

Power Series

Let $(a_n)_{n \geq 0} \subseteq \mathbb{R}$ be a sequence of real numbers. A **power series** is series of functions of the form:

$$\sum_{n \geq 0} a_n x^n,$$

with the convention that the first function in this series, is the constant function a_0 . Thus, for a given $x_0 \in \mathbb{R}$, we get the following series of real numbers

$$\sum_{n \geq 0} a_n x_0^n = a_0 + a_1 x_0 + a_2 x_0^2 + \dots + a_n x_0^n + \dots$$

A point $x_0 \in \mathbb{R}$ is called a **convergence point** of the series of real numbers $\sum_{n \geq 0} a_n x_0^n$, is convergent, thus

$$\sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R}.$$

The set of all convergence points is said to be **the convergence set of the power series** and is denoted by

$$\mathcal{C} = \left\{ x_0 \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R} \right\}.$$

Recall that for each power series

$$0 \in \mathcal{C},$$

due to the fact that

$$\sum_{n=0}^{\infty} a_n 0 = a_0 \in \mathbb{R}.$$

The convergence radius of the power series is

$$R = \frac{1}{\lambda} \quad \text{unde} \quad \lambda = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

According to Cauchy-Hadamard's theorem

$$(-R, R) \subseteq \mathcal{C} \subseteq [-R, R].$$

The particular cases

$$x = -R \quad \text{and} \quad x = R$$

must be analyzed separately in order to determine exactly \mathcal{C} .

Determine the convergence radius and the convergence set for the following power series

Example 1:

$$\sum_{n \geq 0} x^n.$$

Solution:

$$\lambda = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{1} = 1 \implies R = 1.$$

For $x = 1$, the series of real numbers $\sum_{n=0}^{\infty} 1 = \infty$, is divergent, thus $1 \notin \mathcal{C}$.

For $x = -1$, the series of real numbers $\sum_{n \geq 0} (-1)^n$ does not have a sum, due to the fact that the sequence of partial sums is constantly oscillating between 1 and 0, therefore $\nexists \lim_{n \rightarrow \infty} (-1)^n$. Thus $-1 \notin \mathcal{C}$. We conclude that

$$\mathcal{C} = (-1, 1).$$

Remark: The power series may be written as developed around another point in \mathbb{R} , case in which they are stated as

$$\sum_{n \geq 0} a_n (x - x_0)^n.$$

In such a case the convergence radius is computed accordingly to the algorithm above. The convergence radius is computed exactly like in the case above, the only difference relies in the formulation of the convergence set, namely:

$$(x_0 - R, x_0 + R) \subseteq \mathcal{C} \subseteq [x_0 - R, x_0 + R].$$

The cases when $x = x_0 - R$ and $x = x_0 + R$ must be analyzed separately in order to specify completely the convergence set.

Example 2 :

$$\sum_{n \geq 1} \frac{(-1)^n}{n(2n+1)} (x+2)^n.$$

Solution: The power series is expanded about the point $x_0 = -2$, and the generating sequence is $(a_n)_{n \geq 1}$, having the general term

$$a_n = \frac{(-1)^n}{n(2n+1)}, \quad \forall n \in \mathbb{N}.$$

We compute

$$\lambda = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(2n+3)}{n(2n+1)} \right| = 1 \implies R = \frac{1}{1} = 1.$$

Hence

$$(-2-1, -2+1) = (-3, -1) \subseteq \mathcal{C}.$$

Next we check the boundaries:

For $x = -3$, the series of real numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \sim \sum_{n \geq 1} \frac{1}{n^2},$$

is convergent, therefore $-3 \in \mathcal{C}$.

For $x = -1$, the series of real numbers $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} \cdot (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)}$ is alternating. Due to the fact that the sequence of real numbers $\left(\frac{1}{n(2n+1)}\right)_{n \geq 1}$ is decreasing and has the limit 0, according to Leibniz' criterion, we obtain convergence, hence $-1 \in \mathcal{C}$.

In conclusion

$$\mathcal{C} = [-3, -1].$$

Exercises: Determine both the convergence radius and the convergence set for the following power series:

1. $\sum x^n$
2. $\sum_{n \geq 1} \frac{1}{n} x^n$
3. $\sum_{n \geq 1} \frac{1}{n(n+1)} x^n$
4. $\sum_{n \geq 0} \frac{1}{n!} x^n$
5. $\sum_{n \geq 1} n! x^n$
6. $\sum_{n \geq 0} \left(\sqrt[3]{n^2 + n + 1} - \sqrt[3]{n^2 - n - 1} \right)^n x^n$
7. $\sum_{n \geq 0} (n+1)^n x^n$
8. $\sum_{n \geq 0} \frac{(-1)^n}{n} x^n$