Analysis - Homework 7

1. Compute the limits:

a) $\lim_{x\to\infty} x \cdot \cos^2 \frac{x+2}{x}$

$\lim_{x\to\infty} x \cdot \cos^2 \frac{x(1+\overbrace{\frac{2}{x}}^{\to 0})}{x} = \lim_{n\to\infty} x \cos^2(1) = +\infty$

b) $\lim_{x\to 1} \frac{x}{x^2+1} = \frac{1}{1+1} = \frac{1}{2}$

c) $\lim_{x\to-\infty} \frac{x^2+5}{x^3} = \lim_{x\to-\infty} \frac{x^2(1+\overbrace{\frac{5}{x^2}}^{\to 0})}{x^3} = \lim_{x\to-\infty} \frac{1}{x} = 0_-$

d) $\lim_{x\to\infty} \frac{(x+2)(2x+1)}{x^2+3x+5} = \lim_{x\to\infty} \frac{2x^2+5x+2}{x^2+3x+5} = \lim_{x\to\infty} \frac{x^2(2+\overbrace{\frac{5}{x}}^{\to 0}+\overbrace{\frac{2}{x^2}}^{\to 0})}{x^2(1+\underbrace{\frac{3}{x}}_{\to 0}+\underbrace{\frac{5}{x^2}}_{\to 0})} = 2$

e) $\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+1+x)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$

f) $\lim_{x\to 2} \left(\frac{1}{2-x} - \frac{2x}{4-x^2}\right) = \lim_{x\to 2} \frac{4-x^2-2x(2-x)}{(2-x)^2(2+x)} = \lim_{x\to 2} \frac{\cancel{(2-x)}(2+x-2x)}{(2-x)^{\cancel{2}}(2+x)}$

$= \lim_{x\to\infty} \frac{\cancel{2-x}}{(\cancel{2-x})(2+x)} = \frac{1}{4}$

g) $\lim_{x\to 1} \frac{1+x+x^2+\ldots+x^n-(n+1)}{x-1} = \lim_{x\to 1} \frac{1\cdot\frac{x^n-1}{x-1}-(n+1)}{(x-1)} =$

$= \lim_{x\to 1} \frac{(x^n-1)-(x-1)(n+1)}{(x-1)^2} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{n-1}+x^{n-2}+\ldots+1-n-1)}{(x-1)^{\cancel{2}}} = \lim_{x\to 1} \frac{n-\cancel{n}-1}{(x-1)}$ $\rightarrow$

$\left.\begin{array}{l} \lim_{\substack{x\to 1\\ x>1}} \frac{1}{x-1} = \frac{1}{0_+} = +\infty \\ \lim_{\substack{x\to 1\\ x<1}} \frac{1}{x-1} = \frac{1}{0_-} = -\infty \end{array}\right\} \rightarrow \nexists \lim_{x\to 1} \frac{1}{x-1}$

$\rightarrow \nexists \lim_{x\to 1} \frac{1+x+\ldots+x^n-(n+1)}{(x-1)}$

h) $\lim_{x\to 1} \frac{x+x^2+\ldots+x^n-n}{x+x^2+\ldots+x^m-m} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 1} \frac{1+2x+\ldots+nx^{n-1}}{1+2x+\ldots+m\cdot x^{m-1}} \left\langle \begin{array}{l} 0, \ m>n \\ 1, \ m=n \\ \infty, \ n>m \end{array}\right.$

i) $\lim\limits_{x\to 27} \frac{x-27}{\sqrt[3]{x}-3} = \lim\limits_{x\to 27} \frac{\cancel{(\sqrt[3]{x}-3)}((\sqrt[3]{x})^2+3^2+3\sqrt[3]{x})}{\cancel{\sqrt[3]{x}-3}} = [\sqrt[3]{27}]^2+9+3\cdot 3 = 3^2+9+9 = 27$

$(\sqrt[3]{x})^2+1+\sqrt[3]{x}$

j) $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \lim\limits_{x\to 1} \frac{\cancel{(x-1)}\cdot((\sqrt[4]{x})^3+(\sqrt[4]{x})^2+\sqrt[4]{x}+1)}{\cancel{(x-1)}\cdot((\sqrt[3]{x})^2+1+\sqrt[3]{x})} = \frac{1+1+1+1}{1+1+1} = \frac{4}{3}$

k) $\lim\limits_{x\to\infty} (\sqrt[3]{ax^3+x^2+bx+c} - (bx+c)) = \lim\limits_{x\to\infty} \frac{ax^3+x^2+bx+c-(bx+c)^3}{\sqrt[3]{(ax^3+x^2+bx+c)^2}+(bx+c)^2+(bx+c)\sqrt[3]{ax^3+x^2+bx+c}}$

$$= \lim\limits_{x\to\infty} \frac{ax^3+x^2+bx+c-b^3x^3-3b^2x^2c-3bxc-c^3}{x^2\sqrt[3]{(a+\underbrace{\tfrac{1}{x}}_{\to 0}+\underbrace{\tfrac{b}{x^2}}_{\to 0}+\underbrace{\tfrac{c}{x^3}}_{\to 0})^2}+x^2(b+\underbrace{\tfrac{c}{x}}_{\to 0})^2+x^2\sqrt[3]{a+\tfrac{1}{x}+\tfrac{b}{x^2}+\tfrac{c}{x^3}}\cdot(b+\underbrace{\tfrac{c}{x}}_{\to 0})}$$

$$= \lim\limits_{x\to\infty} \frac{x^3(a-b^3)+x^2(1-3b^2c)+x(b-3bc)+c-c^3}{x^2(\sqrt[3]{a^2}+b+b\sqrt[3]{a})} \begin{cases} \infty, & a-b^3\neq 0 \\ \frac{1-3b^2c}{a^{\frac{2}{3}}+b+b\sqrt[3]{a}}, & a-b^3=0 \end{cases}$$

$a=b^3 \to \frac{1-3b^2c}{b^2+b+b\cdot b} = \frac{1-3b^2c}{2b^2+b}$

2. Compute the limits

a) $\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{5x+1}{2x+4}} = \lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{5}{2}} = 0^{\frac{5}{2}} = 0$

$\lim\limits_{x\to\infty} \frac{5x+1}{2x+4} = \lim\limits_{x\to\infty} \frac{x(5+\overbrace{\tfrac{1}{x}}^{\to 0})}{x(2+\underbrace{\tfrac{4}{x}}_{\to 0})} = \frac{5}{2}$

b) $\lim\limits_{x\to 0} \left(\frac{3\cdot\sin x-\tan x}{x}\right)^{\frac{\sin x+2x}{x}} = \lim\limits_{x\to 0} \left(3\cdot\underbrace{\frac{\sin x}{x}}_{\to 1} - \underbrace{\frac{\tan x}{x}}_{\to 1}\right)^{\overbrace{\frac{\sin x}{x}}^{\to 1}+2} = (3-1)^3 = 2^3 = 8$

c) $\lim\limits_{x\to 0} (1+\cos x)^{\frac{1}{x^2}} = (1+\cos 0)^{\lim\limits_{x\to 0}\frac{1}{x^2}} = 2^{\frac{1}{0_+}} = 2^{+\infty} = +\infty$

d) $\lim\limits_{x\to 0} (e^x-x+1)^{\frac{1}{1-\cos x}} = (1-0+1)^{\lim\limits_{x\to 0}\frac{1}{1-\cos x}}$

e) $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}} = \lim\limits_{x\to 0} \left(1+x\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\right)^{\frac{1}{x}} = 1^{\lim\limits_{x\to 0}\frac{1}{x}}$ $\Bigg\} \to \nexists \lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

$\lim\limits_{\substack{x\to 0 \\ x>0}} \frac{1}{x} = \frac{1}{0_+} = +\infty$

$\neq$ $\Bigg\} \nexists \lim\limits_{x\to 0} \frac{1}{x}$

$\lim\limits_{\substack{x\to 0 \\ x<0}} \frac{1}{x} = \frac{1}{0_-} = -\infty$

f) $\lim_{x\to\infty}\left(\frac{x+7}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{x+7-x}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{7}{x}\right)^{\frac{x}{7}}\right]^{\frac{7x}{x}} =$

$= e^7$

3. a) $\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\sin^2 x+\sin^2 2x+\ldots+\sin^2 nx\right)^{\frac{1}{n^3\cdot x^2}}\right] =$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+x^2\cdot\left(\frac{\sin x}{x}\right)^2+(2x)^2\cdot\left(\frac{\sin 2x}{2x}\right)^2+\ldots+(nx)^2\cdot\left(\frac{\sin nx}{nx}\right)^2\right)^{\frac{1}{n^3\cdot x^2}}\right]$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+x^2+(2x)^2+\ldots+(nx)^2\right)^{\frac{1}{n^3\cdot x^2}}\right]$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+x^2(1+2^2+\ldots+n^2)\right)^{\frac{1}{n^3\cdot x^2}}\right]$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+x^2\cdot\frac{n(n+1)(2n+1)}{6}\right)^{\frac{1}{n^3\cdot x^2}}\right]$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+x^2\cdot\frac{n^3(1+\overset{\to 0}{\frac{1}{n}})(2+\overset{\to 0}{\frac{1}{n}})}{6}\right)^{\frac{1}{n^3\cdot x^2}}\right]$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\frac{n^3\cdot x^2}{6}\right)^{\frac{6}{n^3\cdot x^2}\cdot\frac{1}{6}}\right] = e^{\frac{1}{6}}$

b) $\lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+\ldots+\ln(1+nx)\right)^{\frac{1}{n^2\cdot x}}\right] =$

$= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(1+\ln\left[(1+x)(1+2x)\ldots(1+nx)\right]\right)^{\frac{1}{n^2\cdot x}}\right]$

4. a) $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{3x} = \lim\limits_{x\to 0} \dfrac{2x \cdot \dfrac{e^{2x}-1}{2x}}{3x} = \dfrac{2}{3}$ $\to \ln e = 1$

b) $\lim\limits_{x\to 0} \dfrac{e^{x}-\cos x}{3x} = \lim\limits_{x\to 0} \dfrac{e^{x}-1+1-\cos x}{3x} = \lim\limits_{x\to 0} \dfrac{\overbrace{\dfrac{e^{x}-1}{x}}^{\to 1} x + 1 - \cos x}{3x} =$

$= \lim\limits_{x\to 0} \dfrac{x}{3x} + \dfrac{1-\cos x}{3x} = \dfrac{1}{3} + \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x} = \dfrac{1}{3}$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{3x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim\limits_{x\to 0} \dfrac{\sin x}{3} = \lim\limits_{x\to 0} \dfrac{x}{3} = 0$