Limits of Functions

Recall the following:

$$\lim_{x\to\infty} q^x = \begin{cases} +\infty & : q > 1 \\ 1 & : q = 1 \\ 0 : & |q| < 1 \\ \nexists & : q \leq 1 \end{cases}$$

$$\lim_{x\to\infty} q^x = q^{x_0}, \forall q \in (0,\infty) \quad and \quad x_0 \in \mathbb{R}$$

$$\lim_{x\to x_0} \log_a x = \log_a x_0, \forall a \in (0,\infty) \setminus \{1\}, x_0 > 0.$$

$$\lim_{x\to\infty} \log_a x = \begin{cases} +\infty & : a > 1 \\ -\infty & : 0 < a < 1 \end{cases}$$

$$\lim_{x\to 0} \frac{q^x - 1}{x} = \ln q, \forall q > 0 \quad and \quad \lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1.$$

Exercise 1: Compute the limits of the following functions at the specified points:

$$a) \lim_{x\to\infty} x \cos^2 \frac{x+2}{x} \quad b) \lim_{x\to 1} \frac{x}{x^2 + 1} \quad c) \lim_{x\to -\infty} \frac{x^2 + 5}{x^3} \quad d) \lim_{x\to\infty} \frac{(x+2)(2x+1)}{x^2 + 3x + 5}$$

$$e) \lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1} \quad f) \lim_{x\to 2} \left(\frac{1}{2-x} - \frac{2x}{4 - x^2} \right)$$

$$g) \lim_{x\to 1} \frac{1 + x + x^2 + ... + x^n - (n+1)}{x - 1}, n \in \mathbb{N} \quad h) \lim_{x\to 1} \frac{x + x^2 + ... + x^n - n}{x + x^2 + ... + x^m - m}, \forall m, n \in \mathbb{N}.$$

$$i) \lim_{x\to 27} \frac{x - 27}{\sqrt[3]{x} - 3} \quad j) \lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt[4]{x} - 1}$$

$$k) \lim_{x\to\infty} \left(\sqrt[3]{ax^3 + x^2 + bx + c} - (bx + c) \right) \forall a, b, c > 0.$$

Exercise 2: Compute the limits of the following functions at the specified points:

$$a) \lim_{x\to\infty} \left(\frac{1}{x} \right)^{\frac{5x+1}{2x+4}} \quad b) \lim_{x\to 0} \left(\frac{3\sin x - \tan x}{x} \right)^{\frac{\sin x + 2x}{x}}$$

$$c) \lim_{x \to 0} (1 + \cos x)^{\frac{1}{x^2}} \quad d) \lim_{x \to 0} (e^x - x + 1)^{\frac{1}{1 - \cos x}}$$

$$e) \lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}} \quad f) \lim_{x \to \infty} \left(\frac{x + 7}{x}\right)^x$$

Exercise 3:

$$a) \lim_{n \to \infty} \left[\lim_{x \to 0} \left(1 + \sin^2 x + \sin^2 2x + ... + \sin^2 nx\right)^{\frac{1}{n^3 x^2}}\right]$$

$$a) \lim_{n \to \infty} \left[\lim_{x \to 0} \left(1 + \ln(1 + x) + \ln(1 + 2x) + ... + \ln(1 + nx)\right)^{\frac{1}{n^2 x}}\right]$$

Exercise 4: Compute the following limits:

$$a) \lim_{x \to 0} \frac{e^{2x} - 1}{3x}; \quad b) \lim_{x \to 0} \frac{e^x - \cos x}{3x}.$$