12.10.2023

Algebra - Seminar 2

1. $M = \{2, 3, 4, 5, 6\}$

- $x r y \leftrightarrow x < y$

  $R = \{(2,3); (2,4); (2,5); (2,6); (3,4); (3,5); (3,6); (4,5);$
  $(4,6); (5,6)\}$

- $x s y \leftrightarrow x / y$

  $S = \{(2,2); (2,4); (2;6); (3,3); (3,6); (4,4); (5,5), (6,6)\}$

- $x t y \leftrightarrow g.c.d(x,y) = 1$

  $T = \{(2,3); (2;5); (3;4); (3,5); (4;5), (5;6); (5,2); (4;3);$
  $(5;3); (5;4); (6;5); (3,2)\}$

- $x v y \leftrightarrow x \equiv y \pmod 3$

  $V = \{(2,2); (2;5); (3,3), (3,6); (4,4); (5,5); (5,2); (6,6);$
  $(6,3)\}$

2. A, B - sets, $|A| = n$, $|B| = m$, $n, m \in \mathbb{N}^*$

(i) Nr. of relations having the domain A & codom. B

(= nr submulțimilor $A \times B$, $R \subseteq A \times B$)

$2^{|A| \cdot |B|} = 2^{n \cdot m}$

(ii) Nr of homogeneous rel. on A?

$2^{|A| \cdot |A|} = 2^{n^2}$

3. Reflexivity: $R = \{(1,1), (2,2) (3,3), (1,3), (3,2)\}$

$\rightarrow$ (R): because $\Delta_3 \subseteq R$

$\rightarrow$ ✗(T): because we have $(1,3), (3,2)$ but not $(1,2)$

Transitivity: $T = \{(1,2); (2;3); (1;3)\}$ $(x < y)$

Symmetry: $S = \{(1,2); (2;1)\}$

4.
- the strict inequality relations on $\mathbb{R}$: transitivity
- the divisibility relation on $\mathbb{N}$ & $\mathbb{Z}$: reflexivity + trans.
- the perpendicularity rel. of lines in space: symetry
- the paralelism rel. : trans. + sym.
- the congruence + similarity of triangles: all 3

5. $M = \{1, 2, 3, 4\}$; r_1, r_2 - rel.

$R_1 = \Delta_M \cup \{(1,2); (2,1); (1,3); (3,1); (2,3), (3,2)\}$

$R_2 = \Delta_M \cup \{(1,2), (1,3)\}$

$\pi_1 = \{\{1\}, \{2\}, \{3,4\}\}$

$\pi_2 = \{\{1\}, \{1,2\}, \{3,4\}\}$

(i) r_1, r_2 - equivalances on M? If yes → coresponding partition

- r_1
  - (R) ✓, $\Delta_M \subseteq R_1$
  - (T) ✓ $(1,2), (2,1), (1,1)$

    $(1,2), (2,2), (1,2)$

    $(1,2), (2,3) = (1,3)$

    $(2,1), (1,1), (2,1)$

    $(2,1), (1,2), (2,2)$

    $(2,1), (1,3), (2,3)$

    $(1,3), (3,3), (1,3)$

    $(1,3), (3,1), (1,1)$

    $(1,3), (3,2), (1,2)$

    $(3,1), (1,1), (3,1)$

    $(3,1), (1,2), (3,2)$

    $(3,1), (1,3), (3,3)$

    $(2,3), (3,3), = (2,3)$

    $(2,3), (3,1), (2,1)$

    $(2,3), (3,2), (2,2)$

    $(3,2), (2,2), (3,2)$

    $(3,2), (2,1), (3,1)$

    $(3,2), (2,3), (3,3)$
  - (S): ✓ $(1,2), (2,1)$

    $(1,3), (3,1)$

    $(2,3), (3,2)$

r_1 - equivalance → $\pi = \{\{1,2,3\}, \{4\}\}$

- r2
  - (R) ✓ $\Delta_M \subseteq R_2$
  - (T) ✓ $(1,2), (2,2) \quad (1,2)$

    $(1,3), (3,3), \quad (1,3)$
  - (S) X $(1,2) \in R_2$ but $(2,1) \notin R_2 \to r_2$ isn't equivalance

ii) π_1, π_2 - partitions on M? If yes $\to$ corresponding eq. rel.

- $\pi_1 = \{\{1\}, \{2\}, \{3,4\}\}$
  - $\{1\} \cup \{2\} \cup \{3,4\} = \{1,2,3,4\} = M$
  - $\{1\} \cap \{2\} = \emptyset$

    $\{1\} \cap \{3,4\} = \emptyset$

    $\{2\} \cap \{3,4\} = \emptyset$

  π_1 - partition on $M \to R = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$
- $\pi_2 = \{\{1\}, \{1,2\}, \{3,4\}\}$
  - $\{1\} \cup \{1,2\} \cup \{3,4\} = M$
  - $\{1\} \cap \{1,2\} = \{1\} \neq \emptyset \to \pi_2$ - isn't partition

6. $\mathbb{C}$; $\quad z_1 r z_2 \leftrightarrow |z_1| = |z_2|$

$z_1 s z_2 \leftrightarrow \arg z_1 = \arg z_2$ or $z_1 = z_2$

Hw: prove that r & s are equivalance relations

- r
  - (R): $\forall z_1 \in \mathbb{C}$, $|z_1| = |z_1| \to z_1 r z_1$ „T" (1)
  - (T): if $|z_1| = |z_2|$ and $|z_2| = |z_3| \Rightarrow |z_1| = |z_3| \to$

    $\to z_1 r z_2$ and $z_2 r z_3 \to z_1 r z_3$ „T" (2), $z_1, z_2, z_3 \in \mathbb{C}$
  - (S): if $|z_1| = |z_2|$ then $|z_2| = |z_1| \to z_1 r z_2$ & $z_2 r z_1$ (3)

From (1), (2), (3) $\to$ r - equivalance

- s
  - (R): $\forall z_1 \in \mathbb{C}$, $\arg z_1 = \arg z_1$ or $z_1 = z_1 = 0 \to$

    $\to \forall z_1 \in \mathbb{C}$, $z_1 s z_1$ (1)
  - (T): $\left.\begin{array}{l}\arg z_1 = \arg z_2 \text{ and } \arg z_2 = \arg z_3 \to \arg z_1 = \arg z_2 \\ \text{or } z_1 = z_2 = 0 \text{ \& } z_2 = z_3 = 0 \to z_1 = z_3 = 0\end{array}\right\} \to$

    $\to z_1 s z_2$ & $z_2 s z_3 \to z_1 s z_3$, $z_1, z_2, z_3 \in \mathbb{C}$ (2)

• (S): $\arg z_1 = \arg z_2 \to \arg z_2 = \arg z_1$; $z_1 = z_2 = 0 \to z_2 = z_1 = 0$ $\Big\} \to z_1 s z_2$ and $z_2 s z_1$, $z_1, z_2 \in \mathbb{C}$ ③

From ①, ②, ③ → s - equivalance relation

- determine the quotient sets $\mathbb{C}/r$ and $\mathbb{C}/s$

∘ $\mathbb{C}/r$

$r(\mathbb{C}) = \{ b \in \mathbb{C} / \exists x \in \mathbb{C} ; |x| = |b| \}$

$r\langle x \rangle = \{ b \in \mathbb{C} / |x| = |b| \}$

$\mathbb{C}/r = \{ r\langle x \rangle / x \in \mathbb{C} \}$

$\mathbb{C}/r$ represents the set of circles with centers in the origin and of radious $= |x|$

∘ $\mathbb{C}/s$

$s(\mathbb{C}) = \{ b \in \mathbb{C} / \exists x \in \mathbb{C} : \arg x = \arg b$ or $x = b = 0 \}$ ($\arg x = \measuredangle$ care se formează cu Ox)

$s(x) = \{ b \in \mathbb{C} / \arg x = \arg b$ or $x = b = 0 \}$

$\mathbb{C}/s = \{ s(x) / x \in \mathbb{C} \}$

$\mathbb{C}/s$ represents the set of open line segments that starts from the origin and form the angle $\arg x$ with Ox.

Hw: 7. $\mathbb{Z}$, $x \rho_n y \Leftrightarrow n / (x-y)$.

* ρ_n - equivalance relation? + quotient set

* $\rho_0 + \rho_1$

* ρ_n - equivalance rel.

• (R): $\forall x \in \mathbb{Z}$, $x \rho_n x \Leftrightarrow n / (x-x) \Leftrightarrow n / 0$ „T" ①

• (T): if $x \rho_n y$ and $y \rho_n z \to x \rho_n z$?

$x \rho_n z \Leftrightarrow n / (x-z)$

$\left.\begin{array}{l} n / (x-y) \\ n / (y-z) \end{array}\right\} \to n / ((x-y)+(y-z)) \to n / (x-z)$ „T" ②

• (S): $x \rho_n y \to y \rho_n x$

$y \rho_n x \Leftrightarrow n / (y-x)$

$n / (x-y) \to n / -(y-x) \to n / (y-x)$ „T" ③

From ①, ②, ③ $\to$ R_n - equivalence rel.

$\rho_n(\mathbb{Z}) = \{y \in \mathbb{Z} \mid \exists x \in \mathbb{Z} : n \mid (x-y)\}$

$\rho_n(x) = \{y \in \mathbb{Z} \mid n \mid (x-y)\}$

$\mathbb{Z} \setminus \rho_n = \{\rho_n(x) \mid x \in \mathbb{Z}\}$

* For $n = 0$, $x \rho_0 y \leftrightarrow 0 \mid (x-y) \leftrightarrow x = y$

$\rho_0 = (\mathbb{Z}, \mathbb{Z}, \Delta_{\mathbb{Z}})$

For $n = 1$, $x \rho_1 y \leftrightarrow 1 \mid (x-y)$ „T", $\forall x, y \in \mathbb{Z}$

$\rho_1 = u = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z})$

8. $M = \{1, 2, 3\}$. All equivalence rel. on M + partitions

- $R_1 = \Delta_M$, $\pi_1 = \{\{1\}, \{2\}, \{3\}\}$
- $R_2 = \Delta_M \cup \{(1,2), (2,1)\}$, $\pi_2 = \{\{1,2\}, \{3\}\}$
- $R_3 = \Delta_M \cup \{(1,3), (3,1)\}$, $\pi_3 = \{\{1,3\}, \{2\}\}$
- $R_4 = \Delta_M \cup \{(2,3), (3,2)\}$, $\pi_4 = \{\{2,3\}, \{1\}\}$
- $R_5 = \Delta_M \cup \{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$;
  $\pi_5 = \{1, 2, 3\}$

9. $M = \{0, 1, 2, 3\}$, $h = (\mathbb{Z}, M, H)$ - rel., $H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}$. Is h - function?

h - function $\leftrightarrow$ $\forall x \in \mathbb{Z}$, $|r\langle x \rangle| = 1$ $\leftrightarrow$

$\leftrightarrow \forall x \in \mathbb{Z}$, $\exists! y \in M$ s.t. $(x, y) \in H$

$\leftrightarrow \forall x \in \mathbb{Z}$, $\exists! y \in M$ s.t. $x = 4z + y$, $z \in \mathbb{Z}$ „T" because z = quotient and y = reminder in the division of x with 4 and they are unique according to the division theorem (teorema împărțirii cu rest)

Hw: 10. r, s - homogeneous rel. on $\mathbb{N}$. r, s - equivalence rel?

$m r n \leftrightarrow \exists a \in \mathbb{N} : m = 2^a n$

$m s n \leftrightarrow (m = n$ or $m = n^2$ or $n = m^3)$

* r

(R): $\forall x \in \mathbb{N}$, $x r x \leftrightarrow \exists a \in \mathbb{N}: x = 2^a x$

$x = 1 \cdot x \to x = 2^0 \cdot x \to \exists a = 0 \in \mathbb{N}$ s.t. $x = 2^a x \to$

$\to x r x$ (1)

(T): $m r n$ and $n r x \to m r x \leftrightarrow \exists a \in \mathbb{N}: m = 2^a x$

$m r n \leftrightarrow \exists a' \in \mathbb{N}: m = 2^{a'} n$

$n r x \leftrightarrow \exists a'' \in \mathbb{N}: n = 2^{a''} x$

$\to m = 2^{a'} \cdot 2^{a'} \cdot x$

$= 2^{a'+a''} \cdot x$

$a', a'' \in \mathbb{N} \to a' + a' \in \mathbb{N} \to \exists a = a' + a'' \in \mathbb{N}$ s.t. $m = 2^a x \to$

$\to m r x$ (2)

(S): $m r n$ then $n r m \leftrightarrow \exists b \in \mathbb{N}: n = 2^b m$

$\hookrightarrow \exists a \in \mathbb{N}: m = 2^a n \to n = \frac{m}{2^a} = 2^{-a} \cdot m \to$

$\to \exists b = -a$ s.t. $n = 2^b \cdot m$

* s

(R): $\forall x \in \mathbb{N}$, $x = x \to x s x$

(T): $m s n$ & $n s x \to m s x$?

$m = n$ & $n = x \to m = x \to m s x$

$m = n$ & $n = x^2 \to m = x^2 \to m s x$

$m = n$ & $x = n^2 \to x = m^2 \to m s x$

$m = n^2$ & $n = x \to m = x^2 \to m s x$

$m = n^2$ & $n = x^2 \to m = x^4 \to m s x$

$m = n^2$ & $n^2 = x \to m = x \to m s x$

$m^2 = n$ & $n = x \to m^2 = x \to m s x$

$m^2 = n$ & $n = x^2 \to m^2 = x^2 \to m = x \to m s x$

$m^2 = n$ & $n^2 = x \to m^2 = \sqrt{x} \to m s x$

(S): $m s n \to n s m$

$m = n \to n = m \to n s m$

$m = n^2 \to n^2 = m \to n s m$ $\to$ S - equivalance ✓

$m^2 = n \to n = m^2 \to n s m$

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