Series of real numbers

Geometric series

We recall that:

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty & : a > 1 \\ 1 & : a = 1 \\ 0 & : a \in (-1,1) \\ \nexists & : a \le -1 \end{cases}$$

Each series of the type

$$\sum_{n\ge m} q^{n-1},$$

for $q \in \mathbb{R}$ is a **geometric series**. Then:

$$\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} 0 & : q = 0 \\ \frac{1}{1-q} & : q \in (-1,1)\backslash\{0\} \\ +\infty & : q \ge 1 \end{cases}$$

In the case when $q \le -1$, the geometric series does not posses a sum.

Proof:

We study the general term for the sequence of partial sums, given $n \ge 1$.

$$s_n = q^0 + q^1 + ... + q^{n-1} = \frac{1-q^n}{1-q}.$$

Case 1: $q = 0$. In this case $s_n = 0, \forall n \ge 1$, therefore its limit 0 as well, thus the sum of the series is 0.

Case 2: $q \in (-1,1)\backslash\{0\}$. Then: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-0}{1-q} = \frac{1}{1-q}$.

Case 3: $q > 1$. Then: $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1-\infty}{1-q} = \frac{-\infty}{1-q} = \infty$ because $1-q<0$.

Case 4: $q \le -1$. We notice that

$$\lim_{n\to\infty q^n}$$

does not exist, therefore there does not exists the limit $\lim_{n\to\infty} s_n$ as well. Thus, in this case the geometric series does not have a sum.

Telescopic series

If the general term of degree n of the series of real numbers $\sum_{n\geq m} x_n$ is defined as the difference of two successive terms of a sequence of real numbers $(a_n)_{n\geq m}$, i.e.

$$x_n = a_n - a_{n+1}, \forall n \geq m,$$

it is called a **telescopic series**. If the sequence $(a_n)_{n\geq m}$ has the limit

$$l = \lim_{n\to\infty} a_n,$$

then the series $\sum_{n\geq m} x_n$ has a sum, and it:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

Proof: We write the general term of degree n of the sequence of partial sums of the series:

$$s_n = x_m + x_{m+1} + ... + x_n = a_m - a_{m+1} + a_{m+1} - a_{m+2} + ... + a_{n-1} - a_n + a_n - a_{n+1} =$$

$$= a_m - a_{n+1}.$$

In conclusion

$$s_n = a_m - a_{n+1}, \forall n \geq m.$$

We notice that a_m is a constant, thus, due to the fact that the sequence $(a_n)_{n\geq m}$ has a limit, the sequence $(s_n)_{n\geq m}$ has a limit as well, and it is:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_m - a_{n+1}) = a_m - \lim_{n\to\infty} a_{n+1} = a_m - l.$$

Since the limit of the sequence of partial sums is the sum o the series, we reach the conclusion that:

$$\sum_{n=m}^{\infty} x_n = a_m - l.$$

Operations with convergent series

1. Let $\sum_{n\geq 1} x_n$ and $\sum_{n\geq 1} y_n$ be two series of real numbers which have the sums:

$$\sum_{n=1}^{\infty} x_n = x, \quad \sum_{n=1}^{\infty} y_n = y.$$

and let $a, b \in \mathbb{R}$. If $ax + by$ is not an not determined case, then the series

$$\sum_{n\geq 1} ax_n + by_n$$

has a sum and it is:
$$\sum_{n=1}^{\infty}(ax_n+by_n)=ax+by.$$
2. In the case of the sum of a series it is very important to notice the first term of the summation. Thus, if the series $\sum x_n$ has a sum, and if $p\in\mathbb{N}$, then
$$\sum_{n=p}^{\infty}x_n=\sum_{n=1}^{\infty}x_n-\Big(x_1+x_2+...+x_{p-1}\Big).$$

Exercises

Exercise 1: Compute the sums for the following geometric series (if they exists):
$$a)\sum_{n\geq 3}\frac{7}{9^n},\quad b)\sum_{n\geq 4}\frac{3^{n-3}+(-4)^{n+3}}{5^n},\quad c)\sum_{n\geq 5}e^n,\quad d)\sum_{n\geq 2}\Big(-\frac{1}{\pi}\Big)^n\quad e)\sum_{n\geq 3}(-4)^n.$$

Exercise 2: Compute the sums of the following telescopic series:
$$a)\sum_{n\geq 1}\frac{1}{4n^2-1},\quad b)\sum_{n\geq 1}\frac{1}{\sqrt{n}+\sqrt{n+1}},\quad c)\sum_{n\geq 5}\frac{1}{n(n+1)(n+2)}$$
$$d)\sum_{n\geq 1}\ln\Big(1+\frac{1}{n}\Big),\quad e)\sum_{n\geq 2}\frac{\ln\big(1+\frac{1}{n}\big)}{\ln\big(n^{\ln(n+1)}\big)}.$$

Theorem: If the series $\sum x_n$ is convergent, then $\lim_{n\to\infty}x_n=0$.

Consequence: If the $\lim_{n\to\infty}x_n\neq 0$ then the series $\sum x_n$ is divergent.

Remark: If the $\lim_{n\to\infty}x_n=0$ then we have to investigate the series by other means.

Series with positive terms

Theorem: If $\sum x_n$ is spt, then it always has a sum, since the sequence of the intermediate sums is increasing.

C1C: Let $\sum a_n$ and $\sum b_n$ be two spt such that
$$\exists n_0\in\mathbb{N}\quad s.t.\quad \forall n\geq n_0,\quad a_n\leq b_n.$$
Then
$$(1)\sum b_n\quad C\Longrightarrow\sum a_n\quad C:$$
$$(2)\sum a_n\quad D\Longrightarrow\sum b_n\quad D.$$

C2C: Let $\sum a_n$ and $\sum b_n$ be two SPT such that there exists

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \in [0, \infty].$$

Then the following statements hold:

a) If $l = 0$, then the same conclusions of $C1C$ hold.

b) If $l \in (0, \infty)$, then both series have the same nature.

c) If $l = \infty$, then

$$(3) If \sum a_n \quad C \Longrightarrow \sum b_n \quad C$$

$$(4) If \sum b_n \quad D \Longrightarrow \sum a_n \quad D.$$

C3C: Let $\sum a_n$ and $\sum b_n$ be two spt such that

$$\exists n_0 \in \mathbb{N} \quad s.t. \quad \forall n \geq n_0, \qquad \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

Then then the same conclusions of $C1C$ hold.

D'Alambert test and root test: Let $\sum a_n$ be a SPT such that there exists

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Then

$$l < 0 \Longrightarrow \quad \sum a_n \quad is \quad C$$

$$l > 0 \Longrightarrow \quad \sum a_n \quad is \quad D$$

$$l = 0 \Longrightarrow ?$$

Exercise 3: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{n+7}{\sqrt{n^2+7}}, \quad b) \sum_{n\geq 1} \frac{1}{\sqrt[n]{n}}, \quad c) \sum_{n\geq 1} \frac{1}{\sqrt[n]{n!}}, \quad d) \sum_{n\geq 1} \left(1 + \frac{1}{n}\right)^n.$$

Exercise 4: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{2^n + 3^n}{5^n}, \quad b) \sum_{n\geq 1} \frac{2^n}{3^n + 5^n}.$$

Exercise 5: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{1}{2n-1}, \quad b) \sum_{n\geq 1} \frac{1}{(2n-1)^2}, \quad c) \sum_{n\geq 1} \frac{1}{\sqrt{4n^2-1}}, \quad d) \sum_{n\geq 1} \frac{\sqrt{n^2+n}}{\sqrt[3]{n^5-n}}.$$

Exercise 6: Determine the nature (convergence or divergence) of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{100^n}{n!}, \quad b) \sum_{n\geq 1} \frac{2^n n!}{n^n}, \quad c) \sum_{n\geq 1} \frac{3^n n!}{n^n}, \quad d) \sum_{n\geq 1} \frac{(n!)^2}{2^{n^2}}, \quad e) \sum_{n\geq 1} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}.$$

Exercise 7: Determine the nature (convergence or divergence), by discussing the value of the parameter $a > 0$, of the following series of real numbers:

$$a) \sum_{n\geq 1} \frac{a^n}{n^n}, \quad b) \sum_{n\geq 1} \left(\frac{n^2+n+1}{n^2}a\right)^n, \quad c) \sum_{n\geq 1} \frac{3^n}{2^n+a^n}.$$