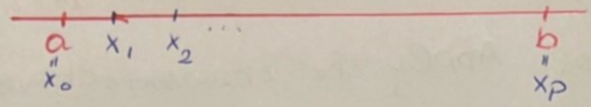
8.12.2025
Algebra - Lecture 9

The Rieman integral

1. Partitions of a compact interval

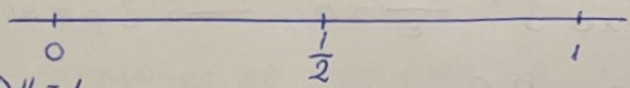
Def: Let $[a,b] \subseteq \mathbb{R}$



each ordered system
$\Delta = (x_0, x_1, \dots, x_p)$ where $a = x_0 < x_1 < \dots < x_p = b$ is called the PARTITION of the interval $[a,b]$.
$Part[a,b] = \{\Delta / \Delta \text{ is a partition of } [a,b]\}$

- THE NORM of the partition Δ is $\|\Delta\| = \max\{x_i - x_{i-1}, \forall i = \overline{1,p}\}$

Example: $[a,b] = [0,1]$



$\Delta_1 = (0,1)$ $\qquad \|\Delta_1\| = 1$

$\Delta_2 = (0, \frac{1}{2}, 1)$ $\qquad \|\Delta_2\| = \max\{1 - \frac{1}{2}; \frac{1}{2} - 0\} = \frac{1}{2}$

$\Delta_3 = (0, \frac{1}{\pi e}; \frac{1}{4}; 1)$ $\qquad \|\Delta_3\| = \max\{\frac{1}{\pi e}; \frac{1}{4} - \frac{1}{\pi e}; 1 - \frac{1}{4}\} = \frac{3}{4}$

$\Delta_2' = (0, \frac{1}{2}; \frac{2}{3}; \frac{3}{4}; 1)$ $\qquad \|\Delta_2'\| = \max\{\frac{1}{2}; \frac{2}{3} - \frac{1}{2}; \frac{3}{4} - \frac{2}{3}, 1 - \frac{3}{4}\} = \frac{1}{2}$

Def: Consider $[a,b] \subseteq \mathbb{R}$
$\Delta = (x_0, x_1, \dots, x_p)$
$\Delta' = (y_0, y_1, \dots, y_q)$ ($x_0 = y_0$, $x_p = y_q$)
Δ is SMOOTHER THAN Δ' if $\{y_0, y_1, \dots, y_q\} \subseteq \{x_0, x_1, \dots, x_p\}$
$\Delta' \subseteq \Delta$

Theorem 1: $\left.\begin{array}{l}\Delta_1, \Delta_2 \subseteq Part[a,b] \\ \Delta_1 \subseteq \Delta_1\end{array}\right\} \rightarrow \|\Delta_2\| \leq \|\Delta_1\|$

Proof: Hw

Property: $\Delta_1, \Delta_2 \in \text{Part}[a,b]$

$\to$ a) $\Delta_1 \subseteq \Delta_1 \cup \Delta_2$

$\Delta_2 \subseteq \Delta_1 \cup \Delta_2$

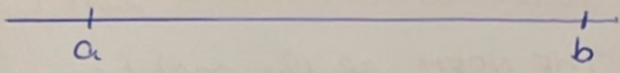
b) $\|\Delta_1 \cup \Delta_2\| \le \|\Delta_1\|$

$\le \|\Delta_2\|$

Proof: Apply the theorem 1

Theorem 2: $\forall \varepsilon > 0 \quad \exists \Delta_\varepsilon \subseteq \text{Part}[a,b]$ with $\|\Delta_\varepsilon\| < \varepsilon$

Proof:



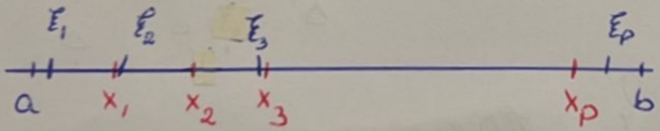
$$\left.\begin{matrix} b-a>0 \\ \varepsilon>0 \end{matrix}\right\} \to \frac{b-a}{\varepsilon} > 0 \xrightarrow{\text{ARe}} \exists\, p \in \mathbb{N} \text{ s.t. } 0 < \frac{b-a}{\varepsilon} < p \iff$$

$$\iff 0 < \frac{b-a}{p} < \varepsilon$$

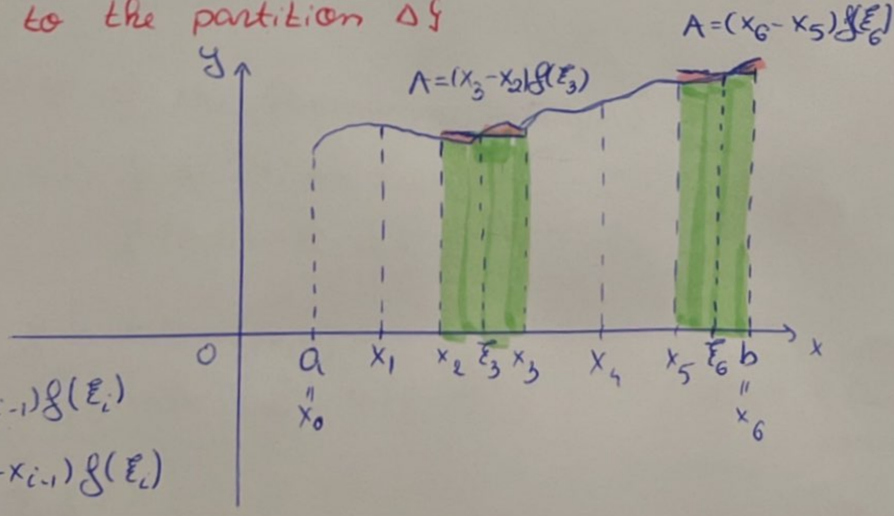
we construct $\Delta = \Big(\underbrace{a,\ a + \frac{b-a}{p}}_{\frac{b-a}{p}},\ \underbrace{\ ,\ a + 2\frac{b-a}{p}}_{\frac{b-a}{p}}, \dots, \underbrace{a + (p-1)\frac{b-a}{p},\ a + p\frac{b-a}{p}}_{\frac{b-a}{p}}\Big)$

$$\to \|\Delta\| = \frac{b-a}{p} < \varepsilon$$

Def: Let $\Delta = (x_0, x_1, \dots, x_n) \in \text{Part}[a,b]$. Then each ordered system $\xi = (\xi_1, \xi_2, \dots, \xi_p)$ s.t. $x_{i-1} \le \xi_i \le x_i \quad \forall i = \overline{1,n}$ is called A SYSTEM OF INTERMEDIATE POINTS ASSOCIATED to the partition Δ.



$\text{IP}(\Delta) = \{\xi \mid \xi$ is a system of intermediate points associated to the partition $\Delta\}$



$A_i = (x_i - x_{i-1}) f(\xi_i)$

$A = \sum_{i=1}^{6} (x_i - x_{i-1}) f(\xi_i)$

Def: Let $[a,b] \subseteq \mathbb{R}$

- $\Delta = (x_0, x_1, \dots, x_p) \in Part[a,b]$
- $\xi = (\xi_1, \xi_2, \dots, \xi_p) \in IP[a,b]$
- $f : [a,b] \to \mathbb{R}$

The RIEMANN SUM ATTACHED TO THE FUNCTION f, THE PARTITION Δ, AND THE SYSTEM OF INTERMEDIATE POINTS ξ IS

$$\sigma(f, \Delta, \xi) = \sum_{i=1}^{p} f(\xi_i)(x_i - x_{i-1})$$

Def: Consider $f : [a,b] \to \mathbb{R}$.

The function f is said to be RIEMANN INTEGRABLE on $[a,b]$ if:

- $\forall (\Delta^n) \subseteq Part([a,b])$ a sequence of partition of $[a,b]$ s.t. $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$
- $\forall (\xi^n) \subseteq IP(\Delta^n)$ a sequence of intermediate points ($\forall n \in \mathbb{N}$, $\xi^n \in IP(\Delta^n)$)

$\exists \lim\limits_{n\to\infty} \sigma(f, \Delta^n, \xi^n) = \lim\limits_{n\to\infty} \sum_{i=1}^{p_n} f(\xi_i)(x_i - x_{i-1}) \boxed{\in \mathbb{R}}$

Theorem (The uniqity of the Riemann integral)

$f : [a,b] \to \mathbb{R}$

Then f is Riemann integrable on $[a,b] \Leftrightarrow \exists\, I \in \mathbb{R}$ s.t.
- $\forall (\Delta^n) \subseteq Part\,[a,b]$ with $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$
- $\forall (\xi^n) \subseteq IP(\Delta^n)$

$\lim\limits_{n\to\infty} \sigma(f, \Delta^n, \xi^n) = I$

Proof: $\to$ individual study (pe site la profa $\to$ manual de licență)

3. Properties of the Riemann integral

Theorem 3.1: $\left.\begin{array}{l} f \text{ is Ri on } [a,b] \\ f : [a,b] \to \mathbb{R} \text{ s.t. } f(x) \geq 0 \end{array}\right\} \to \int_a^b f(x)\,dx \geq 0$

Proof: f is R.i. on $[a,b] \to$

$\to \exists \int_a^b f(x)\,dx = \lim\limits_{n\to\infty} \sigma(f, \Delta^n, \xi^n)$ $\quad \forall (\Delta^n) \subseteq Part[a,b]$ with $\lim\limits_{n\to\infty} \|\Delta^n\| = 0$

$= \lim\limits_{n\to\infty} \sum_{i=1}^{p_n} \underbrace{f(\xi_i)}_{\geq 0} \underbrace{(x_i - x_{i-1})}_{\geq 0} \geq 0$ $\quad \forall (\xi^n),\ \xi^n \subseteq IP(\Delta^n)$

Consequence 1: $f, g : [a,b] \to \mathbb{R}$ s.t. : • f & g are R.i. on $[a,b]$
• $f(x) \geq g(x)$, $\forall x \in [a,b]$

$\to \int_a^b f(x)dx \geq \int_a^b g(x)dx$

Proof: T 3.1. appled for $h = f - g$

Consequence 2: $f : [a,b] \to \mathbb{R}$ R.i on $[a,b]$ s.t. $m \leq f(x) \leq M$, $\forall x \in [a,b] \to$
$\to m \cdot (b-a) \leq \int_a^b f(x)dx \leq M \cdot (b-a)$

Consequence 3: $f : [a,b] \to \mathbb{R}$ is R.i. on $[a,b]$
$\to |f|$ is R.i. on $[a,b]$ and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

4. The antiderivative of a function

Def: $\emptyset \neq A \subseteq \mathbb{R}$
$I \subseteq A$ is an interval
$f : A \to \mathbb{R}$

$\}$ The function f is said to posses an ANTIDERIVATIVE on I if $\exists F : I \to \mathbb{R}$ s.t.:
• F is DIFERENTIABLE on I
• $F'(x) = f(x)$, $\forall x \in I$

The function F is called an ANTIDERIVATIVE of f

Theorem 4.1: $\emptyset \neq I \subseteq \mathbb{R}$ interval
$f : I \to \mathbb{R}$
$F_1, F_2 : I \to \mathbb{R}$ s.t. F_1 & F_2 are antiderivatives of f

Then $\exists c \in \mathbb{R}$ s.t. $F_2(x) = F_1(x) + c$, $\forall x \in I$

Proof: $\left.\begin{array}{l} F_1'(x) = f(x) \\ F_2'(x) = f(x) \end{array}\right\} \forall x \in I \to \underbrace{F_2'(x) - F_1'(x)}_{0} = (F_2 - F_1)'(x) \to$

$\to \exists c \in \mathbb{R}$ s.t. $(F_2 - F_1)(x) = c$

Remark: The condition of I being an interval is essential in T4.1.

Example: $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $\quad f(x) = 0$, $\forall x \in \mathbb{R} \setminus \{0\}$

$F_1, F_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $\quad F_1(x) = e$, $\forall x \in \mathbb{R} \setminus \{0\}$

$$F_2(x) = \begin{cases} e\pi & : x < 0 \\ \pi & : x > 0 \end{cases}$$

$F_1'(x) = f(x) = F_2'(x)$ $\forall x \in \mathbb{R} \setminus \{0\}$

however

$\nexists c \in \mathbb{R}$ s.t. $F_1(x) = F_2(x) + c$ $\qquad \left.\begin{array}{l} e + c = e\pi \\ \quad\;\; = \pi \end{array}\right\}$

Def: $\emptyset \neq I \subseteq \mathbb{R}$

$f: I \to \mathbb{R}$, has antiderivatives

$\to$ THE INDEFINITE INTEGRAL ASSOCIATED to f is

$\int f(x)\,dx = \{F: I \to \mathbb{R} \mid F$ is antiderivative of $f\}$

$= F(x) + \mathcal{C}$

$\mathcal{C} \to$ the set of all constant functions

Def: $f: I \to \mathbb{R}$, I an interval

f is said to be LRI (locally Riemann integrable) if it is Ri on $\forall [c,d] \subseteq I$

Theorem 4.2: (Concerning the existance of antiderivatives for continuous functions)

a) $\emptyset \neq I \subseteq \mathbb{R}$ an interval
$u \in I$
$f: I \to \mathbb{R}$ is LRi on I
b) f is continuous at u

$\Bigg\}$ $\forall a \in I$, the function $F(x) = \int_a^x f(t)\,dt$, $\forall x \in I$ is differentiable at u and $F'(u) = f(u)$

Proof: Chose $a \in I$ random

- $x = a \to F(a) = \int_a^a f(t)\,dt = 0$

We want $F'(u) = f(u) \Leftrightarrow \lim\limits_{x \to u} \dfrac{F(x) - F(u)}{x - u} = f(u)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in I$ with $|x - u| < \delta$, to hold

$\left| \dfrac{F(x) - F(u)}{x - u} - f(u) \right| < \varepsilon$

Choose $\varepsilon > 0$

$-\varepsilon < \dfrac{F(x) - F(u)}{x - u} - f(u) < \varepsilon \quad | + f(u)$

$f(u) - \varepsilon < \dfrac{F(x) - F(u)}{x - u} < f(u) + \varepsilon \quad (1)$

(?) $F'(x) = f(x)$

- f is continuous at u

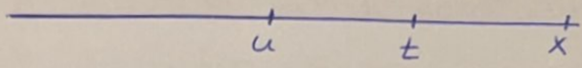
for the chosen $\frac{\varepsilon}{2}$ $\exists \delta > 0$ s.t. $\forall x \in I$ with $|x - u| < \delta$ to hold $|f(x) - f(u)| < \frac{\varepsilon}{2} \Leftrightarrow$

$\Leftrightarrow \quad -\frac{\varepsilon}{2} < f(x) - f(u) < \frac{\varepsilon}{2} \quad | + f(u)$

$\Leftrightarrow \quad f(u) - \frac{\varepsilon}{2} < f(x) < f(u) + \frac{\varepsilon}{2} \quad (2)$

$$\lim_{x\to u} \frac{F(x)-F(u)}{x-u} = F'(u) = f(u)$$

I $x > u$



2, $f(u) - \frac{\varepsilon}{2} < f(t) < f(u) + \frac{\varepsilon}{2}$ $\qquad |t-u| < \delta$

$$\to \int_u^x \left(f(u) - \frac{\varepsilon}{2}\right)dt < \int_u^x f(t)\,dt < \int_u^x \left(f(u) + \frac{\varepsilon}{2}\right)dt$$

$$\left[f(u) - \frac{\varepsilon}{2}\right](x-u) \le \int_u^x f(t)\,dt \le \left[f(u) + \frac{\varepsilon}{2}\right](x-u)$$

$$f(u) - \frac{\varepsilon}{2} \le \frac{\int_u^x f(t)\,dt}{x-u} \le f(u) + \frac{\varepsilon}{2} \quad (3)$$

$$\int_u^x f(t)\,dt = \underbrace{\int_u^a f(t)\,dt}_{= \underbrace{-\int_a^u f(t)\,dt}_{F(u)}} + \underbrace{\int_a^x f(t)\,dt}_{F(x)} = F(x) - F(u)$$

3, $f(u) - \varepsilon < f(u) - \frac{\varepsilon}{2} \le \frac{F(x)-F(u)}{x-u} \le f(u) + \frac{\varepsilon}{2} < f(u) + \varepsilon$

$$\to \left| \frac{F(x)-F(u)}{x-u} - f(x) \right| < \varepsilon \checkmark$$

II $x < u$ Hw

Remark: The converse statement dos not hold

If $F(x) = \int_a^x f(t)\,dt$ is diff. at u and $F'(u) = f(u) \not\Rightarrow f$ is cont. at u

$f : [0,1] \to \mathbb{R}$ $\quad f(x) = [x] = \begin{cases} 0 : x < 1 \\ 1 : x = 1 \end{cases}$

f is nont continuous at 1

$F(x) = \int_a^x f(t)\,dt = \int_a^x 0\,dt = 0 \quad \forall x \in [0,1]$

$F'(a) = \underset{0}{f}(a)$ but f is not c. at 0

Clasical changes of variable in the Riemann integral

TRIGONOMETRICAL SUBSTITUTIONS

- $tg\frac{x}{2} = t \Leftrightarrow \frac{x}{2} = arctg\, t \;|()' \quad dx = \frac{2}{1+t^2}\,dt$

$\sin\left(\frac{x}{2} + \frac{x}{2}\right) = 2\sin x \cos\frac{x}{2}$

$\cos\left(\frac{x}{2} + \frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\left(\frac{x}{2}\right) - \left(1 - \cos^2\frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1$

$\sin^2 x + \cos^2 x = 1 \quad 1 : \cos^2 x$

$tg^2 x + 1 = \frac{1}{\cos^2 x}$

a) $\sin x = 2 \cdot \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \cdot \cos^2\frac{x}{2} = 2\,tg\frac{x}{2} \cdot \cos^2\frac{x}{2} = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}}$

b) $\cos x = 2\,\frac{1}{1 + tg^2\frac{x}{2}} - 1 = \frac{2 - 1 - tg^2\frac{x}{2}}{1 - tg^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}}$

$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt$

Shortcuts:

If your expresion $f(x) = R(\sin x, \cos x)$

- $R(-\sin x, \boxed{\cos x}) = -R(\sin x, \boxed{\cos x}) \qquad \cos x = t$
- $R(\sin x, -\cos x) = -R(\sin x, \cos x) \qquad \sin x = t$
- $R(-\sin x, -\cos x) = R(\sin x, \cos x) \qquad tg\,x = t$

$\cos^2 x = \frac{1}{1+tg^2 x} \to \cos x = \pm\sqrt{\frac{1}{1+t^2}} = \pm \frac{1}{\sqrt{1+t^2}}$

$\sin x = \sqrt{1 - \frac{1}{1+t^2}} = \pm \frac{t}{\sqrt{1+t^2}}$

$\int_0^{\frac{\pi}{4}} \frac{1}{\cos x} dx$

$f(x) = \frac{1}{\cos x} = R(\sin x, \cos x)$

$R(-\sin x, \cos x) = \frac{1}{\cos x} = R(\sin x, \cos x)$ ⊗

$R(\underbrace{\boxed{\sin x}}_{t}, -\cos x) = -\frac{1}{\cos x} = -R(\sin x, \cos x)$ ✓

$\sin x = t \quad |'$ $\qquad \cos x\, dx = dt$

$x = 0 \to \sin x = 0 \to t = 0$

$x = \frac{\pi}{4} \to \sin\frac{\pi}{2} = \frac{\sqrt{2}}{2} \to t = \frac{\sqrt{2}}{2}$

$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{4}} \frac{\cos x\, dx}{\underbrace{\cos^2 x}_{\underbrace{1-\sin^2 x}_{t^2}}} = \int_0^{\frac{\sqrt{2}}{2}} \frac{dt}{1-t^2} = \frac{1}{2} \ln\left|\frac{1-t}{1+t}\right| \Big|_0^{\frac{\sqrt{2}}{2}}$

sau

$u = tg\frac{x}{2} \qquad \frac{x}{2} = arctg\, u$

$x = 2arctg\, u \to dx = \frac{2}{1+u^2} du$

$\cos x = \frac{1-u^2}{1+u^2}$

$\int \frac{dx}{\cos x} = \int \frac{\frac{2}{1+u^2}}{\frac{1-u^2}{1+u^2}} du = \int \frac{2}{1-u^2} du = \ln\left|\frac{1-u}{1+u}\right| + C =$

$= \ln\left|\frac{1-\cos x}{1+\cos x}\right| \Big|_0^{\frac{\pi}{4}} + C = \ln\left|\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\right| - \ln\left|\frac{1-0}{1+0}\right|$

INVERSE TRIGONOMETRICAL SUBSTITUTION

- $\sqrt{a^2 - x^2}$

$x = a\cos t$

or

$x = a\sin t$

$x = a\cos t \quad \sqrt{a^2 - a^2\cos^2 t} = \sqrt{a^2\sin^2 t} = |a\sin t|$

$x = a\sin t \quad \sqrt{a^2 - a^2\sin^2 t} = |a\cos t|$

$$\int_1^{\sqrt{2}} \sqrt{4 - x^2}\, dx$$

$x = 2\cdot\sin t \qquad dx = 2\cos t\, dt$

$\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2 t} = 2|\cos t|$

$x \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right] \to |\cos t| = \cos t$

$x = 1 \qquad t = \arcsin\frac{1}{2} = \frac{\pi}{6}$

$x = \sqrt{2} \qquad t = \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\cdot\cos t\cdot 2\cdot\cos t\, dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\cos^2 t\, dt$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos^2 t\, dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos t\cdot(\sin t)'\, dt = \cos t \sin t\, dt -$$

$$- \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (-\sin t)\sin t\, dt = \alpha + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 - \cos^2 t)\, dt = \alpha + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} dt - I$$

$$\to I = \left(\alpha + \frac{\pi}{4} - \frac{\pi}{6}\right)\cdot\frac{1}{2} = \frac{\left(\frac{1}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right) + \left(\frac{\pi}{4} - \frac{\pi}{6}\right)}{2} = \frac{\frac{2-\sqrt{2}}{4} + \frac{\pi}{12}}{2}$$