Calculus on $\mathbb{R}$ — Anca Grad, Ph.D., Lect. Prof.
Homework 3 — **2023**

Sequences of Real Numbers- part 1

Exercise 1: Study the monotonicity, boundedness and convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, having the general term:

$$a)\quad x_n = \frac{3^n+5^n}{7^n},\quad b)\quad x_n = \frac{(-1)^n}{n},\quad c)\quad x_n = \frac{4^n}{n!},\quad d)\quad x_n = \frac{n}{n^2+1}.$$

Exercise2: Using the characterising theorem with ε prove that

$$a)\lim_{n\to\infty}\frac{2n}{n^2+1} = 0 \qquad b)\lim_{n\to\infty}\frac{2n^2}{-2n+4} = -\infty.$$

Exercise 3: Compute the limit of the sequences of real numbers having the following general terms:

$$a)\quad \frac{3^n+1}{5^n+1},\qquad b)\quad \frac{9^n+(-3)^n}{9^{n-1}+3},\qquad c)\quad \left(\sin\frac{\pi}{10}\right)^n,\qquad d)\sqrt{4n^2+2n+1}-2n,$$

$$e)\quad \left(7+\frac{1-2n^3}{3n^4+2}\right)^2,\qquad f)\quad \sqrt[3]{n^3+n+3}-\sqrt[3]{n^3+1},\qquad g)\quad \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}},$$

$$h)\quad \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n}\right).$$

Exercise 4: Let $t\in\mathbb{R}$.

a) Prove that there exists an decreasing sequence of rational numbers converging to t.

b) Prove that there exists a increasing sequence of irrational numbers converging to t.

Exercise 5: Let $a > 0$ and let $x_0 \in \mathbb{R}$ be such that $0 < x_0 < \frac{1}{a}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers, defined recursively by:

$$x_{n+1} = 2x_n - ax_n^2, \forall n \in \mathbb{N}.$$

Study the convergence of the sequence by following the next steps:

a) Prove by induction that $x_n < \frac{1}{a}, \forall n \in \mathbb{N}$.

b) Prove by induction that $0 < x_n, \forall n \in \mathbb{N}$.

c) By using a) and b) prove that $(x_n)_{n\in\mathbb{N}}$ is increasing.

d) Compute the limit of the sequence.