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Calculus - Lecture 12

Series of functions

Recall:

- $F(A,\mathbb{R}) = \{ f \mid f: A \to \mathbb{R} \} \to$ the set of all real-valued functions whose domain is $A \ (\subseteq \mathbb{R})$

- $h: \mathbb{N} \to F(A,\mathbb{R})$

  $\forall n \in \mathbb{N} \quad h(n) := f_n \qquad f_n : A \to \mathbb{R}$ is a sequence of functions

  $(f_n) = (f_n)_{n \geq 1} = (f_n)_{n \in \mathbb{N}}$

- $\mathcal{C} = \{ x \in A : \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R} \}$ is the convergence set of a sequence of functions

- $f: \mathcal{C} \to \mathbb{R}, \ \forall x \in \mathbb{R} \quad f(x) = \lim\limits_{n\to\infty} f_n(x)$ is called the POINTWISE FUNCTION of the sequence of functions (f_n)

  $f_n \xrightarrow[\mathcal{C}]{} f \ \leftrightarrow \ \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$ $\forall x \in \mathcal{C}$ $|f_n(x) - f(x)| < \varepsilon$

- $f_n \overset{\rightrightarrows}{\mathcal{C}} f$ converges uniformly to f if:

  $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$ $\forall x \in \mathcal{C}$ $|f_n(x) - f(x)| < \varepsilon$

- $\rightrightarrows \ \Rightarrow \ \to$

  $\not\Leftarrow$

Def: Each ordered pair $((f_n), (s_n))$ of two sequences of functions $(f_n), (s_n) \subseteq F(A,\mathbb{R})$ with the property that:

- $\forall x \in \mathcal{C} \quad s_1(x) = f_1(x)$
- $\forall x \in \mathcal{C} \quad s_2(x) = f_2(x)$

  $\vdots$
- $\forall x \in \mathcal{C} \quad s_n(x) = f_1(x) + f_2(x) + \dots + f_n(x)$

is a SERIES of functions

Def: $\bullet$ $\mathcal{C} = \{ x \in A :$ the series $\sum f_n(x)$, of real numbers, is convergent$\} =$

$= \{ x \in A : \sum\limits_{n=1}^{\infty} f_n(x) \in \mathbb{R}$

• If $\mathcal{C} \neq \emptyset$, we define $s: \mathcal{C} \to \mathbb{R}$, $\forall x \in \mathcal{C}$ $s(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{n \to \infty} S_n(x)$ which is the POINTWISE SUM FUNCTION of the series of functions $\sum f_n$

Notation: $\sum f_n \xrightarrow[\mathcal{C}]{} s$

• If $S_n \rightrightarrows s$ then we say that the series of functions $\sum f_n$ converges uniformly to s

Notation: $\sum f_n \underset{\mathcal{C}}{\rightrightarrows} s$

Remark: A particular example of series of functions is the case of a power series

$$\sum_{n \geq 0} a_n \cdot x^n, \text{ where } (a_n) \in \mathbb{R} \quad = \sum_{n \geq 0} f_n(x)$$

$\forall n \in \mathbb{N} \cup \{0\}$ $f_n(x) = a_n \cdot x^n$

• Recall that $(-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

$$R = \frac{1}{\lambda} \text{ and } \lambda = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \text{ or } \sqrt[n]{|a_n|}$$

Uniform convergence criteria for series of functions

T1 (Cauchy)

$$\left.\begin{array}{l}(f_n) \subseteq \mathcal{F}(A, \mathbb{R}) \\ \mathcal{C} \subseteq A\end{array}\right\} \sum f_n \underset{\mathcal{C}}{\rightrightarrows} s \iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \left.\begin{array}{r}\forall n \geq n_\varepsilon \\ \forall p \in \mathbb{N} \\ \forall x \in \mathcal{C}\end{array}\right\} * \ldots$$

$$* \; |f_{n+1}(x) + \ldots + f_{n+p}(x)| < \varepsilon$$

T2 (Weierstrass)

$$\left.\begin{array}{l}(f_n) \subseteq \mathcal{F}(A, \mathbb{R}) \\ \mathcal{C} \subseteq A \\ f: \mathcal{C} \to \mathbb{R} \\ (a_n) \subseteq \mathbb{R}\end{array}\right\} \left.\begin{array}{l}\text{If the series of real numbers} \\ \sum a_n \text{ is C} \\ \text{and } \exists n' \in \mathbb{N} \text{ s.t. } \left.\begin{array}{r}\forall n \geq n' \\ \forall x \in \mathcal{C}\end{array}\right\} |f_n(x)| \leq a_n\end{array}\right\} \sum f_n \underset{\mathcal{C}}{\rightrightarrows}$$

(we only get the nature, not the sum)

Remark: For both sequences and series of functions properties such as:
- continuity
- Rieman Integrability
- differentiability

are inherited from the functions generating either (f_n) or $\sum f_n$

through $\boxed{\rightrightarrows}$ (to f as s)
$\qquad\qquad\qquad\qquad\downarrow \qquad \downarrow$
$\qquad\qquad\qquad\qquad f_n \qquad \sum f_n$

Example: Consider the following series of functions

$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \cdot x^n$ determine its sum.

Solution: $\forall n \in \mathbb{N}$ $f_n(x) = \frac{(-1)^{n+1}}{n} \cdot x^n$, $\forall x \in \mathbb{R}$, $f_n : \mathbb{R} \to \mathbb{R}$

$\to \sum f_n \subseteq F(\mathbb{R}, \mathbb{R})$

- $C = ?$

It is a power series, $a_n = \frac{(-1)^{n+1}}{n}$

$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \to R = 1 \to (-1,1) \subseteq C \subseteq [-1,1]$

- $x = -1 \to \sum_{n\geq 1} \frac{(-1)^{n+1} \cdot (-1)^n}{n} = \sum_{n\geq 1} \frac{-1}{n} = -\sum_{n\geq 1} \frac{1}{n}$ D $\to -1 \notin C$

- $x = 1 \to \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \cdot 1^n = -1 \sum \frac{(-1)^n}{n}$ - C (Leibniz) $\to 1 \in C$

Hence $C = (-1, 1]$

$\to \exists\, s : (-1,1] \to \mathbb{R}$, $\forall x \in (-1,1]$, $s(x) = \sum_{n=1}^{\infty} f_n(x)$

It can be shown than $\sum f_n \rightrightarrows s$ on $(-1,1]$

$\to s'(x) = \sum_{n\geq 1} f_n'(x)$

$s'(x) = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \cdot x^n \right)' = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot n \cdot x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot x^{n-1} =$

$= \sum_{n=1}^{\infty} (-x)^{n-1}$

for $\boxed{x \in C \text{ randomly chosen}}$

$s'(x) = \sum_{n=1}^{\infty} \underbrace{(-x)}_{\substack{\downarrow \\ -x \text{ is a constant}}}{}^{n-1} = \frac{1}{1-(-x)}$ $\forall x \in (-1,1)$ (the g.s is D for $x=1$)

$\to$ geometric series

$= \frac{1}{1+x}$ $\forall x \in (-1,1)$

$\forall n \in \mathbb{N}$ f_n was continuous with f'_n cont $\to \int s'(x)\,dx = \int \frac{1}{1+x}\,dx \to$
s is cont with s' cont

$\rightarrow s(x) = \int \frac{1}{1+x} dx = \ln(1+x) + C$

$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^n \quad \forall x \in (-1,1]$

$S(0) = 0$ (the sequence of the partial sum is constant 0) $\rightarrow$

$\rightarrow \left.\begin{matrix} S(x) = \ln(1+x) + C \\ S(0) = 0 \end{matrix}\right\} \rightarrow \ln(1+0) + C = 0 \rightarrow C = 0$

Hence $S(x) = \ln(1+x) \quad \forall x \in (-1,1)$

$\left.\begin{matrix} f_n \text{ is cont} \\ S \text{ is cont} \end{matrix}\right. \text{ on } (-1,1] \Big\} \rightarrow \lim_{x \to 1} s(x) = s(1) \rightarrow s(1) = \lim_{x\to 1} \ln(1+x) = \ln 2$

$\rightarrow S : (-1,1] \rightarrow \mathbb{R} \quad S(x) = \ln(1+x)$ the sum function of the series of functions generated by the power series $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^n$

$\rightarrow \boxed{\forall x \in (-1,1] \quad \ln(1+x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \cdot x^n}$ (**) the Taylor series extension of $\ln(1+x)$

$$\boxed{\sum_{n=1}^{p} \frac{(-1)^{n+1}}{n} x^n = (T_{p,0} f)(x) \quad f(x) = \ln(1+x)}$$

Therefore $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \cdot 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(1+1) \rightarrow$

$\rightarrow \ln 2 = \lim_{n \to \infty} \left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n}\right)$

Examples for the study of the uniform convergence for series of functions with the help of the weirstrass criterion:

$\left.\begin{matrix} |f_n(x)| \leq a_n \\ \sum a_n \text{ is C} \end{matrix}\right\} \rightarrow \sum f_n(x)$ is UC on B (AC)

a) $\forall n \in \mathbb{N} \quad f_n : \mathbb{R}^* \rightarrow \mathbb{R} \quad f_n(x) = \frac{1}{n! \cdot x^n}$

$\sum_{n \geq 1} \frac{1}{n! \cdot x^n} \qquad |f_n| \leq a_n$

$a_n := \frac{1}{n! \cdot |x|^n}$? $\sum a_n$ is C
is a SPT

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(n+1)! \, |x|^{n+1}} \cdot \frac{n! \, |x|^n}{1} = \lim_{n\to\infty} \frac{1}{|x|} \cdot \frac{1}{n+1} = 0 \to \sum |a_n|$ is C regardless of the values of x

$\to \sum f_n$ is UC

b) $\sum_{n\geq 1} \left(\frac{1}{x^2+n^2} \right)$

$$\left. \begin{array}{l} |f_n(x)| = \frac{1}{x^2+n^2} \leq \frac{1}{n^2} \\ \sum \frac{1}{n^2} \text{ is C} \end{array} \right\} \to \sum f_n \text{ is U.C. on } \mathbb{R}$$

c) $\sum_{n\geq 1} \left(\frac{1}{x^2+2^n} \right)$

$$\left. \begin{array}{l} |f_n(x)| = \frac{1}{x^2+2^n} \leq \frac{1}{2^n} \\ \sum \frac{1}{2^n} \text{ is C} \end{array} \right\} \to \sum f_n \text{ is UC on } \mathbb{R}$$

The bynominal series
- a generalization of Newton's bynomial

$$\sum_{n\geq 1} \frac{k(k-1)\cdots(k-n+1)}{n!} \cdot x^n, \quad \forall n \in \mathbb{N}$$

$k \in \mathbb{R}$ is a fixed value

$a_n = \frac{k(k-1)\dots(k-n+1)}{n!}$ $(a_n) \subseteq \mathbb{R}$

$\to$ it is a power series $\sum a_n \cdot x^n$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{k(k-1)\dots(k-n+1)(k-n)}{(n+1)!} \right| \cdot \left| \frac{n!}{k(k-1)\dots(k-n+1)} \right| =$$

$$= \lim_{n\to\infty} \frac{|k-n|}{n+1} = 1 \to R = \frac{1}{\lambda} = 1 \to (-1,1) \subseteq \mathcal{C} \subseteq [-1,1]$$

Remark: In order to fully determine $\mathcal{C}$ - particular cases for k have to be consider. It cannot be dealt with easyly.

We know for sure $(-1,1) \subseteq \mathcal{C}$

$\sum f_n \underset{(-1,1)}{\rightrightarrows} s$

- $\forall n \in \mathbb{N}$ f_n is con. / dif. / int. $\Big\} \to$ s is c. / d. / int. on $(-1,1)$

$\exists\, s:(-1,1)\to\mathbb{R},\quad s(x) = 1+\sum_{n=1}^{\infty} f_n(x)$

$= 1+\sum_{n=1}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!}\, x^n =$ $a_0 = 1$ for power series

$= 1+\frac{k}{1!}\cdot x + \frac{k(k-1)}{2!}\cdot x^2+\dots+\frac{k(k-1)\dots(k-n+1)}{n!}\cdot x^n+\dots+\dots$

- We differenciate it with respect to x

$s'(x) = k + \frac{k(k-1)}{2}\cdot 2x + \frac{k(k-1)(k-2)}{3!}\cdot 3x^2+\dots+\frac{k(k-1)\dots(k-n+1)}{n!}\cdot n x^{n+1}+\dots$

$= k + \frac{k(k-1)}{1}\cdot x + \frac{k(k-1)(k-2)}{2}\cdot x^2+\dots+\frac{k(k-1)\dots(k-n+1)}{(n-1)!}\cdot x^{n-1}$

$x\cdot s'(x) = \frac{k}{1!}\cdot x + \frac{k(k-1)}{1!}\cdot x^2+\dots+\frac{k(k-1)(k-n+1)}{(n-1)!}\cdot x^n$

$s'(x) + x\cdot s'(x) = k + \frac{k}{1}\cdot x + \frac{k(k-1)}{1}x + \frac{k(k-1)(k-2)}{2!}\cdot x^2 + \frac{k(k-1)}{1!} + x^2+\dots+$

$+ \frac{k(k-1)\dots(k-n+1)}{(n-1)!}x^{n-1} + \frac{k(k-1)\dots(k-n)}{(n-2)!}\cdot x^{n-1}$

$= k + k\cdot x\left(1+\frac{k-1}{1}\right) + \frac{k(k-1)}{1!}\cdot\left(\frac{k-2}{2}+1\right)\cdot x^2+\dots+$

$+ \frac{k(k-1)\dots(k-n)}{(n-1)!}\left(\frac{k-n+1}{n-1}+1\right)x^{n-1}$

$= k + k\cdot x\,(1+k-1) + \frac{k(k-1)}{1!}\left(\frac{k-2+2}{2}\right)\cdot x^2+\dots+\frac{k(k-1)\dots(k-n)}{(n-2)!}\cdot$

$\cdot\Big(\ k-n+$

$= k + kx\cdot\frac{k}{1} + \frac{k(k-1)\cdot k}{2}\cdot x^2 + \frac{k(k-1)(k-2)}{2!}\cdot k\cdot\frac{x^3}{3}+\dots+\frac{k(k-1)(k-2)\dots(k-n+1)}{(n-2)!}\cdot k\frac{x^k}{n}$

$+\dots = k\cdot s(x)\quad \forall x\in(-1,1)$

$\to (1+x)\,s'(x) = k\cdot s(x)$ ***

$$! \left(\frac{S(x)}{(1+x)^k}\right)' = \frac{S'(x)(1+x)^k - S(x)\cdot k(1+x)^{k-1}}{(1+x)^{2k}} = \frac{S'(x)(1+x) - S(x)\cdot k}{(1+x)^{k+1}}$$

$$\xrightarrow{(***)} \forall x \in (-1,1) \quad (1+x)S'(x) - kS(k) = 0 \to \left(\frac{S(x)}{(1+x)^k}\right)' = 0 \to$$

$$\to \exists\, c \in \mathbb{R} \text{ s.t. } \frac{S(x)}{(1+x)^k} = c \Leftrightarrow S(x) = c\cdot(1+x)^k \quad \forall x \in (-1,1)$$

$$? \left.\begin{array}{l} S(0) = c \cdot 1 = c \\ S(0) = 1 \end{array}\right\} \to c = 1 \to S(x) = (1+x)^k \quad \forall x \in (-1,1)$$

$$\forall x \in (-1,1) \quad (1+x)^k = \lim_{n\to\infty}\left(1 + \frac{k}{1}\cdot x + \frac{k(k-1)}{2}\cdot x^2 + \dots + \frac{k(k-1)\dots(k-n+1)}{n!}\cdot x^n\right),$$

$k \in \mathbb{R}$

$$k=-1 \quad \frac{1}{1+x} = (1+x)^{-1} = \lim_{n\to\infty}\left(1 - x + x^2 + \dots - (-1)^n \cdot x^n\right) \quad \forall x \in \underbrace{(-1,1)}_{\text{symetric}}$$

$$x = -t \quad \frac{1}{1-t} = \lim_{n\to\infty}\left(1 + t + t^2 + \dots + t^n\right) \quad \forall x \in (-1,1)$$