Seminar 4

1. Four electronic devices have the property that, for every $i \in \{1,2,3,4\}$, the probability that any i fixed devices are all functional is $\dfrac{1}{4^i}$. Using the inclusion-exclusion principle, compute the probability of the event A:"none of the devices is functional".

A: We will compute $P(\bar{A})$, where $\bar{A}$ is the event that at least one device is functional. For $i \in \{1,2,3,4\}$, let A_i be the event that the ith device is functional. Then $\bar{A} = A_1 \cup A_2 \cup A_3 \cup A_4$. By the inclusion-exclusion principle, we have:

$$\begin{aligned}
P(\bar{A}) = {} & P(A_1) + P(A_2) + P(A_3) + P(A_4) \\
& - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4) \\
& + P(A_2 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_3) \\
& - P(A_1 \cap A_2 \cap A_3 \cap A_4)
\end{aligned}$$

and thus $P(A) = 1 - P(\bar{A}) = 1 - \left(4 \cdot \frac{1}{4} - 6 \cdot \frac{1}{4^2} + 4 \cdot \frac{1}{4^3} - \frac{1}{4^4}\right) = \frac{6 \cdot 16 - 4 \cdot 4 + 1}{4^4} = \frac{81}{256} \approx 0.316\,.$

2. Four antivirus programs are tested by scanning independently an infected file. They detect the virus with corresponding probabilities: $\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{2}{4}, \dfrac{1}{4}$. Compute the probabilities of the following events:
A:"All programs detect the virus."
B:"Exactly one program detects the virus."
C:"Exactly three programs detect the virus."
D:"At most one program detects the virus."
E:"At least one program detects the virus."

A: Let V_n:"The nth program detects the virus.", $k = \overline{1,4}$.

$$P(A) = P(V_1 \cap V_2 \cap V_3 \cap V_4) = P(V_1) \cdot P(V_2) \cdot P(V_3) \cdot P(V_4) = \frac{3}{128} \approx 0.023.$$

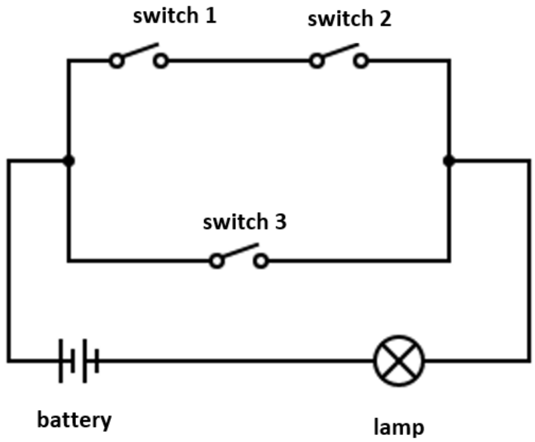
$$\begin{aligned}
P(B) &= P(V_1 \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) + P(\overline{V_1} \cap V_2 \cap \overline{V_3} \cap \overline{V_4}) + P(\overline{V_1} \cap \overline{V_2} \cap V_3 \cap \overline{V_4}) + P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap V_4) \\
&= \frac{3 \cdot 3 \cdot 2 \cdot 3 + 1 \cdot 1 \cdot 2 \cdot 3 + 1 \cdot 3 \cdot 2 \cdot 3 + 1 \cdot 3 \cdot 2 \cdot 1}{256} = \frac{84}{256} = \frac{21}{64} \approx 0.382.
\end{aligned}$$

$$\begin{aligned}
P(C) &= P(\overline{V_1} \cap V_2 \cap V_3 \cap V_4) + P(V_1 \cap \overline{V_2} \cap V_3 \cap V_4) + P(V_1 \cap V_2 \cap \overline{V_3} \cap V_4) + P(V_1 \cap V_2 \cap V_3 \cap \overline{V_4}) \\
&= \frac{1 \cdot 1 \cdot 2 \cdot 1 + 3 \cdot 3 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 2 \cdot 3}{256} = \frac{44}{256} = \frac{11}{64} \approx 0.171.
\end{aligned}$$

$$P(D) = P(B) + P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) = \frac{84}{256} + \frac{1 \cdot 3 \cdot 2 \cdot 3}{256} = \frac{102}{256} = \frac{51}{128} \approx 0.398.$$

$$P(E) = 1 - P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) = 1 - \frac{18}{256} = \frac{238}{256} = \frac{119}{128} \approx 0.929.$$

3. In the diagram below the three switches are either ON or OFF, independently, with probability $\frac{1}{2}$ for each state. Compute the probability that the circuit operates.



A: Let S_i:"Switch i is ON", $i = \overline{1,3}$. Using the independence of the switches, we compute

$$\begin{aligned} P(\text{"circuit operates"}) &= P((S_1 \cap S_2) \cup S_3) = P(S_1 \cap S_2) + P(S_3) - P(S_1 \cap S_2 \cap S_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{2+4-1}{8} = \frac{5}{8}. \end{aligned}$$

4. The owner of three shops decides to give a bonus to the salary of a randomly chosen employee. The first shop has 50 employees and 50% of them are men, the second shop has 75 employees and 60% of them are men and the third shop has 100 employees and 70% are men.
a) Find the probability that the lucky employee works in the third shop, given that the lucky employee is a woman.
b) Find the probability that the lucky employee is a woman, given that the lucky employee works in the third shop.

A: Let S_3:"The lucky employee works in the third shop." and W:"The lucky employee is a woman."
a) $P(S_3|W) = \frac{P(S_3 \cap W)}{P(W)} = \frac{\frac{30}{225}}{\frac{25+30+30}{225}} = \frac{30}{85} = \frac{6}{17}$. b) $P(W|S_3) = \frac{P(W \cap S_3)}{P(S_3)} = \frac{\frac{30}{225}}{\frac{100}{225}} = \frac{30}{100} = \frac{3}{10}$.

5. Three dice are rolled. Let N_k be number that showed on the kth die, $k \in \{1, 2, 3\}$. Find:
a) $P(N_1 = 1, N_2 = 2, N_3 = 3)$.
b) $P(N_1 = N_2 = N_3)$.
c) $P(N_1 + N_2 + N_3 \geq 5)$.
d) $P(N_1 + N_2 + N_3 \geq 5 | N_1 < N_2 < N_3)$.
e) $P(N_1 < N_2 < N_3 | N_1 < N_2)$.
f) $P(N_1 > N_2 < N_3 | N_1 = N_3)$.
g) $P(N_1 = N_2, N_2 > 2 | N_3 > 2)$.

A: a) $P(N_1 = 1, N_2 = 2, N_3 = 3) = P(N_1 = 1)P(N_2 = 2)P(N_3 = 3) = \frac{1}{6^3} = \frac{1}{216}$.

b) $P(N_1 = N_2 = N_3) = \sum_{i=1}^{6} P(N_1 = N_2 = N_3 = i) = \frac{6}{6^3} = \frac{1}{36}$.
c) $P(N_1 + N_2 + N_3 \geq 5) = 1 - P(N_1 + N_2 + N_3 \in \{3, 4\}) = 1 - \frac{4}{6^3} = \frac{212}{216} = \frac{53}{54}$.

d) $P(N_1 + N_2 + N_3 \geq 5 | N_1 < N_2 < N_3) = 1$, because $N_1 < N_2 < N_3 \implies N_1 \geq 1, N_2 \geq 2, N_3 \geq 3 \implies N_1 + N_2 + N_3 \geq 6$.

e) $P(N_1 < N_2 < N_3 | N_1 < N_2) = \frac{P(N_1 < N_2 < N_3)}{P(N_1 < N_2)} = \frac{\frac{C_6^3}{6^3}}{\frac{C_6^2}{6^2}} = \frac{C_6^3}{C_6^2 \cdot 6} = \frac{20}{90} = \frac{2}{9}$.

f) $P(N_1 > N_2 < N_3 | N_1 = N_3) = \frac{P(N_1=N_3>N_2)}{P(N_1=N_3)} = \frac{\frac{C_6^2}{6^3}}{\frac{6}{6^2}} = \frac{C_6^2}{6^2} = \frac{15}{36} = \frac{5}{12}$.

g) $P(N_1 = N_2, N_2 > 2 | N_3 > 2) = \frac{\sum\limits_{i=3}^{6}\sum\limits_{j=3}^{6} P(N_1=i)P(N_2=i)P(N_3=j)}{P(N_3>2)} = \sum\limits_{i=3}^{6} P(N_1 = i)P(N_2 = i) = \frac{4}{36} = \frac{1}{9}$, here we used that $\{N_1 = i\}$, $\{N_2 = i\}$ and $\{N_3 = j\}$ are independent events, for all $i, j \in \{3, 4, 5, 6\}$.

6. A fair coin is tossed infinitely many times. Compute the probability of the events:
a) A:"All tosses show heads."
b) B:"At least one toss shows head."

A: a) Let A_n:"The first n tosses show heads.", $n \in \mathbb{N}^*$. Since the tosses are independent, $P(A_n) = \frac{1}{2^n}$. Since $(A_n)_{n\geq 1}$ is a sequence of decreasing events and $A = \bigcap_{n=1}^{\infty} A_n$, we deduce, by Theorem 4 from the course, that $P(A) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \frac{1}{2^n} = 0$.
b) Let B_n:"At least one toss in the first n tosses show head.", $n \in \mathbb{N}^*$. Since the tosses are independent, $P(B_n) = 1 - P(\overline{B_n}) = 1 - \frac{1}{2^n}$. Since $(B_n)_{n\geq 1}$ is a sequence of increasing events and $B = \bigcup_{n=1}^{\infty} B_n$, we deduce, by Theorem 4 from the course, that $P(B) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$.