Calculus - Homework II

I Multiple integrals over compact intervals

1. $$\int_0^2\int_1^5 \left(x+\frac{1}{y}\right)dx\,dy = \int_1^5\left[\int_0^2\left(x+\frac{1}{y}\right)dx\right] = \int_1^5\left[\left(\frac{x^2}{2}+\frac{1}{y}x\right)\Big|_0^2\right]dy =$$

$$= \int_1^5\left(2+\frac{2}{y}\right)dy = (2y+2\ln y)\Big|_1^5 = 2(5-1)+2(\ln 5-\ln 1) =$$

$$= 8+2\ln 5$$

2. $$\int_0^2\int_1^5 \left(x\cdot\frac{1}{y}\right)dx\,dy = \int_0^2 x\,dx\cdot\int_1^5\frac{1}{y}\,dy = \frac{x^2}{2}\Big|_0^2\cdot\ln y\Big|_1^5 =$$

$$= 2\cdot(\ln 5-\ln 1) = 2\ln 5$$

3. $$\int_0^a\int_0^b\int_0^c \frac{2z}{(x+y+1)^2}\,dx\,dy\,dz = \int_0^a\int_0^b \frac{1}{(x+y+1)^2}\,dx\,dy\cdot\int_0^c 2z\,dz =$$

$$= \int_0^a\int_0^b \frac{1}{(x+y+1)^2}\,dx\,dy\cdot z^2\Big|_0^c = c^2\int_0^a\int_0^b\frac{1}{(x+y+1)^2}\,dx\,dy = \frac{c^2}{-2}\ln\frac{a+b+1}{(1+a)(1+b)}$$

$$I = \int_0^a\int_0^b \frac{1}{(x+y+1)^2}\,dx\,dy = \int_0^b\left[\int_0^a\frac{1}{(x+y+1)^2}\,dx\right] = \int_0^b \frac{1}{-2}\cdot\left[\frac{1}{(y+1+a)}-\frac{1}{y+1}\right]dy$$

$$\int_0^a \frac{1}{(x+y+1)^2}\,dx = \frac{1}{-2(x+y+1)}\Big|_{x=0}^{x=a} = \frac{1}{-2}\left[\frac{1}{(a+y+1)}-\frac{1}{(y+1)}\right]$$

$$I = \frac{1}{-2}\left[\ln(y+1+a)-\ln(y+1)\right]\Big|_0^b = \frac{1}{-2}\left[\ln(b+a+1)-\ln(b+1)-\ln(1+a)+\right.$$

$$\left.+\ln(1)\right] = \frac{1}{-2}\ln\frac{b+a+1}{(b+1)(a+1)}$$

4. $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\pi}(\sin x+\cos y)\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{4}}(\sin x+\cos y)\,dy\right]dx\cdot\int_0^{\pi}dz =$$

$$= \int_0^{\frac{\pi}{2}}(y\cdot\sin x+\sin y)\Big|_{y=0}^{y=\frac{\pi}{4}}dx\cdot z\Big|_0^{\pi} = \int_0^{\frac{\pi}{2}}\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}-\sin 0\,dx\cdot\pi =$$

$$= \pi\int_0^{\frac{\pi}{2}}\frac{\pi}{4}\sin x+\frac{\sqrt{2}}{2}\,dx = \pi\left(-\frac{\pi}{4}\cos x+\frac{\sqrt{2}}{2}x\right)\Big|_0^{\frac{\pi}{2}} = \pi\left(-\frac{\pi}{4}\cos\frac{\pi}{2}+\frac{\pi}{4}\cos 0+\right.$$

$$\left.+\frac{\sqrt{2}}{2}\cdot\frac{\pi}{2}-0\right) =$$

$$= \pi \left(0 + \frac{\pi}{4} + \frac{\pi\sqrt{2}}{4}\right) = \pi \cdot \frac{\pi(1+\sqrt{2})}{4} = \frac{\pi^2(1+\sqrt{2})}{4}$$

5. $$\int_1^6\int_2^3 \frac{1}{(x+y)^2}\,dx\,dy = \int_1^6\left[\int_2^3 \frac{1}{(x+y)^2}\,dy\right]dx = \int_1^6\left(\frac{1}{-2}\cdot\frac{1}{(x+y)}\Big|_{y=2}^{y=3}\right)dx =$$

$$= \frac{1}{-2}\int_1^6\left(\frac{1}{x+3} - \frac{1}{x+2}\right)dx = -\frac{1}{2}\left(\ln(x+3)\Big|_1^6 - \ln(x+2)\Big|_1^6\right) =$$

$$= -\frac{1}{2}\left(\ln 9 - \ln 4 - \ln 8 + \ln 3\right) = -\frac{1}{2}\left(\ln 27 - \ln 32\right) = -\frac{1}{2}\ln\frac{27}{32}$$

6. $$\int_0^1\int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}}\,dx\,dy =$$

$1 + x^2 + y^2 = t \rightarrow x^2 = t - y^2 - 1$

$x = 0 \rightarrow t = 1 + y^2$ $\qquad x = \sqrt{t-y^2-1}$

$x = 1 \rightarrow t = 2 + y^2$

$dx = \frac{1}{2\sqrt{t-y^2-1}}\,dt$

$$\int_0^1\left(\int_{1+y^2}^{2+y^2} \frac{\cancel{\sqrt{t-y^2-1}}}{t^{\frac{3}{2}}}\cdot\frac{1}{2\cancel{\sqrt{t-y^2-1}}}\,dt\right)dy = \int_0^1\left(\int_{1+y^2}^{2+y^2}\frac{1}{2t^{\frac{3}{2}}}\,dt\right)dy =$$

$-\frac{3}{2}+\frac{2}{2} = -\frac{1}{2}$

$$= \int_0^1\left[\int_{1+y^2}^{2+y^2}\frac{1}{2}\left(t^{-\frac{3}{2}}\right)dt\right]dy = \int_0^1 \frac{1}{2}\cdot\left(-\frac{2}{3}\cdot t^{-\frac{1}{2}}\Big|_{1+y^2}^{2+y^2}\right)dy =$$

$$= \int_0^1 -\frac{1}{3}\left(\frac{1}{\sqrt{2+y^2}} - \frac{1}{\sqrt{1+y^2}}\right)dy = -\frac{1}{3}\left[\int_0^1\frac{1}{\sqrt{2+y^2}}\,dy - \int_0^1\frac{1}{\sqrt{1+y^2}}\,dy\right] =$$

$$= -\frac{1}{3}\left[\ln\left(\left|y+\sqrt{y^2+2}\right|\right) - \ln\left(\left|y+\sqrt{1+y^2}\right|\right)\right]\Big|_0^1$$

$$= -\frac{1}{3}\left[\ln(1+\sqrt{3}) - \ln(\sqrt{2}) - \ln(1+\sqrt{2}) + \ln 1\right]$$

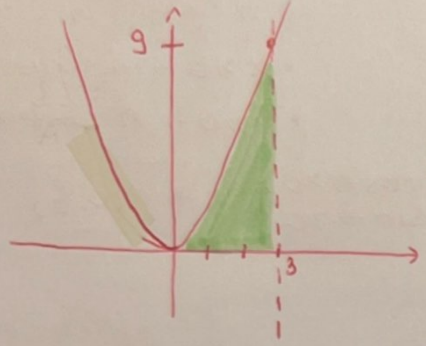
$$= -\frac{1}{3}\left[\ln(1+\sqrt{3}) - \ln(\sqrt{2}+2)\right] = -\frac{1}{3}\ln\frac{1+\sqrt{3}}{\sqrt{2}+2}$$

II Integration over Nonempty Bounded Sets - HW II

1. $\iint_A f(x,y)\,dx\,dy$

$A = \{(x,y) \in \mathbb{R} : (x,y)$ is bounded by $y = x^2$ and $x = 3, x \geq 0, y \geq 0\}$

$f(x,y) = \dfrac{2x}{(1+x^2+y)^2} \quad \forall (x,y) \in A$



$x \geq 0$ 1)

$y \geq 0$ 2)

$x = 3$

$0 < 3 \rightarrow A$ s.t. $x \leq 3$ 3)

• $(3,0)$

$y = 0 < 3^2 = x^2 \rightarrow$ outside the parabola $\rightarrow$

$\rightarrow y \leq x^2$ 4)

$0 \leq x \leq 3$

$0 \leq y \leq x^2 \rightarrow$ y depends on x $\rightarrow$ we start by integrating with respect to y!

$$I = \int_0^3 \int_0^{x^2} \frac{2x}{(1+x^2+y)^2}\,dx\,dy = \int_0^3 \Big( 2x \int_0^{x^2} \underbrace{\frac{1}{(1+x^2+y)^2}}_{c}\,dy \Big)$$

$$T = \int_0^{x^2} \frac{1}{(1+x^2+y)^2}\,dy = \int_0^{x^2} (1+x^2+y)^{-2}\,dy = \int_0^{x^2} (1+x^2+y)^{-2}(1+x^2+y)'_y\,dy =$$

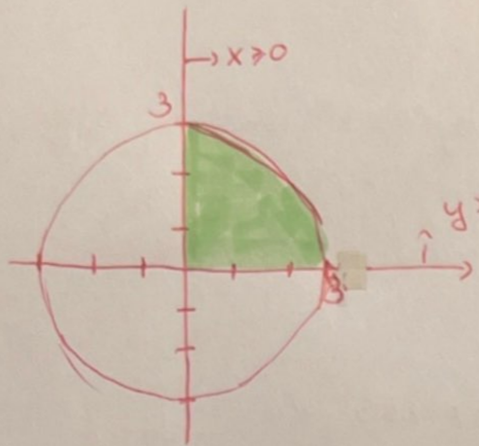
$$= \frac{(1+x^2+y)^{-2+1}}{-2+1}\Big|_0^{x^2} = -(1+2x^2)^{-1} + (1+x^2)^{-1} = \frac{1}{1+x^2} - \frac{1}{1+2x^2}$$

$$I = \int_0^3 2x \cdot T\,dx = \int_0^3 \frac{2x}{1+x^2}\,dx - \int_0^3 \frac{2x}{1+2x^2}\,dx = \int_0^3 \frac{(1+x^2)'}{1+x^2}\,dx - \frac{1}{2}\int_0^3 \frac{(1+2x^2)'}{1+2x^2}\,dx$$

$$= \ln(1+x^2)\Big|_0^3 - \frac{1}{2}\ln(1+2x^2)\Big|_0^3 = \ln 10 - \ln 1 - \frac{1}{2}\ln 19 + \frac{1}{2}\ln 1 =$$

$$= \frac{1}{2}\ln\frac{100}{19} = \ln\sqrt{\frac{100}{19}}$$

2. $\iint_A \frac{x}{1+y^2}\,dx\,dy \qquad A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 9,\ x\ge 0,\ y\ge 0\}$




we change to polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ r\ge 0 \\ \theta\in[0,2\pi] \end{cases}$$

+ merge with A

- $x^2+y^2\le 9 \to$

$\to r^2\cos^2\theta + r^2\sin^2\theta \le 9 \to$

$\to r^2 \le 9 \xrightarrow{r\ge 0} \boxed{r\le 3}$

- $x\ge 0 \to r\cos\theta\ge 0$
- $y\ge 0 \to r\sin\theta \ge 0$

$\left.\right\} \xrightarrow{r\ge 0}$

$\to \left.\begin{matrix}\cos\theta\ge 0\\ \sin\theta\ge 0\end{matrix}\right\} \to \boxed{\theta\in[0,\frac{\pi}{2}]}$

$dx\,dy = r\,dr\,d\theta$

$$I=\int_0^3\int_0^{\frac{\pi}{2}} \frac{r\cos\theta}{1+r^2\sin^2\theta}\cdot r\,dr\,d\theta = \int_0^3\int_0^{\frac{\pi}{2}} \frac{r^2\cos\theta}{1+r^2\sin^2\theta}\,dr\,d\theta =$$

$$T=\int_0^{\frac{\pi}{2}} \frac{r^2\cos\theta}{1+r^2\sin^2\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\frac{1}{r^2}+\sin^2\theta}\,d\theta$$

$$R(\sin\theta,\cos\theta)=\frac{\cos\theta}{\frac{1}{r^2}+\sin^2\theta}$$

$$R(-\sin\theta,\cos\theta)=\frac{\cos\theta}{\frac{1}{r^2}+\sin^2\theta} = R(\sin\theta,\cos\theta)\ \times$$

$$R(\sin\theta,-\cos\theta)=\frac{-\cos\theta}{\frac{1}{r^2}+\sin^2\theta} = -R(\sin\theta,\cos\theta)\ \checkmark$$

$\downarrow$

$t$

$\sin\theta = t \quad |' \to \cos\theta\,d\theta = dt$

$$\int \frac{\cos\theta}{\frac{1}{r^2}+\sin^2\theta}\,d\theta = \int\frac{dt}{\frac{1}{r^2}+t^2} = \frac{1}{\frac{1}{r}}\operatorname{arctg}\frac{t}{\frac{1}{r}} + C = r\operatorname{arctg} rt + C =$$

$$= r\operatorname{arctg} r\sin\theta + C$$

$$T = r\operatorname{arctg} r\sin\theta\Big|_0^{\frac{\pi}{2}} = r[\operatorname{arctg} r - \operatorname{arctg} 0] = r\cdot\operatorname{arctg} r$$

$$I=\int_0^3 r\cdot\operatorname{arctg} r\,dr = \int_0^3\left(\frac{x^2}{2}\right)'\operatorname{arctg} x\,dx = \frac{x^2}{2}\operatorname{arctg} x\Big|_0^3 - \int_0^3\frac{x^2}{2(1+x^2)}\,dx =$$

$$=\frac{9}{2}\operatorname{arctg}3 - \frac{1}{2}\int_0^3 \frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\,dx = \frac{9}{2}\operatorname{arctg}3 - \frac{1}{2}x\Big|_0^3 + \frac{1}{2}\operatorname{arctg}x\Big|_0^3 =$$

$$=\frac{9}{2}\operatorname{arctg}3 - \frac{3}{2} + \frac{1}{2}\operatorname{arctg}3 = 5\operatorname{arctg}3 - \frac{3}{2}$$

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