1. 12. 2023

lgebra - Seminar 11

1. $B = (v_1, v_2, v_3) = ((1,0,1), (0,1,1), (1,1,1))$

$B' = (v_1', v_2', v_3') = ((1,1,0), (-1,0,0), (0,0,1))$

- $T_{BB'}$, $T_{B'B}$
- coordinates of $u = (2,0,-1)$ in both bases

$v_1' = av_1 + bv_2 + cv_3 = (a,0,a) + (0,b,b) + (c,c,c) = (a+c, b+c, a+b+c)$

$$\left.\begin{cases} a+c=1 \\ \left.\begin{array}{l} b+c=1 \\ a+b+c=0 \end{array}\right\} \to a=-1 \end{cases}\right\} \to c=2 \to b=-1$$

$v_2' = (a+c, b+c, a+b+c) = (-1,0,0)$

$$\left.\begin{cases} a+c=1 \\ \left.\begin{array}{l} b+c=0 \\ a+b+c=0 \end{array}\right\} \to a=0 \end{cases}\right\} \to c=1 \to b=-1$$

$v_3' = (a+c, b+c, a+b+c) = (0,0,1)$

$$\begin{cases} a+c=0 \\ \left.\begin{array}{l} b+c=0 \\ a+b+c=1 \end{array}\right\} \to a=1 \to c=-1 \to b=1 \end{cases}$$

$$T_{BB'} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & +1 \\ 2 & -1 & -1 \end{pmatrix}$$

$$T_{B'B} = T_{BB'}^{-1}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & +1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3+2L_1}{\overset{L_2+L_1}{\sim}} \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 & 1 \end{array}\right) \overset{L_3+L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \overset{L_1-L_3}{\sim} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \overset{L_1\cdot(-1)}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \to T_{B'B} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$av_1 + bv_2 + cv_3 = (a+c, b+c, a+b+c) = (2, 0, -1)$

$$\left.\begin{array}{l} a+c = 2 \\ \left.\begin{array}{l} b+c = 0 \\ a+b+c = -1 \end{array}\right\} \to a = -1 \end{array}\right\} \to c = 3 \to b = -3$$

$$[u]_B = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$$

$av_1' + bv_2' + cv_3' = (a-b, a, c) = (2, 0, -1)$

$$\begin{cases} \left.\begin{array}{l} a-b = 2 \\ a = 0 \end{array}\right\} \to b = -2 \\ c = -1 \end{cases}$$

$$[u]_{B'} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

2. Seminar 10

3. $\mathbb{R}$-v.s. $\mathbb{R}_2[X]$, $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$, $B' = (1, X-b, (X-b)^2)$

~~$E = (e_1, e_2, e$~~

• T_{BE} , $T_{EB'}$, $T_{BB'}$

~~• T_{BE}~~

• T_{EB}

$1 = 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2$

$X - a = -a \cdot 1 + 1 \cdot X + 0 \cdot X^2$

$(X-a)^2 = a^2 \cdot 1 - 2a \cdot X + 1 \cdot X^2$

$$T_{EB} = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{T_{BE}} = \left(\begin{array}{ccc|ccc} 1 & -a & a^2 & 1 & 0 & 0 \\ 0 & 1 & -2a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_1 - a^2 L_3}{\overset{L_2 + 2aL_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -a & 0 & 1 & 0 & -a^2 \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_1 + aL_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & a & +a^2 \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$T_{BE} = T_{EB}^{-1} = \begin{pmatrix} 1 & a & +a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}$$

- $T_{BB'}$

$$T_{BB''} = T_{BB'} \cdot T_{B'B''}$$

$$T_{EB'} = \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{BB'} = T_{BE} \cdot T_{EB'}$$

$$T_{BB'} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a-b & b^2+a^2-2ab \\ 0 & 1 & 2(a-b) \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues and eigenvectors

- $f \in End_K(V)$. A non-zero vector $v \in V$ is an eigenvector of f if $\exists\ \lambda \in K$ s.t. $f(v) = \lambda \cdot v$, λ = eigenvalue of f

Consider $A = [f]_B$, B - any basis of V

λ eigenvalue of $f \leftrightarrow \det(A - \lambda I_n) = 0$

$P_A(\lambda) = \det(A - \lambda I_n)$ - characteristic polynomial ($\rightarrow$ eigenvalues = roots of P_A)

4. $f \in End_{\mathbb{R}}(\mathbb{R}^2)$, $f(x,y) = (3x+3y,\ 2x+4y)$

(a) Determine the eigenvalues and eigenvectors of f

$$\left.\begin{array}{l} f(1,0) = (3,2) \\ f(0,1) = (3,4) \end{array}\right\} \rightarrow [f]_E = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

$$P_A(\lambda) = \det([f]_E - \lambda I_n) = \begin{vmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 6 =$$

$$= 12 - 7\lambda + \lambda^2 - 6 = \lambda^2 - 7\lambda + 6 = 0$$

$$\Delta = 49 - 4 \cdot 6 = 49 - 24 = 25$$

$$\lambda_{1,2} = \frac{7 \pm 5}{2} \begin{cases} 6 \\ 1 \end{cases} \rightarrow \lambda \in \{1, 6\}$$

- $\lambda_1 = 1$

$$\begin{pmatrix} 3-1 & 3 \\ 2 & 4-1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 + 3x_2 \\ 2x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2x_1 + 3x_2 = 0 \rightarrow x_2 = -\frac{2}{3}x_1$$

$v(\lambda_1) = \langle 1, -\frac{2}{3} \rangle$

- $\lambda_2 = 6$

$$\begin{pmatrix} 3-6 & 3 \\ 2 & 4-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_1 + 3x_2 \\ 2x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{matrix} -3x_1 + 3x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{matrix} \rightarrow x_2 = x_1$$

$v(\lambda_2) = \langle 1, 1 \rangle$

b) Write a basis B of $\mathbb{R}^2$ consisting of f. Write $[f]_B$

$B = \{ (1, -\frac{2}{3}), (1,1) \}$

$[f]_B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

Compute the eigenvalues and eigenvectors of:

5. Hw

6. $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$\det(A - \lambda I_4) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \overset{L_1}{=} -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= -\lambda(-\lambda^3 + \lambda) - (\lambda^2 - 1) = \lambda^2(\lambda^2 - 1) - (\lambda^2 - 1) = (\lambda^2 - 1)(\lambda^2 - 1) =$$

$$= (\lambda^2 - 1)^2 \Rightarrow \lambda \in \{-1, 1\}$$

$$\lambda_1 = \lambda_2 = 1 \rightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (0,0,0,0)$$

$$\begin{cases} -a + d = 0 \rightarrow d = a \\ -b + c = 0 \rightarrow b = c \\ b - c = 0 \\ a - d = 0 \end{cases}$$

$v(\lambda_1) = v(\lambda_2) = \langle (1,0,0,1), (0,1,1,0) \rangle$

$$\lambda_3 = \lambda_4 = -1 \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (0,0,0,0)$$

$$\left.\begin{cases} a+d=0 \rightarrow a=-d \\ b+c=0 \rightarrow b=-c \end{cases}\right\} \rightarrow V(\lambda_3) = V(\lambda_4) = \langle (1,0,0,-1), (0,1,-1,0) \rangle$$

7. $A = \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix} \quad (x, y \in \mathbb{R}^*)$

$$\det(A - \lambda I_3) = \begin{vmatrix} x-\lambda & 0 & y \\ 0 & x-\lambda & 0 \\ y & 0 & x-\lambda \end{vmatrix} = (x-\lambda)^3 - y^2(x-\lambda) =$$

$$= (x-\lambda)(\underbrace{x^2 - 2x\lambda + \lambda^2}_{(x-\lambda)^2} - y^2) = \cancel{0}\ (x-\lambda)(x-\lambda-y)(x-\lambda+y) = 0 \rightarrow$$

$\rightarrow \quad x = \lambda_1$

$\lambda_2 = x - y$

$\lambda_3 = x + y$

$$\lambda_1 = x \rightarrow \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (0,0,0) \rightarrow \left.\begin{matrix} ya = 0 \\ yc = 0 \end{matrix}\right\} \rightarrow \begin{matrix} a = c = 0 \\ b \in \mathbb{R} \end{matrix}$$

$V(\lambda_1) = \langle (0,1,0) \rangle$

$$\lambda_2 = x - y \rightarrow \begin{pmatrix} y & 0 & y \\ 0 & y & 0 \\ y & 0 & y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (0,0,0)$$

$y(a+c) = 0 \rightarrow a = -c$

$yb = 0 \rightarrow b = 0$

~~$yc = 0 \rightarrow c = 0 \rightarrow a = 0$~~

$V(\lambda_2) = \langle (1,0,-1) \rangle$

$$\lambda_3 = x + y \rightarrow \begin{pmatrix} -y & 0 & -y \\ 0 & -y & 0 \\ -y & 0 & -y \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (0,0,0)$$

$-y(a+c) = 0 \rightarrow a = -c$

$-yb = 0 \rightarrow b = 0$

$V(\lambda_3) = \langle (1,0,-1) \rangle$

8. $A = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$, $x \in \mathbb{R}$

$(K = \mathbb{C})$

$*$ $\sin x \neq 0$ (see what happens when $\sin x = 0$ - Hw)

$\det(A - \lambda I_n) = \begin{vmatrix} \cos x - \lambda & -\sin x \\ \sin x & \cos x - \lambda \end{vmatrix} = (\cos x - \lambda)^2 + \sin^2 x =$

$= \cos^2 x - 2\lambda\cos x + \sin^2 x + \lambda^2 = 1 - 2\lambda\cos x + \lambda^2$

$\Delta = 4\cos^2 x - 4 = \cancel{4(\cos^2 x - 1)} = 4(1 - \sin^2 x) - 4 = -4\sin^2 x$

$\lambda_1 = \dfrac{2\cos x + i \cdot 2\sin x}{2} = \cos x + i\sin x$

$\lambda_2 = \dfrac{2\cos x - i2\sin x}{2} = \cos x - i\sin x$

$\lambda_1 = \cos x + i\sin x \rightarrow \begin{pmatrix} -i\sin x & -\sin x \\ \sin x & -i\sin x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (0,0) \rightarrow$

$\cancel{\rightarrow a(-\sin^2 x) + (\sin x)b =}$

$\begin{cases} a(-i\sin x) + b(-\sin x) = 0 \\ a\sin x + b(-i\sin x) = 0 \end{cases} \Leftrightarrow \begin{cases} -a \cdot i - b = 0 \\ a - bi = 0 \rightarrow a = i \cdot b \end{cases}$

$V(\lambda_1) = \langle (i, 1) \rangle$

λ_2 - Hw

$V(\lambda_2) = \langle (1, i) \rangle$

9. $A \in M_2(\mathbb{R})$, λ_1, λ_2 - eigenvalues of A in $\mathbb{C}$

(i) $\lambda_1 + \lambda_2 = Tr(A)$

$\lambda_1 \cdot \lambda_2 = \det(A)$

Generalization

(ii) A has all eigenval in $\mathbb{R} \Leftrightarrow (Tr(A))^2 - 4\det(A) \geq 0$

(iii) A is a root of its characteristis polynomial

(i) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(A - \lambda I_2) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = ad - \lambda(a+d) + \lambda^2 - bc$

$= \lambda^2 - \lambda(a+d) + (ad-bc)$

Eigenvalues = roots of $P_A \xrightarrow{viet}$

$\lambda_1 + \lambda_2 = -\frac{b}{a} = a+d = Tr(A)$

$\lambda_1 \lambda_2 = \frac{c}{a} = ad - bc = \det(A)$

(ii) $(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 = \lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - 4\lambda_1\lambda_2 = \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 =$

$= (\lambda_1 - \lambda_2)^2 \geq 0 \Leftrightarrow \lambda_1, \lambda_2 \in \mathbb{R}$

(iii) $P_A(A) = O_2 \to A^2 - (a+d)A - (ad-bc) I_2 = O_2$

10. $A \in M_2(\mathbb{R})$ s.t. $\det(A + i \cdot I_2) = 0$. Show that $\det(A + 2I_2) = 5$

$\det(A + i \cdot I_2) = P(-i) = 0 \to -i$ eigenvalue of $A \to i =$ eigenval. of A

$\to Tr(A) = -i + i = 0$

$\det(A) = -i \cdot i = 1$

$\det(A + 2I_2) = P(-2) = (-2)^2 - 2Tr(A) + \det(A) = 4 - 0 + 1 = 5$