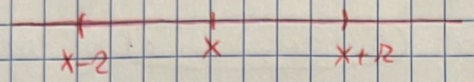
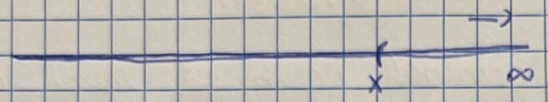
20.10.2023
Analysis - Lecture 3

Sequences of real numbers

1. The topologi on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

Def: Let $x \in \overline{\mathbb{R}}$. • If $x \in \mathbb{R} \Rightarrow B(x, r) = \{y \in \mathbb{R} \mid |y-x| \leq r\} = (x-r, x+r)$





- $B(\infty, r) = (r, \infty]$
- $B(-\infty, r) = [-\infty; -r)$

Def: A set $V \subseteq \overline{\mathbb{R}}$ is said to be a NEIGHBOURHOOD of an element $x \in \overline{\mathbb{R}}$ if $\exists\, r_x > 0$ s.t. $B(x, r_x) \subseteq V$.

Examples

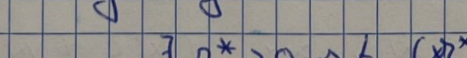
- of neighbourhoods of ∞

* $\overline{\mathbb{R}} \in \mathcal{V}(\infty)$
* $(1, \infty] \in \mathcal{V}(\infty)$
* $\mathbb{R} \notin \mathcal{V}(\infty)$, $\infty \notin \mathbb{R} \to \mathbb{R} \notin \mathcal{V}(\infty)$

  $\mathbb{N}$
  $\mathbb{Q}$
  $\mathbb{R} \setminus \mathbb{Q}$

* $\mathbb{N} \cup \{\infty\} \notin \mathcal{V}(\infty)$ Proof: By contradiction

$\exists\, r^* > 0$ s.t. $(r^*, \infty] \subseteq \mathbb{N} \cup \{\infty\}$ (1)



$r^* < r^* + 1 \xrightarrow{\text{density}} \exists\, t \in \mathbb{R} \setminus \mathbb{Q}$ s.t. (2)

$r^* < t < r^* + 1 \to t \in (r^*, \infty] \cap \mathbb{R} \setminus \mathbb{Q}$

(1) & (2) $\to t \in \mathbb{N} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset \to$ ↯

$\to \mathbb{N} \cup \{\infty\} \notin \mathcal{V}(\infty)$

2. Sequences of real numbers

Def: Each function $f: \mathbb{N} \to \mathbb{R}$ is a SEQUENCE of real numbers

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$\forall n \in \mathbb{N}, \; f(n) := x_n \in \mathbb{R}$

Notations:

- $(x_n)_{n\geq 0} = (x_n)_{n\geq 1} = (x_n)_{n\in\mathbb{N}} = (x_n) \to$

$\to$ notations for a sequence of real numbers

$\hookrightarrow$ a function

- $\forall n \in \mathbb{N}, \; \underset{\in\mathbb{R}}{x_n}$ - is the general term of rank n of the sequence (x_n)

- $\{x_n : x_n$ is a term of $(x_n)\} \to$ the set of all the terms of a sequence (x_n)

Remark: $\mathbb{N}$ may be replaced by $\mathbb{N}_k = \{n \mid n \in \mathbb{N}, n \geq k\}$

When considering limits of functions we have to define them only for ACCUMULATIONS POINTS of the domain.

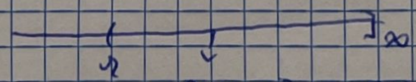
In particular for sequences which are function with the domain $\mathbb{N}$, $\mathbb{N}'$ (in $\overline{\mathbb{R}}$) is $\{\infty\}$

$$\mathbb{N}' = \underset{\text{in } \overline{\mathbb{R}}}{\mathrm{cl}}\, \mathbb{N} \setminus \underset{\text{in } \overline{\mathbb{R}}}{\mathrm{iso}}\, \mathbb{N}$$

$\mathrm{iso}\, \mathbb{N} = \mathbb{N}$

$\underset{\overline{\mathbb{R}}}{\mathrm{cl}}\, \mathbb{N} = \mathbb{N} \cup \{\infty\}$ $\quad (\mathbb{N} \subseteq \mathrm{cl}\, \mathbb{N}$ def$)$

$\hookrightarrow \infty \in \mathrm{cl}\, \mathbb{N} \Leftrightarrow \forall r > 0 \;\; B(\infty, r) \cap \mathbb{N} \neq \emptyset$



$r > 0 : \exists\, n_r \in \mathbb{N}$ s.t. $r < n_r$

(ARCHIMEDES AXIOM)

$\to \underset{\overline{\mathbb{R}}}{\mathrm{cl}}\, \mathbb{N} = \mathbb{N} \cup \{\infty\} \to \mathbb{N}' = \mathbb{N} \cup \{\infty\} \setminus \mathbb{N} = \{\infty\}$

Remark: These notions are consider in the topology on $\overline{\mathbb{R}}$

Def: Consider $(x_n) \subset \mathbb{R}$ to be a sequence of real numbers. An element $\ell \in \overline{\mathbb{R}}$ is said to be the limit of the sequence (x_n) if $\forall V \in \mathcal{V}(\ell)$ $\exists n_\ell \in \mathbb{N}$ s.t. $\forall n \geq n_\ell$, $x_n \in V$.

Remark: In terms of balls, the definition has 3 different approachires.

- $\ell \in \mathbb{R}$, $\ell = \lim\limits_{n \to \infty} x_n \leftrightarrow \forall r > 0, \exists n_r \in \mathbb{N}$ s.t. $\forall n \geq n_r$, $x_n \in B(\ell, r)$

$x_n \in B(\ell, r) \leftrightarrow |x_n - \ell| < r \longleftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $|x_n - \ell| < \varepsilon \longleftrightarrow x_n \in (\ell - \varepsilon, \ell + \varepsilon)$

- $\ell = \infty$, $\infty = \lim\limits_{n \to \infty} x_n \leftrightarrow \forall r > 0, \exists n_r \in \mathbb{N}$ s.t. $\forall n \geq n_r$, $x_n \in B(\infty, r)$

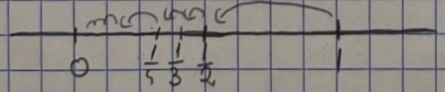
$x_n \in B(\infty, r) \leftrightarrow x_n \in (r; \infty] \leftrightarrow x_n > r \leftrightarrow$

$\leftrightarrow \infty = \lim\limits_{n \to \infty} x_n \leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $x_n > \varepsilon$

- $\ell = -\infty \leftrightarrow -\infty = \lim\limits_{n \to \infty} x_n \leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $x_n < -\varepsilon$

Examples

a) $\forall n \in \mathbb{N}$, $x_n = \frac{1}{n}$, $\lim\limits_{n \to \infty} \frac{1}{n} = 0$



$0 = \lim\limits_{n \to \infty} \frac{1}{n} \overset{\text{def}}{\longleftrightarrow} \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$

$|\frac{1}{n} - 0| < \varepsilon$

Considering $\varepsilon > 0$ randomly chosen

$? \; \left|\frac{1}{n} - 0\right| < \varepsilon \iff 0 < \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$

According to Archimidies Theorem $\exists\, n_\varepsilon \in \mathbb{N}$ s.t.

$0 < \frac{1}{\varepsilon} < n_\varepsilon \; (*) \iff \frac{1}{n_\varepsilon} < \varepsilon$

Consider $n \in \mathbb{N}$, $n \geq n_\varepsilon$ random

$n \geq n_\varepsilon > 0 \iff \frac{1}{n} < \frac{1}{n_\varepsilon}$

$\left|\frac{1}{n} - 0\right| < \frac{1}{n_\varepsilon} \overset{*}{<} \varepsilon$

For a random $\varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \in \mathbb{N}, n \geq n_\varepsilon$,

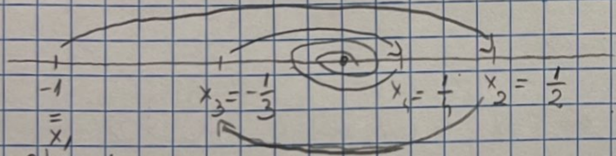
$\left|\frac{1}{n} - 0\right| < \varepsilon$

ε - random $\to \forall \xrightarrow{def} 0 = \lim\limits_{n \to \infty} \frac{1}{n}$

b) $y_n = \frac{(-1)^n}{n}, \; \forall\, n \in \mathbb{N}$

$\downarrow$

$\lim\limits_{n \to \infty} y_n = 0$



$\iff \underbrace{\forall\, \varepsilon > 0}_{\text{step 1}}, \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq n_\varepsilon$, $\underbrace{|y_n - 0| < \varepsilon}_{\text{step 2}}$

S1 $\boxed{\text{Consider } \varepsilon > 0}$ randomly chosen

S2 $?\; |y_n - 0| < \varepsilon \iff \left|\frac{(-1)^n}{n}\right| < \varepsilon \iff \frac{1}{n} < \varepsilon$

in a similar manner like for Example 1, $\boxed{\exists\, n_\varepsilon \in \mathbb{N} \text{ st}}$

$\frac{1}{n_\varepsilon} < \varepsilon$

$\boxed{\forall\, n \geq n_\varepsilon} \; \boxed{\left|\frac{(-1)^n}{n} - 0\right|} = \frac{1}{n} \leq \frac{1}{n_\varepsilon} \boxed{< \varepsilon}$

Thus for a random $\varepsilon > 0$, $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq n_\varepsilon$

$|y_n - 0| < \varepsilon$

ε - random $\to \forall \xrightarrow{def} 0 = \lim\limits_{n \to \infty} y_n$

c) $\forall n \in \mathbb{N}, \; z_n = -n$

$-\infty = \lim_{n\to\infty} z_n \xLeftrightarrow{def} \underbrace{\forall \varepsilon > 0}_{step 1}, \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \underbrace{\forall n \geq n_\varepsilon}_{step 3}, \underbrace{z_n < -\varepsilon}_{step 2}$

Consider $\varepsilon > 0$ randomly chosen

? $z_n < -\varepsilon \Leftrightarrow -n < -\varepsilon \Leftrightarrow \varepsilon < n$

ARCHIMEDES
$\exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \varepsilon < n_\varepsilon$ (**)

Consider $n \in \mathbb{N}$ randomly chosen

$n \geq n_\varepsilon$

$$\left.\begin{array}{l} n_\varepsilon \leq n \\ (**) \quad \varepsilon < n_\varepsilon \end{array}\right\} \to \varepsilon < n$$

For $n \in \mathbb{N}$ randomly chosen, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$,

$\varepsilon < n \Leftrightarrow -n < -\varepsilon \Leftrightarrow z_n < \varepsilon$

ε - random $\to \forall \xrightarrow{def} \lim_{n\to\infty} z_n = -\infty$

d) $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ ($\to \cos n$)

$\left|\frac{\sin n}{n} - 0\right| = \left|\frac{\sin n}{n}\right| \leq \frac{1}{n} < \varepsilon \ldots$

$\sin n, \cos n \in [-1; 1]$

Recall the following classical limits:

- $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$ (Cesaro - Stolz)
- $\lim_{n\to\infty} a^n = \begin{cases} \infty & : a > 1 \\ 1 & : a = 1 \\ 0 & : |a| < 1 \Leftrightarrow a \in (-1; 1) \setminus \{0\} \\ \nexists & : a \leq 1 \end{cases}$

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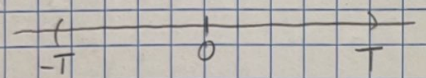
Def: Let $(x_n) \subseteq \mathbb{R}$ be a sequence of real nr. Then it is said to be:

* INCREASING: if $x_n < x_{n+1}$, $\forall n \in \mathbb{N}$
* NONDEACREASING: if $x_n \leq x_{n+1}$, $\forall n \in \mathbb{N}$
* DEACREASING: if $x_n > x_{n+1}$ $\forall n \in \mathbb{N}$
* NONINCREASING: if $x_n \geq x_{n+1}$ $\forall n \in \mathbb{N}$

If any of the adove holds $\to$ (x_n) is said to be MONOTONIC

Def: Let $(x_n) \subseteq \mathbb{R}$. It is said to be

* BOUNDED if $\exists\, T > 0$ s.t. $\forall n \in \mathbb{N}$, $x_n \in B(0, T)$



$\exists$ sharper boundes $\exists\, m, M \in \mathbb{R}$ s.t. $m \leq x_M \leq M$, $\forall n \in \mathbb{N}$

The Weirstrass theorem for monotonic sequences

Let $(x_n) \subseteq \mathbb{R}$

a) If (x_n) is INCREASING then $\exists \lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$

b) If (x_n) is DEACREASING then $\exists \lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

Remark

a) If (x_n) - bounded $\to$ $UB \neq \emptyset$ $\xrightarrow[\text{THE SUP}]{\text{Ax of}}$ $\sup \in \mathbb{R}$

$\xrightarrow[\text{weirstrass}]{\text{classical}}$ INCREASING & BOUNDED $\}$ $\to \exists$ of a finite limit

b) If (x_n) - bounded $\to$ $LB \neq \emptyset$ $\xrightarrow[\text{THE INF}]{\text{Ax of}}$ $\exists \inf \in \mathbb{R}$

$\xrightarrow[\text{Weirstrass}]{\text{classical}}$ DECR & BOUNDED $\}$ $\to \lim \in \mathbb{R}$

Important conceguences of Weirstrass theorem

C1 $\left.\begin{array}{l}(a_n),(x_n)\subseteq\mathbb{R}\\ \ell\in\mathbb{R}\\ \lim\limits_{n\to\infty}a_n=0\\ |x_n-\ell|\le a_n,\ \forall n\in\mathbb{N}\end{array}\right\}\Rightarrow \lim\limits_{n\to\infty}x_n=\ell$

$-a_n\le x_n-\ell\le a_n \quad |+\ell$

$-a_n+\ell\le x_n\le a_n+\ell$

$\to \ell \leftarrow$

C2: $\left.\begin{array}{l}(a_n),(b_n),(c_n)\subset\mathbb{R}\\ \lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}c_n=\ell\\ a_n\le b_n\le c_n\end{array}\right\}\Rightarrow \lim\limits_{n\to\infty}b_n=\ell$

Examples

$z_n=\sqrt{2+\frac{1}{n}}$

$\lim\limits_{n\to\infty}z_n=\sqrt{2}$

- $t_n=\sqrt{2-\frac{1}{n}}$

$\left|\sqrt{2-\frac{1}{n}}-\sqrt{2}\right|\le\frac{1}{n}$

Remarks:

In practice, when we want to prove that $\lim\limits_{n\to\infty}x_n=0$ it is easier to compare it to other sequences that we know for seure to have the limit 0.

According to the QUOTIENT CRITERION (criteriul raportului)

- If $\lim\limits_{n\to\infty}x_n=0$ and $\lim\limits_{n\to\infty}\frac{y_n}{x_n}\in[0;1]\Rightarrow\lim\limits_{n\to\infty}y_n=0$
- If $\lim\limits_{n\to\infty}t_n=\infty$ and $\lim\limits_{n\to\infty}\frac{u_n}{t_n}=1\Rightarrow\lim\limits_{n\to\infty}u_n=\infty$

Thus $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n^2+2n+7}}=\lim\limits_{n\to\infty}\frac{1}{n}=0$

$$\lim_{n\to\infty} \sqrt{n^3+n^2+7} = \lim_{n\to\infty} \sqrt{n^3} = \infty$$

- When we apply Weirstrass

$$\left.\begin{array}{l} |x_n - 0| \le a_n \\ \lim_{n\to\infty} a_n = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} x_n = \frac{\sin n}{n} \\ \left|\frac{\sin n}{n}\right| \le \frac{1}{n} \\ \frac{1}{n} \to 0 \end{array}\right\} \to \dots$$

Series of real numbers

- 

square of side 1, A = 1

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n} + \dots$$

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n}\right) = 1$$

- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} \longrightarrow \infty$

Def: An ordered pair of sequences $((x_n), (s_n))$ is said to be a SERIES of real numbers if

$s_1 = x_1$
$s_2 = x_1 + x_2$
$\vdots$
$s_n = x_1 + x_2 + \dots + x_n$
$\forall n \in \mathbb{N}, n \ge 2$

Notations and terminology:

- (x_n) is the generating sequence
  $\forall n \in \mathbb{N} \to x_n \in \mathbb{R}$ - the general term of rank n
- (s_n) is the sequence of the partial sums
  $\forall n \in \mathbb{N} \to s_n = x_1 + x_2 + \dots + x_n \to$ the PARTIAL SUM of rank n
- $\sum x_n$ or $\sum_{n\ge 1} x^n$ is the notation for the series

Example

$\sum \frac{1}{2^n}$ or $\sum \frac{1}{n}$

Def: Let $\sum x_n$ be a series of real numbers.
If $\exists \lim_{n\to\infty} s_n \in \overline{\mathbb{R}}$ then, it is called THE SUM of THE SERIES and is denoted by $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$ $(\in \overline{\mathbb{R}})$

Example $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$; $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Remarks

* If $\exists \lim_{n\to\infty} s_n \in \mathbb{R}$, the series $\sum x_n$ is said to be CONVERGENT and it is DIVERGENT otherwise:
  - $\exists \lim_{n\to\infty} s_n \in \{-\infty, \infty\}$
  - or
  - $\nexists \lim_{n\to\infty} s_n$

Classical examples of series of real numbers:

I The GEOMETRIC SERIES

$\sum_{n\geq 1} q^{n-1}$, $q \in \mathbb{R}^*$

- $x_n = q^{n-1}$, $\forall n \in \mathbb{N}$ → the general term of the series
- $s_1 = q^0 = 1$

$s_2 = q^0 + q^1 = 1 + q$

...

$s_n = q^0 + q^1 + \dots + q^{n-1} = 1 + q + \dots + q^{n-1} = \begin{cases} \frac{q^n - 1}{q - 1} & ; q \neq 1 \\ n & ; q = 1 \end{cases}$

? $\lim_{n\to\infty} s_n$

I $q = 1$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n = \infty$

II $q > 1 \to \lim_{n\to\infty} q^n = \infty \to \lim_{n\to\infty} \frac{q^n - 1}{q-1} = \frac{\infty}{\underbrace{q-1}_{>0}} = \infty$

III $|q| < 1 \to \lim_{n\to\infty} q^n = 0 \to \lim_{n\to\infty} s_n = \frac{-1}{q-1} = \frac{1}{1-q}$

IV $q \leq -1 \to \nexists$

$$\to \underset{\displaystyle \overset{\parallel}{\sum_{n=1}^{\infty} q^{n-1}}}{\lim_{n\to\infty} s_n} = \begin{cases} \infty & : q \geq 1 \\ \frac{1}{1-q} & : |q| < 1 \\ \nexists & : q \leq -1 \end{cases} \Rightarrow \text{the series } \sum q^{n-1} \text{ is } \begin{cases} \overset{\text{convergent}}{c} \text{ if } |q| < 1 \\ \underset{\text{divergent}}{d} \text{ otherwise} \end{cases}$$

(Hw) ? Nature (C or D) and if $\exists$ the sum $\sum_{n\geq 1} \left(\frac{-3}{4}\right)^{n-1}$

$$\sum_{n\geq 1} \left(\frac{-3}{4}\right)^{n-1}$$

$q = -\frac{3}{4} \to |q| < 1 \to \sum_{n\geq 1} q^{n-1}$ is convergent

$$\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1} = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{1}{1+\frac{3}{4}} = \frac{4}{7}$$