10/31/2025

<u>0.2.3:</u> Project 0 Theory: Quick Angular & Linear Dynamics Review

Ignore Nomenclature!

	Linear	Angular
Position	x	θ
Vel	$\dot{x} = dx/dt$	$\dot{\theta} = d\theta/dt$
Accel	$\ddot{x} = d\dot{x}/dt$	$\ddot{\theta} = d\dot{\theta}/dt$
	m = mass	I = M.O.I.
Force/Torque	$\vec{F} = m\vec{a}$	$\tau = I\ddot{\theta} \longrightarrow \vec{F} = \frac{d\vec{p}}{dt}$
Momentum	$p = mv$	$L = I\dot{\theta}$
Work	$W = Fd$	$W = \tau\theta$
Kin. Energy	$KE = \frac{1}{2}mv^2$	$KE_{rot} = \frac{1}{2}I\dot{\theta}^2$
Power	$P = Fv$	$P = \tau\dot{\theta} \longrightarrow P = \frac{dW}{dt}$

Base Dynamics Nomeclature: From txtbk

V = P.E.

T = K.E.

U = Work

P = Power

ε = Efficiency

M = Moment

$\Sigma M_G = \Sigma(\mathcal{M}_k)_P$

ρ = rad. of curvature

e = Coef. of Restit.

MMOI: $I = \int r^2 dm$

Par.-Ax. Thrm: $I = I_G + md^2$

Rad. O. Gyr.: $k = \sqrt{I/m}$

H = Ang. Momentum

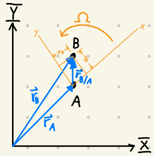
$\Sigma M_o = \dot{H}_o$

M_o = Moment about '0'

Ω = Ang. vel. for axis of interest

L = Lin. Momentum Also referred to as ang.

$\Sigma F = \dot{L}$



Ω = Ang. vel. of local xyz measure from abs. ($\bar{X}\bar{Y}\bar{Z}$)

<u>For Rot. about Z-axis:</u>

If $\Omega = \Omega\hat{k}$, then $\frac{d\hat{i}}{dt} = \Omega \times \hat{i}$ & $\frac{d\hat{j}}{dt} = \Omega \times \hat{j}$

Moment of Inertia (M.O.I.)

$I = \int_m r^2 dm$ — M.O.I. as $\int$(2nd Moment)

$dm = \rho d\forall$ — r = "moment arm"

$I = \int_\forall r^2 \rho d\forall$ — M.M.O.I. (Mass MOI) [$kg \cdot m^2$] or [$slug \cdot ft^2$]

ρ = Density

$I = \rho \int_\forall r^2 d\forall$ w/ const. ρ

$F = ma$

$kg = N/(m/s^2)$

$kg \cdot m^2 = \frac{N \cdot m}{s^2}$

Area M.O.I. (A.M.O.I.)

$I_x = \int_A y^2 dA$ [in^4] or [m^4]

$I_y = \int_A x^2 dA$ [in^4] or [m^4]

$J_o = \int_A r^2 dA = I_x + I_y$ Polar M.O.I.

Statics txtbk

<u>Dyn. txtbk</u>

0.2.3: Project 0 Theory: The Moments, Inertia, & M.O.I.'s

1. **Inertia** is simply a measurement for an object's resistance to change its motion.
2. A **moment** is simply a parameter measuring the output of a quantity as it it is multiplied with its distance away from an axis.
3. A **2nd moment** is just that quantity multiplied by (distance)^2

The above semantics must be understood for being able to differentiate between the following in writing and/or conversation:

1. **Mass Moment of Inertia (M.M.O.I.)** $I = \int r^2 dm$ Rot. Resistance due to m
2. **Area Moment of Inertia (A.M.O.I.)** (Second Moment of Area) $I_{A_x} = \int y^2 dA$ Resistance to bending or deflection
3. **Moments** about axes $M = Fd$ Axis defines pivot point

Statics txtbk
Dyn. txtbk

10/31/2025

0.2.3: Project 0 Theory: First Moment of Qty (M.O.x.)

A first moment can describe the area being multiplied by the distance to its pivot. The first moment of an area can be defined as the measurement of distribution of a shape's area about an axis. First moment of areas are important for finding the centroid's coordinates. Below are notable qualities requiring the calculation of the moment of said parameter.

Center of Gravity

$W = \int dw$

$w = \text{weight}$

$\bar{x} = \frac{\int \tilde{x}\,dw}{\int dw}$

$\bar{y} = \frac{\int \tilde{y}\,dw}{\int dw}$

1st Moments: $Q_x = \int \tilde{y}\,dw$

$Q_y = \int \tilde{x}\,dw$

$(\bar{x}, \bar{y}, \bar{z})$ = Coords of center of Gravity 'G'

$(\tilde{x}, \tilde{y}, \tilde{z})$ = Coords of arb. pnt

Many institutional tables for calculating the location of centroids, and the inertias of various shapes exist, many of which are in our referenced textbooks.

Center of Mass

$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm}$

$\bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}$

$Q_x = \int \tilde{y}\,dm$

$Q_y = \int \tilde{x}\,dm$

Centroid (Center of Volume)

$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV}$

$\bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV}$

$Q_x = \int \tilde{y}\,dm$

$Q_y = \int \tilde{x}\,dm$

or center of Line:

$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL}$

$\bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL}$

Centroid of curved line:

$dL = \left(\sqrt{1 + (dy/dx)^2}\right)dx$

$dL = \left(\sqrt{1 + (dx/dy)^2}\right)dy$

Also centroids of area...

Statics txtbk

Dyn. txtbk

10/31/2025

0.2.3: Project 0 Theory: Area Moment of Inertia (A.M.O.I.)

Sometimes described as the leveraged area squared, the A.M.O.I. measures resistance to bending or deflection. Another name for this is the second moment of area, of which the first moment of area can be described as the leveraged area multiplied by the distance to its pivot.

M.O.I.'s: $I_x = \int_A y^2 dA$
$I_y = \int_A x^2 dA$
$J_o = \int_A r^2 dA = \underbrace{I_x + I_y}_{\text{Perpendicular Axis Theorem}}$
(cannot be negative)

↳since $r^2 = x^2 + y^2$

Parallel Axis Theorem:

$I_x = \bar{I}_{x'} + Ad_y^2$
$I_y = \bar{I}_{y'} + Ad_x^2$
$J_o = \bar{J}_c + Ad^2$

Radius of Gyration:
(...of an area about an axis)

$k_x = \sqrt{I_x/A}$
$k_y = \sqrt{I_y/A}$
$k_o = \sqrt{J_o/A}$

similar to finding M.O.I. for a differential area about an axis. i.e. $I_x = k_x^2 A$ & $dI_x = y^2 dA$

Diving Further:
- Product of Inertia for an area
- M.O.I.'s for an area about inclined axes
- Mohr's Circle for M.O.I.'s
- Principle M.O.I.'s

Statics txtbk
Dyn. txtbk

10/31/2025

0.2.3: Project 0 Theory: Mass Moment of Inertia (M.M.O.I.)

The 2nd moment of Mass

$$I = \int_m r^2 dm$$

For axial sym.: $I = \rho \int_{\forall} r^2 d\forall$

For Comp. Body: $I = I_G + md^2$

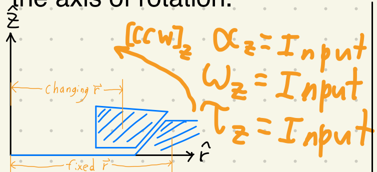
Statics txtbk
Dyn. txtbk

10/31/2025

0.2.3: Project 0 Theory: Passively Triggered Mechanisms

Best described as mechanisms that activate under specific dynamic conditions, passively triggered mechanisms can be designed to trigger transformations that can increase certain efficiencies by shifting designed control parameters. A preliminary example for analysis that can easily be prototyped at home would be a spinning top designed to change shape as a function of torque and/or angular momentum. Diving only a little deeper, it is valid to comment that reducing the M.O.I. of the spinning top while it is in motion will increase the time it will spin for. Two other factors that would increase the efficiency of this system include reducing the friction of its contact with the ground and reducing air drag.

By setting our scope to adjusting the M.O.I. we deduce that two methods for accomplishing this include reducing either the volume or mass distribution within the top. Both can be easily accomplished by realizing that centripetal force causes objects with mass to slide away from the axis of rotation.

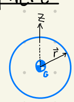


M.M.O.I. will be used here
- Raising $\oplus_G$ reduces stability
- How many nonneglegible increments along $\hat{z}$ are there and what are their states & triggers?

Selecting a top-down analysis of the geometry design profile simplifies introductory analysis.

Statics txtbk
Dyn. txtbk

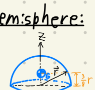
Sphere:



$\forall = \frac{4}{3}\pi r^3$

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m r^2$

Hemisphere:

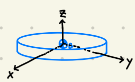


$\forall = \frac{2}{3}\pi r^3$

$I_{xx} = I_{yy} = 0.259 m r^2$

$I_{zz} = \frac{2}{5} m r^2$

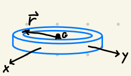
Thin Circ. Disk:



$I_{xx} = I_{yy} = 0.259 m r^2$

$I_{zz} = \frac{2}{5} m r^2$

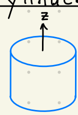
Thin Ring:



$I_{xx} = I_{yy} = \frac{1}{2} m r^2$

$I_{zz} = m r^2$

Cylinder:



$\forall = \pi r^2 h$

$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2)$

$I_{zz} = \frac{1}{2} m r^2$

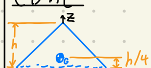
(Eq. 17-15&16):

$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$

$\Sigma F_t = m(a_G)_t = m\alpha r_G$

$\Sigma M_O = I_O \alpha = r_G m(a_G)_t + I_G \alpha$

Cone:



$\forall = \frac{1}{3}\pi r^2 h$

$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2)$

$I_{zz} = \frac{3}{10} m r^2$

These shapes can be used as the primitive building blocks for optimizing the inertial efficiency of spinning tops.

10/31/2025

VOID THIS PAGE!!!

0.2.2: Project 0 Theory: Spinning Top Rigid Body Dynamics

Input Parameters:

$\omega_{max} = \dot{\theta}_{max}$

$\tau = M_{z,applied}$

$= I\alpha_{max}$

$\alpha = \ddot{\theta} = \frac{M_z}{I}$

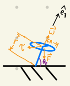
f_k, f_s

Neglect air drag

$\vec{\tau} = \vec{r} \times \vec{F}$

$= I\vec{\alpha}$

$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$



$\vec{\tau} = \vec{r} \times \vec{F} = rmg\cos\theta_e \hat{y}$

$\vec{r} = R\cos\theta_e \hat{x} + r\sin\theta_e \hat{z}$

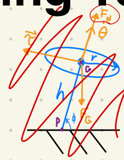
$\vec{F} = -mg\hat{z}$

$\frac{d\vec{L}}{dt} = \cancel{\frac{d\vec{L}}{dt}}^{0} + \vec{\Omega} \times \vec{L}$

$= \Omega \hat{z} \times (\frac{1}{2} mR^2 \omega (\cos\theta_e \hat{x} + \sin\theta_e \hat{z}))$

$\Omega = \frac{2gr}{R^2\omega}$

→ See Euler Angle Theorem



P.E. + K.E. = Const.

$P.E._G = mg\Delta z$

$K.E. = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$dU = \vec{F} \cdot d\vec{r}$

$= \vec{M} \cdot d\vec{\theta}$

$K.E._2 = K.E._1 + \sum_{i=1}^{N} U_{1\to 2}$

$\sum \vec{M}_G = I_G \alpha$

$= \sum \vec{M}_o = \sum (\mathcal{M}_k)_o$

Momentum

$\vec{L}_G = m\vec{v}_G$

$\vec{H}_G = I_G \vec{\omega}$

$\vec{H} = \vec{r} \times \vec{L}$ (Angular Impulse)

$\vec{H}_{G1} + \sum \int \vec{M}_G dt = \vec{H}_{G2}$

$\sum \vec{H}_1 = \sum \vec{H}_2$

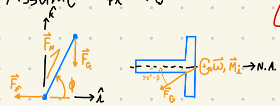
$P = \frac{dU}{dt} = \vec{F} \cdot \vec{v}$

$\varepsilon = \frac{P_{out}}{P_{in}}$

$\vec{H}_{G1} + \sum \int \vec{M}_G dt = \vec{H}_{G2}$

$I_G \cancel{\vec{\omega}_{Gi}}^{0} + \underbrace{\int \vec{M}_i dt_i}_{\text{Momentum from motor}} = I_G \vec{\omega}_G$

Assume $\phi_i \approx 90°$



$\boxed{\text{Does tilt resist spin more than friction?}}$

~~$F_G = -F_N \sin\phi$~~ $\phi \neq 90°$

~~$F_f = -F_N \cos\phi$~~

~~$\hookrightarrow F_G = F_f \frac{\sin\phi}{\cos\phi}$~~

~~$mg = F_f \tan\phi$~~

~~$m = \frac{F_f}{g} \tan\phi$~~

$F_f = fF_N$

~~$fF_N = -F_N \cos\phi$~~

~~$f = -\cos\phi$~~

$m = \frac{fF_N}{g} \tan\theta$

$= \frac{-F_N \cos\phi}{g} \cdot \frac{\sin\phi}{\cos\phi}$

$= -\frac{F_N}{g} \sin\phi$

$F_G = -F_N \sin\phi$ ✓

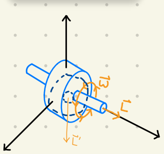
11/01/2025

WARNING: F_N was used in place of internal normal reaction force here and not all the nomenclature on this page is matching. This is redone on the next page for clarity. F_n is usually for normal force

Some textbooks use H for ang. momentum, others use L. Some textbooks use p for lin. momentum and L for ang. momentum.

It is not uncommon for textbooks to use unique symbol traditions, especially in engineering. This is why reviewing one's own notes and keeping engineering journals is so important.

11/01/2025

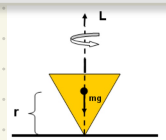


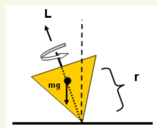
Translational	Rotational
p = Momentum	L = Ang. Momentum
	Ω = consequential ang. vel about abs. $\hat{z}$ and describes the motion of **Precession**
	$\vec{L} = I\vec{\omega}$
	$\vec{\tau} = \vec{r} \times \vec{F}$

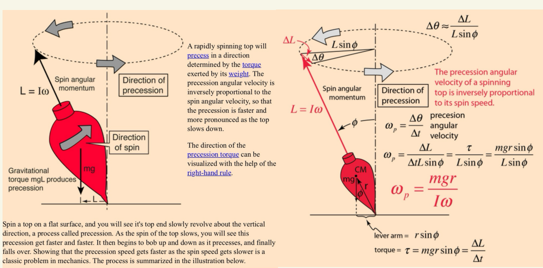
(L) Ang. Momentum & Ext. $\vec{\tau}$:

Assumptions:

- Apply classical mechanics scope (Newtonian/Lagrangian)
- Rigid body spinning about a single axis
- Neglect friction
- The body is symmetric about its rotational axis
- Even the most minuscule angle between the rotational axis and the ground will cause precession which is impossible to avoid







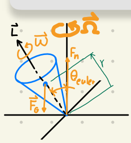
http://hyperphysics.phy-astr.gsu.edu/hbase/top.html

↳ Doesn't rely on Euler Angle Theorem

Statics txtbk

Dyn. txtbk

https://wikis.mit.edu/confluence/display/RELATE/Spinning+Top



$\vec{L} = \vec{r} \times \vec{p}$

$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$ rot. analog of $\sum \vec{F} = d\vec{p}/dt$

$\vec{p} = m\vec{v}$

$\vec{L} = \vec{r} \times m\vec{v}$

Gyroscopic Approximation rules that $L = I\omega$ when $\Omega \ll \omega$

→ Change in L only affects horiz. portion and is defined as:

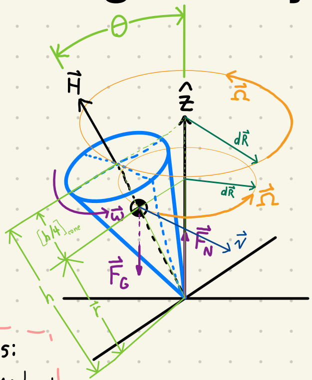
$\left|\frac{d\vec{L}}{dt}\right| = L\Omega \sin\theta = rmg\sin\theta$

$\Omega = \frac{rmg}{L} = \frac{rmg}{I\omega}$

11/01/2025

0.2.2: Project 0 Theory: Gyroscopic Rigid Body Dynamics

Assumptions:
- Apply classical mechanics scope (Newtonian/Lagrangian)
- Rigid body spinning about a single axis
- Neglect friction
- The body is symmetric about its rotational axis
- Even the most minuscule angle between the rotational axis and the ground will cause precession which is impossible to avoid



Note:
There is no energy loss for these conditions, therefore energy and momentum are lost from friction and drag. It is assumed that drag will be negligible for most spinning top designs.

My Nomenclature
- P = Power
- p = pressure
- σ = Stress
- L = Lin. Momentum
- H = Ang. Momentum
- F_G = Grav. Force
- F_N = Normal Force
- F_n = Force normal to N.A.
- τ = Torque (Applied M)
- M = Moment of Force
- m = Mass

Consider making a formula sheet to avoid needing to copy equations like these from page to page.

Vectorial Sys:

Impulse & Momentum:

$\vec{L}_G = m\vec{v}_G$

$\vec{H}_G = I_G\vec{\omega}$

$\vec{L}_{G1} + \sum\int \vec{F}dt = \vec{L}_{G2}$

$\vec{H}_{G1} + \sum\int \vec{M}_G dt = \vec{H}_{G2}$

$\sum \vec{L}_1 = \sum \vec{L}_2$

$\sum \vec{H}_1 = \sum \vec{H}_2$

Scalar Sys:

Work & Energy:

$K.E._2 = K.E._1 + \sum U_i$

$dU = \vec{F}\cdot d\vec{r}$

$= \vec{M}\cdot d\theta$

$K.E. = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

$P.E. = E_G$

$= mg\Delta z$

$P = \frac{dU}{dt} = \vec{F}\cdot\vec{v}$

To make the system of moment equations solvable, the below assumptions must be made in order for this to be classified as being in a state of **steady precession**.
- Constant nutation angle
- Constant precession
- Constant Spin

The system of Moment Eqs in the txtbk, when the above assumptions are applied, simplifies to:

$\sum M_x = \dot{\phi}\sin\theta(I_z\omega_z - I\dot{\phi}\cos\theta)$ 21-31

If Const $\theta, \dot{\phi}, \dot{\psi}$

21-29
$$\begin{cases} \sum M_x = I(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}) \\ \sum M_y = I(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \\ \sum M_z = I_z(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) \end{cases}$$

Given/Inputs: @ $t=0$

Const: m, r, h, g

Dyn. : $\vec{\omega}, \vec{\tau}, \phi, I$

Ini. : $\vec{\omega}_o = \vec{\omega}(t=0)$

$\vec{\tau}_o = \vec{\tau}(t=0)$

$\theta_o = \theta(t=0)$

$I_o = I(t=0)$

Kinematics:

Linear	Angular
s	θ
$v = ds/dt$	$\omega = d\theta/dt$
$a = dv/dt$	$\alpha = d\omega/dt$
$a\,ds = v\,dv$	$\alpha\,d\theta = \omega\,d\omega$

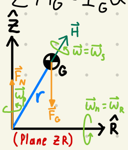
$\vec{v} = \vec{\omega}\times\vec{r}$

$\vec{a} = (\vec{\alpha}\times\vec{r}) - \omega^2\vec{r}$

Statics txtbk

Dyn. txtbk

$\sum \vec{M}_G = I_G\vec{\alpha}$ & $\sum \vec{F} = m\vec{a}$



(Plane ZR)

& see Fig. 21-16 in Dyn. txtbk

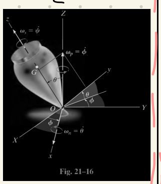


Fig. 21–16

$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$

$= \omega_n\hat{i} + (\omega_p\sin\theta)\hat{j} + (\omega_p\cos\theta + \omega_s)\hat{k}$

$\vec{\Omega} = \Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k}$

$= \omega_n\hat{i} + (\omega_p\sin\theta)\hat{j} + (\omega_p\cos\theta)\hat{k}$

θ = Nutation Angle

$\dot{\phi}$ = Precession

$\dot{\psi}$ = Spin

Conditions for Steady Precession:

$\frac{d\theta}{dt} = 0$

$\frac{d\phi}{dt} = \Omega = \text{constant}$

$\omega_s \ll \Omega$ & $\omega_s \ll \dot{\phi}$

$\Omega \approx \frac{mgr}{I_s\omega_s}$

τ balances $\dot{L}$

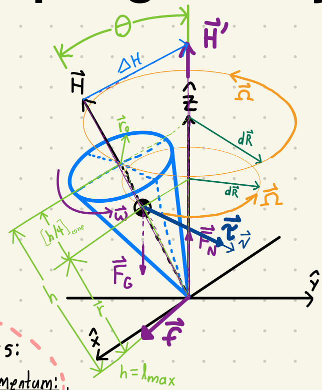
$\tau = \vec{r}\times\vec{F}_G = \dot{\vec{L}}$

These conditions also constrain $\vec{\tau}$ to be perpendicular to the vertical plane (Plane ZR).

11/01/2025

0.2.2: Project 0 Theory: Spinning Top Rigid Body Dynamics

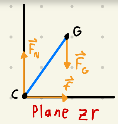
Assumptions:
- Apply classical mechanics scope (Newtonian/Lagrangian)
- Rigid body spinning about a single axis
- The body is symmetric about its rotational axis
- Even the most minuscule angle between the rotational axis and the ground will cause precession which is impossible to avoid

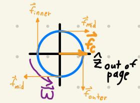


New Assumption:
Since friction and drag are to blame for energy loss, we can assume, for now, that the effects of unsteady precession are negligible. eventually though, the top will fall over.

$f_k = \mu_k F_N$

$f_s = \mu_s F_s$





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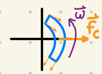


$\sum M_c = 0,\ F_G = mg$

$= M_c - rmg$ $\quad \frac{m \mid kg \mid m}{s^2}$

$(I\alpha)_c = rmg$ $\quad \frac{m^2 \mid kg}{s^2}$

Given/Inputs: @ t=0

Const: m, r, h, g

Dyn. : $\vec{\omega}, \vec{\tau}, \phi, I$

Ini. : $\vec{\omega}_0 = \vec{\omega}(t=0)$

$\vec{\tau}_0 = \vec{\tau}(t=0)$

$\cancel{\theta_0 = \theta(t=0)}$ [ϕ, using Euler Angles]

$I_0 = I(t=0)$

Vectorial Sys:

Impulse & Momentum:

$\vec{L}_G = m\vec{v}_G$

$\vec{H}_G = I_G \vec{\omega}$

$\vec{L}_{G1} + \sum \int \vec{F}\,dt = \vec{L}_{G2}$

$\vec{H}_{G1} + \sum \int \vec{M}_G\,dt = \vec{H}_{G2}$

$\sum \vec{L}_1 = \sum \vec{L}_2$

$\sum \vec{H}_1 = \sum \vec{H}_2$

$\forall = \frac{1}{3}\pi r^2 h$

$I_{xx} = \frac{3}{80} m(4r_0^2 + h^2)$

$I_{yy} = I_{xx}$

$I_{zz} = \frac{3}{10} m r_0^2$

Kinematics:

Linear	Angular
s	θ
$v = ds/dt$	$\omega = d\theta/dt$
$a = dv/dt$	$\alpha = d\omega/dt$
$a\,ds = v\,dv$	$\alpha\,d\theta = \omega\,d\omega$

$\vec{v} = \vec{\omega} \times \vec{r}$

$\vec{a} = (\vec{\alpha} \times \vec{r}) - \omega^2 \vec{r}$

$P = \frac{dU}{dt} = \vec{F} \cdot \vec{v}$

Scalar Sys:

Work & Energy:

$K.E._2 = K.E._1 + \sum U_i$

$dU = \vec{F} \cdot d\vec{r}$

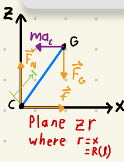
$= \vec{M} \cdot d\theta$

$K.E. = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$P.E. = E_G$

$= mg\Delta z$

11/03/2025



$(I\alpha)_c = \alpha_c I_{xx}$

$\alpha_c = \ddot{\theta} = \frac{rmg}{I_{xx}}$

Suggests nothing opposes gravity from pulling the spinning top over.
→ This must mean the angular momentum of the top is somehow introducing the centripetal accel. ($\vec{a}_c$)

$\sum M_c = 0 \quad (ccw)$

$= M_c - F_G r + m a_c r$

$(I\alpha)_c = m a_c r - mgr$

$\ddot{\theta} = (a_c - g)\frac{mr}{I_{xx}} \quad (ccw)$

$\ddot{\theta} = (a_c - g)\frac{80}{3}\frac{r}{4r_0^2 + h^2} \quad (ccw)$

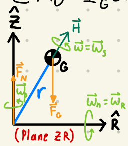
$\ddot{\theta} = (a_c - g)\frac{80r}{12r_0^2 + 3h^2} \quad (ccw)$

$\ddot{\theta} > 0$: Stabilizing, recentering

$\ddot{\theta} < 0$: Destabilizing, falling over

Consider making a formula sheet to avoid needing to copy equations like these from page to page.

$\sum \vec{M}_G = I_G \vec{\alpha}$ & $\sum \vec{F} = m\vec{a}$



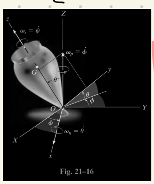


Fig. 21-16

& see Fig. 21-16 in Dyn. txtbk

$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$= \omega_n \hat{i} + (\omega_p \sin\theta)\hat{j} + (\omega_p \cos\theta + \omega_s)\hat{k}$

$\vec{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$

$= \omega_n \hat{i} + (\omega_p \sin\theta)\hat{j} + (\omega_p \cos\theta)\hat{k}$

θ = Nutation Angle

$\dot{\phi}$ = Precession

$\dot{\psi}$ = Spin

Constant for steady precession

Statics txtbk

Dyn. txtbk

Hypothesis:
At the moment of launch, angular momentum will point along the axis of spin and as friction and drag siphon away momentum, the overall rotation of spin will shift more and more to the vertical axis as the top falls over.
- Reducing the I_zz after launch will increase spin.
- Reducing I_xx or I_yy after launch increases how quickly the top loses or gains stability by increasing the rate at which the radius of precession (proportional to the nutation angle) can change.
- The top will hit the ground before its angular momentum vector goes entirely vertical.

$\vec{\tau} = \frac{d\vec{H}}{dt}$

$\vec{H} = I\vec{\omega}$

$\vec{H} + \Delta\vec{H} = \vec{H}'$

$\vec{H} + \underbrace{\sum \int \vec{M}\,dt}_{\text{Friction \& Drag}} = \vec{H}'$

Let $x = y$
→ Assumes circular precession

$\sum F_x = m\ddot{x}$

$= f - m a_c$

$\ddot{x} = \frac{f}{m} - a_c$

$\sum F_z = m\ddot{z}$

$= F_N - F_G$

$\ddot{z} = \frac{F_N}{m} - g$

$\vec{H} + \Delta\vec{H} = \vec{H}'$

$I\vec{\omega} + \Delta(I\vec{\omega}) = I'\omega'$

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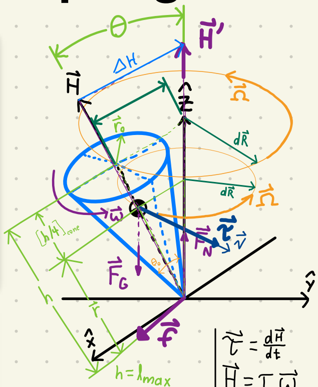

$f_k = \mu_k F_N$
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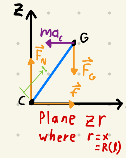

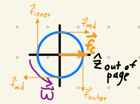

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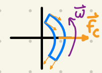


$V = \frac{1}{3}\pi r^2 h$
$I_{xx} = \frac{3}{80}m(4r_0^2 + h^2)$
$I_{yy} = I_{xx}$
$I_{zz} = \frac{3}{10}m r_0^2$

θ = Nutation Angle
$\dot{\phi}$ = Precession
$\dot{\psi}$ = Spin

$\sum M_c = 0 \quad (ccw)$
$= M_c - F_G r + m a_c r$
$(I\alpha)_c = m a_c r - m g r$
$\ddot{\theta} = (a_c - g)\frac{mr}{I_{xx}} \quad (ccw)$

$\vec{\tau} = \frac{d\vec{H}}{dt}$
$\vec{H} = I\vec{\omega}$
$\vec{H} + \Delta\vec{H} = \vec{H}'$
$\vec{H} + \underbrace{\sum\int \vec{M}\,dt}_{\text{Friction \& Drag}} = \vec{H}'$

Let $x = y$
→ Assumes circular precession
$\sum F_x = m\ddot{x}$
$= f - m a_c$
$\ddot{x} = \frac{f}{m} - a_c$

$\sum F_z = m\ddot{z}$
$= F_N - F_G$
$\ddot{z} = \frac{F_N}{m} - g$

$\vec{H} + \Delta\vec{H} = \vec{H}'$
$I\vec{\omega} + \Delta(I\vec{\omega}) = I'\vec{\omega}'$

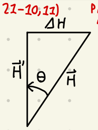
Assume $\Delta\vec{H}$ to be from momentum loss
Assume $\Delta\hat{H}$ to correspond inversely to $\Delta\theta$ (Nutation Angle)
Assume $\phi \ll \theta$

For Const I:
$\Delta(I\vec{\omega}) = I\Delta\vec{\omega}$
$= I(\vec{\omega}' - \vec{\omega})$

A bike wheel suspended one end by a string is a classic example of stable precession.

If the tip of the spinning top is suspended, according to (Eq. 21-32,33)
$\sum M_{c,x} = I_z \Omega_z \omega_z$, where $\omega_z = \dot{\psi}$
$\sum \vec{M}_{c,x} = \vec{\Omega}_z \times \vec{H}$ $\quad \theta = 90°$

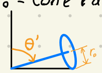
(Eq. 21-10,11) Principle Inertial Axes are used



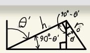
$H_x = I_x \omega_x$
$H_y = I_y \omega_y$
$H_z = I_z \omega_z$

After the top falls over, it will roll along its side.

r_0 = Cone radius of top



Statics txtbk
Dyn. txtbk



$h\sin\theta' = r_0\cos\theta'$
$\tan\theta' = \frac{\sin\theta'}{\cos\theta'}$
$r_0 = h\tan\theta'$

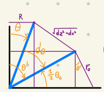
Assume $\hat{H}$ is independent of $\dot{\phi}$

t_1: After launch
$\vec{H} = H_z$
t_2: Imediately before side touches gnd
t_3: Rolling on side

How much H_z becomes H_R after t_2 (toppling on side)?
$H_R = \sqrt{H_x^2 + H_y^2}$

$\ddot{\theta} = (a_c - g)\frac{mr}{I_{xx}}$, $a_c = \frac{f}{m} - \ddot{R}$
$= \frac{d\dot{\theta}}{dt}$
$\int \ddot{\theta}\,dt = \int (a_c - g)\frac{mr}{I_{xx}}\,dt$
$= -gt\frac{mr}{I_{xx}} + \frac{mr}{I_{xx}}\int a_c\,dt$

$\int a_c\,dt = \int(\frac{f}{m} - \ddot{R})\,dt$



$\theta' = \theta + d\theta$

$90° = \frac{1}{2}\theta_0 + \theta + d\theta$
$d\theta = 90° - \frac{1}{2}\theta_0 - \theta$
$\theta' = 90° - \frac{1}{2}\theta_0$
$r_0\sin(\frac{1}{2}\theta_0) = h$

11/04/2025

Non-principal M.O.I.'s?

11/04/2025

0.2.2: Project 0 Theory: Deriving Moments of Various Quantities

$M_n = \sum (r_i)^n Q_i$ Discrete Sys.

$M_n = \int r^n dQ$ Continuous Sys.

Q_i = Val. of Qty or Density @ i
dQ = Qty Differential @ dist. r
(Infinitesimally small prt of Q)
r_i = Distance from ref. axis or pnt to Q_i

3rd, 4th, and nth moments exist too, but only in very specific and specialized contexts.

0th Moments (n = 0) "The Magnitude"

- Equivalent to multiplying the quantity with r^0 which is just 1; useless outside of a foundation for learning moments.

$M_0 = \int r^0 dQ = \int 1 dQ = Q$ 0^{th} Moment of Q

1st Moments (n = 1) "The Mean"

- **Location & Balance**
- Measures the location or asymmetry of the distribution. *(Centers of Distribution)*

i.e. $\vec{M} = \vec{\tau} = \vec{r} \times \vec{F}$ Moment of Force (or Torque)

2nd Moments (n = 2) "The Variance"

- **Inertia & Resistance**
- Measures the spread of the distribution and its resistance to change.

i.e. $I_m = \int r^2 dm$ M.M.O.I. (M.O.I.)
$I_A = \int r^2 dA$ A.M.O.I.

Statics txtbk
Dyn. txtbk

$Q = \vec{F}$: $\vec{M} = \vec{\tau} = \vec{r} \times \vec{F}$ "Torque" or "Moment"
→ "Ang. Force"
→ Summation of Moments
$Q = \int r^2 \rho \, d\forall$ "Quadrupole Moment"
→ Deviation of charge or mass distri. from spher. sym.
→ Rarely, if ever, referred to as the "2nd moment of force"

$Q = m$: $M_1 = \int r(\rho \, d\forall)$, ρ = Density, $\forall$ = Vol.
$\bar{X}_m = \frac{\int r_x dm}{\int dm}$, $\bar{Y}_m = \frac{\int r_y dm}{\int dm}$, $\bar{Z}_m = \frac{\int r_z dm}{\int dm}$ "Center of Mass"
$\bar{X}_G = \frac{\int r_x dW}{\int dW}, \ldots$ "Center of Gravity"

M.M.O.I. $I = \int r^2 dm = \int r^2 \rho \, d\forall$ Dynamic "M.O.I."
→ Resists rotation

$Q = A$: $M_1 = \int r \, dA$
$\bar{X} = \frac{\int r_x dA}{\int dA}, \ldots$ "Centroid" of Area
$\bar{X} = \frac{\int_L r_x dL}{\int_L dL}, \ldots$ "Centroid" of Line

A.M.O.I. $I = \int r^2 dA$ Static "M.O.I."
→ Resists bending and/or deflection

$Q = \forall$: $M_1 = \int r \, d\forall$
$\bar{X} = \frac{\int r_x d\forall}{\int d\forall}, \ldots$ "Centroid" of Volume

$I_\forall = \frac{I_m}{\rho}$
$I = I_\forall = \int r^2 d\forall$

$Q = P$: $E[X] = \int x f(x) dx$
$M_2 = E[(X - \mu)^2]$
} P = Probability

Define Inertia:
Inertia is simply the resistance to change.
- The second moment of area measures a resistance to bending and/or deflection.
- The second moment of mass measures a resistance to angular motion

11/05/2025

<u>0.2.2:</u> Project 0 Theory: Deriving 1st Moments & Centroids

The average value of a quantity over a distribution can be referred to as the first moment. A classic example in physics for a first moment is torque, which is the first moment of force.

$M = \vec{\tau} = \vec{r} \times \vec{F}$ Pnt C_F

$= r \sin\theta \, dF \hat{\tau}$ or C_τ

$\hat{\tau} \perp r$ & $\perp F$ or C_M

θ = Ang. betw. $\vec{r}$ & $\vec{F}$

$\mathcal{M} = M_1$

$\mathcal{M} = \int r \, dm = \int r (\rho d\forall)$ Pnt C_m

$\mathcal{M} = \int r \, dW = \vec{\tau}_G$ Pnt G

$\mathcal{M} = \int r \, d\forall$ Pnt C_{xyz}

$\mathcal{M} = \int r \, dA$ Pnt C_{xy}

$\mathcal{M} = \int r \, dL$ Pnt C_x

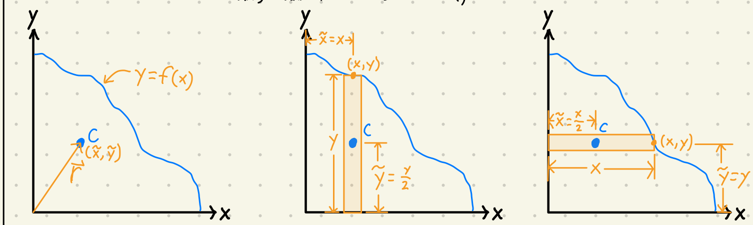
Geometric centers are referred to as "centroids".
First moments are also used for pinpointing the Qty's center.
i.e. $W = \int dW$

$\bar{x} W \int r_x \, dW$, $r^2 = r_x^2 + r_y^2 + r_z^2$

$\bar{x} = \frac{\int r_x dw}{\int dw}$

$(\oplus_G)$ $C_G(x, y, z) = \bar{x}\hat{i} + \bar{y}\hat{j} + \bar{z}\hat{k}$

$= \frac{\int \tilde{x} dw}{\int dw}\hat{i} + \frac{\int \tilde{y} dw}{\int dw}\hat{j} + \frac{\int \tilde{z} dw}{\int dw}\hat{k}$, where $r_x = \tilde{x}, \ldots$



$\bar{x} = \frac{\int \tilde{x} dA}{\int dA}$, $\bar{y} = \frac{\int \tilde{y} dA}{\int dA}$

Combining Composite Centers

$\bar{x} = \frac{\Sigma \tilde{x} A}{\Sigma A}$, $\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A}$

* See Section A in R.C. Hibbeler's Mechanics of Materials

R.C. Hibbeler
Statics txtbk
Dyn. txtbk

11/05/2025

0.2.2: Project 0 Theory: Deriving 2nd Moments

The variance of a quantity over a distribution can be referred to as the second moment of that quantity. In physics, the two widely used second moments are coined as the moments of inertia or M.O.I.

The static M.O.I. is the A.M.O.I. or second moment of area

$$I_x = \int_A y^2 dA$$
$$I_y = \int_A$$
$$J_o = \int_A r^2 dA = I_x + I_y$$

The dynamic M.O.I. is the M.M.O.I. or second moment of mass

$$I = \int_m r^2 dm$$
$$= \rho \int r^2 dV \text{ if } \rho = const.$$

Inertia Tensors

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Note:
It is assumed that the spinning top spins along its principle vertical axis initially.

$$I_{xx} = \int r_x^2 dm = \int (y^2+z^2) dm$$
$$I_{yy} = \int r_y^2 dm = \int (x^2+z^2) dm$$
$$I_{zz} = \int r_z^2 dm = \int (x^2+y^2) dm$$

$$\left.\begin{aligned} dI_{xy} &= xy\,dm \\ I_{xy} = I_{yx} &= \int xy\,dm \\ I_{yz} = I_{zy} &= \int yz\,dm \\ I_{xz} = I_{zx} &= \int xz\,dm \end{aligned}\right\} \text{Products of Inertia}$$

When computing M.O.I. W.R.T. the principal axes of inertia, the tensor simplifies to:

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad \begin{aligned} I_x &= I_{xx} \\ I_y &= I_{yy} \\ I_z &= I_{zz} \end{aligned}$$

Statics txtbk
Dyn. txtbk

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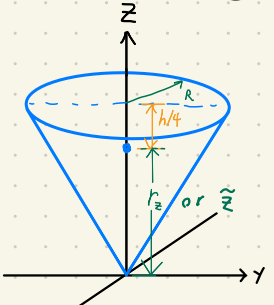
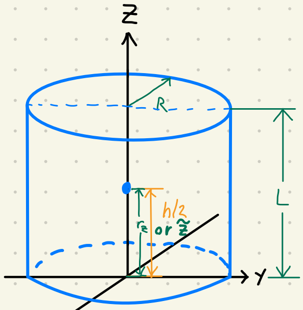
0.2.2: Project 0 Theory: M.M.O.I. of Homogenous Solid Primitives



$\forall = \frac{1}{3}\pi R^2 h$

$I_{xx} = I_{yy} = \frac{3}{80} m(4R^2 + h^2)$

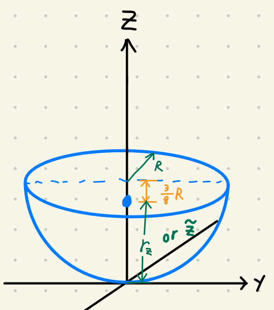
$I_{zz} = \frac{3}{10} mR^2$ ↳ for ⧫



$\forall = \pi R^2 h$

$I_{xx} = I_{yy} = \frac{1}{12} m(3R^2 + h^2)$

$I_{zz} = \frac{1}{2} mR^2$



$\forall = \frac{2}{3}\pi R^3$

$I_{xx} = I_{yy} = 0.259 mR^2$

$I_{zz} = \frac{2}{5} mR^2$ ↳ for ⧫

Statics txtbk

Dyn. txtbk

$\vec{r}$ = centroid coords

$= \frac{M_1}{M_0}$

$= \frac{\int r\,dQ}{\int dQ}$

Let $dQ = dm$

$M_0 = \int dm = m$

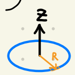
$M_1 = \int r\,dm$

$I = M_2 = \int r^2 dm$

Pnt Mass:

$I = mR^2 = I_{zz}$

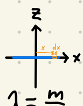
Ring Mass:



$I = R^2 \int dm$

$= mR^2$

Rod Mass:



$\lambda = \frac{m}{L}$

$dm = \lambda dx$

$dI = x^2 dm$

$I = \int_{-L/2}^{L/2} x^2 dm$

Disc:

$\sigma = \frac{m}{\pi R^2}$

Stack ∞ rings of 0 thickness encircling eachother to form a disc.

→Straightening a ring creates a line of length $2\pi R$

$dA = 2\pi R\,dR$

$dm = \sigma\,dA$

$= \sigma(2\pi R)dR$

$dI_{zz} = dm \times R^2$

$= \sigma(2\pi R^3)dR$

$I_{zz} = \int_0^R \sigma(2\pi R^3)dR$

$= 2\pi\sigma \int_0^R R^3 dR$

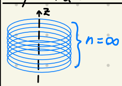
$= \frac{mR^2}{2}$

Hollow Slender Cyl.:

slender ring extended along $\hat{z}$

$I = mR^2$

solid Cylinder M.O.I. (I_{cyl}):



$\rho = \frac{m}{\forall}$

$\rho = \frac{m}{LA} \rightarrow m = \rho L A$

$dm = \rho L\,dA = \rho\,d\forall$

$\underbrace{\rho L\,dA}_{L = const.}$, $\underbrace{\rho\,d\forall}_{L \neq const.}$

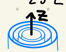
$I = \int R^2 dm$ | $A = \pi R^2 = A_z$

$I_{xx} = \int R^2 \rho L\,dA$ | $dA = \underbrace{2\pi R}_{\text{Ring } \hat{\theta} \text{ length}}\,\underbrace{dR}_{\text{ring } \hat{R} \text{ thickness}} = dA_z$

$= \rho L \int 2\pi R^3 dR$

$= \rho L \left(\frac{2}{4}\right)\pi \left[R^4\right]\Big|_0^R$

$= \frac{\pi}{2}\rho L R^4 = I_{yy}$

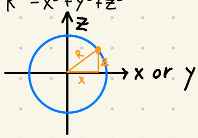


$I_{zz} = \frac{1}{2} mR^2$

Hollow Sphere:

$I = I_x = I_y = I_z$

$R^2 = x^2 + y^2 + z^2$



$dI = dm \times R^2$

$$\begin{cases} dI_x = dm(y^2 + z^2) \\ dI_y = dm(x^2 + z^2) \\ dI_z = dm(x^2 + y^2) \end{cases}$$

$\rightarrow dI_x + dI_y + dI_z = 3dI$

$= dm \times 2(x^2 + y^2 + z^2)$

$= dm \times 2R^2$

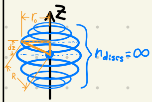
$\rightarrow dI_x = dI_y = dI_z = dI$

$\rightarrow 3 \cdot dI = 2 \cdot (dm \times 2R^2)$

$dI = \frac{2}{3} dm \cdot R^2$

$I = \frac{2}{3} mR^2$

Solid Sphere:



$r_0 = \sqrt{R^2 - z^2}$

$\rho = \frac{m}{\forall}$

$= \frac{m}{(\frac{4}{3}\pi R^3)}$

$dm = \rho \cdot d\forall$

$= \rho \cdot A\,dz$

$= \rho \cdot (\pi r_0^2) \cdot dz$

$dI = dm \cdot \frac{r_0^2}{2}$

$I = \int dI$

$= \int \frac{r_0^2}{2} \rho \cdot (\pi r_0^2) \cdot dz$

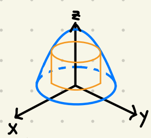
$= \int_{-R}^{R} \left(\frac{\rho\pi}{2}\right)(R^2 - z^2)^2 dz$

$= \frac{\rho\pi}{2}\left(\frac{16R^5}{15}\right)$

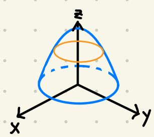
$\rho = \frac{3m}{4\pi R^3}$

$I = \frac{2}{5} mR^2$

Note:
If the quantity isn't orthogonal to r, then use cross products or trigonometry

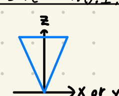


$d\forall_{shell} = (2\pi y)(z)dy$



$d\forall_{disc} = (\pi y^2)dz$

Cone M.O.I. (I_{cone}):



*Review Parallel & Perpendicular Axis Theorems

11/06/2025

0.2.2: Project 0 Theory: The Perpendicular & Parallel Axis Theorems

Perpendicular Axis Theorem

- States that, for flat object of negligible thickness, the M.O.I. about the perpendicular axis is the summation of the M.O.I.'s for the other two orthogonal axes.

$$I_{zz} = I_{xx} + I_{yy}$$

Parallel Axis Theorem

$I_x = \bar{I}_{x'} + A d_y^2$

$I_y = \bar{I}_{y'} + A d_x^2$

$J_o = \bar{J}_c + A d^2$ Polar M.O.I.

$\bar{J}_c = \bar{I}_{x'} + \bar{I}_{y'}$

$I_{xx} = (I_{x'x'})_G + m(y_G^2 + z_G^2)$

$I_{yy} = (I_{y'y'})_G + m(x_G^2 + z_G^2)$

$I_{zz} = (I_{z'z'})_G + m(x_G^2 + y_G^2)$

$I_x = \bar{I}_{x'} + m d_y^2$

When a reference point **(x',y')** or axis does not pass through the centroid or center of quantity, use the **parallel axis theorem** to calculate the **M.O.I.** about a *parallel axis* **d** *distance away.*

Radius of Gyration:

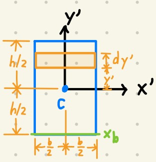
$k_x = \sqrt{I_x/A}$

$k_y = \sqrt{I_y/A}$

$k_o = \sqrt{J_o/A}$

Values reported in some handbooks for quick M.O.I. calcs

Ex 10.1



$\bar{I}_{x'} = \int_A y'^2 dA$

$= \int_{-h/2}^{h/2} y'^2 (b\,dh)$

$= \frac{1}{12} b h^3$

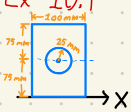
$\rightarrow I_{x_b} = \bar{I}_{x'} + A d_y^2$

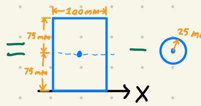
$= \frac{1}{12} b h^3 + bh\left(\frac{h}{2}\right)^2$

$= \frac{1}{3} b h^3$

↳ M.O.I. about axis X_b

Ex 10.4





Circle: $I_x = \bar{I}_{x'} + A d_y^2$

$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2$

$= 11.4\text{E}6 \text{ mm}^4$

Rect: $I_x = \bar{I}_x + A d_y^2$

$= \frac{1}{12}(100)(150)^3$

$+ (100)(150)(75)^2$

$= 112.5\text{E}6 \text{ mm}^4$

Total: $I_x = I_{x,rect} - I_{x,circ}$

$= (112 - 11.4)\text{E}6 \text{ mm}^4$

$= 101\text{E}6 \text{ m}^4$

Statics txtbk

Dyn. txtbk

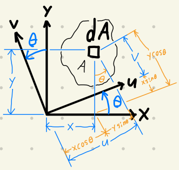
Products of Inertia: $I_{xy} = \int_A xy\,dA$

"Mirroring" an object across x or y axis gives $I_{x'y'} = -I_{xy}$

For Para. Ax: $I_{xy} = \bar{I}_{x'y'} + A d_x d_y$

M.O.I. about Inclined Axis:

Let uv be coplanar with xy but w/ Axes u & v rotated about the origin CCW by ang. θ



$$\begin{cases} u = x\cos\theta + y\sin\theta \\ v = y\cos\theta - x\sin\theta \end{cases}$$

Eq. 10-9:
$$\begin{cases} I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos(2\theta) - I_{xy}\sin(2\theta) \\ I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos(2\theta) + I_{xy}\sin(2\theta) \\ I_{uv} = \frac{I_x - I_y}{2}\sin(2\theta) + I_{xy}\cos(2\theta) \end{cases}$$

$J_o = I_u + I_v = I_x + I_y$

Eq. 10-10: $\tan(2\theta_p) = \frac{-I_{xy}}{(I_x - I_y)/2}$ @ $\theta = \theta_p$ when u & v are the principal axes

Eq. 10-11: $I_{\substack{max \\ min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$

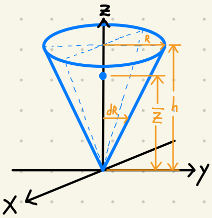
11/07/2025

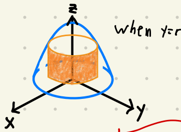
0.2.2: Project 0 Theory: Learning Cone M.O.I.'s

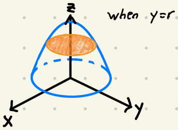
Let $\vec{C} = (\bar{x}, \bar{y})$ Centroid or Cent. of Qty Distr.

$\vec{A} = (\tilde{x}, \tilde{y})$ Loc. of Qty differential

Cone:







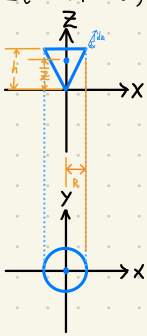
$d\forall_{\text{shell, "tube"}} = \underbrace{(2\pi y)}_{\text{Perimeter}}\underbrace{(z)}_{\text{height}}\underbrace{dy}_{\text{Thickness}}$

$d\forall_{\text{disc}} = \underbrace{(\pi y^2)}_{\text{circle Area}}\underbrace{dz}_{\text{Height}}$

Bounds determine thickness

Only when $r=y$ & $x=0$

Let $\vec{A} = (0,0)$



$\vec{C} = (0,0,\bar{z})$

$\bar{z} = \frac{\int \tilde{z}\, d\forall}{\int d\forall} = \frac{M_1}{M_0}$

$d\forall = \pi r^2 dz$

$= \pi x^2 dz$

$x^2 = R^2 - \cancel{y^2}^{\,0}$

$y^2 = R^2 - x^2$

Let $z = \frac{dz}{dx} x$

$dz = zx\,dx$

$\frac{dz}{dx} = \frac{hx}{R} = zx$

$z = \frac{h}{R}$

$d\forall = \overbrace{\pi x^2}^{A_{circ.}} z\,dx$

$\forall = \int_0^R \pi x^2 z\,dx$

$= \pi \frac{h}{R} \int_0^R x^2 dx$

$= \pi \frac{h}{R} \left(\frac{1}{3}x^3\right)\Big|^R$

$= \frac{1}{3}\pi \frac{h}{R} R^3$

$\forall = \frac{1}{3}\pi h R^2 = M_0$

Using x in place of r can cause confusion unless it is explicitly noted that y=0 if r^2=x^2+y^2

$M_1 = \int z\, d\forall$

$= \int z \cdot \pi x^2 dz$

$\frac{x}{R} = \frac{z}{h}$

$x = \frac{R}{h} z$

$M_1 = \int z \cdot \pi \frac{R^2}{h^2} z^2 dz$

$= \frac{R^2}{h^2}\pi \int_0^h z^3 dz$

$= \frac{R^2}{h^2}\pi \left[\frac{1}{4}z^4\right]\Big|_0^h$

$= \frac{\pi}{4} R^2 h^2$

$\bar{z} = \frac{M_1}{M_0}$

$= \frac{3}{4}h$

$M_2 = I_A$

$= \int z^2 d\forall$

$= \frac{R^2}{h^2}\pi \left[\frac{1}{5}z^5\right]\Big|_0^h$

$= \frac{\pi}{5} R^2 h^3$

$\rho = \frac{m}{\forall} \rightarrow \forall = \frac{m}{\rho}$

$d\forall = \frac{dm}{\rho}$

$I_{x'} = I_{m,x'} = I_{y'}$ @ (0,0)

$= \rho I_A$

$= \rho M_2$

$= \frac{m}{\forall} \cdot \frac{\pi}{5} R^2 h^3$

$= m \frac{3}{5} h^2 = I_{height}$

11/08

$I_{apex} = I_{radius} + I_{height}$

$I_{height} = d_z^2 dm = z^2 dm$

$I_{radius} = \int dI_{cm}$ → int. resis. of disk to rot.

$= \int \frac{1}{4} r^2 dm = I_x = I_y$

$= \frac{1}{2} I_{disk}$ → symm.

$dI_{apex,y} = \frac{1}{4} r^2 dm + z^2 dm$

11/08/2025

~~$M_2 = \frac{R^2}{h^2}\pi \left[\frac{1}{5}z^5\right]\Big|_{-h/4}^{3h/4}$~~

~~$= \frac{R^2}{h^2}\pi \frac{1}{5}\left[\left(\frac{3h}{4}\right)^5 - \left(\frac{h}{4}\right)^5\right]$~~

~~$= \frac{\pi R^2}{5h^2}\left(\frac{3^5 h^5}{4^5} + \frac{h^5}{4^5}\right)$~~

~~$= \frac{\pi R^2 h^3}{5}\left(\frac{3^5+1}{4^5}\right)$~~

~~$= \frac{61\pi R^2 h^3}{1280}$~~

→ $I_{xx} = I_{yy} = \frac{3}{80} m(4R^2 + h^2)$

@ centroid

→ Try $\boxed{\begin{array}{l} dI_{disk} = \frac{1}{2} r^2 dm \\ dI_{shell} = r^2 dm \end{array}}$

M.M.O.I. is just $M_{2,\forall}$ scaled by ρ

$I_m = \rho \cdot I_\forall \rightarrow \rho = \frac{I_m}{I_\forall} = \frac{m}{\forall}$

1) Use dI_{Disk} for cones & cylinders

dI_{shell} for Spheres & thin rods

2) For any continuous shape like a cone, radius (r) must be defined by how it changes w/ the integration variable (z)

$I = \int dI$

$dI = \frac{1}{2} r^2 (\rho\, d\forall)$

$r = f(z) = \frac{R}{h} z$

$I = \int \frac{1}{2}\rho \left(\frac{R}{h}z\right)^2 \left(\pi \left(\frac{R}{h}z\right)^2 dz\right)$

$= \frac{\rho \pi R^4 h}{10}$

$= \frac{3mR^2}{10} = I_z = I_{height}$

$dI_y = dI_{cm} + (dm)d_z^2$, d=dist.

$= dI_x$

M.O.I. about Ax:s of rev. $\quad I = \int \frac{1}{2} r^2 dm = I_{disk}$ $\quad$ use M.O.I of solid disk and define: $dm = \rho \pi r^2 dx$

M.O.I. about Perpen. Axis $\quad I = \int (dI_{cm} + r^2 dm)$ $\quad$ Let $dI_{cm} = dI_{disk}$ & use parallel axis thrm to shift it to the axis of integration

Statics txtbk

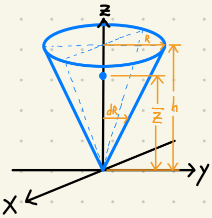
Dyn. txtbk

11/08/2025

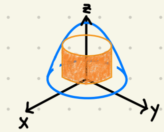
$\underline{0.2.2:}$ Project 0 Theory: Learning Cone M.O.I.'s

Let $\vec{C} = (\bar{x}, \bar{y})$ Centroid or Cent. of Qty Distr.
$\vec{A} = (\tilde{x}, \tilde{y})$ Loc. of Qty differential

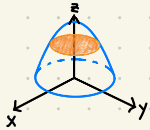
Cone:



Let $\vec{A} = (0,0)$



$d\forall_{shell, \text{"tube"}} = \underbrace{(2\pi r)}_{\text{Perimeter}}\underbrace{(z)}_{\text{height}}\underbrace{dr}_{\text{Thickness}}$



$d\forall_{disk} = \underbrace{(\pi r^2)}_{\text{circle Area}}\underbrace{dz}_{\text{Height}}$

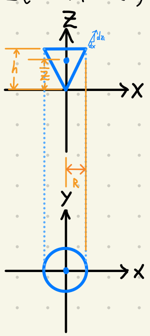
Bounds determine thickness & $\rho = \frac{m}{\forall} = \frac{I_m}{I_\forall}$

$dI_{disk} = \frac{1}{2} r^2 dm$

$dI_{shell} = r^2 dm$

Perpend. Axis Thrm states
$I_z = I_x + I_y$ for each disk

Assume const. ρ & g
$\rightarrow \vec{C} = \vec{C}_M = \vec{C}_G$



$d\forall = \pi x^2 dz$ (disk)

$R\,dz = h\,dx$

$\int d\forall = \int_0^R \pi x^2 \frac{h}{R} dx$

$= \frac{h}{R}\pi \int_0^R x^2 dx$

$= \frac{\pi h}{R}\left[\frac{1}{3}x^3\right]\Big|_0^R$

$\forall = \frac{\pi}{3} h R^2 = M_0$ ✓

$d\forall = 2\pi x z\, dx / 2$

$z\,dx = x\,dz$

$x = \frac{Rz}{h}$

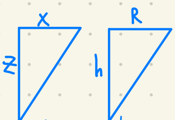
$\int d\forall = \frac{2}{2}\pi \int x^2 dz$

$= \frac{2}{2}\pi \int \left(\frac{Rz}{h}\right)^2 dz$

$= \frac{2\pi R^2}{2h^2}\int_0^h z^2 dz$

$= \frac{\pi R^2}{h^2}\left[\frac{1}{3}z^3\right]\Big|_0^h$

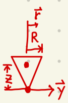
$\forall = \frac{\pi}{3}R^2 h = M_0$ ✓



$\frac{x}{z} = \frac{R}{h} = \frac{dx}{dz}$

$r = x$ when $y = 0$

$dI_y = dI_{cm} + dI_{offset}$



$= dI_r + dI_h$

$= \underbrace{\frac{1}{4}r^2 dm}_{} + z^2 dm$

$\hookrightarrow I_z = I_x + I_y$
$= 2I_y$

for the disk element dist. z away

$\hookrightarrow$ Don't Forget Perpend. Axis Thrm!

dI_cm term is needed when the axis for the moment does not pass through the centroid.

The z bounds controls how much of the volume is being considered for the inertia calculation. This is important to note for composite bodies which is the next step for the spinning top.

$\bar{z}$ is only used for M.O.I. about pnt not on $(x,y) = (0,0)$, z-axis

$\rho = \frac{dm}{d\forall} = \frac{m}{\forall}$

$dm = \rho\, d\forall$

$= \frac{m}{\forall} d\forall$

$= \frac{m}{M_0} d\forall$

$dI_z = \frac{1}{2} r^2 dm$

$= \frac{1}{2}x^2 dm$

$= \frac{1}{2}x^2 m\left(\frac{3}{\pi R^2 h}\right) d\forall$

$= \frac{1}{2}x^2 m\left(\frac{3}{\pi R^2 h}\right)(\pi x^2) dz$

$x = \frac{Rz}{h}$

$dI_z = \frac{1}{2}m\left(\frac{3}{\pi R^2 h}\right)(\pi)\frac{R^4 z^4}{h^4} dz$ Careful when carrying exponents!

$= \frac{3}{2}m\left(\frac{R^2 z^4}{h^5}\right) dz$

$\int dI_z = \frac{3R^2}{2h^5} m \int z^4 dz$

$I_z = \frac{3R^2}{2h^5} m \left[\frac{1}{5}z^5\right]\Big|_0^h$

$= \frac{3R^2}{10h^5} m h^5$

$= \frac{3}{10} m R^2 \cancel{h}$

$dI_z = \frac{1}{2}x^2 dm$

$= \frac{1}{2}x^2 \frac{m}{\forall} d\forall$

$\frac{d\forall}{\forall} = \frac{\pi x^2 dz}{(\pi/3)R^2 h}$, $x = \frac{Rz}{h}$

$= \frac{3dz}{R^2 h}\left(\frac{Rz}{h}\right)^2$

$= \frac{3z^2 dz}{h^3}$

$dI_z = \frac{1}{2}x^2 m \frac{d\forall}{\forall}$

$= \frac{1}{2}\frac{R^2 z^2}{h^2} m \frac{3z^2 dz}{h^3}$

$= \frac{3R^2 z^4}{2h^5} m\, dz$

$\int dI_z = \frac{3R^2 m}{2h^5}\int_0^h z^4 dz$

$I_z = \frac{3R^2 m}{2h^5}\left[\frac{1}{5}z^5\right]\Big|_0^h$

$I_r = \bar{I}_{r'} + m z^2$ Parallel Axis Thrm

$dI_x = dI_y$

$= dI_{cm} + dI_h$

$= \frac{1}{2} d\bar{I}_x + \bar{z}^2 dm$

I still don't really understand why the inertia about CM must be halved.

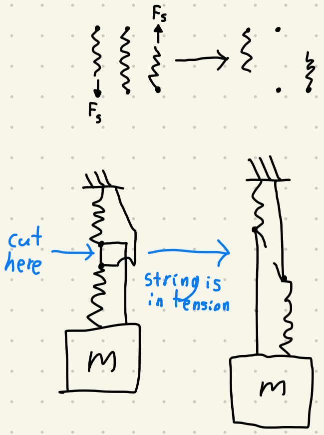
Review Perpendicular Axis Theorem
$\rightarrow I_z = I_x + I_y$
$\rightarrow$ for flat element, no z thickness
$\rightarrow$ Disk element
$\rightarrow I_x$ & I_y are perpendicular to z-axis
$\rightarrow$ Trianle rule

Statics txtbk

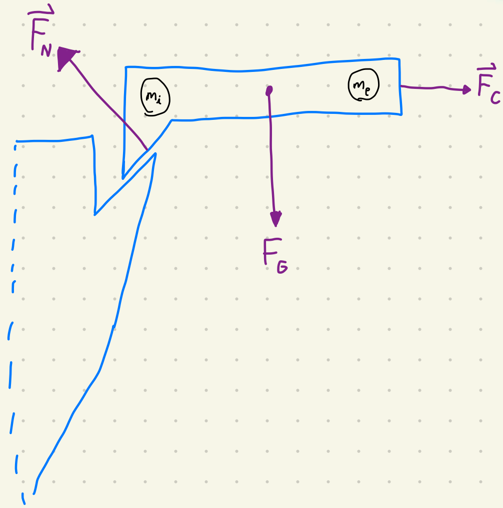
Dyn. txtbk

11/08/2025

Related Brainstorming

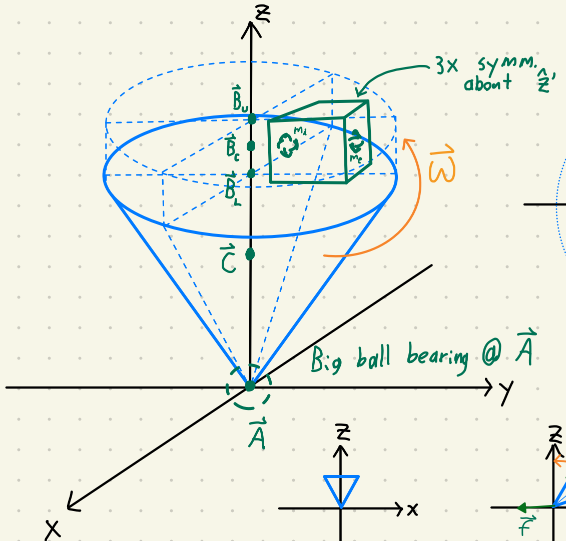


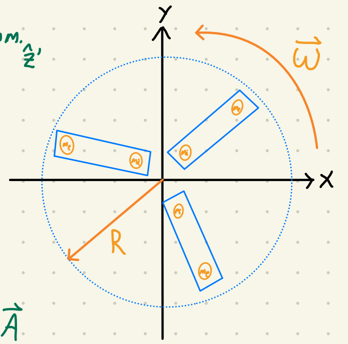
Changes the parallel spring pair into a series.



11/09/2025

Designing Transforming Spinning Tops





$a_c = \frac{v_t^2}{R} = \omega^2 R$

$\Sigma F_c = m a_c = m\omega^2 R$

Let $\omega = \dot{\phi}$

θ = Tilt

ϕ = Spin angle

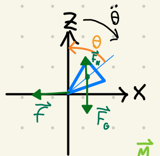
Note (*for FDM printing*):

- Use 0.4 mm clearance for parts requiring smooth glide
- Use 0.2 mm clearance for a high friction fit
- Use (0.05 - 0.15) mm clearance for snug fit depending on print step size relative to nozzle diameter

Assume no slip @ $\vec{A}$

$\vec{A} = (0,0,0)$ = Apex Pnt

F_p = Precession force



$\Sigma F_x = 0$

$F = f$

$= F - F_p$

$\Sigma F_y = 0$

$F_N = F_G$

$f = \mu F_N$

$\Sigma M_A = I\ddot{\theta}$, M_p = Moment from Precession

$I\ddot{\theta} = M_p - F_G Z$

$\Sigma M_A = \Sigma \vec{F}_c \times \vec{Z}$

$\ddot{\theta} = \frac{\Sigma \vec{F}_c \times \vec{Z}}{I}$

$\vec{Z} = \vec{Z}(t)$

$\vec{Z}(x, y, z) = Z_x \hat{i} + Z_y \hat{j} + Z_z \hat{k}$

$= X_c \hat{i} + Y_c \hat{k} + Z_c \hat{k}$

$\vec{H}_0 + \int \vec{M} dt = \vec{H}$

$I_0 \vec{\omega}_0 + \Delta H = I \vec{\Omega}$

$\vec{\Omega} = \Omega_r \hat{r} + \Omega_z \hat{k}$

$= \Omega_r \hat{r} + \omega_0 \hat{k}$

@ $t=0$, $\Omega_z = \omega$

$\omega = \omega_0$

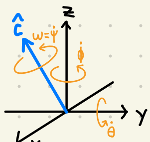
$\vec{H}_0 + \Delta \vec{H} = \vec{H}$

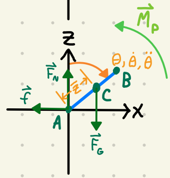
$\hat{H} = \frac{\vec{H}}{H} = \hat{\Omega}$

$\hat{H}_0 = \hat{\omega}_0$

$\theta = \theta_{min}$ @ $t = 0$

$\theta = \theta_{max}$ @ t_{roll}

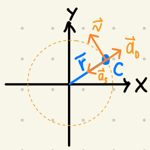




$\vec{H}_0 = I_0 \vec{\omega}_0$ @ $\theta = 0$

$\vec{H}_0 = I_0 \vec{\omega}_0 + I_0 \vec{\dot{\phi}}_0$

$= I_0 \omega \hat{c} + I_0 \dot{\phi}_0 \hat{k}$



$\vec{V}$ = Lin. Vel. @ C

$\vec{a}_s$ = Stabilizing accl @ C

$\vec{a}_D$ = Destabilizing accl @ C

11/10/2025

$\Delta \vec{H} = \vec{H} - \vec{H}_0$

$= \int \vec{M} dt$, $\vec{\tau} = \vec{r} \times \vec{F}$

$= \int_0^t (\vec{Z} \times \vec{F}_c) dt$

Eq. 19-1 $\vec{L} = m\vec{v}$

$\vec{V}_c = \vec{V}_A + \vec{V}_{C/A}$

$= \vec{V}_A + \vec{\omega} \times \vec{Z}$

$\vec{V}_B = \vec{V}_A + \vec{V}_{B/A}$

$= \vec{V}_A + \vec{\omega} \times \vec{C}_{AB}$

$\vec{Z} = \vec{C}_{AC}$

Statics txtbk

Dyn. txtbk

$\vec{H} = (H_A)_c$

$= \vec{Z} \times m\vec{V}_c$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ X_c & Y_c & Z_c \\ m\dot{x}_c & m\dot{y}_c & m\dot{z}_c \end{vmatrix}$

$= (y_c m\dot{z}_c - z_c m\dot{y}_c)\hat{i}$
$- (x_c m\dot{z}_c - z_c m\dot{x}_c)\hat{j}$
$+ (x_c m\dot{y}_c - y_c m\dot{x}_c)\hat{k}$

$= H\cos\theta \hat{r}$
$+ H\cos\psi \hat{c}$
$+ H\cos\phi \hat{z}$

$\vec{H}_0 = H_0 \cos\psi_0 \hat{c}$

Governing Equation

$\vec{H}_0 + \int \underbrace{\vec{\tau}}_{\text{Torque from friction}} dt = \vec{H}$

$H_0 \cos\psi_0 \hat{c} + \vec{\tau} dt = H\cos\theta \hat{r} + H\cos\psi \hat{c} + H\cos\phi \hat{z}$

Initial Coords $\begin{cases} \psi_0 = 0 \\ \phi_0 = 0 \\ \theta_0 \approx 0 \end{cases}$

Final Coords $\begin{cases} \psi, \phi_0 \rightarrow \text{Not needed during spin, roll is outside the current scope} \\ \theta = \theta_{max} = 90^\circ - \alpha_A \end{cases}$

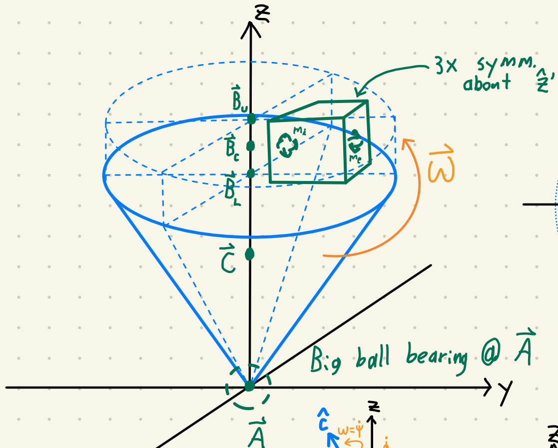
α_A = Apex. Ang.

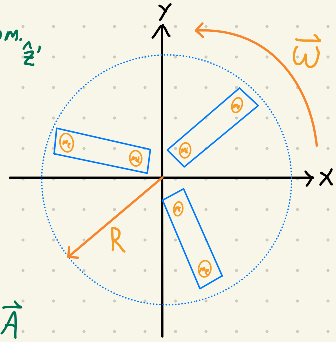
Angular Moment Behavior simplifies the problem:

- During spin, oscillates between x and y since the zr-plane is actively rotating about the z axis.
- The angle for angular momentum is slowly shifting upward between the spin axis and the z axis.
- Torque from friction is orthogonal to the ground and is therefore subtracting from the momentum about the z-axis for each time increment (differential).
- The final momentum for the scope of the problem takes place at the moment the top's side becomes parallel with the ground; it is **assumed** that <u>no significant slipping</u> will occur to cause the apex to shift away from the origin.

11/10/2025

Designing Transforming Spinning Tops





Note *(for FDM printing)*:

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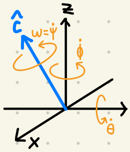
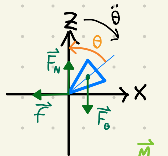
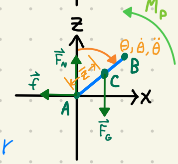
- Use 0.02 in for loose fit
- Use 0.01 in for tighter fit that may or may not be snug





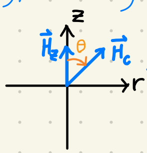


$\sum F_x = 0$
$F = f$
$= F - F_p$
$\sum F_y = 0$
$F_N = F_G$
$f = \mu F_N$

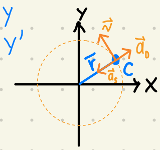
$\sum M_A = I\ddot{\theta}$, M_p = Moment from Precession
$I\ddot{\theta} = M_p - F_G z$
$\sum M_A = \sum \vec{F}_c \times \vec{z}$
$\ddot{\theta} = \frac{\sum \vec{F}_c \times \vec{z}}{I}$
$\vec{z} = \vec{z}(t)$
$\vec{z}(x,y,z) = z_x\hat{i} + z_y\hat{j} + z_z\hat{k}$
$= x_c\hat{i} + y_c\hat{k} + z_c\hat{k}$

$\vec{H}_o = \vec{\tau}_o dt$ $\quad \tau_o$ = Launch Torque
$\vec{H}_o + \Delta\vec{H} = \vec{H}$ $\quad \tau$ = Applied torque during motion
$\vec{H}_o + \int\vec{\tau}dt = \vec{H}$ $\quad$ Assume no energy loss during launch

$\vec{H}_o = H_o\hat{c}$
$\vec{H} = H\cos\theta\hat{r} + H\cos\psi\hat{c} + H\cos\phi\hat{z}$
$H = \sqrt{H_c^2 + H_z^2}$



$\theta: zr$
$\phi: xy$
$\psi: x'y'$



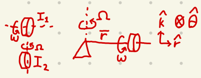
$\vec{V}$ = Lin. Vel. @ C
$\vec{a}_s$ = Stabilizing accl @ C
$\vec{a}_D$ = Destabilizing accl @ C
$\vec{a}_s \approx \vec{a}_D$

$\cancel{H_z = H\cos\phi\hat{z}}$
$\cancel{H_c = H\cos\psi\hat{c} + H\cos\theta\hat{r}}$
$\cancel{H_c = \sqrt{H^2(\cos^2\psi + \cos^2\theta)}}$
$\cancel{= H\sqrt{\cos^2\psi + \cos^2\theta}}$
Let $y=0 \to r=x$
$\to \phi = 0$
$\to H_z = H\cos 0 = H$
$= H_c\cos\theta$
$= I_z\Omega, \; \Omega = \Omega\hat{z}$
$= I_z\dot{\phi}$
$H_c = I_{\bar{z}}\omega$
$= I_{\bar{z}}\dot{\psi}$

$\cancel{\bar{z} = z\cos\theta}$
$\cancel{H_{\bar{z}} = H_c\cos\theta}$
$\cancel{I_z\dot{\phi} = I_{\bar{z}}\dot{\psi}}$

When a gyrscope is released from rest, it begins to rotate (precess) about the absolute vertical axis. The angular momentum gain along the z-axis comes from the initial horizontal dipping motion which can have its angle measured as theta.

$\vec{H}_{orbit} = m\vec{v}_c\bar{r}\hat{k}$
$= m\Omega\bar{r}^2\hat{k}$
$\vec{H}_{spin} = I_1\omega\hat{r}$
$+ I_2\Omega\hat{k}$



$\vec{H} = \underbrace{I_1\omega\hat{r}}_{\text{Spin}} + \underbrace{I_2\Omega\hat{k} + m\Omega\bar{r}^2\hat{k}}_{\text{Assuming constant orbit}}$

Eq. 21-33: $\sum\vec{M}_A = \vec{\Omega}_z \times \vec{H}_c$, $\vec{\Omega}_z = \dot{\vec{\phi}}$, $\vec{H}_c = I_{\bar{z}}\omega_{\bar{z}}$
21-32: $\sum M_A = I_{\bar{z}}\dot{\phi}\,\omega_c$, $\omega_c = \omega = \dot{\psi}$

Eq. 21-30: $\sum M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$
$\sum M_y = 0$ $\quad$ where $\hat{i} \perp [\bar{z} \,\&\, \hat{k}]$
$\sum M_z = 0$

$\vec{M}_p = \vec{\tau}_p = \vec{z} \times (mg)$, $\vec{\tau} \perp [\hat{k} \,\&\, \hat{c}]$
$[\hat{z} \,\&\, \hat{\bar{z}}]$

$\to \dot{\phi}$ Always swings toward the sense of $\sum M_G$

$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$
$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$
$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$

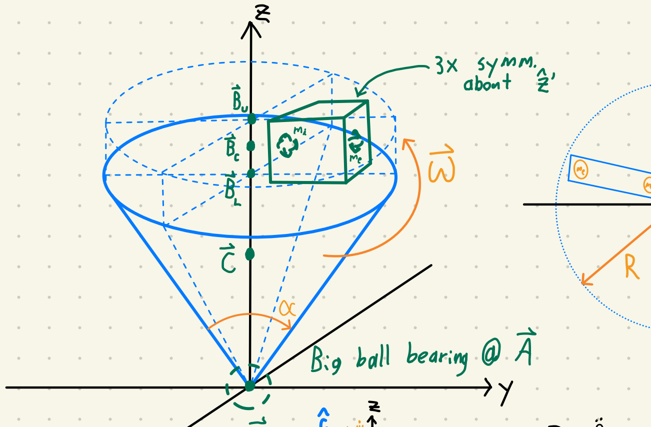
Torque Free Motion
$\theta = \text{const.}$
$\dot{\phi} = \frac{H_G}{I}$
$\dot{\psi} = \frac{I - I_z}{II_z} H_G\cos\theta$

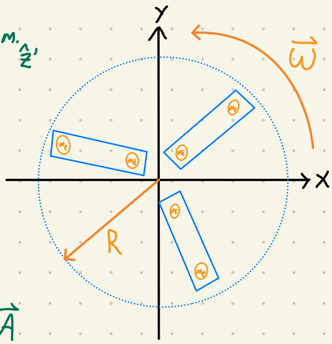
$\vec{H}_A = \underbrace{\vec{H}_{orbit}}_{\text{CM Trans.}} + \underbrace{\vec{H}_{spin}}_{\text{CM Rot.}}$

Statics txtbk
Dyn. txtbk

11/10/2025

Designing Transforming Spinning Tops





Note *(for FDM printing)*:

- Use 0.4 mm clearance for parts requiring smooth glide
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- Use (0.05 - 0.15) mm clearance for snug fit depending on print step size relative to nozzle diameter

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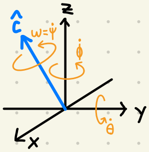
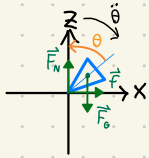
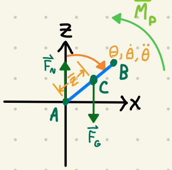




$\vec{Z} = \vec{Z}(t)$

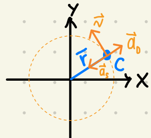
$\vec{Z}(x,y,z) = Z_x\hat{i} + Z_y\hat{j} + Z_z\hat{k}$
$= X_c\hat{i} + Y_c\hat{k} + Z_c\hat{k}$



$\vec{M}_p = \vec{\tau}_p = \vec{Z}\times(mg)$
$\Sigma M_A = I\ddot{\theta} = 0$ if $f=0$
$= \vec{Z}\times\vec{F}_G - I\ddot{\theta}$
$M_p = \vec{Z}\times\vec{F}_G$ = Moment from Precession

$\bar{Z} = \frac{3}{4}h = \frac{M_1}{M_0}$
$f = \mu F_N$
$a_{ctpl} = \frac{v_t^2}{r}$, const.

$\boxed{\vec{H}_A = \vec{H}_c + \vec{r}_{C/A}\times m\vec{v}_c}$ Eq. 21-9



$\vec{V}$ = Lin. Vel. @ C
$\vec{a}_s$ = Stabilizing accl @ C
$\vec{a}_D$ = Destabilizing accl @ C
$\vec{a}_s \approx \vec{a}_D$

Let $f=0 \rightarrow \mu=0$
$\vec{H} = \Sigma M dt$
$\vec{H} = \Sigma(\vec{r}\times\vec{F})dt - \underbrace{\Sigma(\vec{r}\times\vec{f})dt}_{\text{Torque Free Motion when ignored}}$
$\vec{H} = \vec{H}_0 + \Delta\vec{H}$
$\Delta\vec{H} = \vec{\tau}_f dt$
→ Superpose t & t_0 as steady precession states and consider $\Delta\vec{H}$ later

Eq. 21-36
θ_0 & θ are const. w/o $\vec{f}$
$\dot{\phi} = \frac{H_G}{I} \rightarrow I = I_x = I_y$, Let $I = I_T$ [Orbit]
$\dot{\psi} = \frac{I - I_z}{I I_z} H_G \cos\theta$ [Spin]

$\Delta\theta = \theta - \theta_0$
$\theta_{max} = \theta_{roll} = 90^\circ - \alpha$ } Assume no slip
Let $\theta_1 = \theta_{roll}$
Eq. 21-37: $\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta$

$$I_{cm} = \begin{vmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{vmatrix}$$

Statics txtbk
Dyn. txtbk

M.O.I. about G:
$I_{\bar{z}} = \frac{3}{10}mR^2$ [Spin M.O.I.]
$I_{\bar{x}} = I_{\bar{y}} = m\left(\frac{3}{20}R^2 + \frac{3}{5}h^2\right)$ [Transverse M.O.I.]

Parallel Axis Thrm:
$I_z = I_{\bar{z}} + m(0)^2$
$I_z = I_{\bar{z}}$
$I_x = I_{\bar{x}} + m\bar{z}^2$
$= m\left(\frac{3}{20}R^2 + \frac{3}{5}h^2 + \frac{3}{4}h^2\right)$
$= m\left(\frac{3}{20}R^2 + \frac{93}{80}h^2\right)$

If $\dot{\theta} = 0 \rightarrow \Sigma M_{A,r} = 0$,
$\rightarrow I\dot{\theta} = 0, I\ddot{\theta} = 0$

$I = mk^2$ Eq 17-5
$I_x = mk_x^2$ 10-6

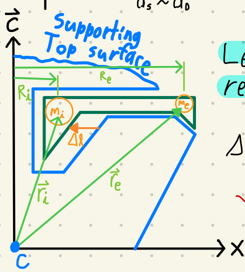
If $I_x = I_{xx}, I_y = I_{yy}, \ldots$
Then $\det(\underline{I}) = I_x I_y I_z$

$I = I_1\alpha_u^2 + I_2\beta_u^2 + I_3\gamma_u^2$

$\Sigma M_A = I\ddot{\theta}$ when $\mu \neq 0$

$\vec{H}_{cm} = \vec{H}_c, \vec{v} = \vec{v}_c$
$= \vec{H}_A - \vec{r}_{C/A}\times m\vec{v}_c$

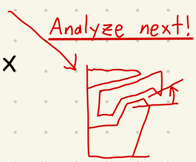
R_i = Horiz. Dist.[$\vec{C}$ & m_i]
R_e = Horiz. Dist.[$\vec{C}$ & m_e]



Let Limb 1 (l_1) reside along $\hat{x}$

$\Delta l = 0$ @ $t = 0$

Analyze next!



Assume pnt mass for ball bearings i & e

$I'_{zz} = mR'^2$
$I'_{xx} = mz'^2 = I'_{yy}$

$I_{iz} = m_i R_i^2$
$I_{ix} = m_i z_i^2$

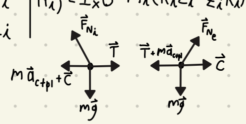
$I_{ez} = m_e R_e^2$
$I_{ex} = m_e z_e^2$

Let $l_r = R_e - R_i$
$l_z = Z_e - Z_i$

$\vec{H}_i = \vec{H}_A - \vec{r}_{i/A}\times m\vec{v}_i$ Local: $\bar{z} = \vec{C}$

$$\vec{r}_{i/A}\times\vec{v}_i = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_{i/A} & y_{i/A} & z_{i/A} \\ \dot{x}_i & \dot{y}_i & \dot{z}_i \end{vmatrix} = \hat{i}(y_{i/A}\dot{z}_i - z_{i/A}\dot{y}_i) - \hat{j}(x_{i/A}\dot{z}_i - \dot{x}_i z_{i/A}) + \hat{k}(x_{i/A}\dot{y}_i - y_{i/A}\dot{x}_i)$$

Let $r = x \rightarrow y = 0$, Let $z_i = z_{i/A} \mp (\bar{Z}_i = z_{i/c})$
$\vec{r}_{i/A}\times\vec{v}_i = 0\hat{i} - \hat{j}(R_i\dot{z}_i - z_i\dot{R}_i) + \hat{k}(0-0)$
$\vec{H}_i = \vec{H}_A + m_i(R_i\dot{z}_i - z_i\dot{R}_i)\hat{j}$
$H_i\hat{j} = I_x\dot{\theta} + m_i(R_i\dot{z}_i - z_i\dot{R}_i)$



11/11/2025

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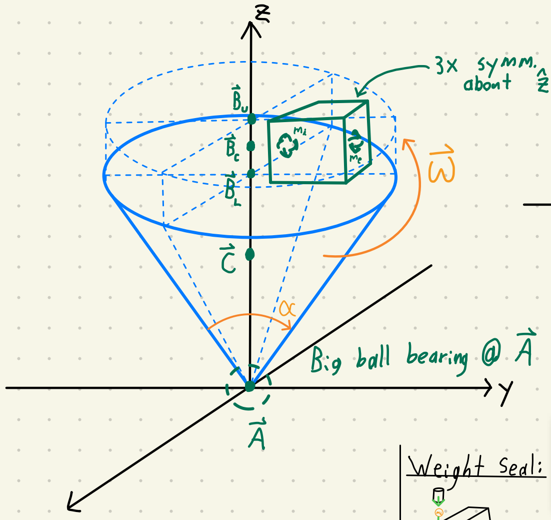

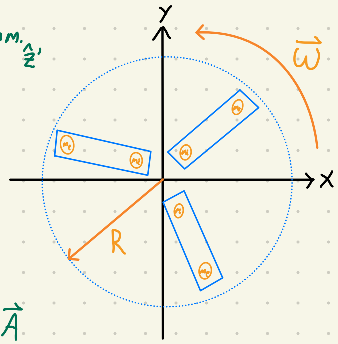

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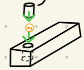
- Use 0.02 in for loose fit
- Use 0.01 in for tighter fit that may or may not be snug

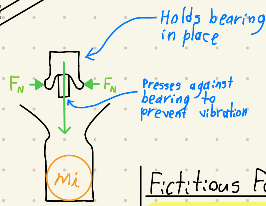
ABS Friction:

- Static friction coef. can range within 0.4 - 0.5
- Dynamic friction coef. can range within 0.3 - 0.45
- Measured friction coef. for dry sliding can range within 0.11 - 0.29

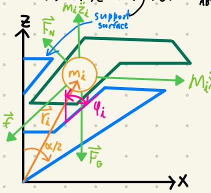
Note (Seal):
At least symm. 2 cantilever tabs exist to hold the seal in place. They must bend without breaking for one-use application.

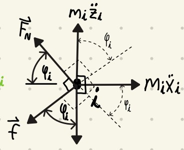
Weight Seal:





Assume higher f_{ABS} → Let $\mu_{ABS}=0.5$





$\rightarrow f = \mu F_N$

Static Analysis: $\Sigma \vec{F}_{real} = m\vec{a}$

$\Sigma F_z = 0 = m_i\ddot{z}_i$

$0 = F_N\sin\varphi_i - F_G - f\cos\varphi_i$

$\rightarrow F_G = F_N\sin\varphi_i - f\cos\varphi_i$

$\Sigma F_x = 0 = m_i\ddot{x}_i$

$0 = -F_N\cos\varphi_i - f\sin\varphi_i$

$\rightarrow F_N\cos\varphi_i = -f\sin\varphi_i$

$\Sigma F_N = 0 \rightarrow$ keep true for during motion too

$= F_N - F_G\sin\varphi_i - m_i\ddot{x}_i\cos\varphi_i + m_i\ddot{z}_i\sin\varphi_i$

$\rightarrow F_N = F_G\sin\varphi_i + m_i\ddot{x}_i\cos\varphi_i - m_i\ddot{z}_i\sin\varphi_i$

$\rightarrow \hat{N}$-axis defined by φ_i

Dynamic Analysis:

By treating the spinning top body as the inertial frame,

$m\vec{a}_i = \Sigma \vec{F}_{real} + \Sigma \vec{F}_{fictitious}$

$\Sigma F_z = m_i\ddot{z}_i$

$m_i\ddot{z}_i = F_N\sin\varphi_i - F_G - f\cos\varphi_i$

$\Sigma F_x = m_i\ddot{x}_i$

$m_i\ddot{x}_i = -F_N\cos\varphi_i - f\sin\varphi_i + m\omega^2 x$

$t_{0\to 1}$: $x = x_{max}$
$\Sigma F_{x_i} \geq 0$
$\Sigma F_{z_i} \geq 0$

$t_{1\to roll}$: $x = x_{min}$
$\Sigma F_{x_i} \leq 0$
$\Sigma F_{z_i} \leq 0$

Let $F_{x,i} = m_i\ddot{x}_i$
$F_{z,i} = m_i\ddot{z}_i$
$F_{n,Top} = F_{z,i}$ @ x_{max}

$\Sigma F_{x_i} = -F_N\cos\varphi_i - \mu F_N\sin\varphi_i + m\omega^2 x$
$= m\omega^2 x - F_N(\cos\varphi + \mu\sin\varphi)$
$\Delta x = \frac{1}{m(\Delta\omega)^2}\left(\Delta F_N(\cos\varphi + \mu\sin\varphi) + M\Delta\ddot{x}\right)$

$F_{G_i} = F_{N_i}\sin\varphi_i - f_i\cos\varphi_i - \Sigma F_{z_i}$
$= F_N(\sin\varphi - \mu\cos\varphi) - \Sigma F_z$

Fictitious Forces:

$F_{Centrifugal} = m\omega^2 r$ Needed

→ $\hat{F}$: away from rot. axis

→ Accounts for object's inertia

→ Tendency of the limb to move outward as the top's horiz. surface accelerates toward the z-axis

→ Always present on objects not sitting @ z-axis

$\vec{F}_{Coriolis} = 2m(\vec{v} \times \vec{\omega})$ Not needed, no Δy, use if $\Delta\theta \neq 0$

→ $\hat{F}$: $\perp(\omega \,\&\, \vec{v})$

→ Accounts for tang. $\Delta\vec{v}_i$ as i moves toward or away from z-axis

→ Not needed → y is constrained

→ Could find $\Sigma F_{N,y}$ w/ this

$\vec{F}_{Euler} = -m(\vec{\alpha} \times \vec{F})$ Not needed, no Δy, use if $\Delta\theta \neq 0$

→ $\hat{F}$: Tangential: $(\hat{j})$

→ Accounts for tang. $\vec{a}$ of non-inertial frame as rot. along z-axis changes rates

All that's left now is to select the ball bearing size (depends on required m) and determine the needed angle for balancing the normal force to allow the limb to extend during launch and descend when the spin velocity drops below a threshold value. The launcher design determines the launch torque and launch spin velocity.

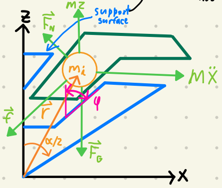
Statics txtbk

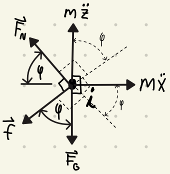
Dyn. txtbk

11/11/2025

Designing Transforming Spinning Tops

Assume higher $f_{ABS} \rightarrow$ Let $\mu_{ABS} = 0.5$





Let $\Delta x = x_{max} - x_{min}$

If $\Delta x = 0$, then static

$\Sigma F_x = 0, \Sigma F_z = 0 \rightarrow \Delta\ddot{x} = 0, \Delta\ddot{z} = 0$

Note *(for FDM printing)*:

- Use 0.4 mm clearance for parts requiring smooth glide
- Use 0.2 mm clearance for a high friction fit
- Use (0.05 - 0.15) mm clearance for snug fit depending on print step size relative to nozzle diameter

- Use 0.02 in for loose fit
- Use 0.01 in for tighter fit that may or may not be snug

ABS Friction:

- Static friction coef. can range within 0.4 - 0.5
- Dynamic friction coef. can range within 0.3 - 0.45
- Measured friction coef. for dry sliding can range within 0.11 - 0.29

$\rightarrow f = \mu F_N$

$\Sigma F_N = 0$

$= F_N - F_G \sin\varphi_i$

$- m_i \ddot{x}_i \cos\varphi_i$

$+ m_i \ddot{z}_i \sin\varphi_i$

$\rightarrow F_N = F_G \sin\varphi_i$

$+ m_i \ddot{x}_i \cos\varphi_i$

$- m_i \ddot{z}_i \sin\varphi_i$

$\rightarrow \hat{N}$-axis defined by φ_i

$m\vec{a}_i = \Sigma \vec{F}_{real} + \Sigma \vec{F}_{fictitious}$

$\Sigma F_z = m_i \ddot{z}_i$

$m_i \ddot{z}_i = F_N \sin\varphi_i - F_G - f\cos\varphi_i$

$\Sigma F_x = m_i \ddot{x}_i$

$m_i \ddot{x}_i = -F_N \cos\varphi_i - f\sin\varphi_i + m\omega^2 x$

$t_{0\rightarrow 1}$: $x = x_{max}$

$\Sigma F_{x_i} \geq 0$

$\Sigma F_{z_i} \geq 0$

$t_{1\rightarrow roll}$: $x = x_{min}$

$\Sigma F_{x_i} \leq 0$

$\Sigma F_{z_i} \leq 0$

Let $F_{x,i} = m_i \ddot{x}_i$

$F_{z,i} = m_i \ddot{z}_i$

$F_{n,Top} = F_{z,i}$

@ x_{max}

$\Sigma F_{x_i} = -F_N \cos\varphi_i - \mu F_N \sin\varphi_i + m\omega^2 x$

$= m\omega^2 x - F_N(\cos\varphi + \mu\sin\varphi)$

$\Delta x = \frac{1}{m(\Delta\omega)^2}\left(\Delta F_N(\cos\varphi + \mu\sin\varphi) + M\Delta\ddot{x}\right)$

$F_{G_i} = F_{N_i}\sin\varphi_i - f_i\cos\varphi_i - \Sigma F_{z_i}$

$= F_N(\sin\varphi - \mu\cos\varphi) - \Sigma F_z$

$$\begin{cases} \Sigma \tau_z = I_z \alpha_z = \vec{r} \times \vec{F} \\ \Sigma H_z = I_z \omega_z = \vec{r} \times \vec{L} = \Sigma H_{z,0} + \int M\,dt \\ \Sigma L_x = m\ddot{x}\,dt = \Sigma L_{x,0} + \int F_x\,dt \end{cases}$$

$\vec{H}_A = \vec{H}_C + \vec{r}_{C/A} \times m\vec{v}_C$ Eq. 21-9

$\cancel{\vec{H}_i = \vec{H}_C + \vec{r}_{C/i} \times m\vec{v}_C}$

$\cancel{\vec{H}_i = \vec{H}_A + \vec{r} \times m\vec{v}_A}$

$\vec{H}_i = \vec{H}_A$

Where $\vec{v}_A$ = Vel. between apex & gnd

$\vec{v}_A = 0$ if no slip

$\Sigma H = (\vec{r} \times \vec{L})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ L_x & L_y & L_z \end{vmatrix} = \hat{i}(yL_z - zL_y) - \hat{j}(xL_z - zL_x) + \hat{k}(xL_y - yL_x)$$

$\Sigma H_z = I_z \omega_z = xL_y - yL_x$

@ $t=0$: $\vec{H}_i = \vec{H}_A = \vec{H}_z = I_z\omega_z = xL_y - yL_x$ when $\hat{z} = \hat{c}$

$m\ddot{x} = \Sigma L_{x,0} + \int F_x\,dt = \Sigma L_x$ $\vec{L}$ @ i

The continuous change in direction for $\vec{v}_t$ requires a const $\vec{a}_{ctpl}$ to maintain a circular trajectory. At every point on & within the spinning top, this acceleration points in toward the spin axis.

$\rightarrow$ Objects without friction that are secured on sides along $\hat{r}$ will slide outward by $\vec{a}_{ctpl} = \frac{v_t}{R}$

$\vec{a}_{ctpl} = \ddot{x}$

$F_{ctpl} = \frac{m_{tot} v_t}{x} = m_i \ddot{x} = -F_N\cos\varphi - f\sin\varphi + m\omega^2 x$ for i where $\omega = \omega_{AB}$

Compare this equation w/

$\rightarrow$ Conservation of Momentum

$\rightarrow$ CAD motion study

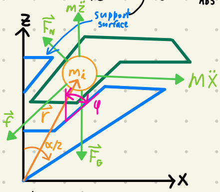
Statics txtbk

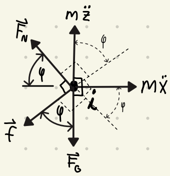
Dyn. txtbk

11/12/2025

Designing Transforming Spinning Tops

Assume higher $f_{ABS} \rightarrow$ Let $\mu_{ABS} = \cancel{0.5}\ 1$





$\vec{a}_{ctpl} = \ddot{x}$

$F_{ctpl} = \frac{m_{net} v_t^2}{x} \cancel{=} m_i \ddot{x} = -F_N \cos\varphi - f \sin\varphi + \underbrace{m\omega^2 x}_{F_{centrif.}}$

If static $\rightarrow \omega = 0,\ \ddot{x} = 0$

$0 = -F_N \cos\varphi - f\sin\varphi$

$f \sin\varphi = -F_N \cos\varphi$

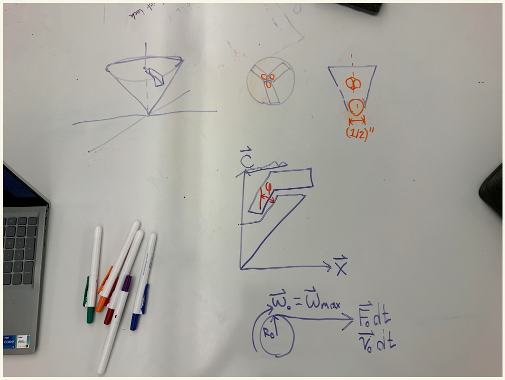
$\mu_{s,max} F_N = -F_N \frac{\cos\varphi}{\sin\varphi}$

$\mu_{s,max} = -\cot(\varphi_{min})$ $\quad \mu_{s,max} = \infty$ @ $\varphi = 0$

Let $\Delta x = x_{max} - x_{min}$

If $\Delta x = 0$, then static

$\Sigma F_x = 0,\ \Sigma F_z = 0 \rightarrow \Delta\dot{x} = 0,\ \Delta\ddot{z} = 0$



Note *(for FDM printing)*:

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- Use 0.02 in for loose fit
- Use 0.01 in for tighter fit that may or may not be snug

ABS Friction:

- Static friction coef. can range within 0.4 - 0.5
- Dynamic friction coef. can range within 0.3 - 0.45
- Measured friction coef. for dry sliding can range within 0.11 - 0.29

11/14/2025

$m\ddot{x} = -F_N(\cos\varphi - \mu\sin\varphi) + m\omega^2 x$

$\ddot{x} = -\frac{F_N}{m}(\cos\varphi - \mu\sin\varphi) + \omega^2 x$

Starting Input Parameters:

We will estimate a child of 8-12 yrs can pull a string with:

- 15 lbf $= F_o$
- 70 in/s $= \dot{x}_o$

$\Sigma H = \hat{i}(yL_z - zL_y) - \hat{j}(xL_z - zL_x) + \hat{k}(xL_y - yL_x)$

$I_z\omega = (\vec{r} \times \vec{L})\hat{k} = \Sigma H_z \longrightarrow \Sigma L_x = \int F_o dt$

$\Sigma H_{x,o} = 0$

$0 = yL_z - zL_y$

$\Sigma H_{y,o} = 0$

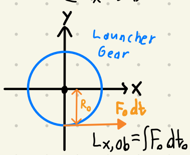
$0 = xL_z - zL_x$

$\Sigma H_{z,o} = I_z\omega$

$I_z\omega = xL_y - yL_x$

$I_z\omega_o = |-R_o F_o dt|$ (ccw)

$\omega_o = \frac{R_o}{I_z}\int_{t_{oa}}^{t_{ob}} F_o dt \quad dt_o =$ spin charge Duration $= t_{ob} - t_{oa}$



$\ddot{x} = \frac{\dot{x}}{dt}$

$F_o = M\ddot{x}_o \quad$ Let $M = \Sigma m$

$= M\frac{\dot{x}_o}{dt}$

$\begin{cases} I_{z,cone} = I_{zz} = I_{z_{cm}} = \frac{3}{10}MR^2 \\ I_{z,pnt} = MR^2 \end{cases}$

$\omega_o = \frac{3R_o}{10R^2}\dot{x}_o$

$\approx \frac{3(1)}{10(3)^2}(70)$

≈ 2.3 rad/s

$\rightarrow \omega_o = \frac{MR_o}{I_z}\dot{x}_{o,Avg}$

Statics txtbk

Dyn. txtbk

$\Sigma F_n = 0$, $\quad$ drm stays on ramp

$F_{N_i} = mg\sin\varphi + m\ddot{x}\cos\varphi - m\ddot{z}\sin\varphi$

$m\ddot{z} = \frac{1}{\sin\varphi}(mg\sin\varphi + m\ddot{x}\cos\varphi - F_N)$

$\Sigma F_z = m\ddot{z}$

$m\ddot{z} = F_N\sin\varphi - mg - \mu F_N\cos\varphi = \frac{1}{\sin\varphi}(mg\sin\varphi + m\ddot{x}\cos\varphi - F_N)$

$m = m_i \approx 8.694\text{E-}4$ lb

Let $\varphi = 45°$

$\mu_{s,max} = \cot 45° = 1$

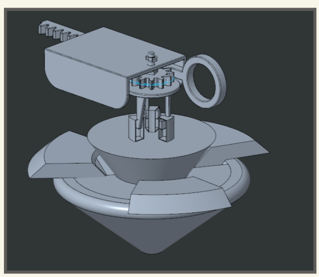
$\ddot{x} = -\frac{F_N}{m}(0) + \omega^2 x$

$\ddot{x}_i = (2.3)^2 x$

$= 5.29\, x_i$

want to move at least 1 in

These parameters should work!



11/21/2025

0.2.2: Project 0 Theory: M.O.I. of Spinning Tops

$M_0 = \int dm = m$
$M_1 = \int r\,dm$
$M_2 = \int r^2 dm = I$
$\vec{r} = \frac{M_1}{M_0}$

$I_{Pnt} = mR^2 = I_\perp$
$= m(x^2+y^2) = I_{zz}$
$= m(x^2+z^2) = I_{yy}$
$= \dots$

$I_{Ring} = \int R^2 dm = mR^2 = I_\perp$
$= m(x^2+y^2) = I_{zz}$
$= \dots$

$= I_{\parallel} \rightarrow$ Perp. Ax. Thrm.
For a planar object, $I_z = I_x + I_y$
$I_\perp = I_\parallel + I_\parallel$

$I_{\parallel, ring} = \frac{1}{2} I_\perp$
$= \frac{1}{2} mR^2$

I_{rod} built from $\infty\, dI_{Pnt}$ using $\lambda = \frac{m}{L}$ thru x
$I_{rod} = \int_{m_1}^{m_2} x^2 dm$, L=Length, $m = m(L)$
$= \frac{1}{12} mL^2$

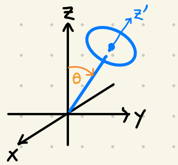
I_{Disk} built from $\infty\, dI_{ring}$ using $\sigma = \frac{m}{A}$ thru R
$I_{Disk} = \frac{1}{2} mR^2$, A=Area

I_{shell} built from $\infty\, dI_{ring}$
$I_{zz} = mR^2$
$I_{xx} = I_{yy}$
$= I_{Ring} + I_{rod}$
$= \underbrace{\frac{1}{2} mR^2}_{\text{radial distr.}} + \underbrace{\frac{1}{12} mL^2}_{\text{L-axis distr.}}$

$\hookrightarrow$ The 'offset' of perp. axis thrm

$$\begin{cases} d\forall_{disk} = (\pi r^2)\,dz \\ dI_{disk} = \frac{1}{2} r^2 dm \\ d\forall_{shell} = (2\pi r)(z)\,dr \\ dI_{shell} = r^2 dm \end{cases}$$

While tilting,
M.O.I. W.R.T. rigid body stays constant
M.O.I. W.R.T. abs. coords are changing
$\rightarrow$ Can be superposed with $H_{z'} \rightarrow H_z$ transfer
Easiest w/ rod & flywheel spinning top



$I_{disk} = \frac{1}{2} mR^2$
$I_{rod} = \frac{1}{12} mL^2$
$I_{xy} = \int_m xy\,dm$
$z' = z\cos\theta + r\sin\theta$
$= z\cos\theta + x\sin\theta$, if $y=0$

$I_{zr} = \int_m zr\,dm$
Eq. 10-10 $\cancel{\tan(2\theta) = \frac{-I_{zr}}{(I_z - I_r)/2}}$
Eq. 10-11 $\cancel{I_{max/min} = \frac{I_z+I_r}{2} \pm \sqrt{\left(\frac{I_z-I_r}{2}\right)^2 + I_{zr}}}$
careful w/ $I_r \equiv I_x$, $I_z \equiv I_x$

$r' = r\cos\theta - z\sin\theta$

Eq. 10-9:
$$\begin{cases} \cancel{I_{z'} = \frac{I_z+I_r}{2} + \frac{I_z-I_r}{2}\cos(2\theta) - I_{zr}\sin(2\theta)} \\ \cancel{I_{r'} = \frac{I_z-I_r}{2}} \\ \cancel{I_{z'r'}} = \\ J_0 = I_{z'} + I_{r'} = I_z + I_r \end{cases}$$

Review Equations 10-9 - 10-11
& inertia tensor transformations

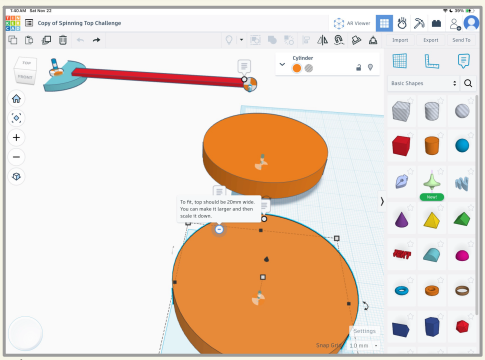
$\bar{I}_{z'} = I_{disk}$
$= \frac{1}{2} mR^2$
$I_{r'} = \frac{1}{2} I_{disk}$ @ h_{disk}
$+ I_{rod}$ @ $\frac{1}{2} h_{disk}$
$\rightarrow$ Apply Parallel Axis Thrm

$\lambda = \frac{m}{L}$
$dm = \lambda\, d\ell$
$m = \int dm$
$= \int_0^{\ell} \lambda\, d\ell$
$I_{rod} = \int_{-\ell}^{\ell} \ell^2 \lambda\, d\ell$
$= \lambda\left[\frac{1}{3}\ell^3\right]$
$= \frac{\lambda}{3}(\ell^3)$
$= \frac{2\lambda h^3}{3}$

Bounds for M.O.I. of entire rod

$I_{z'} = \frac{1}{2} mR^2$ anywhere along z'-axis
$I_{r'} = \frac{1}{4} mR^2 + m(h - h_c) + \cancel{\frac{2\lambda h^3}{3}}$, $\lambda = m_{rod}$ for entire rod
$\cancel{= m\left(\frac{1}{4}R^2 + h - h_c + \frac{2}{3}h^3\right)}$

Tinkercad experiment with timed motor release latch?



$\rightarrow$ Will build more controlled experiment later

R = 10 mm
≈ 0.394 in

3 weeks to perfect my understanding of rigid body spinning tops neglecting aerodynamics.

- **Prototype 0** is the archived group project
- **Prototype 1** is intended to be modular
- **Pretotype 0** will be the classical mechanical scope of the perfect disk and rod spinning tops

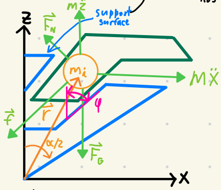
Statics txtbk
Dyn. txtbk

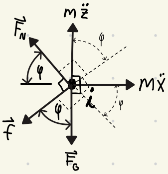
My school group Project CAD Assembly can act as Prototype 0
- The group members didn't want to join the guild

11/25/2025

Transforming Spinning Tops

Assume higher $f_{ABS} \rightarrow$ Let $\mu_{ABS} \in (0.5, 1)$





$\vec{a}_{ctpl} = \ddot{x}$, $\Sigma F_{x_i} = m_i \ddot{x}_i$

$F_{ctpl} = \frac{m_{rot} v_t^2}{x}$ $\pm$ $m_i \ddot{x}_i = -F_N \cos\varphi - f\sin\varphi + \underbrace{m\omega^2 x}_{F_{centrif.}}$

If static $\rightarrow \omega = 0$, $\ddot{x} = 0$

$0 = -F_N \cos\varphi - f\sin\varphi$

$f\sin\varphi = -F_N \cos\varphi$

$\mu_{s,max} F_N = -F_N \frac{\cos\varphi}{\sin\varphi}$

$\mu_{s,max} \geq \cot(\varphi_{min})$

$.5 \geq \cot(\varphi_{min})$

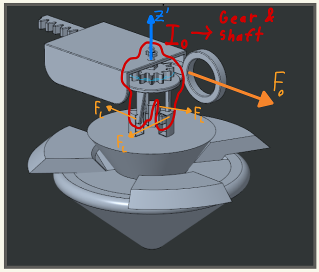
$\operatorname{arccot}\mu = \varphi$

$\arctan(\mu^{-1}) = \varphi$

If $\mu = .5 \rightarrow \varphi \approx 63°$

If $\mu = 1 \rightarrow \varphi = 45$

*Mechanism should work if $\varphi \in (45°, 63°)$

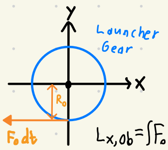


$\Sigma \vec{\tau} = I\vec{\alpha} = \vec{r} \times \vec{F}$

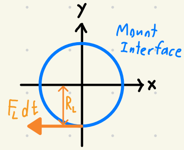
$\Sigma \vec{H} = I\vec{\omega} = \vec{r} \times \vec{L} = \Sigma H_o + \int M dt$

$\Sigma \vec{L} = m\vec{a} dt = \Sigma L_o + \int \vec{F} dt$

& Eq. 21-9 from txtbks



$L_{x,ob} = \int F_o dt_o$



Note:
- Torque is the moment of force that drives acceleration.
- The motor torque is equal to the angular acceleration of the motor and the total summed inertia that is being spun.

$\ddot{x}_i \geq 0$ when extending during charge-up

Let $F_o = 15$ lbf } For ages

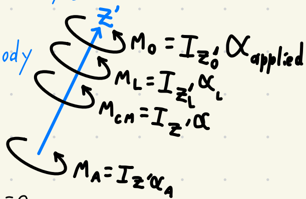
$V_o = 70$ in/s } 8-12 yrs

$M_o = F_o R_o = \tau_o$

$M_o = M_L$ for rigid body

$= F_L R_L$

$\rightarrow F_L = F_o \frac{R_o}{R_L}$



Assuming rigid body

$\rightarrow \Sigma M = \Sigma M_{z'_{o \to A}} = \Sigma FR = 0$

$0 = M_{cm} - I\alpha$

$M_{cm} = \Sigma I_{z'} \ddot{z}'$ @ $t \leq 0$

$\rightarrow \alpha = \alpha_A = \alpha_L = \alpha_{applied}$ @ $t \leq 0$

$\rightarrow \ddot{z}' = \ddot{z}'_A = \ddot{z}'_L = \ddot{z}'_{applied}$

$\Sigma \vec{L} = \vec{F}_o d\vec{t} + \vec{F} dt$

$= \vec{F}_o d\vec{t} - (F_g dt)\hat{z}$

$\Sigma \tau = I\vec{\alpha} = \vec{r} \times \vec{F}$

$\tau_z = \bar{x} F_y - \bar{y} F_x$

$\tau_{z'} = m_i(\ddot{y}_i \bar{x}_i - \ddot{x}_i \bar{y}_i)$

$\Sigma H_z = I_{z'} \omega_{z'} = \vec{r} \times \vec{L}$

$$\vec{r} \times \vec{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \bar{x} & \bar{y} & \bar{z} \\ L_x & L_y & L_z \end{vmatrix}$$

$= (\bar{y}L_z - \bar{z}L_y)\hat{\imath}$

$-(\bar{x}L_z - \bar{z}L_x)\hat{\jmath}$

$+(\bar{x}L_y - \bar{y}L_x)\hat{k}$

$H_z = \bar{x}L_y - \bar{y}L_x$

dist. from z'-axis

Linear Momentum for component

Assumptions:
- No Friction
- No drag or Lift
- Rigid Body

$\tau_{z'} = m_i(\ddot{y}_i \bar{x}_i - \ddot{x}_i \bar{y}_i)$

$= m_i(\ddot{y}_i r_i \cos\phi - \ddot{x}_i r_i \sin\phi)$

$= m_i r_i(\ddot{y}_i \cos\phi - \ddot{x}_i \sin\phi)$

ϕ = Spin Angle

$\tau_z = mr(\ddot{y}\cos\phi - \ddot{x}\sin\phi)$ @ $t \leq 0$ when $\hat{z} = \hat{z}'$, x, y, m of any point of body

$I\alpha = mr(\ddot{r}\sin\ddot{\phi}\cos\phi - \ddot{r}\cos\ddot{\phi}\sin\phi)$ $\quad \ddot{\phi} = \alpha$

$I\ddot{\phi} = mr\ddot{r}(\sin\ddot{\phi}\cos\phi - \underbrace{\cos\ddot{\phi}\sin\phi}_{\text{can I cancel this out?}})$

Eq. 12-19 $a_\perp = \dot{v}$

Eq. 12-20 $\ddot{r} = \frac{v^2}{r}$

12-21 $a = \sqrt{a_\perp^2 + \ddot{r}^2}$

$r^2 = x^2 + y^2$

If $y = 0 \rightarrow r\sin\phi = 0$ @ $\phi = 0$

$\rightarrow I\ddot{\phi} = mr\ddot{r}\sin\ddot{\phi}\cos\phi$

$= mr\ddot{y}\cos\phi$

$= mr\dot{v}\cos\phi$, $\quad \ddot{y} = a_\perp = \dot{v}$

$\vec{L} = m\vec{a}dt$

$H_z = \bar{x}m\ddot{y} - \bar{y}m\ddot{x}$

$\ddot{x} = \frac{\bar{x}m\ddot{y} - H_z}{\bar{y}m}$

$= \frac{1}{\bar{y}}(\bar{x}\ddot{y} - \frac{H_z}{m})$

$= \frac{1}{\bar{y}}(\bar{x}\ddot{y} - \frac{I_z \omega_z}{m})$

$\ddot{x} \propto -\frac{I_z \omega_z}{m_i}$

$\ddot{x}_i = \frac{1}{r\sin\phi}(r\cos\phi\, \ddot{y} - \frac{H_z}{m})$

$= \ddot{y}\cot\phi - \frac{I_z \omega_z}{mr\sin\phi}$

Statics txtbk

Dyn. txtbk

11/25/2025

M.O.I. of Rods vs Cylinders

From txtbk Appendices (I_{cm})

$I_{sphere} = I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$

$\forall = \frac{4}{3}\pi r^3$

$I_{hemis,x} = I_y = 0.259mr^2$

$I_{hemis,z} = \frac{2}{5}mr^2$

$\forall_{hemis} = \frac{2}{3}\pi r^3$

$\bar{r}_{cm} = \frac{3}{8}r\hat{k}$

$I_{rod,xx} = I_{yy} = \frac{1}{12}m\ell^2$

$I_{rod,x'x'} = I_{y'y'} = \frac{1}{3}m\ell^2$

$I_{rod,z'z'} = 0$

$I_{cyl,xx} = I_{yy} = \frac{1}{12}m(3r^2+h^2)$

$I_{cyl,zz} = \frac{1}{2}mr^2$

$\forall_{cyl} = \pi r^2 h$

Note:

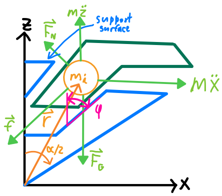
- We want to round out the apex of the spinning top to minimize friction.

Definitions

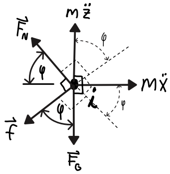
- Rods are simply cylinders or long prisms of any cross-sectional shape that have negligible thickness.

Statics txtbk

Dyn. txtbk



z
$m\ddot{z}$
$\vec{F}_N$
support surface
m_i
$m\ddot{x}$
$\vec{f}$
$\vec{r}$
φ
$\alpha/2$
$\vec{F}_G$
x



$m\ddot{z}$
$\vec{F}_N$
φ
$m\ddot{x}$
i
$\vec{f}$
$\vec{F}_G$