Calculus - Hw 10 - Substitutions in integrals

1. a) $\int \frac{1}{1+\frac{1}{\sin x}} dx$, $x \in (-\pi, \pi)$

$$I = \int \frac{\sin x}{1+\sin x} dx = \int \frac{\frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}}{1+\frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}} dx = \int \frac{2\,tg\frac{x}{2}}{\cancel{1+tg^2\frac{x}{2}}} \cdot \frac{\cancel{1+tg^2\frac{x}{2}}}{1+2tg\frac{x}{2}+tg^2\frac{x}{2}}$$

not: $tg\frac{x}{2} = t \rightarrow \frac{1}{\cos^2\frac{x}{2}} dx = dt$

$$(1+tg^2\tfrac{x}{2})\,dx = dt \rightarrow dx = \frac{dt}{1+t^2}$$

$$I = \int \frac{2t}{1+2t+t^2} \cdot \frac{1}{1+t^2} dt = 2\int \frac{t}{(1+t)^2(1+t^2)} dt$$

$$\frac{t}{(1+t)^2(1+t^2)} = \frac{A}{(1+t)^2} + \frac{Ct+D}{1+t^2}$$

$$t = A + At^2 + Ct + 2Ct^2 + Ct^3 + D + 2Dt + Dt^2$$

$$t = t^3 \cdot C + t^2(A+2C+D) + t(C+2D) + A + D \rightarrow \begin{cases} C=0 \\ A+2C+D=0 \\ C+2D=1 \\ A+D=0 \end{cases} \rightarrow D = \frac{1}{2}$$

$$\rightarrow A = -\frac{1}{2}$$

$$I = 2\int \frac{-\frac{1}{2}}{(1+t)^2} + \frac{\frac{1}{2}}{1+t^2} dt = \int \frac{1}{1+t^2} dt - \int \frac{1}{(1+t)^2} dt =$$

$$= arctg\, t - \int (1+t)^{-2}(1+t)'\, dt = arctg\, t + \frac{1}{1+t} + \mathcal{C} =$$

$$= arctg\, tg\frac{x}{2} + \frac{1}{1+tg\frac{x}{2}} + \mathcal{C} = \frac{x}{2} + \frac{1}{1+tg\frac{x}{2}} + \mathcal{C}$$

b) $\int \frac{1}{3\sin x + 4\cos x} dx = \int \frac{1}{3\cdot\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} + 4\,\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}} dx =$

$$= \int \frac{1+tg^2\frac{x}{2}}{-4tg^2\frac{x}{2}+6tg\frac{x}{2}+4} dx = \int \frac{\cancel{1+t^2}}{-4t^2+6t+4} \cdot \frac{1}{\cancel{1+t^2}} \cdot dt = -\frac{1}{2}\int \frac{1}{2t^2+3t-2}$$

$tg\frac{x}{2} = t$ $\quad \frac{1}{\cos^2 x} dx = dt$

$$(1+tg^2\tfrac{x}{2})dx = dt \rightarrow dx = \frac{dt}{1+t^2}$$

$$= -\frac{1}{2}\int \frac{1}{(t-2)(2t+1)}\,dt = -\frac{1}{2}\int \frac{1}{5}\cdot\frac{1}{t-2} - \frac{2}{5}\cdot\frac{1}{2t+1}\,dt = -\frac{1}{10}\left(\ln|t-2| - \ln|2t+1|\right) + C =$$

$$2t^2-3t-2 = 2t^2-4t+t-2 = 2t(t-2)+(t-2) = (t-2)(2t+1)$$

$$\frac{1}{(t-2)(2t+1)} = \frac{A}{t-2} + \frac{B}{2t+1}$$

$$1 = t(2A+B) + A - 2B \rightarrow 2A+B=0 \rightarrow B=-2A$$

$$A-2B=1 \rightarrow A+4A=1 \rightarrow A=\frac{1}{5} \rightarrow B=-\frac{2}{5}$$

$$\underline{I} = -\frac{1}{10}\cdot\ln\left|\frac{t-2}{2t+1}\right| + C = -\frac{1}{10}\cdot\ln\left|\frac{\operatorname{tg}\frac{x}{2}-2}{2\operatorname{tg}\frac{x}{2}+1}\right| + C$$

c) $$\underline{I} = \int \frac{\sqrt{9-x^2}}{x^2}\,dx = \sqrt{9-x^2}\cdot\left(-\frac{1}{x}\right) - \int\left(-\frac{\cancel{x}}{\sqrt{9-x^2}}\right)\left(-\frac{1}{\cancel{x}}\right)dx =$$

$$f(x) = \sqrt{9-x^2} \rightarrow f'(x) = \frac{1}{2\sqrt{9-x^2}}(-2x) = -\frac{x}{\sqrt{9-x^2}}$$

$$g'(x) = \frac{1}{x^2} \rightarrow g(x) = -\frac{1}{x}$$

$$\underline{I} = -\frac{\sqrt{9-x^2}}{x} - \int\frac{1}{\sqrt{9-x^2}}\,dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$$

d) $$\int\frac{1}{\sqrt{(x^2+1)^3}}\,dx = \int\frac{1+\operatorname{tg}^2t}{(1+\operatorname{tg}^2t)^{\frac{3}{2}}}\,dt = \int\frac{1}{\sqrt{1+\operatorname{tg}^2t}}\,dt = \int\frac{1}{\sqrt{\frac{1}{\cos^2t}}}\,dt =$$

$$x = \operatorname{tg} t \rightarrow t = \operatorname{arctg} x$$

$$dx = (1+\operatorname{tg}^2t)\,dt$$

$$= \int \cos t\,dt = \sin t + C = \frac{\operatorname{tg} t}{\sqrt{1+\operatorname{tg}^2t}} + C = \frac{\operatorname{tg}(\operatorname{arctg} x)}{\sqrt{1+(\operatorname{tg}(\operatorname{arctg}x))^2}} + C = \frac{x}{\sqrt{1+x^2}} + C$$

e) $$\underline{I} = \int\frac{1}{\sqrt{(x^2-8)^3}}\,dx = \int\frac{1}{\sqrt{(x^2-(2\sqrt{2})^2)^3}}\,dx$$

$$x = \frac{2\sqrt{2}}{\cos t} \rightarrow \cos t = \frac{2\sqrt{2}}{x} \rightarrow t = \arccos\frac{2\sqrt{2}}{x}$$

$$dx = 2\sqrt{2}\left(-\frac{1}{\cos^2t}\right)(-\sin t) = 2\sqrt{2}\cdot\frac{\sin t}{\cos^2 t}$$

$$I = \int\frac{1}{\sqrt{\left(\frac{8}{\cos^2t}-8\right)^3}}\cdot 2\sqrt{2}\cdot\frac{\sin t}{\cos^2t}\,dt = 2\sqrt{2}\int\frac{1}{\sqrt{\left(\frac{8-8\cos^2t}{\cos^2t}\right)^3}}\cdot\frac{\sin t}{\cos^2t}\,dt =$$

$$= 2\sqrt{2}\int\frac{\cos^3t}{\sqrt{[8(1-\cos^2t)]^3}}\cdot\frac{\sin t}{\cancel{\cos^2t}}\,dt = 2\sqrt{2}\int\frac{\cancel{\sin t}\cdot\cos t}{8^{\frac{3}{2}}\cdot\cancel{\sin t}}\,dt = \frac{2\sqrt{2}}{(2\sqrt{2})^3}\int\cos t\,dt =$$

$$= \frac{1}{8} \sin t + C = \frac{1}{8}\sqrt{1-\cos^2 t} + C = \frac{1}{8}\sqrt{1-(\cos(\arccos\frac{2\sqrt{2}}{x}))^2} + C =$$

$$= \frac{1}{8}\sqrt{1-\frac{8}{x^2}} + C = \frac{1}{8}\sqrt{\frac{x^2-8}{x^2}} + C = \frac{1}{8} \cdot \frac{\sqrt{x^2-8}}{|x|} + C$$

$$f) \int \sqrt{2x-x^2}\, dx = \int \sqrt{1-(x-1)^2}\, dx = \int \sqrt{1-t^2}\, dt = \int \frac{1-t^2}{\sqrt{1-t^2}}\, dt = \int \frac{1}{\sqrt{1-t^2}}\, dt -$$

$$2x - x^2 = -x^2 + 2x - 1 + 1 = -(x-1)^2 + 1$$

$$x - 1 = t \rightarrow dx = dt$$

$$- \int t \cdot \frac{t}{\sqrt{1-t^2}}\, dt = \arcsin t - \int t(-\sqrt{1-t^2})\, dt =$$

$$= \arcsin t - \left[t(-\sqrt{1-t^2}) - \int (-\sqrt{1-t^2})\, dt \right] + C$$

$$= \arcsin t + t\sqrt{1-t^2} - \int \sqrt{1-t^2}\, dt$$

$$I = \int \sqrt{1-t^2} = \arcsin t + t\sqrt{1-t^2} - \int \sqrt{1-t^2}\, dt + C$$

$$2I = \arcsin(x-1) + (x-1)\sqrt{2x-x^2} + C$$

$$I = \frac{\arcsin(x-1) + (x-1)\sqrt{2x-x^2}}{2} + C$$

$$g) \int \sqrt{4-x^2}\, dx = \int \sqrt{4-4\sin^2 t} \cdot 2\cos t\, dt = \int 2\sqrt{1-\sin^2 t} \cdot 2\cos t\, dt =$$

$$x = 2\sin t \rightarrow t = \arcsin\frac{x}{2}$$

$$dx = 2\cos t$$

$$= \int 4\cos^2 t\, dt = 4\left[\frac{\cos t \sin t}{2} + \frac{1}{2}\int 1\, dt \right] = 2\cos t \sin t + 2t + C =$$

$$= 2\sqrt{1-\sin^2(\arcsin\tfrac{x}{2})} \cdot \sin(\arcsin\tfrac{x}{2}) + 2\arcsin\frac{x}{2} + C$$

$$= 2 \cdot \frac{x}{2}\sqrt{1-\frac{x^2}{4}} + 2\arcsin\frac{x}{2} + C$$

$$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2} + C$$

h) $\int x\sqrt{1+x^2}\,dx = \int tg\,t\,\sqrt{1+tg^2 t}\cdot\frac{1}{\cos^2 t}\,dt = \int tg\,t\sqrt{\frac{1}{\cos^2 t}}\cdot\frac{1}{\cos^2 t}\,dt =$

$x = tg\,t \to t = arctg\,x$

$dx = \frac{1}{\cos^2 t}\,dt$

$= \int tg\,t\cdot\frac{1}{\cos^3 t}\,dt = \int\frac{\sin t}{\cos^4 t}\,dt = -\frac{\cos^{-3}t}{-3}+C = \frac{1}{3}\left(\frac{1}{\sqrt{1+tg^2 t}}\right)^{-3} + C =$

$= \frac{1}{3}\left(\sqrt{1+(tg(arctg\,x))^2}\right)^3 = \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$

2. a) $\int\frac{2x-1}{x^2-3x+2}\,dx = \int\frac{2x-1}{(x-2)(x-1)}\,dx = \int\frac{3}{x-2}-\frac{1}{x-1}\,dx = 3\ln(x-2)-\ln(x-1)+C$

$\frac{2x-1}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1} = \frac{3}{x-2}-\frac{1}{x-1}$

$2x-1 = x(A+B)-A-2B$

$\left.\begin{array}{l} A+B=2 \\ -A-2B=-1 \to A+2B=1 \end{array}\right\} \to 2+B=1 \to B=-1 \to A=3$

b) $I=\int\frac{4}{(x-1)(x+1)^2}\,dx = 4\int\frac{1}{(x-1)(x+1)^2}\,dx = 4\int\frac{1}{4}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x+1}-\frac{2}{4}\cdot\frac{1}{(x+1)^2}\,dx$

$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

$1 = A(x^3+3x^2+3x+1)+B(x-1)(x^2+2x+1)+C(x^2-1)$

$1 = x^3(A+B)+x^2(3A+B+C)+x(3A-B)+A-B-C$

$A+B=0$

$3A+B+C=0 \to -B-C=3A$

$3A-B=0$

$A-B-C=1$

$\left.\begin{array}{l} A+3A=1 \to A=\frac{1}{4} \\ A+B=0 \end{array}\right\} \to B=-\frac{1}{4} \to C=-\frac{3}{4}+\frac{1}{4}=-\frac{2}{4}$

$I = \int\frac{1}{x-1}\,dx - \int\frac{1}{x+1}\,dx - 2\int\frac{1}{(x+1)^2}\,dx$

$= \ln(x-1)-\ln(x+1)+2\cdot\frac{1}{x+1}+C = \ln\left(\frac{x-1}{x+1}\right)+\frac{2}{x+1}+C$

$\int\frac{1}{(x+1)^2}\,dx = \int\frac{1}{t^2}\,dt = -\frac{1}{t} = -\frac{1}{x+1}$

$x+1=t \to x=t-1$

$dx=dt$

c) $\int \frac{1}{x^3-x^4}\,dx = -\int \frac{1}{x^3(x-1)}\,dx = -\int \frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x-1}\,dx =$

$$\frac{1}{x^3(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}$$

$$1 = A(x^6-x^5)+B(x^5-x^4)+C(x^4-x^3)+Dx^6$$

$$= x^6(A+D)+x^5(B-A)+x^4(C-B)-Cx^3$$

$A+D=0 \rightarrow A=-D$

$\left.\begin{matrix} B-A=0 \\ C-B=0 \end{matrix}\right\} \rightarrow A=B=C=-D$

$$I = -\left(\int \frac{1}{x}dx + \int \frac{1}{x^2}dx + \int \frac{1}{x^3}dx - \int \frac{1}{x-1}dx\right)$$

$$= -\ln x + \frac{1}{x} + \frac{1}{2x^2} + \ln x-1 + C = \ln \frac{x-1}{x} + \frac{2x+1}{2x^2} + C$$

d) $\int \frac{2x+5}{x^2+5x+10}\,dx = \int \frac{(x^2+5x+10)'}{x^2+5x+10}\,dx = \ln|x^2+5x+10| + C$

e) $I = \int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

$x+\frac{1}{2}=t$

$dx = dt$

$$I = \int \frac{1}{t^2+\left(\frac{\sqrt3}{2}\right)^2}\,dt = \frac{1}{\frac{\sqrt3}{2}}\,\text{arctg}\,\frac{t}{\frac{\sqrt3}{2}} + C = \frac{2}{\sqrt3}\,\text{arctg}\,\frac{2t}{\sqrt3} + C =$$

$$= \frac{2\sqrt3}{3}\,\text{arctg}\,\frac{2\sqrt3 x+\sqrt3}{3} + C$$

3. a) $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx = \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}\,dx = \int \sqrt{x+1}\,dx - \int \sqrt{x}\,dx =$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C = \frac{2\left[(x+1)^{\frac{3}{2}} - x^{\frac{3}{2}}\right]}{3} + C$$

b) $\int \frac{1}{x+\sqrt{x-1}}\,dx = \int \frac{2t}{t^2+1+t}\,dt = \int \frac{2t+1-1}{t^2+t+1}\,dx =$

$\sqrt{x-1} = t \rightarrow x = t^2+1$

$dx = 2t\,dt$

$$= \int \frac{2t+1}{t^2+t+1}\,dt - \int \frac{1}{t^2+t+1}\,dt = \ln(t^2+t+1) - \int \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}\,dt =$$

$$= \ln(t^2+t+1) - \frac{2}{\sqrt3}\,\text{arctg}\,\frac{2(t+\frac{1}{2})}{\sqrt3} + C = \ln(x-\cancel{1}+\sqrt{x-1}+\cancel{1}) - \frac{2}{\sqrt3}\,\text{arctg}\,\frac{2\sqrt{x-1}+1}{\sqrt3} + C =$$

$$= \ln(x+\sqrt{x-1}) - \frac{2}{\sqrt3}\,\text{arctg}\,\frac{2\sqrt{x-1}+1}{\sqrt3} + C$$

4. a) $\underline{I} = \int \frac{1}{1+\sqrt{x^2+2x-2}}\,dx = \int \frac{1}{1+\sqrt{(x+1)^2-3}}\,dt$

$x+1 = \frac{\sqrt{3}}{\cos t} \rightarrow t = \arccos \frac{\sqrt{3}}{x+1}$

$dx = -\frac{\sqrt{3}}{\sin t}\,dt$

$\underline{I} = \int \frac{1}{1+\sqrt{\frac{3}{\cos^2 t}-3}} \cdot \frac{(-\sqrt{3})}{\sin t}\,dt = (-\sqrt{3})\int \frac{1}{1+\frac{\sqrt{3(1-\cos^2 t)}}{\cos t}} \cdot \frac{1}{\sin t}\,dt =$

$= (-\sqrt{3}) \int \frac{\cos t}{\cos t + \sqrt{3}\sin t} \cdot \frac{1}{\sin t}\,dt$

$= (-\sqrt{3}) \int \frac{\frac{1-tg^2\frac{t}{2}}{1+\cancel{tg^2\frac{t}{2}}} \cdot \frac{\cancel{1+tg^2\frac{t}{2}}}{2tg\frac{t}{2}}}{\frac{1-tg^2\frac{t}{2}+2\sqrt{3}\,tg\frac{t}{2}}{1+tg^2\frac{t}{2}}}\,dt = \int \frac{1-tg^2\frac{x}{2}}{2tg\frac{x}{2}} \cdot \frac{1+tg^2\frac{t}{2}}{1-tg^2\frac{x}{2}+2\sqrt{3}\,tg\frac{t}{2}}\,dt$

$tg\frac{t}{2} = u$

$(1+u^2)\cdot\frac{1}{2}\,dt = du \rightarrow dt = \frac{2du}{(1+u^2)}$

$\underline{I} = \int \frac{1-u^2}{\cancel{2}u} \cdot \frac{\cancel{1+u^2}}{1-u^2+2\sqrt{3}\,u} \cdot \frac{\cancel{2}}{\cancel{1+u^2}}\,du =$

$= \int \frac{1-u^2}{-u^3+2\sqrt{3}u^2+u}\,du = \int \frac{u^2-1}{u(u^2-2\sqrt{3}u+1)}\,du$ -

$\frac{u^2-1}{u(u^2-2\sqrt{3}u-1)} = \frac{A}{u} + \frac{B}{u^2-2\sqrt{3}u-1}$

$u^2-1 = u^2A + u(B-2\sqrt{3}A) - A$

$A = 1$

$B - 2\sqrt{3} = 0 \rightarrow B = 2\sqrt{3}$

$I = \int \frac{1}{u} + \frac{2\sqrt{3}}{u^2-2\sqrt{3}u-1}\,du = \ln u + 2\sqrt{3}\int \frac{1}{(u-\sqrt{3}-2)(u-\sqrt{3}+2)}\,du =$

$$\frac{1}{(u-\sqrt{3}-2)(u-\sqrt{3}+2)} = \frac{A}{u-\sqrt{3}-2} + \frac{B}{u-\sqrt{3}+2}$$

$$1 = u(A+B) - \sqrt{3}(A+B) + 2A - 2B$$

$A+B=0 \to A=-B$

$2A-2B=1 \to -4B=1 \to B=-\frac{1}{4} \to A=\frac{1}{4}$

$$I = \ln u + 2\sqrt{3}\int \frac{1}{4}\cdot\frac{1}{u-\sqrt{3}-2} - \frac{1}{4}\frac{1}{u-\sqrt{3}+2}\,du$$

$$= \ln u + \frac{\sqrt{3}}{2}\left(\ln|u-\sqrt{3}-2| - \ln|u-\sqrt{3}+2|\right) + C$$

$$= \ln|u| + \frac{\sqrt{3}}{2}\ln\left|\frac{u-\sqrt{3}-2}{u-\sqrt{3}+2}\right| + C$$

$$= \ln\left|tg\frac{t}{2}\right| + \frac{\sqrt{3}}{2}\ln\left|\frac{tg\frac{t}{2}-\sqrt{3}-2}{tg\frac{t}{2}-\sqrt{3}+2}\right| + C \qquad \boxed{tg\frac{x}{2} = \frac{1-\cos x}{\sin x}}$$

$$= \ln\left|\frac{1-\cos t}{\sqrt{1-\cos^2 t}}\right| + \frac{\sqrt{3}}{2}\ln\left|\frac{\frac{1-\cos t}{\sqrt{1-\cos^2 t}}-\sqrt{3}-2}{\frac{1-\cos t}{\sqrt{1-\cos^2 t}}-\sqrt{3}+2}\right| + C =$$

$$= \ln\left|\frac{1-\frac{\sqrt{3}}{x+1}}{\sqrt{1-\frac{3}{(x+1)^2}}}\right| + \frac{\sqrt{3}}{2}\ln\left|\frac{\frac{1-\frac{\sqrt{3}}{x+1}}{\sqrt{1-\frac{3}{(x+1)^2}}}-\sqrt{3}-2}{\frac{1-\frac{\sqrt{3}}{x+1}}{\sqrt{1-\frac{3}{(x+1)^2}}}-\sqrt{3}+2}\right| + C$$

$$= \ln\left|\frac{x+1-\sqrt{3}}{\sqrt{(x+1)^2-3}}\right| + \frac{\sqrt{3}}{2}\ln\left|\frac{\frac{x+1-\sqrt{3}}{\sqrt{(x+1)^2-3}}-\sqrt{3}-2}{\frac{x+1-\sqrt{3}}{\sqrt{(x+1)^2-3}}-\sqrt{3}+2}\right| + C$$

b) $$\int\frac{1}{(x+1)\sqrt{-4x^2-x+1}}\,dx = \int\frac{1}{(x+1)\sqrt{-\left((2x)^2-2\cdot 2x\cdot\frac{1}{4}+\frac{1}{16}-\frac{17}{16}\right)}}\,dx =$$

$$= \int\frac{1}{(x+1)\sqrt{\frac{17}{16}-\left(2x+\frac{1}{4}\right)^2}}\,dx = 4\int\frac{1}{(x+1)\sqrt{17-(8x+1)^2}}\,dx$$

$8x+1=t \to x=\frac{t-1}{8}$

$dx=\frac{1}{8}dt$

$$I = 4\int \frac{1}{\left(\frac{t-1}{8}+1\right)\sqrt{17-t^2}} \cdot \frac{1}{8}\, dt = 4\int \frac{8}{(t+7)\sqrt{17-t^2}}\, \frac{1}{8}\, dt$$

$$t = \sqrt{17}\sin u \rightarrow u = \arcsin\frac{t}{\sqrt{17}}$$

$$dt = \sqrt{17}\cos u$$

$$I = 4\int \frac{\cancel{\sqrt{17}\cos u}}{(\sqrt{17}\sin u + 7)\cancel{\sqrt{17(1-\sin^2 u)}}}\, du = 4\int \frac{1}{\sqrt{17}\sin u + 7}\, du =$$

$$= 4\int \frac{1}{\sqrt{17}\,\frac{2tg\frac{u}{2}}{1+tg^2\frac{u}{2}} + 7}\, du = 4\cdot\int \frac{1+tg^2\frac{u}{2}}{7\,tg^2\frac{u}{2} + 2\sqrt{17}\,tg\frac{u}{2} + 7}\, du$$

$$tg\frac{u}{2} = v \rightarrow du = \frac{2dv}{1+v^2}$$

$$I = 4\int \frac{\cancel{1+v^2}}{7v^2 + 2\sqrt{17}v + 7} \cdot \frac{2}{\cancel{1+v^2}}\, dv = 8\int \frac{1}{(\sqrt{7}v)^2 + 2\cdot\sqrt{7}\,\frac{\sqrt{17}}{\sqrt{7}}v + \left(\frac{\sqrt{17}}{\sqrt{7}}\right)^2 + \frac{32}{7}}\, dv =$$

$$= 8\int \frac{1}{\left(\sqrt{7}v + \frac{\sqrt{17}}{\sqrt{7}}\right)^2 + \frac{32}{7}}\, dv = \cancel{8}\cdot\frac{7}{\cancel{32}_4}\cdot arctg\,\frac{7v + \frac{\sqrt{17}}{\sqrt{7}}}{\frac{32}{7}} + C =$$

$$= \frac{7}{4}\, arctg\,\frac{49v + \sqrt{119}}{32} + C = \frac{7}{4}\, arctg\,\frac{49\,tg\frac{u}{2} + \sqrt{119}}{32} + C =$$

$$= \frac{7}{4}\, arctg\,\frac{49\cdot\frac{\sin u}{1+\sqrt{1-\sin^2 u}} + \sqrt{119}}{32} + C$$

$$= \frac{7}{4}\, arctg\,\frac{49\cdot\frac{\frac{t}{\sqrt{17}}}{1+\sqrt{1-\frac{t^2}{17}}} + \sqrt{119}}{32} + C$$

$$= \frac{7}{4}\, arctg\,\frac{49\cdot\frac{t}{\sqrt{17}+\sqrt{17-t^2}} + \sqrt{119}}{32} + C = \frac{7}{4}\, arctg\,\frac{49\,\frac{8x+1}{\sqrt{17}+\sqrt{16-64x^2-16x}} + \sqrt{119}}{32}$$

5. a) $\int_1^2 \frac{1}{x^3+x^2+x+1}\,dx = \int_1^2 \frac{1}{(x+1)(x^2+1)}\,dx$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$1 = x^2(A+B) + x(B+C) + A + C$$

$A+B=0 \to B=-A$

$B+C=0 \qquad \to -A+1-A=0 \to A=\frac{1}{2} \to B=-\frac{1}{2} \to C=+\frac{1}{2}$

$A+C=1 \to C=1-A$

$$I = \frac{1}{2}\int_1^2 \frac{1}{x+1} + \frac{-x+1}{x^2+1}\,dx = \frac{1}{2}\left[\ln(x+1)\Big|_1^2 - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx\right.$$

$$= \frac{1}{2}\left[\ln\frac{3}{2} - \frac{1}{2}\ln(x^2+1)\Big|_1^2 + \operatorname{arctg} x\Big|_1^2\right] = \frac{1}{2}\left[\ln\frac{3}{2} - \frac{1}{2}\ln\frac{5}{2} + \operatorname{arctg} 2 - \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\ln\frac{3}{\sqrt{10}} + \operatorname{arctg} 2 - \frac{\pi}{4}\right] = \ln\frac{\sqrt{3}}{\sqrt[4]{10}} + \frac{\operatorname{arctg} 2}{2} - \frac{\pi}{8}$$

b) $\int_1^3 \frac{1}{x(x^2+9)}\,dx = \frac{1}{10}\int_1^3 \frac{1}{x} + \frac{-x+1}{x^2+9}\,dx$

$$\frac{1}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$1 = x^2(A+B) + x(B+C) + 9A + C$$

$A+B=0 \to B=-A$

$B+C=0 \qquad \to -10A+1=0 \to A=\frac{1}{10} \to B=-\frac{1}{10} \to C=\frac{1}{10}$

$9A+C=1 \to C=1-9A$

$$I = \frac{1}{10}\left[\ln x\Big|_1^3 - \frac{1}{2}\int_1^3 \frac{2x}{x^2+9}\,dx + \int_1^3 \frac{1}{x^2+9}\,dx\right]$$

$$= \frac{1}{10}\left[\ln 3 - \frac{1}{2}\ln(x^2+9)\Big|_1^3 + \frac{1}{3}\operatorname{arctg}\frac{x}{3}\Big|_1^3\right]$$

$$= \frac{1}{10}\left[\ln 3 - \frac{1}{2}\ln\frac{18}{10} + \frac{1}{3}\left(\operatorname{arctg} 1 - \operatorname{arctg}\frac{1}{3}\right)\right]$$

$$= \frac{1}{10}\left[\ln\frac{\cancel{3}}{\frac{\cancel{3}\sqrt{2}}{\sqrt{10}}} + \frac{1}{3}\frac{\pi}{4} - \frac{\operatorname{arctg}\frac{1}{3}}{3}\right]$$

$$= \frac{1}{10}\ln\sqrt{5} + \frac{\pi}{12} - \frac{\operatorname{arctg}\frac{1}{3}}{30}$$

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c) $\int_{-1}^{1} \frac{x^2+1}{x^4+1}\,dx = \int_{-1}^{1} \frac{x^2+1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}\,dx$

$$\frac{x^2+1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

$$x^2+1 = x^3(A+C) + x^2(\sqrt{2}A+B-\sqrt{2}C+D) + x(A+\sqrt{2}B+C-\sqrt{2}D) + B+D$$

$A+C=0 \to A=-C$

$\sqrt{2}A+B-\sqrt{2}C+D=1 \to -2\sqrt{2}C+\underbrace{B+D}_{1}=1 \to -2\sqrt{2}C=0 \to C=A=0$

$A+\sqrt{2}B+C-\sqrt{2}D=0 \to -C+\sqrt{2}(B-D)+C=0 \to B-D=0 \to B=D \Big\} \to D=1-D \to$

$B+D=1 \to B=1-D$ $\qquad \to D=\frac{1}{2} \to B=$

$$I = \frac{1}{2}\int_{-1}^{1} \frac{1}{x^2-\sqrt{2}x+1} + \frac{1}{x^2+\sqrt{2}x+1}\,dx = \frac{1}{2}\left[\int_{-1}^{1} \frac{1}{x^2-2\cdot x\cdot\frac{\sqrt{2}}{2}+\frac{2}{4}+\frac{2}{4}}\,dx + \int_{-1}^{1} \frac{1}{x^2+2x\frac{\sqrt{2}}{2}+\frac{2}{4}+\frac{2}{4}}\,dx\right.$$

$$= \frac{1}{2}\left[\int_{-1}^{1} \frac{1}{(x^2-\frac{1}{2})^2+\frac{1}{2}}\,dx + \int_{-1}^{1} \frac{1}{(x^2+\frac{1}{2})^2+\frac{1}{2}}\,dx\right]$$

$$= \frac{1}{2}\left[\sqrt{2}\,\mathrm{arctg}\,\sqrt{2}\left(x-\frac{1}{2}\right)\Big|_{-1}^{1} + \sqrt{2}\,\mathrm{arctg}\,\sqrt{2}\left(x+\frac{1}{2}\right)\Big|_{-1}^{1}\right]$$

$$= \frac{\sqrt{2}}{2}\left[\mathrm{arctg}\,\frac{\sqrt{2}}{2} - \mathrm{arctg}\left(-\frac{3\sqrt{2}}{2}\right) + \mathrm{arctg}\,\frac{3\sqrt{2}}{2} - \mathrm{arctg}\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{\sqrt{2}}{2}\left[2\,\mathrm{arctg}\,\frac{\sqrt{2}}{2} + 2\,\mathrm{arctg}\,\frac{3\sqrt{2}}{2}\right] = \sqrt{2}\,\mathrm{arctg}\,\frac{\sqrt{2}}{2} + \sqrt{2}\,\mathrm{arctg}\,\frac{3\sqrt{2}}{2}$$

d) $\int_{-1}^{1} \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{2x}{x^2+x+1}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{2x+1}{x^2+x+1}\,dx -$

$$-\frac{1}{2}\int_{-1}^{1} \frac{1}{x^2+x+1}\,dx = \frac{1}{2}\ln(x^2+x+1)\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

$$= \frac{1}{2}\ln 3 - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\,\mathrm{arctg}\,\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Big|_{-1}^{1}$$

$$= \frac{1}{2}\ln 3 - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2x+1}{\sqrt{3}}\Big|_{-1}^{1} = \frac{1}{2}\ln 3 - \frac{1}{\sqrt{3}}\left[\mathrm{arctg}\,\frac{4}{\sqrt{3}} - \mathrm{arctg}\left(-\frac{1}{\sqrt{3}}\right)\right] =$$

$$= \frac{1}{2}\ln 3 - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}$$

6. a) $\displaystyle\int_{-3} \frac{x}{(x+1)(x^2+3)}\,dx = \frac{1}{4}\int_{-3} \frac{-1}{x+1} + \frac{x+3}{x^2+3}\,dx =$

$$\frac{x}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$$

$$x = x^2(A+B) + x(B+C) + 3A + C$$

$A + B = 0 \to B = -A$

$B + C = 1 \to -4A = 1 \to A = -\frac{1}{4} \to B = \frac{1}{4} \to C = \frac{3}{4}$

$3A + C = 0 \to C = -3A$

$$I = \frac{1}{4}\left[-\ln|x+1|\Big|_{-3}^{-2} + \int_{-3}^{-2}\frac{x}{x^2+3}\,dx + \int_{-3}^{-2}\frac{3}{x^2+3}\,dx\right] =$$

$$= \frac{1}{4}\left[-\ln|-1| + \ln|-2| + \frac{1}{2}\ln|x^2+3|\Big|_{-3}^{-2} + \frac{3}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\Big|_{-3}^{-2}\right]$$

$$= \frac{1}{4}\ln 2 + \frac{1}{8}(\ln 7 - \ln 12) + \frac{\sqrt{3}}{4}\left[\operatorname{arctg}\left(-\frac{2}{\sqrt{3}}\right) + \operatorname{arctg}\left(-\frac{3}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{4}\ln 2 + \frac{1}{8}\ln\frac{7}{12} - \frac{\sqrt{3}}{4}\operatorname{arctg}\frac{2\sqrt{3}}{6} - \frac{\sqrt{3}}{4}\cdot\frac{\pi}{3}$$

b) $\displaystyle\int_0^1 \frac{x+1}{(x^2+4x+5)^2}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)-1}{(x^2+4x+5)^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x+4}{(x^2+4x+5)^2}\,dx -$

$$-\int_0^1 \frac{1}{(x^2+4x+5)^2}\,dx = \frac{1}{2}\,\frac{-1}{x^2+4x+5}\Big|_0^1 - \int_0^1 \frac{1}{[(x+2)^2+1]^2}\,dx =$$

$$= \frac{1}{2}\left(\frac{-1}{10} + \frac{1}{5}\right) - \int_2^3 \frac{1}{(t^2+1)^2}\,dt =$$

$x + 2 = t \to dx = dt$

$x = 1 \to t = 3$

$x = 0 \to t = 2$

c) $\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \frac{1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx = \ln x\Big|_1^2 - \frac{1}{2}\ln(x^2+1)\Big|_1^2 =$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = x^2(A+B) + xC + A \rightarrow A+B=0 \rightarrow B=-1$
$C=0$
$A=1$

$= \ln 2 - \frac{1}{2}\ln 5 + \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2 - \frac{1}{2}\ln 5 = \frac{1}{2}\ln\frac{8}{5}$

d) $\int_0^2 \frac{x^3+2x^2+x+4}{(x+1)^2}\,dx = \int_0^2 x + \frac{4}{(x+1)^2}\,dx = \left(\frac{x^2}{2} - \frac{4}{x+1}\right)\Big|_0^2 = \frac{4}{2} - \frac{4}{3} + 4 =$

$= 6 - \frac{4}{3} = \frac{14}{3}$

e) $\int_0^1 \frac{1}{(x+1)(x^2+4)}\,dx = \frac{1}{5}\int_0^1 \frac{1}{x+1} - \frac{x-1}{x^2+4}\,dx =$

$\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$

$1 = x^2(A+B) + x(B+C) + 4A + C$
$A+B=0 \rightarrow B=-A$
$B+C=0 \qquad \rightarrow -5A+1=0 \rightarrow A=\frac{1}{5} \rightarrow B=-\frac{1}{5} \rightarrow C=\frac{1}{5}$
$4A+C=1 \rightarrow C=1-4A$

$I = \frac{1}{5}\left[\ln(x+1)\Big|_0^1 - \int_0^1 \frac{x}{x^2+4}\,dx + \int_0^1 \frac{1}{x^2+4}\,dx\right]$

$= \frac{1}{5}\left[\ln 2 - \frac{1}{2}\ln(x^2+4)\Big|_0^1 + \frac{1}{2}\operatorname{arctg}\frac{x}{2}\Big|_0^1\right]$

$= \frac{1}{5}\left[\ln 2 - \frac{1}{2}(\ln 5 - \ln 4) + \frac{1}{2}\left(\operatorname{arctg}\frac{1}{2} - \operatorname{arctg} 0\right)\right]$

$= \frac{1}{5}\left[\ln 2 - \frac{1}{2}\ln 5 + \ln 2 + \frac{1}{2}\operatorname{arctg}\frac{1}{2}\right]$

$= \frac{1}{5}\left[\ln\frac{4}{\sqrt{5}} + \frac{1}{2}\operatorname{arctg}\frac{1}{2}\right]$

f) $\int_2^3 \frac{2x^3+x^2+2x-1}{x^4-1}dx = \int_2^3 \frac{2x^3+x^2+2x-1}{(x^2-1)(x^2+1)}dx = \int_2^3 \frac{2x^3+x^2+2x-1}{(x-1)(x+1)(x^2+1)}dx$

$$\frac{2x^3+x^2+2x-1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$2x^3+x^2+2x-1 = A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + C(x^3-x) + D(x^2-1)$

$2 = A+B+C$

$1 = A-B+D \rightarrow D = 1-A+B$

$2 = A+B-C$

$-1 = A-B-D$

$A+B+C = A+B-C \rightarrow C=-C \rightarrow C=0 \rightarrow A+B=2$

$\rightarrow A=B=D$

$-1 = A-B-1+A-B \rightarrow A-B=0 \rightarrow A=B$

$\left.\begin{matrix} A=B \\ D=1-A+B \end{matrix}\right\} \rightarrow D=1$

$I = \int_2^3 \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x^2+1}dx = \ln(x-1)\Big|_2^3 + \ln(x+1)\Big|_2^3 + \operatorname{arctg} x\Big|_2^3 =$

$= \ln 2 - \ln 1 + \ln 4 - \ln 3 + \operatorname{arctg} 3 - \operatorname{arctg} 2$

$= \ln\frac{8}{3} + \operatorname{arctg} 3 - \operatorname{arctg} 2$

g) $\int_0^1 \frac{x^3+2}{(x+1)^3}dx = \int_0^1 1 - \frac{3x^2+3x-1}{(x+1)^3}dx = \int_0^1 1 - \frac{3}{(x+1)} + \frac{3}{(x+1)^2} + \frac{1}{(x+1)^3}dx$

$$\frac{3x^2+3x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$3x^2+3x-1 = A(x^2+2x+1) + B(x+1) + C$

$3 = A$

$3 = 2A+B \rightarrow B = 3-6 = -3$

$-1 = A+B+C \rightarrow C=-1$

$I = x\Big|_0^1 - 3\ln(x+1)\Big|_0^1 + \frac{3}{x+1}\Big|_0^1 - \frac{1}{2(x+1)^2}\Big|_0^1$

$= 1 - 3\ln 2 - \frac{3}{2} + 3 - \frac{1}{8} + \frac{1}{2}$

$= 4 - 1 - \frac{1}{8} - 3\ln 2 = \frac{23}{8} - \ln 8$

7. a) $\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}} dx = \arcsin \frac{x}{2} \Big|_{-1}^{1} = \arcsin \frac{1}{2} - \arcsin\left(-\frac{1}{2}\right) =$

$= 2\arcsin \frac{1}{2} = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$

b) $\int_0^1 \frac{1}{\sqrt{x^2+x+1}} dx = \int_0^1 \frac{1}{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}} dx = \ln\left(x + \frac{1}{2} + \underbrace{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}}_{x^2+x+1}\right)\Big|_0^1 =$

$= \ln\left(\frac{3}{2} + \sqrt{1+1+1}\right) - \ln\left(\frac{1}{2} + \sqrt{0+0+1}\right) = \ln\left(\frac{3}{2} + \sqrt{3}\right) - \ln \frac{3}{2} =$

$= \ln\left(\frac{3+2\sqrt{3}}{2} \cdot \frac{2}{3}\right) = \ln\left(\frac{3+2\sqrt{3}}{3}\right)$

c) $\int_{-1}^{1} \frac{1}{\sqrt{4x^2+x+1}} dx = \int_{-1}^{1} \frac{1}{\sqrt{(2x+\frac{1}{4})^2 + \frac{15}{16}}} dx = \ln\left(2x + \frac{1}{4} + \sqrt{(2x+\frac{1}{4})^2 + \frac{15}{16}}\right)\Big|_{-1}^{1}$

$= \ln\left(2x + \frac{1}{4} + \sqrt{4x^2+x+1}\right)\Big|_{-1}^{1} = \ln\left(\frac{9}{4} + \sqrt{6}\right) - \ln\left(\sqrt{4} - \frac{7}{4}\right) =$

$= \ln \frac{9+4\sqrt{6}}{4} - \ln \frac{1}{4} = \ln(9+4\sqrt{6})$

d) $\int_2^3 \frac{x^2}{(x^2-1)\sqrt{x^2-1}} dx = -\ln\left|\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}+x}\right| \Big|_2^3 = \ln\left|\frac{\sqrt{3}-2}{\sqrt{3}+2}\right| - \ln\left|\frac{\sqrt{8}-3}{\sqrt{8}+3}\right|$

$x = \frac{1}{\cos t} \to dx = \frac{\sin t}{\cos^2 t} dt \quad , \cos t = \frac{1}{x} \to t = \arccos \frac{1}{x}$

$\int \frac{x^2}{(x^2-1)\sqrt{x^2-1}} dx = \int \frac{\frac{1}{\cos^2 t} \cdot \frac{\operatorname{tg} t}{\cos t}}{\left(\frac{1}{\cos^2 t} - 1\right)^{\frac{3}{2}}} dt =$

$= \int \frac{\operatorname{tg} t}{\cancel{\cos^3 t}} \cdot \frac{\cancel{\cos^3 t}}{(1-\cos^2 t)^{\frac{3}{2}}} dt = \int \frac{\operatorname{tg} t}{\sin^3 t} dt = \int \frac{\cancel{2\operatorname{tg}\frac{t}{2}}}{1-\operatorname{tg}^2\frac{t}{2}} \cdot \frac{1+\operatorname{tg}^2\frac{t}{2}}{\cancel{2\operatorname{tg}\frac{t}{2}}} dt$

$= \int \frac{1+\operatorname{tg}^2\frac{t}{2}}{1-\operatorname{tg}^2\frac{t}{2}} dt = \int \frac{1}{\cos t} dt = \int \frac{\cos t}{\cos^2 t} dt =$

$= \int \frac{\cos t}{1-\sin^2 t} dt = -\int \frac{\cos t}{\sin^2 t - 1} dt = -\ln\left|\frac{\sin t - 1}{\sin t + 1}\right| + C =$

$= -\ln\left|\frac{\sqrt{1-\cos^2 t} - 1}{\sqrt{1-\cos^2 t} + 1}\right| + C = -\ln\left|\frac{\sqrt{1-\frac{1}{x^2}} - 1}{\sqrt{1-\frac{1}{x^2}} + 1}\right| + C = -\ln\left|\frac{\sqrt{x^2-1} - x}{\sqrt{x^2-1} + x}\right| + C$

8. a) $\int_2^3 \sqrt{x^2+2x-7}\,dx =$

$$I=\int\sqrt{x^2+2x-7}\,dx = \int\sqrt{(x+1)^2-8}\,dx = \int\sqrt{t^2-8}\,dt$$

$$x+1=t \rightarrow dx=dt$$

$$t=\frac{\sqrt8}{\cos u} \rightarrow dt=\sqrt8\cdot\frac{\sin u}{\cos^2 u}\,du \quad , \quad u=\arccos\frac{\sqrt8}{t}$$

$$I=\int\sqrt{\frac{8}{\cos^2u}-8}\cdot\sqrt8\cdot\frac{\sin u}{\cos^2u}\,du = \int\frac{\sqrt{8(1-\cos^2u)}}{\cos u}\cdot\sqrt8\cdot\frac{\sin u}{\cos^2 u}\,du =$$

$$=\int\frac{\sqrt8\cdot\sin u}{\cos u}\cdot\sqrt8\cdot\frac{\sin u}{\cos^2u}\,du = 8\int tg^2u\cdot\frac{1}{\cos u}\,du =$$

$$=8\int tg^2u\,(1-tg^2u)\,du = 8\left[\int tg^2u\,du - \int tg^3u\,du\right] =$$

$$=8\left[-\int -tg^2u\,du + \int tg\,u\cdot tg^2u\,du\right]$$

$$=8\left[-\int 1+tg^2u-1\,du + \int tg\,u\left(\frac{1}{\cos^2u}-1\right)du\right]$$

$$=8\left[-tg\,u + u + \int tg\,u\,(tg\,u)' - tg\,u\,du\right]$$

$$=8\left[-tg\,u + u + \frac{tg^2u}{2} + \ln|\cos u|\right]+C$$

$$=8\left[-\frac{1}{\cos u}+u+\frac12\,\frac{1}{\cos^2u}+\ln|\cos u|\right]+C$$

$$=8\left[-\frac{1}{\frac{\sqrt8}{t}}+\arccos\frac{\sqrt8}{t}+\frac12\,\frac{1}{\frac{8}{t^2}}+\ln\left|\frac{\sqrt8}{t}\right|\right]+C$$

$$=8\left[-\frac{(x+1)}{\sqrt8}+\arccos\frac{\sqrt8}{x+1}+\frac{1}{16}(x+1)^2+\ln\left(\frac{\sqrt8}{x+1}\right)\right]+C$$

$$\int_2^3\sqrt{x^2+2x-7}\,dx = 8\left[\frac{(x+1)}{\sqrt8}+\arccos\frac{\sqrt8}{x+1}+\frac{1}{16}(x+1)^2+\ln\frac{\sqrt8}{x+1}\right]\Bigg|_2^3 =$$

$$=8\left\{\left[-\frac{2}{\sqrt2}+\arccos\frac{\sqrt2}{2}+\cancel{\frac{1}{16}}\cdot\cancel{16}+\ln\frac{\sqrt2}{2}\right]-\left[\frac{-3}{\sqrt8}+\arccos\frac{\sqrt8}{3}+\frac{9}{16}+\ln\frac{\sqrt8}{3}\right]\right\}$$

$$=\left\{\left[\sqrt2+\frac{\pi}{3}+1+\ln\frac{\sqrt2}{2}\right]-\left[\frac{3}{\sqrt8}+\arccos\frac{\sqrt8}{3}+\frac{9}{16}+\ln\frac{\sqrt8}{3}\right]\right\}$$

b) $\int_0 \sqrt{6+4x-2x^2}\cdot dx = \sqrt{2}\,(x-1)\Big|_0$

$$I=\sqrt{2}\int\sqrt{-x^2+2x+3}\;dx=\sqrt{2}\int\sqrt{-(x-1)^2+4}\;dx=\sqrt{2}\int\sqrt{4-t^2}\;dt$$

$x-1=t$

$dx=dt$

$t=2\sin u \rightarrow u=\arcsin\frac{t}{2}$

$dt=2\cos u\,du$

$$I=\sqrt{2}\int\sqrt{4-4\sin^2 u}\cdot 2\cos u\,du=2\sqrt{2}\int\sqrt{4(1-\cos^2 u)}\,\cos u\,du=$$

$$=4\sqrt{2}\int\sin u\cdot\cos u\,du=4\sqrt{2}\cdot\frac{\sin^2 u}{2}=2\sqrt{2}\sin^2 u=2\sqrt{2}\left(\frac{t}{2}\right)^2=\sqrt{2}\,\frac{t^2}{2}=$$

$$=\frac{\sqrt{2}}{2}(x-1)^2$$

c) $\int_0^{\frac{3}{4}}\frac{1}{(x+1)\sqrt{x^2+1}}\,dx=$

$x=tg\,t \rightarrow dx=(1+tg^2 t)\,dt \quad , \; t=arctg\,x$

$$\int\frac{1+tg^2 t}{(1+tg\,t)\sqrt{1+tg^2 t}}\,dt=\int\frac{\frac{1}{\cos^2 t}}{(1+tg\,t)\cdot\frac{1}{\cos t}}\,dt=\int\frac{1}{\cos t\,(1+tg\,t)}\,dt=$$

$$=\int\frac{1}{\frac{1-tg^2\frac{t}{2}}{1+tg^2\frac{t}{2}}\cdot\left(1+\frac{2tg\frac{t}{2}}{1-tg^2\frac{t}{2}}\right)}\,dt=\int\frac{1}{\frac{\cancel{1-tg^2\frac{t}{2}}}{1+tg^2\frac{t}{2}}\cdot\frac{1+2tg\frac{t}{2}-tg^2\frac{t}{2}}{\cancel{1-tg^2\frac{t}{2}}}}\,dt=$$

$$=\int\frac{1+tg^2\frac{t}{2}}{1+2tg\frac{t}{2}-tg^2\frac{t}{2}}\,dt=\int\frac{\cancel{1+u^2}}{1+2u-u^2}\cdot 2\cdot\frac{1}{\cancel{u^2+1}}\,du=-2\int\frac{1}{u^2-2u-1}\,du$$

$tg\frac{t}{2}=u \rightarrow t=2arctg\,u$

$dt=2\cdot\frac{1}{u^2+1}$

$$=-2\int\frac{1}{(u-\sqrt{2}-1)(u+\sqrt{2}-1)}\,du=-\frac{2}{2\sqrt{2}}\left[\int\frac{1}{u-\sqrt{2}-1}\,du-\int\frac{1}{u+\sqrt{2}-1}\,du\right]=$$

$$=-\frac{1}{\sqrt{2}}\left[\ln(u-\sqrt{2}-1)-\ln(u+\sqrt{2}-1)\right]=-\frac{1}{\sqrt{2}}\left[\ln\left(tg\frac{t}{2}-\sqrt{2}-1\right)-\ln\left(tg\frac{t}{2}+\sqrt{2}-1\right)\right]=$$

8.c).

$$I = -\frac{1}{\sqrt{2}}\left[\ln\left|tg\,\frac{arctg\,x}{2} - \sqrt{2} - 1\right| - \ln\left|tg\,\frac{arctg\,x}{2} + \sqrt{2} - 1\right|\right] + C$$

$$I\Big|_0^{\frac{3}{4}} = -\frac{1}{\sqrt{2}}\left\{\left[\ln\left|tg\,\frac{arctg\frac{3}{4}}{2} - \sqrt{2} - 1\right| - \ln\left|tg\,\frac{arctg\frac{3}{4}}{2} + \sqrt{2} - 1\right|\right] - \right.$$

$$\left. - \left[\ln\left|tg\,\frac{arctg\,0}{2} - \sqrt{2} - 1\right| - \ln\left|tg\,\frac{arctg\,0}{2} + \sqrt{2} - 1\right|\right]\right\}$$

$$= \frac{-1}{\sqrt{2}}\left\{\left[\ln\left|\frac{\sqrt{\frac{9}{16}+1} - 1}{\frac{3}{4}} - \sqrt{2} - 1\right| - \ln\left|tg\,\frac{\frac{5}{4} - 1}{\frac{3}{4}} + \sqrt{2} - 1\right|\right] - \right.$$

$$\left. - \left[\ln\left|-\sqrt{2} - 1\right| - \ln\left|\sqrt{2} - 1\right|\right]\right\}$$

$$= -\frac{1}{\sqrt{2}}\left\{\left[\ln\left|\frac{1}{3} - \sqrt{2} - 1\right| - \ln\left|\frac{1}{3} + \sqrt{2} - 1\right|\right] - \left[\ln\left|-\sqrt{2} - 1\right| - \ln\left|\sqrt{2} - 1\right|\right]\right.$$

d) $$\int_2^3 \frac{1}{x\sqrt{x^2-1}}\,dx = \arccos\frac{1}{x}\Big|_2^3 = \arccos\frac{1}{3} - \arccos\frac{1}{2} =$$

$$= \arccos\frac{1}{3} - \frac{\pi}{3}$$

$$x = \frac{1}{\cos t} \rightarrow t = \arccos\frac{1}{t}$$

$$dx = \frac{\sin t}{\cos^2 t}\,dt$$

$$I = \int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{\cos t}{\sqrt{\frac{1}{\cos^2 t} - 1}} \cdot \frac{\sin t}{\cos^2 t}\,dt = \int \frac{\cos t}{\frac{\sin t}{\cos t}}\;\frac{\sin t}{\cos^2 t}\,dt =$$

$$= \int dt = t = \arccos\frac{1}{x}$$

9. a) $$2\sqrt{2} < \int_{-1}^{1} \sqrt{x^2+4x+5}\,dx < 2\sqrt{10}$$

Let $f: [-1,1] \rightarrow \mathbb{R};\ f(x) = \sqrt{x^2+4x+5}$

$$f'(x) = \frac{2x+4}{2\sqrt{x^2+4x+5}} = \frac{x+2}{\sqrt{x^2+4x+5}} > 0,\ \forall x \in [-1,1] \rightarrow f\text{ - increasing on } [-1,1]$$

$$\rightarrow f(-1) < f(x) < f(1),\ \forall\ x \in (-1,1)$$

$$\sqrt{2} < f(x) < \sqrt{10}\quad \Big|\int_{-1}^{1}$$

$$\sqrt{2} \times 1\Big|_{-1}^{1} < \int_{-1}^{1} f(x)\,dx < \sqrt{10} \times 1\Big|_{-1}^{1}$$

$$\sqrt{2}(1-(-1)) < \int_{-1}^{1} f(x)\,dx < \sqrt{10}(1-(-1)) \rightarrow 2\sqrt{2} < \int_{-1}^{1}\sqrt{x^2+4x+5}\,dx < 2\sqrt{10}$$

b) $e^2(e-1) \int_e^{e^2} \frac{x}{\ln x}\,dx < \frac{e^3}{2}(e-1)$

Let $f : [e; e^2] \to \mathbb{R}$, $f(x) = \frac{x}{\ln x}$

$$f'(x) = \frac{\ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - \ln e}{\ln^2 x} = \frac{\ln \frac{x}{e}}{\ln^2 x} > 0, \ \forall x \in (e, e^2] \to$$

$\to f$ - increasing on $(e, e^2]$

$$f(e) < f(x) < f(e^2), \ \forall x \in (e, e^2)$$

$$e < f(x) < \frac{e^2}{2} \ \Big| \int_e^{e^2}$$

$$e \cdot x \Big|_e^{e^2} < \int_e^{e^2} f(x)\,dx < \frac{e^2}{2} x \Big|_e^{e^2}$$

$$e(e^2 - e) < \int_e^{e^2} f(x)\,dx < \frac{e^2}{2}(e^2 - e)$$

$$e^2(e-1) < \int_e^{e^2} \frac{x}{\ln x}\,dx < \frac{e^3}{2}(e-1)$$