22.12.2023
Analysis - Lecture 11

Sequences and series of functions

Def: $\emptyset \neq A \subseteq \mathbb{R}$

$F(A, \mathbb{R}) = \{ f : A \to \mathbb{R} \mid f \text{ is a function} \}$ - the set of all functions whose domain is A with $\mathbb{R}$ as codomain

Each $\gamma : \mathbb{N} \to F(A, \mathbb{R})$ is a sequence of functions

$\forall n \in \mathbb{N}, \ \gamma(n) = f_n$, where $f_n : A \to \mathbb{R}$

- $(f_n)_{n\in\mathbb{N}} = (f_n)_{n\geq 1} = (f_n) \subseteq F(A, \mathbb{R}) \to$ notion for a sequence of functions

Def: Let $(f_n) \subseteq F(A,\mathbb{R})$ be a sequence of functions

- The set $\mathcal{C} = \{ a \in A \mid$ the sequence of real numbers $(f_n(a))$ is convergent $\}$ is called THE CONVERGE SET of the sequence of functions (f_n)
- The function $f : \mathcal{C} \to \mathbb{R}$, $f(a) = \underbrace{\lim_{n\to\infty} f_n(a)}_{*}$ $\forall a \in A$ is called THE POINTWISE LIMIT of the sequence of functions (f_n)

$* \ a \in \mathcal{C} \to \exists \in \mathbb{R}$

Example: 1. $\forall n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = x^n$. Study its converging set and its pointwise limit

! we choose a random $x \in \mathbb{R}$ and try to compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} \infty : x > 1 \\ 1 : x = 1 \\ 0 : |x| < 1 \\ \nexists : x \leq -1 \end{cases}$$

$\to$ the convergence set is $\mathcal{C} = (-1; 1]$

!! We define the function $f : (-1, 1] \to \mathbb{R}$, $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1 : x = 1 \\ 0 : x \in (-1,1) \end{cases}$ which is the pointwise limit of (f_n)

The notation: $\boxed{f_n \xrightarrow[\mathcal{C}]{} f}$

2. $\forall n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{1}{n} \sin(nx)$ $\forall x \in \mathbb{R}$

I choose a random $x \in \mathbb{R}$ and determine $\lim_{n\to\infty} f_n(x) =$

$= \lim_{n\to\infty} \frac{1}{n} \underbrace{\sin(nx)}_{\in[-1,1]} = 0 \to \mathcal{C} = \mathbb{R}$

II we define $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$, $\forall x \in \mathbb{R}$

$$f_n \xrightarrow[\mathbb{R}]{} f$$

3. $\forall n \in \mathbb{N}$, $f_n(x) = \dfrac{nx}{nx+1}$ $\quad f_n : [0,1] \to \mathbb{R}$

I Chose $x \in [0,1]$ random and

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{nx+1} = \begin{cases} 0 : x = 0 \\ 1 : x \in (0,1] \end{cases}$$

$B = [0,1]$

II we define $f: [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 0 : x = 0 \\ 1 : x \in (0,1] \end{cases}$

$$f_n \xrightarrow[[0,1]]{} f$$

Remark: From the example above we notice that the pointwise limit function does not preserve continuity. (See Ex 1 and 3)

Theorem (ε - theorem on pointwise convergence)

Consider $(f_n) \subseteq F(A, \mathbb{R})$ a sequence of functions

$f: B \to \mathbb{R}$ ($B \subseteq A$) another function.

Then $f_n \xrightarrow[B]{} f \Leftrightarrow \underbrace{\forall x \in B, \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon \ |f_n(x) - f(x)| < \varepsilon}_{(*)}$

Proof:

$\boxed{\Rightarrow}$ we know that $f_n \xrightarrow[B]{} f \overset{\text{def}}{\Leftrightarrow} \underline{\forall x \in B}$, $f(x) = \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R}$

$\Updownarrow$ ε-th for limits of sequences of real numbers

$\underline{\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,}$
$\underline{|f_n(x) - f(x)| < \varepsilon} \to (*)$ ✓

$\boxed{\Leftarrow}$ We know $(*)$ and we want $f_n \to f \Leftrightarrow \forall x \in B$, $f(x) = \lim\limits_{n\to\infty} f_n(x)$

Choose $\boxed{x \in B}$ randomly

? $f(x) = \lim\limits_{n\to\infty} f_n(x)$

$\xrightarrow{(*)} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon \ |f_n(x) - f(x)| < \varepsilon \Leftrightarrow$

$$\Leftrightarrow \left.\begin{array}{l} \lim\limits_{n\to\infty} f_n(x) = f(x) \\ x \text{ - random} \to \forall x \in B \end{array}\right\} \to f_n \xrightarrow[B]{} f$$

Def: $(f_n) \subset F(A,\mathbb{R})$, $f \in F(B,\mathbb{R})$, $B \subseteq A$ — The sequence of functions (f_n) is said to CONVERGE UNIFORMLY to the function f on B if:

(**) $\forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$ it holds $\forall x \in B$ $|f_n(x) - f(x)| < \varepsilon$

Notation: $f_n \underset{B}{\rightrightarrows} f$

Recall that

$$f_n \underset{B}{\to} f \quad \underbrace{\forall x \in B, \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon \ |f_n(x)-f(x)|<\varepsilon}_{(*)}$$

Remark: The index $n_\varepsilon \in \mathbb{N}$ depends on: x & ε for $\to$

ε for $\rightrightarrows$

$\rightrightarrows \Rightarrow \to$

$\not\Leftarrow$

In practice:
- first we determine the pointwise limit function
- then we check if it is also the uniform —//—

Remark: For $f: B \to \mathbb{R}$,

$\|f\|_\infty = \sup\limits_{x\in B} |f(x)|$ is called the UNIFORM norm of the function

$f_n \underset{B}{\rightrightarrows} f \iff \lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$

Algorithm for the study of the uniform convergence of a sequence of functions:

$\boxed{\text{Step1.}}$
- Determine $B = \{a \in A : \lim\limits_{n\to\infty} f_n(a) \in \mathbb{R}\}$! Chosing $x \in \mathbb{R}$ random
- Define $f: B \to \mathbb{R}$ $f(x) = \lim\limits_{n\to\infty} f_n(x) \in \mathbb{R}$ $\forall x \in B$ as the poitwise function $f_n \to f$

$\boxed{\text{Step2}}$ $\forall n \in \mathbb{N}$ we determine

$a_n := \|f_n - f\|_\infty = \sup\limits_{x\in B} \{|f_n(x) - f(x)|\} \in [0,\infty]$

$\to$ we generate a sequence of real numbers $(a_n) \in \mathbb{R}$

If $\lim\limits_{n\to\infty} a_n = 0 \to f_n \underset{B}{\rightrightarrows} f$

otherwise $\to f_n \underset{B}{\not\rightrightarrows} f$

Examplee: Study both PC & UC for the following sequence of functions

1. $\forall n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{x^2}{n^2 + x^4}$, $\forall x \in \mathbb{R}$

$\boxed{\text{Step1:}}$ Choose a random $x \in \mathbb{R}$ and compute $\lim\limits_{n\to\infty} f_n(x) =$

$$= \lim_{n\to\infty} \frac{x^2}{n^2+x^4} = 0$$

$C = \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 0$, $\forall x \in \mathbb{R}$ $\qquad f_n \xrightarrow[\mathbb{R}]{} f$

$\boxed{\text{Step2:}}$ Choose $n \in \mathbb{N}$ random
$\downarrow$
constant

$$?\, a_n : \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x^2}{n^2+x^4}\right| = \sup_{x\in\mathbb{R}} \frac{x^2}{n^2+x^4}$$

We consider the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{x^2}{n^2+x^4}$ and study its behaviour with the help of the derivative.

g - differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R} \quad g'(x) = \frac{2x(n^2+x^4) - 4x^3 \cdot x^2}{(n^2+x^4)^2} = \frac{2xn^2 + 2x^5 - 4x^5}{(n^2+x^4)^2} =$$

$$= \frac{2x(n^2 - x^4)}{(n^2+x^4)^2}$$

$$g'(x) = 0 \Leftrightarrow x = 0 \text{ or } n^2 - x^4 = 0 \Leftrightarrow (n - x^2)\underbrace{(n + x^2)}_{>0} = 0 \Leftrightarrow x = \pm\sqrt{n}$$

x	$-\infty$		$-\sqrt{n}$		0		$\sqrt{n}$		$+\infty$
x	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$n-x^2$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$
$g(x)$	$+$	$+$	0	$-$	0	$+$	0	$-$	$-$
		$\nearrow$		$\searrow$		$\nearrow$		$\searrow$	

$\to \forall x \in \mathbb{R} \quad g(x) \leq \max\{g(-\sqrt{n}), g(\sqrt{n})\}$

$$g(-\sqrt{n}) = \frac{(-\sqrt{n})^2}{n^2 + (-\sqrt{n})^4} = \frac{n}{n^2+n^2} = \frac{1}{2n} = g(\sqrt{n})$$

$$\left.\begin{array}{l} \to g(x) \leq \dfrac{1}{2n} \quad \forall n \in \mathbb{N} \\ \qquad\quad \| \\ \qquad g(\sqrt{n}) \end{array}\right\} \to a_n = \frac{1}{2n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n} = 0 \to f_n \overset{\mathbb{R}}{\rightrightarrows} f$$

2. $\forall n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ $f_n(x) = 2n^2 x \cdot e^{\cdots}$ $\forall x \in \mathbb{R}$

$\boxed{\text{Step 1.}}$ Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2n^2 x e^{-n^2x^2} = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \begin{cases} 0 : x = 0 \\ \frac{\infty}{\infty} : x \neq 0 \end{cases}$$

$x \neq 0$

$$\lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = x \lim_{n\to\infty} \frac{2n^2}{e^{n^2x^2}} = 2x \cdot \frac{1}{x^2} \lim_{n\to\infty} \frac{n^2x^2}{e^{n^2x^2}} = \frac{2x}{x^2} \cdot 0 = 0$$

We can define $h : (0,\infty) \to \mathbb{R}$ $h(t) = \frac{t}{e^t}$

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{t}{e^t} \underset{l'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty_+} = 0 \to \lim_{n\to\infty} \frac{n^2x^2}{e^{n^2x^2}} = 0$$

$[(n^2x^2)_{n\geq 1} \subseteq (0,\infty) \to \infty]$
def of the limits of functions

$\lim_{n\to\infty} f_n(x) = 0$ $\forall x \in \mathbb{R} \to C = \mathbb{R}$
$f : \mathbb{R} \to \mathbb{R}$ $f(x) = 0$, $\forall x \in C$ $f_n \rightrightarrows f$

$\boxed{\text{Step 2}}$ Choose $n \in \mathbb{N}$ randomly

$$a_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x) - 0| = \sup_{x\in\mathbb{R}} \left| \frac{2n^2x}{e^{n^2x^2}} \right|$$

We introduce the auxiliary function

$$g : \mathbb{R} \to \mathbb{R}, \ g(x) = \left| \frac{2n^2x}{e^{n^2x^2}} \right| = \frac{2n^2|x|}{e^{n^2x^2}}$$

$\lim_{x\to 0} g(x) = 0 \to g$ is continuous at 0 / on $\mathbb{R}\setminus\{0\}$ $\} \to c$ on $\mathbb{R}$

? is g diff on $\mathbb{R}$? it is only diff on $\mathbb{R}\setminus\{0\}$

$$\lim_{\substack{x\to 0\\x<0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\x<0}} \frac{-2n^2x - 0}{x} = -2n^2$$

$\neq$

$$\lim_{\substack{x\to 0\\x>0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\x>0}} \frac{2n^2x - 0}{x} = 2n^2$$

- $x < 0$

$$g'(x) = \left(\frac{-2n^2x}{e^{n^2x^2}} \right)' = -2n^2 \cdot \frac{x' \cdot e^{n^2x^2} - x \cdot (e^{n^2x^2})'}{(e^{n^2x^2})^2} =$$

$$= -2n^2 \cdot \frac{\cancel{e^{n^2x^2}} (1 - x \cdot (n^2x^2)')}{(e^{n^2x^2})^{\cancel{2}}} = -2n^2 \frac{1 - 2n^2 \cdot x^2}{e^{n^2x^2}}$$

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• $x > 0$ $\quad g'(x) = 2n^2 \cdot \dfrac{1 - 2n^2x^2}{e^{n^2x^2}}$

$g'(x) = 0 \Leftrightarrow 1 - 2n^2x^2 = 0 \rightarrow x = \pm \dfrac{1}{n\sqrt{2}}$

x	$-\infty$		$-\frac{1}{n\sqrt{2}}$		0		$\frac{1}{n\sqrt{2}}$		$+\infty$
$1-2n^2x^2$		$- - -$	0	$+ + +$			0	$- - -$	
$g'(x)$		$+ + +$	0	$- -$	$\vert$	$+ +$	0	$- - -$	
$g(x)$		$\nearrow$	$\sqrt{\frac{2}{e}}\,n$	$\searrow$	0	$\nearrow$	$\sqrt{\frac{2}{e}}\,n$	$\searrow$	

$$g\left(-\frac{1}{n\sqrt{2}}\right) = \frac{2n^2 \left|-\frac{1}{n\sqrt{2}}\right|}{e^{n^2\left(-\frac{1}{n\sqrt{2}}\right)^2}} = \frac{\sqrt{2}n}{e^{\frac{1}{2}}} = \sqrt{\frac{2}{e}}\,n = g\left(\frac{1}{n\sqrt{2}}\right)$$

$\rightarrow a = \sup\limits_{x \in \mathbb{R}} |g(x)| = \sqrt{\frac{2}{e}}\,n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{\frac{2}{e}}\,n = \infty \neq 0 \rightarrow f_n \not\rightrightarrows f$

Uniform convergence criteria

T_1 (Cauchy)

$$\left.\begin{array}{l} (f_n) \subseteq F(A,\mathbb{R}) \\ f \in F(B,\mathbb{R}) \\ B \subseteq A \end{array}\right\} \quad f_n \underset{B}{\rightrightarrows} f \Leftrightarrow \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{R},\ \forall x \in \mathbb{R}$$

$$|f_{n+p}(x) - f_n(x)| < \varepsilon$$

T_2 (Weirstrass)

$$\left.\begin{array}{l} (f_n) \subseteq F(A,\mathbb{R}) \\ f \subseteq F(B,\mathbb{R}) \\ B \subseteq A \\ (a_n) \subseteq A \end{array}\right\} \quad \left.\begin{array}{l} \text{If} \bullet \lim\limits_{n\to\infty} a_n = 0 \\ \bullet \exists n' \in \mathbb{N} \text{ s.t. } |f_n(x) - f(x)| \leq a_n \quad \forall n \geq n',\ \forall x \in B \end{array}\right\} \rightarrow$$

$\rightarrow f_n \rightrightarrows f$

T_3 (Dini)

$$\left.\begin{array}{l} (f_n) \in F(A,\mathbb{R}) \\ A = [a,b] \\ f: [a,b] \to \mathbb{R} \\ f_n, f \text{ are continuous on } [a,b] \\ f_n \to f \\ \forall x \in [a,b],\ (f_n(x)) \text{ is increasing} \end{array}\right\} \text{ then } f_n \rightrightarrows f$$

Theorem:

$$\left.\begin{array}{l} f_n \rightrightarrows f \\ \forall n \in \mathbb{N}\ f_n \text{ is cont.} \end{array}\right\} \to f \text{ is continuous}$$

Proof: $f: B \to \mathbb{R}$. Choose $a \in B$ random and prove that f is cont. at a

$\xleftrightarrow[\varepsilon,\ \delta]{}$ $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\forall x \in B$ with $|x-a| < \delta$, to hold $|f(x) - f(a)| < \varepsilon$

Choose $\varepsilon > 0$ random

? $|f(x) - f(a)| < \varepsilon$

Choose $n \in \mathbb{N}$ random $\xrightarrow{hyp.}$ f_n is continuous at a

$\xrightarrow{\text{TES}}$ $\exists \delta > 0\ \forall x \in B$ with $|x-a| < \delta$ to hold $|f_n(x) - f_n(a)| < \frac{\varepsilon}{3}$

for $\frac{\varepsilon}{3} > 0$ chosen (1)

$$|f(x) - f(a)| = |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)|$$
$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$
$$= |f_n(x) - f(x)| + \underbrace{|f_n(x) - f_n(a)|}_{< \frac{\varepsilon}{3}} + |f_n(a) - f(a)|$$
$$< \underbrace{|f_n(x) - f(x)|}_{? < \frac{\varepsilon}{3}} + \frac{\varepsilon}{3} + \underbrace{|f_n(a) - f(a)|}_{? < \frac{\varepsilon}{3}} \quad (***)$$

hyp. $f_n \rightrightarrows f \leftrightarrow$ for $\frac{\varepsilon}{3} > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$ $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ (I)

Choose $\boxed{n := n_\varepsilon}$ $\to$ $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall x \in B$ (2) from $f_n \rightrightarrows f$

$\downarrow$

$\forall x \in B$ with $|x-a| < \delta$

for $x \in B$ with $|x-a| < \delta$

$$|f(x) - f(a)| < \underbrace{\frac{\varepsilon}{3}}_{\text{from C.}} + \underbrace{|f_{n_\varepsilon}(x) + f(x)|}_{< \frac{\varepsilon}{3}\ (2)} + \underbrace{|f_{n_\varepsilon}(a) - f(a)|}_{< \frac{\varepsilon}{3}\ (2)} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \checkmark \to$$

from $f_n \rightrightarrows f$

$$\to \left.\begin{array}{l} f\text{- cont. at a} \\ \text{a random} \end{array}\right\} \text{on } B$$

Remark:

$$\left.\begin{array}{l} \text{If } f_n \to f \\ (f_n) \text{ is continuous } \forall n \in \mathbb{N} \\ f \text{ is not c.} \end{array}\right\} \to f_n \not\rightrightarrows f$$

Recall: ex 1 & 3

Ex 1. $f_n(x) = x^n$ $\quad f_n \xrightarrow[(-1,1]]{} f$ $\quad f(x) = \begin{cases} 1 : x = 1 \\ 0 : x \in (-1,1) \end{cases}$

$E = (-1,1]$

$\forall n \in \mathbb{N}$ $\left.\begin{array}{l} f_n(x) = x^n \text{ is cont. on } E \\ f \text{ is not c. on } E \end{array}\right\} \to f_n \not\overset{\to}{\underset{E}{\rightrightarrows}} f$

the same holds for ex 3

Remark: $\left.\begin{array}{l} \text{Should } f_n \text{ be cont. } \forall n \in \mathbb{N} \\ f \text{ be cont} \end{array}\right\} \not\to f_n \rightrightarrows f$ (it has to be proved)

$f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$ - cont

f - cont

but $f_n \underset{\mathbb{R}}{\not\rightrightarrows} f$