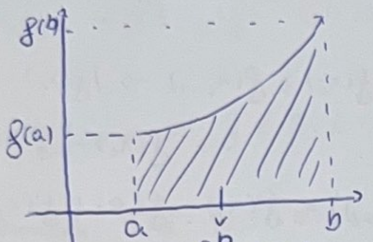
19.01.2024

Calculus - Lecture 13

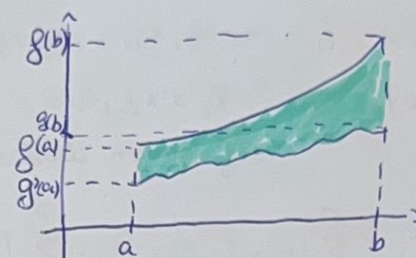
The Reimann Stieltjes Integral

The Reimann integral



$\int_a^b f(x)\,dx$

The Reiman - Stieltjes integral



It generalies the Riemann Itegral

1. Functions of bounded variation (BV-functions)

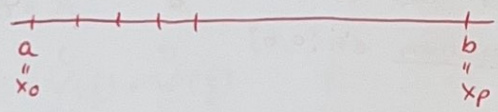
Def: • $f: [a,b] \to \mathbb{R}$ $\quad V_\Delta(f) = \sum_{i=1}^{p} |f(x_i) - f(x_{i-1})|$

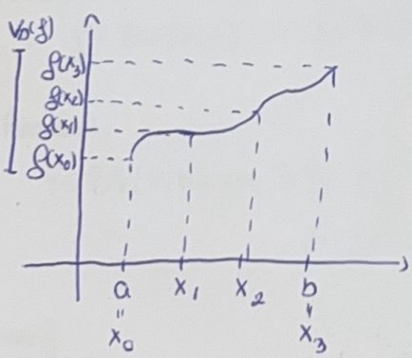
$\Delta \in Part\,[a,b]$

$\hookrightarrow$ variation of f on Δ

where $\Delta = (x_0, x_1, \dots, x_p)$

$a = x_0 < x_1 < \dots < x_p = b$





$V_\Delta(f) = f(b) - f(a)$

• $\overset{b}{\underset{a}{V}}(f) = \sup\{V_\Delta(f) : \Delta \in Part\,[a,b]\}$ $\to$ THE TOTAL VARIATION OF f on $[a,b]$

Example + remark:

$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \\ f \text{ - non decreasing on } [a,b] \end{array}\right\} \to \overset{b}{\underset{a}{V}}(f) < \infty$$

Proof:

Choose $\Delta = (x_0, x_1, x_2, \dots, x_p) \in Part[a,b]$ a random partition of $[a,b]$

$$\left.\begin{array}{l} \forall i \in \{1,\dots p\} \quad x_i > x_{i-1} \\ f \text{ is non decreasing} \end{array}\right\} \to f(x_i) \geq f(x_{i-1}) \to |f(x_i) - f(x_{i-1})| =$$

$$= f(x_i) - f(x_{i-1})$$

$$V_\Delta(f) = \sum_{i=1}^{p} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{p} (f(x_i) - f(x_{i-1})) = f(x_1) - f(a) + f(x_2) - f(x_1) + \dots +$$

$$+ f(x_{p-1}) - f(x_{p-2}) + f(x_p) - f(x_{p-1}) =$$

$$\left.\begin{array}{l} = f(x_p) - f(x_0) = f(b) - f(a) \\ \Delta \text{ - random} \end{array}\right\} \to \forall \Delta \in Part[a,b] \quad V_\Delta(f) = f(b) - f(a) \to$$

$$\to \overset{b}{\underset{a}{V}}(f) = \sup\{V_\Delta(f) : \Delta \in Part[a,b]\} = \sup\{f(b) - f(a)\} = f(b) - f(a) \underline{\in \mathbb{R}}$$

Remark: If g ($g:[a,b]\to\mathbb{R}$) is non-increasing $\overset{b}{\underset{a}{V}} g = g(a) - g(b)$

Def: $\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \\ \overset{b}{\underset{a}{V}}(f) < \infty \end{array}\right\} \to$ f is said to be a function of <u>bounded variation</u> on $[a,b]$

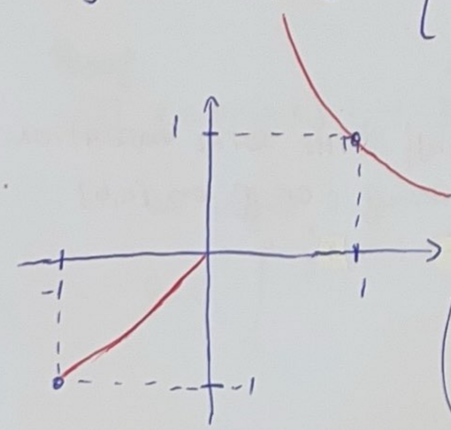
Example: monotonic functions are of bounded variation on random compact sets $[a,b] \subseteq$ their domain

↓

closed + bounded

Example:

$$f:[-1,1]\to\mathbb{R}, \quad f(x) = \begin{cases} x & : x \leq 0 \\ \frac{1}{x} & : x > 0 \end{cases}$$



$f \notin BV[-1,1]$

↓

bounded variation

$$\left(\begin{array}{l} D_n = [-1, 0, \frac{1}{n}, 1] \\ V_{D_n}(f) = f(0) - f(-1) + f(\frac{1}{n}) - f(0) + |f(1) - f(\frac{1}{n})| \end{array} \right.$$

$n \in \mathbb{N}$ random

$\Delta_n = [-1, 0, \frac{1}{n}, 1]$

$$V_{\Delta_n}(f) = |f(0) - f(-1)| + |f(\tfrac{1}{n}) - f(0)| + |f(1) - f(\tfrac{1}{n})|$$

$$= 1 + |\tfrac{1}{\frac{1}{n}} - 0| + |1 - \tfrac{1}{\frac{1}{n}}| = 1 + n + n - 1 = 2n$$

$$\overset{b}{\underset{a}{V}}(f) = \sup\{V_\Delta(f) : \Delta \in Part[a,b]\} \geq \sup\{V_{\Delta_n}(f) : \Delta_n \in Part[a,b]\} =$$

$$= \sup\{2n : n \in \mathbb{N}\} = \infty$$

$\rightarrow \overset{1}{\underset{-1}{V}}(f) = \infty \rightarrow f \notin BV[-1,1]$

Remark: The set on which we consider the bounded variation is <u>important</u>.

Should we change it, the outcome w.r.t. b.v. is random:

For example, for the function above:

$f \in BV[-1,0]$

$f \in BV[\frac{1}{t}, 1]$, $t > 0$, $t \in \mathbb{R}$

Properties of bounded variation functions

1. $\left.\begin{array}{l} f, g \in BV[a,b] \\ \alpha, \beta \in \mathbb{R} \end{array}\right\} \rightarrow \alpha f + \beta g \in BV[a,b]$

2. $\overset{b}{\underset{a}{V}}(f) = \overset{c}{\underset{a}{V}}(f) + \overset{b}{\underset{c}{V}}(f)$, $\forall c \in [a,b]$

3. $f \in BV[a,b] \rightarrow f \in BV[c,d]$, $\forall [c,d] \subseteq [a,b]$

Theorem:

$f \in BV[a,b] \leftrightarrow \exists f_1, f_2 : [a,b] \rightarrow \mathbb{R}$

f_1, f_2 non decreasing s.t. $f = f_1 - f_2$

Proof:

$\boxed{\rightarrow}$ $f \in BV[a,b] \overset{def}{\longleftrightarrow} \overset{b}{\underset{a}{V}}(f) < \infty \rightarrow \forall x \in (a,b]$, $\overset{x}{\underset{a}{V}}(f) < \infty$

We define 2 functions $f_1, f_2 : [a,b] \rightarrow \mathbb{R}$ by

$$f_1(x) = \begin{cases} \overset{x}{\underset{a}{V}}(f) & : x \in (a,b] \\ 0 & : x = a \end{cases} \qquad f_2(x) = f_1(x) - f(x) \ , \forall x \in [a,b]$$

We prove that f_1, f_2 are non-decreasing.

Consider $x < y \in [a,b]$ $(\to y > a)$

- $f_1(y) = \overset{y}{\underset{a}{V}}(f) \underset{P_2}{=\!=} \underbrace{\overset{x}{\underset{a}{V}}(f)}_{f_1(x)} + \overset{y}{\underset{x}{V}}(f) = f_1(x) + \overset{y}{\underset{x}{V}}(f)$

$\left.\begin{array}{l}[x,y] \subseteq [a,b] \\ f \in BV[a,b]\end{array}\right\} \xrightarrow{P_3} f \in BV[x,y] \to \overset{y}{\underset{x}{V}}(f) < \infty$, but $\overset{y}{\underset{x}{V}}(f) = \sup\{V_\Delta(f) : \Delta \in Part_{[x,y]}\}$

$= \sup\{\Sigma| \ |\} \geq 0$

$\to \overset{y}{\underset{x}{V}}(f) = T_x \in [0,\infty)$

$\to \left.f_1(y) = f_1(x) + \underbrace{T_x}_{\geq 0}\right\} \to f_1(y) \geq f_1(x) \to f_1$ is non-decreasing ✓

- $\left.\begin{array}{l}f_2(y) = f_1(y) - f(y) \\ f_2(x) = f_1(x) - f(x)\end{array}\right\} \to f_2(y) - f_2(x) = f_1(y) - f_1(x) + f(x) - f(y) =$

$= \overset{y}{\underset{a}{V}}(f) - \overset{x}{\underset{a}{V}}(f) - (f(y) - f(x)) = \overset{x}{\underset{a}{V}}(f) + \overset{y}{\underset{x}{V}}(f) - \overset{x}{\underset{a}{V}}(x) - (f(y) - f(x)) =$

$= \overset{y}{\underset{x}{V}}(f) - (f(y) - f(x)) \geq 0$

!! $\overset{y}{\underset{x}{V}}(f) = \sup\{V_\Delta(f) : \Delta \in Part[x,y]\} \geq V_{\Delta'}(f) = |f(y) - f(x)|$

$\Delta' = (x, y)$

$$|f(y) - f(x)| = \begin{cases} f(y) - f(x) : f(y) \geq f(x) \\ f(x) - f(y) : f(y) < f(x) \end{cases}$$

I $f(y) < f(x) \to \underbrace{f(y) - f(x)}_{<0} \to -(f(y) - f(x)) > 0 \quad | + \overset{y}{\underset{x}{V}}(f)$

$\overset{y}{\underset{x}{V}}(f) - (f(y) - f(x)) > 0$

II $f(y) \geq f(x)$

$V_{\Delta'} = f(y) - f(x) \to \overset{y}{\underset{x}{V}}(f) = \sup \geq V_{\Delta'} \to \overset{y}{\underset{x}{V}}(f) - (f(y) - f(x)) \geq 0$

$\to f_2$ is non decreasing

! $f = f_1 - f_2$

$\boxed{\Leftarrow}$ f_1, f_2 non decreasing $\to \in BV[a,b]$

$\alpha = 1,\ \beta = -1 \in \mathbb{R} \xrightarrow{P_1} \alpha f + \beta \cdot g \in BV[a,b]$

$\to f_1 - f_2 \in BV[a,b] \to f \in BV[a,b]$

Theorem: $\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ is differentiable} \\ \text{and} \\ f' \text{ is Riemann integrable on } [a,b] \end{array}\right\} \to f \in BV[a,b]$ and $\overset{b}{\underset{a}{V}}(f) = \int_a^b |f'(x)|\,dx$

Proof:

Consider $\Delta = (x_0, x_1, \dots, x_p) \in Part[a,b]$ randomly chosen

$V_\Delta(f) = \sum_{i=1}^{p} |f(x_i) - f(x_{i-1})|$

Choose $i \in \{1, \dots, p\}$ randomly

$\left.\begin{array}{l} f \text{ is cont. on } [x_{i-1}, x_i] \\ f \text{ is diff on } [x_{i-1}, x_i] \end{array}\right\} \xrightarrow[\text{Theorem}]{\text{Lagrange's}} \exists\, \xi_i \in (x_{i-1}, x_i)$ s.t. $f(x_i) - f(x_{i-1}) =$
$= f'(\xi_i)(x_i - x_{i-1})$

i random $\to \forall i\ \exists\, \xi_i \in (x_{i-1}, x_i) \to \xi = (\xi_1, \xi_2, \dots, \xi_p) \in \mathcal{P}(\Delta)$

Thus $V_\Delta(f) = \sum_{i=1}^{p} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{p} |f'(\xi_i)|(x_i - x_{i-1}) =$

$= \sigma(|f'|, \Delta, \xi)$ $(*)$

The Riemann sum associated to $|f'|, \Delta$ and ξ

f' is R.i on $[a,b]$ (hyp) $\to |f'|$ is R.i on $[a,b]$ $\to$

$\to \exists\, I \in \mathbb{R}$ s.t. $I = \lim_{n\to\infty} \sigma(|f'|, \Delta^n, \xi^n) \overset{(*)}{=} \lim_{n\to\infty} V_{\Delta^n}(f) = \overset{b}{\underset{a}{V}}(f) \in \mathbb{R} \to$

- $\forall (\Delta^n) \subseteq Part[a,b]$ with $\lim_{n\to\infty} \|\Delta^n\| = 0$
- $\forall (\xi^n)$ s.t. $\forall n \in \mathbb{N}\ \xi^n \in \mathcal{P}(\Delta^n)$

$\to f \in BV[a,b]$
$\overset{b}{\underset{a}{V}}(f) = \int_a^b |f(x)|\,dx$

Example: f is non-decreasing $\quad \overset{b}{\underset{a}{V}}(f) = f(b) - f(a)$

$f(x) = x \qquad f:[0,1]\to\mathbb{R} \qquad \overset{1}{\underset{0}{V}}(f) = f(1) - f(0) = 1$

$f'(x) = 1 \qquad \int_0^1 |f'(x)|\,dx = \int_0^1 1\,dx = x\Big|_0^1 = 1$

$f:[1,e]\to\mathbb{R},\ f(x)=\ln x$

$\bigvee_1^e(\ln)=\ln e-\ln 1=1-0=1$

$f'(x)=\frac{1}{x}$ $\quad \int_1^e \frac{1}{x}\,dx=\ln x\big|_1^e=\ln e-\ln 1=1$

2. The Riemann - Stieltjes integrals

Def: $\left.\begin{array}{l} f,g:[a,b]\to\mathbb{R} \\ \Delta\in Part[a,b] \\ \xi\in \mathcal{P}(\Delta)\end{array}\right\}$ $\sigma(f,g,\Delta,\xi)=\sum_{i=1}^{p} f(\xi_i)|g(x_i)-g(x_{i-1})|$ is the RIEMAN-STIELTYES INTEGRAL attached to the function f, g, the partition Δ and $\xi\in\mathcal{P}(\Delta)$

Def: $f,g:[a,b]\to\mathbb{R}$
f is said to be Reimann-Stieltjes integrable with respect to g on $[a,b]$ if $\exists\ I\in\mathbb{R}$ s.t. $\cdot\ \forall(\Delta^n)\subseteq Part[a,b]$ with $\lim_{n\to\infty}\|\Delta^n\|=0$
$\cdot\ \forall(\xi^n),\ \forall n\in\mathbb{N},\ \xi^n\in\mathcal{P}(\Delta^n)$

$$I=\lim_{n\to\infty}\sigma(f,g,\Delta^n,\xi^n)$$

Theorem (computing of RSI with the help of the R.i)

$\left.\begin{array}{l}\cdot\ f:[a,b]\to\mathbb{R} \text{ is R.i} \\ \cdot\ g:[a,b]\to\mathbb{R} \left[\begin{array}{l}\text{is dif} \\ g' \text{ is R.i}\end{array}\right.\end{array}\right\} \longrightarrow$ f is RSI with respect to g and $\int_a^b f(x)\,dg(x)=\int_a^b f(x)\cdot g'(x)\,dx$

Properties:

- liniarity w.r.t. both functions
  - $\int_a^b (f_1+f_2)\,dg=\int_a^b f_1\,dg+\int_a^b f_2\,dg$
  - $\int_a^b f\,d(g_1+g_2)=\int_a^b f\,dg_1+\int_a^b f\,dg_2$
- additivity w.r.t. $[a,b]$

  $\int_a^c f\,dg+\int_c^b f\,dg=\int_a^b f\,dc \quad ,\forall c\in(a,b)$
- reversability

  $\int_a^b g\,df=f\,g\big|_a^b-\int_a^b f\,dg$ (side-parts applications)

Examples: a) $\int_0^1 x\,dx^2$, $f, g : [0,1] \to \mathbb{R}$, $f(x) = x$
$g(x) = x^2$

f - c → R.i
g is diff
$g'(x) = 2x$ → R.i

$\xrightarrow{T} \int_0^1 x\,dx^2 = \int_0^1 x \cdot 2x\,dx = 2\int_0^1 x^2\,dx =$
$= 2 \cdot \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$

b) $\int_0^{\pi} x\,d\cos x$, $f; g : [0,\pi] \to \mathbb{R}$ $f(x) = x$
$g(x) = \cos x$

f is cont → R.i
g is diff
$g'(x) = -\sin x$ - is cont → R.i

$\xrightarrow{T} \int_0^{\pi} x\,d\cos x = \int_0^{\pi} x \cdot (-\sin x)\,dx =$
$= x \cdot \cos x\Big|_0^{\pi} - \int_0^{\pi} \cos x\,dx =$
$= \pi\cos\pi - 0 - \sin x\Big|_0^{\pi}$
$= \pi(-1) - \sin\pi + \sin 0$
$= -\pi - 0 + 0 = -\pi$