuous at at a point $a \in A$, one option is to emphasize a sequence

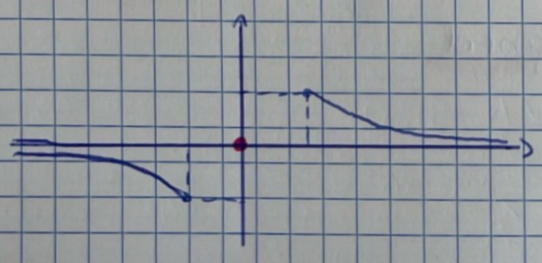
$$\left.\begin{matrix} (x_n) \subseteq A \\ \lim\limits_{n \to \infty} x_n = a \end{matrix}\right\} \text{ but for which } \lim_{n \to \infty} f(x_n) \neq f(a)$$

or we could use (like for limits) two sequences

$$\left.\begin{matrix} (a_n) \in A \\ (b_n) \in A \\ \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} b_n = a \end{matrix}\right\} \text{ but for which } \lim_{n \to \infty} f(a_n) \neq \lim_{n \to \infty} f(b_n)$$

b) Study the continuity of the function

$$f(x) = \begin{cases} \frac{1}{x} : & |x| > 1 \\ 0 : & x = 0 \end{cases}$$



- f is continuous on its domain $\mathbb{R} \setminus [-1,1] \cup \{0\}$

Theorem

$$\left.\begin{array}{l} \emptyset \neq A \subseteq \mathbb{R} \\ f: A \to \mathbb{R} \\ a \in \operatorname{Izo} A \end{array}\right\} \to f \text{ is } \underline{\text{continuous}} \text{ at } a$$

Proof: We want f - continuous at a

$\xleftrightarrow[\text{neigh.}]{T}$ $\forall V \in \mathcal{V}(f(a)), \exists U \in \mathcal{V}(a)$ s.t. $\forall x \in U \cap A$, $f(x) \in V$

Consider $\boxed{V \in \mathcal{V}(f(a)) \text{ random}}$

? $f(x) \in V$ [I know that $f(a) \in V$]

$a \in \operatorname{Izo} A \to \boxed{\exists\, T \in \mathcal{V}(a)}$ s.t. $T \cap A = \{a\}$

if $\boxed{x \in T \cap A} \to x = a \to \boxed{f(x) = f(a) \in V}$

V - random $\to \forall$

$\to f$ - continuous at a

Remark: Each function is continuous at all the points from its isolated points of the domain

$$\left.\begin{array}{l} f: A \to \mathbb{R} \\ a \in \operatorname{Izo} A \end{array}\right\} \to f \text{ is continuous at } A$$

Revisit the example: $f(x) = \begin{cases} \frac{1}{x}, & |x| > 1 \\ 0, & x = 0 \end{cases}$

$A = (-\infty, -1) \cup \{0\} \cup \{1, \infty\}$

$0 \in \operatorname{Izo} A$ $\left(\left(-\frac{1}{2}; \frac{1}{2}\right) \cap A = \{0\} \right) \to f$ is continuous at 0

!!! $\underline{0 \notin A'}$, the notion of the $\underline{\text{limit of } f \text{ at } 0}$ is $\underline{\text{not well defined}}$

Remark: The conection between limits of functions & continuity

Framework: $\emptyset \neq A \subseteq \mathbb{R}$

$f : A \to \mathbb{R}$

$a \in \overline{\mathbb{R}}$

- $a \in \text{Izo}\, A$ → f is continuous at a

  → $\lim\limits_{x \to a} f(x)$ is not well defined ($a \notin A'$)

- $a \in A' \setminus A$ → $\lim\limits_{x \to a} f(a)$ is well defined

  → the notion of continuity at a is not well defined ($a \notin A$)

- $a \in A \cap A'$ → both notions are well defined

If $\exists \lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} f(x) = f(a)$ then f is continuous at a.