Substitutions in Integrals

1 Euler's substitions

Sometime when in the formulation of the function to be integrated we encounter

$$\sqrt{ax^2+bx+c},$$

where $a, b, c \in \mathbb{R}$, we consider a new variable t, in one of the following cases:

$$\sqrt{ax^2+bx+c} = \begin{cases} \pm\sqrt{a}x \pm t & \text{if } a > 0 \\ \pm x \cdot t \pm \sqrt{c} & \text{if } c > 0 \\ t(x-x_0) & \text{if } x_0 \text{ is a solution of the equation } ax^2+bx+c=0. \end{cases}$$

2 Weirstras' (trigonometric) substitutions

For functions in whose formulations are involved trigonometric functions, there is a usual substitution, namely:

$$tg\frac{x}{2} = t.$$

If we denote by $R(\sin x, \cos x)$ the expression to be integrated, sometimes we may consider other substitutions, which might lead us faster to the expected solution. Hence:P

- If $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, then choose $\cos x = t$.
- If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $\sin x = t$.
- If $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$, then choose $tg\ x = t$.

Recall the following trigonometric identities:

$$\cos^2 x = \frac{1}{1+tg^2x} \qquad \sin^2 x = \frac{tg^2x}{1+th^2x}.$$

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} \qquad \cos x = \frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

3 Other tigonometric substitution

Sometimes, when the integrating function contains square roots of second degree polynomials (alternatively to using Euler's substitutions) we may pass to trigonometric functions, in the following situations:

- When $\int R(x, \sqrt{r^2 - x^2}dx$ choose $x = r\sin$ or $x = r\cos t$.
- When $\int R(x, \sqrt{r^2 + x^2}dx$ choose $x = rtgt$ or $x = rctgt$.
- When $\int R(x, \sqrt{x^2 - r^2}dx$ choose $x = \frac{r}{\cos x}$ or $x = \frac{r}{\sin x}$.

Exercise 1:

a) $\int \frac{1}{1+\frac{1}{sinx}}dx, \quad x \in (\pi, \pi);$

b) $\int \frac{1}{3\sin x + 4\cos x}dx \quad x \in (\pi, \pi);$

c) $\int \frac{\sqrt{9-x^2}}{x^2}dx, \quad x \in (-3, 3);$

d) $\int \frac{1}{\sqrt{(x^2+1)^3}}dx, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$

e) $\int \frac{1}{\sqrt{(x^2-8)^3}}dx \quad x \in (-\sqrt{8}, \sqrt{8});$

f) $\int \sqrt{2x - x^2}dx \quad x \in (0, 2);$

g) $\int \sqrt{4 - x^2}dx \quad x \in (-2, 2);$

h) $\int x\sqrt{1 + x^2}dx.$

Exercise 2:

Determine

$$a) \int \frac{2x-1}{x^2-3x+2}\mathrm{d}x, \;\; x \in]2, +\infty[;$$

b) $\displaystyle\int \frac{4}{(x-1)(x+1)^2}\mathrm{d}x, \quad x>1;$

c) $\displaystyle\int \frac{1}{x^3-x^4}\mathrm{d}x, \quad x>1;$

d) $\displaystyle\int \frac{2x+5}{x^2+5x+10}, \ x\in\mathbb{R};$

e) $\displaystyle\int \frac{1}{x^2+x+1}, \ x\in\mathbb{R}.$

Exercise 3:

Determine:

a) $\displaystyle I=\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\mathrm{d}x, \quad x\in]0,+\infty[;$

b) $\displaystyle I=\int \frac{1}{x+\sqrt{x-1}}\mathrm{d}x, \quad x\in]1,+\infty[.$

Exercise 4:

Determine:

a) $\displaystyle I=\int \frac{1}{1+\sqrt{x^2+2x-2}}\mathrm{d}x, \quad x\in]\sqrt{3}-1,+\infty[;$

b) $\displaystyle I=\int \frac{1}{(x+1)\sqrt{-4x^2-x+1}}\mathrm{d}x, \quad x\in]\frac{-1-\sqrt{17}}{8},\frac{\sqrt{17}-1}{8}[.$

Exercise 5:

Determine:

a) $\displaystyle\int_1^2 \frac{1}{x^3+x^2+x+1}\mathrm{d}x;$ *b)* $\displaystyle\int_1^3 \frac{1}{x(x^2+9)}\mathrm{d}x;$

c) $\displaystyle\int_{-1}^1 \frac{x^2+1}{x^4+1}\mathrm{d}x;$ *d)* $\displaystyle\int_{-1}^1 \frac{x}{x^2+x+1}\mathrm{d}x.$

Exercise 6:

Determine:

$a)\ \displaystyle\int_{-3}^{-2} \frac{x}{(x+1)(x^2+3)}\mathrm{d}x;\qquad b)\ \displaystyle\int_{0}^{1} \frac{x+1}{(x^2+4x+5)^2}\mathrm{d}x;$

$c)\ \displaystyle\int_{1}^{2} \frac{1}{x^3+x}\mathrm{d}x;\qquad d)\ \displaystyle\int_{0}^{2} \frac{x^3+2x^2+x+4}{(x+1)^2}\mathrm{d}x.e)\ \displaystyle\int_{0}^{1} \frac{1}{(x+1)(x^2+4)}\mathrm{d}x;$

$f)\ \displaystyle\int_{2}^{3} \frac{2x^3+x^2+2x-1}{x^4-1}\mathrm{d}x;\qquad g)\ \displaystyle\int_{0}^{1} \frac{x^3+2}{(x+1)^3}\mathrm{d}x.$

Exercise 7:

Determine:

$a)\ \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\mathrm{d}x;\qquad b)\ \displaystyle\int_{0}^{1} \frac{1}{\sqrt{x^2+x+1}}\mathrm{d}x;$

$c)\ \displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4x^2+x+1}}\mathrm{d}x;\qquad d)\ \displaystyle\int_{2}^{3} \frac{x^2}{(x^2-1)\sqrt{x^2-1}}\mathrm{d}x.$

Exercise 8:

Determine

$a)\ \displaystyle\int_{2}^{3} \sqrt{x^2+2x-7}\mathrm{d}x;\qquad b)\ \displaystyle\int_{0}^{1} \sqrt{6+4x-2x^2}\mathrm{d}x;$

$c)\ \displaystyle\int_{0}^{3/4} \frac{1}{(x+1)\sqrt{x^2+1}}\mathrm{d}x;\qquad d)\ \displaystyle\int_{2}^{3} \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x.$

Exercise9:

Determine:

$a)\ 2\sqrt{2} < \displaystyle\int_{-1}^{1} \sqrt{x^2+4x+5}\mathrm{d}x < 2\sqrt{10};$

$b)\ \mathrm{e}^2(\mathrm{e}-1) < \int_{\mathrm{e}}^{\mathrm{e}^2} \frac{x}{\ln x}\mathrm{d}x < \frac{\mathrm{e}^3}{2}(\mathrm{e}-1).$