23.11.2023

Algebra - Seminar 8

1. $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

A - invertable ?, $A^{-1} = ?$, Solve $AX = B$

2. Using the Kronecker - Capelli Th decide if the following sistems are compatible and solve the compatible ones:

(i) $S: \begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$

$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix}$

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = 1 - 6 - 4 - 2 - 6 - 2 = -19 \neq 0 \rightarrow rang(A) = 3$

$A| = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 3 \end{pmatrix}$

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -19 \neq 0 \rightarrow rang(A|) = 3$

$rang(A) = rang(A|) = 3 \rightarrow S$ - compatible

Let $x_4 = \alpha \in \mathbb{R}$

$\begin{cases} x_1 + x_2 + x_3 = 5 + 2\alpha \\ 2x_1 + x_2 - 2x_3 = 1 - \alpha \\ 2x_1 - 3x_2 + x_3 = 3 - 2\alpha \end{cases} \rightarrow x_1 = 5 + 2\alpha - x_2 - x_3 \rightarrow \begin{cases} 10 + 4\alpha - 2x_2 - 2x_3 + x_2 - 2x_3 = 1 - \alpha \\ 10 + 4\alpha - 2x_2 - 2x_3 - 3x_2 + x_3 = 3 - 2\alpha \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -x_2 - 4x_3 = -9 - 5\alpha \\ -5x_2 - x_3 = -7 - 6\alpha \end{cases} \begin{matrix} \rightarrow x_2 = -4x_3 + 9 + 5\alpha \\ \rightarrow +20x_3 - 45 - 25\alpha - x_3 = 7 - 6\alpha \end{matrix}$

$19x_3 = 52 + 19\alpha \rightarrow x_3 = \frac{52}{19} + \alpha$

$x_2 = -\frac{208}{19} - 4\alpha + 9 + 5\alpha = -\frac{37}{19} + \alpha$

$x_1 = 5 + \cancel{2\alpha} + \frac{37}{19} - \cancel{\alpha} - \frac{52}{19} - \cancel{\alpha} = 5 - \frac{15}{19}$

(ii) $S = \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix}$$

$$d_3 = \begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -2 - 2 - 2 + 2 + 2 + 2 = 0$$

$$d_3' = \begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 1 & -2 & 5 \end{vmatrix} = -\cancel{10} - \cancel{2} + \cancel{2} + 2 - \cancel{2} + \cancel{10} = 0$$

$$d_3'' = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

$$d_2 = \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix} = -2 + 2 = 0$$

$$\left.\begin{aligned} & d_2' = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0 \to \text{rang}(A) = 2 \\ & \bar{A} = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix} = A \to \text{rang}(\bar{A}) = 2 \end{aligned}\right\} \to S\text{-compat.}$$

Let $x_1 = \alpha$, $x_2 = \beta$

$$\begin{cases} x_3 + x_4 = 1 - \alpha + 2\beta \\ x_3 - x_4 = -1 - \alpha + 2\beta \end{cases} \quad +$$

$2x_3 = -2\alpha + 4\beta \to x_3 = -\alpha + 2\beta$

$x_4 = 1 - \alpha + 2\beta + \alpha - 2\beta = 1$

$S = \{(\alpha, \beta, -\alpha + 2\beta, 1) \mid \alpha, \beta \in \mathbb{R}\}$

(iii) $S = \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \qquad \bar{A} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix}$$

$$\left.\begin{array}{l} d_2 = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0 \\ \text{toți } det_3 \text{ a lui } A = 0 \end{array}\right\} \rightarrow rang(A) = 2$$

$det\ \bar{A} = 0$ (2 coloane egale)

$$d_3 = \begin{vmatrix} -1 & 1 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 4 \end{vmatrix} = -8-1+0-0+3+4 = -9+7 = -2 \neq 0 \rightarrow rang(\bar{A}) = 3$$

$\rightarrow rang(A) \neq rang(\bar{A}) \rightarrow$ S is not compat.

3. (iii) $d_p = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0$

$$d_{c_1} = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 3-6-1+3-2+3 = 9-9 = 0$$

$$d_{c_2} = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 4-3+0-0-1+4 = 8-4 = 4 \neq 0 \rightarrow \text{incompat}$$

4. $\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$

$$A = \begin{pmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{pmatrix}$$

$$det(A) = \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = 0+0+0-0-abc-abc = -2abc \neq 0,\ a,b,c \neq 0$$

$$d_x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = 0+0+a^3-0-ac^2-ab^2 = a^3-ac^2-ab^2$$

$$x = \frac{d_x}{det(A)} = \frac{a^3-ac^2-ab^2}{-2abc} = \frac{1-c^2-b^2}{-2bc}$$

$$y = \frac{a^2+c^2-b^2}{-2ac}$$

$$z = \frac{a^2+b^2-c^2}{-2ab}$$

8. $x, y, z > 0$

$$\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \dfrac{z}{xy} = 81 \end{cases}$$

$$\frac{x^3y^2z^2}{x^2y^2z^2} = \frac{27}{1^2} \rightarrow x = 27 \longrightarrow yz = \frac{1}{27}$$

$$\frac{z}{y} \cdot \frac{1}{27} = 81 \rightarrow \frac{z}{y} \cdot yz = 81 \rightarrow z^2 = 81 \rightarrow \left.\begin{array}{l} z = \pm 9 \\ z > 0 \end{array}\right\} \rightarrow z = 9$$

$z = 9$

$$\frac{9}{27 \cdot y} = 81 \rightarrow y = \frac{9}{\cancel{81}^{9} \cdot 27} = \frac{1}{243} = \frac{1}{3^5}$$

~~$z = -9$~~

$$\cancel{\frac{-9}{27y} = 81 \rightarrow -\frac{1}{3y} = 81 \rightarrow y = \frac{-1}{3 \cdot 81} = -\frac{1}{3^5}}$$

5. i) $\begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

(iii) $\begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$

$$\bar{A} = \left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_2 \leftrightarrow L_1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{L_3 - L_1}{\overset{L_2 - 2L_1}{\sim}}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$S: \begin{cases} x + 2y - z = 3 \\ \quad\; y + 3z = 1 \end{cases} \rightarrow y = 1 - 3\alpha$

$\alpha \in \mathbb{R}, \ z = \alpha$

$x + 2 - 6\alpha - \alpha = 3 \rightarrow x = 1 + 7\alpha$

$S = \{(1 + 7\alpha, 1 - 3\alpha, \alpha) \mid \alpha \in \mathbb{R}\}$

G-J: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \dots$

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$ $(\lambda \in \mathbb{R})$

$$\bar{A} = \left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \underset{L_3 - L_1}{\overset{L_2 - 2L_1}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda - 2 \end{array}\right) \overset{L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda - 5 \end{array}\right)$$

$\lambda \neq 5 \rightarrow$ incompatible system

$\lambda = 5$

$\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ \quad -3x_2 + 3x_3 - 7x_4 = -3 \end{cases}$

$x_3, x_4 \in \mathbb{R} \rightarrow x_1 = x_3 + \frac{2}{3}x_4$

$x_2 = 1 + x_3 - \frac{7}{3}x_4$

G-J HW

7. $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$, $(a \in \mathbb{R})$

$$\bar{A} = \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \overset{L_2 \leftrightarrow L_1}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right) \underset{L_3 - L_1}{\overset{L_2 - aL_1}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & a^2-a \end{array}\right) \underset{\frac{1}{1-a} L_3,\ a \neq 1}{\overset{\frac{1}{1-a} L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1+a & 1 & 1+a \\ 0 & 1 & -1 & -a \end{array}\right) \overset{L_2 \leftrightarrow L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 1+a & -1 & -a \end{array}\right) \overset{L_3 - (a+1)L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 2+a & (1+a)^2 \end{array}\right)$$

$a \neq -2$ $\begin{cases} x + ay + 1 = a \\ y - z = -a \\ (2+a)z = (1+a^2) \end{cases}$ $\rightarrow z = \dfrac{(1+a)^2}{a+2}$, $y = \dfrac{1}{a+2}$, $x = \dfrac{a+1}{a+2}$

$a = -2 \rightarrow$ incompat. system

$a = 1 \rightarrow x + y + z = 1 \rightarrow S = \{(1-y-z, y, z) \mid \forall y, z \in \mathbb{R}\}$