Calculus - Homework 12 - Sequences of functions

Study the pointwise convergence and the uniform convergence

1. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\cos nx}{n^\alpha}$, $\alpha > 0$

- Choose a random $x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\cos nx}{n^\alpha} = 0 \to B = \mathbb{R}$$

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad f_n \xrightarrow[\mathbb{R}]{} f$$

- Choose $n \in \mathbb{N}$ - random

$$a_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{|\cos nx|}{n^\alpha}$$

$$0 \le |\cos nx| \le 1 \ \Big|\cdot \frac{1}{n^\alpha} \to 0 \le \frac{|\cos nx|}{n^\alpha} \le \frac{1}{n^\alpha} \to a_n = \frac{1}{n^\alpha}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^\alpha} = 0 \to f_n \underset{B}{\rightrightarrows} f$$

2. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x(1+n^2)}{n^2}$

- Choose $x \in [0,1]$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x(1+n^2)}{n^2} = x \in [0,1] \to B = [0,1]$$

$$f : [0,1] \to \mathbb{R},\ f(x) = x \qquad f_n \xrightarrow[B]{} f$$

- Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left| \frac{x(1+n^2)}{n^2} - x \right| = \sup_{x\in[0,1]} \left| \frac{x(1+n^2-n^2)}{n^2} \right| =$$

$$= \sup_{x\in[0,1]} \left| \frac{x}{n^2} \right| = \sup_{x\in[0,1]} \frac{x}{n^2} = \frac{1}{n^2}$$

$$0 \le x \le 1$$

$$0 \le \frac{x}{n^2} \le \frac{1}{n^2} \to a_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0 \to f_n \rightrightarrows f$$

3. $f_n : \mathbb{R} \to \mathbb{R}, \; f_n(x) = \frac{x^2}{x^4+n^2}$

- choose $x \in \mathbb{R}$ random

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{x^4+n^2} = 0 \to \mathcal{C} = \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}, \; f(x) = 0, \forall x \in \mathbb{R} \quad f_n \xrightarrow[\mathbb{R}]{} f$

- choose $n \in \mathbb{N}$ random

$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \left|\frac{x^2}{x^4+n^2}\right| = \sup_{x\in\mathbb{R}} \frac{x^2}{x^4+n^2}$

$g: \mathbb{R} \to \mathbb{R}, \; g(x) = \frac{x^2}{x^4+n^2}$

g - diff on $\mathbb{R}$

$g'(x) = \frac{2x(x^4+n^2) - x^2 \cdot 4x^3}{(x^4+n^2)^2} = \frac{2x^5 + 2xn^2 - 4x^5}{(x^4+n^2)^2} =$

$= \frac{-2x(x^4-n^2)}{(x^4+n^2)^2} =$

$g'(x) = 0 \leftrightarrow x = 0$ or $x^4 - n^2 = 0 \leftrightarrow x = \pm\sqrt{n}$

x	$-\infty$		$-\sqrt{n}$		0		$\sqrt{n}$		$+\infty$
$-2x$		+ + +	+	+	0	− −	−	− −	
x^4-n^2		+ + +	0	− − −	−	− −	0	+ + +	
$g'(x)$		+ +	0	− −	0	+ +	0	− −	
$g(x)$		$\nearrow$	$\frac{1}{2n}$	$\searrow$		$\nearrow$	$\frac{1}{2n}$	$\searrow$	

$\to \forall x \in \mathbb{R} \; g(x) \le \max\{g(-\sqrt{n}), g(\sqrt{n})\}$

$g(-\sqrt{n}) = \frac{n}{n^2+n^2} = \frac{1}{2n} = g(\sqrt{n}) \to g(x) \le \frac{1}{2n}, \forall x \in \mathbb{R} \to a_n = \frac{1}{2n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n} = 0 \in \mathbb{R} \to f_n \overset{\mathbb{R}}{\rightrightarrows} f$

4. $f_n : [0,\infty) \to \mathbb{R}, \; f_n(x) = \frac{1}{1+nx}$

- choose $x \in \mathbb{R}$ random

$\lim_{n\to\infty} f_n(x) = 0 \to \mathcal{C} = [0,\infty)$

$f: [0,\infty) \to \mathbb{R}, \; f(x) = 0: \; f_n \xrightarrow[\mathcal{C}]{} f$

- choose $n \in \mathbb{N}$ random

$a_n := \sup_{x\in\mathcal{B}} |f_n(x) - f(x)| = \sup_{x\in\mathcal{B}} \left|\frac{1}{1+nx}\right| = \sup_{x\in\mathcal{B}} \frac{1}{1+nx}$

$g: [0,\infty) \to \mathbb{R}, \quad g(x) = \dfrac{1}{1+nx}$

g - cont. on $[0,\infty)$,

$g'(x) = \dfrac{-n}{(1+nx)^2}$

$g'(x) \neq 0, \ \forall x \in [0,\infty)$

$g'(x) < 0$

x	$[0$	∞
$g'(x)$	$- - - - - - -$	
$g(x)$	$\searrow \searrow \searrow$	

$\to g(x) < g(0), \ \forall x \in [0,\infty)$

$g(x) < 1, \ \forall x \in [0,\infty) \to a_n = 1 \to \lim\limits_{n\to\infty} a_n = 1 \neq 0 \to f_n \not\rightrightarrows f$

5. $f_n: \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \dfrac{2n^2x}{e^{n^2x^2}}$

- Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \begin{cases} 0 : x = 0 \\ \frac{\infty}{\infty} : x \neq 0 \end{cases}$$

$x \neq 0$

$$\lim_{n\to\infty} f_n(x) = x \lim_{n\to\infty} \frac{2n^2}{e^{n^2x^2}} = 2x \cdot \frac{1}{x^2} \cdot \lim_{n\to\infty} \frac{n^2x^2}{e^{n^2x^2}} = \frac{2x}{x^2} \cdot 0 = 0 \to \boxed{C = \mathbb{R}}$$

Let $h: (0,\infty) \to \mathbb{R}, \ h(t) = \dfrac{t}{e^t}$

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{t}{e^t} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty_+} = 0 \to \lim_{n\to\infty} \frac{n^2x^2}{e^{n^2x^2}} = 0$$

$f: \mathbb{R} \to \mathbb{R}, \ f(x) = 0 \qquad f_n \xrightarrow[\mathbb{R}]{} f$

- Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - 0| = \sup_{x\in\mathbb{R}} \left| \frac{2n^2x}{e^{n^2x^2}} \right| = \sup_{x\in\mathbb{R}} \frac{2n^2|x|}{e^{n^2x^2}}$$

Let $g: \mathbb{R} \to \mathbb{R}, \ g(x) = \dfrac{2n^2|x|}{e^{n^2x^2}}$

$\lim\limits_{x\to 0} g(x) = 0 \to g$ - continuous at 0
— // — on $\mathbb{R}\setminus\{0\}$
$\Big\} \to g$ - cont. on $\mathbb{R}$

$$\lim_{\substack{x\to 0\\x<0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\x<0}} \frac{-2n^2x}{x} = -2n^2$$

$$\lim_{\substack{x\to 0\\x>0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\x>0}} \frac{2n^2x}{x} = 2n^2$$

$\Bigg\} \to \lim\limits_{\substack{x\to 0\\x<0}} \dfrac{g(x)-g(0)}{x-0} \neq \lim\limits_{\substack{x\to 0\\x>0}} \dfrac{g(x)-g(0)}{x-0} \to$

$\to g$ is not diff. at 0

• $x < 0$

$$g'(x) = \left(\frac{-2n^2x}{e^{n^2x^2}}\right)' = \frac{-2n^2 \cdot e^{n^2x^2} + 2n^2x\, e^{n^2x^2}(2n^2x)}{e^{2n^2x^2}} =$$

$$= \frac{\cancel{e^{n^2x^2}}(-2n^2 + 4n^4x^2)}{e^{\cancel{2}n^2x^2}} = \frac{4n^4x^2 - 2n^2}{e^{n^2x^2}} = 2n^2 \frac{2n^2x^2 - 1}{e^{n^2x^2}}$$

• $x > 0$

$$g'(x) = \left(\frac{2n^2x}{e^{n^2x^2}}\right)' = \frac{2n^2e^{n^2x^2} - 4n^2x^2\, e^{n^2x^2}}{e^{2n^2x^2}} = -2n^2 \frac{2n^2x^2 - 1}{e^{n^2x^2}}$$

$$g'(x) = 0 \Leftrightarrow 2n^2x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{2n^2} \Leftrightarrow x = \pm\frac{1}{\sqrt{2}n}$$

x	$-\infty$	$-\frac{1}{n\sqrt{2}}$		0		$\frac{1}{n\sqrt{2}}$		∞
$2n^2x^2-1$	$+ +$	0	$- -$		$- -$	0	$+ +$	
$g'(x)$	$+ +$	$+0$	$- -$	$\vert$	$++$	0	$- - -$	
$g(x)$	$\nearrow$	$\sqrt{\frac{2}{e}}n$	$\searrow$	0	$\nearrow$	$\sqrt{\frac{2}{e}}n$	$\searrow$	

$$g\left(-\frac{1}{n\sqrt{2}}\right) = \frac{2n^{\cancel{2}} \cdot \left|-\frac{1}{\cancel{n}\sqrt{2}}\right|}{e^{\frac{\cancel{n^2}}{\cancel{n^2}\cdot 2}}} = \frac{\sqrt{2}n}{\sqrt{e}} = \sqrt{\frac{2}{e}}n$$

$$g\left(\frac{1}{n\sqrt{2}}\right) = \frac{2n^2\left|\frac{1}{n\sqrt{2}}\right|}{e^{\frac{n^2}{n^2\cdot 2}}} = \sqrt{\frac{2}{e}}n$$

$g(x) \le g\left(\frac{1}{n\sqrt{2}}\right), \forall x \in \mathbb{R} \to g(x) \le \sqrt{\frac{2}{e}}n, \forall x \in \mathbb{R} \to a_n = \sqrt{\frac{2}{e}}n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{\frac{2}{e}}n = \infty \neq 0 \to f_n \not\rightrightarrows f$

6. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$

• Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \frac{x}{x^2} \lim_{n\to\infty} \frac{n}{\frac{1}{x^2} + n^2} = \frac{1}{x}\cdot 0 = 0 \to C = \mathbb{R}$$

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ $\qquad f_n \xrightarrow[\mathbb{R}]{} f$

• Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\frac{nx}{1+n^2x^2}\right| = \sup_{x\in\mathbb{R}} \frac{n|x|}{1+n^2x^2}$$

Let $g: \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{n \cdot |x|}{1+n^2x^2}$

$\lim_{x \to 0} g(x) = 0 \to g$ - continuous on $\mathbb{R}$

$$\lim_{\substack{x \to 0 \\ x<0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x \to 0 \\ x<0}} \frac{-nx}{1+n^2x^2} = \frac{0}{1} = 0$$

$$\lim_{\substack{x \to 0 \\ x>0}} \frac{g(x)-g(0)}{x-0} \lim_{\substack{x \to 0 \\ x>0}} \frac{nx}{1+n^2x^2} = \frac{0}{1} = 0$$

$\to g'(0) = 0$

$\to g$ is diff. at $0 \to$

$\to g$ is diff on $\mathbb{R}$

- $x<0$

$$g'(x) = \left(\frac{-nx}{1+n^2x^2}\right)' = \frac{-n(1+n^2x^2)+nx(2n^2x)}{(1+n^2x^2)} = \frac{-n-n^3x^2+2n^3x^2}{(1+n^2x^2)^2} =$$

$$= \frac{n^3x^2-n}{(1+n^2x^2)^2}$$

- $x>0$

$$g'(x) = \left(\frac{nx}{1+n^2x^2}\right)' = \frac{n(1+n^2x^2)-nx(2n^2x)}{(1+x^2n^2)^2} = -\frac{n^3x^2-n}{(1+x^2n^2)^2}$$

$g'(x) = 0$ $\to x = 0$

$\to n^3x^2 - n = 0 \to x^2 = \frac{1}{n^2} \to x = \pm\frac{1}{n}$

x	$-\infty$	$-\frac{1}{n}$	0	$\frac{1}{n}$	$+\infty$
n^3x^2-n	$+ + +$	0	$- - - -$	0	$+ + +$
$g'(x)$	$+ + +$	0	$- - 0 + +$	0	$- - -$
$g(x)$	$\nearrow$	$\frac{1}{2}$	$\searrow 0 \nearrow$	$\frac{1}{2}$	$\searrow$

$\to g(x) \le g(\frac{1}{n}), \forall x \in \mathbb{R} \to$

$$g\left(-\frac{1}{n}\right) = g\left(\frac{1}{n}\right) = \frac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} = \frac{1}{2}$$

$\to g(x) \le \frac{1}{2} \to a_n = \frac{1}{2} \to \lim_{n \to \infty} a_n = \frac{1}{2} \ne 0 \to f_n \not\rightrightarrows f$

7. $f_n: \mathbb{R} \to \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$

- choose $x \in \mathbb{R}$ random

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \sqrt{x^2 + \underbrace{\frac{1}{n^2}}_{0}} = \sqrt{x^2} = |x| \in \mathbb{R} \to C = \mathbb{R}$$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ $\quad f_n \xrightarrow[\mathbb{R}]{} f$

• Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right|$$

$$g: \mathbb{R} \to \mathbb{R},\ g(x) = \sqrt{x^2+\frac{1}{n^2}} - |x| = \begin{cases} \sqrt{x^2+\frac{1}{n^2}} + x : x < 0 \\ \sqrt{x^2+\frac{1}{n^2}} - x : x > 0 \end{cases}$$

$$\lim_{x\to 0} g(x) = \frac{1}{n} \to g \text{ - cont on } \mathbb{R}$$

$$\lim_{\substack{x\to 0\\ x<0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{\sqrt{x^2+\frac{1}{n^2}} + x - \frac{1}{n}}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{\frac{\sqrt{n^2x^2+1}}{n} + x - \frac{1}{n}}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{\frac{1}{\cancel{n}} \cdot \frac{n^{\cancel{2}} \cdot \cancel{2}x}{\cancel{2}\sqrt{n^2x^2+1}} + 1}{1} = \lim_{\substack{x\to 0\\ x<0}} \frac{nx}{\sqrt{n^2x^2+1}} + 1 = 0 + 1 = 1$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{\frac{\sqrt{n^2x^2+1}}{n} - x - \frac{1}{n}}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{\frac{nx}{\sqrt{n^2x^2+1}} - 1}{1} = -1$$

$\to$ g is not dif at 0

• $x < 0$

$$g'(x) = \left(\sqrt{x^2+\frac{1}{n^2}} + x\right)' = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+\frac{1}{n^2}}} + 1 = \frac{nx}{\sqrt{x^2n^2+1}} + 1$$

• $x > 0$

$$g'(x) = \left(\sqrt{x^2+\frac{1}{n^2}} - x\right)' = \frac{nx}{\sqrt{x^2n^2+1}} - 1$$

$$g'(x) = 0 \iff nx + \sqrt{x^2n^2+1} = 0 \to nx = -\sqrt{x^2n^2+1} \ \big|^2 \to n^2x^2 = x^2n^2+1 \to x \in \emptyset$$

$$nx - \sqrt{x^2n^2+1} = 0 \to nx = \sqrt{x^2n^2+1} \to x \in \emptyset$$

$\to g'(x) \neq 0$

$$\frac{nx}{\sqrt{x^2n^2+1}} + 1 \ \boxed{>}\ 0 \qquad \frac{nx}{\sqrt{x^2n^2+1}} - 1 \ \boxed{<}\ 0$$

$$\frac{nx}{\sqrt{x^2n^2+1}} \ \boxed{>}\ -1 \qquad nx \ \boxed{<}\ \sqrt{x^2n^2+1}$$

$$nx \ \boxed{>}\ -\sqrt{x^2n^2+1}$$

$$n^2x^2 \ \boxed{<}\ x^2n^2+1$$

x	$-\infty$	0	$+\infty$
$g'(x)$	$+ \quad + \quad + \quad +$	0	$- \quad - \quad - \quad - \quad -$
$g(x)$	$\nearrow$	$\frac{1}{n}$	$\searrow$

$\rightarrow g(x) \leq g(0), \forall x \in \mathbb{R}$

$g(x) \leq \frac{1}{n}, \forall x \in \mathbb{R} \rightarrow a_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} a_n = 0 \rightarrow f_n \rightrightarrows f$

8. $f_n : \overset{[0,\infty)}{\mathbb{R}} \to \mathbb{R}, \; f_n(x) = n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right)$

- Choose $x \in \mathbb{R}$ random

$$\lim_{n\to\infty} n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) = \lim_{n\to\infty} n\left(\frac{\sqrt{nx+1}}{\sqrt{n}} - \sqrt{x}\right) = \lim_{n\to\infty} n \cdot \frac{\sqrt{nx+1} - \sqrt{nx}}{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \sqrt{n}\left(\sqrt{nx+1} - \sqrt{nx}\right) = \lim_{n\to\infty} \sqrt{n} \cdot \frac{nx+1-nx}{\sqrt{nx+1}+\sqrt{nx}} = \lim_{n\to\infty} \cancel{\sqrt{n}} \cdot \frac{1}{\cancel{\sqrt{n}}\underbrace{\left(\sqrt{x+\frac{1}{n}}+\sqrt{x}\right)}_{>0}} =$$

$$= \frac{1}{2\sqrt{x}} \in \mathbb{R} \rightarrow \mathcal{C} = [0,\infty)$$

$f : \overset{[0,\infty)}{\mathbb{R}} \to \mathbb{R}, \; f(x) = \frac{1}{2\sqrt{x}} \qquad f_n \overset{\rightrightarrows}{\underset{\mathbb{R}}{}} f$

- choose $n \in \mathbb{N}$ random

$$a_n = \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| n\left(\sqrt{x+\frac{1}{n}} - \sqrt{x}\right) - \frac{1}{2\sqrt{x}} \right| = \sup_{x\in[0,\infty)} \left| \sqrt{n(nx+1)} - n\sqrt{x} - \frac{1}{2\sqrt{x}} \right|$$

$g : [0,\infty) \to \mathbb{R}, \; g(x) = \left| \sqrt{n(nx+1)} - n\sqrt{x} - \frac{1}{2\sqrt{x}} \right|$

9. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{nx}{e^{nx^2}}$

- choose $x \in [0,1]$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = x \lim_{n\to\infty} \frac{n}{e^{nx^2}} = \frac{x}{x^2} \lim_{n\to\infty} \frac{nx^2}{e^{nx^2}} = \frac{1}{x} \lim_{t\to\infty} \frac{t}{e^t} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}}$$

$$= \frac{1}{x} \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{x} \cdot 0 = 0 \to C = [0,1]$$

$f : [0,1] \to \mathbb{R},\ f(x) = 0$

- choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left| \frac{nx}{e^{nx^2}} \right| = \sup_{x\in[0,1]} \frac{nx}{e^{nx^2}}$$

$g : [0,1] \to \mathbb{R},\ g(x) = \dfrac{nx}{e^{nx^2}}$

$$g'(x) = \frac{n \cdot e^{nx^2} - nx \cdot e^{nx^2} \cdot 2nx}{e^{2nx^2}} = \frac{n - 2n^2x^2}{e^{nx^2}}$$

$$g'(x) = 0 \to n - 2n^2x^2 = 0 \to x^2 = \frac{1}{2n} \to \left. \begin{array}{l} x = \pm \frac{1}{\sqrt{2n}} \\ x \in [0,1] \end{array} \right\} \to x = \frac{1}{\sqrt{2n}}$$

x	0		$\frac{1}{\sqrt{2n}}$		1
$g'(x)$	$+$	$+$ $+$	0	$-$ $-$ $-$	$-$
$g(x)$		$\nearrow$	$\frac{\sqrt{n}}{\sqrt{2e}}$	$\searrow$	

$$g\left(\frac{1}{\sqrt{2n}}\right) = \frac{\frac{1}{\sqrt{2}}\sqrt{n}}{e^{\cancel{n} \cdot \frac{1}{2n}}} = \frac{\sqrt{n}}{\sqrt{2e}} \to g(x) \le g\left(\frac{1}{\sqrt{2n}}\right),\ \forall x \in [0,1]$$

$$\to g(x) \le \frac{\sqrt{n}}{\sqrt{2e}},\ \forall x \in [0,1] \to a_n = \frac{\sqrt{n}}{\sqrt{2e}} \to \lim_{n\to\infty} a_n = \infty \ne 0 \to f_n \not\rightrightarrows f$$

10. $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \dfrac{x(1+n^2)}{n^2}$

- choose $x \in [0,1]$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x(1+n^2)}{n^2} = x \lim_{n\to\infty} \frac{1+n^2}{n^2} = x \in [0,1] \to C = [0,1]$$

$f : [0,1] \to \mathbb{R},\ f(x) = x$

- choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left| \frac{x + xn^2}{n^2} - x \right| = \sup_{x\in[0,1]} \left| \frac{x + xn^2 - xn^2}{n^2} \right| =$$

$$= \sup_{x\in[0,1]} \frac{x}{n^2}$$

$$0 \le x \le 1$$

$$0 \le \frac{x}{n^2} \le \frac{1}{n^2} \to a_n = \frac{1}{n^2} \to \lim_{n\to\infty} a_n = 0 \to f_n \overset{\rightrightarrows}{\underset{C}{}} f$$

11. $f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{1+n^2x^2}$

• choose $x \in [-1,1]$ random

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+n^2x^2} = 0 \in \mathbb{R} \to \mathcal{C} = [-1,1]$$

$$f : [-1,1] \to \mathbb{R},\quad f(x) = 0 \qquad f_n \to f.$$

• Choose $n \in \mathbb{N}$ random

$$a_n := \sup_{x\in[-1,1]} \left| f_n(x) - f(x) \right| = \sup_{x\in[-1,1]} \frac{|x|}{1+n^2x^2}$$

$$g : [-1,1] \to \mathbb{R},\quad g(x) = \frac{|x|}{1+n^2x^2} = \begin{cases} \dfrac{-x}{1+n^2x^2} & : x < 0 \\ \dfrac{x}{1+n^2x^2} & : x > 0 \end{cases}$$

$\lim\limits_{x\to 0} g(x) = 0 \to g$ - cont. on $[-1,1]$

$$\left.\begin{aligned} &\lim_{\substack{x\to 0\\ x<0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{\frac{-x}{1+n^2x^2}}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{-1}{1+n^2x^2} = -1 \\ &\lim_{\substack{x\to 0\\ x>0}} \frac{g(x)-g(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{x}{x(1+n^2x^2)} = \lim_{\substack{x\to 0\\ x>0}} \frac{1}{1+n^2x^2} = 1 \end{aligned}\right\} \to g \text{ is not dif at } 0$$

• $x < 0$

$$g'(x) = \frac{-(1+n^2x^2) + x \cdot 2n^2x}{(1+n^2x^2)^2} = \frac{-1-n^2x^2+2n^2x^2}{(1+n^2x^2)^2} = \frac{n^2x^2-1}{(1+n^2x^2)^2}$$

• $x > 0$

$$g'(x) = \frac{1+n^2x^2 - 2n^2x^2}{(1+n^2x^2)^2} = \frac{1-n^2x^2}{(1+n^2x^2)^2} = -\frac{n^2x^2-1}{(1+n^2x^2)^2}$$

$$g'(x) = 0 \to n^2x^2 - 1 = 0 \to x = \pm\frac{1}{n}$$

x	$-\infty$	$[-1$		$-\frac{1}{n}$		0		$\frac{1}{n}$		$1]$	$+\infty$
$g'(x)$	$+$	$+$	$+$	0	$-$	$\vert$	$+$	0	$-$	$-$	$-$
$g(x)$			$\nearrow$	$\frac{1}{2n}$	$\searrow$		$\nearrow$	$\frac{1}{2n}$	$\searrow$		

$$g\left(\tfrac{1}{n}\right) = g\left(-\tfrac{1}{n}\right) = \frac{\frac{1}{n}}{1+n^2\cdot\frac{1}{n^2}} = \frac{1}{2n} \to g(x) \leq g\left(\tfrac{1}{n}\right),\ \forall x \in [-1,1] \to$$

$$\to g(x) \leq \frac{1}{2n},\ \forall x\in[-1,1] \to a_n = \frac{1}{2n} \to \lim_{n\to\infty} a_n = 0 \to f_n \underset{\mathcal{C}}{\rightrightarrows} f$$