14.11.2023

Algebra - Seminar 6

1. $v_1 = (1,-1,0),\quad v_2 = (2,1,1),\quad v_3 = (1,5,2)$

(i) v_1, v_2, v_3 - lin. dep. and det. rel.

(ii) v_1, v_2 - lin. indep.

(i) $k_1 v_1 + k_2 v_2 + k_3 v_3 = k_1(1,-1,0) + k_2(2,1,1) + k_3(1,5,2) =$

$= (k_1 + 2k_2 + k_3,\ -k_1 + k_2 + 5k_3,\ 0k_1 + k_2 + 2k_3) = (0,0,0) \Leftrightarrow$

$$\Leftrightarrow S \begin{cases} k_1 + 2k_2 + k_3 = 0 \\ -k_1 + k_2 + 5k_3 = 0 \\ 0 \cdot k_1 \quad k_2 + 2k_3 = 0 \end{cases}$$

$$\det(S) = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \\ 0 & 1 & 2 \end{vmatrix} = 2 - 1 + 0 - 0 - 5 + 4 = 6 - 6 = 0 \rightarrow$$

$\rightarrow$ The system has multiple sol. $\rightarrow v_1, v_2, v_3$ - lin. dep.

$k_3 = \alpha \in \mathbb{R}$

$$\begin{cases} k_1 + 2k_2 = -\alpha \\ -k_1 + k_2 = -5\alpha \end{cases} +$$

$3k_2 = -6\alpha \rightarrow k_2 = -2\alpha$

$k_1 - 4\alpha = -\alpha \rightarrow k_1 = 3\alpha$

$\alpha = 1 \rightarrow 3v_1 - 2v_2 = -v_3$

(ii) $k_1 v_1 + k_2 v_2 = (k_1 + 2k_2,\ -k_1 + k_2,\ 0 + k_2) = (0,0,0)$

$$\left.\begin{cases} k_1 + 2k_2 = 0 \\ -k_1 + k_2 = 0 \quad \rightarrow -k_1 + 0 = 0 \rightarrow k_1 = 0 \\ k_2 = 0 \end{cases}\right\} \rightarrow v_1, v_2 \text{ - lin. indep.}$$

n-vectors in K^n $<$ $\det = 0 \rightarrow$ lin. dep. ; $\det \neq 0 \rightarrow$ lin indep

2 vectors in K^n : $\frac{1}{2} \neq \frac{-1}{1} \neq \frac{0}{1} \rightarrow$ lin indep (

2. - Hw

3. $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$

$a = ?$ s.t. v_1, v_2, v_3 - lin indep

$$\begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 1 & 0 & a \end{vmatrix} = a + 0 + a - 0 - 0 - a^3 = -a^3 + 2a = a(-a^2 + 2)$$

v_1, v_2, v_3 - lin indep $\Leftrightarrow$ $a(-a^2+2) \neq 0$ $\begin{cases} a \neq 0 \\ -a^2 + 2 \neq 0 \rightarrow a^2 \neq 2 \end{cases} \rightarrow$

$\rightarrow a \in \mathbb{R} \setminus \{-\sqrt{2}, 0, \sqrt{2}\}$

4. $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$

$a = ?$ s.t. v_1, v_2, v_3 - lin. dep.

v_1, v_2, v_3 - lin dep $\rightarrow$ $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ and $k_{1,2,3}$ not all

$(k_1 + 2k_2 + 0k_3;\ -2k_1 + k_2 + ak_3,\ 0k_1 + k_2 + k_3,\ -k_1 + 0k_2 + 2k_3) =$
$= (0, 0, 0, 0)$

$$\Leftrightarrow S = \begin{cases} k_1 + 2k_2 = 0 \\ -2k_1 + k_2 + ak_3 = 0 \\ k_2 + k_3 = 0 \rightarrow k_2 = -k_3 \\ -k_1 + 2k_3 = 0 \rightarrow k_1 = 2k_3 \end{cases}$$

$\rightarrow -4k_3 - k_3 + ak_3 = 0$

$k_3(a - 5) = 0$

k_3 can't be 0 because if $k_3 = 0 \rightarrow k_1 = k_2 = 0 \rightarrow$ lin indep.

$\rightarrow a - 5 \neq 0 \rightarrow a = 5$

5 $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$

(i) (v_1, v_2, v_3) - basis $_{\mathbb{R}}\mathbb{R}^3$

(ii) express (e_1, e_2, e_3) as a linear combination of v_1, v_2, v_3

(iii) Det $u = (1, -1, 2)$ in each of the 2 bases

$V = $ v.s. over K

$B = (v_1, \ldots, v_n)$ in V is a basis for V if:

(i) $v_1, \ldots, v_n$ are lin indep

(ii) $v_1, \ldots, v_n$ are a system of generators for V, $V = \langle B \rangle$

• $B = (v_1, \dots, v_n)$ is a bases of V if every vector in V can be uniquely written as a lin combination of $v_1, \dots, v_n$.

$\forall v \in V, \exists! k_1, \dots, k_n$ s.t. $v = k_1 v_1 + \dots + k_n v_n$

$k_1, \dots, k_n$ = coordinates of v in the basis B

(i) Let $k_1, k_2, k_3 \in \mathbb{R}$ & $(x, y, z) \in \mathbb{R}^3$

$(x, y, z) = k_1 v_1 + k_2 v_2 + k_3 v_3$

$(x, y, z) = (k_1 - k_2 + k_3, \ k_1 + k_3, \ 2k_2 + k_3)$

$$\begin{cases} k_1 - k_2 + k_3 = x \\ k_1 + k_3 = y \\ 2k_2 + k_3 = z \end{cases} \rightarrow k_2 = k_1 + k_3 - x = y - x$$

$k_3 = z - 2(y - x) = z - 2y + 2x$

$k_1 = y - (z - 2y + 2x) = -z + 3y + 2x$

$\rightarrow$ unique sol $\rightarrow$ (v_1, v_2, v_3) - basis of $\mathbb{R}^3$

(ii) $e_1 = (1, 0, 0) = k_1 v_1 + k_2 v_2 + k_3 v_3 = -2v_1 - v_2 + 2v_3$

$k_1 = -0 + 3 \cdot 0 - 2 \cdot 1 = -2$

$k_2 = 0 - 1 = -1$

$k_3 = 0 - 0 + 2 = 2$

$e_2 = (0, 1, 0) = k_1' v_1 + k_2' v_2 + k_3' v_3 = 3v_1 + v_2 - 2v_3$

$k_1' = -0 + 3 \cdot 1 - 2 \cdot 0 = 3$

$k_2' = 1 - 0 = 1$

$k_3' = 0 - 2 \cdot 1 + 2 \cdot 0 = -2$

$e_3 = (0, 0, 1) = k_1'' v_1 + k_2'' v_2 + k_3'' v_3 = -v_1 + 0v_2 + v_3$

$k_1'' = -1 + 0 - 0 = -1$

$k_2'' = 0 - 0 = 0$

$k_3'' = 1 - 0 + 0 = 1$

(iii) $u = (1, -1, 2)$

$u = k_1 v_1 + k_2 v_2 + k_3 v_3 = -7 v_1 - 2 v_2 + 6 v_3$

$k_1 = 3 \cdot (-1) - 2 - 2 \cdot 1 = -7$

$k_2 = -1 - 1 = -2$

$k_3 = 2 - 2 \cdot (-1) + 2 \cdot 1 = 2 + 2 + 2 = 6$

$u = k_1' e_1 + k_2' e_2 + k_3' e_3 = (k_1', k_2', k_3') \to$

$\to k_1' = 1$

$k_2' = -1$

$k_3' = 2$

6. $n \in \mathbb{N}^*$

$v_1 = (1, \dots, 1, 1, 1)$

$v_2 = (1, \dots, 1, 1, 2)$

$v_3 = (1, \dots, 1, 2, 3)$

$v_n = (1, 2, \dots, n-1, n)$

form a basis in $\mathbb{R}^n$ and write the coord of $x = (x_1, \dots, x_n)$ in this base

$x = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$

$$\begin{cases} k_1 + k_2 + \dots + k_n = x_1 & e_1 \\ k_1 + k_2 + \dots + k_{n-1} + 2k_n = x_2 & e_2 \\ k_1 + k_2 + \dots + k_{n-2} + 2k_{n-1} + 3k_n = x_3 & e_3 \\ \dots \\ k_1 + 2k_2 + \dots + nk_n = x_n & e_n \end{cases}$$

$e_2 - e_1 \to k_n = x_2 - x_1$

$e_3 - e_2 \to k_{n-1} + k_n = x_3 - x_2 \Rightarrow k_{n-1} = x_3 - x_2 - (x_2 - x_1)$

$= x_3 - 2x_2 + x_1$

$e_4 - e_3 \to k_{n-2} = x_4 - 2x_3 + x_2$

$$\begin{cases} k_j = x_{n-j+2} - 2x_{n-j+1} + x_{n-j}, & j = \overline{1, n-1} \\ k_n = x_2 - x_1 \end{cases}$$ (astea sunt si coordonatele lui x)

$\rightarrow$ unique sol $\rightarrow$ basis

9. Det. the nr. of bases of the v.s. $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$

$\mathbb{Z}_2^3 = \{(\hat{0}, \hat{0}, \hat{0}), (\hat{0}, \hat{0}, \hat{1}), (\hat{0}, \hat{1}, \hat{0}), (\hat{1}, \hat{0}, \hat{0}), (\hat{0}, \hat{1}, \hat{1}), (\hat{1}, \hat{0}, \hat{1}), (\hat{1}, \hat{1}, \hat{0}), (\hat{1}, \hat{1}, \hat{1})\}$

$card(\mathbb{Z}_2^3) = 8$

We have to determine in how many ways we can choose 3 linearly independent vectors

$B = (v_1, v_2, v_3)$, v_1, v_2, v_3 - lin. indep.

$v_1 \rightarrow 7$ options ($(0,0,0)$ nu poate fi în nici o bază pentru că dacă ar fi, mulțimea ar fi lin. dependent)

$v_2 \rightarrow 6$ op. , $v_2 \neq \underset{0,1}{\hat{k}} \cdot v_1 \rightarrow v_2 \neq v_1$

$v_3 \rightarrow 4$ op , $v_3 \neq \underset{0,1}{\hat{k_1}} v_1 + \underset{0,1}{\hat{k_2}} v_2$

total basis $= 4 \cdot 6 \cdot 7 = 168$

10. the nr of el. $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3 - matrices over $\mathbb{Z}_2$

$\boxed{\text{A is invertible} \leftrightarrow \text{its columns are lin indep.}}$

Columns of A are the vectors from ex 9 $\rightarrow$ 168 elem.

7. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$(E_i)_{i=\overline{1,4}} \rightarrow$ basis of ${}_{\mathbb{R}}M_2(\mathbb{R})$

$(A_i)_{i=\overline{1,4}} \rightarrow$ — // —

Det coord of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in this bases

$\forall \begin{pmatrix} x & y \\ z & t \end{pmatrix} = k_1E_1 + k_2E_2 + k_3E_3 + k_4E_4 =$

$= \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$

$\begin{cases} x = k_1 \\ y = k_2 \\ z = k_3 \\ t = k_4 \end{cases} \rightarrow k_1, k_2, k_3, k_4$ - unique $\rightarrow (E_i)_{i=\overline{1,4}}$ - basis

$\forall \begin{pmatrix} x & y \\ z & t \end{pmatrix} = k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \begin{pmatrix} k_1+k_2+k_3+k_4 & k_2+k_3+k_4 \\ k_3+k_4 & k_1+k_4 \end{pmatrix}$

$\begin{cases} x = k_1 + k_2 + k_3 + k_4 \\ y = k_2 + k_3 + k_4 \\ z = k_3 + k_4 \\ t = k_1 + k_4 \end{cases}$

$k_1 = x - y$

$k_2 = y - z$

$k_3 = x - y + z - t$

$k_4 = t - k_1 = t - x + y$

k_1, k_2, k_3, k_4 - unique $\rightarrow (A_i)_{i=\overline{1,4}}$ - basis

$(E_i)_{i=\overline{1,4}}: \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ $k_1 = 2$

$k_2 = 1$

$k_3 = 1$

$k_4 = 0$

$(A_i)_{i=\overline{1,4}}$ $k_1 = 2 - 1 = 1$

$k_2 = 1 - 1 = 0$

$k_3 = 2 - 1 + 1 - 0 = 2$

$k_4 = 0 - 2 + 1 = -1$

Scanned with CamScanner

2. (i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$

$$\begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 2 - 2 + 0 - 12 - 1 - 0 = -13 \neq 0 \to v_1, v_2, v_3 - \text{lin indep}$$

(ii) $v_1 = (1, 2, 3, 4)$; $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \overset{C_1 + C_2 + C_3 + C_4 \to C_1}{=} \begin{vmatrix} 10 & 2 & 3 & 4 \\ 10 & 3 & 4 & 1 \\ 10 & 4 & 1 & 2 \\ 10 & 1 & 2 & 3 \end{vmatrix} = 10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{vmatrix} \overset{\substack{L_2 - L_1 \\ L_3 - L_1 \\ L_4 - L_1}}{=}$$

$$= 10 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & 1 & -1 \end{vmatrix} \overset{\substack{C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 - 4C_1}}{=} 10 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & 1 & -1 \end{vmatrix} = 10 \cdot (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & 1 & -1 \end{vmatrix} =$$

$$= 10 \cdot (\cancel{2} - \cancel{6} + 2 + \cancel{6} + 2 + 2) = 40 \neq 0 \to v_1, v_2, v_3, v_4 - \text{lin indep.}$$

8. $\mathbb{R}_2[x] = \{ f \in \mathbb{R}[x] \mid \text{degree}(f) \leq 2 \}$

$E = (\underset{e_0}{1}, \underset{e_1}{x}, \underset{e_2}{x^2})$

$B = (1, x - a, (x-a)^2)$, $a \in \mathbb{R}$

- E basis of $\mathbb{R}_2[x]$

$\leftrightarrow \forall\, a + bx + cx^2 \in \mathbb{R}_2[x]$, $a + bx + cx^2 = k_1 \cdot 1 + k_2 \cdot x + k_3 \cdot x^2 \to$

$\to k_1 = a$

$k_2 = b$

$k_3 = c$

$$f = a_0 + a_1 x + a_2 x^2 \to f = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = a_0 \cdot e_0 + a_1 e_1 + a_2 e_2$$

- B basis of $\mathbb{R}_2[x]$

$\forall\, a + bx + cx^2 \in \mathbb{R}_2[x]$, $a + bx + cx^2 = k_1 \cdot 1 + k_2 (x - a') + k_3 (x - a')^2$

$\leftrightarrow a + bx + cx^2 = k_1 + k_2 x - k_2 a' + k_3 x^2 - 2k_3 x \cdot a' + k_3 a'^2$

$\leftrightarrow a + bx + cx^2 = (k_1 - k_2 a' + k_3 a'^2) + (k_2 - 2k_3 a') x + k_3 x^2$

$k_1 - k_2 a' + k_3 a'^2 = a \to k_1 = a + ba' + 3c(a')^2$

$k_2 - 2k_3 a' = b \to k_2 = b + 2c \cdot a'$

$k_3 = c$

$$f = a_0 + a_1 x + a_2 x^2 \Rightarrow \begin{cases} k_1 = a_0 + a_1 a' + 3 a_2 (a')^2 \\ k_2 = a_1 + 2 a_2 a' \\ k_3 = a_2 \end{cases} \rightarrow$$

$$\rightarrow f = (a_0 + a_1 a' + 3 a_2 (a')^2)\, b_1 + (a_1 + 2 a_2 a')\, b_2 + a_2 b_3$$

Scanned with CamScanner