24.11.2023
Analysis - Lecture 8

TAYLOR'S POLINOMIAL
- 2nd part -

Framework:

$\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$
$a \in A$ s.t. f is n times differentiable at a

$T_{n;a} f: \mathbb{R} \to \mathbb{R}$, $\underbrace{(T_{n;a} f)}_{\text{degree} \leq n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k, \forall x \in \mathbb{R}$

Recall: $\forall t \in \{0, \dots, n\}$, $(T_{n;a} f)^{(t)}(a) = f^{(t)}(a)$
$(T_{n;a} f)^{(t)}(x) = (T_{n-t;a} f)(x) \to$ inductively

TAYLOR'S FORMULA

Def: Framework: (the same to Taylor's polinomial).
TAYLOR'S REMINDER of RANK n, ATTACHED to THE FUNCTION f about the point a is
$R_{n;a} f: A \to \mathbb{R}$, given by: $f(x) = (T_{n;a} f)(x) + (R_{n;a} f)(x)$ $\forall x \in A$

Property: $\emptyset \neq A \subseteq \mathbb{R}$
$f: A \to \mathbb{R}$ $a \in \text{int} A$ and f - n time diff. at a.
Then $R_{n;a} f$ is n times diff. at a.

Proof: $\forall t \in \{0, \dots, n\}$ both f and $T_{n;a} f$ are t times diff at $a \to$
$\to (R_{n;a} f)$ is diff at a as difference of differentiable functions

Moreover: $\forall t \in \{0, \dots, n\}$ $(R_{n;a} f)^{(t)}(a) = f^{(t)}(a) - (T_{n;a} f)^{(t)}(a) = 0$

Property 2: In the hypothesis of property 1

$$\exists \lim_{x \to a} \frac{(R_{n;a} f)(x)}{(x-a)^n} = 0$$

Proof: $\lim_{x \to a} \frac{(R_{n;a} f)(x)}{(x-a)^n} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x \to a} \frac{f'(x) - (T_{n;a} f)(a)}{n(x-a)^{n-1}} \overset{\frac{0}{0}}{\underset{l'H}{=}} \dots =$

$= \frac{0}{n!} = 0$

Remark: We may define a new function

$(\alpha_{n;a} f)(x) = \begin{cases} \frac{(R_{n;a} f)(x)}{(x-a)^n} & : x \neq a \\ \dots \end{cases}$, $\alpha_{n;a} f: A \to \mathbb{R}$ - it is continuous on A $\to$

→ one way to express Taylor's formula is

$$f(x) = (T_{n;a} f)(x) + \underbrace{(x-a)^n \cdot (\alpha_{n;a} f)(x)}_{o(x^n),\ \lim_{x \to a} o(x) = 0} \rightarrow$$

→ PARETO'S FORM of the reminder

TAYLOR'S THEOREM ON THE FORM OF THE REMINDER

$$\begin{cases} \emptyset \neq J \subseteq \mathbb{R} \text{ - interval} \\ f \text{ is } (n+1) \text{ times differentiable on } J \\ a \in J \end{cases}$$

Then: $\forall x \in J$

$\forall p \in \mathbb{N}$ $\exists k \in \mathbb{R}$ s.t. $f(x) = (T_{n;a} f)(x) + (x-a)^p \cdot k$

$\exists c_x$ between x and a s.t.

s.t. $k = \dfrac{f^{(n+1)}(c_x)}{(n+1)!} \cdot (x - c_x)^{n-p+1}$

Rewriting it:

$\forall x \in J$, $\forall p \in \mathbb{N}$, $\exists c_x$ between x and a s.t.

$$\underbrace{(R_{n;a} f)(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)^p \cdot (x - c_x)^{n-p+1}}_{\text{SCHLÖMLICH - ROCHE reminder}}$$

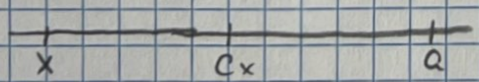
If $p = n+1$ → LAGRANGE'S REMINDER

$$(R^L_{n;a} f)(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)^{n+1}$$

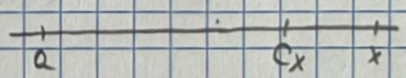
If $p = 1$ → CAUCHY'S REMINDER

$$(R^C_{n;a} f)(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)(x - c_x)^n$$

Remark: c_x is between x and a (due to Rolle's th.) because we know not for sure if $x < a$ or $x > a$



or



$f:[0,1]\to\mathbb{R}$, $f(t)=(1-t)x+ty$ → CONVEX COMBINATION of x and y

$f(0)=x$

$f(1)=y$

$c_x \to \exists t'\in(0,1)$ s.t. $c_x=(1-t')x+t'y$

$\exists c_x$ between x and 0 $\leftrightarrow$ $\exists\theta\in(0,1)$ s.t. $c_x=\theta x=$

$=\theta\cdot 0+\theta x \to c_x\in(0,x)$ or $(x,0)$

$\exists c_x$ between x & a $\to$ $\exists\theta\in(0,1)$ s.t. $c_x=\theta x+(1-\theta)a$

$\exists c_x$ between x & a $\leftrightarrow$ $\exists\theta\in(0,1)$ s.t. $c_x=\theta x+(1-\theta)a$

We get: MACLAURIN'S formula: ($a=0$ and L)

$\forall x\in A$, $\exists\theta\in(0,1)$ s.t. $f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+\frac{f^{(n+1)}(\theta x)}{(n+1)!}\cdot x^{n+1}$

Examples:

a) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$ – $f^{(n)}(x)=e^x$, $\forall n\in\mathbb{N}$ →

→ $f^{(n)}(0)=1$, $\forall n\in\mathbb{N}$

$\forall n\in\mathbb{N}$, $\forall x\in A$ $\exists\theta\in(0,1)$ s.t. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}$

$=1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^{n+1}}{(n+1)!}\cdot e^{\theta x}$

b) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$

$\forall n\in\mathbb{N}$ $f^{(n)}(x)=\sin\left(x+n\cdot\frac{\pi}{2}\right)$

$\forall n\in\mathbb{N}$, $\forall x\in\mathbb{R}$ $\to\exists\theta\in(0,1)$ s.t.

$\sin x=\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots-?+\frac{\sin(\theta x+(n+1)\frac{\pi}{2})}{(n+1)!}\cdot x^{n+1}$

TAYLOR'S SERIES EXPENSIONS

Def: $\emptyset\neq J\subset\mathbb{R}$

$f:J\to\mathbb{R}$ → indefinite diff on J

$a\in J$

$x\in J$

The function f may be expended in a Taylor series at the point x if

$\exists\lim_{n\to\infty}(R_{n,a}f)(x)=0$

Scanned with CamScanner

In this case $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$

If $a=0$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Applications: a) Study if the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ may be extended in a Taylor series at a random $x \in \mathbb{R}$

Choose $x \in \mathbb{R}$ random

Consider $(R_{n,0} f)(x) = e^{\theta x} \cdot \frac{x^{n+1}}{(n+1)!}$, $\theta \in (0,1)$

Because $\left.\begin{matrix} \theta \in (0,1) \\ x \in \mathbb{R} \to \text{constant} \end{matrix}\right\}$ θx is a constant

$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$

$a_n = \frac{x^n}{(x+1)!}$ $x \neq 0 \to \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{x^{n+1}} = \frac{|x|}{n+2} < 1 \to$

$\to \lim_{n\to\infty} a_n = 0$

$e^{\theta x}$ - constant $\to \lim_{n\to\infty} (R_{n,0} f)(x) = e^{\theta x} \cdot 0 = 0 \to$

$\to f = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \leftrightarrow e^x = \lim_{n\to\infty} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right)$, $\forall x \in \mathbb{R}$

$\boxed{x=1}$ $e = \lim_{n\to\infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right)$

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$

$(R^{L}_{n,0} f)(x) = \sin\left(\theta x + n\frac{\pi}{2}\right) \cdot \frac{x^{n+1}}{(n+1)!}$

$\left.\begin{matrix} \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \\ \left|\sin\left(\theta x + n\frac{\pi}{2}\right)\right| \leq 1 \end{matrix}\right\}$ $\lim_{n\to\infty} (R^{L}_{n,0} f)(x) = 0$

Dem Consider $x_n = a_n \cdot b_n$, $b_n = \frac{x^{n+1}}{(n+1)!}$

$a_n = \sin\left(\theta x + n \cdot \frac{\pi}{2}\right)$

We prove that $\lim_{n\to\infty} x_n = 0$

$\leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $|x_n - 0| < \varepsilon$

Choose $\varepsilon > 0$ random

? $\left.\begin{matrix} |x_n - 0| < \varepsilon \leftrightarrow |a_n| \cdot |b_n| < \varepsilon \\ |a_n| < 1 \end{matrix}\right\}$ $\to |b_n| < \varepsilon$

$\lim_{n\to\infty} b_n = 0 \xrightarrow{\text{for } \varepsilon} \exists n_\varepsilon$ s.t. $\forall n \geq n_\varepsilon$, $|b_n| < \varepsilon$

$\to |a_n| \cdot |b_n| < \varepsilon \quad \to \forall x \in \mathbb{R} \quad \sin x = \lim_{n\to\infty} \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$

$\boxed{x=0} \quad \sin 0 = 0$

$x = \frac{\pi}{2} \quad 1 = \sin \frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{\pi}{2} - \frac{\pi}{2^3 3!} + \ldots + \right)$

In a similar way $f: (-1, \infty) \to \mathbb{R}$, $f(x) = \ln(x+1)$ may be expanded in a Taylor series only on $(-1, 1)$.

Applications for limits of functions (wich do not with L'Hopital)

Recall that

$$\left. \begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \end{aligned} \right\} \forall x \in \mathbb{R}$$

Compute: $\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)}$

$$e^{x^4} = 1 + \frac{x^4}{1!} + \frac{x^8}{2!} + \frac{x^{12}}{3!} + \underbrace{O(x^4)}_{\downarrow \lim_{x\to 0} 0}$$

$$\sin(x^4) = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} + \underbrace{O(x^4)}_{\downarrow \lim_{x\to 0} 0}$$

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} + \underbrace{O(x^6)}_{\downarrow x\to 0 \; 0}$$

$$\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \lim_{x\to 0} \frac{\cancel{1} - \frac{x^4}{2!} + \frac{x^8}{4!} + O(\;)^{\to 0} - \cancel{1} - \frac{x^4}{1!} - \frac{x^8}{2!} - \frac{x^{12}}{3!} - O(\;)^{\to 0}}{x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} + O(\;)_{\to 0}}$$

$(x^4$

$$= \lim_{x\to 0} \frac{-x^4\left(\frac{1}{2!} + \frac{1}{1!}\right) + x^8\left(\frac{1}{4!} - \frac{1}{2!}\right) - \frac{x^{13}}{3!}}{x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!}}$$

$$= \lim_{x\to 0} \frac{-\frac{3}{2} + x^4(\;) - \frac{x^8}{3!}}{1 - \frac{x^8}{3!} + \frac{x^{16}}{5!}} = -\frac{3}{2}$$

POWER SERIES

Def: Consider $(a_n) \subseteq \mathbb{R}$. A series of real numbers of the form $\sum_{n\geq 0} a_n x^n$ is called A POWER SERIES. $x \in \mathbb{R}$

Remark: By convention if $x = 0$, $\sum_{n=0}^{\infty} a_n \cdot x^n = a_0$

Def: Consider $\sum a_n x^n$ a POWER SERIES. The set:
$\mathcal{C} = \{c \in \mathbb{R} \mid$ the series $\sum_{n\geq 0} a_n c^n$ is $C\}$ is called the CONVERGENT SET of the POWER SERIES

Remark: $\mathcal{C} \neq \emptyset$ because $0 \in \mathcal{C}$

Example: $a_n = (-1)^n \to \sum_{n\geq 0} (-1)^n \cdot x^n$

- $x = \frac{1}{2} \to \left.\begin{array}{l}(x^n)_{n\geq 0} \text{ is decreasing} \\ \lim_{n\to\infty} x^n = 0\end{array}\right\} \xrightarrow{\text{Leibnitz}} \sum_{n\geq 0} (-1)^n \cdot x_n - C \to$

$\to \frac{1}{2} \in \mathcal{C}$

- $x = 1 \to \sum_{n\geq 0} (-1)^n \cdot 1 = \sum_{n\geq 0} (-1)^n$ is $D \to 1 \notin \mathcal{C}$

Theorem (Abel for power series)

$\sum_{n\geq 0} a_n x^n$ - PS, then

a) If $0 \neq t \in \mathcal{C} \to \sum a_n \cdot u^n$ is A.C., $\forall u$ s.t. $|u| < |t|$

b) If $t \notin \mathcal{C} \to \sum a_n \cdot v^n$ is D, $\forall v$ s.t. $|t| < |v|$

Proof: We know $\sum a \cdot t^n$ is C ①

We want: $\forall u$ s.t. $|u| < |t|$ for $\sum a_n u^n$ to be AC

Consider u s.t. $|u| < |t|$, random

② $\sum |a_n \cdot u^n|$ is C $\leftrightarrow$ its sequence of partial sum is bounded

① $\to \exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $|a_n \cdot t^n| < M$

$|a_n \cdot u^n| = |a_n| \cdot |u_n|^n = |a_n t^n \cdot \frac{u^n}{t^n}| = |a_n \cdot t^n| \cdot \left|\frac{u}{t}\right|^n < M \cdot \left|\frac{u}{t}\right|^n = M \cdot \alpha^n$

$\alpha \overset{\text{not}}{=} \left|\frac{u}{t}\right| < 1$ fixed

$\to |a_n \cdot u^n| < \underbrace{M \cdot \alpha^n}_{y_n}$

$\sum y_n = M \cdot \underbrace{\sum \alpha^n}_{\text{geom. series of } q \in (0,1)} - C$

$\left.\right\} \to \sum |a_n \cdot u^n|$ is C $\leftrightarrow$

$\leftrightarrow \sum a_n \cdot u^n$ is AC $\to$

u-random, $|u| < |t|$

b) We know $t \notin \mathcal{C}$

Choose a random $v \in \mathbb{R}$ s.t. $|t| < |v|$ and assume that $\sum a_n \cdot v^n$ is $C \underset{|t|<|v|}{\overset{a)}{\Rightarrow}} \sum a_n \cdot t^n$ is C - contradiction
$\hookrightarrow t \in \mathcal{C}$

Remark: From Abel's theorem on power series we get that if $t \in \mathcal{C} \to (-|t|, |t|) \in \mathcal{C}$

Def: $R = \sup \mathcal{C} \in \overline{\mathbb{R}}$ is called THE CONVERGENCE RADIUS of the power series.

Remark: $\mathcal{C} \neq \emptyset \to \exists \sup \mathcal{C} \in \overline{\mathbb{R}}$

From Abel's Th we get that $(-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

the cases $x = -R$ and $x = R$ have to be studied separately

T (Cauchy Hadamard's Theorem)

If $\sum a_n \cdot x^n$ is a power series for which

$$\exists \lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|} \left(= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \right)$$

$$\to R = \frac{1}{\lambda} = \begin{cases} \infty, & \lambda = 0_+ \\ \frac{1}{\lambda}, & \lambda \in (0, \infty) \\ 0, & \lambda = \infty \end{cases}$$

$$\left(-\frac{1}{\lambda}; \frac{1}{\lambda}\right) \subseteq \mathcal{C} \subseteq \left[-\frac{1}{\lambda}; \frac{1}{\lambda}\right]$$

!! $x = -\frac{1}{\lambda}$ & $x = \frac{1}{\lambda}$ to be studied separately

Example

$$\sum_{n\geq 1} \frac{(-1)^n}{n} \cdot x^n \qquad a_n = \frac{(-1)^n}{n}, \ \forall n \geq 1, \quad a_0 = 0$$

determin $\mathcal{C}$

$$! \ \lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 \to R = \frac{1}{1} = 1$$

$$(-1, 1) \subseteq \mathcal{C} \subseteq [-1, 1]$$

$$!! \quad x = -1 \to \sum_{n\geq 1} \frac{(-1)^n}{n} (-1)^n = \sum \frac{(-1)^{2n}}{n} = \sum \frac{1}{n} - D \to -1 \notin \mathcal{C}$$

$$x = 1 \to \sum_{n\geq 1} \frac{(-1)^n}{n} \cdot 1 = \sum \frac{(-1)^n}{n} - C \ (\text{Leibnitz}) \to 1 \in \mathcal{C}$$

thus $\mathcal{C} = (-1; 1]$