Seminar 6

1. A pair of dice - one white die and one red die - is rolled two times. Compute the probability that the two pairs of numbers, obtained after the two rolls, are equal. (Example of favorable case: the white die shows number 2 and the red die shows number 4, both after the first roll and the second roll; example of unfavorable case: first roll "2 on white die, 4 on red die", second roll "4 on white die, 2 on red die".)

A: Let (W_k, R_k) be the pair of numbers obtained by the white die, respectively, the red die after the kth roll, $k \in \{1, 2\}$. The desired probability is

$$p = \sum_{i=1}^{6}\sum_{j=1}^{6} P(\{W_1 = i\} \cap \{R_1 = j\} \cap \{W_2 = i\} \cap \{R_2 = j\}) = \sum_{i=1}^{6}\sum_{j=1}^{6} \frac{1}{6^4} = \frac{6^2}{6^4} = \frac{1}{36},$$

where we use the independence of the events $\{W_1 = i\}, \{R_1 = j\}, \{W_2 = i\}, \{R_2 = j\}$, $\forall\, i, j \in \{1, \ldots, 6\}$. Alternatively, by the formula of total probability,

$$p = \sum_{i,j=1}^{6} P\big(\{W_2 = i\} \cap \{R_2 = j\}\big|\{W_1 = i\} \cap \{R_1 = j\}\big) P(\{W_1 = i\} \cap \{R_1 = j\}) = \sum_{i,j=1}^{6} \frac{1}{6^2} \cdot \frac{1}{6^2} = \frac{1}{36}.$$

2. A computer center has three printers A, B, and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A, B, and C are 0.5, 0.3, and 0.2, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers A, B, and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed when a printer jams. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

A: Let A_i, $i = \overline{1,3}$ denote the events that the program was routed to printers A, B and C, respectively, and let E denote the event that the program was destroyed. Then $\{A_1, A_2, A_3\}$ form a partition and we have

$$P(A_1) = 0.5,\ P(A_2) = 0.3,\ P(A_3) = 0.2,$$

and

$$P(E|A_1) = 0.02,\ P(E|A_2) = 0.06,\ P(E|A_3) = 0.1.$$

By the formula of total probability we have

$$P(E) = P(E|A_1) P(A_1) + (E|A_2) P(A_2) + (E|A_3) P(A_3) = 0.5 \cdot 0.02 + 0.3 \cdot 0.06 + 0.2 \cdot 0.1 = 0.048.$$

By Bayes' formula, we get

$$P(A_1|E) = \frac{0.5 \cdot 0.02}{0.048} \approx 0.2083; P(A_2|E) = \frac{0.3 \cdot 0.06}{0.048} = 0.375; P(A_3|E) = \frac{0.2 \cdot 0.1}{0.048} \approx 0.4166.$$

3. a) Let $(S, \mathcal{K}, P)$ be a probability space and $B \in \mathcal{K}$ such that $P(B) > 0$. Prove that $(B, \mathcal{K}_B, P(\cdot|B))$ is a probability space, where $\mathcal{K}_B := \{B \cap A : A \in \mathcal{K}\}$ and $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)}, A \in \mathcal{K}_B$.

b) Give examples, by considering a random experiment and its corresponding probability space $(S, \mathcal{K}, P)$, for the probability space $(B, \mathcal{K}_B, P(\cdot|B))$ from **a)**.

A: a) Since $(S, \mathcal{K})$ is a measurable space, we deduce that $(B, \mathcal{K}_B)$ is a measurable space (note that $\mathcal{K}_B$ has the properties of a σ-field):

(i) $\mathcal{K}_B = B \cap \mathcal{K} \neq \emptyset$, because $B \neq \emptyset$ and $\mathcal{K} \neq \emptyset$;
(ii) if $A \in \mathcal{K}_B$, then $\bar{A} \in \mathcal{K}_B$, because there is $C \in \mathcal{K}$ such that $A = B \cap C$ and thus $\bar{A} = B \setminus A = B \setminus (B \cap C) = B \setminus C = B \cap \bar{C} \in B \cap \mathcal{K}$;
(iii) if $A_n \in \mathcal{K}_B$, $n \in \mathbb{N}^*$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{K}_B$, because there are $C_n \in \mathcal{K}$, $n \in \mathbb{N}^*$, such that $A_n = B \cap C_n$, so $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty}(B \cap C_n) = B \cap \left(\bigcup\limits_{n=1}^{\infty} C_n\right) \in \mathcal{K}_B$.
Since P is a probability, we deduce that $P(\cdot|B)$ is a probability:
(1) $P(B|B) = \frac{P(B)}{P(B)} = 1$;
(2) $P(A|B) = \frac{P(A)}{P(B)} \geq 0$, for $A \in \mathcal{K}_B$;
(3) if $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint events from $\mathcal{K}_B$, then

$$P(\bigcup_{n=1}^{\infty} A_n|B) = \frac{P(\bigcup\limits_{n=1}^{\infty} A_n)}{P(B)} = \frac{\sum\limits_{n=1}^{\infty} P(A_n)}{P(B)} = \sum_{n=1}^{\infty} P(A_n|B).$$

b) Consider the experiment of rolling a die. Then we can choose $S = \{1,2,3,4,5,6\}, \mathcal{K} = \mathcal{P}(S), P(A) = \frac{\#A}{6}, A \in \mathcal{K}$. Next, let $B = \{2,4,6\}$ (i.e., an even number shows up on the die). Then $\mathcal{K}_B = B \cap \mathcal{K} = \mathcal{P}(\{2,4,6\})$ and $P(A|B) = \frac{P(A)}{P(B)} = \frac{\#A}{3}$, $A \in \mathcal{K}_B$.

Theoretical part

The binomial probabilistic model

Repeated independent trials of an experiment such that there are only two possible outcomes for each trial - which we classify as either *success* or *failure* - and their probabilities remain the same throughout the trials are called **Bernoulli trials**. The binomial model describes the *number of successes* in a series of independent Bernoulli trials:

- *success* appears with probability p, *failure* with probability $1-p$;
- the experiment is repeated n times;
- the probability that success occurs k times in n trials for $k \in \mathbb{N}$, $k \in \{0,\ldots,n\}$ is $\boldsymbol{C_n^k p^k (1-p)^{n-k}}$.

▶ $C_n^k p^k (1-p)^{n-k}$ represents the coefficient of x^k in the expansion $(px+1-p)^n$ for $k \in \{0,1,\ldots,n\}$.

▶ This model corresponds to the binomial distribution $Bino(n,p), n \in \mathbb{N}^*, p \in (0,1)$.

▶ **Example:** A die is rolled 10 times. The probability that the number 6 shows up 3 times is $C_{10}^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$.

The multinomial probabilistic model

Consider $n \in \mathbb{N}^*$ independent trials such that each trial can have several possible mutually exclusive outcomes $O_1,\ldots,O_j$ ($j \in \mathbb{N}^*$) with $P(O_i) = p_i \in (0,1)$, $i \in \{1,\ldots,j\}$. Obviously, $p_1 + \cdots + p_j = 1$. The probability that O_i occurs n_i times in n trials for $n_i \in \mathbb{N}$, $i \in \{1,\ldots,j\}$ and $n_1 + \cdots + n_j = n$ is $\boldsymbol{\frac{n!}{n_1!n_2!\ldots n_j!} p_1^{n_1} p_2^{n_2} \ldots p_j^{n_j}}$.

▶ $\frac{n!}{n_1!n_2!\ldots n_j!} p_1^{n_1} p_2^{n_2} \ldots p_j^{n_j}$ represents the coefficient of $x_1^{n_1} \cdot \ldots \cdot x_j^{n_j}$ in the expansion of $(p_1x_1 + \ldots + p_jx_j)^n$.

▶ This model corresponds to the multinomial distribution $Multino(n, p_1, \ldots, p_j)$, $n \in \mathbb{N}^*$, $p_1, \ldots, p_j \in (0,1)$, $p_1 + \ldots + p_j = 1$.

▶ **Example:** Suppose that an urn contains 2 red marbles, 1 yellow marble and 3 blue marbles. 7 marbles are drawn randomly with replacement from the urn (each drawn marble is put back into the urn). The probability that there are drawn 3 red marbles, 2 yellow marbles and 2 blue marble is $\frac{7!}{3!2!2!}\left(\frac{2}{6}\right)^3\left(\frac{1}{6}\right)^2\left(\frac{3}{6}\right)^2$.

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4. Let S be the set of all positive integers less or equal than 50, with exactly 2 digits such that one is an even digit and the other is an odd digit. A number is randomly extracted from S. Let X be the sum of its digits. Write the probability distribution of X.

A: Let Y be the extracted number. We have:

- $X = 1$, if $Y \in \{10\}$.
- $X = 3$, if $Y \in \{12, 21, 30\}$.
- $X = 5$, if $Y \in \{14, 41, 23, 32, 50\}$.
- $X = 7$, if $Y \in \{16, 25, 34, 43\}$.
- $X = 9$, if $Y \in \{18, 27, 36, 45\}$.
- $X = 11$, if $Y \in \{29, 38, 47\}$.
- $X = 13$, if $Y \in \{49\}$.

So, $X \sim \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ \frac{1}{21} & \frac{3}{21} & \frac{5}{21} & \frac{4}{21} & \frac{4}{21} & \frac{3}{21} & \frac{1}{21} \end{pmatrix}$.

5. The probability that a chipset is defective equals 0.06. A circuit board has 12 such independent chipsets and it's functional if at least 11 chipsets are operating. 4 independent such circuit boards are installed in a computer unit. Compute the probabilities of the following events:
B:"A circuit board is functional."
C:"Exactly two circuit boards are functional in the computer unit."
D:"At least a circuit board is functional in the computer unit."
A: We use the binomial model: $p = P(B) = C_{12}^{11}(0.94)^{11}0.06 + (0.94)^{12}$; $P(C) = C_4^2 p^2(1-p)^2$; $P(D) = \sum_{k=1}^{4} C_4^k p^k (1-p)^{4-k} = 1 - (1-p)^4$.

6. Let (X, Y) be a discrete random vector with the joint probability distribution given by the following contingency table

$X \backslash Y$	-2	1	2
1	0.2	0.1	0.2
2	0.1	0.1	0.3

a) Find the probability distributions of X and Y.
b) Compute the probability that $|X - Y| = 1$, given that $Y > 0$.
c) Are the events $\{X = 2\}$ and $\{Y = 1\}$ independent?
d) Are the random variables X and Y independent?

A: a) $X \sim \begin{pmatrix} 1 & 2 \\ 0.5 & 0.5 \end{pmatrix}, Y \sim \begin{pmatrix} -2 & 1 & 2 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$.
b) $P(|X-Y|=1 \big| Y>0) = \frac{P(|X-Y|=1, Y>0)}{P(Y>0)} = \frac{P(X=1,Y=2)+P(X=2,Y=1)}{P(Y>0)} = \frac{0.2+0.1}{0.7} = \frac{3}{7}$.
c) $P(X=2, Y=1) = 0.1 = 0.5 \cdot 0.2 = P(X=2) \cdot P(Y=1) \implies$ the events $\{X=2\}$ and $\{Y=1\}$ are independent.
d) $P(X=2, Y=2) = 0.3 \neq 0.25 = 0.5 \cdot 0.5 = P(X=2) \cdot P(Y=2) \implies$ the random variables X and Y are not independent.