Analysis - Homework 4

1. Compute the sum (if the exist)

a) $\sum_{n\geq 3} \frac{7}{9^n} = 7 \sum_{n\geq 3} \frac{1}{9^n} \to 7 \cdot \sum_{n=3}^{\infty} \frac{1}{9^n} = 7 \cdot \frac{41}{72} = \frac{287}{72}$

$\sum_{n\geq 3} \frac{1}{9^n}$ - geometric series with $q = \frac{1}{9} < 1$

$$\sum_{n=3}^{\infty} \frac{1}{9^n} = \sum_{n=1}^{\infty} \frac{1}{9^n} - \left(\frac{1}{9} + \frac{1}{9^2}\right) = \frac{1}{1-\frac{1}{9}} - \left(\frac{1}{9} + \frac{1}{18}\right) =$$

$$= \frac{9}{8} - \frac{10}{18} = \frac{81-40}{72} = \frac{41}{72}$$

b) $\sum_{n\geq 4} \frac{3^{n-3} + (-4)^{n+3}}{5^n} = \sum_{n\geq 4} \frac{3^{n-3}}{5^n} + \sum_{n\geq 4} \frac{(-4)^{n+3}}{5^n}$

$$= \sum_{n\geq 4} \left(\frac{3}{5}\right)^{n-3} \frac{1}{5^3} + \sum_{n\geq 4} \left(\frac{-4}{5}\right)^n \cdot (-4)^3 = \frac{1}{125} \sum_{n\geq 4} \left(\frac{3}{5}\right)^{n-3} + (-64) \sum_{n\geq 4} \left(\frac{-4}{5}\right)^n$$

$$= \frac{1}{125} \cdot \frac{53}{18} + (-64) \cdot \frac{1381}{1125}$$

$\sum_{n\geq 4} \left(\frac{3}{5}\right)^{n-3}$ - geometric series with $q = \frac{3}{5} \to \left|\frac{3}{5}\right| < 1 \to$

$$\to \sum_{n=4}^{\infty} \left(\frac{3}{5}\right)^{n-3} = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{n-3} - \left[\left(\frac{3}{5}\right)^{1-3} + \left(\frac{3}{5}\right)^{2-3} + \left(\frac{3}{5}\right)^{3-3}\right] =$$

$$= \frac{1}{1-\frac{3}{5}} - \left(\frac{25}{9} + \frac{5}{3} + \frac{1}{1}\right) = \frac{5}{2} - \frac{25+15+9}{9} = \frac{45-98}{18} = \frac{53}{18}$$

$\sum_{n\geq 4} \left(\frac{-4}{5}\right)^n$ - geometric series with $q = \left(-\frac{4}{5}\right) \to |q| < 1 \to$

$$\to \sum_{n=4}^{\infty} \left(\frac{-4}{5}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{4}{5}\right)^n - \left[\left(\frac{-4}{5}\right)^1 + \left(\frac{-4}{5}\right)^2 + \left(\frac{-4}{5}\right)^3\right] =$$

$$= \frac{1}{1 - \frac{-4}{5}} - \left(\frac{-4}{5} + \frac{16}{25} - \frac{64}{125}\right) = \frac{5}{9} - \frac{-100+80-64}{125} =$$

$$= \frac{5}{9} - \frac{-84}{125} = \frac{625+756}{1125} = \frac{1381}{1125}$$

c) $\sum_{n\geq 5} e^n$ - gemetric series with $q = e \to |q| > 1 \to$

$\to \sum_{n\geq 5} e^n = \infty - (e + e^2 + e^3 + e^4) = +\infty$

d) $\sum_{n\geq 2} \left(-\frac{1}{\pi}\right)^n$ - geometric series with $q = -\frac{1}{\pi} \to |q| < 1 \to$

$\to \sum_{n=2}^{\infty} \left(-\frac{1}{\pi}\right)^n = \frac{1}{1-\left(-\frac{1}{\pi}\right)} - \left(-\frac{1}{\pi}\right) = \frac{\pi}{\pi+1} + \frac{1}{\pi} = \frac{\pi^2+\pi+1}{\pi^2+\pi}$

e) $\sum_{n\geq 3} (-4)^n$ - geometric series with $q = -4 < -1 \to$ without sum

2. a) $\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1} \frac{1}{(2n)^2-1} = \sum_{n\geq 1} \frac{1}{(2n-1)(2n+1)} = \sum_{n\geq 1} \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1}$

$\frac{1}{(2^n-1)(2^n+1)} = \frac{A}{2^n-1} + \frac{B}{2^n-1}$

$1 = A(2^n+1) + B(2^n-1) = 2^n(A+B) + 1(A-B)$

$A + B = 0 \to A = -B$

$A - B = 1 \to -2B = 1 \to B = -\frac{1}{2} \to A = \frac{1}{2}$

$a_n = \frac{\frac{1}{2}}{2n-1} \to \lim_{n\to\infty} a_n = 0 \Big\} \to$

- telescopic series

$\to \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = a_1 - 0 = \frac{\frac{1}{2}}{2-1} - 0 = \frac{1}{2}$

b) $\sum_{n\geq 1} \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sum_{n\geq 1} \frac{\sqrt{n}-\sqrt{n+1}}{n-(n+1)} = -\sum_{n\geq 1} (\sqrt{n}-\sqrt{n+1})$

$a_n = \sqrt{n} \to \lim_{n\to\infty} a_n = \infty$

$\sum_{n=1}^{\infty} (\sqrt{n}-\sqrt{n+1}) = a_1 - a_n = 1 - \infty = -\infty$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = -(-\infty) = +\infty$

c) $\sum_{n\geq 5} \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n\geq 5}\frac{1}{n} - \sum_{n\geq 5}\frac{1}{n+1} - \frac{3}{2}\sum_{n\geq 5}\frac{1}{n+2}$

$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$

$1 = (n+1)(n+2)A + n(n+2)B + n(n+1)C$

$1 = (n^2+3n+2)A + (n^2+2n)B + (n^2+n)C$

$1 = n^2(A+B+C) + n(3A+2B+C) + 2A$

$2A = 1 \rightarrow A = \frac{1}{2}$

$A+B+C = 0 \rightarrow B+C = -\frac{1}{2} \rightarrow C = -\frac{1}{2} - B$

$3A+2B+C = 0 \rightarrow 2B+C = -\frac{3}{2} \rightarrow 2B - \frac{1}{2} - B = -\frac{3}{2}$

$B = -1 \rightarrow C = -\frac{1}{2} - 1 = -\frac{3}{2}$

d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right) = \sum_{n\geq 1} \ln \frac{n+1}{n} = \sum_{n\geq 1} \ln(n+1) - \ln n$

$a_n = \ln n \rightarrow \lim_{n\to\infty} a_n = +\infty$

$\sum_{n=1}^{\infty} a_{n+1} - a_n = \lim_{n\to\infty} a_n - a_1 = +\infty - 0 = +\infty$

e) $\sum_{n\geq 2} \frac{\ln(1+\frac{1}{n})}{\ln(n^{\ln(n+1)})} = \sum_{n\geq 2} \frac{\ln(n+1) - \ln n}{\ln(n+1)\cdot \ln n}$

$\frac{\ln(n+1) - \ln n}{\ln(n+1)\cdot \ln n} = \frac{A}{\ln(n+1)} \cdot \frac{B}{\ln n} = \frac{-1}{\ln(n+1)} + \frac{1}{\ln n} = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$

$\ln(n+1) - \ln n = A \ln n + B \cdot \ln(n+1) \rightarrow A = -1, \quad B = 1$

$a_n = \frac{1}{\ln n} \rightarrow \lim_{n\to\infty} a_n = 0$

$\sum_{n=2}^{\infty} a_n - a_{n+1} = a_2 - \lim_{n\to\infty} a_n = \ln 2$

3. Det. the nature

a) $\sum\limits_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}$

$x_n = \frac{n+7}{\sqrt{n^2+7}} \to \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{n+7}{\sqrt{n^2+7}} = \lim\limits_{n\to\infty} \frac{n(1+\overset{\to 0}{\frac{7}{n}})}{n\sqrt{1+\frac{7}{n}}} = 1 > 0 \to$

$\to \sum\limits_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}} - D$

b) $\sum\limits_{n\geq 1} \frac{1}{\sqrt[n]{n}}$

$x_n = \frac{1}{\sqrt[n]{n}} \to \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0 \to$

$\to \sum\limits_{n\geq 1} x_n - D$

c) $\sum\limits_{n\geq 1} \frac{1}{\sqrt[n]{n!}}$

$x_n = \frac{1}{\sqrt[n]{n!}} =$

e) $\sum\limits_{n\geq 1} \left(1+\frac{1}{n}\right)^{\infty}$

$x_n = \left(1+\frac{1}{n}\right)^{\infty} \to \lim\limits_{n\to\infty} x_n = e > 1 \to \sum\limits_{n\geq 1} x_n - D$

4. Determine the nature

a) $\sum\limits_{n\geq 1} \frac{2^n+3^n}{5^n}$

$x_n = \frac{2^n+3^n}{5^n} > 0 \ \forall n\geq 1 \to \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{3^n\left(\overset{\to 0}{\left(\frac{2}{3}\right)^n}+1\right)}{5^n} = \lim\limits_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$

$\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{5^{n+1}} \cdot \frac{5^n}{2^n+3^n} = \lim\limits_{n\to\infty} \frac{3^{n+1}}{5^{n+1}} \cdot \frac{5^n}{3^n} = \frac{3}{5} < 1 \to \sum\limits_{n\geq 1} x_n - C$

b) $\sum_{n\geq 1} \frac{2^n}{3^n+5^n}$

$$x^n = \frac{2^n}{3^n+5^n} > 0, \forall n \geq 1$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1}+5^{n+1}} \cdot \frac{3^n+5^n}{\cancel{2^n}} =$$

$$= \lim_{n\to\infty} \frac{2}{5^{\cancel{n+1}}\left[\underbrace{\left(\frac{3}{5}\right)^{n+1}}_{\to 0}+1\right]} \cdot \frac{\cancel{5^n}\left[\left(\frac{3}{5}\right)^n+1\right]}{1} = \frac{2}{5} < 1 \xrightarrow{D'A} \sum_{n\geq 1} x_n - C$$

5. Det. the nature

a) $\sum_{n\geq 1} \frac{1}{2n-1}$ - SPT

$$x_n = \frac{1}{2n-1} > \frac{1}{2n} = y_n$$

$$\left.\begin{aligned} y_n = \frac{1}{2n} \to \alpha = 1 \xrightarrow{g.h.s.} y_n - D \\ y_n < x_n \end{aligned}\right\} \xrightarrow{CIC} \sum_{n\geq 1} x_n - D$$

b) $\sum_{n\geq 1} \frac{1}{(2n-1)^2}$ - SPT

$$x_n = \frac{1}{(2n-1)^2} < \frac{1}{n^2} = y$$

$$\left.\begin{aligned} y = \frac{1}{n^2}, \alpha = 2 > 1 \xrightarrow{g.h.s} y_n - C \\ x_n < y_n \end{aligned}\right\} \xrightarrow{CIC} x_n - C$$

c) $\sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}$ - SPT

$$x_n = \frac{1}{\sqrt{4n^2-1}} > \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}$$

$$\left.\begin{aligned} \frac{1}{2n} - D \\ \frac{1}{2n} < x_n \end{aligned}\right\} \xrightarrow{CIC} \sum_{n\geq 1} x_n - D$$

d) $\sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$

$$x_n = \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}$$

6. Det. the nat.

a) $\sum_{n\geq 1} \frac{100^n}{n!}$

$$x_n = \frac{100^n}{n!} > 0$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{100^{n+1}}{\underbrace{(n+1)!}_{(n+1)}} \cdot \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1 \xrightarrow{D'A}$$

$\to \sum x_n - C$

b) $\sum_{n\geq 1} \frac{2^n \cdot n!}{n^n}$

$$x_n = \frac{2^n \cdot n!}{n^n} > 0$$

$$\to \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{n+1} \cdot (n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{2^n \cdot n!} = \lim_{n\to\infty} \frac{2(n+1) \cdot n^n}{(n+1) \cdot (n+1)^n} =$$

$$= 2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 2 \lim_{n\to\infty} \left[\left(1 + \frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}} = 2e^{-1} = \frac{2}{e} < 1 \to C$$

c) $\sum_{n\geq 1} \frac{3^n \cdot n!}{n^n}$

$$x_n = \frac{3^n \cdot n!}{n^n} > 0 \to \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{3}{e} > 1 \to D$$

d) $\sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}}$

$x_n = \frac{(n!)^2}{2^{n^2}}$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{[n!]^2} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n^2+2n+1}} \cdot 2^{n^2}$

$= \lim_{n\to\infty} (n+1)^2 \cdot 2^{n^2-n^2-2n-1} = \lim_{n\to\infty} (n+1)^2 \cdot 2^{-2n-1} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} = 0 < 1 \to C$

(2^{2n} - infinit mai puternic)

e) $\sum_{n\geq 1} \frac{n^2}{(2+\frac{1}{n})^n}$

$x_n = \frac{n^2}{(2+\frac{1}{n})^n}$

$\lim_{n\to\infty} \frac{n^2}{(2+\frac{1}{n})^n} = \lim_{n\to\infty} \frac{n^2}{2^n} = 0$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(2+\frac{1}{n+1})^{n+1}} \cdot \frac{(2+\frac{1}{n})^n}{n^2} =$

$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{1}{2} < 1 \to C$

7. Det the nature

a) $\sum_{n\geq 1} \frac{a^n}{n^n}$

$x_n = \frac{a^n}{n^n} > 0 \to \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a^n} = a \lim_{n\to\infty} \frac{n^n}{n^{n+1}(1+\frac{1}{n})^{n+1}} =$

$= 0 < 1 \to C$

b) $\sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2} \cdot a \right)^n$

$x_n = \left(\frac{n^2+n+1}{n^2} \cdot a \right)^n$

$a \neq 1 \to \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left\{ \frac{n^2(1+\underbrace{\frac{1}{n}}_{\to 0}+\underbrace{\frac{1}{n^2}}_{\to 0})}{n^2} \cdot a \right\}^n = \lim_{n\to\infty} a^n \overset{a>0}{=} \begin{cases} 0, & a\in(0,1) \\ \infty, & a>1 \end{cases} \to$

$\to \sum x_n - D, \ a>1$

$a=1 \to \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(1+\frac{\cancel{n^2}+n+1-\cancel{n^2}}{n^2}\right)^n = \lim_{n\to\infty} \left\{ \left(1+\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}} \right\}^{\frac{n+1}{n^2}\cdot n} =$

$= e \neq 0 \to \sum x_n - D, \ a=1$

$a\in(0,1) \to \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{a^n} = a \neq 0 \to \sum x_n - D$

$\to \sum x_n - D, \ \forall \ a>0$

c) $\sum_{n\geq 1} \frac{3^n}{2^n+a^n}$

I $a<2$

$\lim_{n\to\infty} \frac{3^n}{2^n+a^n} = \lim_{n\to\infty} \frac{3^n}{2^n(1+\underbrace{(\frac{a}{2})^n}_{\to 0})} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty \neq 0 \to$

$\to \sum x_n - D$

II $a=2$

$\lim_{n\to\infty} \frac{3^n}{2\cdot 2^n} = \frac{1}{2} \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty \to \sum x_n - D$

III $a>2$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3^n}{a^n(\underbrace{(\frac{2}{a})^n}_{\to 0}+1)} = \lim_{n\to\infty} \left(\frac{3}{a}\right)^n$

a) $a<3 \to \lim_{n\to\infty} x_n = +\infty \to \sum x_n - D$

b) $a=3 \to \lim_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n - D$

c) $a_n > 3 \rightarrow \lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{3^{\cancel{n+1}}}{a^{n+1}+2^{n+1}} \cdot \frac{a^n+2^n}{\cancel{3^n}} =$$

$$= 3 \cdot \lim_{n\to\infty} \frac{a^n\left(1+\left(\frac{2}{a}\right)^n\right)}{a^{n+1}\left(1+\left(\frac{2}{a}\right)^{n+1}\right)} = \frac{3}{a} < 1 \rightarrow \sum x_n = C$$

$a > 3$

$$\sum x_n - \begin{cases} D, & a \leq 3 \\ C, & a > 3 \end{cases}$$