Analysis - Homework 3

1. a) $x_n = \frac{3^n + 5^n}{7^n}$

$$x_{n+1} - x_n = \frac{3^{n+1} + 5^{n+1}}{7^{n+1}} - \frac{3^n + 5^n}{7^n}$$

$$= \frac{3^{n+1} - 7\cdot 3^n + 5^{n+1} - 7\cdot 5^n}{7^{n+1}}$$

$$= \frac{3^n(3-7) + 5^n(5-7)}{7^{n+1}}$$

$$= \frac{-4\cdot 3^n - 2\cdot 5^n}{7^{n+1}} = \frac{-2\overbrace{(2\cdot 3^n + 5^n)}^{>0}}{\underbrace{7^{n+1}}_{>0}} < 0 \to$$

$\to (x_n)$ - deacreasing $\xrightarrow{\text{Weierstrass T.}} \exists \lim\limits_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

(x_n) - deacreasing $\to x_1 < x_2 < x_3 < \dots < x_n$ $\Big\} \to$

$x_1 = \frac{3^1 + 5^1}{7^1} = \frac{8}{7}$

$\to \frac{8}{7} < x_n$

Let $T > 0$ $(> \frac{8}{7})$ s.t. $\forall\, n \in \mathbb{N}$ $x_n \in B(0,T) \to$

$\to (x_n)$ - bounded
(x_n) - decreasing $\Big\} \xrightarrow{\text{Weierstrass}} \exists \lim\limits_{n\to\infty} x_n \in \mathbb{R} \to$

$\to x_n$ - convergent

b) $x_n = \frac{(-1)^n}{n}$

$$x_{n+1} - x_n = \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} = \frac{n(-1)^{n+1} - (-1)^n\cdot(n+1)}{n(n+1)} =$$

$$= \frac{(-1)^n[-n-n-1]}{n(n+1)} = \frac{(-1)^n[-2n-1]}{n(n+1)} = \frac{(-1)^{n+1}[2n+1]}{n(n+1)} \to$$

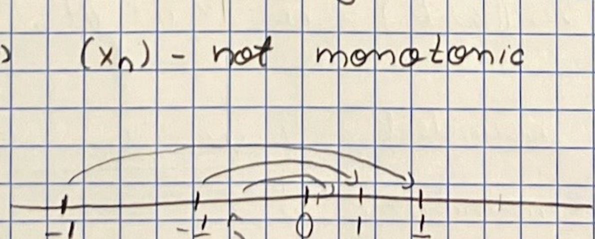
$\to$ varies according to the value of n $\to$ doesn't have a constant sign $\to$ (x_n) - not monotonic

$x_1 = \frac{-1}{1} = -1$

$x_2 = \frac{1}{2}$

$x_3 = \frac{-1}{3}$

$x_4 = \frac{1}{4}$



$\forall\, n \in \mathbb{N}$ $\quad x_1 \le x_n \le x_2 \to x_n \in [-1; \frac{1}{2}] \to$

$\to (x_n)$ - bounded

$\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0 \Leftrightarrow \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall n \geq n_\varepsilon,\ |x_n - 0| < \varepsilon$

Consider $\varepsilon > 0$ randomly chosen

$? \ |x_n - 0| < \varepsilon \Leftrightarrow \left|\frac{(-1)^n}{n}\right| < \varepsilon \Leftrightarrow 0 < \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$

According to Archimedes Theorem $\exists n_\varepsilon \in \mathbb{N}$ s.t. $0 < \frac{1}{\varepsilon} < n_\varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{n_\varepsilon} < \varepsilon$ (*)

Consider $n \in \mathbb{N}$, $n \geq n_\varepsilon$ randomly chosen

$$\left.\begin{array}{l} n \geq n_\varepsilon > 0 \Leftrightarrow \frac{1}{n} < \frac{1}{n_\varepsilon} \xrightarrow{(*)} \frac{1}{n} < \varepsilon \\ \varepsilon, n \text{ - random} \end{array}\right\} \rightarrow$$

$\rightarrow \forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon,\ |x_n - 0| < \varepsilon \rightarrow$

$\rightarrow \lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0 \in \mathbb{R} \rightarrow (x_n)$ - convergent

c) $x_n = \frac{4^n}{n!}$

$$x_{n+1} - x_n = \frac{4^{n+1}}{(n+1)!} - \frac{4^n}{n!}^{\,n+1} = \frac{4^{n+1} - (n+1)\cdot 4^n}{(n+1)!} = \frac{4^n(4-n-1)}{(n+1)!} =$$

$$= \frac{\overbrace{4^n}^{>0}(3-n)}{\underbrace{(n+1)!}_{>0}}$$

$3 - n \leq 0$, $n \geq 3 \rightarrow x_{n+1} - x_n \leq 0$, $n \geq 3 \rightarrow (x_n)$ - nonincreasing for $n \geq 3$

$3 - n > 0$, $n < 3 \rightarrow x_{n+1} - x_n > 0$, $n < 3 \rightarrow (x_n)$ - increasing for $n < 3$

$\rightarrow (x_n)$ - monotonic

$$\left.\begin{array}{l} (x_n) \text{ - increasing for } n < 3 \\ (x_n) \text{ - nonincreasing for } n \geq 3 \end{array}\right\} \rightarrow x_n \leq x_3 = \frac{4^3}{3!} = \frac{64}{6} = \frac{32}{3} \rightarrow$$

$\rightarrow \exists T > 0 \left(\geq \frac{32}{3}\right)$ s.t. $\forall n \in \mathbb{N},\ x_n \in B(0, T) \rightarrow$

$$\left.\begin{array}{l} \rightarrow (x_n) \text{ - bounded} \\ (x_n) \text{ - monotonic} \end{array}\right\} \xrightarrow{\text{Weierstrass}} \exists \lim\limits_{n\to\infty} x_n \in \mathbb{R} \rightarrow (x_n) \text{ - convergent}$$

d) $x_n = \frac{n}{n^2+1}$

$$x_{n+1} - x_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1}$$

$$= \frac{(n+1)(n^2+1) - [(n+1)^2+1]n}{(n^2+1)[(n+1)^2+1]}$$

$$= \frac{(n+1)(n^2+1) - n(n+1)^2 - n}{(n^2+1)[(n+1)^2+1]}$$

$$= \frac{(n+1)(n^2+1-n^2-n) - n}{(n^2+1)[(n+1)^2+1]}$$

$$= \frac{(1+n)(1-n) - n}{\underbrace{(n^2+1)[(n+1)^2+1]}_{>0}} < 0 \to (x_n) \text{ - decreasing}$$

$n \in \mathbb{N} \to 1 \le n \quad 1-n \le 0 \to (1+n)(1-n) \le 0$

$n \ge 1$

$-n \le -1$ } →

$\to (1+n)(1-n) - n < 0$

$\to (x_n)$ - decreasing $\to x_1 > x_2 > \dots > x_n$

$x_1 = \frac{1}{2}$ } →

$\to \exists\ T > 0\ (> \frac{1}{2})$ s.t. $\forall\, n \in \mathbb{N},\ x_n \in B(0,T) \to$

$\to (x_n)$ - bounded

(x_n) - decreasing } $\xrightarrow{\text{Weierstrass}} \exists \lim_{n\to\infty} x_n \in \mathbb{R} \to$

$\to (x_n)$ - convergent

2) a) $\lim_{n\to\infty} \frac{2n}{n^2+1} = 0$

$$\lim_{n\to\infty} \frac{2n}{n^2+1} = \lim_{n\to\infty} \frac{2n}{n^2\left(1+\left(\frac{1}{n}\right)^2\right)} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} =$$

$$= 2 \lim_{n\to\infty} \frac{1}{n} = 0$$

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^2 = 0 \leftrightarrow \lim_{n\to\infty} \frac{1}{n} = 0 \leftrightarrow \forall \varepsilon > 0, \exists\, n_\varepsilon \in \mathbb{N}$ s.t.

$\forall\, n \ge n_\varepsilon,\ \left|\frac{1}{n} - 0\right| < \varepsilon$

Consider $\varepsilon > 0$ randomly chosen

$? \ \left|\frac{1}{n} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$

According to Archimedes Theorem $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n_\varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{n_\varepsilon} < \varepsilon \quad (*)$

Consider $n \geq n_\varepsilon$, $n \in \mathbb{N}$ - randomly chosen

$n \geq n_\varepsilon \rightarrow \frac{1}{n} < \frac{1}{n_\varepsilon} \xrightarrow{(*)} \frac{1}{n} < \varepsilon \rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$ $\left.\begin{matrix} \\ \varepsilon, n\text{-random} \end{matrix}\right\} \rightarrow$

$\rightarrow \forall \varepsilon > 0\ \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall\, n \geq n_\varepsilon$, $\left|\frac{1}{n} - 0\right| < \varepsilon \rightarrow$

$\rightarrow \lim\limits_{n \to \infty} \frac{1}{n} = 0$

b) $\lim\limits_{n \to \infty} \frac{2n^2}{-2n+4} = -\infty$

$$\lim_{n \to \infty} \frac{2n^2}{-2n+4} = \lim_{n \to \infty} \frac{2n^2}{n\left(-2+\frac{4}{n}\right)} \overset{(*)}{=} \lim_{n \to \infty} \frac{2n^2}{-2n} = \lim_{n \to \infty} -n \overset{(**)}{=} -\infty$$

$(*)$ $\lim\limits_{n \to \infty} \frac{4}{n} = 4 \lim\limits_{n \to \infty} \frac{1}{n} = 4 \cdot 0 = 0 \Leftrightarrow \forall \varepsilon > 0, \exists\, n_\varepsilon \in \mathbb{N}$, s.t. $\forall n \geq n_\varepsilon$
$\left|\frac{1}{n} - 0\right| < \varepsilon$

Consider $\varepsilon > 0$ randomly chosen

$? \ \left|\frac{1}{n} - 0\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$

According to Archimedes Theorem $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} > n_\varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{n_\varepsilon} < \varepsilon \quad *$

Consider $n \in \mathbb{N}$, $n \geq n_\varepsilon$ randomly chosen

$n \geq n_\varepsilon \rightarrow \frac{1}{n} < \frac{1}{n_\varepsilon} \xrightarrow{(*)} \frac{1}{n} < \varepsilon \rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$ $\left.\begin{matrix} \\ \varepsilon, n\text{-random} \end{matrix}\right\} \rightarrow$

$\rightarrow \forall \varepsilon > 0, \exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\left|\frac{1}{n} - 0\right| < \varepsilon \rightarrow \lim\limits_{n \to \infty} \frac{1}{n} = 0$

$\circledast\ast$ $\lim_{n\to\infty} -n = -\infty \leftrightarrow \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $-n < -\varepsilon$

Consider $\varepsilon > 0$ randomly chosen

? $-n < -\varepsilon \leftrightarrow \varepsilon < n$

According to Archimedes Theorem $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\varepsilon < n_\varepsilon$ $(*)$

Consider $n \in \mathbb{N}$, $n \geq n_\varepsilon$ randomly chosen

$$\left.\begin{array}{l} n \geq n_\varepsilon \leftrightarrow n_\varepsilon \leq n \\ (*)\ \varepsilon < n_\varepsilon \end{array}\right\} \to \varepsilon < n \to \left.\begin{array}{l} -n < -\varepsilon \\ \varepsilon, n\text{ - random} \end{array}\right\} \to$$

$\to \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $-n < -\varepsilon \to \lim_{n\to\infty} -n = -\infty$

3) a) $\lim_{n\to\infty} \frac{3^n+1}{5^n+1} = ?$

$$\lim_{n\to\infty} \frac{3^n+1}{5^n+1} = \lim_{n\to\infty} \frac{3^n\left[1+\left(\frac{1}{3}\right)^n\right]}{5^n\left[1+\left(\frac{1}{5}\right)^n\right]} = \lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$$

$\left|\frac{1}{3}\right| < 1 \to \lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0$

$\left|\frac{1}{5}\right| < 1 \to \lim_{n\to\infty} \left(\frac{1}{5}\right)^n = 0$

$\left|\frac{3}{5}\right| < 1 \to \lim_{n\to\infty} \left(\frac{3}{5}\right)^n = 0$

b) $\lim_{n\to\infty} \frac{9^n+(-3)^n}{9^{n-1}+3} = ?$

$$\lim_{n\to\infty} \frac{9^n+(-3)^n}{9^{n-1}+3} = \lim_{n\to\infty} \frac{9^n\left[1+\left(\frac{-3}{9}\right)^n\right]}{9^{n-1}\left[1+\underbrace{\frac{3}{9^{n-1}}}_{\searrow 0}\right]} = \lim_{n\to\infty} \frac{9^n}{9^{n-1}} = 9$$

$\left|-\frac{3}{9}\right| = \left|-\frac{1}{3}\right| < 1 \to \lim_{n\to\infty} \left(\frac{-3}{9}\right)^n = 0$

c) $\lim_{n\to\infty} \left(\sin\frac{\pi}{10}\right)^n = ?$

$0 < \frac{\pi}{10} < \frac{\pi}{2} \to \left|\sin\frac{\pi}{10}\right| < 1 \to \lim_{n\to\infty} \left(\sin\frac{\pi}{10}\right)^n = 0$

d) $\lim\limits_{n\to\infty} \left(\sqrt{4n^2+2n+1}-2n\right) = ?$

$$\lim_{n\to\infty} \left(\sqrt{4n^2+2n+1}-2n\right) = \lim_{n\to\infty} \frac{\cancel{4n^2}+2n+1-\cancel{4n^2}}{\sqrt{4n^2+2n+1}+2n} =$$

$$= \lim_{n\to\infty} \frac{2n+1}{|2n|\sqrt{1+\frac{1}{2n}+\frac{1}{4n^2}}+2n} = \lim_{n\to\infty} \frac{2n\left(1+\frac{1}{2n}\right)}{4n} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{2n} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n} \overset{(**)}{=} \frac{1}{2}\cdot 0 = 0$$

$$\lim_{n\to\infty} \frac{1}{4n^2} = \lim_{n\to\infty}\left(\frac{1}{2n}\right)^2 = 0^2 = 0$$

e) $\lim\limits_{n\to\infty} \left(7+\frac{1-2n^3}{3n^4+2}\right)^2 = ?$

$$\lim_{n\to\infty}\left(\overset{3n^4+2}{7}+\frac{1-2n^3}{3n^4+2}\right)^2 = \lim_{n\to\infty}\left(\frac{21n^4+14+1-2n^3}{3n^4+2}\right)^2 =$$

$$= \lim_{n\to\infty} \frac{n^4\left(21+\frac{15}{n^4}\overset{\to 0}{}-\frac{2}{n}\overset{\to 0}{}\right)}{n^4\left(3+\underbrace{\frac{2}{n^4}}_{\to 0}\right)} = \frac{21}{3} = 7$$

$$\lim_{n\to\infty}\frac{15}{n^4} = 15\lim_{n\to\infty}\left(\frac{1}{n}\right)^2 \overset{(**)}{=} 0$$

$$\lim_{n\to\infty}\frac{2}{n} = 2\lim_{n\to\infty}\frac{1}{n} \overset{(**)}{=} 0$$

$$\lim_{n\to\infty}\frac{2}{n^4} = 2\lim_{n\to\infty}\left(\frac{1}{n}\right)^4 \overset{(**)}{=} 0$$

$\lim\limits_{n\to\infty}\frac{1}{n} = 0 \Leftrightarrow \forall \varepsilon>0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon, \left|\frac{1}{n}-0\right|<\varepsilon$

Consider $\varepsilon>0$ randomly chosen

(?) $\left|\frac{1}{n}-0\right|<\varepsilon \Leftrightarrow \frac{1}{n}<\varepsilon \Leftrightarrow \frac{1}{\varepsilon}<n$

According to Archimedes Theorem $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon}<n_\varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{n_\varepsilon}<\varepsilon$ (*)

Consider $n \geq n_\varepsilon$, $n\in\mathbb{N}$ randomly chosen

$n \geq n_\varepsilon \to \frac{1}{n} < \frac{1}{n_\varepsilon} \xrightarrow{(*)} \frac{1}{n}<\varepsilon$

n, ε - random $\Big\} \to \forall \varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$ s.t. $\forall n\geq n_\varepsilon, \left|\frac{1}{n}-0\right|<\varepsilon \to$

$\to \lim\limits_{n\to\infty}\frac{1}{n} = 0$ (**)

f) $\lim\limits_{n\to\infty} \left(\sqrt[3]{n^3+n+3} - \sqrt[3]{n^3+1}\right) = ?$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^3+n+3-(n^3+1)}{\sqrt[3]{(n^3+n+3)^2}+\sqrt[3]{(n^3+n+3)(n^3+1)}+\sqrt[3]{(n^3+1)^2}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}+n+3-\cancel{n^3}-1}{\sqrt[3]{n^6+n^2+9+2n^4+6n^3+6n}+\sqrt[3]{n^6+n^4+4n^3+n+3}+\sqrt[3]{n^6+2n^3+1}}$$

$$= \lim_{n\to\infty} \frac{n+2}{n^2\sqrt[3]{1+\underbrace{\frac{1}{n^4}}_{\to 0}+\underbrace{\frac{9}{n^6}}_{\to 0}+\underbrace{\frac{2}{n^2}}_{\to 0}+\underbrace{\frac{6}{n}}_{\to 0}+\underbrace{\frac{6}{n^3}}_{\to 0}}+n^2\sqrt{1+\underbrace{\frac{1}{n^2}}_{\to 0}+\underbrace{\frac{4}{n}}_{\to 0}+\underbrace{\frac{1}{n^5}}_{\to 0}+\underbrace{\frac{3}{n^6}}_{\to 0}}+n^2\sqrt{1+\underbrace{\frac{2}{n^3}}_{\to 0}+\underbrace{\frac{1}{n^6}}_{\to 0}}}$$

$$= \lim_{n\to\infty} \frac{n(1+\overbrace{\frac{2}{n}}^{\to 0})}{3n^2} = \lim_{n\to\infty} \frac{1}{3n} = 0$$

g) $\lim\limits_{n\to\infty} \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}} = ?$

$$\lim_{n\to\infty} \left(\frac{n^3+5n+1}{n^2-1}\right)^{\frac{1-5n^4}{6n^4+1}} = \lim_{n\to\infty} \left[\frac{n^3(1+\overbrace{\frac{5}{n^2}}^{\to 0}+\overbrace{\frac{1}{n^3}}^{\to 0})}{n^2(1-\underbrace{\frac{1}{n^2}}_{\to 0})}\right]^{\frac{n^4[\overbrace{\frac{1}{n^4}}^{\to 0}-5]}{n^4[6+\underbrace{\frac{1}{n^4}}_{\to 0}]}} =$$

$$= \lim_{n\to\infty} n^{-\frac{5}{6}} = \lim_{n\to\infty} \frac{1}{n^{\frac{5}{6}}} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{5}{6}} = 0^{\frac{5}{6}} = 0$$

h) $\lim\limits_{n\to\infty} \left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)\right] = ?$

$$x_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right) = \frac{2-1}{2}\cdot\frac{3-1}{3}\cdot\ldots\cdot\frac{n-1}{n} = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\ldots\cdot\frac{\cancel{n-1}}{n} =$$

$$= \frac{1}{n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

4. $t \in \mathbb{R}$

a) Let (x_n) s.t. $\forall n \in \mathbb{N}$, $x_n = \frac{[nt]}{n} \in \mathbb{Q}$

$$nt \leq [nt] \leq nt+1 \quad |:n$$

$$t \leq \frac{[nt]}{n} \leq \frac{nt+1}{n}$$

$\to t \leftarrow$ $\xrightarrow{\text{Weierstrass Consequence}}$ $\lim\limits_{n\to\infty} x_n = t \to (x_n)$ - converging to t

$$\lim_{n\to\infty} \frac{nt+1}{n} = \lim_{n\to\infty} \frac{n(t+\overbrace{\frac{1}{n}}^{\to 0})}{n} = \lim_{n\to\infty} \frac{nt}{n} = t$$

b) Let (x_n) s.t. $\forall n \in \mathbb{N}$ $x_n = \frac{[nt]}{n} \in \mathbb{Q}$

$nt - 1 \leq [nt] \leq nt \quad |:n$

$\frac{nt-1}{n} \leq \frac{[nt]}{n} \leq t$

$\to t \leftarrow$ weirstrass consequence $\Rightarrow \lim_{n\to\infty} x_n = t \Rightarrow$

$\to (x_n)$ converging to t

$\lim_{n\to\infty} \frac{nt-1}{n} = \lim_{n\to\infty} \frac{n(t-\overset{\to 0}{\frac{1}{n}})}{n} = t$

$* *$

5. $a > 0$

$x_0 \in \mathbb{R}$ s.t. $0 < x_0 < \frac{1}{a}$

$(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = 2x_n - a x_n^2$, $\forall n \in \mathbb{N}$

a) $x_n < \frac{1}{a}$, $\forall n \in \mathbb{N}$?

I Verification

hypothesis $\to$ $x_0 < \frac{1}{a}$

$x_1 = 2 \cdot x_0 - a \cdot x_0^2 \overset{x_0 < \frac{1}{a}}{<} 2 \cdot \frac{1}{a} - a \cdot \frac{1}{a^2} = \frac{2-1}{a} = \frac{1}{a}$ "T"

II $P(k) \to P(k+1)$

$P(k)$: $x_k = 2 \cdot x_{k-1} - a x_{k-1}^2 < \frac{1}{a}$

$P(k+1)$: $x_{k+1} = 2x_k - a x_k^2 \overset{?}{<} \frac{1}{a}$

$x_{k+1} = 2x_k - a x_k^2 \overset{x_k < \frac{1}{a}}{<} 2 \cdot \frac{1}{a} - a \cdot \left(\frac{1}{a}\right)^2 = \frac{2}{a} - \frac{1}{a} = \frac{1}{a} \Rightarrow$

$\Rightarrow x_{k+1} < \frac{1}{a} \Rightarrow x_n < \frac{1}{a}$, $\forall n \in \mathbb{N}$

b) $0 < x_n$, $\forall n \in \mathbb{N}$?

I Verification

hipothesis $\to$ $0 < x_0$

$x_1 = 2 x_0 - a x_0^2 \overset{x_0 > 0}{>} 2 \cdot 0 - a \cdot 0^2 = 0 \to x_1 > 0$

II $P(k) \to P(k+1)$

$P(k)$: $x_k > 0$

$P(k+1)$: $x_{k+1} = 2x_k - a \cdot x_k^2 > 0$?

$$x_{k+1} = 2x_k - a x_k^2 \overset{x_k>0}{>} 2\cdot 0 - a\cdot 0^2 = 0 \to x_{k+1} > 0 \to$$

$\to 0 < x_n, \forall n \in \mathbb{N}$

c) (x_n) - increasing?

$$x_{n+1} - x_n = 2\cdot x_n - a x_n^2 - x_n = x_n - a x_n^2 = x_n(1 - a x_n)$$

$0 < x_n < \frac{1}{a} \quad | \cdot a$

$0 < a x_n < 1 \quad | \cdot (-1)$

$0 > -a x_n > -1 \quad | +1$

$\left.\begin{array}{l} 1 > 1 - a x_n > 0 \\ \frac{1}{a} > x_n > 0 \end{array}\right\} \to x_n(1 - a x_n) > 0 \to x_{n+1} - x_n > 0 \to$

$\to (x_n)$ - increasing

d) $\lim\limits_{n\to\infty} x_n = ?$

$\left.\begin{array}{l} (x_n) \text{ - increasing} \\ 0 < x_n < \frac{1}{a} \end{array}\right\} \to 0 < x_0 < x_1 < \dots < x_n < \frac{1}{a}$

$0 < x_n < \frac{1}{a}, \forall n \in \mathbb{N} \to \left.\begin{array}{l} (x_n) \text{ - bounded} \\ (x_n) \text{ - increasing} \end{array}\right\} \xrightarrow{\text{Weierstrass}}$

$\to \exists \lim\limits_{n\to\infty} x_n = \sup\{x_n : x \in \mathbb{N}\} = \frac{1}{a}$

4. b) (**) Let (y_n) s.t. $\forall n \in \mathbb{N}$, $y_n = \frac{\pi}{n+1}$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{\pi}{n+1} = 0$$

Let (z_n), $z_n = x_n + y_n$, $\forall n \in \mathbb{N}$, $(z_n) \in \mathbb{R} \setminus \mathbb{Q}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = t + 0 = t \in \mathbb{R} \to$$

$\to (z_n)$ - converging to t