16.11.2023

Algebra - Seminar 7

1. Det. a basis and the dim:

$A = \{(x, y, z) \mid z = 0\}$

$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$C = \{(x, y, z) \in \mathbb{R} \mid x = y = z\}$

$* A = \{(x, y, 0) \mid x, y \in \mathbb{R}\} = \{(x, 0, 0) + (0, y, 0) \mid x, y \in \mathbb{R}\} =$

$= \{x(1,0,0) + y(0,1,0) \mid x, y \in \mathbb{R}\} = \underbrace{\langle (1,0,0), (0,1,0) \rangle}_{\text{generators for } A}$

$k_1, k_2 \in \mathbb{R}, \quad k_1(1,0,0) + k_2(0,1,0) = (0,0,0) \rightarrow$

$\rightarrow k_1 = k_2 = 0 \rightarrow (1,0,0), (0,1,0)$ - lin independent $\rightarrow$

$\rightarrow \{(1,0,0), (0,1,0)\}$ - basis for A $\rightarrow \dim A = 2$

$* B = \{(-y-z, y, z) \mid y, z \in \mathbb{R}\} = \{(-y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{y(-1, 1, 0) + z(-1, 0, 1) \mid y, z \in \mathbb{R}\} = \langle (-1,1,0), (-1,0,1) \rangle$ gener. for B

$k_1, k_2 \in \mathbb{R}, \quad k_1(-1,1,0) + k_2(-1,0,1) = (0,0,0) \rightarrow$

$\rightarrow \left.\begin{array}{l} -k_1 - k_2 = 0 \\ k_1 = 0 \\ k_2 = 0 \end{array}\right\} \rightarrow (-1,1,0), (-1,0,1)$ - lin. indep. $\rightarrow$

$\rightarrow \{(-1,1,0), (-1,0,1)\}$ - basis for B $\rightarrow \dim B = 2$

$* C = \{(x, x, x) \mid x \in \mathbb{R}\} = \{x(1,1,1) \mid x \in \mathbb{R}\} = \langle (1,1,1) \rangle$ - gen for C

$k_1 \in \mathbb{R}, \quad k_1(1,1,1) = (0,0,0) \leftrightarrow k_1 = 0 \rightarrow$ lin indep $\rightarrow$

$\rightarrow \{(1,1,1)\}$ - basis for C $\rightarrow \dim C = 1$

2. $S = \{(x_1, \ldots, x_n) \in K^n \mid x_1 + \ldots + x_n = 0\}$

(i) $S \leq K^n$?

(ii) det the basis & dim of S

(i) Let $k_1, k_2 \in K$ & $v_1, v_2 \in S$

$k_1 v_1 + k_2 v_2 = k_1(x_1, \dots, x_n) + k_2(x_1', \dots, x_n') =$

$= (k_1 x_1 + k_2 x_1', \dots, k_1 x_n + k_2 x_n') \in S \to S \le K^n$

$(k_1 x_1 + k_2 x_1') + \dots + (k_1 x_n + k_2 x_n') = k_1 \underbrace{(x_1 + \dots + x_n)}_{\overset{\shortparallel}{0},\ v_1 \in S} + k_2 \underbrace{(x_1' + \dots + x_n')}_{\overset{\shortparallel}{0},\ v_2 \in S} = 0$

$(0, 0, \dots, 0) \in S \to S \neq \emptyset$

$\to S \le K^n$

(ii) $x_1 = -x_2 - \dots - x_n$

$S = \{(-x_2 - \dots - x_n, x_2, \dots, x_n)\}$

$= \{(-x_2, x_2, \dots, 0) + \dots + (-x_n, 0, \dots, x_n)\}$

$= \langle (-1, 1, \dots, 0) \dots (-1, 0, \dots 1) \rangle$ - gen. for S

$k_1, \dots, k_{n-1} \in K$, $k_1(-1, 1, \dots, 0) + \dots + k_{n-1}(-1, 0, \dots 1) = (0, \dots, 0)$

$\Leftrightarrow$ $k_1 = k_2 = \dots = k_{n-1} = 0 \to$ lin indep

$\{(-1, 1, \dots, 0) \dots (-1, 0, \dots 1)\}$ - basis for $S \to \dim S = n-1$

3. basis & dim ${}_{\mathbb{C}}\mathbb{C}$, ${}_{\mathbb{R}}\mathbb{C}$

• $\{1, i\}$ → lin dep in ${}_{\mathbb{C}}\mathbb{C}$
→ lin indep in ${}_{\mathbb{R}}\mathbb{C}$

${}_{\mathbb{C}}\mathbb{C} = \{z \mid z \in \mathbb{C}\} = \{z \cdot 1 \mid z \in \mathbb{C}\} = \langle 1 \rangle$ - gen for ${}_{\mathbb{C}}\mathbb{C}$

$k_1 \cdot 1 = 0 \Leftrightarrow k_1 = 0 \to (1)$ - lin indep

$\{(1)\}$ - basis for ${}_{\mathbb{C}}\mathbb{C} \to \dim_{\mathbb{C}} \mathbb{C} = 1$

${}_{\mathbb{R}}\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} = \langle 1, i \rangle$ - gen for ${}_{\mathbb{R}}\mathbb{C}$

$\langle 1, i \rangle$ - basis of ${}_{\mathbb{R}}\mathbb{C} \to \dim_{\mathbb{R}} \mathbb{C} = 2$

$\exists\, k_1, k_2 \in \mathbb{C}$, $k_1 = 1$, $k_2 = i \to k_1 \cdot 1 + k_2 \cdot i = 1 - 1 = 0 \to$

$\to \{1, i\}$ - lin dep in ${}_{\mathbb{C}}\mathbb{C}$

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$k_1, k_2 \in \mathbb{R}$, $k_1 \cdot 1 + k_2 \cdot i = 0 \cdot 1 + 0 \cdot i \Leftrightarrow k_1 = k_2 = 0 \Rightarrow$

$\Rightarrow \{1, i\}$ - lin indep in ${}_{\mathbb{R}}\mathbb{C}$

4. $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (y, -x)$

f - linear map?

basis & dim of $\operatorname{Ker} f$ & $\operatorname{Im} f = ?$

$$\left.\begin{array}{l} \forall u, v \in \mathbb{R}^3 \\ f(u+v) = f(u) + f(v) \\ \forall k \in \mathbb{R} \\ f(k \cdot u) = k \cdot f(u) \end{array}\right\} \Rightarrow f \text{ - linear map}$$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in \mathbb{R}^3$

$f(u+v) = f(x_1+x_2, y_1+y_2, z_1+z_3) = (y_1+y_2, -x_1-x_2) =$

$= (y_1, -x_1) + (y_2, -x_2) = f(u) + f(v)$ (1)

Let $k \in \mathbb{R}$

$f(k \cdot u) = f(kx_1, kxy_1, kz_1) = (ky_1, -kx_1) = k(y_1, -x_1) =$

$= k \cdot f(u)$ (2)

From (1) & (2) $\Rightarrow$ f - linear map

$\operatorname{Ker} f = \{u \in \mathbb{R}^3 \mid f(u) = (0,0)\} = \{(0, 0, z) \mid z \in \mathbb{R}\}$

$f(u) = (0,0) \Rightarrow (y, -x) = (0,0) \Rightarrow y = x = 0$

$\operatorname{Ker} f = \{z(0,0,1) \mid z \in \mathbb{R}\} = \langle (0,0,1) \rangle$ - gen.

$k \in \mathbb{R}$, $k \cdot (0,0,1) = (0,0,0) \Leftrightarrow k = 0 \Rightarrow (0,0,1)$ - lin indep.

$\Rightarrow \{(0,0,1)\}$ - basis of $\operatorname{Ker} f \Rightarrow \dim \operatorname{Ker} f = 1$

$f: V \to V'$, lin map

$\dim(V) = \dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f)$

$\left.\begin{array}{l} \dim(\mathbb{R}^3) = 3 \\ \dim(\mathrm{Ker}\, f) = 1 \end{array}\right\} \to \dim(\mathrm{Im}\, f) = 2$

$\mathrm{Im}\, f = \{(y, -x) \mid y, x \in \mathbb{R}\} = \{(y, 0) + (0, -x) \mid y, x \in \mathbb{R}\} =$

$= \langle (1,0), (0,-1) \rangle$ - gen

$k_1, k_2 \in \mathbb{R} \to k_1(1,0) + k_2(0,-1) = (0,0) \Leftrightarrow \left.\begin{array}{l} k_1 = 0 \\ k_2 = 0 \end{array}\right\} \to$ lin inde

$\to \{(1,0), (0,-1)\}$ - basis of $\mathrm{Im}\, f$

5. - Hw

6. Complete the bases from Ex 1 to some bases of ${}_{\mathbb{R}}\mathbb{R}^3$

V = v. space over K, $\underline{\underline{\dim V = n}}$

$(u_1, \dots, \underline{\underline{u_n}})$ in V. Then if $u_1, \dots, u_n$ - lin indep. $\Leftrightarrow$

$\Leftrightarrow u_1, \dots, u_n$ - generators

$A: \{(1,0,0), (0,1,0)\}$

$\dim(\mathbb{R}^3) = 3 \to$ we have to add only one vector

$\to$ add a vector s.t. the determinant $\neq 0$

For example $(0,0,1)$

$B, C \to$ homework

7. Det. a complement for: $A = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in ${}_{\mathbb{R}}\mathbb{R}^3$

$A = \{(-2y - 3z, y, z) \mid y, z \in \mathbb{R}\} = \{(-2y, y, 0) + (-3z, 0, z)\} =$

$= \langle (-2, 1, 0), (-3, 0, 1) \rangle$ -gen

$k_1, k_2 \in \mathbb{R}$, $k_1(-2,1,0) + k_2(-3,0,1) = (0,0,0) \Leftrightarrow k_1 = k_2 = 0$

$\{(-2,1,0), (-3,0,1)\}$ - bases

we need to add another vector s.t. det $\neq 0$

$$\Delta = \begin{vmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0+0+2-0+4+6 = 12 \neq 0 \Rightarrow$$

$\rightarrow \bar{A} = \langle (2,2,2) \rangle = \{(x,x,x) \mid x \in \mathbb{R}\}$

$B = \{ax + bx^3 \mid a, b \in \mathbb{R}\}$ in $\mathbb{R}_3[x]$
$L = \langle x, x^3 \rangle$
$\hookrightarrow \langle 1, x, x^2, x^3 \rangle$

$\bar{B} = \langle 1, x^2 \rangle = \{a + bx^2 \mid a, b \in \mathbb{R}\}$

8. $V =$ v. s. over K, $S, T, U \leq V$

$\dim(S \cap U) = \dim(T \cap U)$

$\dim(S + U) = \dim(T + U)$

Prove that, if $S \subseteq T$, then $S = T$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$\dim(S+U) = \dim S + \dim U - \dim(S \cap U)$

$\dim(T+U) = \dim T + \dim U - \dim(T \cap U)$ $\ominus$

$\underbrace{\dim(S+U) - \dim(T+U)}_{0} = \dim S - \dim T \rightarrow$

$\rightarrow \left.\begin{array}{l} \dim S = \dim T \\ S \subseteq T \\ S, T \leq V \end{array}\right\} \rightarrow S = T$

9. $S = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$

$T = \langle (0,1,1), (1,1,0) \rangle$

$S \cap T = ?$ and show that $S + T = \mathbb{R}^3$

$S = \{(0,y,z) \mid y, z \in \mathbb{R}\} = \{(0,y,0) + (0,0,z) \mid y, z \in \mathbb{R}\} =$

$= \langle (0,1,0), (0,0,1) \rangle$

$T = \{ (a, a+b, b) \mid a, b \in \mathbb{R} \}$

$S \cap T = \{ (0, b, b) \mid b \in \mathbb{R} \}$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$= 2 + 2 - 1 = 3$

$$\left.\begin{array}{l} = 2+2-1 = 3 \\ \dim \mathbb{R}^3 = 3 \\ S + T \subseteq \mathbb{R}^3 \end{array}\right\} \to S + T = \mathbb{R}^3$$

10. Det. the dimensions of:

$S, T, S+T, S \cap T$ of ${}_{\mathbb{R}}M_2(\mathbb{R})$

$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

$T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

$\dim({}_K M_{m,n}(K)) = m \cdot n$

$\dim S = 2$ (Hw: prove they are lin indep S, T)

$\dim T = 2$

$S = \left\{ \begin{pmatrix} a+b & a \\ b & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

$T = \left\{ \begin{pmatrix} 0 & x \\ x+y & y \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

$a = x \to a = 0$

$a + b = 0 \to 0 + b = 0 \to b = 0 \to y = 0$

$\left.\begin{array}{l} b = x + y \\ b = y \end{array}\right\} \to x = 0$

$S \cap T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$\dim(S \cap T) = 0$

$\dim(S+T) = \dim S + \dim T - \dim S \cap T = 2 + 2 - 0 = 4$