17.11.2023

Analysis - Lecture 7

Continuous functions

~ 2^{nd} part ~

Example (DIRICHLET'S TYPE FUNCTIONS)

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 : x \in \mathbb{Q} \\ 0 : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Study its continuity:

This is an example of a function nowhere continuous.

Choose a random $x \in \mathbb{R}$

$\exists (a_n) \subseteq \mathbb{Q}$ s.t. $\lim\limits_{n\to\infty} a_n = x$ $\quad \lim\limits_{n\to\infty} f(a_n) = 1$

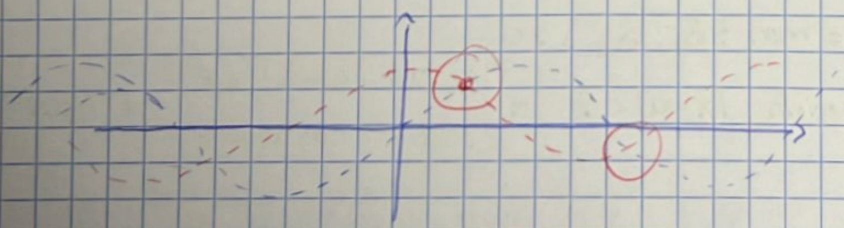
$\exists (b_n) \subseteq \mathbb{R}\setminus\mathbb{Q}$ s.t. $\lim\limits_{n\to\infty} b_n = x$ $\quad \lim\limits_{n\to\infty} f(b_n) = 0$

$\to f$ - discontinuous on $\mathbb{R}$

$A' = \overline{\mathbb{R}}$

Example:

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \sin x, & x \in \mathbb{Q} \\ \cos x, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$



$B = \{x \in \mathbb{R} : \sin x = \cos x\}$

$= \{x \in \mathbb{R} : \operatorname{tg} x = 1\} = \{\frac{\pi}{4} + 2k\pi : k \in \mathbb{Z}\}$

I We prove that f is not continuous on $\mathbb{R}\setminus B$

choose a random $x \in \mathbb{R}\setminus\mathbb{Q}$

$\exists (a_n) \subseteq \mathbb{Q}$ s.t. $\lim\limits_{n\to\infty} a_n = x \to \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} \sin(a_n) = \sin x$

$\exists (b_n) \subseteq \mathbb{R}\setminus\mathbb{Q}$ s.t. $\lim\limits_{n\to\infty} b_n = x \to \lim\limits_{n\to\infty} f(b_n) = \lim\limits_{n\to\infty} \cos(b_n) = \cos x$

$\forall x \notin B \to \sin x \neq \cos x \Rightarrow$

$\rightarrow$ f is not continuous at x
x - random
$\Big\} \rightarrow$ f is not continuous on $\mathbb{R} \setminus B$.

II We prove that f is continuous on B

Choose $a \in B$ random

We use the ε, δ theorem for continuity

We prove that $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$ with $|x-a| < \delta$ to hold $|f(x) - f(a)| < \varepsilon$

Chose $\varepsilon > 0$ - random

$$? \; |f(x) - f(a)| = \begin{cases} |\sin x - f(a)| : x \in \mathbb{Q} \\ |\cos x - f(a)| : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \overset{?}{<} \varepsilon$$

$a \in B \rightarrow \sin a = \cos a = f(a)$

$$|f(x) - f(a)| = \begin{cases} |\sin x - \sin a| : x \in \mathbb{Q} \\ |\cos x - \cos a| : x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \overset{?}{<} \varepsilon$$

sin and cos are continuous functions on $\mathbb{R}$

for $\varepsilon > 0$ already chosen

$\rightarrow \exists \delta_1 > 0$ s.t. $\forall x \in \mathbb{R}$ with $|x-a| < \delta_1$ that holds $|\sin x - \sin a| < \varepsilon$

$\rightarrow \exists \delta_2 > 0$ s.t. $\forall x \in \mathbb{R}$ with $|x-a| < \delta_2$ that holds $|\cos x - \cos a| < \varepsilon$

Choose $\delta = \min\{\delta_1, \delta_2\} > 0$

$$\rightarrow \forall x \in \mathbb{R} \text{ with } |x-a| < \delta \rightarrow \begin{cases} |\sin x - \sin a| < \varepsilon \\ |\cos x - \cos a| < \varepsilon \end{cases} \rightarrow |f(x) - f(a)| < \varepsilon$$

ε - random $\rightarrow \forall \rightarrow$ f is continuous at a
a - random
$\Big\} \rightarrow$ f is continuous on B

Example - HW.

$$f(x) = \begin{cases} -3x^2 + 4 ; & x \leq -5 \\ 7 : & x = -4 \\ e^x : & x > -3, \; x < 0 \\ \sin x : & x \geq 0, \; x \in \mathbb{Q} \\ \cos x : & x \geq 0, \; x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

!! $f(x)$ is continuous at -4

$-4 \in J_{2\varnothing}$ (domain)

Remark: Each elementary function and compositions of elementary functions are continuous on their domain

V. DIFFERENTIABLE FUNCTIONS

Framework:

$$\left.\begin{array}{l} \emptyset \neq A \subseteq \mathbb{R} \\ f: A \to \mathbb{R} \\ a \in A \cap A' \end{array}\right\}$$

- f is said to have a derivative at a if $\exists \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} \overset{den}{=} f'(a) \in \overline{\mathbb{R}}$
- f is said to be DIFFERENTIABLE at a if $\exists f'(a) \in \mathbb{R}$

Remark: a) $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, $x - a \overset{not}{=} h$

b) Side derivatives may be consindered and they obey the same rule as side limits.

c) If $\text{Izo}\, A \neq \emptyset$, $\forall y \in \text{Izo}\, A$ the notion of a derivative is not defined, since $y \notin A'$

Example:

Study the differentiability of the function

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x| = \begin{cases} x ; & x \geq 0 \\ -x ; & x < 0 \end{cases}$

- I $a < 0$

Choose $a < 0$ randomly $\to f(a) = |a| = -a$

Study $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a} \dfrac{-x - (-a)}{x - a} = -1$

? $f(x) = -x$

$\exists B(a, r)$ s.t. $x < 0$

$\lim\limits_{x \to a} f(x) = \lim\limits_{\substack{x \to a \\ x < a}}$

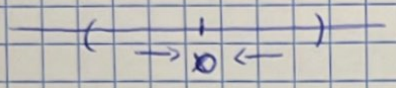
II $a > 0$

Choose $a > 0$ randomly

$\lim \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a} \dfrac{x - a}{x - a} = 1$

$\rightarrow f'(x) = \begin{cases} -1 : x < 0 \\ 1 : x > 0 \end{cases}$

III We study the differentiability at 0



$0 \in \mathbb{R} \cap \mathbb{R}'$

$$\left.\begin{array}{l} \exists \lim\limits_{\substack{x \to 0 \\ x < 0}} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{\substack{x \to 0 \\ x < 0}} \dfrac{-x - 0}{x - 0} = -1 \\ \exists \lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{x - 0}{x} = 1 \end{array}\right\} \rightarrow \neq \rightarrow \nexists \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \rightarrow$$

$\rightarrow f$ is not differentiable at 0

Theorem

$$\left.\begin{array}{l} \emptyset \neq A \subseteq \mathbb{R} \\ f : A \to \mathbb{R} \\ a \in A \cap A' \end{array}\right\} \text{ if } f \text{ is differentiable at } a \rightarrow f \text{ is continuous at } a$$

Remark: $\not\leftarrow$

Example: $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} x \cdot \sin \frac{1}{x} : x \neq 0 \\ 0 : x = 0 \end{cases}$

a) We prove that f is continuous at 0 with $Th_{\varepsilon,\delta}$

(*) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}$ with $\boxed{|x - 0| < \delta}$ to hold $|f(x) - f(0)| < \varepsilon$

Choose $\varepsilon > 0$ random

? $|f(x) - f(0)| = \begin{cases} |x \cdot \sin \frac{1}{x} - 0| = |x| \, |\sin \frac{1}{x}| \leq \boxed{|x| \overset{?}{<} \varepsilon} \; (**) \\ 0 < \varepsilon \checkmark \end{cases}$

Choose $\delta = \varepsilon$

Then for the $\varepsilon > 0$ randomly chosen $\exists \delta = \varepsilon > 0$ s.t. $\forall x \in \mathbb{R}$ with $|x - 0| < \delta$ to hold $|f(x) - f(0)| \overset{(**)}{<} |x| < \varepsilon$ $\Big\}$ ε - random

$\rightarrow \forall$

$\rightarrow f$ is continuous at 0

I & II f is continuous $\mathbb{R} \setminus \{0\}$ on a decomposition of

$\rightarrow f$ is continuous on $\mathbb{R}$

$\forall x \in \mathbb{R},\ x \neq 0 \quad f'(x) = \left(x \sin \frac{1}{x}\right)' = \sin \frac{1}{x} + x \cdot \cos \frac{1}{x} \cdot \left(\frac{-1}{x^2}\right) =$

$= \sin x - \frac{1}{x} \cdot \cos \frac{1}{x}$

If $a = 0$ we study the differentiability with the definition

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \cdot \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin \frac{1}{x}$$

$a_n = \frac{1}{2n\pi} \to 0 \qquad \sin(a_n) = 0$

$\neq$

$b_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0 \qquad \sin(b_n) = 1$

$\rightarrow f$ is not diff at 0

$\rightarrow f$ is diff on $(-\infty, 0) \cup (0, \infty)$

Recall: Rolle, Lagrange, Cauchy, Fermat

TAYLOR'S POLINOMIAL

Framework:

$\emptyset \neq A \subseteq \mathbb{R}$

$a \in A \cap A'$ $\quad + f$ is n times differentiable at $\boxed{a}$

$f : A \rightarrow \mathbb{R}$

Def: Taylor's polinomial of rank n, attached to the function f, about the point a is the polinomial function $T_{n:a} f : \mathbb{R} \rightarrow \mathbb{R}, \quad (T_{n:a} f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k =$

$$= f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Remark Taylor's polinomial is on $\mathbb{R}$ while f has the domain just A.

Example:

a) $f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = e^x$

$\hookrightarrow f$ is indefinite differentiable on $\mathbb{R}$ ($\forall x \in \mathbb{R} \ \exists f^{(n)}(x) = e^x$)
$\forall n \in \mathbb{N}$

$\forall n \in \mathbb{N} \quad f^{(n)}(0) = e^0 = 1$

$(x-0)^n = x^n$

For a random $n \in \mathbb{N}$, $(T_{n:0} f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k =$

$= \sum_{k=0}^{n} \frac{1}{k!} \cdot x^k = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} \sin x : n = 4k \\ \cos x : n = 4k+1 \\ -\sin x : n = 4k+2 \\ -\cos x : n = 4k+3 \end{cases}$$

- $\sin(x + \frac{\pi}{2}) = \sin x \underbrace{\cos\frac{\pi}{2}}_{0} + \underbrace{\sin\frac{\pi}{2}}_{1} \cdot \cos x = \cos x = f'(x)$ ①

- $\sin(x + \pi) = \sin x \underbrace{\cos \pi}_{-1} + \underbrace{\sin \pi}_{0} \cdot \cos x = -\sin x = f''(x)$

- $\sin(x + 3 \cdot \frac{\pi}{2}) = -\cos x = f'''(x)$

For $n \in \mathbb{N}$

$P(n)$: $\boxed{\sin^{(n)}(x) = \sin(x + n \cdot \frac{\pi}{2})}$

We prove by induction that $P(n)$ is true $\forall n \in \mathbb{N}$

I $n = 1$ ✓

II Choose $k \in \mathbb{N}$, $k > 1$. Assume that $P(k)$ is True and prove $P(k+1)$.

$P(k)$: $\sin^{(k)}(x) = \sin(x + k \cdot \frac{\pi}{2}) \mid '$

$\sin^{(k+1)}(x) = \cos(x + k \cdot \frac{\pi}{2}) \overset{①}{=} \sin(x + k\frac{\pi}{2} + \frac{\pi}{2}) =$

$= \sin(x + (k+1) \cdot \frac{\pi}{2}) \to P(k+1)$ is True

I & II $\to \forall n \in \mathbb{N}$, $P(n)$ is true

For $a = 0$

$$\sin^{(n)}(0) = \sin(0 + n \cdot \frac{\pi}{2}) = \sin(n \cdot \frac{\pi}{2}) = \begin{cases} 1 : n = 4k+1 \\ 0 : n = 2k \\ -1 : n = 4k+3 \end{cases} =$$

$$= \begin{cases} (-1)^k : n = 2k+1 \\ 0 : n = 2k \end{cases}$$

Scanned with CamScanner

We will write Taylor's polynomial of rank $n = 2k+1$ attached to the function $f(x) = \sin x$, about the point $a = 0$.

$$(T_{n;0} f)(x) = \sum_{t=0}^{2k+1} \frac{\sin^{(t)} 0}{t!} \cdot x^t =$$

$$= \sum_{t=0}^{2k+1} \frac{\sin(0 + t\cdot\frac{\pi}{2})}{t!} \cdot x^t = 0 + \frac{1}{1!}\cdot x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 -$$

$$- \frac{1}{7!}\cdot x^7 + \dots + \frac{(-1)^k}{(2k+1)!}\cdot x^{2k+1}$$

Remark: Since $\sin^{(2k+2)}(x) = 0$, $(T_{2k+1;0} \sin) = (T_{2k+2;0} \sin)$

$$\sin x \sim \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)}$$

Seminar:

$$\cos x \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \cdot \frac{x^{2k}}{(2k)!}$$

Theorem:

Framework a) $(T_{n;a} f)$ is an indefinite differentiable function on $\mathbb{R}$

b) $\forall\, t \in \{0, 1, \dots, n\}$, $(T_{n;a} f)^{(t)}(a) = f^{(t)}(a)$

c) $\forall\, t > n$, $(T_{n;a} f)^{(t)}(x) = 0 \quad \forall\, x \in \mathbb{R}$

a) $\to$ it is a polynomial

c) degree $(T_{n;a} f) \leq n \to$ Each derivative of $(T_{n;a} f)$ of rank $\geq n+1$ $\to 0$

b) $\circ\ t = 0$

$$(T_{n;a} f)(a) = f(a) + \frac{f'(a)}{1!}\underbrace{(a-a)}_{x-a} + \dots + \frac{f^{(n)}(a)}{n!}(a-a) =$$

$$= f(a)$$

$$(T_{n;a} f)'(x) = \underbrace{(f(a))'}_{0} + \left[\frac{f'(a)}{1!}(x-a)\right]' + \left[\frac{f''(a)}{2!}(x-a)^2\right]' +$$

$$+ \dots + \left[\frac{f^{(n)}(a)}{n!}(x-a)^n\right]' =$$

$$= 0 + f'(a) + \frac{f''(a)}{\cancel{2!}\ 1!}\cancel{2}(x-a) + \frac{f'''(a)}{\cancel{3!}\ 2!}\cdot\cancel{3}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{\cancel{n!}\ (n-1)!}\cdot$$

$$\cdot \cancel{n}(x-a)^{n-a} = f'(a) + \frac{f''(a)}{1!}\cdot(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

$= (T_{n-1;a}\, f')(x)$

$(T_{n;a}\, f')'(a) = f'(a) + 0$

$(T_{n;a}\, f)''(x) = (T_{n-1;a}\, f')'(x) = (T_{n-2;a}\, f'')(x)$

... inductively

$(T_{n;a}\, f)'(x) = (T_{n-1;a}\, f')(x)$

$(T_{n;a}\, f)^{(k)}(x) = (T_{n-k;a}\, f^{(k)})(x)$