Analysis - Homework 5

1. Study both the absolute convergence & the convergence:

a) $\sum_{n\geq 1} (-1)^n \cdot \frac{(n+1)^{n+1}}{n^{n+2}}$

$x_n = (-1)^n \cdot \frac{(n+1)^{n+1}}{n^{n+2}} \to |x_n| = \frac{(n+1)^{n+1}}{n^{n+2}}$

Let $y_n = \frac{1}{n^\alpha}$

$\lim_{n\to\infty} \frac{|x_n|}{y_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^{n+2}} \cdot n^\alpha = \lim_{n\to\infty} \frac{n^{n+1}\overbrace{(1+\frac{1}{n})^{n+1}}^{\to 0} \cdot n^\alpha}{n^{n+2}} =$

$= \lim_{n\to\infty} \frac{n^{n+1+\alpha}}{n^{n+2}} = \begin{cases} 0, & n+1+\alpha < n+2 \\ 1, & n+1+\alpha = n+2 \\ \infty, & n+1+\alpha > n+2 \end{cases} \Bigg\} \to$

In oder to apply C2C b) we need $\lim_{n\to\infty} \frac{|x_n|}{y_n} \in (0,\infty)$

$\to n+1+\alpha = n+2 \to \alpha = 1 \to \lim_{n\to\infty} \frac{|x_n|}{y_n} = 1$

$\sum_{n\geq 1} |x_n|$ - SPT

$\sum_{n\geq 1} y_n$ - SPT

$\xrightarrow[b]{C2C} \sum_{n\geq 1} |x_n| \sim \sum_{n\geq 1} y_n$ - D $\to$

$\to \sum_{n\geq 1} |x_n|$ - D $\to \sum_{n\geq 1} x_n$ - is not A.C.

Let $a_n = \frac{(n+1)^{n+1}}{n^{n+2}}$

$a_{n+1} - a_n = \frac{(n+2)^{n+2}}{(n+1)^{n+3}} - \frac{(n+1)^{n+1}}{n^{n+2}} = \frac{(n^2+2n)^{n+2} - (n+1)^{2n+4}}{n^{n+2} \cdot (n+1)^{n+3}}$

$= \frac{(n^2+2n)^{n+2} - (n^2+2n+1)^{n+2}}{n^{n+2} \cdot (n+1)^{n+2}} < 0 \to (a_n)$ - decreasing

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{n+1}\overbrace{(1+\frac{1}{n})^{n+1}}^{\to 0}}{n^{n+2}} = \lim_{n\to\infty} \frac{1}{n} = 0$

$\xrightarrow{\text{Leibnitz's Criterion}}$

$\to \sum_{n\geq 1} (-1)^n \cdot a_n$ is C.

b) $\sum\limits_{n\geq 1} (-1)^n \dfrac{2n+1}{3^n}$

$x_n = (-1)^n \cdot \dfrac{2n+1}{3^n} \to |x_n| = \dfrac{2n+1}{3^n}$

$\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \dfrac{2n+3}{3^{n+1}} \cdot \dfrac{3^n}{2n+1} = \lim\limits_{n\to\infty} \dfrac{2n+3}{3(2n+1)} = \lim\limits_{n\to\infty} \dfrac{n(2+\overset{\to 0}{\frac{3}{n}})}{n(6+\underset{\to 0}{\frac{3}{n}})} = \dfrac{2}{6} = \dfrac{1}{3} < 1 \to$

$\xrightarrow{D'A} \sum\limits_{n\geq 1} |x_n| - C \to \sum\limits_{n\geq 1} x_n - A.C \to \sum\limits_{n\geq 1} x_n - C$

c) $\sum\limits_{n\geq 1} (-1)^n \cdot \dfrac{1}{\ln n}$

$x_n = (-1)^n \cdot \dfrac{1}{\ln n} \to |x_n| = \dfrac{1}{\ln n}$

$\left.\begin{array}{l} \ln n < n \to \dfrac{1}{\ln n} > \dfrac{1}{n} \\ \sum \dfrac{1}{n} - D \end{array}\right\} \xrightarrow{CIC} \sum\limits_{n\geq 1} \dfrac{1}{\ln n} - D \to \sum\limits_{n\geq 1} x_n$ - is not AC

Let $a_n = \dfrac{1}{\ln n}$

$\left.\begin{array}{l} a_{n+1} - a_n = \dfrac{1}{\ln(n+1)} - \dfrac{1}{\ln n} = \dfrac{\overbrace{\ln n - \ln(n+1)}^{<0}}{\underbrace{\ln(n+1)\cdot \ln n}_{>0}} < 0 \to (a_n) \text{ - decreasing} \\ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} = \dfrac{1}{\infty} = 0 \end{array}\right\} \xrightarrow{\text{Leibniz}}$

$\to \sum\limits_{n\geq 1} (-1)^n \cdot a_n - C$

d) $\sum\limits_{n\geq 1} (-1)^n \cdot \dfrac{1}{\sqrt{n(n+1)}}$

$x_n = (-1)^n \cdot \dfrac{1}{\sqrt{n(n+1)}} \to |x_n| = \dfrac{1}{\sqrt{n(n+1)}}$

$\left.\begin{array}{l} \dfrac{|x_{n+1}|}{|x_n|} = \dfrac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} = \sqrt{\dfrac{n}{n+2}} < 1 \to (|x_n|) \text{ - decreasing} \\ \sum |x_n| - SPT \end{array}\right\} \xrightarrow[\text{criterion}]{\text{Cauchy's condensation}}$

$\to \sum\limits_{n\geq 1} |x_n| \sim \sum\limits_{n\geq 1} 2^n \cdot |x_{2^n}|$

$2^n \cdot |x_{2^n}| = \dfrac{2^n}{\sqrt{2^n(2^n+1)}}$

$\lim\limits_{n\to\infty} \dfrac{2^n}{\sqrt{2^{2n}+2^n}} = \lim\limits_{n\to\infty} \dfrac{2^n}{2^n\sqrt{1+\underset{\to 0}{\frac{1}{2^n}}}} = 1 \neq 0 \to \sum\limits_{n\geq 1} 2^n |x_{2^n}| - D \to \sum\limits_{n\geq 1} |x_n| - D \to$

$\to \sum\limits_{n\geq 1} x_n$ is not AC

Let $a_n = \frac{1}{\sqrt{n(n+1)}}$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n(n+1)}}{\sqrt{(n+1)(n+2)}} = \sqrt{\frac{n(n+1)}{(n+1)(n+2)}} = \sqrt{\frac{n}{n+2}} < 1 \to (a_n) - \text{decreasing}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Leibniz $\to$

$\to \sum_{n\geq 1} (-1)^n \cdot a_n$ is C

e) $\sum_{n\geq 1} (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!}$

$$x_n = (-1)^n \cdot \frac{(2n-1)!!}{(2n)!!} \to |x_n| = \frac{(2n-1)!!}{(2n)!!}$$

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n \cdot n!}$$

$$(2n)!! = 2\cdot 4 \cdot \ldots \cdot 2n = 2\cdot 1 \cdot 2\cdot 2 \cdot \ldots \cdot 2\cdot n = 2^n (1\cdot 2 \cdot \ldots \cdot n) = 2^n \cdot n!$$

$$|x_n| = \frac{\frac{(2n)!}{2^n\, n!}}{2^n\, n!} = \frac{(2n)!}{2^{n+1}\cdot (n!)^2}$$

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\overbrace{\cancel{(2n+2)!}}^{(2n+1)(2n+2)}}{\underbrace{\cancel{2^{n+2}}}_{2}\underbrace{\cancel{[(n+1)!]}^2}_{(n+1)^2}} \cdot \frac{\cancel{2^{n+1}}\ \cancel{(n!)}^2}{\cancel{(2n)}!} = \lim_{n\to\infty} \frac{\cancel{2}(\cancel{n+1})(2n+1)}{\cancel{2}(n+1)^{\cancel{2}}} =$$

$= 2 > 1 \to \sum_{n\geq 1} |x_n| - D \to \sum_{n\geq 1} x_n$ is not AC

g) $\sum_{n\geq 1} (-1)^{\frac{n(n+1)}{2}} \cdot \frac{n^{100}}{2^n}$

$$x_n = (-1)^{\frac{n(n+1)}{2}} \cdot \frac{n^{100}}{2^n} \to |x_n| = \frac{n^{100}}{2^n}$$

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{(n+1)^{100}}{2^{n+1}} \cdot \frac{2^n}{n^{100}} = \lim_{n\to\infty} \frac{\cancel{n^{100}}(1+\overbrace{\tfrac{1}{n}}^{\to 0})^{100}\ \cancel{2^n}}{2\cdot \cancel{2^n}\ \cancel{n^{100}}} = \frac{1}{2} < 1 \to$$

$\xrightarrow{D'A} \sum_{n\geq 1} |x_n| - C \to \sum_{n\geq 1} x_n$ - is A.C. $\to \sum_{n\geq 1} x_n$ is C

g) $\sum_{n\geq 1} (-1)^{\frac{n(n+1)}{2}} \cdot \sin\frac{\pi}{n\sqrt{n+1}}$

$x_n = (-1)^{\frac{n(n+1)}{2}} \sin\frac{\pi}{n\sqrt{n+1}} \to |x_n| = \left|\sin\frac{\pi}{n\sqrt{n+1}}\right|$

Let $y_n = \frac{\pi}{n\sqrt{n+1}} > 0$

$\lim_{n\to\infty} \frac{|x_n|}{y_n} = \lim_{n\to\infty} \left| \frac{\sin\frac{\pi}{n\sqrt{n+1}}}{\frac{\pi}{n\sqrt{n+1}}} \right| = 1 \in (0;\infty)$

$\sum |x_n|$ - Spt

$\sum y_n$ - Spt

$\xrightarrow[b]{C2C} \sum |x_n| \sim \sum y_n$

$\sum y_n = \pi \sum \underbrace{\frac{1}{n\sqrt{n+1}}}_{z_n}$

Let $t_n = \frac{1}{n^\alpha}$

$\lim_{n\to\infty} \frac{z_n}{t_n} = \lim_{n\to\infty} \frac{n^\alpha}{n\sqrt{n+1}} = \lim_{n\to\infty} \frac{n^\alpha}{n^{\frac{3}{2}}} = \begin{cases} 0, & \alpha < \frac{3}{2} \\ 1, & \alpha = \frac{3}{2} \\ \infty, & \alpha > \frac{3}{2} \end{cases}$

In order to apply C2C b for z_n & t_n then $\alpha = \frac{3}{2}$

$\lim_{n\to\infty} \frac{z_n}{t_n} = 1 \in (0,\infty) \xrightarrow[b]{C2C} \sum z_n \sim \sum t_n$

$t_n = \frac{1}{n^{\frac{3}{2}}}$, $\frac{3}{2} > 1$ $\to \sum t_n$ - C

$\to \sum z_n$ - C $\to$

$\to \sum y_n$ - C $\to \sum |x_n|$ - C $\to \sum x_n$ - AC $\to \sum x_n$ - C

2 a, b > 0. Study the nature

a) $\sum_{n\geq 1} \underbrace{\frac{a(2a+1)(3a+1)\dots(na+1)}{b(2b+1)(3b+1)\dots(nb+1)}}_{x_n}$

x_n, $a, b > 0$ $\to \sum x_n$ - Spt

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a(2a+1)\dots(na+1)}(na+a+1)}{\cancel{b(2b+1)\dots(nb+1)}(nb+b+1)} \cdot \frac{\cancel{b(2b+1)\dots(nb+1)}}{\cancel{a(2a+1)\dots(na+1)}} =$

$= \lim_{n\to\infty} \frac{na+a+1}{nb+b+1} = \frac{a}{b}$

I $0 < \frac{a}{b} < 1$

$0 < a < b \to \sum_{n\geq 1} x_n - C$

II $\frac{a}{b} > 1$

$a > b \to \sum_{n\geq 1} x_n - D$

III $\frac{a}{b} = 1$

$a = b \to \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 \to ?$

$a = b \to x_n = \frac{a(2a+1)\ldots(na+1)}{a(2a+1)\ldots(na+1)} = 1 \to \lim_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n - D$

$$\sum_{n\geq 1} x_n \text{ is } \begin{cases} C, & 0 < a < b \\ D, & a \geq b \end{cases}$$

b) $\sum_{n\geq 1} \underbrace{\frac{a(a+1)\ldots(a+n)}{n!} \cdot \frac{1}{n^b}}_{x_n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a(a+1)\ldots(a+n)}(a+n+1)}{\cancel{(n+1)!}\cdot(n+1)^b} \cdot \frac{\cancel{n!}\cdot n^b}{\cancel{a\cdot(a+1)\ldots(a+n)}} =$$

$$= \lim_{n\to\infty} \frac{(a+n+1)n^b}{(n+1)^{b+1}} = \lim_{n\to\infty} \frac{n^b \cdot n\left(\overset{\to 0}{\frac{a}{n}} + 1 + \overset{\to 0}{\frac{1}{n}}\right)}{n^{b+1}\underbrace{\left(1+\frac{1}{n}\right)^{b+1}}_{\to 0}} = 1$$

Rabe - Duhamel

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot \frac{(n+1)^{b+1} - n^b(a+n+1)}{(a+n+1)n^b} =$$

$$= \lim_{n\to\infty} n\cdot \frac{n^{b+1}\left(1+\overset{\to 0}{\frac{1}{n}}\right)^{b+1} - n^{b+1} - n^b(a+1)}{n^b(a+n+1)} =$$

$$= \lim_{n\to\infty} n\cdot \frac{\cancel{n^{b+1}} - \cancel{n^{b+1}} - n^b(a+1)}{n^b(a+n+1)} = \lim_{n\to\infty} \frac{-n^{b+1}(a+1)}{n^b\cdot n\left(\underbrace{\frac{a}{n}}_{\to 0} + 1 + \underbrace{\frac{1}{n}}_{\to 0}\right)} = -(a+1)$$

I $-(a+1) < 1$

$a+1 > -1$

$a > -2 \to \sum x_n - D$

$a > 0 > -2$

II $-(a+1) > 1 \to \sum x_n - C$

$a < -2$, but $a > 0 \to$ imposible

$\to \sum_{n\geq 1} x_n - D$

III $-(a+1) = 1 \to ?$

$a = -2$, but $a > 0$ $\} \to$ imosible

c) $\sum_{n\geq 1} \frac{2^n}{a^n+b^n}$

$\ell = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2\cdot \cancel{2^n}}{a^{n+1}+b^{n+1}} \cdot \frac{a^n+b^n}{\cancel{2^n}} = 2 \lim_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}}$

I $a>b$

$\ell = 2\lim_{n\to\infty} \frac{a^n\left[1+\left(\frac{b}{a}\right)^n\right]}{a^{n+1}\left[1+\left(\frac{b}{a}\right)^{n+1}\right]} = \frac{2}{a}$ ($\left(\frac{b}{a}\right)^n \to 0$)

1. $\frac{2}{a} < 1$

$2 < a \to \sum_{n\geq 1} x_n - C$

2. $\frac{2}{a} > 1$

$2 > a > 0 \to \sum_{n\geq 1} x_n - D$

3. $\frac{2}{a} = 1$

$a = 2 \to x_n = \frac{2^n}{2^n + b^n}$

$\lim_{n\to\infty} x_n \overset{2=a>b}{=} \lim_{n\to\infty} \frac{2^n}{2^n\left[1+\left(\frac{b}{2}\right)^n\right]} = 1 \neq 0 \to \sum_{n\geq 1} x_n - D$ ($\left(\frac{b}{2}\right)^n \to 0$)

II $b>a$

$\ell = \frac{2}{b}$

1. $\frac{2}{b} < 1$

$2 < b \to \sum x_n - C$

2. $\frac{2}{b} > 1$

$2 > b > 0 \to \sum x_n - D$

3. $\frac{2}{b} = 1$

$2 = b \to x_n = \frac{2^n}{a^n + 2^n}$

$\lim_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n - D$

III $a=b$

$\ell = 2 \cdot \lim_{n\to\infty} \frac{2\cdot a^n}{2\cdot a^{n+1}} = \frac{2}{a}$

1,2 - same as I 1,2

3. $a = b = 2$

$x_n = \frac{2^n}{2\cdot 2^n} = \frac{1}{2} \to \lim_{n\to\infty} x_n = \frac{1}{2} \neq 0 \to \sum x_n - D$

$\sum_{n\geq 1} x_n$ is $\begin{cases} C \text{ when } a>b \text{ \& } a>2;\ b>a \text{ \& } b>2,\ a=b>2 \\ D \text{ when } a>b \text{ \& } a\leq 2;\ b>a \text{ \& } b\leq 2,\ a=b\leq 2 \end{cases}$

d) $\sum_{n\geq 1} \frac{a^n \cdot b^n}{a^n+b^n}$

$$\ell = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a^{n+1}}\cdot \cancel{b^{n+1}}}{a^{n+1}+b^{n+1}} \cdot \frac{a^n+b^n}{\cancel{a^n}\cdot\cancel{b^n}} = a\cdot b \cdot \lim_{n\to\infty} \frac{a^n+b^n}{a^{n+1}+b^{n+1}}$$

I $a>b$

$$\ell = a\cdot b \cdot \lim_{n\to\infty} \frac{a^n\left[1+\left(\frac{b}{a}\right)^n\right]}{a^{n+1}\left[1+\left(\frac{b}{a}\right)^{n+1}\right]} = \frac{a\cdot b}{a} = b$$

1. $0<b<1 \to \sum x_n - C$
2. $a>b>1 \to \sum x_n - D$
3. $b=1 \to a>1$

$$x_n = \frac{a^n}{a^n+1}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a^n}{a^n\left[1+\left(\frac{1}{a}\right)^n\right]} = 1 \neq 0 \to \sum x_n - D$$

II $a<b$

$\ell = a$

1. $0<a<1 \to \sum x_n - C$
2. $b>a>1 \to \sum x_n - D$
3. $a=1 \to b>1$

$$x_n = \frac{b^n}{b^n+1} \to \lim_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n - D$$

III $a=b$

$$x_n = \frac{a^{2n}}{2\cdot a^n} = \frac{a^n}{2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{\cancel{2}} \cdot \frac{\cancel{2}}{a^n} = a$$

1. $a<1 \to \sum x_n - C$
2. $a>1 \to \sum x_n - D$
3. $a=1 \to x_n = \frac{1}{2} \to \lim_{n\to\infty} x_n = \frac{1}{2} \neq 0 \to \sum x_n - D$

$\sum_{n\geq 1} x_n$ is $\begin{cases} C, \text{ when } a>b \text{ \& } b<1;\ a<b \text{ \& } a<1 \text{ \& } a=b<1 \\ D, \text{ when } a>b \text{ \& } b\geq 1;\ a<b \text{ \& } a\geq 1 \text{ \& } a=b\geq 1 \end{cases}$