5. 10. 2023

Algebra - Seminar 1

~~partial~~

Exercices

1. Addition: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Subtraction: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Multiplication: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Division: NONE because they contain 0 (nu putem împărți la 0)
but $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

2. $A = \{a_1, a_2, a_3\}$

(i) no. of operations on A = ?

operation = $f: A \times A \to A$

no of functions $f: A \to B = |B|^{|A|}$, where $|A| = \text{card}\, A$

no of op = $\cancel{|A \times A|}\ |A|^{|A \times A|} = 3^9$

generalization: n^{n^2}

(ii) no of commutative operations on $A = |A|^{\frac{|A|(|A|+1)}{2}}$

↳ $f(a_1, a_2) = f(a_2, a_1)$

	a_1	a_2	a_3
a_1	\	*	·
a_2	*	\	o
a_3	·	o	\

generalization: $n^{\frac{n(n+1)}{2}}$

(ii) operations on A with identity element

if $e = a_1 \to$ we can complete line and column 1 $\to$

$\to f: \{(a_i, a_j) \mid i, j \in \{2, 3\}\} \to A \to$ we have $|A|^4 = 3^4$

	a_1	a_2	a_3
a_1	a_1	a_2	a_3
a_2	a_2		
a_3	a_3		

we can have 3 different elements $= e \to$

$\to$ no $= 3 \cdot 3^4 = 3^5$

generalization: $|A| \cdot |A|^{(|A|-1)^2} = n \cdot n^{(n-1)^2}$

3. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ - group with the addition or multiplication?

Let "$*$" - operation on A. $(A, *)$ - group if

1. „$*$" - associative
2. we have identity element
3. every element is invertible

- addition: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
- multiplication: none (because of 0)

4. $x * y = x + y + xy$

(i) $(\mathbb{R}, *)$ - commutative monoid

- commutative $\forall x, y \in \mathbb{R}$, $x * y = y * x$

$y * x = y + x + yx = x + y + xy = x * y$ „T" ①

- associative $\forall x, y, z \in \mathbb{R}$ $x * (y * z) = (x * y) * z$

$x * (y * z) = x * (y + z + yz) = x + y + z + yz + x(y + z + yz) =$

$= x + y + z + yz + xy + xz + xyz$

$= x + y + xy + z + z(x + y + xy)$

$= (x * y) + z + z(x * y) = (x + y) * z \to$ „T" ②

- identity element $\forall x \in \mathbb{R}$ $\exists e \in \mathbb{R}$ s.t. $x * e = e * x = x$

„$*$" - commutative $\to$ $x * e = e * x$, $\forall x, e \in \mathbb{R}$

$x * e = x + e + xe = x \to e(1 + x) = 0 \to e = 0 \in \mathbb{R}$ ③

From ①, ②, ③ $\to$ $(\mathbb{R}, *)$ - comutative ~~element~~ monoid

(ii) $[-1; \infty)$ - stabel ~~set~~ subset of $(\mathbb{R}, *)$?

- $[-1; \infty)$ - stabel subset $\leftrightarrow$ $\forall x, y \in [-1; \infty)$ $x * y \in [-1; \infty)$

$x * y = x + y + xy = x(1 + y) + y + 1 - 1 = (y + 1)(x + 1) - 1$

$x, y \in [-1; \infty) \to$ $-1 \le x < \infty$ $| +1$

$\to$

$-1 \le y < \infty$ $| +1$

$\to$ $0 \le x + 1 < \infty$

$0 \le y + 1 < \infty$

$0 \le (x+1)(y+1) < \infty$ $| -1$ $\to$ $-1 \le (x+1)(y+1) - 1 < \infty$ $\to$

$\to$ $-1 \le x * y < \infty$ $\to$ $x * y \in [-1; \infty)$

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5. $x * y = g.c.d(x, y)$, $x, y \in \mathbb{N}$
↳ greatest common divisor = cmmdc

(i) $(\mathbb{N}, *)$ - commutative monoid?

- commutative $\forall x, y \in \mathbb{N}$, $x * y = y * x$

$y * x = gcd(y, x) = gcd(x, y) = x * y$ "T" ①

~~- assoc~~

- association $\forall x, y, z \in \mathbb{N}$ $x * (y * z) = (x * y) * z$

$x * (y * z) = (x * y) * z \Leftrightarrow$

$\Leftrightarrow x * gcd(y, z) = gcd(x, y) * z$

$\Leftrightarrow \underbrace{gcd(x, gcd(y, z))}_{d} = gcd(gcd(x, y), z)$

$d \rightarrow d / x$ & $d / gcd(y, z)$

Let d' be anoter divisor of x and $gcd(y, z) \rightarrow$

$\rightarrow d' < d$

$d / gcd(y, z) \rightarrow d / y$ and d / z

d / y and $d / x \rightarrow d / gcd(x, y)$, d / z $\Big\} \rightarrow d / gcd(gcd(x, y), z)$

Let d'' be a div of $gcd(x, y)$ and $z \rightarrow$

$\rightarrow d'' / gcd(x, y) \rightarrow d'' / x$ and d'' / y

d'' / z

but d / x and d / y and $d / z \rightarrow gcd(gcd(x, y), z) = d \rightarrow$

$\rightarrow$ "$*$" - associative ②

- identity element $= 0$ ③

①, ②, ③ $\rightarrow (\mathbb{N}, *)$ - commutative element

(ii) $D_n = \{x \in \mathbb{N} / x / n\}$ - stable subset of $(\mathbb{N}, *)$

$(D_n, *)$ - commutative monoid

D_n - stable subset $\Leftrightarrow \forall x, y \in D_n$, $x * y \in D_n$

$x, y \in D_n \rightarrow x / n$ and y / n
$x * y = gcd(x, y) = \alpha \rightarrow \alpha / x$, α / y $\Big\} \rightarrow \alpha / n \rightarrow \alpha \in D_n$, $\forall x, y \in D_n$

(ii) If D_n - stabel subset of $(\mathbb{N}, *)$ →

→ association & commut. holds in D_n } → $(D_n, *)$ - comm. monoid

identity element = n

(iii) $D_6 = \{1, 2, 3, 6\}$

$*$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

6. finite stable subset of $(\mathbb{Z}, \cdot)$

A - finite stable subset of $(\mathbb{Z}, \cdot)$ → $x \in A$, $x^n \in A$ →

→ $x \in \{-1; 0; 1\}$

A: $\emptyset$, $\{0\}$; $\{1\}$, $\{-1; 1\}$; $\{0; 1\}$; $\{-1; 0; 1\}$

8. A, $X, Y \subseteq A$

power set $P(A)$ → the set of all subsets of A

$X * Y = \{x \cdot y \mid x \in X, y \in Y\}$

(i) If $(A, \cdot)$ - monoid, then $(P(A), *)$ - monoid

- associativity: $\forall X, Y, Z \in P(A)$, $(X * Y) * Z = X * (Y * Z)$

$(X * Y) * Z = (\{x \cdot y \mid x \in X, y \in Y\}) * Z =$

$= \{(x \cdot y) \cdot z \mid x \in X, y \in Y, z \in Z\} \xrightarrow[\to A\text{-assoc}]{A\text{-mon}\to}$

$= \{x \cdot (y \cdot z) \mid x \in X, y \in Y, z \in Z\} = X * (\{y \cdot z \mid y \in Y, z \in Z\}) =$

$= X * (Y * Z)$

- identity element

Let $E = \{1\}$, where 1 = identity element for $(A, \cdot)$ →

→ $X * E = \{x \cdot 1 \mid x \in X, 1 \in E\} = \{x \mid x \in X\} = X = E * X$