Calculus - Homework 10

1. Determine all local extrema

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^3 - 3x + y^2 + z^2$

for all $(x,y,z) \in \mathbb{R}^3$ it holds:

$$\frac{\partial f}{\partial x}(x,y,z) = 3x^2 - 3$$

$$\frac{\partial f}{\partial y}(x,y,z) = 2y$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z$$

$$\begin{cases} 3x^2 - 3 = 0 \to x^2 = 1 \to x = \pm 1 \\ 2y = 0 \to y = 0 \\ 2z = 0 \to z = 0 \end{cases}$$

$\to (-1,0,0)$ & $(1,0,0)$ - stationary points of f

For all $(x,y,z) \in \mathbb{R}^3$ it holds:

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 6x \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 0 \qquad \frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(x,y,z) = 2 \qquad \frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0$$

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = 2$$

- $Hf(-1,0,0) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\left.\begin{aligned} \Delta_1 &= |-6| = -6 = (-1)^1 \cdot 6 \\ \Delta_2 &= \begin{vmatrix} -6 & 0 \\ 0 & 2 \end{vmatrix} = -6 \cdot 2 - 0 = -12 = (-1)^1 \cdot 12 \\ \Delta_3 &= \begin{vmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -24 \end{aligned}\right\}$$

$\to (-1,0,0)$ is not a local extremum, it's a saddle point

- $Hf(1,0,0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\left.\begin{aligned} \Delta_1 &= |6| = 6 > 0 \\ \Delta_2 &= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 > 0 \\ \Delta_3 &= \begin{vmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 24 > 0 \end{aligned}\right\}$$

$\to Hf(1,0,0)$ - positive definite $\to$

$\to (1,0,0)$ - local minimum point

$f(1,0,0) = 1 - 3 + 0 + 0 = -2$

b) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=x^4+y^4-4(x-y)^2$
$=x^4+y^4-4x^2+8xy-4y^2$

For all $(x,y)\in\mathbb{R}^2$ it holds

$$\frac{\partial f}{\partial x}(x,y)=4x^3-8x+8y$$

$$\frac{\partial f}{\partial y}(x,y)=4y^3-8y+8x$$

$$\begin{cases}4x^3-8x+8y=0 & |:4\\ 4y^3-8y+8x=0 & |:4\end{cases} \iff \begin{cases}x^3-2x+2y=0\\ y^3-2y+2x=0\end{cases} \oplus$$

$$x^3+y^3=0 \to x^3=-y^3 \to x=-y$$

$$x^3-2x+2(-x)=0 \iff x^3-4x=0 \iff x(x^2-4)=0 \to \left.\begin{array}{l}x_1=0\\ x_2=2\\ x_3=-2\\ x=-y\end{array}\right\} \to$$

$\to (0,0);\ (2,-2),\ (-2,2)$ - stationary points of f

For all $(x,y)\in\mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y)=12x^2-8 \qquad \frac{\partial^2 f}{\partial x\partial y}(x,y)=8 \qquad \frac{\partial^2 f}{\partial y^2}(x,y)=12y^2-8$$

* $Hf(0,0)=\begin{pmatrix}-8 & 8\\ 8 & -8\end{pmatrix}$

  - $\Delta_1=-8<0$
  - $\Delta_2=\begin{vmatrix}-8 & 8\\ 8 & -8\end{vmatrix}=64-64=0 \to$ we can't apply Silvester's Th

$$\Phi(h_1,h_2)=-8h_1^2+8h_1h_2+8h_1h_2-8h_2^2=-8h_1^2-8h_2^2+16h_1h_2=$$
$$=-8(h_1^2-2h_1h_2+h_2^2)=-8(h_1-h_2)^2\le 0,\ \forall h_1,h_2\in\mathbb{R} \to \text{semidefinite}$$

* $Hf(2,-2)=\begin{pmatrix}40 & 8\\ 8 & 40\end{pmatrix}=H(-2,2)$

  - $D_1=|40|=40>0$
  - $D_2=\begin{vmatrix}40 & 8\\ 8 & 40\end{vmatrix}=1600-64>0$

$\Big\}\to H(-2,2)$ & $H(2,-2)$ - positive definite $\to$
$\to(-2,2)$ & $(2,-2)$ local minimum points

$$f(-2,2)=16+16-4(-2-2)^2=32-4\cdot 16=-32=$$
$$=f(2,-2)$$

c) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2(1+xy) + xy$
$= z^2 + xyz^2 + xy$

For all $(x,y,z) \in \mathbb{R}^3$ it holds:

$\frac{\partial f}{\partial x}(x,y,z) = yz^2 + y$

$\frac{\partial f}{\partial y}(x,y,z) = xz^2 + x$

$\frac{\partial f}{\partial z}(x,y,z) = 2z + 2z \cdot xy$

$$\begin{cases} yz^2 + y = 0 \\ xz^2 + x = 0 \\ 2z(1+xy) = 0 \end{cases} \to \begin{cases} z_1 = 0 \to x = y = 0 \\ xy = -1 \to x = -\frac{1}{y} \to \begin{cases} yz^2 + y = 0 \\ -\frac{1}{y}z^2 - \frac{1}{y} = 0 \mid \cdot y \end{cases} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} yz^2 + y = 0 \\ -z^2 - 1 = 0 \end{cases} \to z^2 = -1 \text{ - impos. } z \in \mathbb{R}$$

$\to (0,0,0)$ - stationary point of f

For all $(x,y,z) \in \mathbb{R}^3$ it holds

$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 0$ $\quad \frac{\partial^2 f}{\partial x \partial y} = z^2 + 1$ $\quad \frac{\partial^2 f}{\partial x \partial z} = 2zy$

$\frac{\partial^2 f}{\partial y^2}(x,y,z) = 0$ $\quad \frac{\partial^2 f}{\partial y \partial x} = z^2 + 1$ $\quad \frac{\partial^2 f}{\partial y \partial z} = 2zx$

$\frac{\partial^2 f}{\partial^2 z^2}(x,y,z) = 2 + 2xy$ $\quad \frac{\partial^2 f}{\partial z \partial x} = 2zy$ $\quad \frac{\partial^2 f}{\partial z \partial y} = 2zx$

$$Hf(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- $\Delta_1 = 0 \to$ we can't apply Sylvester's Th.
- $\Delta_2 = -1$
- $\Delta_3 = -2$

$\Phi(h_1, h_2, h_3) = h_1h_2 + h_2h_1 + 2h_3^2 = 2h_1h_2 + 2h_3^2$

$$\left.\begin{array}{l} \Phi(0,0,1) = 2 > 0 \\ \Phi(-1,1,0) = -2 < 0 \end{array}\right\} \to Hf(0,0,0) \text{ - indefinite} \to (0,0,0) \text{ - saddle point}$$

d) $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^3 + 3xy^2 - 15x - 12y$

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15$$

$$\frac{\partial f}{\partial y}(x,y) = 6xy - 12$$

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \to x^2 + y^2 - 5 = 0 \to x^2 + y^2 = 5 \\ 6xy - 12 = 0 \to xy = 2 \end{cases}$$

$\to x^2 + 2xy + y^2 = 9$

$(x+y)^2 = 3^2 \to |x+y| = 3$ $\begin{cases} \to x = 3 - y \\ \to x = -3 - y \end{cases}$

- $y(3-y) = 2$

$-y^2 + 3y - 2 = 0$

$\Delta = 9 - 4 \cdot 2 = 9 - 8 = 1$

$y_{1,2} = \frac{-3 \pm 1}{-2}$ $\begin{cases} 1 \to x = 2 \\ 2 \to x = 1 \end{cases}$

- $y(-3-y) = 2$

$-y^2 - 3y - 2 = 0 \to y^2 + 3y + 2 = 0$

$\Delta = 9 - 8 = 1$

$y_{3,4} = \frac{-3 \pm 1}{2}$ $\begin{cases} -1 \to x = -2 \\ -2 \to x = -1 \end{cases}$

$(-1,-2); (-2,-1); (1,2), (2,1)$ - stationary points

For all $(x,y) \in \mathbb{R}^2$ it holds

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 6x \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 6y$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 6x$$

* $Hf(-1,-2) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix}$

$\left.\begin{array}{l} \bullet\ D_1 = -6 < 0 \\ \bullet\ D_2 = 36 - 144 = -108 < 0 \end{array}\right\} \to Hf(-1,-2)$ - indefinite $\to (-1,-2)$ - saddle point

* $Hf(-2,-1) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$

$\left.\begin{array}{l} \Delta_1 = -12 = (-1)\cdot 12 \rightarrow (-1)\cdot\Delta_1 = (-1)^2 \cdot 12 = 12 > 0 \\ \Delta_2 = 144 - 36 = 108 = (-1)^2 \cdot 108 \rightarrow (-1)^2 \cdot \Delta_2 = (-1)^4 \cdot 108 = 108 > 0 \end{array}\right\} \rightarrow$

$\rightarrow Hf(-2,-1)$ - negative definite $\rightarrow (-2,-1)$ - local maximum of f

* $Hf(1,2) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix}$

$\left.\begin{array}{l} \bullet\ \Delta_1 = 6 > 0 \\ \bullet\ \Delta_2 = 36 - 144 = -108 < 0 \end{array}\right\} \rightarrow Hf(1,2)$ - indefinite $\rightarrow (1,2)$ - saddle point

* $Hf(2,1) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$

$\left.\begin{array}{l} \bullet\ \Delta_1 = 12 > 0 \\ \bullet\ \Delta_2 = 144 - 36 = 108 > 0 \end{array}\right\} \rightarrow Hf(2,1)$ - positive definite $\rightarrow (2,1)$ - local minimum of f