3.11.2023
Analysis - Lecture 5

Săptămâna 7 - Examen

SPT

Recall:

D'Alembert: Let $\sum x_n$ SPT s.t. $\exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0,\infty]$

Then $\ell<1 \to \sum x_n - C$

$\ell>1 \to \sum x_n - D$

$\ell=1 \to ?$ [go back to the hypothesis (with something)
go to RAABE DUHAMEL CRITERION

RAABE DUHAMEL CRITERION

Let $\sum x_n$ SPT s.t. $\exists$ $R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$. Then

$R<1 \to \sum x_n = D$

$R>1 \to \sum x_n = C$

$R=1 \to ?$ [go back to hy
go to KUMMER'S CRITERION

Example:

1. Let $a>0$ $\sum_{n\geq 1} \frac{a(a+1)(a+2)\dots(a+n)}{(n+1)!}$ Study its nature

Solution

SPT

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a(a+1)(a+2)\dots(a+n)}(a+n+1)}{\cancel{(n+2)!}_{n+2}} \cdot \frac{\cancel{(n+1)!}}{\cancel{a(a+1)\dots(a+n)}}$$

$$= \lim_{n\to\infty} \frac{a+n+1}{n+2} = 1 \not\to \text{hyp.}$$

GO TO R-D

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+2}{a+n+1} - 1\right) = \lim_{n\to\infty} n \frac{\cancel{n}+2-\cancel{n}-a-1}{a+n+1} =$$

$$= \lim_{n\to\infty} \frac{(1-a)n}{a+n+1} = 1-a$$

$1-a<1 \to \sum x_n\ D$

$1-a>1 \to \sum x_n\ C$

$\boxed{1-a}=1 \to ?$

$\hookrightarrow a=0$! ~~impossible~~ $\to$ $\sum x_n\ C$ for $a<0$ this in not the case from hyp

$\sum x_n\ D$ for $a>0$

$\to \sum x_n$ is D

2. $\sum_{n\geq 1} \frac{a(a+1)\dots(a+n-1)}{(n+1)!}$ - SPT $a>0$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a(a+1)\dots(a+n-1)(a+n)}{\underbrace{(n+2)!}_{n+2}} \cdot \frac{(n+1)!}{a(a+1)\dots(a+n-1)} =$$

$$= \lim_{n\to\infty} \frac{a+n}{n+2} = 1 \nrightarrow \text{hyp}$$

Go to R-D

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(\frac{n+2}{a+n}-1\right) = \lim_{n\to\infty} n\cdot\frac{\cancel{n}+2-a-\cancel{n}}{n+a} =$$

$$= \lim_{n\to\infty} \frac{(2-a)n}{n+a} = 2-a$$

$2-a<1 \to \sum x_n\ D \quad \to \sum x_n\ D$ for $a>1$

$a>0$

$2-a>1 \to \sum x_n\ C \quad \sum x_n\ C$ for $a<1$

$2-a=1 \to ?$

$2-a=1 \to a=1 \to x_n = \frac{1\cdot(1+1)(1+2)\cdot\dots\cdot(1+n-1)}{(n+1)!} = \frac{n}{(n+1)!} = \frac{1}{n+1} \to$

$\to \sum x_n - D,\ a=1$

$\sum x_n - D,\ a\geq 1$ ✓

$\sum x_n - C,\ 0<a<1$ ✓

$\frac{x_{n+1}}{x_n} \to 1$

$n\left(\frac{x_n}{x_{n+1}}-1\right) \to 2-a$ <1 D

>1 C

$2-a=1 \to ? \to a=1 \to x_n = \frac{1\cdot 2\cdot\ \cdot n}{(n+1)!} = \frac{1}{n+1}$

$\sum \frac{1}{n+1} \sim \sum \frac{1}{n}$ D

Theorem - CAUCHY'S CONDENSATION CRITERION FOR SPT

$\sum x_n$ - SPT with (x_n) decreasing $\to \sum x_n \sim \sum 2^n x_{2^n}$

Example

Study the nature of the series

$$\sum_{n\geq 2} \frac{1}{\ln(\ln n)}$$

Method I

$\ln(\ln n) < n$

$$\left.\begin{array}{l} \underbrace{\frac{1}{n}}_{y_n} < \underbrace{\frac{1}{\ln(\ln n)}}_{x_n} \\ \sum y_n - D \end{array}\right\} \to \sum_{n\geq 2} \frac{1}{\ln(\ln n)} - D$$

+

bonus $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)} = +\infty$

Method II

Apply the condensation criterion :

$$x_n = \frac{1}{\ln(\ln n)} \Rightarrow x_{2^n} = \frac{1}{\ln(\ln 2^n)} = \frac{1}{\ln(n \ln 2)} = \frac{1}{\ln n + \ln(\ln 2)}$$

$$\underbrace{\boxed{2^n \cdot x_{2^n}}}_{t_n} = \frac{2^n}{\ln n + \ln(\ln 2)}$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{2^n}{\ln n + \ln(\ln 2)} = +\infty \neq 0 \to \sum t_n \text{ is } D$$

Important example: Study the nature of the generalized HARMONIC SERIES $\sum_{n\geq 1} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$

$x_n = \frac{1}{n^\alpha} \to$ case I: $\alpha = 0 \to x_n = \frac{1}{n^0} = 1, \forall n \in \mathbb{N} \to$

$\to (x_n)$ is the constant sequence 1

with $\lim_{n\to\infty} x_n = 1 \neq 0 \to \sum x_n$ is D

Case 2: $\boxed{\alpha \neq 0}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} n^{-\alpha} = \begin{cases} \infty, & -\alpha > 0 \\ 0, & -\alpha < 0 \end{cases} = \left.\begin{cases} \infty, & \alpha < 0 \\ 0, & \alpha > 0 \end{cases}\right\} (*)$$

$$\to \sum x_n \text{ is } D \text{ if } \left.\begin{matrix} \alpha < 0 \\ + \\ \text{case } 1 \end{matrix}\right\} \alpha \leq 0$$

If $\alpha \leq 0$ $\sum \frac{1}{n^\alpha}$ is D and $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \infty$

Subcase 2.A $\boxed{\alpha > 0}$

$$\lim_{n\to\infty} x_n = 0 \to ?$$

$$\frac{x_{n+1}}{x_n} = \frac{1}{(n+1)^\alpha} \cdot n^\alpha = \left(\frac{n}{n+1}\right)^\alpha < 1 \to \boxed{\begin{matrix} (x_n) \text{ is decreasing} \\ \sum x_n - SPT \end{matrix}}$$

$$\xrightarrow{\text{CAUCHY'S CONDENSATION CRITERION}} \sum x_n \sim \sum 2^n x_{2^n}$$

$$t_n = 2^n \cdot x_{2^n} = 2^n \cdot \frac{1}{(2^n)^\alpha} = 2^n \cdot \frac{1}{2^{n\alpha}} = 2^n \cdot 2^{-n\alpha} = 2^{n(1-\alpha)} =$$

$$= \{\underbrace{2^{(1-\alpha)}}_{q}\}^n$$

We may regard $\sum t_n$ as a geometric series of ratio $q = 2^{1-\alpha}$, $\sum q^n$ is $\begin{cases} C, & |q| < 1 \\ D, & q \geq 1 \end{cases}$

$$2^{1-\alpha} = 1 \Leftrightarrow$$

$$\Leftrightarrow 2^{1-\alpha} = 2^0 \Leftrightarrow 1-\alpha = 0 \Leftrightarrow \alpha = 1$$

$$\begin{matrix} 1-\alpha < 0 \to C \\ 1-\alpha \geq 0 \to D \end{matrix} \Leftrightarrow \begin{cases} 1 < \alpha, & C \\ \boxed{1 \geq \alpha, \ D} \end{cases} (**)$$

$$\alpha > 0$$

$$(*) \ (**) \to \sum \frac{1}{n^\alpha} \text{ is } \begin{cases} C, & \alpha > 1 \\ D, & \alpha \leq 1 \end{cases}$$

Series with random terms (SRT)

Def: SRT $\sum x_n$ with $x_n \neq 0$

Theorem 1: ABEL'S CRITERION

Considering $(a_n), (u_n) \subseteq \mathbb{R}$ s.t.

- (a_n) - decreasing
- $\lim_{n\to\infty} a_n = 0$
- (u_n) has its sequence of partial sums bounded

Then $\sum_{n\geq 1} u_n \cdot a_n$ is Convergent

C (LEIBNIZ'S CRITERION)

$\sum_{n\geq 1} (-1)^n \cdot a_n$ If: $\left.\begin{array}{l} \cdot (a_n) \text{ - decreasing} \\ \cdot \lim_{n\to\infty} a_n = 0 \end{array}\right\} \to \sum (-1)^n \cdot a_n$ is C

Proof:

T_1 $\quad u_n = (-1)^n$

$t_1 = u_1 = (-1)^1 = -1$

$t_2 = u_1 + u_2 = -1 + (-1)^2 = -1 + 1 = 0$

$\to t_n = \begin{cases} -1 : n \text{ is odd} \\ 0 : n \text{ is even} \end{cases} \to |t_n| \leq 1 \to (t_n) \text{ - bounded} \xrightarrow{T_1}$

$\xrightarrow{T_1} \sum (-1)^n a_n$ is C

Example:

1. Study the nature of the series $\sum_{n\geq 1} \frac{\sin n}{n(n+1)} \cdot \frac{1}{n!}$

Solution: $x_n = \frac{\sin n}{n(n+1)} \neq 0$ $\quad$ SRT

$a_n = \frac{1}{n!} \to (a_n)$ - decreasing ✓

$\lim_{n\to\infty} a_n = 0$ ✓

$u_n = \frac{\sin n}{n(n+1)}$

We need its sequence of partial sums bounded

$\Leftrightarrow \exists B > 0$ s.t. $|t_n| < B$

$|t_n| = \left| \frac{\sin 1}{1\cdot 2} + \frac{\sin 2}{2\cdot 3} + \dots + \frac{\sin n}{n\cdot(n+1)} \right| \le \left| \frac{\sin 1}{1\cdot 2} \right| + \left| \frac{\sin 2}{2\cdot 3} \right| + \dots +$

$+ \left| \frac{\sin n}{n\cdot(n+1)} \right| \le \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1} =$

$= 1 - \frac{1}{n+1} < 1$

$\rightarrow |t_n| < 1 \rightarrow (u_n)$ has its sequence of partial sums bounded ✓

$\xrightarrow[\text{Criterion}]{\text{Abel's}} \sum_{n\ge 1} u_n \cdot a_n$ is C

2. Study the nature of the series $\sum_{n\ge 1} (-1)^n \cdot \frac{1}{n}$

$a_n = \frac{1}{n} \rightarrow (a_n)$ - decreasing, $\lim_{n\to\infty} a_n = 0$ $\Big\} \xrightarrow{\text{Leibnitz}} \sum_{n\ge 1} (-1)^n \cdot \frac{1}{n}$ is C

$\rightarrow \exists\, \ell = \lim_{n\to\infty} \left(-1 + \frac{1}{2} - \frac{1}{3} + \dots + \frac{(-1)^n}{n} \right) \in \mathbb{R}$

!!! $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) = \infty$

Absolutely convergent series

Def: Let $\sum x_n$ be a series of real numbers. It is said to be ABSOLUTELY CONVERGENT if $\sum |x_n|$ is C

THEOREM

Consider $\sum x_n$ SRT. If $\sum x_n$ is A.C. $\to$ $\sum x_n$ is C

Remark The converse statement does not hold

$\sum x_n$ - C $\not\to$ $\sum x_n$ - A.C

Example

1. $\sum\limits_{n\geq 1} \frac{(-1)^n}{n^3\sqrt{n^2+4}}$. Study both C & AC

Remark: When we have to study both C & AC it is adviceble to start with AC.

If AC $\to$ C

If ~~AC~~ $\to$ study C separately

$$x_n = \frac{(-1)^n}{n^3\sqrt{n^2+4}} \; ; \; |x_n| = \frac{1}{n^3\sqrt{n^2+4}}$$

$\sum |x_n|$ is SPT $\underset{\substack{C2C \\ b}}{\sim}$ $\sum \frac{1}{n^4}$ is C

Thus $\sum |x_n|$ is C $\to$ $\sum x_n$ is A.C. $\xrightarrow{Th}$ $\sum x_n$ is C

2. Study both AC & C for $\sum \frac{(-1)^n}{n}$

$$x_n = \frac{(-1)^n}{n} \; , \; |x_n| = \frac{1}{n}$$

$\sum |x_n| = \sum \frac{1}{n}$ is D $\to$ $\sum |x_n|$ is D thus $\sum x_n$ is not AC $\to$?

We have to study separetly the C

$\sum (-1)^n \cdot \frac{1}{n}$ directly

Ex. at Libnitz → it was C → $\sum x_n$ is C but not AC

Până aici e mat pt examen

Scanned with CamScanner

IV LIMITS OF FUNCTIONS

Remark: Limits of functions are considered only for points situated in the set of the accumulation points of the domain of the function considered in the topology of $\overline{\mathbb{R}}$

If $x \in \overline{\mathbb{R}}$ and $r > 0$ • $B(x, r) = \{y \in \overline{\mathbb{R}} : |y - x| < r\} =$
$= (x - r, x + r)$ if $x \in \mathbb{R}$

• $B(\infty, r) = (r, \infty]$

• $B(-\infty, r) = [-\infty, r)$

$V \in \mathcal{V}(x)$ if $\exists\, r_V > 0$ s.t. $B(x, r_V) \subseteq V$

If $A \neq \emptyset$, $A \subseteq \mathbb{R}$

$A'_{(in\ \overline{\mathbb{R}})} = \{y \in \overline{\mathbb{R}} : \forall V \in \mathcal{V}(y),\ V \cap A \setminus \{y\} \neq \emptyset\}$
$= \{y \in \overline{\mathbb{R}} : \forall r > 0,\ B(y, r) \cap A \setminus \{y\} \neq \emptyset\}$

$A' = \mathrm{Cl}A \setminus \mathrm{Iso}\,A$

Examples:

A	A' (in $\overline{\mathbb{R}}$)
$[-1, 2] \cup \{3\}$	$[-1, 2]$
$(-\infty, 4) \cup (5, 6, 7)$	$[-\infty, 4]$
$(-\infty, 4) \cup \mathbb{N}$	$[-\infty, 4] \cup \{\infty\}$
$\mathbb{N}$	$\{\infty\}$
$\mathbb{Z}$	$\{-\infty, \infty\}$
$\mathbb{Q}$	$\overline{\mathbb{R}}$
$\mathbb{R} \setminus \mathbb{Q}$	$\overline{\mathbb{R}}$
$\mathbb{R}$	$\overline{\mathbb{R}}$

THEOREM

- Let $\emptyset \neq A \subseteq \mathbb{R}$.
- An element $a \in \overline{\mathbb{R}}$

$a \in A' \Leftrightarrow \exists (a_n) \subseteq A \setminus \{a\}$ s.t. $\lim_{n\to\infty} a_n = a$

PROOF

$\boxed{\text{"}\Rightarrow\text{"}}$ We know that $a \in A' \overset{\text{def}}{\Leftrightarrow} \forall r > 0,\ B(a, r) \cap A \setminus \{a\} \neq \emptyset$ (1)

$a + \frac{1}{n} \to a$ this should guide us

$\overset{?}{\subseteq} A$

$\boxed{\forall n \in \mathbb{N}}$, $r_n = \frac{1}{n} > 0$

$\xrightarrow{(1)} \underbrace{B(a, \tfrac{1}{n}) \cap A \setminus \{a\} \neq \emptyset}$

$\Downarrow$

$\boxed{\exists t_n \in A \setminus \{a\}}$

and

$t_n \in B(a, \frac{1}{n}) \Leftrightarrow \boxed{|t_n - a| \leq \frac{1}{n}}$

Thus $\forall n \in \mathbb{N}\ \exists t_n \in A \setminus \{a\}$ with $|t_n - a| \leq \frac{1}{n} \to$

$\to \exists (t_n) \subseteq A \setminus \{a\}$ s.t. $\left.\begin{array}{l} \forall n \in \mathbb{N}\ \ |t_n - a| \leq \frac{1}{n} \\ \lim_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \xrightarrow{\text{weierstrass}} \lim_{n\to\infty} t_n = a$

$\boxed{\text{"}\Leftarrow\text{"}}$ We know

(2) $\exists (a_n) \subseteq A \setminus \{a\}$ s.t. $\lim_{n\to\infty} a_n = a$

we want (1)

Case 1: $a \in \mathbb{R}$

Freeze

Case 2: $a = \infty$. We want $(a_n) \subseteq A \setminus \{a\}$ s.t. $\lim_{n\to\infty} a_n = \infty$

$a_n = n \quad \forall n \in \mathbb{N}$ $\boxed{B(\infty, r) \cap A \setminus \{a\} \neq \emptyset}$

$\Downarrow$

$\boxed{\exists t_n} \in B(\infty, r) \cap A \setminus \{a\} \neq \emptyset \to$

$\to (t_n) \subseteq A \setminus \{a\}$

$\forall n \in \mathbb{N},\ t_n \in B(\infty, r) \Leftrightarrow t_n > n \to$

$\to \boxed{\lim_{n\to\infty} t_n = \infty}$

HW: Case 3: $a = -\infty$