7.12.2023

Algebra - Seminar 9

Compute by applying elementary operations the rank of

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 - 2L_1}{\overset{L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 - L_3}{\sim}$

$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \text{rank} = 3$

2. $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{L_3 + L_1}{\overset{L_2 + 2L_1}{\sim}} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & 9 & 3 \end{pmatrix} \overset{L_3 + 2L_2}{\sim}$

$\sim \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 12 & 5 \end{pmatrix} \to \text{rank} = 3$

3. $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{C_2 \leftrightarrow C_4}{\overset{C_1 \leftrightarrow C_3}{\sim}} \begin{pmatrix} 3 & 4 & \beta & 1 \\ 3 & 3 & 1 & \alpha \\ 4 & 7 & 2 & 3\alpha \end{pmatrix} \underset{L_3 - \frac{4}{3}L_1}{\overset{L_2 - L_1}{\sim}} \begin{pmatrix} 3 & 4 & \beta & 1 \\ 0 & -1 & 1-\beta & \alpha - 1 \\ 0 & \underbrace{7 - \frac{16}{3}}_{\frac{5}{3}} & 2 - \frac{4}{3}\beta & 3\alpha - \frac{4}{3} \end{pmatrix}$

$\overset{L_3 + \frac{5}{3}L_2}{\sim} \begin{pmatrix} 3 & 4 & \beta & 1 \\ 0 & -1 & 1-\beta & \alpha - 1 \\ 0 & 0 & \frac{11}{3} - 3\beta & \frac{14}{3}\alpha - 3 \end{pmatrix}$

$\frac{11}{3} - 3\beta = 0 \to \beta = \frac{11}{9}$

$\frac{14}{3}\alpha - 3 = 0 \to \alpha = \frac{9}{14}$

$\text{rank} = \begin{cases} 3, & \alpha \in \mathbb{R} \setminus \{\frac{9}{14}\} \text{ or } \beta \in \mathbb{R} \setminus \{\frac{11}{9}\} \\ 2, & \alpha = \frac{9}{14},\ \beta = \frac{11}{9} \end{cases}$

Compute by applying elem. op. the inverses of

4. $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 - 2L_1}{\overset{L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 - 2L_2}{\sim}$

$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \underset{\frac{1}{9}L_3}{\overset{-\frac{1}{3}L_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \sim$

$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{6}{9} & -\frac{3}{9} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \overset{L_2 - 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{9} & -\frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \quad L_1 - 2L_2 - 2L_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & [illegible] & [illegible] & [illegible] \\ 0 & 1 & 0 & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 1 & [illegible] & [illegible] & [illegible] \end{array}\right)$$

A^{-1}

5. $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3-3L_1}{\overset{L_2-2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_1+L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$\overset{-\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3+12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \overset{5L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_2-\frac{3}{5}L_3}{\overset{L_1+L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 8 & -11 & 5 \\ 0 & 1 & 0 & -\frac{25}{5} & \frac{35}{5} & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 8 & -11 & 5 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_1+L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

6. K - field

$B = (e_1, e_2, e_3, e_4)$ - basis

$X = (v_1, v_2, v_3)$ in ${}_K K^4$ v.s.

$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4$

$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4$

$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4$

Matrix of X in the bases B, det an echelon for it and deduce that X is lin. dep.

$v_1 = (3,0,0,0,0) + (0,2,0,0) + (0,0,-5,0) + (0,0,0,4) = (3,2,-5,4)$

$v_2 = (3,-1,3,-3)$

$v_3 = (3,5,-13,11)$

$$[X]_B = \begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \underset{L_3-L_1}{\overset{L_2-L_1}{\sim}} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \overset{L_3+L_2}{\sim} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$L_3 = (0,0\,0\,0) \to X$ - lin. dep.

7. $v_1=(1,0,4),\ v_2=(2,1,0),\ v_3=(1,5,-36),\ v_4=(2,10,-72)$

$\dim\langle X\rangle$, basis for X?

$$\begin{pmatrix}1&0&4\\2&1&0\\1&5&-36\\2&10&-72\end{pmatrix}\underset{L_3-L_1}{\overset{L_2-2L_1}{\sim}}\begin{pmatrix}1&0&4\\0&1&-8\\0&5&-40\\2&10&-72\end{pmatrix}\overset{L_4-2L_1}{\sim}\begin{pmatrix}1&0&4\\0&1&-8\\0&5&-10\\0&10&-80\end{pmatrix}\underset{L_4-10L_2}{\overset{L_3-5L_2}{\sim}}$$

$$\sim\begin{pmatrix}1&0&4\\0&1&-8\\0&0&0\\0&0&0\end{pmatrix}\to \dim\langle X\rangle=2\to \quad \text{basis}: \{(1,0,4),(0,1,-8)\}$$

8. $v_1=(1,0,4,3);\ v_2=(0,2,3,1);\ v_3=(0,4,6,2)$

$\dim\langle X\rangle$, basis for X?

$$\begin{pmatrix}1&0&4&3\\0&2&3&1\\0&4&6&2\end{pmatrix}\overset{L_3-2L_2}{\sim}\begin{pmatrix}1&0&4&3\\0&2&3&1\\0&0&0&0\end{pmatrix}\to \dim\langle X\rangle=2$$

basis: $\{v_1, v_2\}$

Determine the dimension of $S, T, S+T, S\cap T$ and a basis of $S, T, S+T, S\cap T$

9. $S=\langle(1,0,4),(2,1,0),(1,1,-4)\rangle$

$T=\langle(-3,-2,4),(5,2,4),(-2,0,-8)\rangle$

$$S=\begin{pmatrix}1&0&4\\2&1&0\\1&1&-4\end{pmatrix}\underset{L_3-L_1}{\overset{L_2-2L_1}{\sim}}\begin{pmatrix}1&0&4\\0&1&-8\\0&1&-8\end{pmatrix}\overset{L_3-L_2}{\sim}\begin{pmatrix}1&0&4\\0&1&-8\\0&0&0\end{pmatrix}\to \dim\langle S\rangle=2$$

basis: $\{(1,0,4),(0,1,-8)\}$

$$T=\begin{pmatrix}-3&-2&4\\5&2&4\\-2&0&-8\end{pmatrix}\overset{L_2+L_1+L_2}{\sim}\begin{pmatrix}-3&-2&4\\0&0&0\\-2&0&-8\end{pmatrix}\overset{L_2\leftrightarrow L_3}{\sim}\begin{pmatrix}-3&-2&4\\-2&0&-8\\0&0&0\end{pmatrix}\overset{L_2-\frac{2}{3}L_1}{\sim}$$

$$\sim\begin{pmatrix}-3&-2&4\\0&\frac{4}{3}&-\frac{32}{3}\\0&0&0\end{pmatrix}\to \dim\langle T\rangle=2$$

basis: $\{(-3,-2,4),(0,\frac{4}{3},-\frac{32}{3})\}$

$$S+T=\langle S\cup T\rangle=\begin{pmatrix}1&0&4\\0&1&-8\\0&0&0\\-3&-2&4\\0&\frac{4}{3}&-\frac{32}{3}\\0&0&0\end{pmatrix}\underset{L_3\leftrightarrow L_5}{\overset{L_2\leftrightarrow L_4}{\sim}}\begin{pmatrix}1&0&4\\-3&-2&4\\0&\frac{4}{3}&-\frac{32}{3}\\0&1&-8\\0&0&0\\0&0&0\end{pmatrix}\overset{3L_5}{\sim}\begin{pmatrix}1&0&4\\-3&-2&4\\0&4&-32\\0&1&-8\\0&0&0\\0&0&0\end{pmatrix}\overset{L_2+3L_1}{\sim}$$

$$\sim\begin{pmatrix}1&0&4\\0&1&-8\\0&4&-32\\0&1&-8\\0&0&0\\0&0&0\end{pmatrix}\underset{L_4-L_2}{\overset{L_3-4L_2}{\sim}}\begin{pmatrix}1&0&4\\0&1&-8\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}\to \dim\langle S+T\rangle=2$$

basis $=\{(1,0,4),(0,1,-8)\}$

$\dim\langle S\cap T\rangle=\dim\langle S\rangle+\dim\langle T\rangle-\dim\langle S+T\rangle=2+2-2=2$

10. $S = \langle (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) \rangle$

$T = \langle (2, 5, -6, -5), (-1, 2, -7, -3) \rangle$

$$S = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \underset{L_3 + L_1}{\overset{L_2 - 3L_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 6 \\ 0 & 2 & 0 & -3 \end{pmatrix} \overset{L_3 + \frac{2}{5}L_2}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 6 \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \end{pmatrix} \rightarrow$$

$\rightarrow \dim \langle S \rangle = 3$

basis : $\{ (1, 2, -1, -2), (0, -5, 4, 6), (0, 0, \frac{8}{5}, \frac{3}{5}) \}$

$$T = \begin{pmatrix} 2 & 5 & -6 & -5 \\ -1 & 2 & -7 & -3 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 2 & 5 & -6 & -5 \end{pmatrix} \overset{L_2 + 2L_1}{\sim} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix} \rightarrow$$

$\rightarrow \dim \langle T \rangle = 2$

basis: $\{ (-1, 2, -7, -3); (0, 9, -20, -11) \}$

$$S + T = \langle S \cup T \rangle = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 6 \\ 0 & 0 & \frac{8}{5} & \frac{-1}{5} \\ -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix} \overset{5L_3}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 6 \\ 0 & 0 & 8 & -1 \\ -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix} \underset{L_3 \leftrightarrow L_5}{\overset{L_2 \leftrightarrow L_4}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \\ 0 & -5 & 4 & 6 \\ 0 & 0 & 8 & -1 \end{pmatrix} \overset{L_2 + L_1}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 4 & -8 & -5 \\ 0 & 9 & -20 & -11 \\ 0 & -5 & 4 & 6 \\ 0 & 0 & 8 & -1 \end{pmatrix} \ldots$$

$\rightarrow \dim \langle S+T \rangle = 3$

basis = $\{ (1, 2, -1, -2), (91, -4, -1), (0, 0, -16, 2)$

$\dim \langle S \cap T \rangle = 3 + 2 - 3 = 2$