

Linear Algebra Theorems and Definitions

Alexander J. Clarke

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All theorems, corollaries, lemmas, remarks, and asides are direct quotes from Linear Algebra,
4th Edition, by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence

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Chapter 0

List of Symbols

A_{ij}	the ij -th entry of the matrix A
A^{-1}	the inverse of the matrix A
A^\dagger	the pseudoinverse of the matrix A
A^*	the adjoint of the matrix A
\tilde{A}_{ij}	the matrix A with row i and column j deleted
A^t	the transpose of the matrix A
$(A B)$	the matrix A augmented by the matrix B
$B_1 \bigoplus \cdots \bigoplus B_k$	the direct sum of matrices B_1 through B_k
$\mathcal{B}(V)$	the set of bilinear forms on V
β^*	the dual basis of β
β_x	the T -cyclic basis generated by x
\mathbb{C}	the field of complex numbers
\mathbb{C}_i	the i th Gerschgorin disk
$\text{cond}(A)$	the condition number of the matrix A
$C^n(\mathbb{R})$	set of functions f on \mathbb{R} with $f^{(n)}$ continuous
C^∞	set of functions with derivatives of every order
$C(\mathbb{R})$	the vector space of continuous functions on \mathbb{R}
$C([0, 1])$	the vector space of continuous functions on $[0, 1]$
C_x	the T -cyclic subspaces generated by x
D	the derivative operator on C^∞
$\det(A)$	the determinant of the matrix A
δ_{ij}	the Kronecker delta
$\dim(V)$	the dimension of V
e^A	$\lim_{m \rightarrow \infty} \left(I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!} \right)$
e_i	the i th standard vector of \mathbb{F}^n

E_λ	the eigenspace of T corresponding to λ
\mathbb{F}	a field
$f(A)$	the polynomial $f(x)$ evaluated at the matrix A
F^n	the set of n -tuples with entries in a field \mathbb{F}
$f(T)$	the polynomial $f(x)$ evaluated at the operator T
$\mathcal{F}(S, \mathbb{F})$	the set of functions from S to a field \mathbb{F}
H	space of continuous complex functions on $[0, 2\pi]$
I_n or I	the $n \times n$ identity matrix
\mathbb{I}_V or \mathbb{I}	the identity operator on V
K_λ	generalized eigenspace of T corresponding to λ
K_ϕ	$\{x \mid (\phi(T))^p(x) = 0, \text{ for some positive integer } p\}$
L_A	left-multiplication transformation by matrix A
$\lim_{m \rightarrow \infty} A_m$	the limit of a sequence of matrices
$\mathcal{L}(V)$	the space of linear transformations from V to V
$\mathcal{L}(V, W)$	the space of linear transformations from V to W
$M_{m \times n}(\mathbb{F})$	the set of $m \times n$ matrices with entries in \mathbb{F}
$v(A)$	the column sum of the matrix A
$v_j(A)$	the j th column sum of the matrix A
$N(T)$	the null space of T
nullity (T)	the dimension of the null space of T
O	the zero matrix
per (M)	the permanent of the 2×2 matrix M
$P(\mathbb{F})$	the space of polynomials with coefficients in \mathbb{F}
$P_n(\mathbb{F})$	the polynomials in $P(\mathbb{F})$ of degree at most n
ϕ_β	the standard representation with respect to basis β
\mathbb{R}	the field of real numbers
rank (A)	the rank of the matrix A
rank (T)	the rank of the linear transformation T
$\rho(A)$	the row sum of the matrix A
$\rho_i(A)$	the i th row sum of the matrix A
$R(T)$	the range of the linear transformation T
$S_1 + S_2$	the sum of sets S_1 and S_2
span (S)	the span of the set S
S^\perp	the orthogonal complement of the set S
$[T]_\beta$	the matrix representation of T in basis β
$[T]_\beta^\gamma$	the matrix representation of T in bases β and γ
T^{-1}	the inverse of the linear transformation T

T^\dagger	the pseudoinverse of the linear transformation T
T^*	the adjoint of the linear operator T
T_0	the zero transformation
T^t	the transpose of the linear transformation T
T_θ	the rotation transformation by θ
T_W	the restriction of T to a subspace W
$\text{tr}(A)$	the trace of the matrix A
V^*	the dual space of the vector space V
V/W	the quotient space of V modulo W
$W_1 + \cdots + W_k$	the sum of subspaces W_1 through W_k
$\sum_{i=1}^k W_i$	the sum of subspaces W_i through W_k
$W_1 \oplus W_2$	the direct sum of subspaces W_1 and W_2
$W_1 \oplus \cdots \oplus W_k$	the direct sum of subspaces W_1 through W_k
$\ x\ $	the norm of the vector \vec{x}
$[x]_\beta$	the coordinate vector of x relative to β
$\langle x, y \rangle$	the inner product of \vec{x} and \vec{y}
\mathbb{Z}_2	the field consisting of 0 and 1
\bar{z}	the complex conjugate of \vec{z}
$\vec{0}$	the zero vector

Chapter 1

Vector Spaces

1.1 Introduction

1.2 Vector Spaces

1.3 Subspaces

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1.5 Linear Dependence and Linear Independence

1.6 Bases and Dimension

1.7 Maximal Linearly Independent Subsets

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- 3.2 The Rank of a Matrix and Matrix Inverses
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Determinants

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4.3 Properties of Determinants

4.4 Summary – Important Facts about Determinants

4.5 A Characterization of the Determinant

Chapter 5

Diagonalization

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5.3 Matrix Limits and Markov Chains

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Chapter 7

Canonical Forms

7.1 The Jordan Canonical Form I

7.2 The Jordan Canonical Form II

7.3 The Minimal Polynomial

7.4 The rational Canonical Form