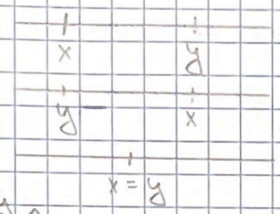
6. 10. 2023
Analysis - Lecture 1

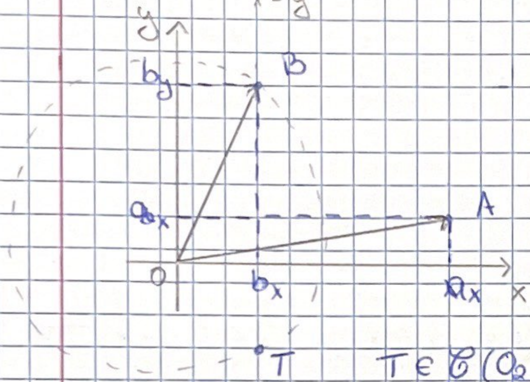
Topologi on $\mathbb{R}$

$(\mathbb{R}, +, \cdot) \to$ COMUTATIVE / ABELIAN FIELD (corp)

$\to$ TOTALLY ORDERED: $\forall\, x, y \in \mathbb{R}$, $x \le y$ or $y \le x$



If $x \le y$ and $y \le x \to \boxed{x = y}$



$d(O_2, B) < d(O_2, A)$

$d(O_x, A) < d(O_x, B)$

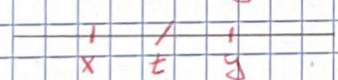
$T \in \mathcal{C}(O_2, OB)$, $T \neq B$

$d(O_2, T) = d(O_2, B)$ but $T \neq B \to$ on $\mathbb{R}^2$ the order is not TOTAL

Axioms on $\mathbb{R}$

ASE (AXIOM OF THE SEPARATIN ELEMENT)

$\forall$ ~~$x \in$~~ $x < y \in \mathbb{R}$ $\exists\, t \in \mathbb{R}$ such that $x < t < y$



REMARK: $\mathbb{N}$ & $\mathbb{Z}$ do not enjoy the property above

ARCHIMEDE'S AXIOM (for $\mathbb{N}$)

$\forall\, x > 0 \in \mathbb{R}$ $\exists\, n_x \in \mathbb{N}$ s.t. $x < n_x$

$\Updownarrow$ $0 < x < n_x \leftrightarrow 0 < \frac{1}{n_x} < \frac{1}{x} = y$

$\forall\, y > 0 \in \mathbb{R}$ $\exists\, n_y \in \mathbb{N}$ s.t. $0 < \frac{1}{n_y} < y$

These axioms characterises the fact that $\mathbb{N}$ is infinite

Def: Let $\emptyset \neq A \subseteq \mathbb{R}$. An element:

- $x \in \mathbb{R}$ is said to be a LOWER BOUND (MINORANT) of the set A if $x \leq a$, $\forall a \in A$
- $y \in \mathbb{R}$ is said to be an UPPER BOUND (MAJORANT) of the set A if $y \geq a$, $\forall a \in A$

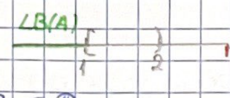
Notations:

$LB(A) = \{x \in \mathbb{R} \mid x \leq a, \forall a \in \mathbb{R}\} \rightarrow$ the set of all lower bounds

$UB(A) = \{y \in \mathbb{R} \mid y \geq a, \forall a \in \mathbb{R}\} \rightarrow$ the set of all upper bounds

Examples:

$A = [1; 2)$



$LB(A) = (-\infty; 1] \subseteq \mathbb{R}$

$UB(A) = [2; +\infty)$

$? \exists t \in UB(A)$ s.t. $t < 2$

Assume by contradiction $\exists \boxed{t \in UB(A)}$ s.t. $\boxed{1 < t < 2} \rightarrow$

$\Leftrightarrow a \leq t, \forall a \in A$ (1) $\qquad \Leftrightarrow t \in (1; 2), t \neq 2$

$\rightarrow$ ASE: $\exists h \in A$ s.t. $t < h < 2$ (2)

$\left.\begin{array}{l} (1) \rightarrow h \leq t \\ (2) \rightarrow t < h \end{array}\right\} \rightarrow t < h \leq t \rightarrow t < t$ – CONTRADICTION!

$\left.\begin{array}{l} [2; \infty) \subseteq UB(A) \\ \nexists\, t < 2 \text{ s.t. } t \in UB(A), \cancel{t \in} \end{array}\right\} \rightarrow UB(A) = [2; +\infty)$

AXIOMS OF INF AND SUP

Let $\emptyset \neq A \subseteq \mathbb{R}$

a) If $LB(A) \neq \emptyset$: $\exists$ the GREATEST LOWER BOUND of A (named the INFINUM of a set) denoted by $\inf A$ and $\inf A \in \mathbb{R}$

b) ~~$\exists$~~ If $UB(A) \neq \emptyset$: $\exists$ the LEAST UPPER BOUND of A (named the SUPREMUM of A) ~~and~~ denoted by $\sup A$ and $\sup A \in \mathbb{R}$

REMARK

1. ASE $\Leftrightarrow$ AI $\Leftrightarrow$ AS
2. If $LB(A) = \emptyset$, by convention we set $\inf A = -\infty$
3. If $UB(A) = \emptyset$, by convention we set $\sup A = +\infty$

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Thus $\forall\ \emptyset \neq A \subseteq \mathbb{R}$ $\left.\begin{array}{l}\exists \inf A \\ \exists \sup A\end{array}\right\} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty; +\infty]$

4. When $A = \emptyset$ by convention $\inf A = +\infty$ and $\sup A = -\infty$

Def: Let $\emptyset \neq A \subset \mathbb{R}$

a) If $\inf A \in A \to$ it is called the MINIMUM of $A \overset{not}{=} \min A$

b) If $\sup A \in A \to$ it is called the MAXIMUM of $A \overset{not}{=} \max A$

Example:

1. $A = [1; 2) \neq \emptyset$

$LB(A) = (-\infty; 1] \to \inf A = 1 \in A \to \min A = 1$

$UB(A) = [2; +\infty) \to \sup A = 2 \notin A \to \nexists \max A$

2. $A = [1; 2) \cup \{3\}$

$\inf A = 1 \in A \to \min A = 1 \to LB(A) = (-\infty; 1]$

$\sup A = 3 \in A \to \max A = 3 \to UB(A) = [3; +\infty)$

3. $A = (-\infty; 1] \cup [7; 8]$

$\inf A = -\infty \to \nexists \min A$ and $LB(A) = \emptyset$

$\sup A = 8 \in A \to \exists \max A = 8$ and $UB(A) = [8; +\infty)$

4. $A = (-\infty; 1) \cup \mathbb{N}$

$\inf A = -\infty \to \nexists \min A$ and $LB(A) = \emptyset$

$\sup A = +\infty \to \nexists \max A$ and $UB(A) = \emptyset$

Elements of TOPOLOGY

The distance in space: $d(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$