Calculus - Homework 9

Det. the convergence radius & the convergence set.

1. $\sum_{n\geq 0} x^n$, $a_n = 1$

$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{1} = 1 \to R = 1$

$(-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

$(-1, 1) \subseteq \mathcal{C}$

- $x = 1 \to \sum_{n=0}^{\infty} 1^n = \infty \to \sum_{n\geq 0} 1^n$ - divergent $\to 1 \notin \mathcal{C}$
- $x = -1 \to \sum_{n\geq 0} (-1)^n$ - doesn't have a sum $\to -1 \notin \mathcal{C}$

$\to \mathcal{C} = (-1, 1)$

2. $\sum_{n\geq 1} \frac{1}{n} x^n \to a_n = \frac{1}{n}$

$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \to R = 1$

$(-1, 1) \subseteq \mathcal{C} \subseteq [-1, 1]$

- $x = 1 \to \sum_{n=1}^{\infty} \frac{1}{n} 1^n = \sum_{n=1}^{\infty} \frac{1}{n}$ - D $\to 1 \notin \mathcal{C}$
- $x = -1 \to \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n$

$\left.\begin{array}{l} \frac{1}{n} \text{ - decreasing} \\ \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \xrightarrow{\text{Leb.}} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ - C $\to -1 \in \mathcal{C}$

$\mathcal{C} = [-1; 1)$

3. $\sum_{n\geq 1} \frac{1}{n(n+1)} x^n \to a_n = \frac{1}{n(n+1)}$

$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n^2+n}{n^2+3n+2} = 1 \to R = 1$

$(-1, 1) \subseteq \mathcal{C} \subseteq [-1, 1]$

- $x = 1 \to \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \cdot 1^n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sim \sum_{n\geq 1} \frac{1}{n^2}$ - C $\to 1 \in \mathcal{C}$
- $x = -1 \to \sum_{n\geq 1} \frac{1}{n(n+1)} (-1)^n$

$\left.\begin{array}{l} \frac{1}{n(n+1)} \text{ - decreasing} \\ \lim_{n\to\infty} \frac{1}{n(n+1)} = 0 \end{array}\right\} \xrightarrow{\text{Leibnitz}} \sum_{n\geq 1} \frac{(-1)^n}{n(n+1)}$ - C $\to -1 \in \mathcal{C}$

$\to \mathcal{C} = [-1, 1]$

4. $\sum_{n\geq 0} \frac{1}{n!}\cdot x^n \to a_n = \frac{1}{n!}$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0_+ \to R = \frac{1}{\lambda} = \frac{1}{0_+} = \infty$$

$(-\infty, \infty) \subseteq \mathcal{C} \to \mathcal{C} = (-\infty, \infty)$

5. $\sum_{n\geq 1} n!\, x^n \to a_n = n!$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = +\infty \to R = 0$$

$\cdot\, x = 0 \to \sum_{n=1}^{\infty} n!\cdot 0^n = 0 \in \mathbb{R} \to C \to \mathcal{C} = \{0\}$

6. $\sum_{n\geq 0} (\sqrt[3]{n^2+n+1} - \sqrt[3]{n^2-n-1})^n x^n \to a_n = (\sqrt[3]{n^2+n+1} - \sqrt[3]{n^2-n-1})^n$

$$\cancel{\lambda = \lim_{n\to\infty} \frac{\sqrt[3]{(n+1)^2+n+2} - \sqrt[3]{(n+1)^2-n-2}}{\sqrt[3]{n^2+n+1} - \sqrt[3]{n^2-n-1}}}$$

$$\cancel{= \lim_{n\to\infty} \frac{[(n+1)^2+n+2 - (n+1)^2+n+2]\cdot[\sqrt[3]{(n^2+n+1)^2} + \sqrt[3]{(n^2+n+1)(n^2-n-1)} + \sqrt[3]{(n^2+n+1)^2}]}{[n^2+n+1-n^2+n+1][\sqrt[3]{((n+1)^2+n+2)^2} + \sqrt[3]{((n+1)^2+n+2)((n+1)^2-n-1)} + \sqrt[3]{((n+1)^2-n-2)^2}]}}$$

$$\cancel{= \lim_{n\to\infty} \frac{(2n+4)\cdot n^{\frac{4}{3}}[\sqrt[3]{(1+\frac{1}{n}+\frac{1}{n^2})^2} + \sqrt[3]{(1-\frac{1}{n^2}-\frac{1}{n^3})} + \sqrt[3]{(1+\frac{1}{n}+\frac{1}{n^2})^2}]}{(2n+2)\cdot n^{\frac{4}{3}}[\sqrt[3]{\ldots}}}$$

$$\cancel{= \lim_{n\to\infty} \frac{2n^{\frac{7}{3}}}{2n^{\frac{7}{3}}} = 1}$$

$$\lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (\sqrt[3]{n^2+n+1} - \sqrt[3]{n^2-n-1}) \overset{\infty-\infty}{=}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n+1-\cancel{n^2}+n+1}{\sqrt[3]{(n^2+n+1)^2} + \sqrt[3]{(n^4-n^2-1)} + \sqrt[3]{(n^2-n-1)^2}} =$$

$$= \lim_{n\to\infty} \frac{2n+2}{n\cdot n^{\frac{1}{3}}\left(\sqrt[3]{(1+\frac{1}{n}+\frac{1}{n^2})^2} + \sqrt[3]{1-\frac{1}{n^2}-\frac{1}{n^3}} + \sqrt[3]{(1-\frac{1}{n}-\frac{1}{n^2})^2}\right)} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+\overset{\to 0}{\frac{2}{n}})}{\cancel{n}\cdot n^{\frac{1}{3}}} = \lim_{n\to\infty} \frac{2}{\sqrt[3]{n}} = 0_+ \to R = \frac{1}{0_+} = +\infty \to \mathcal{C} = (-\infty, \infty)$$

7. $\sum_{n\geq 0} (n+1)^n x^n \to a_n = (n+1)^n$

$\lambda = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n+1 = \infty \to R = \frac{1}{\infty} = 0$

$x = 0 \to \sum_{n=0}^{\infty} (n+1)^n \cdot 0^n = 0 \in \mathbb{R} \to \sum_{n\geq 0} (n+1)^n x^n - C \to \mathcal{C} = \{0\}$

8. $\sum_{n\geq 0} \frac{(-1)^n}{n} x^n \to a_n = \frac{(-1)^n}{n}$

$\lambda = \lim_{n\to\infty} \frac{(-1)^{\cancel{n+1}}}{n+1} \cdot \frac{n}{\cancel{(-1)^n}} = -\lim_{n\to\infty} \frac{n}{n+1} = -1 \to R = -1$

$(-1,1) \subseteq \mathcal{C} \subseteq [-1,1]$

- $x = 1 \to \sum_{n\geq 0} \frac{(-1)^n}{n}$

  $\left.\begin{array}{l} \frac{1}{n} - \text{decreasing} \\ \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \xrightarrow{\text{Leibnitz}} \sum_{n\geq 0} \frac{(-1)^n}{n} - C \to 1 \in \mathcal{C}$

- $x = -1 \to \sum_{n\geq 0} \frac{(-1)^n}{n} (-1)^n = \sum_{n\geq 0} \frac{(-1)^{2n}}{n} = \sum_{n\geq 0} \frac{1}{n} - D \to -1 \notin \mathcal{C}$

$\to \mathcal{C} = (-1,1)$