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Analysis - Lecture 4

Series of real numbers

Ex1: $\sum\limits_{n\geq 1} \left(-\frac{3}{4}\right)^{n-1}$ → a geometric series of rational nr

$q = -\frac{3}{4}$

$|q| = \frac{3}{4} < 1$ → C (convergent)

and $\sum\limits_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1} = \dfrac{1}{1-\left(-\frac{3}{4}\right)} = \frac{4}{7}$

Ex2: $\sum\limits_{n\geq 1} 3^n$ → geometric series of nat. nr. of ratio $q = 3 > 1$ → divergent but $\sum\limits_{n=1}^{\infty} 3^n = +\infty$

Ex3: $\sum\limits_{n\geq 1} (-4)^n$ → geometric series of ratio $q = -4 < -1$ → divergent → without a sum

2^{nd} classical example - TELESCOPIC SERIES

Def: $\sum x_n$ it is said to be a TELESCOPIC SERIES if $\exists (a_n) \in \mathbb{R}$ s.t. $x_n = a_n - a_{n+1}$

Proposition

If $\sum x_n$ is a telescopic series ($x_n = a_n - a_{n+1}$) and $\exists \lim\limits_{n\to\infty} a_n = \ell \in \overline{\mathbb{R}}$ then $\sum\limits_{n\geq k}^{\infty} x_n = a_k - \ell$

Proof:

- $\forall n \in \mathbb{N}$, $n \geq k$: $x_n = a_n - a_{n+1}$
- $\forall n \in \mathbb{N}$, $n \geq k$: $S_k = x_k = a_k - a_{k+1}$

$S_{k+1} = x_k + x_{k+1} = a_k - a_{k+1} + a_{k+1} - a_{k+2}$

$= a_k - a_{k+2}$

...

$S_n = x_k + \dots + x_n = a_k - \cancel{a_{k+1}} + \cancel{a_{k+1}} - \cancel{a_{k+2}} + \dots + \cancel{a_n} - a_{n+1} = a_k - a_{n+1}$

$\rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} (a_k - a_{n+1}) = a_k - \lim_{n\to\infty} a_{n+1} = a_k - \ell$

$\rightarrow \exists \sum_{n=k}^{\infty} x_n = a_k - \ell$

Remark: $\boxed{\text{If } \left.\begin{array}{l} y_n = b_{n+1} - b_n \\ \exists \ell = \lim_{n\to\infty} b_n \in \overline{\mathbb{R}} \end{array}\right\} \rightarrow \sum_{n=k}^{\infty} y_n = \ell - b_k}$

Example: $\sum_{n\geq 5} \frac{1}{n(n+1)}$

$$x_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \underbrace{\frac{1}{n}}_{a_n} - \underbrace{\frac{1}{n+1}}_{a_{n+1}}$$

We consider the sequence $(a_n) \subseteq \mathbb{R}$ having the general term $a_n = \frac{1}{n}$. Because $\lim_{n\to\infty} a_n = 0$

$$\sum_{n=5}^{\infty} x_n = a_5 - \lim_{n\to\infty} a_n = \frac{1}{5} - 0 = \frac{1}{5}$$

Remark: Basic algebraic operations such as adition, substraction, multiplication with a scalar, may be considered even for series, the results are imidiat as long as we are not in a nondetermination case.

$$\left.\begin{array}{l} \text{If } \exists \sum_{n=k}^{n} a_n = a \in \overline{\mathbb{R}} \\ \quad \exists \sum_{n=k}^{n} b_n = b \in \overline{\mathbb{R}} \\ \quad \alpha, \beta \in \mathbb{R} \\ \quad \text{and} \\ \quad \alpha \cdot a + \beta \cdot b \text{ is NOT a NONDEDERMINATION CASE} \end{array}\right\} \rightarrow$$

$$\rightarrow \exists \sum_{n=k}^{n} (\alpha \cdot a_n + \beta \cdot b_n) = \alpha \cdot a + \beta \cdot b$$

Example: 1. Study $\sum_{n\geq 1}\left(3\left(-\frac{1}{4}\right)^{n-1} + 4\cdot\left(\frac{2}{5}\right)^{n-1}\right)$

We consider $a_n = \left(-\frac{1}{4}\right)^{n-1}$ $\forall n \in \mathbb{N}$

$b_n = \left(\frac{2}{5}\right)^{n-1}$, $\forall n \in \mathbb{N}$

We study separetly $\sum a_n$ & $\sum b_n$

- $\sum a_n$ is a geometric series of ratio $q = -\frac{1}{4} \rightarrow$
$\rightarrow |q| < 1 \rightarrow$ convergent and $\sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{5} \in \mathbb{R}$

- $\sum b_n$ is a geometric series of ratio $q = \frac{2}{5} \rightarrow C.$
and $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n-1} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3} \in \mathbb{R}$

Because $\underbrace{3\cdot\frac{4}{5} + 4\cdot\frac{5}{3}}$ is not a NONDET. CASE

$\rightarrow \exists \sum_{n=1}^{\infty}(3\cdot a_n + 4\cdot b_n) = \frac{12}{5} + \frac{20}{3} = \frac{136}{15}$

Example 2: $\sum_{n\geq 1}\left(3\cdot\underbrace{\left(-\frac{1}{4}\right)^{n-1}}_{a_n} + 5\cdot\underbrace{2^n}_{c_n}\right)$, $c_n > 1 \rightarrow \sum c_n$ is D.

$a_n \downarrow$ like Ex 1

and $\sum_{n=1}^{\infty} 2^n = \infty$

$3\cdot\frac{4}{5} + 4\cdot\infty = \infty$ - it is not a nondet. case $\rightarrow$

$\rightarrow \sum_{n=1}^{\infty}(3a_n + 4\cdot c_n) = \infty$ (this series is D)

Remark: Unlike when working with sequences, the behaviour of the first few terms of a series matters when we compute the sum of the series.

The nature of the series is nevertheless unbothered by the first few terms.

Property:

- Let $\sum x_n$ be a series of real nr. and consider $k \in \mathbb{N}$, $k > 1$.

If $\sum_{n=1}^{\infty} x_n = \ell \in \overline{\mathbb{R}} \rightarrow \sum_{n=k+1}^{\infty} x_n = \ell - (x_1 + x_2 + \dots + x_k)$

• The nature of the series is not altered by this behaviour

Examples:

$$\sum_{n\geq 4}\left(-\frac{1}{4}\right)^{n-1}$$

$$\sum_{n=4}^{\infty}\left(-\frac{1}{4}\right)^{n-1} = \sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} - (x_1 + x_2 + x_3) =$$

$$x_n = \left(-\frac{1}{4}\right)^{n-1}$$

$$= \frac{1}{1-\frac{-1}{4}} - \left(1 - \frac{1}{4} + \frac{1}{16}\right) = \frac{4}{5} - \frac{16-4+1}{16} = \frac{4}{5} - \frac{13}{16} = \frac{64-65}{80} = -\frac{1}{80}$$

$$\to \sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} \neq \sum_{n=4}^{\infty}\left(-\frac{1}{4}\right)^{n-1}$$ but THEY HAVE THE SAME NATURE (C)

SERIES WITH POSITIVE TERMS (SPT)

Def: Let $\sum x_n$ be a series of real numbers. It is said to be a SPT if $\boxed{x_n > 0}$ $\forall n \in \mathbb{N}$

Properties

• $\sum x_n$ SPT $\to$ (S_n) is increasing

$$\Downarrow \text{weierstrass}$$

$$\exists \lim_{n\to\infty}(S_n) \in \overline{\mathbb{R}}$$

• $\leftrightarrow$ If $\sum x_n$ SPT, it always has a sum

• If $\left.\begin{array}{l}\sum x_n \text{ SPT} \\ + \\ D\end{array}\right\} \to \sum_{n=1}^{\infty} x_n = \infty$

Proof:

• For a random $n \in \mathbb{N}$ $S_{n+1} - S_n = x_1 + x_2 + \ldots + x_n + x_{n+1} - (x_1 + x_2 + \ldots + x_n) = x_{n+1} > 0 \to$

$\to (S_n)$ - increasing $\xrightarrow{W}$ $\exists \lim_{n\to\infty} S_n$ ⊛

· If $\sum x_n$ SPT + D $\Bigg\} \to$ $\nexists \lim\limits_{n\to\infty} s_n$ or $\exists \lim\limits_{n\to\infty} s_n \in \{-\infty, \infty\}$

but from (*) $\to \exists \lim\limits_{n\to\infty} s_n$
(s_n) - increasing $\Bigg\} \to \lim\limits_{n\to\infty} s_n = \infty$

thus $\sum\limits_{n=1}^{\infty} x_n = \infty$

Theorem

Let $\sum x_n$ be a $\boxed{\text{random}}$ series of real numbers.

If $\sum x_n$ is C $\longrightarrow \lim\limits_{n\to\infty} x_n = 0$

Remark

The reverse statement of the theorem does not hold

$\lim\limits_{n\to\infty} x_n = 0 \longrightarrow$?

Examples:

$\sum \left(\frac{1}{2}\right)^n$ has $\lim\limits_{n\to\infty} x_n = 0$ and is C.

$\sum \frac{1}{n}$ has $\lim\limits_{n\to\infty} x_n = 0$ but is D.

Remark

In practice, we find it very usefull to apply the equivalant negation of the theorem.

$p \to q \leftrightarrow \bar{q} \to \bar{p}$

If $\lim\limits_{n\to\infty} x_n \neq 0 \to \sum x_n$ is D.

Examples:

1. $\sum\limits_{n\geq 1} \frac{1}{\sqrt[n]{n}}$

$x_n = \frac{1}{\sqrt[n]{n}}$ and $\lim\limits_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n$ is D

Bonus! $\left.\begin{array}{c}\sum x_n \text{ is SPT} \\ + \\ D\end{array}\right\} \to \exists \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$

As a conceguence

$$\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}\right)=\infty$$

2. $\sum_{n\geq 1}\left(1+\frac{1}{n}\right)^n$

$x_n = \left(1+\frac{1}{n}\right)^n$

$\lim_{n\to\infty} x_n = e \neq 0 \to \left.\begin{array}{l}\sum x_n - D \\ \text{SPT}\end{array}\right\} \to \exists \sum_{n=1}^{\infty} x_n = \infty$

! This theorem applies for random types of series

3. $\sum_{n\geq 1} (-1)^n$

$\nexists \lim_{n\to\infty} (-1)^n \neq 0$ (nu se aplica) $\to \sum x_n$ is D (without a sum)

because $\lim_{n\to\infty} x_{2n} = 1$

&

$\lim_{n\to\infty} x_{2n+1} = -1$

comperison criteria for SPT

criterion - sg

criteria - pl.

Remark: When appling comperison criteria for SPT we usually consider:

- either the geometric series $\sum_{n\geq 1} q^{n-1} \begin{cases} C: |q|<1 \\ D: \text{oth.}\end{cases}$

- or the generalized harmonic series $\sum_{n\geq 1} \frac{1}{n^\alpha} \begin{cases} C: \alpha>1 \\ D: \alpha\leq 1\end{cases}$

C1C: Let $\sum a_n$ and $\sum b_n$ be two SPT s.t. $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $a_n \leq b_n$.

Then ① $\sum b_n - C \rightarrow \sum a_n - C$

② $\sum a_n - D \rightarrow \sum b_n - D$

Remark: C1C does not cover 2 cases:

$$\boxed{\begin{array}{l} \cdot \sum a_n - C \\ \cdot \sum b_n - D \end{array}}$$

C2C: Consider $\sum a_n$ and $\sum b_n$ SPT s.t. $\exists\, \ell = \lim_{n\to\infty} \frac{a_n}{b_n} \in [0;\infty]$

(a) If $\ell = 0 \rightarrow$ ①, ② - C1C

(b) If $\ell \in (0;\infty) \rightarrow \boxed{\sim}$ $\sum a_n$ & $\sum b_n$ has the same nat.

(c) If $\ell = \infty \rightarrow$ ③ $\sum a_n - C \rightarrow \sum b_n - C$ (C1C reversed)

④ $\sum b_n - D \rightarrow \sum a_n - D$

C3C Let $\sum a_n$ & $\sum b_n$ SPT s.t. $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$

$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \Rightarrow \boxed{C1C}$

Theorem: (D'Alambert criterion for series)

Let $\sum a_n$ be a ~~series~~ SPT s.t. $\exists \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $(\in [0;\infty])$

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \rightarrow \sum a_n$ is C

$> 1 \rightarrow \sum a_n$ is D

!!! Remark: (Root CRITERION)

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Remark: D'Alambert & the root criterion are used if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ case in which we have to consider ~~one~~ other approches

Examples: Study the nature of the series:

a) $\sum_{n\geq 1} \frac{3^{n-1}}{4^{n-1}+5^{n-1}}$

Approach 1

$x_n = \frac{3^{n-1}}{4^{n-1}+5^{n-1}} > 0 \to$ SPT

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{3^n}{4^n+5^n} \cdot \frac{4^{n-1}+5^{n-1}}{3^{n-1}} = 3\cdot \lim_{n\to\infty} \frac{4^{n-1}+5^{n-1}}{4^n+5^n}$

$= 3 \lim_{n\to\infty} \frac{5^{n-1}\left(\frac{4}{5}\right)^{n-1}}{5^n\left(\frac{4}{5}\right)^n} = \frac{3}{5} < 1$

$\xrightarrow{D'A} \sum x_n$ is C.

Remark When in the formulation of x_n we encounter constants at the power of n we could try a comparison to the geometric series

Approach 2

- $x_n = \frac{3^{n-1}}{4^{n-1}+5^{n-1}} \leq \underbrace{\left(\frac{3}{4}\right)^{n-1}}_{y_n}$

- $\sum y_n$ is a geometric series of ratio $q = \frac{3}{4} \to C$
  $x_n \leq y_n$

$\xrightarrow{C/C} \sum x_n - C$

b) $\sum_{n\geq 1} \frac{1}{\sqrt{n^3+7n^2+5}}$

$\frac{1}{\sqrt{n^3+7n^2+5}} \leq \underbrace{\frac{1}{\sqrt{n^3}}}_{y_n} > 0, \quad x_n, y_n -$ SPT

$\sum y_n = \sum \frac{1}{n^{\frac{3}{2}}}$, $\frac{3}{2} > 1$ HARMONIC SERIES $\to \sum y_n - C$

$x_n \leq y_n$

$\xrightarrow[(1)]{C/C} \sum x_n - C$ ✓

! $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 \to (?)$

! $\lim_{n\to\infty} x_n = 0 \to (?)$

! $x_n < \frac{1}{n}$ D C/C does not cover

Approch 2

$\boxed{C2C\ b}$ is the most complete criteria to be appl[e]ed since it covers all possible cases

When in the formulation of the general term we have n^{constant} we try $\boxed{C2C\ b)}$ with $b_n = \frac{1}{n^\alpha}$

$$? \lim_{n\to\infty} \frac{x_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3+7n^2+5}}}{\frac{1}{n^\alpha}} = \lim_{n\to\infty} \frac{n^\alpha}{\sqrt{n^3+7n^2+5}} \underset{\text{only here}}{\overset{\alpha=\frac{3}{2}}{=}} 1$$

!! My goal is to have the limit $\in (0, \infty)$

I set α s.t. the lim $\in (0,\infty)$

$$\lim_{n\to\infty} \frac{x_n}{b_n} = 1 \in (0,\infty) \xrightarrow[b]{C2C} \sum x_n \sim \sum b_n = \sum \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}}$ is the harmonic series with $\alpha = \frac{3}{2} > 0 \to C \to$

$\to \sum x_n - C$

c) $\sum_{n\geq 1} \frac{1}{n \cdot n!}$

$\lim_{n\to\infty} x_n = 0 \to ?$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1}{(n+1)\cancel{(n+1)!}_{\,n+1}} \cdot \frac{n \cdot \cancel{n!}}{1} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{n+1} = 0 < 1 \to C$$

$\frac{1}{n \cdot n!} < \frac{1}{n^2}$ C...

d) $\sum_{n\geq 1} \left(\frac{n^2+3n+4}{n^2-2n-6}\right)^n$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{n^2+3n+4}{n^2-2n-6}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\cancel{n^2}+3n+4-\cancel{n^2}+2n+6}{n^2-2n-6}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{5n+10}{n^2-2n-6}\right)^{\frac{n^2-2n-6}{5n+10}}\right]^{n \cdot \frac{5n+1}{n^2-2n-6}} =$$

$$= e^{\lim_{n\to\infty} \frac{5n^2+1}{n^2-2n-6}} = e^5 \neq 0 \to \sum_{n\geq 1} x_n - D$$

BONUS $\sum_{n=1}^{\infty} x_n = \infty$

When in x_n we spot $f(n)^{g(n)}$ we ~~wa~~ might apply the root criterion (not for this case)

$$\lim_{n\to\infty} \sqrt[n]{x_n} = 1 \longrightarrow ?$$