Calculus - Homework 8

1. Det. the n-th derivate:

a) $f: (-1;\infty) \to \mathbb{R}$, $f(x) = (1+x)^r$, $r \in \mathbb{R}$

$f'(x) = r(1+x)^{r-1}$

$f''(x) = r(r-1)(1+x)$

I $r < n \to f^{(n)}(x) = 0$

II $r = n \to f^{(n)}(x) = n!$

III $r > n \to f^{(n)}(x) = r(r-1)(r-2)\dots(r-n+1)(1+x)^{r-n} = \frac{r!}{(r-n)!}(1+x)^{r-n}$

b) $f: (-1,\infty) \to \mathbb{R}$, $f(x) = x \cdot \ln(1+x)$

Let $g(x) = x$, $h(x) = \ln(1+x)$, $g, h: (-1,\infty) \to \mathbb{R}$

$g'(x) = 1$

$g''(x) = 0$

$\to g^{(n)}(x) = \begin{cases} 1, & n=1 \\ 0, & n>1 \end{cases}$

$h'(x) = \frac{1}{1+x}$

$h''(x) = -\frac{1}{(1+x)^2}$

$h'''(x) = (-1)^2 \cdot \frac{2(1+x)}{(1+x)^4} = \frac{(-1)^2 \cdot 2}{(1+x)^3}$

$h^{(4)}(x) = (-1)^3 \cdot \frac{2 \cdot 3 \cdot (1+x)^2}{(1+x)^6} = \frac{(-1)^3 \cdot 3!}{(1+x)^4}$

$h^{(5)}(x) = (-1)^4 \cdot \frac{3! \cdot 4 \cdot (1+x)^3}{(1+x)^8} = \frac{(-1)^4 \cdot 4!}{(1+x)^5}$

$\to h^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$f^{(n)}(x) = [g(x)h(x)]^{(n)} = [h(x) \cdot g(x)]^{(n)} = \sum_{k=0}^{n} C_n^k h^{(n-k)}(x) + g^{(k)}(x) =$

$= C_n^0 h^{(n)}(x) \cdot g(x) + C_n^1 h^{(n-1)}(x) \cdot g'(x) + \sum_{k=2}^{n} C_n^k h^{(n-k)}(x) \underbrace{g^{(k)}(x)}_{0}$

$= \frac{n!}{n!} \cdot \frac{(-1)^{(n-1)}(n-1)!}{(1+x)^n} \cdot x + \frac{n!}{(n-1)!} \cdot \frac{(-1)^{n-2}(n-2)!}{(1+x)^{n-1}}$

$= \frac{(-1)^{n-1}(n-1)!\,x}{(1+x)^n} + \frac{n(-1)^{n-2}(n-2)!}{(1+x)^{n-1}} = \frac{(-1)^{n-1}(n-1)!\,x + (-1)^{n-2} n(n-2)!(1+x)}{(1+x)^n} =$

$= \frac{(-1)^{n-2}(n-2)!\,[-(n-1)x + n(1+x)]}{(1+x)^n} = \frac{(-1)^{n-2}(n-2)!\,[-\cancel{nx} + x + n + \cancel{nx}]}{(1+x)^n}$

$= \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(1+x)^n}$

c) $f:(-\infty,-1)\to\mathbb{R},\ f(x)=x\cdot\ln(1-x)$

Let $g,h:(-\infty,-1)\to\mathbb{R},\ g(x)=x,\ h(x)=\ln(1-x)$

$$g^{(n)}(x)=\begin{cases}1, & n=1\\ 0, & n>1\end{cases}$$

$$h'(x)=-\frac{1}{1-x}$$

$$h''(x)=(-1)^2\,\frac{(-1)}{(1-x)^2}$$

$$h'''(x)=(-1)^3\cdot\frac{2(-1)(1-x)}{(1-x)^4}=\frac{(-1)^4\cdot 2}{(1-x)^3}$$

$$h^{(4)}(x)=(-1)^5\cdot\frac{2\cdot 3(1-x)^2(-1)}{(1-x)^6}=\frac{(-1)^6\cdot 3!}{(1-x)^4}$$

$$h^{(5)}(x)=(-1)^7\cdot\frac{3!\cdot 4(1-x)^3\cdot(-1)}{(1-x)^8}=\frac{(-1)^8\cdot 4!}{(1-x)^5}$$

$$h^{(n)}(x)=\begin{cases}\frac{-1}{1-x}, & n=1\\ \frac{-1}{(1-x)^2}, & n=2\\ \frac{(n-1)!}{(1-x)^n}, & n>2\end{cases}$$

$$f^{(n)}(x)=[h(x)\cdot g(x)]^{(n)}=\sum_{k=0}^{n}C_n^k\cdot h^{(n-k)}(x)\,g^{(k)}(x)=C_n^0\,h^{(n)}(x)\,g(x)+C_n^1\,h^{(n-1)}(x)\cdot g'(x)+$$

$$+\sum_{k=2}^{n}C_n^k\,h^{(n-k)}(x)\,\underbrace{g^{(k)}(x)}_{0}=\frac{(n-1)!}{(1-x)^n}\cdot x+n\cdot\frac{(n-2)!}{(1-x)^{n-1}}=\frac{(n-1)!\cdot x+n(n-2)!(1-x)}{(1-x)^n}=$$

$$=\frac{(n-2)![(n-1)x+n(1-x)]}{(1-x)^n}=\frac{(n-2)![\cancel{nx}-x+n-\cancel{nx}]}{(1-x)^n}=\frac{(n-2)!\,(n-x)}{(1-x)^n}$$

d) $f:(-1,1)\to\mathbb{R},\ f(x)=\sqrt{3x+4}$

$$f'(x)=\frac{3}{2\sqrt{3x+4}}=\frac{3}{2}(3x+4)^{-\frac{1}{2}}$$

$$f''(x)=\frac{3}{2}\cdot\left(-\frac{1}{2}\right)\cdot(3x+4)^{-\frac{1}{2}-1}\cdot 3=-\frac{9}{4}(3x+4)^{-\frac{3}{2}}$$

$$f'''(x)=\left(-\frac{9}{4}\right)\cdot\left(-\frac{3}{2}\right)3\cdot(3x+4)^{-\frac{3}{2}-1}=\frac{27}{8}\cdot 3\cdot(3x+4)^{-\frac{5}{2}}$$

$$f^{(4)}(x)=-\frac{27}{8}\cdot 3\cdot\left(-\frac{5}{2}\right)\cdot 3\,(3x+4)^{-\frac{7}{2}}=-\frac{81}{16}\cdot 3\cdot 5\,(3x+4)^{-\frac{7}{2}}$$

$$f^{(5)}(x)=-\frac{81}{16}\cdot 3\cdot 5\cdot\left(-\frac{7}{2}\right)\cdot 3\cdot(3x+4)^{-\frac{9}{2}}=\frac{3^5}{2^5}\cdot 3\cdot 5\cdot 7\,(3x+4)^{-\frac{9}{2}}=$$

$$=\frac{3^5}{2^5}\cdot 7!!\,(3x+4)^{-\frac{1}{2}(2\cdot 5-1)}$$

$$f^{(n)}(x)=(-1)^{n+1}\frac{3^n}{2^n}\cdot(3x+4)^{-\frac{1}{2}(2\cdot n-1)}\quad(2n-3)!!$$

e) $f:(-\frac{1}{2};\infty)\to\mathbb{R},\ f(x)=\frac{1}{\sqrt{2x+1}}=(2x+1)^{-\frac{1}{2}}$

$f'(x)=\left(-\frac{1}{2}\right)(2x+1)^{-\frac{1}{2}-1}\cdot 2=-\left(\frac{1}{\cancel{2}}\right)\cdot\cancel{2}\,(2x+1)^{-\frac{3}{2}}=(-1)(2x+1)^{-\frac{3}{2}}$

$f''(x)=(-1)\left(-\frac{3}{\cancel{2}}\right)\cdot(2x+1)^{-\frac{3}{2}-1}\cdot\cancel{2}=(-1)^2\cdot 3\cdot(2x+1)^{-\frac{5}{2}}$

$f'''(x)=(-1)^2\cdot 3\left(-\frac{5}{\cancel{2}}\right)\cdot\cancel{2}(2x+1)^{-\frac{7}{2}}=(-1)^3\cdot 3\cdot 5\cdot(2x+1)^{-\frac{7}{2}}$

$f^{(4)}(x)=(-1)^3\, 3\cdot 5\cdot\left(-\frac{7}{\cancel{2}}\right)\cdot\cancel{2}\cdot(2x+1)^{-\frac{9}{2}}=(-1)^4\cdot 3\cdot 5\cdot 7\cdot(2x+1)^{-\frac{9}{2}}$

$f^{(n)}(x)=(-1)^n\cdot(2n-1)!!\cdot(2x+1)^{-\frac{(2n+1)}{2}}$

2. Det. the n-th derivate

a) $f:\mathbb{R}\setminus\{\frac{b}{a}\}\to\mathbb{R},\ f(x)=\frac{1}{ax+b}=(ax+b)^{-1}$

$f'(x)=(-1)(ax+b)^{-2}\cdot a$

$f''(x)=(-1)^2\cdot 2\cdot a^2\,(ax+b)^{-3}$

$f'''(x)=(-1)^3\cdot 2\cdot 3\cdot a^3(ax+b)^{-4}$

$f^{(4)}(x)=(-1)^4\, 2\cdot 3\cdot 4\cdot a^4\,(ax+b)^{-5}$

$f^{(n)}(x)=\frac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}$

b) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin(ax+b)$

$f'(x)=\cos(ax+b)\cdot a$

$f''(x)=a^2\cdot(-\sin(ax+b))$

$f'''(x)=a^3\,(-\cos(ax+b))$

$$f^{(n)}(x)=\begin{cases} a^n\sin(ax+b): n=4k\\ a^n\cos(ax+b): n=4k+1\\ -a^n\sin(ax+b): n=4k+2\\ -a^n\cos(ax+b): n=4k+3\end{cases}=a^n\cdot\sin\left(ax+b+\frac{n\pi}{2}\right)$$

c) $f:\mathbb{R}\to\mathbb{R},\ f(x)=\cos(ax+b)$

$f'(x)=-a\sin(ax+b)$

$f''(x)=-a^2\cos(ax+b)$

$f'''(x)=a^3\sin(ax+b)$

$f^{(4)}(x)=a^4\cos(ax+b)$

$$\to f^{(n)}(x)=\begin{cases}-a^n\cos(ax+b): n=4k\\ -a^n\sin(ax+b): n=4k+1\\ -a^n\cos(ax+b): n=4k+2\\ a^n\sin(ax+b): n=4k+3\end{cases}$$

d) $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^{ax+b}$

$f'(x)=(ax+b)'\cdot e^{ax+b}=a\cdot e^{ax+b}$

$f''(x)=a(ax+b)'e^{ax+b}=a^2e^{ax+b}$

$f^{(n)}(x)=a^ne^{ax+b}$

3. Compute the derivatives

a) $f:(0,\infty)\to\mathbb{R},\ f(x)=x^x=e^{x\ln x}$

$f'(x)=(x\ln x)'e^{x\ln x}=(\ln x+\frac{x}{x})e^{x\ln x}=e^{x\ln x}(1+\ln x)=x^x(1+\ln x)$

b) $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{\frac{1}{x}}=e^{\frac{1}{x}\ln x}$

$f'(x)=\left(\frac{1}{x}\ln x\right)'\cdot e^{\frac{1}{x}\ln x}=\left(-\frac{1}{x^2}\ln x+\frac{1}{x^2}\right)\cdot x^{\frac{1}{x}}=x^{\frac{1}{x}}\cdot\frac{1}{x^2}(1-\ln x)=$

$=x^{\frac{1}{x}-2}(1-\ln x)=x^{\frac{1-2x}{x}}(1-\ln x)$

c) $f:(0;\pi)\to\mathbb{R},\ f(x)=\sin x^x$

$f'(x)=(x^x)'\cdot\cos x^x=x^x(1+\ln x)\cdot\cos x^x$

d) $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{\sin x}=e^{\sin x\cdot\ln x}$

$f'(x)=(\sin x\cdot\ln x)'\cdot e^{\sin x\ln x}=\left(\cos x\cdot\ln x+\frac{\sin x}{x}\right)x^{\sin x}$

4. $f:\mathbb{R}\to\mathbb{R},\ f(x)=x+|x-1|=\begin{cases}x-x+1: & x-1<0\\ x+x-1: & x-1\geq 0\end{cases}=\begin{cases}1: & x<1\\ 2x-1: & x\geq 1\end{cases}$

a) prove that f has side derivatives at $x_0=1$

$f'(x)=\begin{cases}0, & x<1\\ 2, & x\geq 1\end{cases}\to\exists$ side derivatives at $x_0=1$

b) sides limit of f at $x_0=1$ = ?

$f_l'(x)=0$

$f_r'(x)=2$

c) f is continuous on the left & on the right of $x_0=1\to f$ is differentiable on the left & on the right of $x_0=1$

d) f dosn't have derivative at $x_0 = 1$ because the side derivates at $x_0 = 1$ aren't esual ($f_l'(1) = 0$, $f_r'(1) = 2$)

e) $\lim\limits_{x \nearrow 1} f(x) = 1$

$\lim\limits_{x \searrow 1} f(x) = \lim\limits_{x \searrow 1} 2x - 1 = 1$

$\left.\right\} \to \lim\limits_{x \nearrow 1} f(x) = \lim\limits_{x \searrow 1} f(x) \to f$ is continuous at $x_0 = 1 \to f$ is differentiable at $x_0 = 1$