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Algebra - Seminar 10

Theory

1. The matrix of a linear map

(A) $f: V \to V'$ - K-linear map, $B = (v_1, \dots, v_n)$ basis for V, $B' = (v_1', \dots, v_m')$ basis for V'. We can uniquely write the vectors in $f(B)$ as a linear combination of the vectors in B'

$$f(v_1) = a_{11}v_1' + \dots + a_{m1}v_m'$$
$$\dots$$
$$f(v_n) = a_{1n}v_1' + \dots + a_{mn}v_m'$$

The matrix of the linear map is:

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

! coordinates of the matrix of:
* a list of vectors → as rows
* a lin. map → as columns

If $V = V' \to [f]_{BB'} = [f]_B$

(B) $v \in V$: $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$

(C) $\text{rank}(f) = \text{rank}([f]_{BB'})$

(D) V, V', V'' - v.s. over K

$\dim V = n$, $\dim V' = m$, $\dim V'' = p$

$B = (v_1, \dots, v_n)$; $B' = (v_1', \dots, v_m')$; $B'' = (v_1'', \dots, v_p'')$ - bases of V, V', V''

$\forall f, g \in Hom_K(V, V')$, $\forall h \in Hom_K(V', V'')$, $\forall k \in K$

$$[f+g]_{BB'} = [f]_{BB'} + [g]_{BB'}$$
$$[k \cdot f]_{BB'} = k \cdot [f]_{BB'}$$
$$[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}$$

(E) $f \in End_K(V)$, then $f \in Aut_K(V) \Leftrightarrow \det([f]_B) \neq 0$, for B-basis of V

2. Change of basis

V - v.s. over K; $B = (v_1, \dots, v_n)$, $B' = (v_1', \dots, v_n')$ - bases of V. Then we can uniquely write:

$$\begin{cases} v_1' = t_{11} \cdot v_1 + t_{21} \cdot v_2 + \dots + t_{n1} v_n \\ \vdots \\ v_n' = t_{1n} v_1 + t_{2n} v_2 + \dots + t_{nn} v_n \end{cases}$$

* The matrix that has as columns the coord. of the vectors in B' in the basis B = change matrix from B to B'.

Not: $T_{BB'}$

- $T_{BB'}^{-1} = T_{B'B}$
- $[v]_B = T_{BB'} \cdot [v]_{B'}$
- $f \in End_K(V)$, B, B' - bases of V:

! $\boxed{[f]_{B'} = T_{BB'}^{-1} \cdot [f]_B \cdot T_{BB'}}$

Exercices

1. $f \in End_{\mathbb{R}}(\mathbb{R}^3)$, $f(x,y,z) = (x+y, y-z, 2x+y+z)$

$[f]_E = ?$, E = canonical basis of $\mathbb{R}^3$

$f(e_1) = f(1,0,0) = (1, 0, 2)$

$f(e_2) = f(0,1,0) = (1, 1, 1)$

$f(e_3) = f(0,0,1) = (0, -1, 1)$

$$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

2. $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$, $f(x,y,z) = (y, -x)$

bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$

$B' = (v_1', v_2') = ((1,1), (1,-2))$

$E' \to$ canonical basis of $\mathbb{R}^2$

$[f]_{BE'}$, $[f]_{BB'} = ?$

$f(v_1) = f(1,1,0) = (1,-1)$

$f(v_2) = f(0,1,1) = (1,0)$

$f(v_3) = f(1,0,1) = (0,-1)$

$[f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$

$f(v_1) = (1,-1) = a \cdot v_1' + b \cdot v_2' = a(1,1) + b(1,-2) = (a+b,\ a-2b)$

$\begin{cases} a+b = 1 \\ a-2b = -1 \end{cases}$ $-$

$3b = 2 \rightarrow b = \frac{2}{3} \rightarrow a = 1 - \frac{2}{3} = \frac{1}{3}$

$f(v_2) = (1,0) = (a+b,\ a-2b)$

$\begin{cases} a+b = 1 \\ a-2b = 0 \end{cases}$ $-$

$3b = 1 \rightarrow b = \frac{1}{3} \rightarrow a = \frac{2}{3}$

$f(v_3) = (0,-1) = (a+b,\ a-2b)$

$\begin{cases} a+b = 0 \quad \rightarrow a = -b \\ a-2b = -1 \end{cases}$

$3a = -1 \rightarrow a = -\frac{1}{3} \rightarrow b = \frac{1}{3}$

$[f]_{BB'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

3. $f \in Hom_K(\mathbb{R}^3, \mathbb{R}^4)$

$f(e_1) = (1,2,3,4)$

$f(e_2) = (4,3,2,1)$

$f(e_3) = (-2,1,4,1)$

(i) $f(v)$ - for every $v \in \mathbb{R}^3$

(ii) the matrix of f in the canonical basis

(iii) basis + dim. of $\ker f$, $\operatorname{Im} f$'.

(i) $v = (x,y,z)$

$v = x \cdot e_1 + y \cdot e_2 + z \cdot e_3$

$f(v) = x f(e_1) + y f(e_2) + z f(e_3)$

$= x(1,2,3,4) + y(4,3,2,1) + z(-2,1,4,1)$

$= (x+4y-2z,\ 2x+3y+z,\ 3x+2y+4z,\ 4x+y+z)$

(ii) $[f]_{EE'} = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix}$

(iii) $[f(v)]_{EE'} = [f]_{EE'} \cdot [v]_E$

$$[f]_{EE'} \cdot [v]_E = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 4 & -2 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 1 & 1 & 0 \end{pmatrix} \underset{\substack{L_3-3L_1 \\ L_4-4L_1}}{\overset{L_2-2L_1}{\sim}} \begin{pmatrix} 1 & 4 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -15 & 9 & 0 \end{pmatrix} \underset{L_4-3L_2}{\overset{L_3-2L_2}{\sim}} \begin{pmatrix} 1 & 4 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{pmatrix}$$

$-6z = 0 \to z = 0$

$-5y + 5z = 0 \to -5y = 0 \to y = 0$

$x + 4y - 2z = 0 \to x = 0$

$\operatorname{Ker} f = \{(0,0,0)\} \to \dim(\operatorname{Ker} f) = 0 \to$ basis of $\operatorname{Ker} f = \emptyset$

$\operatorname{Im} f = \langle (1,2,3,4), (4,3,2,1), (-2,1,4,1) \rangle$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -2 & 1 & 4 & 1 \end{pmatrix} \underset{L_3+2L_1}{\overset{L_2-4L_1}{\sim}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & 5 & 10 & 9 \end{pmatrix} \overset{L_3+L_2}{\sim} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & 0 & 0 & -6 \end{pmatrix} \to \operatorname{rank} = 3 \to$$

$\to \dim(\operatorname{Im} f) = 3$

basis of $\operatorname{Im} f = \{(1,2,3,4), (0,-5,-10,-15), (0,0,0,-6)\}$

4. Homework

5. $\mathbb{R}_2[X]$

bases: $E = (1, X, X^2)$

$B = (1, X-1, X^2+1)$

$f \in \operatorname{End}_{\mathbb{R}}(\mathbb{R}_2[X])$, $f(a_0 + a_1 X + a_2 X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2$

$[f]_E$, $[f]_B$

$f(1) = (1+0) + (0+0)X + (1+0)X^2 = 1 + X^2 = e_1 + e_3$

$f(X) = (0+1) + (1+0)X + (0+0)X^2 = 1 + X = e_1 + e_2$

$f(X^2) = (0+0) + (0+1)X + (0+1)X^2 = X + X^2 = e_2 + e_3$

$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$[\varphi]_B = T_{EB}^{-1} \cdot [\varphi]_E \cdot T_{EB}$

for T_{EB} we write the vectors of B in the base E

$b_1 = 1 = a \cdot e_1 + b \cdot e_2 + c \cdot e_2 = a \cdot 1 + b \cdot x + c \cdot x^2 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$
$= 1 \cdot e_1$

$b_2 = x-1 = (-1) \cdot 1 + 1 \cdot x + 0 \cdot x^2$

$b_3 = x^2 + 1 = 1 \cdot 1 + 0 \cdot x + 1 \cdot x^2$

$$T_{EB} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{EB}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ → rez, calcule - hw

$$[\varphi]_B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$ → calcule hw.

6. ${}_{\mathbb{R}}\mathbb{R}^2$ - v.s., bases: $B = (v_1, v_2) = ((1,2), (1,3))$
$B' = (v_1', v_2') = ((1,0), (2,1))$

$f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$

$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $\quad [g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$

$[2 \cdot f]_B = ?$, $[f+g]_B$, $[f \circ g]_{B'}$

$[2 \cdot f]_B = 2 \cdot [f]_B = 2 \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_B$

$[g]_B = T_{B'B}^{-1} \cdot [g]_{B'} \cdot T_{B'B}$

$T_{B'B}$ → vectors of B in B'

$v_1 = (1,2) = a(1,0) + b(2,1) = (a+2b, b)$ → $b = 2$
$a + 4 = 1$ → $a = -3$

$v_2 = (1,3) = (c+2d, d)$ → $d = 3$
$c = 1 - 6 = -5$

$T_{BB'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ $\quad T_{BB'}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$[f\circ g]_{B'} = [f]_{B'} \cdot [g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}\begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

$$[f]_{B'} = T_{BB'}^{-1} \cdot [f]_B \cdot T_{BB'} = T_{B'B} \cdot [f]_B \cdot T_{B'B}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$T_{BB'}^{-1} = T_{B'B} \qquad = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

7. $f: \mathbb{R}^2 \to \mathbb{R}^2$ - endom. , $f(x,y) = (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$

$[f]_E = ?$, f-automorphism?

$f(e_1) = f(1,0) = (\cos\alpha,\ \sin\alpha)$
$f(e_2) = f(0,1) = (-\sin\alpha,\ \cos\alpha)$

$$[f]_E = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\det([f]_E) = \cos^2\alpha + \sin^2\alpha = 1 \neq 0 \to f$-automorphism

8. V - v.s. of dim. 2 over the field $K = \mathbb{Z}_2$

Det: $|V|$, $|End_K(V)|$, $|Aut_K(V)|$

Use: $\varphi: End_K(V) \to M_2(K)$, $\varphi(f) = [f]_B$ is an isomorphism $\to$

$\to |End_K(V)| = |M_2(K)|$

$|V| = |K|^{\dim(V)} = 2^2 = 4$

$|M_2(K)| = 2^4 = 16 = |End_K(V)|$

- $Aut_K(V) \leftrightarrow \det([f]_B) \neq 0$

$ad - bc \neq 0 \to ad \neq bc$

I $a = \hat{1}$ $d = \hat{1}$, $b = \hat{0} \to c \in \{\hat{0}, \hat{1}\}$
$c = \hat{0} \to b \in \{\hat{0}, \hat{1}\}$

II $a = \hat{1}$ $d = \hat{0} \to b,c = \hat{1}$

III $a = \hat{0}$ $d \in \{\hat{0}, \hat{1}\}$, $b = c = \hat{1}$

$|Aut_K(V)| = 6$

lin. indep. columns.

I col : 3
II col : 2

$3 \cdot 2 = 6$

$v_2 \notin \langle v_1 \rangle$