Seminar 7 (2024)

1. It was found that the probability to log on to a computer from a remote terminal is 0.7. Let X denote the number of attempts, which are independent, that must be made to gain access to the computer:
a) Write the probability distribution of X.
b) Write the cumulative distribution function of X.
c) Compute the probability that at most 4 attempts must be made to gain access to the computer.
d) Compute the probability that at least 3 attempts must be made to gain access to the computer.

A: a) $X \sim \begin{pmatrix} k \\ (0.7)(0.3)^{k-1} \end{pmatrix}_{k\in\{1,2,3,\ldots\}}$.
Note that, $X-1$ has a geometric distribution with parameter $p = 0.7$.
b) The cumulative distribution function is $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = P(X \le x) = \begin{cases} 0, & \text{if} \quad x < 1 \\ 0.7, & \text{if} \quad 1 \le x < 2 \\ (0.7)[1 + (0.3)], & \text{if} \quad 2 \le x < 3 \\ & \ldots \\ (0.7)[1 + (0.3) + \cdots + (0.3)^{k-1}], & \text{if} \quad k \le x < k+1 \\ & \ldots \end{cases}.$$

In particular, using the formula for the sum of terms in geometric progression, we get

$$F(k) = P(X \le k) = 1 - (0.3)^k, \text{ for } k \in \{1, 2, \ldots\}.$$

c) $P(X \le 4) = F_X(4) = 1 - (0.3)^4$.

d) $P(X \ge 3) = 1 - P(X \le 2) = 1 - F_X(2) = (0.3)^2$.

2. The time, in minutes, it takes to reboot a certain system is a continuous variable with the density function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) \;=\; \begin{cases} c(4-x)^2, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute the constant c.
(b) Compute the probability that the system takes between 1 and 2 minutes to reboot.
(c) Compute the probability that the system takes at least 1 minute to reboot.

A: (a) Using the property (of density functions) that $\int_{\mathbb{R}} f(x)dx = 1$, we get

$$1 = \int_0^4 c(4-x)^2 dx = -c\frac{(4-x)^3}{3}\bigg|_0^4 = c \cdot \frac{64}{3} \implies c = \frac{3}{64}.$$

(b) $P(1 \le X \le 2) = \int_1^2 f(x)dx = -\frac{3}{64}\frac{(4-x)^3}{3}\Big|_1^2 = \frac{3}{64} \cdot \frac{27-8}{3} = \frac{19}{64}$.
(c) $P(X \ge 1) = \int_1^\infty f(x)dx = -\frac{3}{64}\frac{(4-x)^3}{3}\Big|_1^4 = \frac{3}{64} \cdot \frac{27}{3} = \frac{27}{64}$.

3. Find the density function of the volume V of a cube, whose edge X is a random variable uniformly distributed on $[0,2]$.

$$X \sim Unif[0,2] \Longleftrightarrow f(x) = \begin{cases} \frac{1}{2}, & x \in [0,2] \\ 0, & x \notin [0,2] \end{cases} \text{ is the density function of the } Unif[0,2] \text{ distribution}$$

A: The volume of the cube is the random variable $V = X^3$, where $X \sim Unif[0,2]$. We compute first the cumulative distribution function of V

$$F_V(v) = P(V \leq v) = P(X^3 \leq v) = \begin{cases} 0, & \text{if } v < 0 \\ P(X \leq \sqrt[3]{v}), & \text{if } 0 \leq v. \end{cases}$$

For $0 \leq \sqrt[3]{v} < 2$ we have $P(X \leq \sqrt[3]{v}) = \displaystyle\int_0^{\sqrt[3]{v}} \frac{1}{2} dx = \frac{\sqrt[3]{v}}{2}$, and for $\sqrt[3]{v} \geq 2$ we obtain $P(X \leq \sqrt[3]{v}) = \displaystyle\int_0^2 \frac{1}{2} dx = 1$. Then, for $0 < v < 8$: $F'_V(v) = \dfrac{1}{6\sqrt[3]{v^2}}$ and for $v \in \mathbb{R} \setminus [0,8]$: $F'_V(v) = 0$. Observe that F_V is not derivable at 0 and 8. It is known that $f_V(v) = F'_V(v)$, if F_V is derivable at v. Therefore, the density function of V is $f_V(v) = \begin{cases} \dfrac{1}{6\sqrt[3]{v^2}}, & \text{if } v \in (0,8) \\ 0, & \text{otherwise.} \end{cases}$

Note, that V has *not* a $Unif[0, 2^3]$ distribution.

4. The time to failure T, in hours of operating time, of a television set subject to random voltage surges has exponential $Exp(\frac{1}{500})$ distribution.
(a) Compute the cumulative distribution function of T.
(b) Compute the probability that the unit operates successfully more than 400 hours.
(c) Suppose the unit has operated successfully for 400 hours. What is the (conditional) probability it will operate for another 500 hours?

$$T \sim Exp\left(\frac{1}{500}\right) \Longleftrightarrow f_T(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{1}{500} e^{-\frac{t}{500}}, & \text{if } t > 0\,. \end{cases}$$

A: (a) The cumulative distribution function of T is

$$F_T(x) = \int_{-\infty}^{x} f_T(t)dt = \begin{cases} 0, & x \leq 0, \\ \int_0^x \frac{1}{500} e^{-\frac{t}{500}} dt = -e^{-\frac{t}{500}}\Big|_0^x = 1 - e^{-\frac{x}{500}}, & x > 0. \end{cases}$$

(b) $P(T > 400) = F(400) = e^{-0.8}$.
(c) $P(T > 400 + 500 | T > 400) = \frac{P(T>900)}{P(T>400)} = \frac{e^{-1.8}}{e^{-0.8}} = e^{-1}$.

5. A random number generator produces independently a sequence of numbers between 2 and 5. Each of these can be considered an observed value of a random variable uniformly distributed on the interval $[2,5]$. Ten numbers are generated. What is the probability that seven or more numbers are less than or equal to 4.7?

A: Let X be the random variable that shows how many of the generated random numbers are less than or equal to 4.7. Then $X \sim Bino(10, p)$, where $p = \int_{-\infty}^{4.7} f(x)dx$ is the probability that a randomly generated number is less than or equal to 4.7, where $f(x) = \begin{cases} \frac{1}{5-2}, & x \in [2,5] \\ 0, & x \notin [2,5] \end{cases}$ is the density function of the $Unif[2,5]$ distribution. We have $p = \int_2^{4.7} \frac{1}{3} dx = \frac{2.7}{3} = 0.9$ and thus $X \sim Bino(10, 0.9)$. So, $P(X \geq 7) = \sum_{k=7}^{10} C_{10}^k (0.9)^k (0.1)^{10-k}$.

6. Six identical electronic devices are installed at one time. The units fail independently, and the time to failure, in days, of each is a random variable with exponential distribution $Exp(\frac{1}{30})$. A maintenance check is made at fifteen days. What is the probability that at least four are still operating at the maintenance check?

A: Let X be number of operating devices at 15 days. Then $X \sim Bino(6, p)$, where $p = \int_{15}^{\infty} \frac{1}{30} e^{-\frac{t}{30}} dt = 1 - e^{-0.5}$ is the probability that the failure time of a device is more than 15 days. So, $P(X \geq 4) = \sum_{k=4}^{6} C_6^k (1 - e^{-0.5})^k (e^{-0.5})^{6-k}$.

7. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \begin{cases} 0, & \text{if } x < -4 \\ \dfrac{a(x+4)}{|x|+b}, & \text{if } x \geq -4, \end{cases}$$

where $a, b \in \mathbb{R}$ are parameters. For what values of $a, b \in \mathbb{R}$ the function F is the cumulative distribution function of a continuous random variable X? Find the density function of X when $P(-1 < X < 1) = 0.4$.

A: We use the properties of a distibution function. The condition $\lim_{x \to -\infty} F(x) = 0$ is verified, while $\lim_{x \to \infty} F(x) = 1$ implies $a = 1$. The function F is right-continuous. The derivative of F is a.s.

$$f(x) = \begin{cases} 0, & \text{if } x < -4, \\ \dfrac{b+4}{(b-x)^2}, & \text{if } -4 < x < 0, \\ \dfrac{b-4}{(b+x)^2}, & \text{if } 0 < x. \end{cases}$$

The function F is monotone increasing, if $F'(x) \geq 0$ for a.e. $x \in \mathbb{R}$. Therefore, $b \geq 4$. So, for $a = 1$ and $b \geq 4$ the function F is a cumulative distribution function, having the density function f.
$0.4 = \frac{2}{5} = P(-1 < X < 1) = F(1) - F(-1) = \frac{5}{1+b} - \frac{3}{1+b} = \frac{2}{1+b} \implies b = 4$. Hence,

$$f(x) = \begin{cases} \frac{8}{(x-4)^2}, & x \in (-4, 0), \\ 0, & x \notin (-4, 0). \end{cases}$$