Multiple Integrals

1 Multiple Integrals Over Compact Intervals

Let $a_1 < b_1$, $a_2 < b_2$,...,$a_n < b_n$ be real numbers. Then, the product of intervals

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

is a **compact interval** in the vector space $\mathbb{R}^n$.

For example, in $\mathbb{R}^2$, such a compact interval is a rectangle, while in $\mathbb{R}^3$, such an interval is a rectangular parallelepiped.

Continuous functions defined on compact intervals from $\mathbb{R}^n$ can be Riemann integrated, with a theory similar to the one in the real case. Thus, the Riemann integral is the limit of a sequence of Riemann sums, defined in terms of the values of the functions at certain intermediate points, and the measure of some partitions. (For details see the lecture notes)

In practice, when computing multiple integrals over compact sets (much like when computing partial derivatives), we integrate with respect to one variable (at the same time consider all the others as constant), and continue the procedure, until we are done integrating with respect to all variables.

The order in which we integrate is random, so we are free to start with whichever variable we like. We just have to make sure that we do not forget the domain it is considered on. Moreover, as we will see in some exercises, sometimes the exercise is easier to solve when starting with one of the variables, and harder if when starting with the other one. (This depends on the particular expression of the function we have to integrate)

Exercise 1:

Determine the following double integral

$$\int_0^2 \int_1^5 \left(x + \frac{1}{y} \right) dxdy.$$

Remark: If the function to integrate may be written as a product of function with separable variable, so if, for

$$A \subseteq \mathbb{R}^2, \quad f : A \to \mathbb{R}, \quad \exists g, h : R \to \mathbb{R}, \quad f(x, y) = g(x) \cdot h(y), \quad \forall (x, y) \in A,$$

then

$$\int \int f(x, y)dxdy = \int g(x)dx \quad \cdot \quad \int h(y)dy.$$

The same happens for more than two variables. This situation is encountered in the following example.

Exercise 2:

Determine the following double integral

$$\int_0^2 \int_1^5 \left(x \cdot \frac{1}{y}\right) dxdy.$$

Remark: Notice that in **Exercise 1**, the function does not have separable variables, so the procedure adopted for Exercise 2 in terms of the multiplication of simple integrable does not apply in that case.

Remark: The procedure when dealing with triple integrals is quite the same.

Exercise 3:

Determine the following double integral, where $a, b, c > 0$

$$\int_0^a \int_0^b \int_0^c \frac{2z}{(x+y+1)^2} dxdydz.$$

Remark: Be careful not to forget to integrate with respect to all variables, even in the cases when in the explicit expression of f, not all the variables appear. This is the case of the following example

Exercise 4:

Determine the following triple integral, where $a, b, c > 0$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\pi} (\sin x + \cos y) dxdydz.$$

Exercise 5:

Determine the following double integral

$$\int_1^6 \int_2^3 \frac{1}{(x+y)^2} dxdy.$$

Exercise 6:

Determine the following double integral

$$\int_0^1 \int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dxdyd.$$

2 Integration over Nonempty Bounded Sets

In $\mathbb{R}^n$ there are many bounded sets, which are not generalized compact intervals in $\mathbb{R}^n$. ON such set, integration is possible. In $\mathbb{R}^2$ we are able to deliver the value of the Riemann integral, when the set is a normal domain with respect to one of the axes (ox or oy - we talked such details during the live seminar on Friday 08.05.2020)

A set $A \subset \mathbb{R}^2$ is said to be normal w.r.t to the $0y$ axis, if $x \in [a, b]$, thus x is in a compact interval $[a, b]$ and y can be expressed as bounded by two functions depending on x, hence, there exist the functions ψ_1 and $\psi_2 : [a, b] \to \mathbb{R}$ such that

$$\psi_1(x) \leq y \leq \psi_2(x).$$

In this case

$$\int\int_A f(x,y)dxdy = \int_{x=a}^{x=b}\int_{y=\psi_1(x)}^{y=\psi_2(x)} f(x,y)dxdy$$

and in order to solve if, **we have to** start integrating first with respect to the dependent variable, which in this case is **variable** y. Therefore

$$\int\int_A f(x,y)dxdy = \int_{x=a}^{x=b}\left(\int_{y=\psi_1(x)}^{y=\psi_2(x)} f(x,y)dy\right)dx$$

.

A set $B \subset \mathbb{R}^2$ is said to be normal w.r.t to the $0x$ axis, if $y \in [c, d]$, thus y is in a compact interval $[c, d]$ and x can be expressed as bounded by two functions depending on y, hence, there exist the functions ϕ_1 and $\phi_2 : [c, d] \to \mathbb{R}$ such that

$$\phi_1(y) \leq x \leq \phi_2(y).$$

In this case

$$\int\int_B f(x,y)dxdy = \int_{x=\phi_1(y)}^{x=\phi_2(y)}\int_{y=c}^{y=d} f(x,y)dxdy$$

and in order to solve if, **we have to** start integrating first with respect to the dependent variable, which in this case is **variable** x. Therefore

$$\int\int_B f(x,y)dxdy = \int_{y=c}^{y=d}\left(\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)dx\right)dy$$

.

Remark: In practice the same set A my be normal with respect to both axes or just one. The picture attached to the set is extremely useful in finding a good approach. In the case when A is normal w.r.t. to both axes we are free to fix any of the variables. In this case the choice depends on the structure of A and on the explicit formula of the function f.

Exercise 1:

Determine the following double integral

$$\int\int_A f(x,y)dxdy$$

where

$$A = \{(x,y) \in \mathbb{R}^{:}(x,y) \textbf{ is bounded by } y = x^2 and x = 3\},$$

and

$$f(x,y) = \frac{2x}{(1+x^2+y)^2}, \quad \forall (x,y) \in A.$$

Exercise 2:

Determine

$$\int\int_A \frac{x}{1+y^2} dxdy$$

where

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9, x \geq 0, y \geq 0\}.$$